

WORKING PAPER

Welfare analysis of discrete choices based on a direct utility function

Abstract

Discrete choice analysis is a cornerstone of modern day transportation economics. It facilitates the analysis and prediction of individual's transportation choices as well as the computation of welfare and willingness to pay (WTP) metrics for economic appraisal.

In this paper, I develop an approach which enables the derivation of welfare and willingness to pay (WTP) measures for various specifications of the deterministic parts of a random utility specification of a discrete-choice model. Unlike previous approaches such as Small and Rosen (1981), my approach is based on a *direct* utility function. I can show that using this approach, I am able to derive the so-called Logsum measure and the WTP measure used for Multinomial Logit models (MNL) in a very natural way and thus much more easily, when compared to the current approach which is based on an indirect utility function. Moreover, I can show that the approach based on an indirect utility function as proposed by Small and Rosen (1981) and others is not consistent with the direct utility function foundations used in mainstream microeconomic models. I show that the assumptions concerning the impact of the income and the prices on the indirect utility function in Small and Rosen (1981) are too lax.

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1. Introduction

Small and Rosen (1981), in a landmark paper, derive a welfare economic basis for discrete choice analysis. They do so by first considering the discrete-continuous case where households can choose to consume a commodity that is available in a continuous quantity but only exist in a small number of mutually exclusive varieties. This means that one good is consumed in a positive amount whereas the others are not consumed. One application would be to choose a specific drink from a menu in a restaurant and subsequently order a specific amount of it. Second, they examine the discrete-choice case where households choose between mutually exclusive goods that are only available in a fixed quantity. They derive the choice probability and the so-called Logsum formula using the same modelling framework that they used for the discrete-continuous choice case, by imposing some additional assumptions. The whole procedure for proving these formulas involves splitting the demand function into a probability function and a conditional demand function, applying the “Roy’s identity” theorem to derive the conditional uncompensated demand function, applying two other theorems, imposing some additional assumptions and computing an integral.

In contrast to their approach, which is based on the notion of an indirect utility function, I show that the choice probability as well as the Logsum formula for a discrete choice Logit model can, alternatively, be derived in a much more straightforward way if the analysis is based upon a direct utility function. Moreover, I show that the function that Small and Rosen used to describe the deterministic component of the household’s utility is too lax in terms of the restrictions imposed when incorporating the household’s income and the prices of the alternatives. I show that the income and the price of the alternative must enter in terms of their difference in the corresponding deterministic component and that they must enter always in the exact same way by a function that increases and is concave or weakly concave in its arguments.

Section 2 presents the theoretical framework concerning which specific welfare utility function is chosen. Section 3 shows how the welfare measure, the Logsum function, is derived based upon this theoretical framework. Section 4 highlights and discusses the difference of the results compared to Small and Rosen (1981). In Section 5 I compare approaches based on an indirect utility function with the approach I am proposing that is based on a direct utility function. Section 6 draws conclusions.

2. Theoretical framework

In this case, the theoretical framework is based upon a direct utility function. It is assumed that the household maximises its utility subject to its budget constraint. I assume that the household’s income is exogenous; thus, it does not try to change its income as a result of changes in the wage rate or changes in consumption due to changes in goods’ prices or qualities. Generally, the utility has some general properties, namely that it is strictly increasing in all goods and that it is (quasi) concave. This includes the notion that the marginal utility of one good can be affected by the level of consumption of another good. In this case, the utility function only valuates the utility of a set of discrete goods $x_1, x_2, \dots, x_J$ and the so-called numeraire good x_n , which is the bundle of all goods apart from these discrete goods and has the price one, $p_n = 1$. It is assumed that the household spends its entire income y . Therefore, the household spends the income net the cost p_i for the discrete good x_n that it chooses, $y - p_i \cdot x_i$. Note that the utility of the numeraire good can also be considered as the utility obtainable from the remaining income. The utility function in its general form is as follows:

$$u = u(y - p_1 x_1 - p_2 x_2 - \dots - p_J x_J, x_1, x_2, \dots, x_J). \quad (1)$$

The household then chooses the discrete good x_i such that the utility (1) is at its maximum. Note that in the context of this model, it may only choose one option $x_i = 1$ out of $x_1, x_2, \dots, x_J$, while all the others will be zero. Thus, $x_1, x_2, \dots, x_J$ are all dummy variables. Therefore, the household maximizes the following function:

$$\max_i u_i, \quad (2)$$

$$\text{where } u_i = u(x_n = y - p_i x_i, x_1 = 0, x_2 = 0, \dots, x_{i-1} = 0, x_i = 1, x_{i+1} = 0, \dots, x_J = 0). \quad (2a)$$

Note that despite the fact that u_i is solely a function of the income y and the price p_i , it is not an indirect utility function conditional upon the choice of the good x_i in the general microeconomic sense. If it was an indirect utility function, this would imply that this function would result from maximizing a utility function subject to a budget constraint by choosing the optimum amount of consumption of the goods x_i and x_n . However, in this case the household may not choose the optimum amount, since it may only choose good i , $x_i = 1$ or not choose it, $x_i = 0$. Since only one of the goods can be consumed, the utility function u will only take as many levels as there are different goods. Therefore, the utility function (2) can be formulated in an additive form without a loss of generality.

$$U = u_n(y - p_1 X_1 - p_2 X_2 - \dots - p_J X_J) + (W_1 + \sigma \varepsilon_1) \cdot X_1 + (W_2 + \sigma \varepsilon_2) \cdot X_2 + \dots + (W_J + \sigma \varepsilon_J) \cdot X_J, \quad (3)$$

Given that the household chooses the alternative i , the utility function (3) collapses to the conditional utility function:

$$U_i = V_i + \sigma \varepsilon_i, \quad (4)$$

$$\text{where } V_i = u_n(y - p_i) + W_i, \quad (4a)$$

$$\text{and } W_i = W_i(z_i, s). \quad (4b)$$

The term V_i is the deterministic component of the utility specific to the good i . It comprises attributes related to the alternative i , z_i , and sociodemographic variables s specific to household, which enter to V_i via a function W_i that is in the general case specific to the good type i , and the term $u_n(y - p_i)$, which maps the household's income and the price to the term V_i . It is important that the term $y - p_i$ must enter into V_i in the same way for each good type i . Generally, $u_n(\bullet)$ may be a non-linear function increasing function with decreasing returns. Note that $u_n(\bullet)$ and $W_i(\bullet)$ are parametrized functions and their parameters will be estimated by using observed data. In many cases, the functions $W_i(\bullet)$ are chosen to be a linear function of the arguments. A commonly-used specification is that the attributes specific to the alternative i , z_i , enter linearly and by the same weights β in $W_i(\bullet)$.

$$W_i = W(z_i, s) = \beta \cdot z_i + \gamma_i \cdot s. \quad (5)$$

The random variables ε_i account for the fact that the researcher cannot know the exact preferences of the households and thus he can only predict the household's choice with a certain probability.

$$P_i = P\left(U_i > \max_{j \in J_i} U_j\right) = P\left(\varepsilon_i > \max_{j \in J_i} \left\{\sigma^{-1}V_j - \sigma^{-1}V_i + \varepsilon_j\right\}\right), J_i = \{1..J\} \setminus i. \quad (6)$$

As researchers, we want to explain as much of the household's decision by the deterministic component V_i . This means that the standard deviation of the error term should be as small as possible. In (6), the standard deviation can be defined by the use of σ when estimating the parameters.⁹ Imagine that, apart from the standard deviation, the error terms ε_i might also have a certain mean I would like to calibrate. However, from (6), it can be seen that the mean cannot be identified, since only the differences between the error terms matter. This means that the probabilities P_i remain unchanged if I change the mean of all ε_i . I assume that the error terms ε_i are iid standard Gumbel distributed, which corresponds to the Multinomial Logit model (MNL). Thus, the following probability function follows:

$$P_i = \frac{e^{\sigma^{-1}V_i}}{\sum_{j=1}^J e^{\sigma^{-1}V_j}}. \quad (7)$$

Note that the parameter σ is often not estimated explicitly; for instance, in the very commonly-used choice of the functional form (4) where $W_i(\cdot)$ is linear in the parameters as in (5), estimating σ can be omitted, since in this case all the parameters β and γ_i will simply increase by factor σ , resulting in the exact same impact of all z_i and S on $W_i(\cdot)$. By contrast, in this case the parameter σ may never be omitted when estimating the parameters defining the function $u_n(\cdot)$, unless $u_n(\cdot)$ is already parametrized such that one of the parameters enters as a multiplier into $u_n(\cdot)$. For instance, this is the case where the non-linearity of $u_n(\cdot)$ is modelled by spline regression, or as a constant multiplied by a non-linear function such as the square root. Note that some authors use a linear approximation for $u_n(\cdot)$; for instance, Train (1993; 30) uses the term $\theta \cdot y^{-1} \cdot p_i$, which approximates $\theta \cdot u_n(y - p_i) = \theta \cdot \ln(y - p_i)$,¹⁰ where the parameter θ measures the term σ^{-1} . This is feasible if the prices of the different alternatives are sufficiently small compared to the income level. This also includes the case where the researcher assumes that $u_n(\cdot)$ is linear, in which case $\theta \cdot u_n(y - p_i)$ reduces to $\theta \cdot p_i$.

⁹ Note that the expression $\varepsilon_i > \left\{\sigma^{-1}V_j - \sigma^{-1}V_i + \varepsilon_j\right\}_{j \in J_i}$ is equivalent to $\sigma\varepsilon_i > \left\{V_j - V_i + \sigma\varepsilon_j\right\}_{j \in J_i}$. From this it can be seen that σ defines the standard deviation of the random term $\sigma\varepsilon_i$.

¹⁰ Note that the correct approximation is in fact $u_n(y - p_i) = \sigma^{-1} \cdot \ln(y - p_i) \approx \sigma^{-1} \cdot (\ln(y) + y^{-1} \cdot (y - p_i)) \approx \sigma^{-1} \cdot y^{-1} \cdot p_i$. However, since the term $\sigma^{-1} \cdot (\ln(y) + y^{-1} \cdot y) = \sigma^{-1} \cdot (\ln(y) + 1)$ does not create any difference in the term $\varepsilon_i > \left\{\sigma^{-1}V_j - \sigma^{-1}V_i + \varepsilon_j\right\}_{j \in J_i}$ used in (5), it can be omitted.

3. Deriving the welfare measures

In the following, I will compute the welfare measures Logsum and the WTP. They are derived directly from the direct utility framework (3), which proves to be a rather straightforward procedure. Note that in contrast, Small and Rosen (1981) derive the choice probability and the so-called Logsum formula using the same modelling framework that they used for the discrete-continuous choice case, by imposing some additional assumptions. The whole procedure for proving these formulas involves splitting the demand function into a probability function and a conditional demand function, applying the “Roy’s identity” theorem to derive the conditional uncompensated demand function, applying two other theorems, imposing some additional assumptions and computing an integral.

The starting point of the approach that I use is based upon the direct utility function (4). I use the fact that U_i will be the utility provided to the households in the event that the household will choose the good i . This will be the case if U_i is greater than any other utility of the choice set, $U_i > \max_{j \in J_i} U_j$. Thus, the utility the household gains conditional on ε is as follows:

$$U(\varepsilon) = U_1(\varepsilon_1) \cdot I\left(U_1(\varepsilon_1) > \max_{j \in J_1} U_j(\varepsilon_j)\right) + U_2(\varepsilon_2) \cdot I\left(U_2(\varepsilon_2) > \max_{j \in J_1} U_j(\varepsilon_j)\right) + \dots \quad (8)$$

$$\dots + U_J(\varepsilon_J) \cdot I\left(U_J(\varepsilon_J) > \max_{j \in J_J} U_j(\varepsilon_j)\right).$$

Note that (7) is identical to the maximum of the sequence $\{U_j(\varepsilon_j)\}_{j \in 1 \dots J}$, which has the following distribution:

$$U(\varepsilon) = \max\{U_j(\varepsilon_j)\}_{j \in 1 \dots J}, \text{ with } F = \exp(-e^{(z-D)}), D = \ln\left(\sum_{j=1}^J e^{\sigma^{-1}v_j}\right). \quad (9)^{11}$$

Computing the expectation value of (9) yields the Log-sum formula:

$$\text{Logsum} = E(U(\varepsilon)) = \gamma + \ln\left(\sum_{j=1}^J e^{\sigma^{-1}v_j}\right), \quad (10)$$

where $\gamma = 0.5772156649\dots$ is the Euler-Mascheroni constant.

I now want to compute the effect on welfare and thus the Logsum if an attribute of a good marginally changes, given that the individuals can freely chose an alternative. I will relate this value to the change in income necessary to compensate for the change in Logsum.

¹¹ For the distribution of the maximum of two independent Gumbel distributed random variables, see Ben-Akiva and Lerman (1985; 121).

$$\frac{dy}{dz_{ik}} = \left(\frac{\partial u_n}{\partial y} \right)^{-1} \cdot P_i \cdot \frac{\partial W_i}{\partial z_{ik}}, \quad (11)^{12}$$

where the index ik indicates the k -th attribute of the i -th alternative.

What is known as the willingness to pay (WTP) refers to the case where the Logsum measure is kept constant by changing the price of the alternative for which an attribute is changed.

$$wtp_{ik} = \frac{dp_i}{dz_{ik}} = - \frac{\partial W_i}{\partial z_{ik}} \bigg/ \frac{\partial u_n}{\partial p_i}. \quad (12)$$

Note that if the change in W_i is compensated by a change in the price p_i , the choice probabilities also remain unchanged. For the case where W_i is linear in the parameters as in (5), the term $\partial W_i / \partial z_{ik}$ reduces to β_k . Furthermore, if the function u_n is linear, the result (11) is equivalent to the negative of the ratio of parameter β_k and the parameter β_p relating to the prices p_i , $-\beta_k / \beta_p$; see for instance Train (2003; 47, 49). Note that in the context of the modelling structure that I present here, the parameter β_p corresponds to σ^{-1} .

One question to answer is whether it is feasible to use this framework to compare the utility of an initial situation of a number of J goods with a situation where one good is modified, since one of its attributes has been modified. Note that we must apply the same utility function to compare the utility of two different situations. The answer to this question is yes: imagine the case where W_i depends only on attributes that relate to the alternative i , $W_i = W_i(z_i, s)$. In the initial situation the utility function captures all goods $i=1..J$ plus one good $J+1$, which corresponds to a good that has the same attributes as x_i apart from the attribute k , which has the level $z_{ik} + dz_{ik}$. Subsequently, in the initial situation, the household may only consider choosing from the choice set $j=1..J$ and in the new situation from the choice set $\{1..J\} \setminus i, J+1$. Since the utility is measured by use of the same utility function in both situations, the comparison is feasible.

4. Differences from Small and Rosen (1981)

Small and Rosen (1981) propose the following general indirect utility, conditional upon the choice of good i , upon which their welfare analysis is based:

$$u_i = v_i(p_i, y, z_i, s, \varepsilon_i) = L(y) + W_i(p_i, y, z_i, s) + \varepsilon_i. \quad (13)^{13}$$

$$^{12} - \frac{\partial E(U(\varepsilon)) / \partial z_{ik}}{\partial E(U(\varepsilon)) / \partial y} = - \frac{e^{\sigma^{-1} V_i} \cdot \frac{1}{\sigma} \cdot \frac{\partial W_i(\cdot)}{\partial z_{ik}}}{\sum_{i=1}^J e^{\sigma^{-1} V_i}} \bigg/ \frac{\sum_{i=1}^J e^{\sigma^{-1} V_i} \cdot \frac{1}{\sigma} \cdot \frac{\partial u_n(\cdot)}{\partial y}}{\sum_{i=1}^J e^{\sigma^{-1} V_i}} = - \frac{P_i \cdot \frac{\partial W_i(\cdot)}{\partial z_{ik}}}{\sum_{i=1}^J P_i \cdot \frac{\partial u_n(\cdot)}{\partial y}} = - \frac{P_i \cdot \frac{\partial W_i(\cdot)}{\partial z_{ik}}}{\partial u_n(\cdot) / \partial y}.$$

¹³ This is slightly adapted from Small and Rosen (1981; 123), formula (5.1), in order to keep the notation consistent with the notation I use.

Note that there is a difference in the approach proposed in this paper compared to the assumptions of Small and Rosen: The income y and the price of the alternative p_i may not enter as a separate argument into the deterministic component of the utility $\sigma^{-1}V_i$. Their assumption that the marginal effect of the income is independent of the price p_i also does not change this: the marginal effect of the income on u_i may generally be different from minus the marginal effect of the price p_i , which would need to be the case of (4a).

The conclusion from the modelling framework that I propose is that – in contrast to Small and Rosen – the Logsum welfare measure is only compatible with the traditional microeconomic consumer theory, if the economic variables y and p_i may only enter as a difference $y - p_i$ and by the same function in the deterministic part of the utility of an alternative.

5. The use of indirect utility functions versus use of direct utility functions

In this section, I want to examine whether it is preferable to use an approach based upon an indirect utility function or a direct utility function, as well as whether it is feasible to use an approach based upon an indirect utility function. Indeed, I start by answering the latter question.

A conditional indirect utility function with respect to good i can be regarded as indicating the maximum utility that a household can achieve if it can only consume the numeraire good x_n and good x_i of the other goods $\{x_j\}_{j=1..J}$. In this case, it is assumed that the household maximizes a utility function $u_i = u(x_n, x_1=0, x_2=0, \dots, x_{i-1}=0, x_i, x_{i+1}=0, \dots, x_J=0)$ subject to the budget constraint $y = x_n + p_i \cdot x_i$ by choosing the optimum quantity for x_i . If the household may only choose one good x_i , it will choose the option that provides the highest utility. The problem is that in this case the resulting utility level is the result of a choice of x_i on a continuum, whereas in the case of discrete-choice analysis the value x_i might either be zero or one. However, there is a special case of a conditional indirect utility function that is compatible with a household always choosing $x_i=1$, namely (4).¹⁴ Therefore, the model and the conditional indirect utility function (4) that follows from this is the only possible specification that has additive random terms and is compatible with microeconomic assumptions and the corresponding welfare measures. Therefore, there is no contradiction between the approach based upon the direct utility function (3) and an approach based upon an indirect utility function, given that the indirect utility function is (4).

6. Conclusions

The main finding from this paper is that the choice of the functional form of the indirect utility form is restricted. I found that from this approach – which is based upon a direct utility function – the implied indirect utility function differs from the one proposed by Small and Rosen (1981): The difference is that – in contrast to Small and Rosen (1981) – the economic variables y and p_i may only enter as a difference $y - p_i$ and by the same function in the deterministic part of the utility of an alternative.

¹⁴ For a proof I apply Roy's identity: $x_i(y, p_i) = -\frac{\partial U_i / \partial p_i}{\partial U_i / \partial y} = -\frac{\partial (u_n(y - p_i) + W_i + \sigma \varepsilon_i) / \partial p_i}{\partial (u_n(y - p_i) + W_i + \sigma \varepsilon_i) / \partial y} = -\frac{u'_n(y - p_i)}{u'_n(y - p_i) \cdot (-1)} = 1$.

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