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THE HYDRODYNAMIC COEFFICIENTS OT TWO PARALLEL IDENTICAL CYLINDERS OSCILLATING IN THE FREE SURFACE.

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THE HYDRODYNAMIC COEFFICIENTS OF TWO PARALLEL IDENTICAL CYLINDERS OSCILLATING IN THE FREE SURFACE*)

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Summary.

In the present report expressions are derived for the hydrodynamic coefficients of two identical rigidly connected parallel cylinders of infinite length which perform harmonic oscillations in the free fluid surface. The cylinders are supposed to have only connections above the free surface.

The method applied in this report is in fact an extension of Ursell's method for the corresponding problem of one cylinder.

Preface.

In this report basic data are given for the determination according to the strip theory of the hydrodynamic coefficients of catamarans performing harmonic oscillations.

The reader is supposed to be familiar to a certain extent with Ursell's method to evaluate the hydrodynamic coefficients of a single cylinder, oscillating in the free surface. This method is in fact the starting point for the determination of the hydrodynamic coefficients of a 'single-body' vessel according to the strip theory. For this the reader is referred to the papers of Ursell [1, 2], Tasai [3, 4] or Porter [5].

Their results have been compiled and supplemented by the present author in his earlier report [6]. It will be seen that many methods. which are eeded for the solution of the double-cylinder problem, are identical to those of the single-cylinder problem and, therefore, they will be mentioned in this report only very concisely. In order to aid the reader who wants to acquaint himself with these methods, the present author will, at some places in this report, refer with much detail to his above-mentioned report.

Introduction.

The last few years several catamarans have been constructed for all kinds of purposes. This is largely due to the possibility to construct these vessels in such a way that the resistance for high F_n is much lower as compared with conventional ships with the same deck area. In this connection it should be observed that we can give the floating bodies of the catamaran shapes which can't exist as separate single body vessels. However, experiences with recent designs showed that catamarans have larger heaving and pitching motions as compared with conventional ships. These phenomena justify a theoretical analysis of the motions of a catamaran. The present report gives a basis to determine the hydrodynamic coefficients of such a ship with the strip theory method which has proved to give very useful results for single body vessels. Analogous to the single body vessel the catamaran is divided up into a number of sections and for each section, which is taken to have a constant profile, the hydrodynamic properties are determined, assuming that the disturbances in the fluid due to the motions of the sections only propagate in the direction perpendicular to its longitudinal axes.

The catamaran is assumed to be composed of two identical floating bodies which have been rigidly connected above the free surface. Consequently, for the apllication of the strip theory method we need expressions for the hydrodynamic coefficients of a system of two infinitely long identical parallel cylinders which have been rigidly connected above the free surface at a given finite distance.

Analogous to the single cylinder, this problem

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is formulated as a linearized boundary value problem from the potential theory which is achieved by assuming the fluid to be inviscid incompressible and irrotational and the amplitudes of the motions of the system to be very small.

The velocity potential is again composed of a linear combination of a source or dipole potential and a number of multipole potentials such that the boundary conditions are satisfied. However, the expressions for the source, dipole and multipole potentials have to be adapted to the conditions which result from the presence of two cylinders.

Further, analogous to the single-cylinder problem, also for the problem of two cylinders we have to determine a conformal transformation which maps a circular cylinder into the cross-section of one of these cylinders. Since suitable numerical techniques are available to devise such a transformation, we will consider in this report cylinders with an arbitrary shape, [see 6, section 4.1].

1. Formulation of the problem.

We assume that a system of two parallel identical infinitely long cylinders, which are rigidly connected above the free surface at a distance l from each other, carries out a harmonic one-dimensional oscillation with frequency σ, while the mean position of the axes of both cylinders is in the undisturbed free surface of the fluid. The origin of the rectangular Cartesian coordinates (x, y) is taken in the mean position of the axis of the right-hand cylinder, (Figure 1.1). The x-axis is horizontal and perpendicular to the axis of the cylinder, the y-axis vertical, positive in downward direction.

As possible modes of oscillation we consider heaving, swaying and rolling about the point P which is in the free surface and in the symmetry

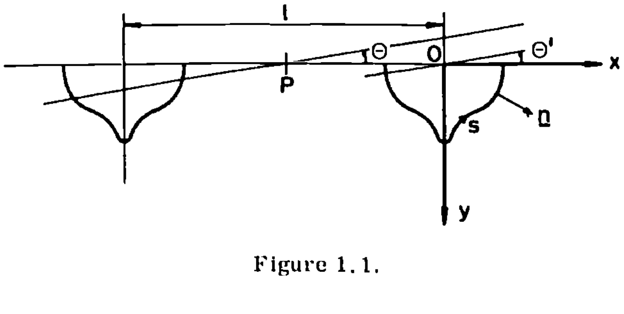

Figure 1.1.

plane $x = -\frac{1}{2}l$ of the system. With respect to the rolling motion, represented by $\Theta = \Theta_a \cos(\sigma t+\gamma)$, it is readily seen that this motion involves a combined heaving, swaying and rolling motion of the separate cylinders. For the right-hand cylinder we find the following ordinates of these components, respectively:

$$y_\Theta = \frac{1}{2} l \sin\Theta$$
$$x_\Theta = \frac{1}{2} l (\cos\Theta - 1) \qquad (1.1)$$
$$\Theta' = \Theta$$

Expanding the sine and cosine functions in these formulas and retaining only the first term we obtain:

$$y_\Theta = \frac{1}{2} l\Theta$$
$$x_\Theta = \frac{1}{4} l\Theta^2 \qquad (1.2)$$
$$\Theta' = \Theta$$

Since the swaying component is a second order quantity, we shall conceive in this report the rolling motion of the double cylinder system as a combined heaving and rolling motion of the separate cylinders. The velocity potential for the rolling motion of the system will consist of a component due to the heaving and another due to the rolling motion of the separate cylinders. Analogous to the single cylinder problem [6, ch 1], a velocity potential $\Phi(x, y, t)$ has to be determined which is a solution of a linearized boundary value problem from the potential theory. Consequently, we may write:

$$\Phi(x, y, t) = -i\varphi(x, y)\, e^{i\sigma t} \qquad (1.3)$$

where $\varphi(x, y)$ is a solution of the equation of Laplace:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad (1.4)$$

and satisfies, in addition, the following conditions:

(i) the linearized free-surface condition:

$$k\varphi + \frac{\partial \varphi}{\partial y} = 0 \quad \text{when } y = 0 \qquad (1.5)$$

in which $k = \frac{\sigma^2}{g}$ represents the wave number.

(ii) the radiation condition:

$$
\begin{aligned}
\varphi &\rightarrow C_1 e^{-ky-ikx}\,(1 + e^{-ikl}) \quad \text{as } x \rightarrow \infty \\
\varphi &\rightarrow C_2 e^{-ky+ikx}\,(1 + e^{-ikl}) \quad \text{as } x \rightarrow -\infty
\end{aligned}
\qquad (1.6)
$$

where C_1 and C_2 are constants.

This condition implies that every disturbance in the fluid vanishes when $y \rightarrow \infty$

(iii) the boundary condition on the cylinder contour:

$$\frac{\partial\varphi}{\partial n} = U_n\,(x, y) \quad \text{when } y = 0 \qquad (1.7)$$

where U_n denotes the normal outward velocity on the cylinder surface, (Figure 1.1). We observe that the above condition refers to the mean position of the system, since the linearized case is considered.

(iv) when the system is carrying out a heaving motion, the potential $\varphi(x,y)$ has to be a symmetric function with respect to the plane $x = -\frac{1}{2}l$, for swaying and rolling the potential is skew-symmetric.

(v) for $l = 0$ the potentials for the heaving, swaying and rolling motion of the double-cylinder system become equivalent with the potentials for the respective motions of one cylinder.

2. Outline of the method of solution.

Analogous to the single-cylinder problem the velocity potential for the two cylinders is also synthesized of a source or dipole potential and a linear combination of multipole potentials. The source, dipole and multipole potentials are chosen such that the conditions (i), (ii), (iv) and (v) are satisfied by each potential separately, while the condition (iii) is satisfied by choosing the linear combination in an appropriate way. In the following chapters we will derive for each mode of oscillation of the double-cylinder system adequate expressions for the above mentioned potentials. It turns out that these potentials are easily derived from the corresponding single-cylinder potentials. It is well-known that the expressions for the multipole potentials, (see (3.2) and (4.2)), for the single-cylinder problem depend on the parameters a, a_1, a_3, ... which are the coefficients in the following transformation formulas, [6; eq. (4.1.8)]:

$$
\begin{aligned}
x &= a\left\{ r\sin\theta + \sum_{n=0}^{M} (-1)^n \frac{a_{2n+1}}{r^{2n+1}} \sin(2n+1)\theta \right\} \\
y &= a\left\{ r\cos\theta + \sum_{n=0}^{M} (-1)^{n+1} \frac{a_{2n+1}}{r^{2n+1}} \cos(2n+1)\theta \right\}
\end{aligned}
\qquad (2.1)
$$

The coefficients a, a_1, a_3, ... are determined such that the semi-unit circle ($r=1$, $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$) in the reference plane (ζ-plane), in which the polar coordinate system (r, θ) has been defined, is mapped onto the cross-section in the physical plane (z-plane) with cartesian coordinates (x, y), [6; section 4.1]. The number of terms M, considered in the equations (2.1), determines the accuracy of the transformation.

The formulas (2.1) can also be interpreted as defining a curvilinear coordinate system (r, θ) in the physical plane such that one of the coordinate lines ($r=1$) coincides with the cross-section.

In addition to the rectangular coordinate

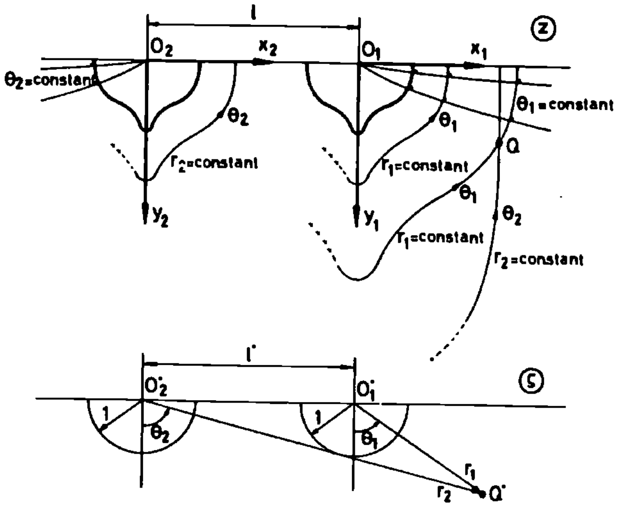

Figure 2.1.

system (x,y), as defined in the previous chapter, which has its origin O in the mean position of the axis of the right-hand cylinder, we introduce in this chapter an identical coordinate system which has its origin in the mean position of the axis of the left-hand cylinder. The first-mentioned system is denoted here by (x_1, y_1) with origin O_1, the other by (x_2, y_2) with origin O_2, (Figure 2.1). The corresponding polar coordinate systems in the ζ-plane or curvilinear coordinate systems in the z-plane are denoted by (r_1, θ_1) and (r_2, θ_2), respectively. According to (2.1) the following relations are valid between the rectangular and curvilinear coordinates in the z-plane:

$$x_i = a\left\{ r_i \sin\theta_i + \sum_{n=0}^{M} (-1)^n \frac{a_{2n+1}}{r_i^{2n+1}} \sin(2n+1)\theta_i \right\}$$

$$y_i = a\left\{ r_i \cos\theta_i + \sum_{n=0}^{M} (-1)^{n+1} \frac{a_{2n+1}}{r_i^{2n+1}} \cos(2n+1)\theta_i \right\} \qquad i = 1,2. \qquad (2.2)$$

The distance l' between the two semi-circles can be determined from the relation which is obtained by inserting in the first equation of (2.2) either $x_2 = l$, $\theta_2 = \frac{\pi}{2}$ and $r_2 = l'$ or $x_1 = -l$, $\theta_1 = -\frac{\pi}{2}$ and $r_1 = l'$. This yields:

$$l = a\left\{ l' + \sum_{n=0}^{M} \frac{a_{2n+1}}{l'^{2n+1}} \right\} \qquad (2.3)$$

where $l' > 2$

Consider a point Q in the z-plane, which is the image point of the point Q' in the ζ-plane. The point Q is represented by the rectangular coordinates (x_1, y_1) and the curvilinear coordinates (r_1, θ_1), which both refer to the origin O_1. The image point Q' in the ζ-plane is given by the polar coordinates (r_1, θ_1) with origin O'_1. However, this point can also be represented by the polar coordinates (r_2, θ_2) with origin O'_2. It is easy to see from Figure 2.1 that the following relations are valid between the polar coordinate systems (r_1, θ_1) and (r_2, θ_2):

$$\theta_2 = \arctan \frac{l' + r_1 \sin\theta_1}{r_1 \cos\theta_1}$$

$$r_2 = \sqrt{r_1^2 + l'^2 + 2 r_1 l' \sin\theta_1} \qquad (2.4)$$

It is clear that the equations (2.4) can also be interpreted as representing the relations between the curvilinear coordinate systems (r_1, θ_1) and (r_2, θ_2) in the z-plane which are very useful in our future calculations.

3. Added mass and damping for the heaving motion.

It is clear that the source and the multipole potentials, which are used for the solution of the heaving problem of one cylinder, can't be used for the solution of our problem here since the symmetry condition (iv) is not satisfied. However, it will be seen that the set of potentials, which satisfy each the condition (iv) are easily derived from the single-cylinder potentials. The source potential φ^s and the symmetric multipole potentials φ^s_{2m} used for the single-cylinder problem are given by:

$$\varphi^s(x,y) = \frac{gb}{\pi\sigma}\left\{ \int_0^\infty e^{-\beta|x|} \frac{(k \sin\beta y - \beta \cos\beta y)}{k^2 + \beta^2}\, d\beta + i\pi e^{-ky - ik|x|} \right\} \qquad (3.1)$$

and

$$\varphi^s_{2m}(r,\theta) = \frac{\cos 2m\theta}{r^{2m}} + ka\left\{ \frac{\cos(2m-1)\theta}{(2m-1)r^{2m-1}} + \sum_{n=0}^{N} (-1)^n \frac{(2n+1)a_{2n+1}\cos(2m+2n+1)\theta}{(2m+2n+1)r^{2m+2n+1}} \right\} \qquad (3.2)$$

$m = 1, 2, 3, \ldots$

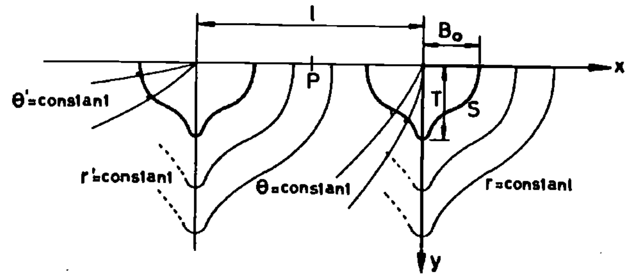

Figure 3.1.

In the first formula b represents the wave height at infinity due to the oscillation of the cylinder while the parameters $a, a_1, a_3, \ldots\ldots$ in the second formula represent the coefficients in the transformation formulas (2.1).

The corresponding potentials for the double-cylinder problem, which will be denoted by the superscript c, are defined by:

$$ {}^{c}\varphi^{s}(x,y) = \frac{1}{2}\left\{ \varphi^{s}(x,y) + \varphi^{s}(x+l,y) \right\} \tag{3.3}$$

and

$$ {}^{c}\varphi^{s}_{2m}(r,\theta) = \frac{1}{2}\left\{ \varphi^{s}_{2m}(r,\theta) + \varphi^{s}_{2m}(r',\theta') \right\} \tag{3.4}$$

where, according to (2.4), by identifying (r',θ') with (r_2,θ_2) and (r,θ) with (r_1,θ_1):

$$\theta' = \arctan\frac{l' + r\sin\theta}{r\cos\theta}$$

$$r' = \sqrt{r^2 + l'^2 + 2l' r\sin\theta} \tag{3.5}$$

in which l' and l satisfy the relation (2.3).

Since the conditions (i) and (ii) are satisfied by the potentials φ^s and φ^s_{2m} separately, the corresponding potentials ${}^{c}\varphi^{s}$ and ${}^{c}\varphi^{s}_{2m}$ for the double-cylinder problem satisfy also these conditions. Further, we can easily verify that these potentials satisfy the symmetry condition (iv) and the condition (v) for $l=0$.

By using the Cauchy-Riemann relations we determine the conjugate streamfunctions ψ^s and ψ^s_{2m} of the potentials φ^s and φ^s_{2m}, respectively. In rectangular coordinates these relations have the form:

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}$$

$$\frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x} \tag{3.6}$$

and in polar coordinates:

$$\frac{\partial\varphi}{\partial r} = -\frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} \tag{3.7}$$

consequently:

$$\psi^s(x,y) = \frac{gb}{\pi\sigma}\left\{ \pm\int_0^\infty e^{-\beta|x|}\cdot\frac{k\cos\beta + \beta\sin\beta y}{k^2+\beta^2}\,d\beta \mp \pi e^{-ky-ik|x|}\right\} \qquad x \gtrless 0 \tag{3.8}$$

and

$$\psi^s_{2m}(r,\theta) = \frac{\sin 2m\theta}{r^{2m}} + ka\left\{ \frac{\sin(2m-1)\theta}{(2m-1)r^{2m-1}} + \sum_{n=0}^{N}(-1)^n \frac{(2n+1)a_{2n+1}\sin(2m+2n+1)\theta}{(2m+2n+1)r^{2m+2n+1}}\right\} \tag{3.9}$$

The streamfunctions ${}^{c}\psi^{s}(x,y)$ and ${}^{c}\psi^{s}_{2m}(r,\theta)$ for the double-cylinder system are given by:

$$ {}^{c}\psi^{s}(x,y) = \frac{1}{2}\left\{ \psi^{s}(x,y) + \psi^{s}(x+l,y) \right\} \tag{3.10}$$

and

$$ {}^{c}\psi^{s}_{2m}(r,\theta) = \frac{1}{2}\left\{ \psi^{s}_{2m}(r,\theta) + \psi^{s}_{2m}(r',\theta') \right\} \tag{3.11}$$

Expression (3.10) is clear without further preface.

The validity of (3.11) is easy to show by observing that:

$$\frac{\partial\varphi^s_{2m}(r',\theta')}{\partial r} = \frac{\partial\varphi^s_{2m}(r',\theta')}{\partial r'}\cdot\frac{r + l\sin\theta}{r'} \tag{3.12}$$

and

$$\frac{\partial \psi^{s}_{2m}(r',\theta')}{\partial \theta} = \frac{\partial \psi^{s}_{2m}(r',\theta')}{\partial \theta'} \cdot \frac{r(r+l\sin\theta)}{r'^{2}} \qquad (3.13)$$

where $\psi^{s}_{2m}(r',\theta')$ is defined such, that the Cauchy-Riemann relation

$$\frac{\partial \varphi^{s}_{2m}(r',\theta')}{\partial r} = \frac{1}{r}\frac{\partial \psi^{s}_{2m}(r',\theta')}{\partial \theta} \quad \text{holds.}$$

Then it can be shown that also:

$$\frac{\partial \varphi^{s}_{2m}(r',\theta')}{\partial r'} = \frac{1}{r'}\frac{\partial \psi^{s}_{2m}(r',\Theta')}{\partial \theta'} \qquad (3.14)$$

which proves that (3.11) is correct.

The velocity potential for the heaving motion of the system is written in the form:

$$^{c}\Phi(r,\theta,t) = \frac{gb}{\pi\sigma}\left[\left\{ {}^{cs}\varphi_{c} + \sum_{m=1}^{N} p_{2m}\, {}^{cs}\varphi_{2m} \right\}\cos\sigma t + \left\{ {}^{cs}\varphi_{s} + \sum_{m=1}^{N} q_{2m}\, {}^{cs}\varphi_{2m} \right\}\sin\sigma t\right] \qquad (3.15)$$

where

$$^{cs}\varphi_{c}(x,y) = \frac{1}{2}\left\{ {}^{s}\varphi_{c}(x,y) + {}^{s}\varphi_{c}(x+l,y) \right\}$$

and

$$^{cs}\varphi_{s}(x,y) = \frac{1}{2}\left\{ {}^{s}\varphi_{s}(x,y) + {}^{s}\varphi_{s}(x+l,y) \right\} \qquad (3.16)$$

while

$$^{s}\varphi_{c}(x,y) = \pi e^{-ky}\cos kx$$

$$^{s}\varphi_{s}(x,y) = \pi e^{-ky}\sin k|x| - \int_{o}^{\infty} \frac{e^{-\beta|x|}}{k^{2}+\beta^{2}} (\beta\cos\beta y - k\sin\beta y)\,d\beta \qquad (3.17)$$

in which the coordinates (x,y) and (r,θ) are related by the formulas (2.1).

For the conjugate streamfunction $^{c}\Psi(r,\theta,t)$ we find:

$$^{c}\Psi(r,\theta,t) = \frac{gb}{\pi\sigma}\left[\left\{ {}^{cs}\psi_{c} + \sum_{n=1}^{N} p_{2m}\, {}^{cs}\psi_{2m} \right\}\cos\sigma t + \left\{ {}^{cs}\psi_{s} + \sum_{n=1}^{N} q_{2m}\, {}^{cs}\psi_{2m} \right\}\sin\sigma t\right] \qquad (3.18)$$

where

$$^{cs}\psi_{c}(x,y) = \frac{1}{2}\left\{ {}^{s}\psi_{c}(x,y) + {}^{s}\psi_{c}(x+l,y) \right\} \qquad (3.19)$$

and

$$^{cs}\psi_{s}(x,y) = \frac{1}{2}\left\{ {}^{s}\psi_{s}(x,y) + {}^{s}\psi_{s}(x+l,y) \right\}$$

in which:

$$^{s}\psi_{c}(x,y) = \pi e^{-ky}\sin kx$$

$$^{s}\psi_{s}(x,y) = \pm \int_{o}^{\infty} \frac{e^{-\beta|x|}}{k^{2}+\beta^{2}} \left\{ \beta\sin\beta y + k\cos\beta y \right\} d\beta \mp \pi e^{-ky}\cos kx \quad x \gtrless o \qquad (3.20)$$

The values of the coefficients p_{2m} and q_{2m} in (3.15) and (3.18) are determined in such a way that the boundary condition (iii) on the contour of the cylinders is satisfied. The value of N determines the accuracy of the approximation of the velocity potential by (3.15).

In virtue of the symmetry of the system with respect to $x = -\frac{1}{2}l$, it is sufficient to restrict our discussions with respect to the determination of p_{2m} and q_{2m} to the right-hand cylinder.

The boundary condition on the contour of this cylinder is given by

$$\frac{\partial\, {}^{c}\Phi}{\partial n} = \frac{dy}{dt}\cos\alpha \quad \text{or} \quad -\frac{\partial\, {}^{c}\Psi}{ds} = \frac{dy}{dt}\frac{\partial x}{\partial s} \qquad (3.21)$$

where α is the angle between the positive normal on the cross-section and the positive y-axis, (Figure 3.2).

Analogous to the single-cylinder problem [6, (4.2.8)..........(4.2.13)], this relation can be reduced to

$$^{c}\Psi(r=1,\theta) = -\frac{dy}{dt}\, x(r=1,\theta) \qquad (3.22)$$

Substituting $\theta = 0$, yields:

$$^{c}\Psi(1,\frac{\pi}{2}) = -\frac{dy}{dt} B_{o} \qquad (3.23)$$

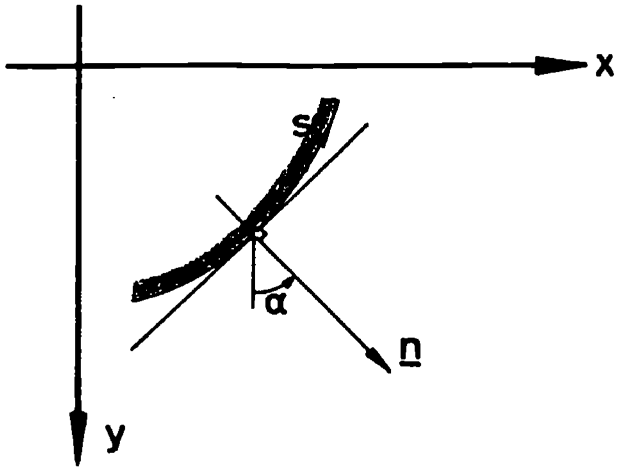

Figure 3.2.

Eliminating $\frac{dy}{dt}$ from (3.22) and (3.23), we obtain:

$$ {}^{c}\psi(1,\theta) = \frac{x(1,\theta)}{B_o}\, {}^{c}\psi(1,\tfrac{\pi}{2}) \qquad (3.24)$$

Substituting (3.18) in this expression, we find a set of 2N linear equations for the coefficients p_{2m} and q_{2m}:

$$ {}^{cs}\psi_c(1,\theta) - \frac{x(1,\theta)}{B_o}\, {}^{cs}\psi_c(1,\tfrac{\pi}{2}) = \sum_{m=1}^{N} p_{2m} f_{2m}(1,\theta)$$

$$ {}^{cs}\psi_s(1,\theta) - \frac{x(1,\theta)}{B_o}\, {}^{cs}\psi_s(1,\tfrac{\pi}{2}) = \sum_{m=1}^{N} q_{2m} f_{2m}(1,\theta) \qquad (3.25)$$

where

$$ f_{2m}(1,\theta) = \frac{x(1,\theta)}{B_o}\, {}^{cs}\psi_{2m}(1,\tfrac{\pi}{2}) - {}^{cs}\psi_{2m}(1,\theta) \qquad (3.26)$$

It is observed that the set of equations (3.25) has to be solved for the range $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$.

The velocity potential at the contour of the cylinder (r=1) is written as:

$$ {}^{c}\Phi(1,\theta,t) = \frac{gb}{\pi\sigma}(M \sin\sigma t + N\cos\sigma t) \qquad (3.27)$$

where

$$ M(\theta) = {}^{cs}\varphi_s(1,\theta) + \sum_{m=1}^{N} q_{2m}\, {}^{cs}\varphi_{2m}(1,\theta)$$

$$ N(\theta) = {}^{cs}\varphi_s(1,\theta) + \sum_{m=1}^{N} p_{2m}\, {}^{cs}\varphi_{2m}(1,\theta) \qquad (3.28)$$

According to the relation $p(r,\theta) = -\rho\frac{\partial\, {}^{c}\Phi}{\partial t}$, the pressure distribution along the contour of the cylinder can be written as:

$$ p(1,\theta) = -\frac{\rho g b}{\pi}(M\cos\sigma t - N\sin\sigma t) \qquad (3.29)$$

We define:

$$ \frac{dy}{dt} = \frac{gb}{\pi\sigma B_o}(-A\cos\sigma t - B\sin\sigma t) \qquad (3.30)$$

where in virtue of (3.23):

$$ A = {}^{cs}\psi_c(1,\tfrac{\pi}{2}) + \sum_{m=1}^{N} p_{2m}\, {}^{cs}\psi_{2m}(1,\tfrac{\pi}{2})$$

$$ B = {}^{cs}\psi_s(1,\tfrac{\pi}{2}) + \sum_{m=1}^{N} q_{2m}\, {}^{cs}\psi_{2m}(1,\tfrac{\pi}{2}) \qquad (3.31)$$

Then the pressure along the cylinder contour can be written in the following form:

$$ p(1,\theta) = \rho B_o \frac{MB+NA}{A^2+B^2}\ddot{y} + \rho B_o \sigma \frac{MA-NB}{A^2+B^2}\dot{y} \qquad (3.32)$$

The total vertical force per unit length on both cylinders becomes:

$$ F_y = -2\int_{S(-\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2})} p(1,\theta)\cos\alpha\, ds \qquad (3.33)$$

Analogous to the single-cylinder problem [6; eqs. (4.2.21),..........(4.2.26)], this expression can be reduced to:

$$ F_y = -2\rho B_o^2 \frac{M_o B + N_o A}{A^2+B^2}\ddot{y} - 2\rho B_o^2 \sigma \frac{M_o A - N_o B}{A^2+B^2}\dot{y} \qquad (3.34)$$

where

$$ M_o = \int_{-\pi/2}^{\pi/2} M(\theta)\frac{W(\theta)}{G}\, d\theta \qquad (3.35)$$

$$N_o = \int_{-\pi/2}^{\pi/2} N(\theta)\frac{W(\theta)}{G}d\theta \tag{3.35}$$

in which

$$W(\theta) = \cos\theta + \sum_{n=o}^{N} (-1)^n (2n+1)\, a_{2n+1} \cos(2n+1)\theta$$

$$G = 1 + \sum_{n=o}^{N} a_{2n+1} \tag{3.36}$$

For the added mass M_y and damping N_y per unit length we find:

$$M_y = 2\rho B_o^2 \frac{M_o B + N_o A}{A^2 + B^2} \tag{3.37}$$

$$N_y^2 = 2\rho\sigma B_o^2 \frac{M_o A - N_o B}{A^2 + B^2} \tag{3.38}$$

4. Added mass and damping for swaying; coupling coefficients of swaying into rolling.

The dipole and skew-symmetric multipole potentials φ^d and φ^a_{2m}, used for finding the velocity potential for the single-cylinder problem, are given by:

$$\varphi^d(x,y) = \frac{gb}{\pi\sigma}\left\{ \frac{x}{k(x^2+y^2)} \mp \int_o^\infty e^{-\beta|x|} \frac{[k\cos\beta y + \beta\sin\beta y]}{k^2+\beta^2} d\beta \pm \pi e^{-ky-ik|x|} \right\} \quad x \gtrless o \tag{4.1}$$

$$\varphi^a_{2m}(r,\theta) = \frac{\sin(2m+1)\theta}{r^{2m+1}} + ka\left\{ \frac{\sin 2m\theta}{2mr^{2m}} + \sum_{n=o}^{N} \frac{(-1)^n a_{2n+1}(2n+1)\sin(2m+2n+2)\theta}{(2m+2n+2)r^{2m+2n+2}} \right\} \tag{4.2}$$

The conjugate streamfunctions of these potentials are given by:

$$\psi^d(x,y) = \frac{gb}{\pi\sigma}\left\{ -\frac{y}{k(x^2+y^2)} - \int_o^\infty e^{-\beta|x|} \frac{\beta\cos\beta y - k\sin\beta y}{k^2+\beta^2} d\beta + i\pi e^{-ky-ik|x|} \right\} \tag{4.3}$$

$$\psi^a_{2m}(r,\theta) = \frac{-\cos(2m+1)\theta}{r^{2m}} - ka\left\{ \frac{\cos 2m\theta}{2mr^{2m}} + \sum_{n=o}^{N} \frac{(-1)^n a_{2n+1}(2n+1)\cos(2m+2n+2)\theta}{(2m+2n+2)r^{2m+2n+2}} \right\} \tag{4.4}$$

The rectangular coordinates (x,y) and curvilinear coordinates (r,θ) in these formulas relate to the right-hand cylinder of the system as indicated in Figure 3.1. The dipole potential ${}^c\varphi^d(x,y)$ and the multipole potentials ${}^c\varphi^a_{2m}(r,\theta)$ for the double-cylinder problem are now defined by:

$${}^c\varphi^d(x,y) = \frac{1}{2}\left\{ \varphi^d(x,y) + \varphi^d(x+l,y) \right\} \tag{4.5}$$

$${}^c\varphi^a_{2m}(r,\theta) = \frac{1}{2}\left\{ \varphi^a_{2m}(r,\theta) + \varphi^a_{2m}(r',\theta') \right\} \tag{4.6}$$

where (r,θ) and (r',θ') represent the curvilinear coordinate systems which relate, respectively, to the right-hand and left-hand cylinder of the system in the physical plane (Figure 3.1).

Between the coordinate systems (r,θ) and (r',θ') exist the relations (3.5). Analogous to the heaving problem we find for the corresponding streamfunctions:

$${}^c\psi^d(x,y) = \frac{1}{2}\left\{ \psi^d(x,y) + \psi^d(x+l,y) \right\} \tag{4.7}$$

$${}^c\psi^a_{2m}(r,\theta) = \frac{1}{2}\left\{ \psi^a_{2m}(r,\theta) + \psi^a_{2m}(r',\theta') \right\} \tag{4.8}$$

For the complete potential ${}^c\Phi$ and conjugate streamfunction ${}^c\psi$ we write:

$$ {}^{c}\Phi(r,\theta,t) = \frac{gb}{\pi\sigma}\left[\left\{ {}^{cd}\varphi_{c} + \sum_{m=1}^{N} p_{2m}\,{}^{ca}\varphi_{2m}\right\}\cos\sigma t + \left\{ {}^{cd}\varphi_{s} + \sum_{m=1}^{N} q_{2m}\,{}^{ca}\varphi_{2m}\right\}\sin\sigma t\right] \tag{4.9}$$

and

$$ {}^{c}\Psi(r,\theta,t) = \frac{gb}{\pi\sigma}\left[\left\{ {}^{cd}\Psi_{c} + \sum_{m=1}^{N} p_{2m}\,{}^{ca}\Psi_{2m}\right\}\cos\sigma t + \left\{ {}^{cd}\Psi_{s} + \sum_{m=1}^{N} q_{2m}\,{}^{ca}\Psi_{2m}\right\}\sin\sigma t\right] \tag{4.10}$$

where:

$$\begin{aligned}
{}^{cd}\varphi_{c}(x,y) &= \frac{1}{2}\left\{ {}^{d}\varphi_{c}(x,y) + {}^{d}\varphi_{c}(x+1,y)\right\}\\
{}^{cd}\varphi_{s}(x,y) &= \frac{1}{2}\left\{ {}^{d}\varphi_{s}(x,y) + {}^{d}\varphi_{s}(x+1,y)\right\}\\
{}^{cd}\Psi_{c}(x,y) &= \frac{1}{2}\left\{ {}^{d}\Psi_{c}(x,y) + {}^{d}\Psi_{c}(x+1,y)\right\}\\
{}^{cd}\Psi_{s}(x,y) &= \frac{1}{2}\left\{ {}^{d}\Psi_{s}(x,y) + {}^{d}\Psi_{s}(x+1,y)\right\}
\end{aligned} \tag{4.11}$$

in which

$$ {}^{d}\varphi_{c}(x,y) = -\pi e^{-ky}\sin kx$$

$$ {}^{d}\varphi_{s}(x,y) = \pm\pi e^{-ky}\cos kx \mp \int_{0}^{\infty} e^{-\beta|x|}\frac{k\cos\beta y + \beta\sin\beta y}{k^{2}+\beta^{2}}\,d\beta + \frac{x}{k(x^{2}+y^{2})} \qquad x \gtrless 0 \tag{4.12}$$

and

$$ {}^{d}\Psi_{c}(x,y) = \pi e^{-ky}\cos kx$$

$$ {}^{d}\Psi_{s}(x,y) = \pi e^{-ky}\sin k|x| - \int_{0}^{\infty} e^{-\beta|x|}\frac{\beta\cos\beta y - k\sin\beta y}{k^{2}+\beta^{2}}\,d\beta - \frac{y}{k(x^{2}+y^{2})}$$

The coefficients p_{2m} and q_{2m} are chosen such that the boundary condition on the contour of the cylinder is satisfied. It is easy to verify that expression (4.9) satisfies also the other condition mentioned in chapter 1.

In virtue of the skew-symmetry of the system with respect to $x = -\frac{1}{2}l$, it is again sufficient to restrict our discussions to the right-hand cylinder. From Figure 3.2 we derive that the boundary condition on the surface of this cylinder is given by:

$$\frac{\partial\,{}^{c}\Phi}{\partial n} = \frac{dx}{dt}\sin\alpha \quad\text{or}\quad \frac{\partial\,{}^{c}\Psi}{\partial s} = \frac{dx}{dt}\frac{dy}{ds} \tag{4.13}$$

Analogous to the single-cylinder problem [6, section 5.2], we can reduce this relation to

$$ {}^{c}\Psi(r=1,\theta) - {}^{c}\Psi(r=1,\tfrac{\pi}{2}) = \frac{dx}{dt}\,y(r=1,\theta) \tag{4.14}$$

Substituting $\theta = o$ yields:

$$ {}^{c}\Psi(1,0) - {}^{c}\Psi(1,\tfrac{\pi}{2}) = \frac{dx}{dt}T \tag{4.15}$$

where T is the draft of the cylinder.

Eliminating $\frac{dx}{dt}$ from (4.14) and (4.15) yields:

$$\frac{1}{y(1,\theta)}\left\{ {}^{c}\Psi(1,\theta) - {}^{c}\Psi(1,\tfrac{\pi}{2})\right\} = \frac{1}{T}\left\{ {}^{c}\Psi(1,0) - {}^{c}\Psi(1,\tfrac{\pi}{2})\right\} \tag{4.16}$$

Inserting (4.10), we obtain the following set of linear equations for the coefficients p_{2m} and q_{2m}:

$$\begin{aligned}
&\left\{ {}^{cd}\Psi_{c}(1,\theta) - {}^{cd}\Psi_{c}(1,\tfrac{\pi}{2})\right\} - \frac{y(1,\theta)}{T}\left\{ {}^{cd}\Psi_{c}(1,0) - {}^{cd}\Psi_{c}(1,\tfrac{\pi}{2})\right\} = \sum_{m=1}^{N} p_{2m} f_{2m}\\
&\left\{ {}^{cd}\Psi_{s}(1,\theta) - {}^{cd}\Psi_{s}(1,\tfrac{\pi}{2})\right\} - \frac{y(1,\theta)}{T}\left\{ {}^{cd}\Psi_{s}(1,0) - {}^{cd}\Psi_{s}(1,\tfrac{\pi}{2})\right\} = \sum_{m=1}^{N} q_{2m} f_{2m}
\end{aligned} \tag{4.17}$$

where

$$ f_{2m} = \frac{y(1,\theta)}{T}\left\{ {}^{ca}\Psi_{2m}(1,0) - {}^{ca}\Psi_{2m}(1,\tfrac{\pi}{2})\right\} -$$

$$- \left\{ {}^{c}\psi^{a}_{2m}(1,\theta) - {}^{c}\psi^{a}_{2m}(1,\tfrac{\pi}{2}) \right\} \qquad (4.18)$$

Next, we define:

$$\frac{dx}{dt} = \frac{gb}{\pi\sigma T}\left\{ - A\cos\sigma t - B\sin\sigma t \right\} \qquad (4.19)$$

Then, according to (4.10) and (4.15):

$$A = {}^{c}\psi^{d}_{c}(1,\tfrac{\pi}{2}) - {}^{c}\psi^{d}_{c}(1,0) + \sum_{m=1}^{N} p_{2m}\left\{ {}^{c}\psi^{a}_{2m}(1,\tfrac{\pi}{2}) - {}^{c}\psi^{a}_{2m}(1,0) \right\}$$

$$B = {}^{c}\psi^{d}_{s}(1,\tfrac{\pi}{2}) - {}^{c}\psi^{d}_{s}(1,0) + \sum_{m=1}^{N} q_{2m}\left\{ {}^{c}\psi^{a}_{2m}(1,\tfrac{\pi}{2}) - {}^{c}\psi^{a}_{2m}(1,0) \right\} \qquad (4.20)$$

The potential along the contour of the right-hand cylinder is defined in the following way:

$${}^{c}\Phi(1,\theta) = \frac{gb}{\pi\sigma}(M\sin\sigma t + N\cos\sigma t) \qquad (4.21)$$

Then

$$M = {}^{c}\varphi^{d}_{s}(1,\theta) + \sum_{m=1}^{N} q_{2m}\, {}^{c}\varphi^{a}_{2m}(1,\theta)$$

$$N = {}^{c}\varphi^{d}_{c}(1,\theta) + \sum_{m=1}^{N} p_{2m}\, {}^{c}\varphi^{a}_{2m}(1,\theta) \qquad (4.22)$$

The pressure along this cylinder can be written in the form:

$$p(1,\theta) = -\frac{\rho g b}{\pi}(M\cos\sigma t - N\sin\sigma t) \qquad (4.23)$$

or by using (4.19):

$$p(1,\theta) = \rho T\frac{MB+NA}{A^2+B^2}\ddot{x} + \rho T\sigma\frac{MA-NB}{A^2+B^2}\dot{x} \qquad (4.24)$$

The total horizontal hydrodynamic force on the system is given by:

$$F_x = -2\int_{S(-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2})} p(1,\theta)\sin\alpha\, ds \qquad (4.25)$$

Analogous to the single-cylinder problem, [6, section 5.2], this formula can be reduced to:

$$F_x = -2B_0\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p(1,\theta)\frac{V(\theta)}{G}\,d\theta \qquad (4.26)$$

where

$$V(\theta) = -\sin\theta + \sum_{n=0}^{N}(-1)^n a_{2n+1}(2n+1)\sin(2n+1)\theta$$

$$G = 1 + \sum_{n=0}^{N} a_{2n+1} \qquad (4.27)$$

Substituting (4.24), we obtain:

$$F_x = -2\rho T B_0\frac{M_0B+N_0A}{A^2+B^2}\ddot{x} - 2\rho T B_0\sigma\frac{M_0A-N_0B}{A^2+B^2}\dot{x} \qquad (4.28)$$

where

$$N_0 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} N(\theta)\frac{V(\theta)}{G}\,d\theta$$

$$M_0 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} M(\theta)\frac{V(\theta)}{G}\,d\theta \qquad (4.29)$$

For the added mass of the system per unit length we find:

$$M_x = 2\rho T B_0\frac{M_0B+N_0A}{A^2+B^2} \qquad (4.30)$$

and for the damping:

$$N_x = 2\rho T B_0\sigma\frac{M_0A-N_0B}{A^2+B^2} \qquad (4.31)$$

We consider now the rolling moment on the system about the point P due to the swaying motion:

$$M_{RS} = -2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p(1,\theta)\left\{ (\tfrac{1}{2}l + x)\frac{\partial x}{\partial\theta} + y\frac{\partial y}{\partial\theta} \right\} d\theta \qquad (4.32)$$

where the moment is considered to be positive when it is in clockwise direction. Substituting (4.23), we obtain:

$$M_{RS} = \frac{2B_o^2 \rho g b}{\pi} \left\{ -X_R \sin\sigma t + Y_R \cos\sigma t \right\} \quad (4.33)$$

where

$$X_R = \frac{1}{B_o^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} N(\theta) \left\{ (x + \frac{1}{2}l)\frac{\partial x}{\partial \theta} + y\frac{\partial y}{\partial \theta} \right\} d\theta$$

$$Y_R = \frac{1}{B_o^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} M(\theta) \left\{ (x + \frac{1}{2}l)\frac{\partial x}{\partial \theta} + y\frac{\partial y}{\partial \theta} \right\} d\theta \quad (4.34)$$

Using (4.19), we can write (4.33) in the following form:

$$M_{RS} = -2\rho T B_o^2 \frac{BY_R + AX_R}{A^2 + B^2}\ddot{x} - 2\rho\sigma T B_o^2 \frac{AY_R - BX_R}{A^2+B^2}\dot{x} \quad (4.35)$$

Hence, for the added moment of inertia and the damping per unit length for the rolling motion produced by swaying we obtain:

$$I_{RS} = 2\rho T B_o^2 \frac{BY_R + AX_R}{A^2 + B^2} \quad (4.36)$$

$$N_{RS} = 2\rho\sigma T B_o^2 \frac{AY_R - BX_R}{A^2 + B^2} \quad (4.37)$$

5. Added moment of inertia and damping for rolling; coupling coefficients of rolling into swaying.

As mentioned before in chapter 1, we can conceive in the linearized case the rolling motion of the system, $\Theta = \Theta_a \cos(\sigma t + \gamma)$, as a combined heaving and rolling motion of the separate cylinders, accoring to

$$y_\Theta = \frac{1}{2} l \Theta$$
$$\Theta' = \Theta \quad (5.1)$$

The heaving motions of the two separate cylinders have 180° phase difference while the rolling motions are in phase.

Contribution of the heaving component.

In contrast with the case of pure heaving, considered in chapter 3, the source potential ${}^c\varphi^s$ and multipole potentials ${}^c\varphi^s_{2m}$ have to be skew-symmetric with respect to $x = -\frac{1}{2}l$.

Therefore, we define:

$${}^c\varphi(x,y) = \frac{1}{2}\left\{ \varphi^s(x,y) - \varphi^s(x+l,y) \right\}$$
$${}^c\varphi_{2m}(r,\theta) = \frac{1}{2}\left\{ \varphi^s_{2m}(r,\theta) - \varphi^s_{2m}(r',\theta') \right\} \quad (5.2)$$

where φ^s and φ^s_{2m} are defined by (3.1) and (3.2).

In an analogous manner we define the complete potential by

$${}^c\Phi(r,\theta,t) = \frac{gb}{\pi\sigma}\left[\left\{ {}^c\varphi_c + \sum_{m=1}^{N} p_{2m}\, {}^c\varphi_{2m} \right\} \cos\sigma t + \left\{ {}^c\varphi_s + \sum_{m=1}^{N} q_{2m}\, {}^c\varphi_{2m} \right\} \sin\sigma t \right] \quad (5.3)$$

in which

$${}^c\varphi_c(x,y) = \frac{1}{2}\left\{ \varphi^s_c(x,y) - \varphi^s_c(x+l,y) \right\}$$
$${}^c\varphi_s(x,y) = \frac{1}{2}\left\{ \varphi^s_s(x,y) - \varphi^s_s(x+l,y) \right\} \quad (5.4)$$

where φ^s_c and φ^s_s are defined by (3.17).

The conjugate streamfunction becomes:

$${}^c\Psi(r,\theta,t) = \frac{gb}{\pi\sigma}\left[\left\{ {}^c\psi_c + \sum_{m=1}^{N} p_{2m}\, {}^c\psi_{2m} \right\} \cos\sigma t + \left\{ {}^c\psi_s + \sum_{m=1}^{N} q_{2m}\, {}^c\psi_{2m} \right\} \sin\sigma t \right] \quad (5.5)$$

where

$$
\begin{aligned}
{}^{c}\psi_{c}(x,y) &= \frac{1}{2}\left\{ {}^{s}\psi_{c}(x,y) - {}^{s}\psi_{c}(x+1,y) \right\} \\
{}^{c}\psi_{s}(x,y) &= \frac{1}{2}\left\{ {}^{s}\psi_{s}(x,y) - {}^{s}\psi_{s}(x+1,y) \right\} \qquad (5.6) \\
{}^{c}\psi_{2m}(r,\theta) &= \frac{1}{2}\left\{ {}^{s}\psi_{2m}(r,\theta) - {}^{s}\psi_{2m}(r',\theta') \right\}
\end{aligned}
$$

in which ${}^{s}\psi_{c}$ and ${}^{s}\psi_{s}$ are defined by (3.20) and ${}^{s}\psi_{2m}$ by (3.9).

The coefficients p_{2m} and q_{2m} in the expressions (5.2) and (5.4) are determined from a set of linear equations similar to (3.25). However, the streamfunctions ${}^{s}\psi_{c}$, ${}^{s}\psi_{s}$ and ${}^{s}\psi_{2m}$ in these equations have to be replaced by ${}^{c}\psi_{c}$, ${}^{c}\psi_{s}$ and ${}^{c}\psi_{2m}$, respectively, which are defined by (5.5).

Analogous to expression (3.32), we find for the pressure distribution along the cylinder contour:

$$
p_H(1,\theta) = \rho B_o \frac{MB+NA}{A^2+B^2}\ddot{y} + \rho B_o \sigma \frac{MA-NB}{A^2+B^2}\dot{y} \qquad (5.7)
$$

where:

$$
\begin{aligned}
A &= {}^{c}\psi_{c}(1,\tfrac{\pi}{2}) + \sum_{m=1}^{N} p_{2m}\, {}^{c}\psi_{2m}(1,\tfrac{\pi}{2}) \\
B &= {}^{c}\psi_{s}(1,\tfrac{\pi}{2}) + \sum_{m=1}^{N} q_{2m}\, {}^{c}\psi_{2m}(1,\tfrac{\pi}{2}) \\
M &= {}^{c}\varphi_{s}(1,\theta) + \sum_{m=1}^{N} q_{2m}\, {}^{c}\varphi_{2m}(1,\theta) \qquad (5.8) \\
N &= {}^{c}\varphi_{c}(1,\theta) + \sum_{m=1}^{N} p_{2m}\, {}^{c}\varphi_{2m}(1,\theta)
\end{aligned}
$$

Inserting, according to (5.1), $\frac{dy}{dt} = \frac{1}{2} l \dot{\Theta}$ and $\frac{d^2y}{dt^2} = \frac{1}{2} l \ddot{\Theta}$ we find:

$$
p_H(1,\theta) = \frac{\rho B_o l}{2}\frac{MB+NA}{A^2+B^2}\ddot{\Theta} + \frac{\rho B_o \sigma l}{2}\frac{MA-NB}{A^2+B^2}\dot{\Theta} \qquad (5.9)
$$

The rolling moment on the system about P is found to be:

$$
M_R = -2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p_H(1,\theta)\left\{ (x+\tfrac{1}{2}l)\frac{\partial x}{\partial \theta} + y\frac{\partial y}{\partial \theta} \right\} d\theta \qquad (5.10)
$$

Substituting (5.9), we find:

$$
M_R = -\rho B_o l \frac{M_oB+N_oA}{A^2+B^2}\ddot{\Theta} - \rho B_o \sigma l \frac{M_oA-N_oB}{A^2+B^2}\dot{\Theta} \qquad (5.11)
$$

where

$$
\begin{aligned}
M_o &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} M(\theta)\left\{ (x+\tfrac{1}{2}l)\frac{\partial x}{\partial \theta} + y\frac{\partial y}{\partial \theta} \right\} d\theta \\
N_o &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} N(\theta)\left\{ (x+\tfrac{1}{2}l)\frac{\partial x}{\partial \theta} + y\frac{\partial y}{\partial \theta} \right\} d\theta
\end{aligned} \qquad (5.12)
$$

The added moment of inertia I_R^H and damping N_R^H per unit length due to the heaving component is given by:

$$
I_R^H = \rho B_o l \frac{M_oB+N_oA}{A^2+B^2} \qquad (5.13)
$$

$$
N_R^H = \rho B_o \sigma l \frac{M_oA-N_oB}{A^2+B^2} \qquad (5.14)
$$

Contribution of the rolling component.

It is readily seen that in this case the potentials and streamfunctions can be chosen identically to those of the pure swaying motion, defined by (4.5), ..., (4.8).

The complete potential and streamfunction have the form (4.9) and (4.10). However, here the coefficients p_{2m} and q_{2m} have to be determined such that the following boundary condition on the contour of the right-hand cylinder is valid, [6, section 5.3]:

$$\frac{\partial\,{}^c\Phi}{\partial n} = R\frac{d\Theta}{dt}\frac{dR}{ds} \quad \text{or} \quad -\frac{\partial\,{}^c\Psi}{\partial s} = \frac{d\Theta}{dt}\frac{d}{ds}\left(\frac{1}{2}R^2\right) \qquad (5.15)$$

where $R = \left\{ x^2(1,\theta) + y^2(1,\theta) \right\}^{\frac{1}{2}}$ is the distance between the origin O and the point (x, y) on the contour of the right-hand cylinder and s the line coordinate along this contour (Figure 1.1).

In the usual manner we reduce (5.15) to:

$${}^c\Psi(1,\theta) - {}^c\Psi(1,\tfrac{\pi}{2}) = -\frac{1}{2}\frac{d\Theta}{dt}\left\{ x^2(1,\theta) + y^2(1,\theta) - B_o^2 \right\} \qquad (5.16)$$

Analogous to the method, expounded in [6, section 5.3], we represent the rolling motion by $\Theta = \Theta_a\cos(\sigma t+\gamma)$ and substitute $\frac{d\Theta}{dt} = -\Theta_a\,\sigma\sin(\sigma t+\gamma)$ in above formula. Then we obtain:

$$\frac{\pi\sigma}{gb}\left\{ {}^c\Psi(1,\theta) - {}^c\Psi(1,\tfrac{\pi}{2}) \right\} = g(\theta)\,(p_o\cos\sigma t + q_o\sin\sigma t) \qquad (5.17)$$

where

$$g(\theta) = \frac{x^2(1,\theta) + y^2(1,\theta) - B_o^2}{B_o^2} \qquad (5.18)$$

$$p_o = \frac{\pi\Theta_a K B_o^2}{2b}\sin\gamma$$

and

$$q_o = \frac{\pi\theta_a K B_o^2}{2b}\cos\gamma \qquad (5.18)$$

This can be reduced to the following set of linear equations for p_{2m} and q_{2m}:

$${}^c_c\psi^d(1,\theta) - {}^c_c\psi^d(1,\tfrac{\pi}{2}) = \sum_{m=0}^{N} p_{2m}\,f_{2m}(\theta)$$

$${}^c_s\psi^d(1,\theta) - {}^c_s\psi^d(1,\tfrac{\pi}{2}) = \sum_{m=0}^{N} q_{2m}\,f_{2m}(\theta) \qquad (5.19)$$

in which

$$f_o = g(\theta) = \frac{x^2(1,\theta) + y^2(1,\theta) - B_o^2}{B_o^2}$$

$$f_{2m} = {}^c_{2m}\psi^a(1,\tfrac{\pi}{2}) - {}^c_{2m}\psi^a(1,\theta),\quad m \neq o \qquad (5.20)$$

For the pressure along the contour of the right-hand cylinder due to the rolling motion of the separate cylinders we find, analogous to (4.23):

$$p_R(1,\theta) = \frac{-\rho g b}{\pi}(M\cos\sigma t - N\sin\sigma t) \qquad (5.21)$$

where we substitute for the coefficients p_{2m} and q_{2m}, which are found in the expressions (4.22) for M and N, the values which satisfy the set of equations (5.19).

Analogous to (4.32) and (4.33), we find for the hydrodynamic moment on the system.

$$M_R = \frac{2B_o^2\rho g b}{\pi}\left\{ -X_R\sin\sigma t + Y_R\cos\sigma t \right\} \qquad (5.22)$$

Combining this with the relation:

$$\frac{d\Theta}{dt} = -\Theta_a\,\sigma\sin(\sigma t+\gamma) = \frac{2bg}{\pi\sigma B_o^2}(-q_o\sin\sigma t - p_o\cos\sigma t) \qquad (5.23)$$

which is derived from (5.18), we find:

$$M_R = -\rho B_o^4\frac{Y_R q_o + X_R p_o}{p_o^2 + q_o^2}\ddot{\Theta} - \rho\sigma B_o^4\frac{Y_R p_o - X_R q_o}{p_o^2 + q_o^2}\dot{\Theta} \qquad (5.24)$$

For the added moment of inertia I_R^R and damping N_R^R per unit length due to rolling component we find:

$$I_R^R = \rho B_o^4 \frac{Y_R q_o + X_R p_o}{p_o^2 + q_o^2} \tag{5.25}$$

$$N_R^R = \rho\sigma B_o^4 \frac{Y_R p_o - X_R q_o}{p_o^2 + q_o^2} \tag{5.26}$$

where the expressions for the quantities X_R and Y_R are given by (4.34), in which we substitute for the coefficients p_{2m} and q_{2m}, which are found in the expressions (4.22) for M and N the values, satisfying the set of linear equations (5.19).

Finally, we find from (5.13), (5.14), (5.25) and (5.26) for the total added moment of inertia I_R and the total damping N_R of the system per unit length.

$$I_R = \rho B_o l \frac{M_o B + N_o A}{A^2 + B^2} + \rho B_o^4 \frac{Y_R q_o + X_R p_o}{p_o^2 + q_o^2} \tag{5.27}$$

$$N_R = \rho B_o \sigma l \frac{M_o A - N_o B}{A^2 + B^2} + \rho\sigma B_o^4 \frac{Y_R p_o - X_R q_o}{p_o^2 + q_o^2} \tag{5.28}$$

Analogous to (4.26) the swaying force on the system due to the rolling motion is determined from:

$$F_{SR} = -2B_o \int_{-\pi/2}^{\pi/2} \left\{ p_H(1,\theta) + p_R(1,\theta) \right\} \frac{V(\theta)}{G} d\theta \tag{5.29}$$

For $p_H(1,\theta)$ we substitute (5.9) while for $p_R(1,\theta)$ the following expression is inserted:

$$p_R(1,\theta) = \frac{\rho B_o^2}{2} \frac{Mq_o + Np_o}{p_o^2 + q_o^2} \ddot{\theta} + \frac{\rho\sigma B_o^2}{2} \frac{p_o M - q_o N}{p_o^2 + q_o^2} \dot{\theta} \tag{5.30}$$

which is obtained by combining (5.21) and (5.23).

Then:

$$F_{SR} = -\rho B_o^2 l \frac{\bar{M}_o B + \bar{N}_o A}{A^2 + B^2} \ddot{\theta} - \rho\sigma l B_o^2 \frac{\bar{M}_o A - \bar{N}_o B}{A^2 + B^2} \dot{\theta} - \rho B_o^3 \frac{\tilde{M}_o q_o + \tilde{N}_o p_o}{p_o^2 + q_o^2} \ddot{\theta} - \rho\sigma B_o^3 \frac{\tilde{M}_o p_o - \tilde{N}_o q_o}{p_o^2 + q_o^2} \dot{\theta} \tag{5.31}$$

The quantities $\bar{M}_o$ and $\bar{N}_o$ in the first two terms of this expression are given by:

$$\bar{M}_o = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} M(\theta) \frac{V(\theta)}{G} d\theta$$

$$\bar{N}_o = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} N(\theta) \frac{V(\theta)}{G} d\theta \tag{5.32}$$

in which for M and N the expressions are inserted which are given by (5.8). The quantities $\tilde{M}_o$ and $\tilde{N}_o$ in the last two terms of (5.31) are obtained by replacing in (5.32) the functions $M(\theta)$ and $N(\theta)$ in the integrands by the similar functions which are found in (5.21).

So, finally, for the added mass and damping per unit length for swaying produced by the rolling motion we obtain:

$$M_{SR} = \rho B_o^2 l \frac{\bar{M}_o B + \bar{N}_o A}{A^2 + B^2} + \rho B_o^3 \frac{\tilde{M}_o q_o + \tilde{N}_o p_o}{p_o^2 + q_o^2} \tag{5.33}$$

$$N_{SR} = \rho B_o^2 \sigma l \frac{\bar{M}_o A - \bar{N}_o B}{A^2 + B^2} + \rho\sigma B_o^3 \frac{\tilde{M}_o p_o - \tilde{N}_o q_o}{p_o^2 + q_o^2} \tag{5.34}$$

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Report no. 268

LABORATORIUM VOOR SCHEEPSBOUWKUNDE

TECHNISCHE HOGESCHOOL DELFT

THE HYDRODYNAMIC COEFFICIENTS OF TWO PARALLEL IDENTICAL CYLINDERS OSCILLATING IN THE FREE SURFACE

by

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June 1970

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Summary

In the present report expressions are derived for the hydrodynamic coefficients of two identical rigidly connected parallel cylinders of infinite length, which perform harmonic oscillations in the free fluid surface. The cylinders are supposed to have only connections above the free surface.
The method applied in this report is in fact an extension of Ursell's method for the corresponding problem of one cylinder.

Preface

In this report basic data are given for the determination according to the strip theory, of the hydrodynamic coefficients of catamarans performing harmonic oscillations.

The reader is supposed to be familiar to a certain extent with Ursell's method to evaluate the hydrodynamic coefficients of a single cylinder, oscillating in the free surface. This method is in fact the starting point for the determination of the hydrodynamic coefficients of a "single-body" vessel according to the strip theory. For this the reader is referred to the papers of Ursell [1,2], Tasai [3,4], or Porter [5].

Their results have been compiled and supplemented by the present author in his earlier report [6]. It will be seen that many methods, which are needed for the solution of the double-cylinder problem, are identical to those of the single-cylinder problem and, therefore, they will be mentioned in this report only very concisely. In order to aid the reader who wants to acquaint himself with these methods, the present author will, at some places in this report, refer with much detail to his above-mentioned report.

Introduction

The last few years several catamarans have been constructed for all kinds of purposes. This is largely due to the possibility to construct these vessels in such a way that the resistance for high F_n is much lower as compared with convential ships with the same deck area. In this connection it should be observed that we can give, the floating bodies of the catamaran shapes, which can't exist as separate single body vessels. However, experiences with recent designs showed that catamarans have larger heaving and pitching motions as compared with convential ships. These phenomena justify a theoretical analysis of the motions of a catamaran. The present report gives a basis to determine the hydrodynamic coefficients of such a ship with the strip theory method, which has proved to give very useful results for single body vessels. Analogous to the single body vessel the catamaran is divided up into a number of sections and for each section, which is taken to have a constant profile, the hydrodynamic properties are determined, assuming that the disturbances in the fluid due to the motions of the sections only propagate in the direction perpendicular to its longitudinal axes.

The catamaran is assumed to be composed of two identical floating bodies, which have been rigidly connected above the free surface. Consequently for the application of the striptheory method we need expressions for the hydrodynamic coefficients of a system of two infinitely long identical parallel cylinders, which have been rigidly connected above the free surface at a given finite distance.

Analogous to the single cylinder, this problem is formulated as a linearized boundary value problem from the potential theory, which is achieved by assuming the fluid to be inviscid incompressible and irrotational and the amplitudes of the motions of the system to be very small.

The velocity potential is again composed of a linear combination of a source or dipole potential and a number of multipole potentials such that the boundary conditions are satisfied. However, the expressions for the source,dipole and multipole potentials have to be adapted to the conditions which result from the presence of two cylinders.

Further, analogous to the single-cylinder problem also for the problem of two cylinders we have to determine a conformal transformation, which maps a circular cylinder into the cross-section of one of these cylinders. Since suitable numerical techniques are available to devise such a transformation, we will consider in this report cylinders with an arbitrary shape, [See 6; section 4.1].

1 Formulation of the problem

We assume that a system of two parallel identical infinitely long cylinders, which are rigidly connected above the free surface at a distance l from each other, carries out a harmonic one-dimensional oscillation with frequency σ, while the mean position of the axes of both cylinders is in the undisturbed free surface of the fluid. The origin of the rectangular Cartesian coordinates (x,y) is taken in the mean position of the axis of the eight-hand cylinder (Fig. 1.1). The x-axis is horizontal and perpendicular to the axis of the cylinder, the y-axis vertical, positive in downward direction.

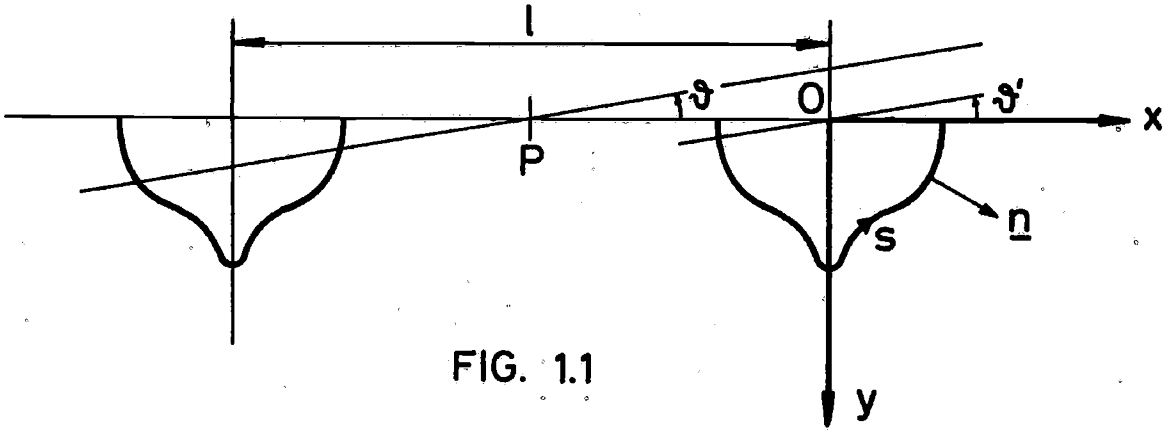

FIG. 1.1

Its possible modes of oscillation we consider heaving, swaying and rolling about the point P , which is in the free surface and in the symmetry plane $x = -\frac{1}{2}l$ of the system. With respect to the rolling motion, represented by $\vartheta = \vartheta_a \cos(\sigma t+\gamma)$, it is readily seen that this motion involves a combined heaving, swaying and rolling motion of the separate cylinders. For the right-hand cylinder we find the following ordinates of these components respectively:

$$
\begin{aligned}
y_\vartheta &= \tfrac{1}{2}l \sin\vartheta \\
x_\vartheta &= \tfrac{1}{2}l (\cos\vartheta - 1) \\
\vartheta' &= \vartheta
\end{aligned}
\qquad (1.1)
$$

Expanding the sine and cosine functions in these formulas and retaining only the first term we obtain:

$$
\begin{aligned}
y_\vartheta &= \tfrac{1}{2}l\vartheta \\
x_\vartheta &= \tfrac{1}{4}l\vartheta^2 \\
\vartheta' &= \vartheta
\end{aligned}
\qquad (1.2)
$$

Since the swaying component is a second order quantity, we shall conceive in this report the rolling motion of the double cylinder system as a combined heaving and rolling motion of the separate cylinders. The velocity potential for the rolling motion of the system will consist of a component due to the heaving and another due to the rolling motion of the separate cylinders. Analogous to the single cylinder problem [6; ch 1], a velocity potential $\Phi(x,y,t)$ has to be determined, which is a solution of a linearized boundary value problem from the potential theory. Consequently we may write:

$$\Phi(x, y, t) = -i\phi(x, y)\, e^{i\sigma t} \qquad (1.3)$$

where $\phi(x,y)$ is a solution of the equation of Laplace

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad (1.4)$$

and satisfies, in addition the following conditions:

(i) the linearized free-surface condition:

$$k\,\phi + \frac{\partial\phi}{\partial y} = 0 \qquad \text{when } y = 0 \qquad (1.5)$$

in which $k = \frac{\sigma^2}{g}$ represents the wave number.

(ii) the radiation condition:

$$\phi \rightarrow C_1 e^{-ky-ikx} \qquad \text{as } x \rightarrow \infty$$
$$\phi \rightarrow C_2 e^{-ky+ikx} \qquad \text{as } x \rightarrow -\infty \qquad (1.6)$$

where C_1 and C_2 are constants

This condition implies that every disturbance in the fluid vanishes when $y \rightarrow \infty$

(iii) the boundary condition on the cylinder contour:

$$\frac{\partial\phi}{\partial n} = U_n(x, y) \qquad \text{when } y = 0 \qquad (1.7)$$

where U_n denotes the normal outward velocity on the cylinder surface (Fig.1.1). We observe that the above condition refers to the mean position of the system, since the linearized case is considered.

(iv) when the system is carrying out a heaving motion, the potential $\phi(x,y)$ has to be a symmetric function with respect to the plane $x = -\frac{1}{2}\,l$, for swaying and rolling the potential has to be a skew-symmetric one.

(v) for l = 0 the potentials for the heaving, swaying and rolling motion of the double-cylinder system become equivalent with the potentials for the respective motions of one cylinder.

2 Outline of the method of solution

Analogous to the single-cylinder problem the velocity potential for the two cylinders is also synthesized of a source or dipole potential and a linear combination of multipole potentials. The source, dipole and multipole potentials are chosen such that the conditions (i), (ii), (iv) and (v) are satisfied by each potential separately, while the condition (iii) is satisfied by choosing the linear combination in an appropriate way. In the following chapters we will derive for each mode of oscillation of the double-cylinder system adequate expressions for the above mentioned potentials. It turns out, that these potentials are easily derived from the corresponding single-cylinder potentials. It is well-known that the expressions for the multipole potentials (see (2.2) and (3.2)) for the single-cylinder problem depend on the parameters a, a_1, a_3,.... which are the coefficients in the following transformation formulas, [6;eq(4.1.8)]:

$$x = a\left\{ r \sin\Theta + \sum_{n=0}^{N} (-1)^n \frac{a_{2n+1}}{r^{2n+1}} \sin(2n+1)\Theta \right\}$$

$$y = a\left\{ r \cos\Theta + \sum_{n=0}^{N} (-1)^{n+1} \frac{a_{2n+1}}{r^{2n+1}} \cos(2n+1)\Theta \right\} \qquad (2.1)$$

The coefficients a, a_1, a_3,......are determined such that the semi-unit circle ($r=1$, $-\frac{\pi}{2} \leq \Theta \leq \frac{\pi}{2}$) in the reference plane (ζ -plane), in which the polar coordinate system (r,Θ) has been defined, is mapped into the cross-section in the physical plane (z - plane) with cartesian coordinates (x,y), [6; section 4.1]. The number of terms M, considered in the equations (2.1), determines the accuracy of the transformation.

The formulas (2.1) can also be interpreted as defining a curvilinear coordinate system (r,Θ) in the physical plane such that one of the coordinate lines ($r=1$) coincides with the cross-section.

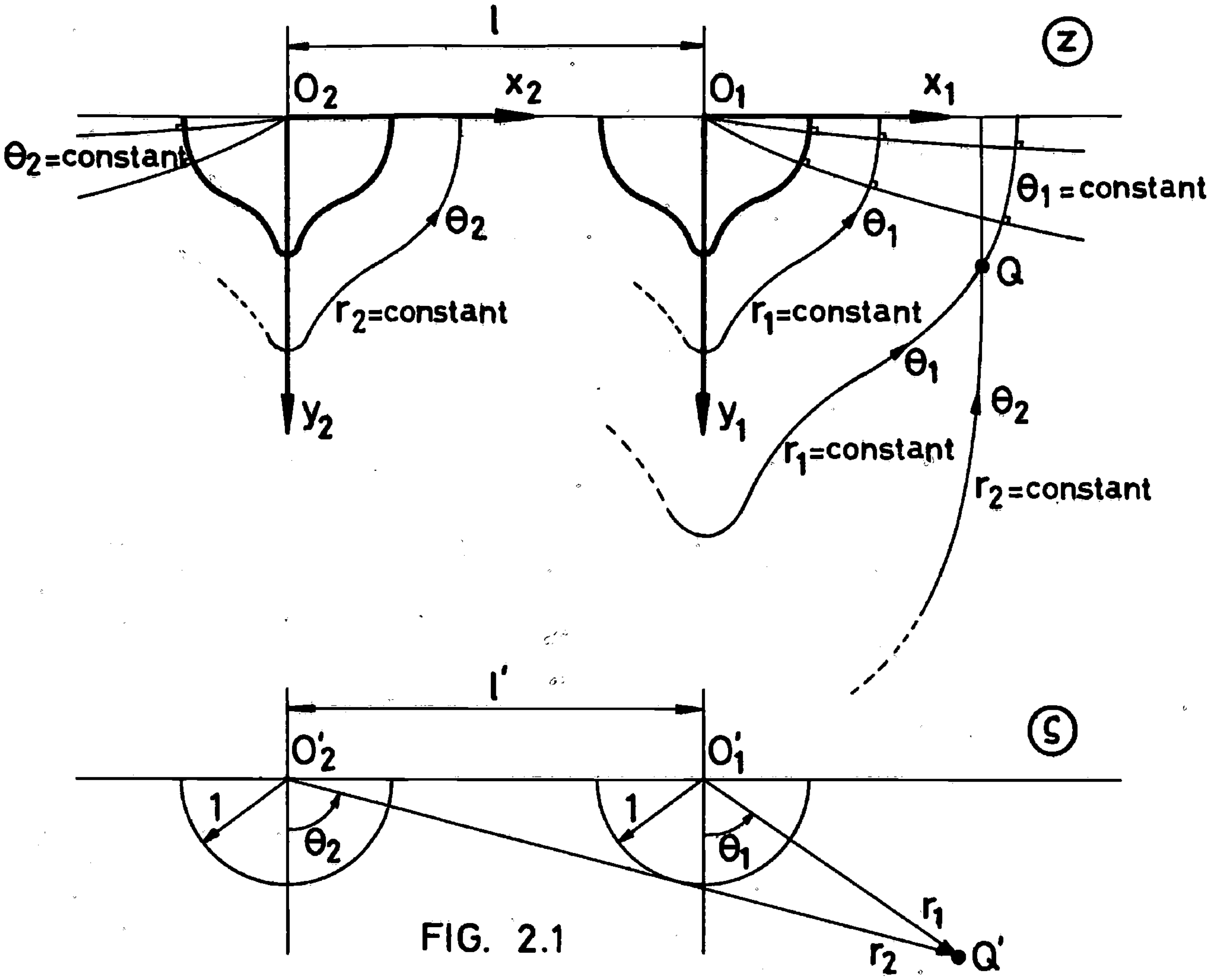

FIG. 2.1

In addition to the rectangular coordinate system (x,y), as defined in the previous chapter, which has its origin O in the mean position of the axis of the right-hand cylinder, we introduce in this chapter an identical coordinate system, which has its origin in the mean position of the axis of the left-hand cylinder. The first-mentioned system is denoted here by (x_1, y_1) with origin O_1, the other by (x_2, y_2) with origin O_2, (Fig. 2.1). The corresponding polar coordinate systems in the ζ -plane or curvilinear coordinate systems in the z-plane are denoted by (r_1, θ_1) and (r_2, θ_2) respectively. According to (2.1) the following relations are valid between the rectangular and curvilinear coordinates in the z -plane:

$$x_i = a \left\{ r_i \sin \theta_i + \sum_{n=0}^{M} (-1)^n \frac{a_{2n+1}}{r_i^{2n+1}} \sin (2n+1)\theta_i \right\}$$

$$y_i = a \left\{ r_i \cos \theta_i + \sum_{n=0}^{M} (-1)^{n+1} \frac{a_{2n+1}}{r_i^{2n+1}} \cos (2n+1)\theta_i \right\} \qquad (2.2)$$

$i = 1, 2.$

Consider a point Q in the z -plane, which is the image point of the point Q' in the ζ-plane. The point Q is represented by the rectangular coordinates

(x_1, y_1) and the curvilinear coordinates (r_1, θ_1), which both refer to the origin O_1. The image point Q' in the ζ-plane is given by the polar coordinates (r_1, θ_1) with origin O_1'. However, this point can also be represented by the polar coordinates (r_2, θ_2) with origin O_2'. It is easily seen from Fig. 2.1., that the following relations are valid between the polar coordinate systems (r_1, θ_1) and (r_2, θ_2):

$$\theta_2 = \arctan \frac{l' + r_1 \sin \theta_1}{r_1 \cos \theta_1} \qquad (2.3)$$

$$r_2 = \sqrt{r_1^2 + l'^2 + 2 r_1 l' \sin \theta_1}$$

The distance l' between the two semi-circles can be determined from the relation which is obtained by inserting in the first equation of (2.2) either $x_2 = 1$, $\theta_2 = \frac{\pi}{2}$ and $r_2 = l'$ or $x_1 = -1$, $\theta_1 = -\frac{\pi}{2}$ and $r_1 = l'$. This yields:

$$1 = a \left\{ l' + \sum_{n=0}^{M} \frac{a_{2n+1}}{l'^{2n+1}} \right\} \qquad (2.4)$$

where $l' > 2$

It is clear that the equations (2.3) can also be interpreted as representing the relations between the curvilinear coordinate systems (r_1, θ_1) and (r_2, θ_2) in the z-plane, which are very useful in our future calculations.

3 Added mass and damping for the heaving motion

It is clear, that the source and the multipole potentials, which are used for the solution of the heaving problem of one cylinder can't be used for the solution of our problem here since the symmetry condition (iv) is not satisfied. However, it will be seen, that the set of potentials, which satisfy each the condition (iv) are easily derived from the single-cylinder potentials. The source potential ϕ^s and the symmetric multipole potentials φ^s_{2m}, used for the single-cylinder problem are given by:

$$\phi^s(x, y) = \frac{gb}{\pi\sigma}\left\{\int_0^\infty e^{-\beta|x|}\frac{(k\sin\beta y - \beta\cos\beta y)}{k^2+\beta^2}\,d\beta + i\pi e^{-ky-ik|x|}\right\} \qquad (3.1)$$

and

$$\varphi^s_{2m}(r, \theta) = \frac{\cos 2m\theta}{r^{2m}} + ka\left\{\frac{\cos(2m-1)\theta}{(2m-1)r^{2m-1}} + \sum_{n=0}^{N}(-1)^n\frac{(2n+1)a_{2n+1}\cos(2m+2n+1\theta)}{(2m+2n+1)r^{2m+2n+1}}\right\} \qquad (3.2)$$

$m = 1, 2, 3 \ldots\ldots$

In the first formular b represents the wave height at infinity due to the oscillation of the cylinder while the parameters a, a_1, a_3, ... in the second formula represent the coefficients in the transformation formulas (2.1)

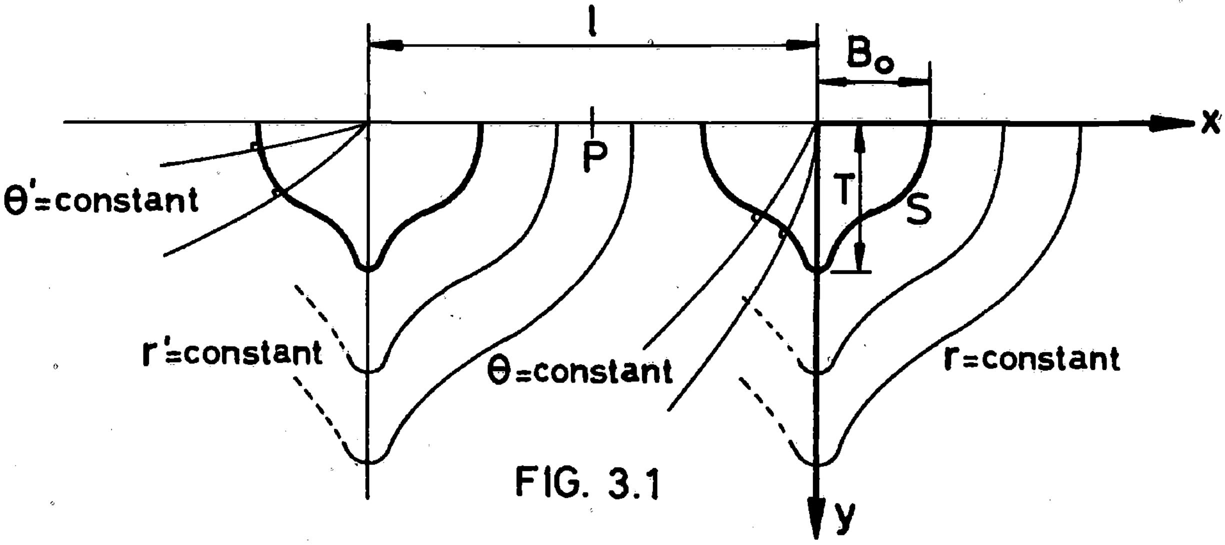

FIG. 3.1

The corresponding potentials for the double-cylinder problem, which will be denoted by the superscript c, are defined by:

$${}^c\phi^s(x, y) = \tfrac{1}{2}\left\{\phi^s(x, y) + \phi^s(x+l, y)\right\} \qquad (3.3)$$

and

$${}^c\varphi^s_{2m}(r, \theta) = \tfrac{1}{2}\left\{\varphi^s_{2m}(r, \theta) + \varphi^s_{2m}(r', \theta')\right\} \qquad (3.4)$$

where, according to (2.3), by identifying (r',θ') with (r_2,θ_2) and $(r\ ,\theta\ )$ with (r_1,θ_1):

$$\theta' = \arctan \frac{l' + r \sin\theta}{r\cos\theta}$$
$$r' = \sqrt{r^2 + l'^2 + 2l'r\sin\theta} \tag{3.5}$$

in which l' and l satisfy the relation (2.4).

Since the conditions (i) and (ii) are satisfied by the potentials ϕ^s and φ^s_{2m} separately, the corresponding potentials ${}^c\phi^s$ and ${}^c\varphi^s_{2m}$ for the double-cylinder problem satisfy also these conditions. Further, we can easily verify, that these potentials satisfy the symmetry condition (iv) and the condition (v) for $l=0$.

By using the Cauchy-Riemann relations we determine the conjugate streamfunctions ψ^s and ψ^s_{2m} of the potentials ϕ^s and φ^s_{2m} respectively. In rectangular coordinates these relations have the form

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$$
$$\frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \tag{3.6}$$

and in polar coordinates:

$$\frac{\partial\phi}{\partial r} = -\frac{1}{r}\frac{\partial\psi}{\partial\theta}$$
$$\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \tag{3.7}$$

consequently:

$$\psi^s(x,\ y) = \frac{gb}{\pi\sigma}\left\{ \pm \int_0^\infty e^{-\beta|x|}\frac{k\cos\beta + \beta\sin\beta y}{k^2+\beta^2}\, d\beta \mp \pi e^{-ky-ik|x|}\right\} \quad x \gtrless 0 \tag{3.8}$$

and

$$\psi^s_{2m}(\ r,\ \theta) = \frac{\sin 2m\theta}{r^{2m}} + ka\left\{ \frac{\sin(2m-1)\theta}{(2m-1)r^{2m-1}} + \sum_{n=0}^{N}(-1)^n \frac{a^{(2n+1)}_{2n+1}\sin(2m+2n+1)\theta}{(2m+2n+1)r^{2m+2n+1}} \right\} \tag{3.9}$$

The streamfunctions ${}^c\psi^s(x,\ y)$ and ${}^c\psi^s_{2m}(r,\theta)$ for the double-cylinder system are given by:

$${}^c\psi^s(x,\ y) = \tfrac{1}{2}\left\{ \psi^s(x,\ y) + \psi^s(x+l,y)\right\} \tag{3.10}$$

and

$${}^c\psi^s_{2m}(r,\ \theta) = \tfrac{1}{2}\left\{ \psi^s_{2m}(r,\ \theta) + \psi^s_{2m}(r',\ \theta')\right\} \tag{3.11}$$

Expression (3.10) is clear without further preface.
The validity of (3.11) is easily shown by observing, that:

$$\frac{\partial \varphi^s_{2m}(r', \theta')}{\partial r} = \frac{\partial \varphi^s_{2m}(r', \theta')}{\partial r'} \cdot \frac{r + l\sin\theta}{r'} \tag{3.12}$$

and

$$\frac{\partial \psi^s_{2m}(r', \theta')}{\partial \theta} = \frac{\partial \psi^s_{2m}(r', \theta')}{\partial \theta'} \cdot \frac{r(r + l\sin\theta)}{r'^2} \tag{3.13}$$

where $\Psi^s_{2m}(r', \theta')$ is defined such, that the Cauchy-Riemann relation

$$\frac{\partial \varphi^s_{2m}(r', \theta')}{\partial r} = \frac{1}{r}\frac{\partial \psi^s_{2m}(r', \theta')}{\partial \theta}$$

holds

Then it can be shown that also:

$$\frac{\partial \varphi^s_{2m}(r', \theta')}{\partial r'} = \frac{1}{r'}\frac{\partial \psi^s_{2m}(r', \theta')}{\partial \theta'} \tag{3.14}$$

which proves, that (3.11) is correct.
The velocity potential for the heaving motion of the system is written in the form:

$$^c\Phi(r, \theta, t) = \frac{gb}{\pi\sigma}\left[\left\{{}^c\phi^s_c + \sum_{m=1}^{N} p_{2m}\, {}^c\varphi^s_{2m}\right\}\cos\sigma t + \left\{{}^c\phi^s_s + \sum_{m=1}^{N} q_{2m}\, {}^c\varphi^s_{2m}\right\}\sin\sigma t\right] \tag{3.15}$$

where

$${}^c\phi^s_c(x, y) = \tfrac{1}{2}\left\{\phi^s_c(x, y) + \phi^s_c(x + 1, y)\right\}$$

and

$${}^c\phi^s_s(x, y) = \tfrac{1}{2}\left\{\phi^s_s(x, y) + \phi^s_s(x + 1, y)\right\} \tag{3.16}$$

while

$$\phi^s_c(x, y) = \pi e^{-ky}\cos kx$$

$$\phi^s_s(x, y) = \Pi e^{-ky}\sin k|x| - \int_0^\infty \frac{e^{-\beta|x|}}{k^2+\beta^2}(\beta\cos\beta y - k\sin\beta y)\,d\beta \tag{3.17}$$

in which the coordinates (x,y) and (r,θ) are related by the formulas (2.1)
For the conjugate streamfunction $^c\Psi(r, \theta, t)$ we find:

$$^c\Psi(r,\theta,t) = \frac{gb}{\pi\sigma}\left[\left\{{}^c\psi^s_c + \sum_{n=1}^{N} p_{2m}\, {}^c\psi^s_{2m}\right\}\cos\sigma t + \left\{{}^c\psi^s_s + \sum_{n=1}^{N} q_{2m}\, {}^c\psi^s_{2m}\right\}\sin\sigma t\right. \tag{3.18}$$

where

$${}^c\psi^s_c(x, y) = \tfrac{1}{2}\left\{\psi^s_c(x, y) + \psi^s_c(x+1, y)\right\} \tag{3.19}$$

and

$${}^c\psi^s_s(x, y) = \tfrac{1}{2}\left\{\psi^s_s(x, y) + \psi^s_s(x+1, y)\right\}$$

in which:

$$\psi^s_c(x, y) = \pi e^{-ky}\sin kx$$

$$\psi_s^s(x, y) = \pm \int_0^\infty \frac{e^{-\beta|x|}}{k^2+\beta^2} \cdot \left\{ \beta \sin \beta y + k \cos \beta y \right\} d\beta \mp \pi e^{-ky} \cos kx \qquad x \gtrless 0 \tag{3. 20}$$

The values of the coefficients p_{2m} and q_{2m} in (3.15) and (3.18) are determined in such a way that the boundary condition (iii) on the contour of the cylinders is satisfied. The value of N determines the accuracy of the approximation of the velocity potential by (3.15).

In virtue of the symmetry of the system with respect to $x = -\frac{1}{2}l$ it is sufficient to restrict our discussions with respect to the determination of p_{2m} and q_{2m} to the right-hand cylinder.

The boundary condition on the contour of this cylinder is given by

$$\frac{\partial\, {}^c\Phi}{\partial n} = \frac{dy}{dt} \cos \alpha \qquad \text{or} - \frac{\partial\, {}^c\Psi}{\partial s} = \frac{dy}{dt} \frac{\partial x}{\partial s} \tag{3. 21}$$

where α is the angle between the positive normal on the cross-section and the positive y-axis (Fig. 3.2)

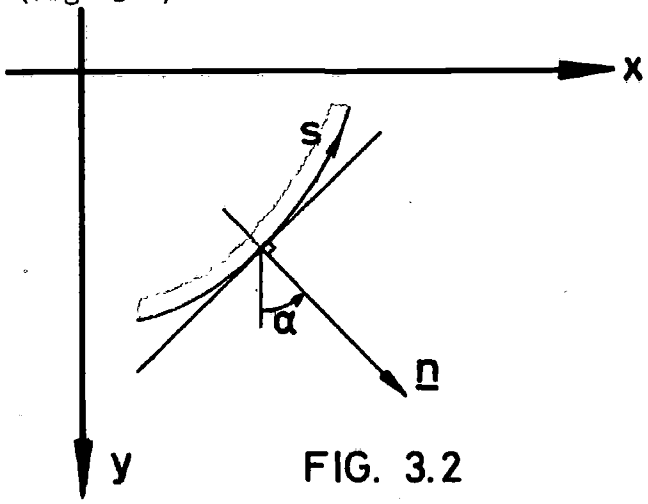

FIG. 3.2

Analogous to the single-cylinder problem [6, (4.2.8)............(4.2.13)], this relation can be reduced to

$${}^c\Psi(r=1, \theta) = -\frac{dy}{dt}\, x\, (r = 1, \theta) \tag{3. 22}$$

Substituting $\theta = 0$, yields:

$${}^c\Psi(1, \tfrac{\pi}{2}) = -\frac{dy}{dt}\, B_o \tag{3. 23}$$

Eliminating $\frac{dy}{dt}$ from (3.22) and (3.23), we obtain:

$${}^c\Psi(1, \theta) = \frac{x(1,\theta)}{B_o}\, {}^c\Psi(1, \tfrac{\pi}{2}) \tag{3. 24}$$

Substituting (3.18) in this expression we find a set of 2N linear equations for the coefficients p_{2m} and q_{2m}

$$ {}^{c}\psi^{s}_{c}(1,\theta) - \frac{x(1,\theta)}{B_o}\,{}^{c}\psi^{s}_{c}(1,\tfrac{\pi}{2}) = \sum_{m=1}^{N} p_{2m}\, f_{2m}(1,\theta) $$

$$ {}^{c}\psi^{s}_{s}(1,\theta) - \frac{x(1,\theta)}{B_o}\,{}^{c}\psi^{s}_{s}(1,\tfrac{\pi}{2}) = \sum_{m=1}^{N} q_{2m}\, f_{2m}(1,\theta) \qquad (3.\ 25) $$

where

$$ f_{2m}(1,\theta) = \frac{x(1,\theta)}{B_o}\,{}^{c}\psi^{s}_{2m}(1,\tfrac{\pi}{2}) - {}^{c}\psi^{s}_{2m}(1,\theta) \qquad (3.\ 26) $$

It is observed, that the set of equations (3.25) has to be solved for the range $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$

The velocity potential at the contour of the cylinder (r=1) is written as:

$$ {}^{c}\Phi(1,\theta,t) = \frac{gb}{\pi\sigma}(M\sin\sigma t + N\cos\sigma t) \qquad (3.\ 27) $$

where

$$ M(\theta) = {}^{c}\phi^{s}_{s}(1,\theta) + \sum_{m=1}^{N} q_{2m}\,{}^{c}\varphi^{s}_{2m}(1,\theta) $$

$$ N(\theta) = {}^{c}\phi^{s}_{s}(1,\theta) + \sum_{m=1}^{N} p_{2m}\,{}^{c}\varphi^{s}_{2m}(1,\theta) \qquad (3.\ 28) $$

According to the relation $p(r,\theta) = -\rho\frac{\partial\,{}^{c}\Phi}{\partial t}$, the pressure distribution along the contour of the cylinder can be written as:

$$ p(1,\theta) = -\frac{\rho g b}{\pi}(M\cos\sigma t - N\sin\sigma t) \qquad (3.\ 29) $$

We define:

$$ \frac{dy}{dt} = \frac{gb}{\pi\sigma B_o}(-A\cos\sigma t - B\sin\sigma t) \qquad (3.\ 30) $$

where, in virtue of (3.23):

$$ A = {}^{c}\psi^{s}_{c}(1,\tfrac{\pi}{2}) + \sum_{m=1}^{N} p_{2m}\,{}^{c}\psi^{s}_{2m}(1,\tfrac{\pi}{2}) $$

$$ B = {}^{c}\psi^{s}_{s}(1,\tfrac{\pi}{2}) + \sum_{m=1}^{N} q_{2m}\,{}^{c}\psi^{s}_{2m}(1,\tfrac{\pi}{2}) \qquad (3.\ 31) $$

Then the pressure along the cylinder contour can be written in the following form:

$$ p(1,\theta) = \rho B_o \frac{MB + NA}{A^2 + B^2}\,\ddot{y} + \rho B_o \sigma \frac{MA - NB}{A^2 + B^2}\,\dot{y} \qquad (3.\ 32) $$

The total vertical force per unit length on both cylinders becomes:

$$F_y = -2\int_{S\,(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})} p(1,\theta)\cos\alpha\, ds \tag{3. 33}$$

Analogous to the single-cylinder problem [6; eqs(4.2.21),.........., (4.2.26)], this expression can be reduced to:

$$F_y = -2\rho B_o^2 \frac{M_o B + N_o A}{A^2 + B^2}\ \ddot{y} - 2\rho B_o^2 \sigma \frac{M_o A - N_o B}{A^2+B^2}\ \dot{y} \tag{3. 34}$$

where

$$M_o = \int_{-\pi/2}^{\pi/2} M(\theta)\,\frac{W(\theta)}{G}\,d\theta$$

$$N_o = \int_{-\pi/2}^{\pi/2} N(\theta)\,\frac{W(\theta)}{G}\,d\theta \tag{3. 35}$$

in which

$$W(\theta) = \cos\theta + \sum_{n=o}^{N} (-1)^n\,(2n+1)\,a_{2n+1}\cos(2n+1)\theta$$

$$G = 1 + \sum_{n=o}^{N} a_{2n+1} \tag{3. 36}$$

For the added mass M_y and damping N_y per unit length we find:

$$M_y = 2\rho B_o^2\,\frac{M_o B + N_o A}{A^2 + B^2} \tag{3. 37}$$

$$N_y^2 = 2\rho\,\sigma B_o^2\,\frac{M_o A - N_o B}{A^2 + B^2} \tag{3. 38}$$

4 Added mass and damping for swaying; coupling coefficients of swaying into rolling

The dipole and asymmetric multipole potentials ϕ^d and φ^a_{2m}, used for finding the velocity potential for the single-cylinder problem are given by:

$$\phi^d(x,\ y)=\frac{gb}{\pi\sigma}\left\{\frac{x}{k(x^2+y^2)}\mp\int_0^\infty\frac{e^{-\beta|x|}\{k\cos\beta y+\beta\sin\beta y\}}{k^2+\beta^2}d\beta\pm\pi e^{-ky-ik|x|}\right\}\qquad x\gtrless 0 \tag{4. 1}$$

$$\varphi^a_{2m}(r,\ \theta)=\frac{\sin(2m+1)\theta}{r^{2m+1}}+ka\left\{\frac{\sin 2m\theta}{2mr^{2m}}+\sum_{n=o}^{N}\frac{(-1)^n a_{2n+1}(2n+1)\sin(2m+2n+2)\theta}{(2m+2n+2)r_\circ^{2m+2n+2}}\right\} \tag{4. 2}$$

The conjugate streamfunctions of these potentials are given by:

$$\psi^d(x,\ y)=\frac{gb}{\pi\sigma}\left\{-\frac{y}{k(x^2+y^2)}-\int_0^\infty e^{-\beta|x|}\frac{\beta\cos\beta y-k\sin\beta y}{k^2+\beta^2}d\beta+i\pi e^{-ky-ik|x|}\right\} \tag{4. 3}$$

$$\psi^a_{2m}(r,\ \theta)=\frac{-\cos(2m+1)\theta}{r^{2m}}-ka\left\{\frac{\cos 2m\theta}{2mr^{2m}}+\sum_{n=o}^{N}\frac{(-1)^n a_{2n+1}(2n+1)\cos(2m+2n+2)\theta}{(2m+2n+2)r^{2m+2n+2}}\right. \tag{4. 4}$$

The rectangular coordinates (x,y) and curvilinear coordinates (r, θ) in these formulas relate to the right-hand cylinder of the system as indicated in Fig3.1 The dipole potential ${}^c\phi^d_\circ(x,y)$ and the multipole potentials ${}^c\varphi^a_{2m}(r,\theta)$ for the double-cylinder problem are now defined by:

$${}^c\phi^d(x,\ y)=\tfrac{1}{2}\left\{\phi^d(x,\ y)+\phi^d(x+l,\ y)\right\} \tag{4. 5}$$

$${}^c\varphi^a_{2m}(r,\ \theta)=\tfrac{1}{2}\left\{\varphi^a_{2m}(r,\ \theta)+\varphi^a_{2m}(r',\ \theta')\right\} \tag{4. 6}$$

where (r, θ) and (r',θ') represent the curvilinear coordinate systems, which relate respectively to the right-hand and left-hand cylinder of the system in the physical plane (Fig. 3.1).

Between the coordinate systems (r, θ) and (r', θ') exist the relations (3.5). Analogous to the heaving problem we find for the coresponding streamfunctions:

$${}^c\psi^d(x,\ y)=\tfrac{1}{2}\left\{\psi^d(x,\ y)+\psi^d(x+l,\ y)\right\} \tag{4. 7}$$

$${}^c\psi^a_{2m}(r,\ \theta)=\tfrac{1}{2}\left\{\psi^a_{2m}(r,\ \theta)+\psi^a_{2m}(r',\ \theta')\right\} \tag{4. 8}$$

For the complete potential ${}^c\Phi$ and conjugate streamfunction ${}^c\Psi$ we write:

$$^{c}\Phi(r,\ \theta,\ t)=\frac{gb}{\pi\sigma}\left[\left\{{}^{c}\phi_{c}^{d}+\sum_{m=1}^{N}p_{2m}\,{}^{c}\varphi_{2m}^{a}\right\}\cos\sigma t+\left\{{}^{c}\phi_{s}^{d}+\sum_{m=1}^{N}q_{2m}\,{}^{c}\varphi_{2m}^{a}\right\}\sin\sigma t\right] \quad (4.\ 9)$$

and

$$^{c}\Psi(r,\ \theta,\ t)=\frac{gb}{\pi\sigma}\left[\left\{{}^{c}\psi_{c}^{d}+\sum_{m=1}^{N}p_{2m}\,{}^{c}\psi_{2m}^{a}\right\}\cos\sigma t+\left\{{}^{c}\psi_{s}^{d}+\sum_{m=1}^{N}q_{2m}\,{}^{c}\psi_{2m}^{a}\right\}\sin\sigma t\right] \quad (4.\ 10)$$

where:

$$\begin{aligned}
{}^{c}\phi_{c}^{d}(x,\ y) &= \tfrac{1}{2}\{\phi_{c}^{d}(x,\ y)+\phi_{c}^{d}(x+l,\ y)\}\\
{}^{c}\phi_{s}^{d}(x,\ y) &= \tfrac{1}{2}\{\phi_{s}^{d}(x,\ y)+\phi_{s}^{d}(x+l,\ y)\}\\
{}^{c}\psi_{c}^{d}(x,\ y) &= \tfrac{1}{2}\{\psi_{c}^{d}(x,\ y)+\psi_{c}^{d}(x+l,\ y)\}\\
{}^{c}\psi_{s}^{d}(x,\ y) &= \tfrac{1}{2}\{\psi_{s}^{d}(x,\ y)+\psi_{s}^{d}(x+l,\ y)\}
\end{aligned} \quad (4.\ 11)$$

in which

$$\phi_{c}^{d}(x,\ y) = -\ \pi e^{-ky}\sin kx$$

$$\phi_{s}^{d}(x,\ y) = \pm\ \pi e^{-ky}\cos kx \mp \int_{0}^{\infty} e^{-\beta|x|}\cdot\frac{k\cos\beta y+\beta\sin\beta y}{k^{2}+\beta^{2}}\,d\beta + \frac{x}{k(x^{2}+y^{2})} \quad x \gtrless 0$$

and (4. 12)

$$\psi_{c}^{d}(x,\ y) = \pi e^{-ky}\cos kx$$

$$\psi_{s}^{d}(x,\ y) = \pi e^{-ky}\sin k|x| - \int_{0}^{\infty} e^{-\beta|x|}\cdot\frac{\beta\cos\beta y-k\sin\beta y}{k^{2}+\beta^{2}}\,d\beta - \frac{y}{k(x^{2}+y^{2})}$$

The coefficients p_{2m} and q_{2m} are chosen such that the boundary condition on the contour of the cylinder is satisfied. It is easily verified that expression (4.9) satisfies also the other condition mentioned in chapter 1.

In virtue of the skew-symmetry of the system with respect to $x = -\frac{1}{2}l$, it is again sufficient to restrict our discussions to the right-hand cylinder.

From Fig. 3.2. we derive, that the boundary condition on the surface of this cylinder is given by:

$$\frac{\partial\,{}^{c}\Phi}{\partial n} = \frac{dx}{dt}\sin\alpha \qquad \text{or} \qquad \frac{\partial\,{}^{c}\Psi}{\partial s} = \frac{dx}{dt}\frac{dy}{ds} \quad (4.\ 13)$$

Analogous to the single-cylinder problem [6; section 5.2], we can reduce this relation to

$$^{c}\Psi(r=1,\ \theta) - {}^{c}\Psi(r=1,\ \tfrac{\pi}{2}) = \frac{dx}{dt}\,y(r=1,\ \theta) \quad (4.\ 14)$$

Substituting $\theta = o$ yields:

$$^{c}\Psi(1,\ 0) - {}^{c}\Psi(1,\ \tfrac{\pi}{2}) = \frac{dx}{dt}\,T \quad (4.\ 15)$$

where T is the draft of the cylinder.

Eliminating $\frac{dx}{dt}$ from (4.14) and (4.15) yields:

$$\frac{1}{y(1,\ \theta)}\left\{{}^{c}\Psi(1,\ \theta) - {}^{c}\Psi(1,\ \tfrac{\pi}{2})\right\} = \frac{1}{T}\left\{{}^{c}\Psi(1,\ 0) - {}^{c}\Psi(1,\ \tfrac{\pi}{2})\right\} \quad (4.\ 16)$$

Inserting (4.10), we obtain the following set of linear equations for the coefficients p_{2m} and q_{2m} :

$$\left\{{}^{c}\psi_c^d(1,\theta) - {}^{c}\psi_c^d(1,\tfrac{\pi}{2})\right\} - \frac{y(1,\theta)}{T}\left\{{}^{c}\psi_c^d(1,0) - {}^{c}\psi_c^d(1,\tfrac{\pi}{2})\right\} = \sum_{m=1}^{N} p_{2m} f_{2m}$$

$$\left\{{}^{c}\psi_s^d(1,\theta) - {}^{c}\psi_s^d(1,\tfrac{\pi}{2})\right\} - \frac{y(1,\theta)}{T}\left\{{}^{c}\psi_s^d(1,0) - {}^{c}\psi_s^d(1,\tfrac{\pi}{2})\right\} = \sum_{m=1}^{N} q_{2m} f_{2m} \qquad (4.\ 17)$$

where

$$f_{2m} = \frac{y(1,\theta)}{T}\left\{{}^{c}\psi_{2m}^a(1,0) - {}^{c}\psi_{2m}^a(1,\tfrac{\pi}{2})\right\} - \left\{{}^{c}\psi_{2m}^a(1,\theta) - {}^{c}\psi_{2m}^a(1,\tfrac{\pi}{2})\right\} \qquad (4.\ 18)$$

Next, we define:

$$\frac{dx}{dt} = \frac{gb}{\Pi\sigma T}\left\{ - A\cos\sigma t - B\sin\sigma t\right\} \qquad (4.\ 19)$$

Then, according to (4,10) and (4.15):

$$A = {}^{c}\psi_c^d(1,\tfrac{\pi}{2}) - {}^{c}\psi_c^d(1,0) + \sum_{m=1}^{N} p_{2m}\left\{{}^{c}\psi_{2m}^a(1,\tfrac{\pi}{2}) - {}^{c}\psi_{2m}^a(1,0)\right\}$$

$$B = {}^{c}\psi_s^d(1,\tfrac{\pi}{2}) - {}^{c}\psi_s^d(1,0) + \sum_{m=1}^{N} q_{2m}\left\{{}^{c}\psi_{2m}^a(1,\tfrac{\pi}{2}) - {}^{c}\psi_{2m}^a(1,0)\right\} \qquad (4.\ 20)$$

The potential along the contour of the right-hand cylinder is defined in the following way:

$${}^{c}\phi(1,\theta) = \frac{gb}{\pi\sigma}(M\sin\sigma t + N\cos\sigma t) \qquad (4.\ 21)$$

Then

$$M = {}^{c}\phi_s^d(1,\theta) + \sum_{m=1}^{N} q_{2m}\, {}^{c}\varphi_{2m}^a(1,\theta)$$

$$N = {}^{c}\phi_c^d(1,\theta) + \sum_{m=1}^{N} p_{2m}\, {}^{c}\varphi_{2m}^a(1,\theta) \qquad (4.\ 22)$$

The pressure along this cylinder can be written in the form:

$$p(1,\theta) = -\frac{\rho g b}{\pi}(M\cos\sigma t - N\sin\sigma t) \qquad (4.23)$$

or, by using (4.19):

$$p(1,\theta) = \rho T\frac{MB+NA}{A^2+B^2}\ddot{x} + \rho T\sigma\frac{MA-NB}{A^2+B^2}\dot{x} \qquad (4.24)$$

The total horizontal hydrodynamic force in the system is given by:

$$F_x = -2\int_{S(-\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2})} p(1,\theta)\sin\alpha ds \qquad (4.25)$$

Analogous to the single-cylinder problem [6;section 5.2], this formula can be reduced to:

$$F_x = -2B_o \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p(1,\ \theta)\ \frac{V(\theta)}{G}\ d\theta \qquad (4.\ 26)$$

where

$$V(\theta) = -\sin\theta + \sum_{n=0}^{N} (-1)^n\ a_{2n+1}(2n+1)\sin(2n+1)\theta$$
$$G = 1 + \sum_{n=0}^{N} a_{2n+1} \qquad (4.\ 27)$$

Substituting (4.24), we obtain:

$$F_x = -2\rho TB_o \frac{M_oB+N_oA}{A^2+B^2}\ddot{x} - 2\rho TB_0\sigma \frac{M_oA-N_0B}{A^2+B^2}\dot{x} \qquad (4.\ 28)$$

where

$$N_o = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} N(\theta)\ \frac{V(\theta)}{G}\ d\theta$$
$$M_o = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} M(\theta)\ \frac{V(\theta)}{G}\ d\theta \qquad (4.\ 29)$$

For the added mass of the system per unit lenth, we find:

$$M_x = 2\rho TB_o \frac{M_oB+N_oA}{A^2+B^2} \qquad (4.\ 30)$$

and for the damping:

$$N_x = 2\rho TB_o\sigma \frac{M_oA-N_oB}{A^2+B^2} \qquad (4.\ 31)$$

We consider now the rolling moment on the system about the point P due to the swaying motion:

$$M_{RS} = -2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p(1,\ \theta)\left\{(\tfrac{1}{2}1+x)\frac{\partial x}{\partial\theta} + y\frac{\partial y}{\partial\theta}\right\} d\theta \qquad (4.\ 32)$$

where the moment is considered to be positive when it is in clockwise direction. Substituting (4.23), we obtain:

$$M_{RS} = \frac{2B_o^2\ \rho gb}{\pi}\left\{-X_R\sin\sigma t + Y_R\cos\sigma t\right\} \qquad (4.\ 33)$$

where

$$X_R = \frac{1}{B_o^2}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} N(\theta)\left\{(x+\tfrac{1}{2}1)\frac{\partial x}{\partial\theta} + y\frac{\partial y}{\partial\theta}\right\} d\theta \qquad (4.\ 34)$$

$$Y_R = \frac{1}{B_o^2} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} M(\theta) \left\{ (x+\tfrac{1}{2}l) \frac{\partial x}{\partial \theta} + y \frac{\partial y}{\partial \theta} \right\} d\theta \qquad (4.\ 34)$$

Using (4.19), we can write (4.33) in the following form:

$$M_{RS} = -\ 2\rho T B_o^2 \ \frac{BY_R + AX_R}{A^2+B^2}\ \ddot{x} - 2\rho\sigma T B_o^2 \ \frac{AY_R - BX_R}{A^2+B^2}\ \dot{x} \qquad (4.\ 35)$$

Hence, for the added moment of inertia and the damping per unit length for the rolling motion produced by swaying, we obtain:

$$I_{RS} = 2\rho T B_o^2 \ \frac{BY_R + AX_R}{A^2+B^2} \qquad (4.\ 36)$$

$$N_{RS} = 2\rho\sigma T B_o^2 \ \frac{AY_R - BX_R}{A^2+B^2} \qquad (4.\ 37)$$

5 Added moment of inertia and damping for rolling, coupling coefficients of rolling into swaying.

As mentioned before in chapter 1, we can conceive in the linearized case the rolling motion of the system, $\vartheta = \vartheta_a \cos(\sigma t + \gamma)$, as a combined heaving and rolling motion of the separate cylinders, according to

$$y_\vartheta = \tfrac{1}{2} l \vartheta \qquad \vartheta' = \vartheta \tag{5.1}$$

The heaving motions of the two separate cylinders have 180^o phase difference while the rolling motions are in phase.

Contribution of the heaving component

In contrast with the case of pure heaving, considered in chapter 3, the source potential ${}^c\phi^s$ and multipole potentials ${}^c\varphi^s_{2m}$ have to be skew-symmetric with respect to $x = -\tfrac{1}{2}l$

Therefore, we define:

$$\begin{aligned} {}^c\phi(x, y) &= \tfrac{1}{2}\{ \phi^s(x, y) - \phi^s(x+l, y) \} \\ {}^c\varphi_{2m}(r, \theta) &= \tfrac{1}{2}\{\varphi^s_{2m}(r, \theta) - \varphi^s_{2m}(r', \theta')\} \end{aligned} \tag{5.2}$$

where ϕ^s and φ^s_{2m} are defined by (3.1) and (3.2).

In an analogous manner we define the complete potential by

$${}^c\Phi(r, \theta, t) = \frac{gb}{\pi\sigma}\left[\left\{ {}^c\phi_c + \sum_{m=1}^{N} p_{2m}\, {}^c\varphi_{2m}\right\}\cos \sigma t + \left\{ {}^c\phi_s + \sum_{m=1}^{N} q_{2m}\, {}^c\varphi_{2m}\right\}\sin \sigma t \right] \tag{5.3}$$

in which

$$\begin{aligned} {}^c\phi_c(x, y) &= \tfrac{1}{2}\{\phi^s_c(x, y) - \phi^s_c(x+l, y)\} \\ {}^c\phi_s(x, y) &= \tfrac{1}{2}\{\phi^s_s(x, y) - \phi^s_s(x+l, y)\} \end{aligned} \tag{5.4}$$

where ϕ^s_c and ϕ^s_c are defined by (3.17)

The conjugate streamfunction becomes:

$${}^c\Psi(r, \theta, t) = \frac{gb}{\pi\sigma}\left[\left\{{}^c\psi_c + \sum_{m=1}^{N} p_{2m}\, {}^c\psi_{2m}\right\}\cos \sigma t + \left\{ {}^c\psi_s + \sum_{m=1}^{N} q_{2m}\, {}^c\psi_{2m}\right\} \sin \sigma t\right] \tag{5.5}$$

where

$$ {}^{c}\psi_c(x, y) = \tfrac{1}{2}\left\{ \psi_c^s(x, y) - \psi_c^s(x+1, y) \right\} $$

$$ {}^{c}\psi_s(x, y) = \tfrac{1}{2}\left\{ \psi_s^s(x, y) - \psi_s^s(x+1, y) \right\} \qquad (5.\ 6) $$

$$ {}^{c}\psi_{2m}(r, \theta) = \tfrac{1}{2}\left\{ \psi_{2m}^s(r, \theta) - \psi_{2m}^s(r', \theta') \right\} $$

in which ψ_c^s and ψ_s^s are defined by (3.20) and ψ_{2m}^s by (3.9)
The coefficients p_{2m} and q_{2m} in the expressions (5.2) and (5.4) are determined from a set of linear equations similar to (3.25). However, the streamfunctions ${}^{c}\psi_c^s$, ${}^{c}\psi_s^s$ and ${}^{c}\psi_{2m}^s$ in these equations have to be replaced by ${}^{c}\psi_c$, ${}^{c}\psi_s$ and ${}^{c}\psi_{2m}$ respectively, which are defined by (5.5)
Analogous to expression (3.32), we find for the pressure distribution along the cylinder contour:

$$ P_H(1, \theta) = \rho B_o \frac{MB + NA}{A^2+B^2}\ddot{y} + \rho B_o \sigma \frac{MA - NB}{A^2+B^2}\dot{y} \qquad (5.\ 7) $$

where:

$$ A = {}^{c}\psi_c(1, \tfrac{\pi}{2}) + \sum_{m=1}^{N} p_{2m}\ {}^{c}\psi_{2m}(1, \tfrac{\pi}{2}) $$

$$ B = {}^{c}\psi_s(1, \tfrac{\pi}{2}) + \sum_{m=1}^{N} q_{2m}\ {}^{c}\psi_{2m}(1, \tfrac{\pi}{2}) \qquad (5.\ 8) $$

$$ M = {}^{c}\phi_s(1, \theta) + \sum_{m=1}^{N} q_{2m}\ {}^{c}\varphi_{2m}(1, \theta) $$

$$ N = {}^{c}\phi_c(1, \theta) + \sum_{m=1}^{N} p_{2m}\ {}^{c}\varphi_{2m}(1, \theta) $$

Inserting, according to (5.1), $\frac{dy}{dt} = \frac{1}{2}\, l\dot{\vartheta}$ and $\frac{d^2y}{dt^2} = \frac{1}{2}\, l\ddot{\vartheta}$, we find:

$$ P_H(1, \theta) = \frac{\rho B_o l}{2}\ \frac{MB + NA}{A^2+B^2}\ddot{\vartheta} + \frac{\rho B_o \sigma l}{2}\ \frac{MA - NB}{A^2+B^2}\dot{\vartheta} \qquad (5.\ 9) $$

The rolling moment on the system about P is found to be:

$$ M_R = -2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_H(1, \theta)\left\{ (x+\tfrac{1}{2}l)\frac{\partial x}{\partial\theta} + y\frac{\partial y}{\partial\theta} \right\} d\theta \qquad (5.\ 10) $$

Substituting (5.9), we find:

$$ M_R = -\rho B_o l\ \frac{M_oB+N_oA}{A^2+B^2}\ddot{\vartheta} - \rho B_o \sigma l\ \frac{M_oA-N_oB}{A^2+B^2}\dot{\vartheta} \qquad (5.\ 11) $$

where

$$ M_o = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} M(\theta)\left\{ (x+\tfrac{1}{2}l)\frac{\partial x}{\partial\theta} + y\frac{\partial y}{\partial\theta} \right\} d\theta \qquad (5.\ 12) $$

$$N_o = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} N(\theta) \left\{ (x+\tfrac{1}{2}l) \frac{\partial x}{\partial \theta} + y \frac{\partial y}{\partial \theta} \right\} d\theta \qquad (5.\ 12)$$

The added moment of inertia I_R^H and damping N_R^H per unit length due to the heaving component is given by:

$$I_R^H = \rho B_o l \frac{M_o B + N_o A}{A^2+B^2} \qquad (5.\ 13)$$

$$N_R^H = \rho B_o \sigma l \frac{M_o A - N_o B}{A^2+B^2} \qquad (5.\ 14)$$

Contribution of the rolling component

It is readily seen, that in this case the potentials and streamfunctions can be chosen identically to those of the pure swaying motion, defined by (4.5), ..., (4.8).

The complete potentil and streamfunction have the form (4.9) and (4.10). However, here the coefficients p_{2m} and q_{2m} have to be determined such that the following boundary condition on the contour of the right-hand cylinder is valid, [6, section 5.3]:

$$\frac{\partial\, {}^c\Phi}{\partial n} = R \frac{d\vartheta}{dt} \frac{dR}{ds} \quad \text{or} \quad -\frac{\partial\, {}^c\Psi}{\partial s} = \frac{d\vartheta}{dt} \frac{d}{ds} (\tfrac{1}{2}R^2) \qquad (5.\ 15)$$

where $R = \left\{x^2(1,\ \theta) + y^2(1,\ \theta)\right\}^{\frac{1}{2}}$ is the distance between the origin O and the point (x,y) on the contour of the right-hand cylinder and s the line coordinate along this contour (fig. 1.1).

In the usual manner we reduce (5.15) to:

$${}^c\Psi(1,\ \theta) - {}^c\Psi(1,\ \tfrac{\pi}{2}) = -\tfrac{1}{2} \frac{d\vartheta}{dt} \left\{ x^2(1,\ \theta) + y^2(1,\ \theta) - B_o^2 \right\} \qquad (5.\ 16)$$

Analogous to the method, expounded in [6; section 5.3], we represent the rolling motion by $\vartheta = \vartheta_a \cos(\sigma t + \gamma)$ and substitute $\frac{d\vartheta}{dt} = -\vartheta_a\, \sigma \sin(\sigma t + \gamma)$ in above formula. Then we obtain:

$$\frac{\pi\sigma}{gb} \left\{ {}^c\Psi(1,\ \theta) - {}^c\Psi(1,\ \tfrac{\pi}{2}) \right\} = g(\theta)\ (p_o \cos \sigma t + q_o \sin \sigma t) \qquad (5.\ 17)$$

where

$$g(\theta) = \frac{x^2(1,\ \theta) + y^2(1,\ \theta) - B_o^2}{B_o^2}$$

$$p_o = \frac{\pi \vartheta_a K\, B_o^2}{2b} \sin \gamma \qquad (5.\ 18)$$

and

$$q_o = \frac{\pi \vartheta_a K B_o^2}{2b} \cos \gamma \tag{5.18}$$

This can be reduced to the following set of linear equations for p_{2m} and q_{2m} :

$$\begin{aligned} {}^{c}\psi_c^d(1, \theta) - {}^{c}\psi_c^d(1, \tfrac{\pi}{2}) &= \sum_{m=0}^{N} p_{2m} f_{2m}(\theta) \\ {}^{c}\psi_s^d(1, \theta) - {}^{c}\psi_s^d(1, \tfrac{\pi}{2}) &= \sum_{m=0}^{N} q_{2m} f_{2m}(\theta) \end{aligned} \tag{5.19}$$

in which

$$\begin{aligned} f_o = g(\theta) &= \frac{x^2(1, \theta)+y^2(1, \theta)-B_o^2}{B_o^2} \\ f_{2m} &= {}^{c}\psi_{2m}^a(1, \tfrac{\pi}{2})-{}^{c}\psi_{2m}^a(1, \theta) \quad , m \neq o \end{aligned} \tag{5.20}$$

For the pressure along the contour of the right-hand cylinder due to the rolling motion of the separate cylinders, we find analogous to (4.23):

$$p_R(1, \theta) = \frac{-\rho g b}{\pi}(M \cos \sigma t - N \sin \sigma t) \tag{5.21}$$

where we substitute for the coefficients p_{2m} and q_{2m} , which are found in the expressions (4.22) for M and N, the values, which satisfy the set of equations (5.19).

Analogous to (4.32) and (4.33), we find for the hydrodynamic moment on the system.

$$M_R = \frac{2B_o^2 \rho g b}{\pi} \left\{ -X_R \sin \sigma t + Y_R \cos \sigma t \right\} \tag{5.22}$$

Combining this with the relation:

$$\frac{d\vartheta}{dt} = - \vartheta_a \sigma \sin(\sigma t+\gamma) = \frac{2bg}{\pi \sigma B_o^2}(-q_o \sin \sigma t - p_o \cos \sigma t) \tag{5.23}$$

which is derived from (5.18), we find:

$$M_R = - \rho B_o^4 \frac{Y_R q_o + X_R p_o}{p_o^2 + q_o^2} \ddot{\vartheta} - \rho \sigma B_o^4 \frac{Y_R p_o - X_R q_o}{p_o^2 + q_o^2} \dot{\vartheta} \tag{5.24}$$

For the added moment of inertia I_R^R and damping N_R^R per unit length due to rolling component we find:

$$I_R^R = \rho B_o^4 \frac{Y_R q_o + X_R p_o}{p_o^2 + q_o^2} \tag{5.25}$$

$$N_R^R = \rho \sigma B_o^4 \frac{Y_R p_o - X_R q_o}{p_o^2 + q_o^2} \tag{5.26}$$

where the expressions for the quantities X_R and Y_R are given by (4.34), in which we substitute for the coefficients p_{2m} and q_{2m}, which are found in the expressions (4.22) for M and N the values, satisfying the set of linear equations (5.19).

Finally, we find from (5.13), (5.14), (5.25) and (5.26) for the total added moment of inertia I_R and the total damping N_R of the system per unit length.

$$I_R = \rho B_o l \frac{M_o B + N_o A}{A^2 + B^2} + \rho B_o^4 \frac{Y_R q_o + X_R p_o}{p_o^2 + q_o^2} \tag{5.27}$$

$$N_R = \rho B_o \sigma l \frac{M_o A - N_o B}{A^2 + B^2} + \rho \sigma B_o^4 \frac{Y_R p_o - X_R q_o}{p_o^2 + q_o^2} \tag{5.28}$$

Analogous to (4.26) the swaying force on the system due to the rolling motion is determined from:

$$F_{SR} = -2B_o \int_{-\pi/2}^{\pi/2} \left\{ p_H(1,\theta) + p_R(1,\theta) \right\} \frac{V(\theta)}{G} \, d\theta \tag{5.29}$$

For $p_H(1,\theta)$ we substitute (5.9) while for $p_R(1,\theta)$ the following expression is inserted:

$$p_R(1,\theta) = \frac{\rho B_o^2}{2} \frac{M q_o + N p_o}{p_o^2 + q_o^2} \ddot{\vartheta} + \frac{\rho \sigma B_o^2}{2} \frac{p_o M - q_o N}{p_o^2 + q_o^2} \dot{\vartheta} \tag{5.30}$$

which is obtained by combining (5.21) and (5.23).
Then:

$$\begin{aligned} F_{SR} = & -\rho B_o^2 l \frac{\bar{M}_o B + \bar{N}_o A}{A^2 + B^2} \ddot{\vartheta} - \rho \sigma l B_o^2 \frac{\bar{M}_o A - \bar{N}_o B}{A^2 + B^2} \dot{\vartheta} \\ & - \rho B_o^3 \frac{\tilde{M}_o q_o + \tilde{N}_o p_o}{p_o^2 + q_o^2} \ddot{\vartheta} - \rho \sigma B_o^3 \frac{\tilde{M}_o p_o - \tilde{N}_o q_o}{p_o^2 + q_o^2} \dot{\vartheta} \end{aligned} \tag{5.31}$$

The quantities $\bar{M}_o$ and $\bar{N}_o$ in the first two terms of this expression are given by:

$$\begin{aligned} \bar{M}_o &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} M(\theta) \frac{V(\theta)}{G} \, d\theta \\ \bar{N}_o &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} N(\theta) \frac{V(\theta)}{G} \, d\theta \end{aligned} \tag{5.32}$$

in which for M and N the expressions are inserted, which are given by (5.8).
The quantities $\tilde{M}_o$ and $\tilde{N}_o$ in the last two terms of (5.31) are obtained by

replacing in (5.32) the functions $M(\theta)$ and $N(\theta)$ in the integrands by the similar functions, which are found in (5.21).
So, finally, for the added mass and damping per unit length for swaying produced by the rolling motion we obtain:

$$M_{SR} = \rho B_o^2 l \frac{\bar{M}_o B + \bar{N}_o A}{A^2 + B^2} + \rho B_o^3 \frac{\tilde{M}_o q_o + \tilde{N}_o p_o}{p_o^2 + q_o^2} \tag{5. 33}$$

$$N_{SR} = \rho B_o^2 \sigma l \frac{\bar{M}_o A - \bar{N}_o B}{A^2 + B^2} + \rho\sigma B_o^3 \frac{\tilde{M}_o p_o - \tilde{N}_o q_o}{p_o^2 + q_o^2} \tag{5. 34}$$

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