Dynamics of patterns subject to noise

Joris van Winden

DYNAMICS OF PATTERNS SUBJECT TO NOISE

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The struggle itself toward the heights is enough to fill a man's heart.

Albert Camus, *The myth of Sisyphus*

Contents

Summary

Patterns occur naturally in many physical and biological systems. By pattern, we mean a structure which has a complicated spatial dependence, but retains its shape as time passes. Prototypical examples are water waves, traveling pulses in neurons, and tropical cyclones. This dissertation is concerned with the analysis of such patterns when they are subjected to a random environment, which we refer to as *noise*. The key questions which we address are stability, noise-induced motion, and long-time behavior of patterns.

In Chapter 1, we give a general introduction to the topic. Results from the literature and from later chapters are demonstrated with a concrete example: a pulse in the stochastic FitzHugh–Nagumo equation. The chapter also contains an outlook, consisting of some open problems and potential future research directions.

In Chapter 2, we study a specific pattern: a solitary wave in a parametrically forced stochastic nonlinear Schrödinger equation, which models a signal propagating through an optical fiber. We show that the wave is stable on an exponentially long time scale, much longer than previous results in similar settings. The main challenge is to deal with the dispersive nature of the equation, which is accomplished by using deterministic and stochastic Strichartz estimates to control the nonlinearities. As a result, we can work in the natural L^2-based solution space and with noise which has low regularity.

In Chapter 3, we prove long-time stability of patterns in an abstract framework, unifying and improving upon many existing results. We explain the degrees of freedom of noise-induced motion by comparing the symmetries of the pattern with those of the equation itself. A key novelty is that we systematically treat symmetry groups which may be noncommutative. On a technical level, we introduce a new tracking mechanism which is minimally simple and bypasses technical difficulties associated with other tracking methods. This results in a simple and flexible proof of stability and allows us to treat rougher noise than before in a general setting.

In Chapter 4, we prove synchronization by noise for traveling pulses. Synchronization by noise means that the position of the pulse after a long time becomes nearly independent of the initial position, and instead is almost entirely determined by the noise. The proof uses the method of phase reduction, which yields an autonomous SDE describing the position of the pulse. The SDE is valid on a time scale which is long enough for it to accurately capture the long-time behavior of the pulse. The proof method allows for highly degenerate noise, which is out of reach when using conventional methods to show synchronization for SPDEs.

In Chapter 5, we prove sharp moment bounds for the supremum of a sequence of stochastic integrals. By combining this with a generalized Ciesielski-type embedding which identifies C^α with a weighted subspace of ℓ^∞, we obtain new regularity estimates for SPDEs. The results have applications for numerical approximations to SPDEs and stability of noisy patterns.

SAMENVATTING

Patronen komen voor in veel biologische en natuurkundige systemen. Met de term patroon bedoelen we een structuur die zijn vorm behoudt naarmate tijd voorbij gaat. Typische voorbeelden zijn watergolven, elektrische signalen in zenuwcellen, en tropische cyclonen. Dit proefschrift betreft de analyse van patronen die worden beïnvloed door *ruis*, wat bestaat uit kleine storingen afkomstig van de omgeving. Centrale kwesties zijn stabiliteit, beweging veroorzaakt door ruis, en het gedrag van patronen na lange tijd.

Hoofdstuk 1 bevat een algemene inleiding over het onderwerp. Resultaten uit de literatuur en de andere hoofdstukken worden gedemonstreerd aan de hand van een concreet voorbeeld: een puls in de stochastische FitzHugh–Nagumo vergelijking. Het hoofdstuk bevat ook enkele open vragen en richtingen voor toekomstig onderzoek.

In Hoofdstuk 2 bestuderen we een specifiek patroon: een eenlinggolf in een parametrisch gedreven stochastische niet-lineaire Schrödingervergelijking. We bewijzen dat de golf stabiel is op een exponentieel lange tijdschaal, veel langer dan eerdere resultaten in een vergelijkbare setting. De voornaamste uitdaging is om met de dispersie van de vergelijking om te gaan. We bereiken dit door de niet-lineaire termen te temmen met behulp van deterministische en stochastische Strichartz-afschattingen. Hierdoor kunnen we met de natuurlijke L^2-gebaseerde oplossingen werken, en hebben we geen extra regulariteitsaannames nodig voor de ruis.

In Hoofdstuk 3 bewijzen we stabiliteit van patronen, op een lange tijdschaal, in een algemene setting. Hiermee verenigen en verbeteren we een aantal resultaten uit de literatuur. Ook verklaren we de vrijheidsgraden van een patroon aan de hand van de symmetrieën van het patroon en van de vergelijking zelf. Een belangrijke innovatie is dat we op systematische wijze patronen met niet-commutatieve symmetriegroepen behandelen. Daarnaast introduceren we een simpele fase-tracking methode, waardoor we een aantal technische complicaties inherent aan andere methodes vermijden. Dit resulteert in een simpel en flexibel bewijs waarmee we ruis kunnen behandelen wat ruiger is dan voorheen.

In Hoofdstuk 4 bewijzen we dat ruis leidt tot synchronisatie van bewegende pulsen. Dit houdt in dat de positie van een puls na lange tijd amper afhangt van de beginpositie, en dus vrijwel volkomen bepaald wordt door de ruis. Voor het bewijs gebruiken we een fase-reductie, met als resultaat een vergelijking die de positie van de puls beschrijft. Omdat deze vergelijking nauwkeurig is op een voldoende lange tijdschaal, kunnen we het gedrag van de puls na lange tijd karakteriseren. In tegenstelling tot conventionele methodes kunnen we met ons bewijs sterk ontaarde ruis behandelen.

In Hoofdstuk 5 bewijzen we scherpe momentafschattingen voor rijtjes van stochastische convoluties die geïndexeerd zijn door een discrete parameter. Door deze afschattingen te combineren met een gegeneraliseerde Ciesielski-inbedding die C^α identificeert met een deelruimte van ℓ^∞, leiden we nieuwe regulariteitsafschattingen af voor SPDVs. De resultaten hebben toepassingen voor numerieke benaderingen en voor stabiliteit van stochastische patronen.

Preface

This dissertation was written during my time as a PhD candidate in the Analysis group at the Delft Institute of Applied Mathematics, funded by a DIAM fast-track scholarship. I am grateful to the institute for funding my research proposal, which has allowed me to pursue an accelerated PhD as an extension of the research initiated in my MSc thesis.

I would like to express my gratitude to the people who contributed in any way toward the dissertation and the projects on which it is based. Firstly, I want to thank my advisor and copromotor Manuel Gnann for his support, and for introducing me to this topic and the welcoming community of researchers working on it. I appreciate his encouragement, thoughtful advice, careful proofreading, and the occasional reminder that a PhD is not about how many projects you can start, but rather how many you can finish. I also want to thank my promotor Mark Veraar for many interesting discussions, valuable comments, and for encouraging me to continue the line of research which led to Chapter 5.

A large part of the dissertation is based on collaborative works, for which I want to thank my coauthors Sonja Cox, Manuel Gnann, Christian Kuehn, and Rik Westdorp. Ongoing projects with Björn de Rijk and Max Sauerbrey did not make it into the dissertation, but still resulted in insights which were helpful during the writing of Chapter 1. I am grateful to Christian Kuehn and Max Sauerbrey for inviting me to visit TU Munich and the Max Planck Institute for Mathematics in the Sciences, respectively. I would like to thank Stefan Geiss for constructive suggestions on the material of Chapter 5. I am grateful to Manuel Gnann, Esmée Theewis, and Mark Veraar for reading parts of the dissertation and providing helpful comments.

I have greatly enjoyed my time as a PhD candidate, and for this I have many people to thank. Firstly my office mates Bálint, Joshua, Max, and Noé: you brought life to our shared office and made it more than just a place to prove inequalities. More broadly, I am grateful to my fellow MaPhyA members as well as everyone in the Analysis and the Mathematical Physics groups for creating a friendly and engaging environment. Finally I want to thank my friends, my family, and Xiaoxia, for their unconditional love and support.

Joris van Winden
Delft, October 2025

1

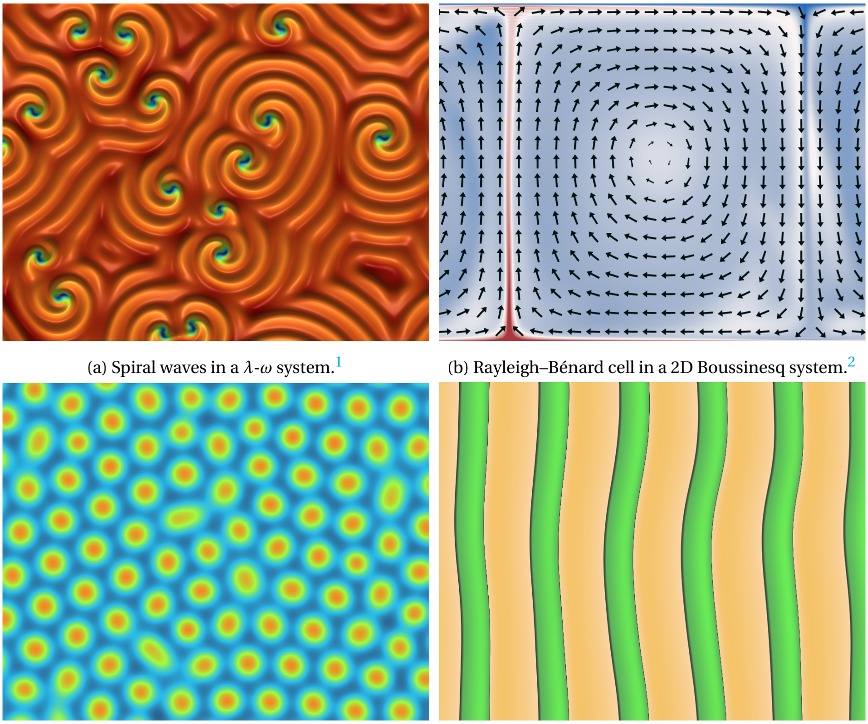
INTRODUCTION

Evolutionary partial differential equations, typically taking the form

$$\partial_t u = f(u, \nabla u, D^2 u, \dots), \tag{PDE}$$

have proven to be immensely useful for modeling the behavior of physical systems. Many of these equations admit solutions which have a complex spatial structure but evolve in time in a simple and predictable way. A selection of examples shows that these solutions, which we refer to as *patterns*, are important to a wide range of research areas:

- Neuroscience: traveling waves in the Hodgkin–Huxley [117], Nagumo [176], and FitzHugh–Nagumo [89] models for neural pulse propagation.
- Cardiac dynamics: spiral waves in the heart muscle [68], which have been proposed as a potential cause of cardiac arrhythmias.
- Population dynamics: traveling waves in invasion models with logistic growth [88, 140] and models subject to an Allee effect [155].
- Chemistry: spiral, target and wave patterns in autocatalytic reactions [197, 235].
- Fluids: Korteweg–de Vries solitons [143], Camassa–Holm peakons [49], Rayleigh–Bénard convection cells [198].
- Optics: solitons in the nonlinear Schrödinger equation [6].



(a) Spiral waves in a λ-ω system.[1]

(b) Rayleigh–Bénard cell in a 2D Boussinesq system.[2]

(c) Spots in a Brusselator model.[3]

(d) Planar wave in a Klausmeier model.[4]

Figure 1.1: Solutions of pattern forming systems, made with the VisualPDE interactive solver [223]. Simulations are adapted from examples on `https://visualpde.com/explore.html`.

However, modeling real-world phenomena using equations such as (PDE) can sometimes lead to incorrect predictions, since they fail to account for the influence of the surrounding environment. To remedy this, it is often appropriate to model this influence using a random forcing term ξ which is commonly referred to as *noise*. This results in a stochastic PDE of the form

$$\partial_t u = f(u, \nabla u, D^2 u, \ldots) + \sigma \cdot g(u, \nabla u, D^2 u, \ldots)\xi. \tag{SPDE}$$

The advantage of this approach is that it requires only a statistical description of ξ, which could be derived physically (e.g. by a separation of scales or a fluctuation-dissipation relation) or could be phenomenological. Either way, SPDEs have seen many important applications:

- Landau–Lifschitz theory of fluctuating hydrodynamics [153, Chapter 9].
- Turbulence: Kolmogorov's K41 theory for fully developed turbulence [141, 142],

[1]Interactive simulation available at `https://visualpde.com/sim/?mini=-lT0uJEY`.
[2]Interactive simulation available at `https://visualpde.com/sim/?mini=Yx9QKP8r`.
[3]Interactive simulation available at `https://visualpde.com/sim/?mini=cMXgM83e`.
[4]Interactive simulation available at `https://visualpde.com/sim/?mini=0AOMVvj5`.

Obukhov–Corrsin theory for scalar turbulence [61, 189], Kraichnan model [144, 145].

- Fluctuations in conservative systems [70, 134].
- Random surface growth models [110, 131].
- Stochastic quantization of Euclidean quantum field theories [64, 109, 190].

Aside from this, stochastic models have provided insight into the Gordon–Haus effect of jittering solitons [100], droplet rupture and formation time scales for thin films [103], and reliability of neural signaling [79]. Motivated by the abundance of pattern solution to PDEs and the modeling relevance of SPDEs, the central question underlying this dissertation is the following:

How do patterns behave in the presence of noise?

To make this question more tractable, we restrict our study to the regime of small noise. For this reason, (SPDE) includes a parameter $0 < \sigma \ll 1$ which is used to control the amplitude of the noise. Interestingly, we shall see that the noise significantly affects the dynamics even in the limiting case where this amplitude approaches zero.

1.1. NOISY PATTERNS

The behavior of noisy patterns has been widely studied in the physics literature. Although the pattern shape, domain geometry and physical scales vary greatly, the following phenomena have been observed: [45, 100, 136, 208, 209]

- Patterns are stable (i.e., observable) on a macroscopic time scale T_s.
- Noise-induced motion occurs on macroscopic time scale T_m.
- Macroscopic motion occurs before the pattern breaks down (i.e., $T_m \ll T_s$).
- The random motion may be biased towards certain directions on average.
- Noninteracting patterns may synchronize when subjected to common noise.

The random motion of patterns has been of particular interest, and various attempts have been made to give a mathematical description. However, due to the roughness of the noise this is a delicate matter, and as a result there has been disagreement about which equations accurately describe the random motion [215, 234]. One of the difficulties is that certain terms which naively appear to be of higher order contribute meaningfully to the motion, and hence cannot be neglected. This motivates a mathematically rigorous treatment which aims to answer the following questions.

1. (stability) How can we quantify T_s and T_m in terms of the noise amplitude σ?
2. (effective motion) Which equations correctly describe the effective motion?
3. (long-term behavior) How do patterns move for $T_m \ll t \ll T_s$?

By now there is a considerable body of mathematical work concerned with these questions, which we briefly review.

Before we proceed, let us remark that more detailed mathematical formulations of these questions and their answers are stated in the upcoming Section 1.2, where a concrete example is treated.

1.1.1. Stability

The matter of stability has by far received the greatest amount of attention from mathematicians, and is treated for instance in [37, 39, 40, 74, 112, 113, 115, 121, 147, 148, 154]. However, a limitation shared by all these works is that they only establish $T_s \gtrsim T_m$. Results which show the stronger result $T_s \gg T_m$ (important for applications and in line with empirical observations) are more scarce, but can be found in [12, 13, 114, 161].

1.1.2. Effective motion

It is a near-universal feature of the aforementioned works that they include a *phase-tracking* mechanism, which contains a mathematical description of the pattern motion. In our context, the term *phase* refers to the (generalized) position of the pattern. A significant challenge is that there is no canonical definition of position for a noisy pattern, since the solution only approximately resembles the deterministic 'perfect' profile. This is reflected by the literature, in which we can identify at least five distinct tracking methods.

- Stochastic freezing phase [33, 35, 112, 113, 114, 115, 227, 228].
- Variational phase [121, 150, 161, 162].
- Collective coordinate approach [51, 52].
- Phase-lag method [74, 147, 148, 202].
- Isochronal phase [1, 4].

This classification is not comprehensive, and does not cover [12, 13, 31, 37, 39, 40, 158]. However, after examining these tracking mechanisms, one finds that all of them require full knowledge of the solution u to (SPDE) to determine the phase for $t \gg T_m$. This is not problematic for a stability proof, but does have the unfortunate consequence that the tracking mechanisms do not provide effective (i.e., decoupled from (SPDE)) equations of motion for $t \gg T_m$. As a result, the resulting insight into the long-time motion of the patterns is limited.

It is only recently that truly effective equations of motions have been derived which are valid for $t \gg T_m$ and also completely decoupled from the full solution u. A key development has been the introduction of the *isochronal phase*, which was already widely applied to nonlinear oscillators [105, 230], to this context by Adams [1] and Adams and MacLaurin [4]. Although the isochronal phase is a priori defined in terms of the full solution, a natural approximation results in a fully autonomous SDE for the phase which is valid on a time scale strictly larger than T_m (see Section 4.3 of Chapter 4). Moreover, the coefficients of this SDE can be directly computed from the noise correlation and the isochron map, which in turn is computable from the deterministic dynamics.

1.1.3. LONG-TIME BEHAVIOR

Even more recently, rigorous results on long-time behavior have started to appear. Since most research has focused on one-dimensional traveling waves, a key quantity of interest has been the noise-induced correction to the (average) wave speed. Although a formula for the speed correction was conjectured in [113], the first rigorous proof was given in [4] using ergodic properties of the isochronal phase. The later work [34] showed how to compute averages of more general quantities, such as the wave shape. The isochronal phase is not used in [34], but the proof instead heavily relies on a (statistical) translational invariance of the noise.

Remark 1.1.1. It is not obvious that the rigorously established formula for the average speed in [4] agrees with the conjectured formula in [113, §2.3], and the connection between the formulas has not been remarked on in the literature. However, a second-order version of the calculation from the proof of Lemma 4.3.5 in Chapter 4 shows that the formulas agree.

1.2. CASE STUDY: FITZHUGH–NAGUMO EQUATION

We will now review the essential mathematical ideas and results concerning noisy patterns. For the sake of concreteness, this is done by treating an important example: the stochastic FitzHugh–Nagumo equation with linear multiplicative noise:

$$\begin{aligned} \partial_t u &= \partial_{xx} u + u(1-u)(u-a) - v + \sigma u \xi, \\ \partial_t v &= \epsilon(u - \gamma v), \end{aligned} \tag{1.2.1}$$

for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. We shall write $w(t,x) = (u(t,x), v(t,x))^\top$.

1.2.1. EXISTENCE AND LINEAR STABILITY

Our starting point is the existence of a stable fast pulse solution to (1.2.1) in the deterministic case (i.e., $\sigma = 0$ and there is no noise). The parameter $\epsilon \ll 1$ gives rise to a slow-fast structure, which allows one to construct the pulse using geometric singular perturbation theory. In the nonasymptotic case $\epsilon = 0.01$, existence and stability of the pulse have been shown using computer-assisted techniques [14].

Theorem 1.2.1 (Deterministic stable pulse [60, 124, 233])**.** *For* $0 < a < 1/2$*,* $\gamma > 0$*, and* $\epsilon \ll 1$*, there exists a pulse speed* $c > 0$ *and a pulse profile* $w^* = (u^*, v^*)^\top$ *which solves the* traveling wave ODE*:*

$$\begin{aligned} -c\partial_x u^* &= \partial_{xx} u^* + u^*(1-u^*)(u^*-a) - v^*, \\ -c\partial_x v^* &= \epsilon(u^* - \gamma v^*). \end{aligned} \tag{1.2.2}$$

The profile satisfies $\lim_{|x|\to\infty} w^*(x) = 0$ *and is spectrally stable.*

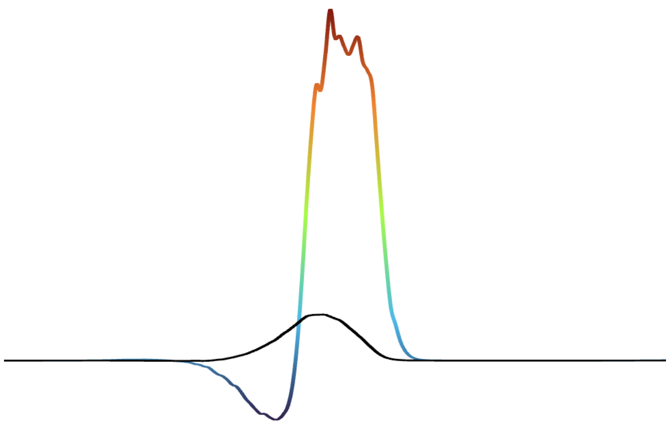


Figure 1.2: Noisy fast pulse in the stochastic FitzHugh–Nagumo equation with $a = 0.1$, $\epsilon = 0.01$, $\gamma = 5$, made with the VisualPDE interactive solver[5] [223]. Existence and stability of a deterministic pulse with these parameters is proven in [14].

Using (1.2.2) it is easily seen that the *traveling pulse* $\hat{w}(t,x) := w^*(x-ct)$ solves (1.2.1) with $\sigma = 0$. Spectral stability in Theorem 1.2.1 is defined in terms of the operator

$$\mathscr{L}^* = \begin{pmatrix} \partial_{xx} - 3u^{*2} + 2(a+1)u^* - a & -1 \\ \epsilon & -\gamma \end{pmatrix} + c\partial_x,$$

which is the linearization of (1.2.1) around $\hat{w}$ in a co-moving coordinate frame. By differentiating (1.2.2) with respect to x, it is seen that $\partial_x w^*$ is an eigenfunction of $\mathscr{L}^*$ with eigenvalue 0. Spectral stability then entails that the following two statements hold:

- (simple eigenvalue) 0 is a simple eigenvalue of $\mathscr{L}^*$.
- (spectral gap) There exists $\alpha > 0$ such that $\operatorname{Re}\sigma(\mathscr{L}^*) \setminus \{0\} \leq -\alpha$.

Note that the spectrum of $\mathscr{L}^*$ might depend on the choice of function space, which we have not specified yet. However, since w^* is smooth the above statements holds true for a wide range of spaces including C_{ub}, L^2, and H^n.

1.2.2. DETERMINISTIC ORBITAL STABILITY

From now on, we will fix a, γ, ϵ which satisfy the conditions of Theorem 1.2.1, and let c and w^* be the resulting pulse speed and profile. We will write $\|\cdot\|$ to denote an appropriate function space norm; an admissible choice would be the Bessel space $H^{s,p}$ with $p \geq 2$ and $s > 1/p$.

From the spectral stability and sectoriality of $\mathscr{L}^*$, the following nonlinear stability result can be deduced.

Theorem 1.2.2 (Deterministic orbital stability [80])**.** *Let $w(t,x)$ be the solution to* (1.2.1) *with $\sigma = 0$ and initial condition $w(0,x) = z(x)$. If $\|z(\cdot) - w^*(\cdot - \gamma_0)\|$ is sufficiently small*

[5] Interactive simulation available at https://visualpde.com/sim/?mini=NXg0lHMz.

for some $\gamma_0 \in \mathbb{R}$*, then there exists a unique* $\pi(z) \in \mathbb{R}$ *such that*

$$\lim_{t\to\infty} \| w(t,\cdot) - w^*(t,\cdot - ct - \pi(z))\| = 0. \tag{1.2.3}$$

In other words, a solution starting close to a pulse will converge to a shifted version of the pulse as $t \to \infty$. From (1.2.3) we also see that the long-time behavior of pulse-like solutions is completely encoded in the functional π, which is called the *isochronal phase*. We will see that π (or rather, its functional derivatives) also plays a crucial role in characterizing the stochastic dynamics.

1.2.3. STOCHASTIC STABILITY

We now turn our attention to the more dynamic case $\sigma > 0$. Before we do so, let us briefly comment on the properties of ξ. We will assume throughout that ξ is a Gaussian random field which is white in time and colored in space, meaning that it (formally) satisfies the correlation relation

$$\mathbb{E}\left[\xi(t,x)\xi(t',x')\right] = \delta(t-t')g(x,x'), \tag{1.2.4}$$

for some suitable spatial correlation function g. Such a noise can also be represented as

$$\xi(t,x) = \sum_{k\in\mathbb{N}} e_k(x)\dot{\beta}_k(t),$$

where $(e_k)_{k\in\mathbb{N}}$ is a sequence of functions and $(\beta_k)_{k\in\mathbb{N}}$ is a sequence of independent standard Brownian motions. In this setting, the well-known semigroup approach to SPDEs [66] provides a robust solution theory for (1.2.1).

In the noisy case $\sigma > 0$, statements like Theorem 1.2.2 typically do not hold and it is generally expected that the noise will destroy the pulse sooner or later. To quantify this, we introduce stopping times τ_σ^ε which measure the first time that a solution to (1.2.1) leaves an ε-neighborhood of the set of translates of w^*:

$$\tau_\sigma^\varepsilon := \inf\{t \geq 0 : \inf_{\gamma\in\mathbb{R}} \| w(t,\cdot) - w^*(\cdot - \gamma)\| \geq \varepsilon\}. \tag{1.2.5}$$

The ε in (1.2.5) is not to be confused with ϵ in (1.2.1), which will play no further role from now on. The statement 'the pulse is stochastically orbitally stable' is then interpreted to mean 'the exit time τ_σ^ε is large with high probability'. The following theorem gives quantitative meaning to this statement. A proof of (1.2.6a) can be found in Chapter 3 or in [161], and a proof of (1.2.6b) is contained in Section 4.3 of Chapter 4.

Theorem 1.2.3 (Stochastic orbital stability)**.** *Let* w_σ *be the solution to* (1.2.1) *with initial condition* $w_\sigma(0,x) = w^*(x)$*. There exists* $C > 0$ *such that*

$$\mathbb{P}[\tau_\sigma^\varepsilon < T] \lesssim (1+T)\exp(-C(\varepsilon/\sigma)^2). \tag{1.2.6a}$$

Moreover, for $\varepsilon \ll 1$ *we have*

$$w_\sigma(t) = w^*(\cdot - \pi(w_\sigma(t))) + \mathscr{O}(\varepsilon), \qquad t \leq \tau_\sigma^\varepsilon. \tag{1.2.6b}$$

By balancing T against σ with $\varepsilon \ll 1$ fixed on the right-hand side of (1.2.6a), it is seen that the characteristic time scale of stability satisfies $\log T_s \gtrsim \sigma^{-2}$. Moreover, this scaling is expected to be optimal by analogy with classical Freidlin–Wentzell theory [92].

To understand the dynamics of w_σ, it now suffices by (1.2.6a)-(1.2.6b) to characterize the evolution of the isochronal phase $\pi(w_\sigma)$. Applying the Itô formula to $\pi(w_\sigma)$ and using the approximation (1.2.6b) together with translational symmetry of (1.2.1), it follows that $\pi(w_\sigma)$ can be approximated by a process γ which satisfies the following SDE[6] (see Section 4.3 of Chapter 4 or [4] for a derivation):

$$\begin{aligned} \mathrm{d}\gamma = c\,\mathrm{d}t + \sigma \sum_{k\in\mathbb{N}} \pi'(w^*)[e_k(\cdot+\gamma)w^*]\,\mathrm{d}\beta_k \\ + \tfrac{1}{2}\sigma^2 \sum_{k\in\mathbb{N}} \pi''(w^*)[e_k(\cdot+\gamma)w^*, e_k(\cdot+\gamma)w^*]\,\mathrm{d}t. \end{aligned} \tag{1.2.7}$$

The advantage of this SDE is that it is autonomous and decoupled from (1.2.1). Moreover, the coefficients can be computed directly from w^*, e_k, and functional derivatives of π evaluated at w^*. An explicit expression for $\pi'(w^*)$ in terms of the adjoint zero eigenfunction of $\mathscr{L}^*$ is even available (Lemma 4.3.5). An equally simple characterization for π'' does not seem to exist.

Equation (1.2.7) also reveals that the natural time scale for noise-induced motion satisfies $T_m \sim \sigma^{-2}$.

The following theorem quantifies the accuracy of the approximation $\gamma \approx \pi(w_\sigma)$.

Theorem 1.2.4 (Effective motion)**.** *We have the moment estimate*

$$\mathbb{E}\Big[\sup_{t\in[0,T\wedge\tau_\sigma^\varepsilon]} \|\pi(w_\sigma(t)) - \gamma_\sigma(t)\|^2\Big] \lesssim \varepsilon^2\sigma^2 T e^{\sigma^2 T}. \tag{1.2.8}$$

As a consequence, there exist $\delta, \mu, \nu > 0$ such that upon introducing the time scale

$$t_\sigma = \sigma^{-2}\log(\sigma^{-\delta}), \tag{1.2.9}$$

we have the estimate

$$\mathbb{P}\Big[\sup_{t\in[0,t_\sigma]} \|w_\sigma(t,\cdot) - w^*(\cdot-\gamma_\sigma(t))\| \gtrsim \sigma^\mu\Big] \lesssim \sigma^\nu, \qquad \sigma \ll 1. \tag{1.2.10}$$

The most important feature of (1.2.10) is the time scale of the validity given by (1.2.9). Since $t_\sigma \gg T_m \sim \sigma^{-2}$ by the extra logarithmic factor, we can safely state that (1.2.7) correctly describes the noise-induced motion. As a consequence, the long-time behavior of w_σ is governed by the long-time behavior of (1.2.7) (with respect to its own natural time scale $T_m \sim \sigma^{-2}$).

Let us now assume that ξ is translation invariant in a statistical sense, meaning that the laws of $\xi(\cdot - b)$ and $\xi(\cdot)$ are identical for every $b \in \mathbb{R}$. In that case, after defining

$$c_1 := \tfrac{1}{2} \sum_{k\in\mathbb{N}} \pi''(w^*)[e_k(\cdot+b)w^*, e_k(\cdot+b)w^*],$$

[6] The SDE is only valid if we interpret (1.2.1) and (1.2.7) in the Itô sense. When interpreting (1.2.1) in the Stratonovich sense, an additional term would be present in (1.2.7). Remarkably, this term does not disappear when interpreting (1.2.7) in the Stratonovich sense.

it follows from symmetry considerations that c_1 does not depend on b. By grouping terms together in (1.2.7), we can interpret $c_1\sigma^2$ as the leading-order correction to the average wave speed. The following theorem, first established in [4], provides a rigorous basis for this interpretation.

Theorem 1.2.5 (Correction to the wave speed)**.** *Let t_σ be as in* (1.2.9)*, and define $c_2(\sigma)$ via*

$$\frac{\pi(w_\sigma(t_\sigma))}{t_\sigma} = c + (c_1 + c_2(\sigma))\sigma^2. \tag{1.2.11}$$

Then $c_2(\sigma)$ converges to zero in probability as $\sigma \to 0$.

Clearly, c_1 cannot be replaced by any other (deterministic) value in Theorem 1.2.5. This uniqueness is exactly what allows us to interpret c_1 as the leading-order correction. Since it can be difficult to compute c_1 analytically, a numerical method is proposed in the recent preprint [170].

The final theorem is the main result from Chapter 4, and concerns two pulses which start from different initial positions but which experience the same noise realization ξ. Somewhat surprisingly, we find that these pulses end up at the same position for times $t \gg T_m$ with high probability, regardless of the initial positions. To formulate the statement, we write w_σ^b for the solution to (1.2.1) with initial condition $w_\sigma^b(0,\cdot) = w^*(\cdot - b)$.

Theorem 1.2.6 (Synchronization by noise)**.** *Let t_σ be as in* (1.2.9)*, and suppose that ξ is periodic with period L and satisfies a (rather weak) nondegeneracy condition (see Assumption 4.3). For any $x, y \in \mathbb{R}$ and $\varepsilon > 0$ it holds that*

$$\lim_{\sigma\to 0} \mathbb{P}\left[\inf_{n\in\mathbb{Z}} \| w_\sigma^x(t_\sigma,\cdot) - w_\sigma^y(t_\sigma,\cdot - nL)\| \geq \varepsilon\right] = 0. \tag{1.2.12}$$

The infimum over n indicates that synchronization happens modulo integer translations (i.e., pulses whose positions differ by an integer multiple of L are considered to be the same). Since ξ has period L by assumption, it always holds that $w_\sigma^{x+L}(t,\cdot) = w_\sigma^x(t,\cdot - L)$, which shows that the infimum cannot be removed from the theorem statement. The proof of Theorem 1.2.6 is contained in Chapter 4.

1.3. OVERVIEW OF RESULTS

We now briefly summarize the main contributions of the dissertation, which are contained in the upcoming chapters. For more detailed theorem statements, we refer to the individual chapter introductions.

Chapter 2, based on [A3], treats orbital stability of a solitary wave in a stochastic parametrically forced nonlinear Schrödinger equation. Stability on a time scale $t \sim \exp(\sigma^{-2})$ is achieved using a short-time phase tracking method combined with a resetting procedure. Using deterministic and stochastic Strichartz estimates, we can work in the natural L^2-solution space and do not require regularity of the noise beyond that necessary for well-posedness.

In Chapter 3, which is based on [P2], the resetting procedure is extended to a much more general setting. By encoding the symmetries of the pattern and the equation in a Lie group and using a minimally simple short-time tracking method, we obtain a comparatively simple proof of stability which recovers and improves upon many existing results. Important novelties are that our proof includes patterns and equations with general noncommutative symmetry groups, and does not require the state space to be Hilbert.

In Chapter 4, which is based on [A1], we prove synchronization by noise for a general class of traveling pulses. This is done by performing a phase reduction using the isochronal phase and carefully analyzing the resulting SDE which describes the position of the pulse. An explicit characterization of the first variation of the isochron map allows us to formulate a weak nondegeneracy condition which ensures that the pulse synchronizes.

Chapter 5 is a slight departure from the main topic. Based on [A2], it contains new moment inequalities for sequences of indexed martingales and stochastic convolutions. After identifying C^α with a subspace of ℓ^∞ by means of a general Kolmogorov-type chaining argument, the inequalities allow us to prove moment estimates for stochastic integrals in C^α with only a logarithmic loss in the modulus of continuity. As a second application, we deduce moment estimates for stochastic convolutions with exponentially stable semigroups. Corresponding tail estimates, which follow as an immediate consequence, have previously been used to show stability of noisy patterns on long time scales.

1.4. OUTLOOK

Taken together, Theorems 1.2.3-1.2.6 paint a clear picture of the dynamics of stochastic pulse solutions to (1.2.1), and it seems that this situation is now well-understood. Moreover, the theorems hold true in much greater generality than stated above. Nevertheless, many questions involving more complicated settings remain open. We now outline some future directions and sketch several possible projects in varying levels of detail.

1.4.1. ABSENCE OF A SPECTRAL GAP

A near-universal assumption in works on stability of noisy patterns is the presence of a spectral gap of the linearized dynamics. However, there are many types of patterns which generically have continuous spectrum touching or lying on the imaginary axis, such as:

- Spatially periodic wave trains, already in one dimension [199].
- Planar wave fronts in $\mathbb{R}^d$ with $d \geq 2$ [35, 128].
- Viscous shock waves in systems of hyperbolic conservation laws [236].
- Spiral waves [204].
- Solitons in integrable (Hamiltonian) systems [101, 102].
- Pulled fronts in invasion processes [19].

In the deterministic setting, nonlinear stability has been shown using a wide variety of techniques such as weighted function spaces [127, 205], variational methods [101, 102], or pointwise bounds on the Green's function [236]. Currently, applying any of these methods to stochastic settings seems challenging, and it is unclear whether existing stability results are optimal.

Aside from obstructing stability proofs, essential spectrum near the imaginary axis potentially allows for noise to have a bigger effect on the dynamics. An interesting example is the Fisher–Kolmogorov–Petrovsky–Piskunov (F-KPP) equation, which admits a family of traveling fronts with different speeds, all of which are stable against localized perturbations. However, there is a distinguished front with (in this case, minimal) speed c^* which is selected by the dynamics for generic steep initial data [86]. This front is marginally stable, in the sense that the essential spectrum touches the imaginary axis in an optimally weighted space. For F-KPP with noise of amplitude $\sigma \ll 1$ it was conjectured by Brunet and Derrida [45] and proven by Mueller, Mytnik, and Quastel [175] that the correction to the observed speed is of order $|\log\sigma|^{-2}$, much larger than the order σ^2 correction in Theorem 1.2.5 which applied to waves with a spectral gap. Since marginal stability of the selected front is a generic feature of pulled invasion processes according to the *marginal stability conjecture* [201], we raise the following question:

> *Does the $\mathcal{O}(|\log\sigma|^{-2})$ Brunet–Derrida correction to the wave speed for F-KPP apply to generic noisy pulled invasion fronts?*

Since the marginal stability conjecture has been proven recently for higher-order scalar equations [19], and even more recently for systems of equations [18], we expect that this question could be answered at the same level of generality.

1.4.2. WEAKLY INTERACTING PATTERNS

A feature shared by many of the aforementioned patterns is that they are localized in the sense that the spatial profile converges, typically at an exponential rate, to an asymptotic rest state. Thus, it seems plausible that a profile which is a superposition of multiple widely-separated individual patterns may be (meta-)stable on large time scales. For deterministic fronts and pulses, this was studied in [72, 73], and we believe that a stochastic extension of these results would be worthwile and could result in interesting dynamics. As an example, we conjecture that patterns which attract each other deterministically (possibly leading to their eventual destruction) could remain separated in the presence of spatially correlated noise due to the synchronization effect demonstrated in Theorem 1.2.6.

A closely related open problem concerns the stochastic complex Ginzburg–Landau equation, which has applications in the study of superconductivity. For the deterministic equation it is known that, under an appropriate rescaling of time and space, the dynamics are characterized by certain point defects called *vortices*, which carry a topological charge and move according to a system of ODEs [123]. In the stochastic case, vortices have been identified and characterized in [57], and the authors conjecture that the vortices move according to a system of SDEs. A proof of this fact, together with a detailed analysis of the SDE and its long-time behavior, would certainly be an interesting result.

1.4.3. HIGHER-DIMENSIONAL PATTERNS AND CHAOS

By Theorem 1.2.4, the dynamics of (1.2.1) near the pulse are essentially one-dimensional, and are accurately described by (1.2.7). Thus, any possible chaotic behavior is immediately ruled out by monotonicity. However, similar phase reductions can also be applied to patterns with more degrees of freedom, such as rotating and spiral waves [150] and multidimensional traveling waves [33, 35] (see also Chapter 3). In these cases the reduced SDEs are multidimensional, so chaotic behavior might occur as opposed to synchronization. We expect that the sign of the top Lyapunov exponent (which commonly characterizes chaos/synchronization) could be determined using either analytical [23] or computer-assisted techniques [42, 43].

1.4.4. SINGULAR SPDE

Another line of investigation would be to consider the effect of singular (i.e., rough) noise on the motion of patterns. The theory of singular SPDE has seen rapid development in the last decade, and by now several solution concepts based on regularity structures [109, 156], paracontrolled calculus [104], or energy solutions [99] are available. However, as of the writing of this dissertation, it seems there exists only a single preprint [232] containing results on stability and motion of patterns in a singular setting.

As a step in this direction, we propose to study a stochastic Allen–Cahn equation in a two-dimensional cylindrical domain:

$$\partial_t u = (\partial_{xx} + \partial_{yy})u + (1-u^2)(u-a) + \sigma\xi, \qquad (t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{T}, \tag{1.4.1}$$

where ξ is space-time white noise and $|a| < 1$. For $\sigma = 0$ it is well-known that the equation admits a planar traveling front solution $u(t,x,y) = u^*(x-ct)$ with a spectral gap.[7] Moreover, when the front is perturbed, the leading-order dynamics are captured by the Ansatz $u(t,x,y) = u^*(t, x-ct-\phi(t,y))$ [168], where ϕ satisfies

$$\phi_t = \Delta\phi + \frac{c}{2}|\nabla\phi|^2 \tag{1.4.2}$$

For $\sigma > 0$ and $c \neq 0$, a formal calculation similar to [209] shows that the dynamics of the front position ϕ are given by the Kardar–Parisi–Zhang (KPZ) equation. Interestingly, the counterterm necessary to renormalize KPZ [110] can then be interpreted as a diverging correction to the wave speed, showing that microscopic rough noise can have a macroscopic effect on wave propagation.

To highlight the feasibility of such a project we remark that among singular SPDE, the equations above are some of the most well-studied ones. In fact, both the stochastic Allen–Cahn and KPZ equations have solution concepts which predate regularity structures, based on the so-called 'Da Prato–Debussche trick' [64] and the Cole–Hopf transform [27] respectively. We also mention that $a = 0$ could serve as an easier test case because it implies $c = 0$, so that ϕ should satisfy a stochastic heat equation instead of KPZ.

[7] Here it is crucial that $y \in \mathbb{T}$. If $y \in \mathbb{R}$, the spectral gap disappears and the dynamics are significantly different already in the deterministic setting, as is seen from the results in [167, 168]. In the preprint [P1] we extend these results to planar waves in general reaction–diffusion systems.

1.5. MATHEMATICAL TOOLS AND PREREQUISITES

The topic of this dissertation lies at the intersection of dynamical systems, probability, and partial differential equations, and requires tools and concepts from all three fields. We now list the main tools which are used and provide some references.

Throughout the chapters, familiarity with functional analysis (spectral theory, function spaces), stochastic processes (martingales, Brownian motion, Itô calculus), and the semigroup approach to PDEs is assumed. Also, some harmonic analysis and interpolation theory will occasionally be used. In particular, we make use of the Bessel potential spaces $H^{s,p}$. In terms of dynamical systems, advanced theory is not needed as most arguments will be done 'by hand'. The existence and linear stability of patterns, which typically involve advanced dynamical systems techniques, are either assumed or taken from existing literature.

1.5.1. STOCHASTIC INTEGRATION IN BANACH SPACES

The theory of stochastic integration, pioneered by Kiyosi Itô, provides an effective way to interpret and solve SDEs forced by Gaussian noise which is white in time. This theory has since been been extended to infinite dimensions and has been combined with the theory of C_0-semigroups, resulting in the semigroup approach to SPDEs [66]. This approach provides a robust solution concept, essentially based on the classical method of variation of parameters, which has been applied in many settings. The main idea is that the solution to the stochastic PDE

$$\mathrm{d}u = Au\,\mathrm{d}t + f(t)\,\mathrm{d}W(t), \tag{1.5.1a}$$

can be represented as a *stochastic convolution* via

$$u(t) = \int_0^t S(t-s)f(s)\,\mathrm{d}W(s). \tag{1.5.1b}$$

Here, A is the generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on a Banach space X, f is an $\mathscr{L}(H;X)$-valued adapted integrand, and $(W(t))_{t\geq 0}$ is an H-cylindrical Wiener process, which is the infinite-dimensional generalization of a Brownian motion on a Hilbert space H.

When X is a Hilbert space, the stochastic integration theory used to give meaning to (1.5.1b) generalizes straightforwardly from the finite-dimensional setting. When X is only a Banach space, the situation is more delicate and two theories are available: one based on the UMD (unconditional martingale differences) property and one based on 2-smoothness, a geometric condition which is satisfied e.g. by L^p with $2 \leq p < \infty$. The latter is also referred to as the martingale type 2 or M-type 2 theory of stochastic integration. A theorem of Pisier [196] shows that a Banach space has martingale type 2 if and only if it is isomorphic to a space which is 2-smooth.

For an introduction to both theories we refer to the survey [187]. The theory based on 2-smoothness is the one used throughout the dissertation.

1.5.2. ADDITIONAL PREREQUISITES

Chapter 4 makes use of the theory of random dynamical systems in the sense of Arnold. The standard reference on this topic is [15], and we recommend the lecture notes [76] as an accessible introduction. In Chapter 3, the basic notions of Lie groups and Lie algebras are used to formulate stability results for patterns with general symmetries.

2

SOLITARY WAVES IN A STOCHASTIC PFNLS EQUATION

This chapter is based on the article

[A3] M.V. Gnann, R.W.S. Westdorp, and J. van Winden. "Solitary waves in a stochastic parametrically forced nonlinear Schrödinger equation". In: *SIAM Journal on Applied Dynamical Systems* 24.4 (2025), pp. 3012–3044. [doi].

Early results on well-posedness of the equation and stability of the solitary wave (on shorter time scales and with smoother noise) can be found in the second and third authors' MSc theses [226, 229].

Abstract. *We study a parametrically forced nonlinear Schrödinger (PFNLS) equation, driven by multiplicative translation-invariant noise. We show that a solitary wave in the stochastic equation is orbitally stable on a time scale which is exponential in the inverse square of the noise strength. We give explicit expressions for the phase shift and fluctuations around the shifted wave which are accurate to second order in the noise strength on a fixed time scale. This is done by developing a new perspective on the phase-lag method introduced by Krüger and Stannat. Additionally, we show well-posedness of the equation in the fractional Bessel space* H^s *for any* $s \in [0,\infty)$*, demonstrating persistence of regularity.*

2.1. INTRODUCTION

Optic fibers that act as wave guides for electromagnetic signals form the basis for systems of fiber-optic communications, enabling long-distance communication at high bandwidth [5]. The behavior of a pulse propagating through an optic fiber is governed by the nonlinear Schrödinger (NLS) equation [6], which is an archetypal example of a nonlinear dispersive equation that is known to support solitary waves. The NLS equation has many applications in physics, for instance in the description of Bose–Einstein condensates [44], deep-water waves [220], and plasma oscillations [210]. In these applications, the NLS equation describes the complex amplitude of a wave packet propagating through a nonlinear medium. We refer to [213] for a detailed treatment of the physical background.

In optic fibers, the nonlinear behavior arises due to a response of the refractive index of the fiber to an applied electric field known as the Kerr effect, leading to a cubic nonlinear term in the equation. Effective signal transmission in optic communication systems may be obstructed by the presence of linear loss in the fiber, weakening the signal as it propagates. Kutz et al. [151] proposed a method of compensating loss using periodic phase-sensitive amplification, which has since become a popular approach for increasing feasible transmission lengths. The approach is modeled by the parametrically forced nonlinear Schrödinger (PFNLS) equation:

$$\mathrm{d}u = (i\Delta u - i\nu u - \epsilon(\gamma u - \mu\overline{u}))\mathrm{d}t + i\kappa|u|^2 u\mathrm{d}t \quad \text{for } (x,t) \in \mathbb{R}\times\mathbb{R}^+. \tag{2.1.1}$$

Here, the complex-valued function $u(x,t)$ denotes the envelope of the electric field in an optic fiber, t is the distance along the fiber, and x denotes time in a translating frame that moves with the group velocity of light. The constants $\gamma > 0$ and $\mu > 0$ model linear loss in the fiber and phase-sensitive amplification, respectively. The constant $\nu \in \mathbb{R}$ models a phase advance of the signal carrier, and the constant $\kappa > 0$ denotes the strength of the Kerr effect in the fiber. In this model, the local effect of the periodically spaced phase-sensitive amplifiers is averaged over the spacing length of the amplifiers. This description assumes that the amplifiers are closely spaced, which is valid for long propagation lengths [171]. In particular, the model applies well to a re-circulating loop used for long-term storage of pulses in optical networks.

In case that $\mu > \gamma$, i.e. enough amplification is supplied, equation (2.1.1) admits stationary solutions $u(t,x) = u^*(x)$ called *solitary waves*, with the profile u^* given by

$$u^*(x) = \sqrt{\frac{2(\nu + \epsilon\mu\sin(2\theta))}{\kappa}}\operatorname{sech}(\sqrt{\nu+\epsilon\mu\sin(2\theta)}\,x)e^{i\theta}, \tag{2.1.2}$$

where θ is a solution to $\cos(2\theta) = \gamma/\mu$. This can be seen from [130, equation (1.8)] after scaling in κ by setting $\phi = \frac{1}{2}\sqrt{\kappa}u$. As equation (2.1.1) is translation invariant, shifting the solitary waves by an arbitrary constant $a \in \mathbb{R}$ produces a family of solutions. The solitary waves for which $\sin(2\theta) > 0$ were shown to be orbitally exponentially stable by Kapitula and Sandstede [130]: small perturbations of the solitary wave converge at an exponential rate to a suitable translate of the solitary wave. Solitary waves for which $\sin(2\theta) < 0$ are known to be unstable [152]. Here and throughout the rest of this chapter, the 'orbit' in orbital stability refers to the orbit of u^* under the action of the group of spatial translations.

We briefly note that in the physical application of optic fiber loops, the term standing wave is misleading, as the equation describes the electric field in a moving frame. The standing waves (2.1.2) represent traveling pulses, and their stability is crucial for attaining long transmission lengths of signals and for the feasibility of long-time storage.

The stability analysis in [130] relies on computing the spectrum of the (real-)linear operator

$$\mathscr{L} v = i\Delta v - i\nu v - \epsilon(\gamma v - \mu\overline{v}) + i\kappa(2|u^*|^2 v + (u^*)^2\overline{v})$$

on $L^2(\mathbb{R};\mathbb{C})$ associated with the linearization of (2.1.1) around the solitary wave. It is known that the spectrum of the linearization is located at an $\mathscr{O}(\epsilon)$ distance to the left of the imaginary axis, except for a simple eigenvalue at zero [11, 130]. This eigenvalue arises due to the translation invariance of (2.1.1). For $\epsilon = \nu = 0$, the operator $\mathscr{L}$ corresponds to the linearization around the primary soliton in the NLS equation, and has continuous spectrum on the imaginary axis. The primary NLS soliton is also orbitally stable, but no exponential decay of perturbations can be expected [173, 225]. As such, parametric forcing entails stronger linear stability.

2.1.1. THE STOCHASTIC EQUATION

Mecozzi et al. [171] discuss two mechanisms that further inhibit signal transmission by introducing noise in the system, thereby transforming the description of pulse propagation into a stochastic partial differential equation. In this chapter, we study the evolution of the solitary wave u^* (2.1.2) in the stochastic parametrically forced nonlinear Schrödinger (SPFNLS) equation:

$$\mathrm{d}u = (i\Delta u - i\nu u - \epsilon(\gamma u - \mu\overline{u}))\,\mathrm{d}t + i\kappa|u|^2 u\,\mathrm{d}t - iu\circ(\phi * \mathrm{d}W) \quad \text{for } (x,t) \in \mathbb{R}\times\mathbb{R}^+. \tag{2.1.3}$$

The symbol W denotes a cylindrical Wiener process in the Hilbert space $L^2(\mathbb{R},\mathbb{R})$, meaning that $\xi := \frac{\mathrm{d}W}{\mathrm{d}t}$ is a (formal) space-time white noise, and $\circ$ denotes the Stratonovich product. Here, ϕ is a real-valued function which serves to regularize the noise. Thus, u is multiplied by a noise process $\zeta := \phi * \xi$ which is white in time, and formally satisfies the covariance relation

$$\mathbb{E}\big[\zeta(t,x)\zeta(t,y)\big] = \int_{\mathbb{R}} \phi(z)\phi(z+x-y)\,\mathrm{d}z$$

in space. Because the covariance only depends on $x-y$, equation (2.1.3) preserves the physically relevant symmetry of translation invariance (in a statistical sense). This is highly relevant to our study of the motion of solitary waves.

The multiplicative noise term that we consider in (2.1.3) models phase noise induced by the coupling of light with the thermally excited acoustical modes of the fiber known as guided acoustic-wave Brillouin scattering (GAWBS) [171]. We use the Stratonovich product, as it is more realistic for physical applications. Indeed, in the absence of parametric forcing, it allows for conservation of the $L^2(\mathbb{R})$-norm [36, Proposition 4.1]. Because our variable x corresponds to physical time, our noise is correlated in time, which is a natural assumption in the context of GAWBS phase noise. The other noise effect proposed in [171] is due to quantum effects and results in an additive noise term. We focus in the present chapter only on the multiplicative GAWBS phase noise.

2.1.2. WELL-POSEDNESS

Our first result concerns well-posedness of the stochastic equation (2.1.3). We show that for any $s \geq 0$, ϕ in the fractional Bessel space $H^s(\mathbb{R};\mathbb{R})$ and $u(0) \in H^s_x$, equation (2.1.3) has a unique mild solution u taking values in the space $C([0,T];H^s_x) \cap L^r(0,T;L^p_x)$ for every $T > 0$ and certain pairs (p,r) (see Theorem 2.3.1 and Definition 2.2.2).

The 'standard' SNLS equation with linear multiplicative noise (corresponding to the case $\epsilon = \nu = \gamma = \mu = 0$) was first shown to be well-posed in the spaces L^2_x (corresponding to $s = 0$) [36] and H^1_x (corresponding to $s = 1$) [38]. A proof of the L^2_x well-posedness using stochastic Strichartz estimates is given in [118]. Since the PFNLS equation differs from the NLS equation by linear terms, our proof of well-posedness is very similar. The main novelties are well-posedness in $H^s(\mathbb{R};\mathbb{R})$ for $s \in [0,\infty) \setminus \{0,1\}$ and the use of translation-invariant noise. The translation-invariant noise, aside from being motivated by physical symmetries, is relevant to our subsequent study of the solitary waves and is not directly covered by previous results. The well-posedness in H^s_x shows that, like its deterministic counterpart, the SPFNLS (and by extension, the one-dimensional cubic SNLS) equation has *persistence of regularity*, meaning that regularity of the solution is the same as the minimum of that of the noise and the initial data. Previous results on stochastic versions of these equations have mainly been concerned with the cases $s = 0$ and $s = 1$.

2.1.3. ORBITAL STABILITY

With the well-posedness of (2.1.3) firmly established, we turn to the stability of the solitary wave u^* with $\sin(2\theta) > 0$ (see the discussion following (2.1.2)) in the stochastic equation. We establish that the solitary wave is orbitally stable under the multiplicative stochastic forcing in (2.1.3) on a time scale $T \sim \exp(\sigma^{-2})$, where σ denotes the strength of the noise. We describe the solution to (2.1.3) with initial condition close to u^* using the decomposition

$$u(x,t) = u^*(x + a(t)) + v(x,t),$$

where a is a real-valued stochastic process that tracks the wave position, and v an infinite-dimensional perturbation which is small when measured in the L^2_x-norm. In a parabolic setting, such problems are well-studied (see e.g. [112, 121, 148, 161]). Rigorous results in a dispersive setting are more scarce but available [37, 39, 40, 41, 227]. However, these works are limited to the time scale $T \sim \sigma^{-2}$, and stability for $T \sim \exp(\sigma^{-2})$ has not been shown before in this setting as far as we are aware.

We give explicit expressions for $a(t)$ and $v(t)$ which are accurate to second order in σ. Second-order results in this setting are scarce, and mostly consist of formal computations [154]. By developing a new perspective on an established phase-tracking method (see Section 2.1.4) we rigorously and efficiently prove accuracy of the second-order expressions for the first time.

To first order, the phase process $a(t)$ behaves like a Brownian motion with variance proportional to $t\sigma^2$, and the perturbation $v(t,x)$ behaves like an infinite-dimensional Ornstein–Uhlenbeck process. In particular, v satisfies an estimate of the form

$$\mathbb{E}\left[\|v(t)\|^2_{L^2_x}\right]^{1/2} \leq C\sigma(e^{-at}\|v(0)\|_{L^2} + \min\{t^{\frac{1}{2}},1\}) + \mathcal{O}(\sigma^2) \tag{2.1.4}$$

(see Theorem 2.3.6). Using such bounds to control the development of a perturbation over short time scales combined with a resetting procedure, we show that there exists a stochastic process $a(t)$ and constants $C, k, \varepsilon' > 0$ such that

$$\mathbb{P}\Big[\sup_{t\in[0,T]} \|u(\cdot,t) - u^*(\cdot + a(t))\|_{L_x^2} \geq \varepsilon\Big] \leq CTe^{-k\sigma^{-2}\varepsilon^2}$$

for all $T > 0$ and $0 < \sigma \leq \varepsilon \leq \varepsilon'$ (Proposition 2.3.7 and Theorem 2.3.8). This shows stability on a time scale $T \sim e^{k\sigma^{-2}\varepsilon^2}$. By a scaling argument, this is (up to better constants) the longest time for which the solitary wave can be expected to be stable, and matches the best results obtained in different settings, such as [114, 161].

2.1.4. PHASE TRACKING

When showing stochastic orbital stability, there are several different ways of defining and tracking the phase process $a(t)$ (see e.g. [112, 121, 148]). Our method is closely related to the one developed by Krüger and Stannat [147, 148], which has also been applied by Eichinger, Gnann, and Kuehn to the FitzHugh–Nagumo equation [74]. Briefly, this method consists of defining an approximation process $a_m(t)$ using the random ODE

$$\frac{\mathrm{d}a_m(t)}{\mathrm{d}t} = -m\frac{\partial\|u(t,x) - u^*(x + a_m(t))\|_{L_x^2}^2}{\partial a_m},$$

and computing an SDE for $\frac{\mathrm{d}a_m(t)}{\mathrm{d}t}$. The idea is that $a_m(t)$ will dynamically move towards a minimizer of $a \mapsto \|u(t,x) - u^*(x+a)\|_{L_x^2}^2$, so that $u^*(x + a_m(t))$ is expected to serve as a good approximation to $u(t,x)$. By approximating the SDE to first order in σ and taking $m \to \infty$, orbital stability can be shown on time scales of the order $T \sim \sigma^{-2}$.

Our method obtains a similar phase process via a completely different route, which we briefly summarize. Before introducing our phase process, we first prove an asymptotic expansion of the form

$$u(t,x) = u^*(x) + \sigma v_1(t,x) + \sigma^2 v_2(t,x) + \mathcal{O}(\sigma^3) \tag{2.1.5}$$

(Proposition 2.3.2). This results in explicit representations of v_1 and v_2, as well as exact estimates relating to the validity of the expansion. On the linear level, the dynamics of v_1 and v_2 are governed by an operator $\mathcal{L}$ which is the linearization of (2.1.1) around u^* (see Theorem 2.2.6). Since the PFNLS equation is not parabolic, we rely on dispersive estimates to control the nonlinear terms. We also require Gaussian tail estimates on the remainder terms, for which we use a result by Seidler [207] to estimate L_Ω^p-norms of stochastic integrals with a constant which is $\mathcal{O}(\sqrt{p})$.

The next step is to introduce the following decomposition of v_1 and v_2:

$$v_1(t,x) = w_1(t,x) + a_1(t)u_x^*(x), \tag{2.1.6a}$$

$$v_2(t,x) = w_2(t,x) + a_2(t)u_x^*(x) + \tfrac{1}{2}a_1(t)^2 u_{xx}^*(x), \tag{2.1.6b}$$

where w_1 and w_2 should be regarded as being determined by (2.1.6) for a given choice of a_1 and a_2. The decomposition (2.1.6) is motivated by the fact that u_x^* is an eigenfunction of $\mathcal{L}$ with eigenvalue 0 (which is a consequence of the fact that (2.1.1) has a translational symmetry). By applying the spectral projection from Theorem 2.2.6 to (2.1.6) it is seen that there is a *unique* choice of a_1, a_2 such that w_1, w_2 both avoid the (secular) zero eigenmode. Since $\mathscr{L}$ has a spectral gap as shown in [130], the linear dynamics of w_1 and w_2 are then exponentially stable. We use this fact to control the nonlinear terms and prove Theorem 2.3.6, which captures the slower growth rates of w_1 and w_2 compared to v_1, v_2.

Directly combining the asymptotic expansion (2.1.5) with the decomposition (2.1.6) using a Taylor expansion finally results in

$$u(t,x) = u^*(x + \sigma a_1(t) + \sigma^2 a_2(t)) + \sigma w_1(t,x) + \sigma^2 w_2(t,x) + \mathcal{O}(\sigma^3),$$

which, combined with smallness of w_1 and w_2, shows orbital stability on a time scale for which the asymptotic expansion (2.1.5) is valid.

Asserting stability on longer time scales requires additional effort. The main issue is that (2.1.5) is a linearization around u^*, but after time t the solution is close to the translated wave $u^*(x + a(t))$. Thus, when $a(t)$ gets large enough (which happens on a time scale $T \sim \sigma^{-2}$), the linearization becomes completely inaccurate. We remedy this by resetting the linearization after a fixed time T, by linearizing around the shifted wave $u^*(x + a(T))$ instead. This makes it possible to combine the short-term estimates on each time interval $[NT, (N+1)T]$ to obtain long-term stability (Theorem 2.3.8). The cost of this procedure is that we incur a discontinuity in the phase process each time we reset, and our explicit representation is only valid in between resetting. We are not aware of any methods to obtain *explicit* descriptions of the phase which are accurate on long time scales. Surprisingly, the resetting procedure suggests that it is possible to show stability on long time scales without accurately tracking the phase on short time scales. This is investigated in more detail and in a much more general setting in Chapter 3.

2.1.5. OUTLINE

In Section 2.2 we specify our notation and introduce the preliminaries necessary to state and prove the main results (Theorems 2.3.1, 2.3.6, and 2.3.8), which are contained in Section 2.3. The proof of well-posedness of (2.1.3) is given in Section 2.4, followed by the proof of the stability results in Section 2.5. Appendices 2.A and 2.B contain some auxiliary results needed for the proofs.

2.2. PRELIMINARIES

We now give the preliminaries required to state and prove the main results, as well as some notational shorthands. We give a rigorous meaning to (2.1.3), and formulate the Strichartz estimates which are used to show well-posedness. Afterwards we state the deterministic stability of the solitary wave, along with additional Strichartz estimates related to the linearization around the solitary wave, which are needed for our stochastic stability results.

2.2.1. NOTATION AND CONVENTIONS

We denote the norm of general normed spaces X by $\|\cdot\|_X$, and the inner product of general inner product spaces H by $\langle \cdot, \cdot \rangle_H$. In the case where H is complex, we take the inner product to be conjugate-linear in the second variable. The space of bounded linear operators from a Banach space X to a Banach space Y is denoted by $\mathscr{L}(X;Y)$, and the space of Hilbert–Schmidt operators between separable Hilbert spaces H and $\tilde{H}$ as $\mathscr{L}_2(H;\tilde{H})$.

If X is a Banach space, we will write $C([0,T];X)$ for the space of continuous X-valued functions. For $p \in [1,\infty]$, we write $L^p(S;X)$ for the usual Bochner spaces defined on a measure space $(S,\mathscr{F},\mu)$ (which coincide with the Lebesgue spaces if $X=\mathbb{C}$ or $X=\mathbb{R}$). If $p=2$, and H is a Hilbert space, then $L^2(S;H)$ is a Hilbert space with the inner product given by

$$\langle f,g\rangle_{L^2(S;X)} = \int_S \langle f,g\rangle_H \,\mathrm{d}\mu.$$

For $z \in \mathbb{C}$, we write $\overline{z}$ for its complex conjugate. For $p \in [1,\infty]$, we write p' for its Hölder conjugate, which is the unique $p' \in [1,\infty]$ such that $\frac{1}{p}+\frac{1}{p'}=1$. Throughout the chapter, all random variables will be defined on a complete probability space $(\Omega,\mathscr{F},\mathbb{P})$ equipped with a complete and right-continuous filtration $\mathbb{F}=(\mathscr{F}_t)_{t\in[0,\infty)}$. We will make use of the following abbreviations:

$$\begin{aligned} L^p_x &:= L^p(\mathbb{R};\mathbb{C}),\\ L^p_\Omega(X) &:= L^p(\Omega;X),\\ L^p(T,T';X) &:= L^p([T,T'];X), \end{aligned}$$

where $\mathbb{R}$ and $[T,T']$ are equipped with the usual Lebesgue measure.

The weak derivative of a weakly differentiable function $f \in L^p_x$ is denoted by $\partial_x f$ and we write $\Delta = \partial_x^2$ for the Laplacian on the real line. We write u^*_x and u^*_{xx} for the first and second spatial derivatives of u^*. For $s \in [0,\infty)$ and $p \in (1,\infty)$, the Bessel space $H^{s,p}_x$ consists of the functions $f \in L^p_x$ for which the quantity

$$\|f\|_{H^{s,p}_x} = \|(1-\Delta)^{\frac{s}{2}} f\|_{L^p_x}$$

is finite. Here, the fractional power $(1-\Delta)^{\frac{s}{2}}$ is defined using the Fourier multiplier with symbol $\xi \mapsto (1+|\xi|^2)^{\frac{s}{2}}$. The space $H^{s,p}_x$ is a Banach space and we have continuous embeddings $H^{s_1,p}_x \hookrightarrow H^{s_2,p}_x$ if $s_1 \geq s_2$. When k is a nonnegative integer, the Bessel space $H^{k,p}_x$ is isomorphic to the classical Sobolev space $W^{k,p}_x$, which consists of the function in L^p_x for which all partial derivatives of order k or less are also in L^p_x. Proofs of these statements rely on the theory of singular integrals, and can for example be found in [211, Chapter 3]. We also note that $H^{s,2}_x$ is a Hilbert space with inner product $\langle f,g\rangle_{H^{s,2}_x} = \langle (1-\Delta)^{\frac{s}{2}} f, (1-\Delta)^{\frac{s}{2}} g\rangle_{L^2_x}$. In this case we will write $H^s_x := H^{s,2}_x$.

Lastly, we denote by $\{S(t)\}_{t\in\mathbb{R}}$ the C_0-group on L^2_x generated by $i\Delta : L^2_x \supset H^2_x \to L^2_x$, which acts at $t \in \mathbb{R}$ as a Fourier multiplier with symbol $\xi \mapsto e^{-4\pi^2 i|\xi|^2 t}$. Using Plancherel's theorem, it can be seen that $S(t)$ is unitary on L^2_x. Since the Fourier multiplier of $S(t)$ commutes with that of $(1-\Delta)^{\frac{s}{2}}$, it is immediate that $S(t)$ is also a unitary group on H^s_x for any s.

2.2.2. STOCHASTIC SET-UP

We let $W(t)$ be an $L^2(\mathbb{R};\mathbb{R})$-cylindrical Wiener process on Ω, which is adapted to $\mathbb{F}$. Then $W(t)$ has an interpretation as the time integral from 0 to t over a space-time white noise.

To regularize the noise, fix some $\phi \in L^2(\mathbb{R};\mathbb{R})$ and define $\Phi\colon L^2(\mathbb{R};\mathbb{R}) \to L^\infty_x$ and $\beta \in \mathbb{R}$ as

$$\Phi f := \phi * f, \tag{2.2.1a}$$
$$\beta := \|\phi\|_{L^2_x}. \tag{2.2.1b}$$

We now convert (2.1.3) into an equivalent formulation in Itô form. Formally applying an Itô–Stratonovich correction to (2.1.3) results in

$$\mathrm{d}u = [i\Delta u - i\nu u - \epsilon(\gamma u - \mu\overline{u}) + i\kappa|u|^2 u]\,\mathrm{d}t - \tfrac{1}{2}Fu\,\mathrm{d}t - iu\Phi\,\mathrm{d}W, \tag{2.2.2}$$

with F being defined as

$$F := \sum_{k\in\mathbb{N}} (\Phi e_k)^2, \tag{2.2.3}$$

where e_k is an orthonormal basis of $L^2(\mathbb{R};\mathbb{R})$. Let us collect some facts about Φ and F which will be used throughout. The proof of Proposition 2.2.1 is contained in Appendix 2.A.

Proposition 2.2.1. *Let $\phi \in L^2(\mathbb{R};\mathbb{R})$ and $u \in L^2_x$. Then the series in (2.2.3) is well-defined and we have the identities*

$$F = \beta^2, \tag{2.2.4a}$$
$$\|u\Phi\|_{\mathscr{L}_2(L^2(\mathbb{R};\mathbb{R});L^2_x)} = \beta\|u\|_{L^2_x}. \tag{2.2.4b}$$

If additionally $\phi \in H^s(\mathbb{R};\mathbb{R})$ and $u \in H^s_x$ for some $s \in [0,\infty)$, then we have the estimate

$$\|u\Phi\|_{\mathscr{L}_2(L^2(\mathbb{R};\mathbb{R});H^s_x)} \le C_s\|\phi\|_{H^s_x}\|u\|_{H^s_x} \tag{2.2.4c}$$

for some $C_s > 0$ which depends only on s.

Substituting (2.2.4a) into (2.2.2), the stochastic PFNLS equation in Itô form reads

$$\mathrm{d}u = [i\Delta u - i\nu u - \epsilon(\gamma u - \mu\overline{u}) + i\kappa|u|^2 u]\,\mathrm{d}t - \tfrac{1}{2}\beta^2 u\,\mathrm{d}t - iu\Phi\,\mathrm{d}W. \tag{2.2.5}$$

From the definition of Φ (2.2.1a), it is clear that this operator commutes with translation. Furthermore, since $\xi = \frac{\mathrm{d}W}{\mathrm{d}t}$ formally represents a white noise, its statistics are also invariant under translation. Thus, the noise terms do not break the temporal- and spatial translation symmetries inherent to (2.1.1) (in a statistical sense).

Before we proceed with the mathematical analysis, we give a meaningful interpretation to our noise. Since ξ formally has a covariance operator on $L^2(\mathbb{R};\mathbb{R})$ equal to the identity, it can be seen using (2.2.1a) that $\Phi\xi$ formally satisfies the covariance relation

$$\begin{aligned}\mathbb{E}\Big[(\Phi\xi)(t,x)\cdot(\Phi\xi)(t',x')\Big] &= \mathbb{E}\Big[\langle\Phi\xi(t),\delta_x\rangle_{L^2_x}\langle\Phi\xi(t'),\delta_{x'}\rangle_{L^2_x}\Big]\\ &= \delta_0(t-t')\langle\Phi^*\delta_x,\Phi^*\delta_{x'}\rangle_{L^2_x}\\ &= \delta_0(t-t')(\tilde{\phi}*\phi)(x-x'),\end{aligned}$$

where δ_a denotes a Dirac mass at the point $x = a$, and $\tilde{\phi}$ is defined via $\tilde{\phi}(x) := \phi(-x)$. Therefore, $g := \tilde{\phi}*\phi$ can be interpreted as the spatial correlation function of our noise. Note that g is an even function, so that the correlation only depends on $|x-x'|$. The variance at any point is given by $g(0) = \beta^2$, which means this quantity can be viewed as the strength of the noise.

2.2.3. STRICHARTZ ESTIMATES

In the analysis of nonlinear Schrödinger equations, the dispersion displayed by the linear Schrödinger equation plays a major role. In our context, this dispersion manifests in the form of *Strichartz estimates*. These estimates give control over certain space-time mixed Lebesgue norms of solutions to the linear Schrödinger equation. In our one-dimensional setting, they take the following form.

Definition 2.2.2. A pair (r,p) with $r \in [4,\infty]$, $p \in [2,\infty]$ is called *admissible* if it satisfies

$$\frac{2}{r} + \frac{1}{p} = \frac{1}{2}. \tag{2.2.6}$$

Theorem 2.2.3 (Strichartz estimates). *Let $s \in [0,\infty)$, and let $(r,p) \neq (4,\infty)$ and (α,δ) be admissible. There exists a constant C, such that the estimates*

$$\|S(\cdot)f\|_{L^r(0,T;H_x^{s,p})} \leq C\|f\|_{H_x^s}, \tag{2.2.7a}$$

$$\left\|\int_0^\cdot S(\cdot - t')g(t')\,\mathrm{d}t'\right\|_{L^r(0,T;H_x^{s,p})} \leq C\|g\|_{L^{\alpha'}(0,T;H_x^{s,\delta'})}, \tag{2.2.7b}$$

$$\left\|\int_0^\cdot S(\cdot - t')h(t')\Phi\,\mathrm{d}W(t')\right\|_{L^q_\Omega(L^r(0,T;H_x^{s,p}))} \leq C\sqrt{q}\|\phi\|_{H_x^s}\|h\|_{L^q_\Omega(L^2(0,T;H_x^s))}, \tag{2.2.7c}$$

hold for every $q \in [2,\infty)$, $T \in (0,\infty]$, $f \in H_x^s$, $g \in L^{\alpha'}(0,T;H_x^{s,\delta'})$, $h \in L^q_\Omega(L^2(0,T;H_x^s))$, and $\phi \in L^2(\mathbb{R};\mathbb{R}) \cap H_x^s$ (recall (2.2.1)).

Remark 2.2.4. In the case $(r,p) = (\infty,2)$, the relevant processes in Theorem 2.2.3 have continuous versions, and the L^∞-norm on the left-hand side of (2.2.7) can be replaced by $C([0,T])$. We will always use these continuous versions. This also applies to (2.2.14) further below.

Remark 2.2.5. Estimates (2.2.7a) and (2.2.7b) still hold in the case $(r,p) = (4,\infty)$. This also applies to (2.2.14a), (2.2.14b), (2.2.18a), and (2.2.18b) further below.

Estimates (2.2.7a), (2.2.7b), and (2.2.7c) are commonly referred to as the *homogeneous, convolution,* and *stochastic* Strichartz estimates respectively. The homogeneous and convolution Strichartz estimates are well-known and can be found in [54, Theorem 2.3.3] or [135]. The stochastic Strichartz estimate is more recent, and was first shown in [46] for the case $r = q$. The proof of our formulation of (2.2.7c), which is contained in Appendix 2.B, follows the same idea as [46], except that we use [207, Theorem 1.1] to obtain a constant which is $\mathcal{O}(\sqrt{q})$.

2.2.4. SOLITARY WAVES AND LINEAR STABILITY

We now fix a set of parameters $\nu \in \mathbb{R}$, $\epsilon, \gamma, \mu > 0$ which satisfy $\mu > \gamma$. We additionally let $\theta \in [0,\pi)$ be the unique solution to $\cos(2\theta) = \frac{\gamma}{\mu}$ which satisfies $\sin(2\theta) > 0$. This ensures that the deterministic equation (2.1.1) has a stable solitary wave solution u^*, explicitly given by

$$u^*(x) = \sqrt{\frac{2(\nu + \epsilon\mu\sin(2\theta))}{\kappa}}\operatorname{sech}(\sqrt{\nu + \epsilon\mu\sin(2\theta)}\,x)e^{i\theta} \tag{2.2.8}$$

(see [130, equation (1.8)]). Since (2.1.1) is preserved under the transformation $u \mapsto -u$, it follows that $-u^*$ is also a stable solitary wave. Alternatively, this profile could be obtained by letting $\theta \in \mathbb{R}$. In that case the above conditions only select θ up to an additive multiple of π, giving rise to both u^* and $-u^*$ via the term $e^{i\theta}$ in (2.2.8).

We remark that u^* is infinitely often differentiable, and all of its derivatives are rapidly decaying.

We will frequently make use of expansions around the solitary wave u^*. Due to the cubic term in (2.1.3), this will require expansions of terms like $|a+b|^2(a+b)$. Here, the absolute value prevents the use of convenient multinomial expansion formulas. To remedy this, we introduce the following notation, which we call the *triple bracket*:

$$\begin{aligned} \{\cdot,\cdot,\cdot\}&: \mathbb{C}\times\mathbb{C}\times\mathbb{C} \to \mathbb{C} \\ \{a,b,c\} &= ab\overline{c} + a\overline{b}c + \overline{a}bc. \end{aligned} \tag{2.2.9}$$

Observe that the triple bracket is symmetric, (real-)trilinear and that $|u|^2u = \frac{1}{3}\{u,u,u\}$. Therefore, we can compactly write binomial expansions like

$$|u+v|^2(u+v) = \tfrac{1}{3}\{u+v,u+v,u+v\} = \tfrac{1}{3}\{u,u,u\} + \{u,u,v\} + \{u,v,v\} + \tfrac{1}{3}\{v,v,v\}.$$

This notation is particularly useful when using multinomial expansions with more terms. For readability, we abbreviate

$$Lu := -i\nu u - \epsilon(\gamma u - \mu\overline{u}). \tag{2.2.10}$$

Combining our new notation, we may compactly rewrite (2.1.1) as

$$\partial_t u = i\Delta u + Lu + \tfrac{1}{3}i\kappa\{u,u,u\}.$$

Using the additivity of the triple bracket, it is now straightforward to see that the operator

$$\mathscr{L}\colon\ v \mapsto i\Delta v + Lv + i\kappa\{u^*,u^*,v\} \tag{2.2.11}$$

corresponds to the linearization of (2.1.1) around the solitary wave u^*. The linear stability of the solitary wave (2.2.8) is captured in the following theorem, which has been shown in [130].

Theorem 2.2.6. *The operator $\mathscr{L}$ has the following properties:*

1. *$\mathscr{L}$ is the generator of a strongly continuous semigroup on L^2_x, denoted by $P(t)$.*

2. *u^*_x is an eigenfunction of $\mathscr{L}$ with eigenvalue 0, which has algebraic multiplicity one.*

3. *The spectrum of $\mathscr{L}$ on L^2_x is contained in $\{z \in \mathbb{C} : \operatorname{Re} z \le -b\} \cup \{0\}$ for some $b > 0$. Thus, the Riesz spectral projection*

$$\Pi^0 := \frac{1}{2\pi i}\oint_C (\lambda I - \mathscr{L})^{-1}\, d\lambda,$$

is well-defined if C is a sufficiently small contour encircling 0 counterclockwise.

4. *If we additionally define* $\Pi := I - \Pi^0$*, then there exist constants* M *and* $a > 0$ *such that the inequalities*

$$\|P(t)\Pi\|_{\mathscr{L}(L_x^2)} \leq Me^{-at}, \qquad \|P(t)\|_{\mathscr{L}(L_x^2)} \leq M, \tag{2.2.12}$$

hold for all $t \in [0,\infty)$.

Remark 2.2.7. The operator $\mathscr{L}$ is not complex-linear, and the same applies to $P(t)$, Π^0, and Π. Additionally, Π^0 projects onto the real span of u_x^* as opposed to the complex span. Thus, in the context of the linearization we should regard $L_x^2 \simeq L^2(\mathbb{R};\mathbb{R}^2)$ as a real vector space.

Using Π, we also define the linear operator $\mathscr{P}$ as follows:

$$\mathscr{P}\colon f \mapsto \frac{\langle f - \Pi f, u_x^* \rangle_{L_x^2}}{\|u_x^*\|_{L_x^2}^2}. \tag{2.2.13}$$

Proposition 2.2.8. *The operator* $\mathscr{P}$ *is bounded from* L_x^2 *to* $\mathbb{R}$*, and for every* $f \in L_x^2$ *we have the decomposition*

$$f = \Pi f + \mathscr{P}(f) u_x^*.$$

Proof. The boundedness of $\mathscr{P}$ follows from the boundedness of Π and the Cauchy–Schwarz inequality. Now fix $f \in L_x^2$. Since $I = \Pi + \Pi^0$ and Π^0 projects onto the span of u_x^*, there exists a unique $a \in \mathbb{R}$ such that

$$f = \Pi f + \Pi^0 f = \Pi f + a u_x^*.$$

Rearranging this equation, taking inner products with u_x^* and dividing by $\|u_x^*\|_{L_x^2}^2$ shows that $a = \mathscr{P}(f)$. □

We now formulate appropriate Strichartz estimates for the semigroups $P(\cdot)\Pi$ and $P(\cdot)\Pi^0$ separately. Using the decomposition $P(t) = P(t)\Pi + P(t)\Pi^0$, we now also obtain Strichartz estimates for $P(t)$.

Proposition 2.2.9 (Strichartz estimates for $P(\cdot)\Pi$)**.** *Let* $(r,p) \neq (4,\infty)$ *be admissible. There exists a constant* C*, such that the estimates*

$$\|P(\cdot)\Pi f\|_{L^r(0,T;L_x^p)} \leq C\|f\|_{L_x^2}, \tag{2.2.14a}$$

$$\left\|\int_0^\cdot P(\cdot - t')\Pi g(t')\,\mathrm{d}t'\right\|_{L^r(0,T;L_x^p)} \leq C\|g\|_{L^1(0,T;L_x^2)}, \tag{2.2.14b}$$

$$\left\|\int_0^\cdot P(\cdot - t')\Pi h(t')\Phi\,\mathrm{d}W(t')\right\|_{L_\Omega^q(L^r(0,T;L_x^p))} \leq C\sqrt{q}\,T^{\frac{1}{2}-\frac{1}{q+\varepsilon}}\beta\|h\|_{L_\Omega^q(L^{q+\varepsilon}(0,T;L_x^2))}, \tag{2.2.14c}$$

hold for all $q \in [2,\infty)$, $\varepsilon \in (0,\infty)$, $T \in (0,\infty)$, $f \in L_x^2$, $g \in L^1(0,T;L_x^2)$, $h \in L_\Omega^q(L^{q+\varepsilon}(0,T;L_x^2))$, *and* $\phi \in L^2(\mathbb{R};\mathbb{R})$ *(recall* (2.2.1)*).*

Proof. We first show (2.2.14a). Consider for some $f \in L^2_x$ the evolution equation

$$\begin{aligned} \mathrm{d}u &= [i\Delta u - i\nu u - \epsilon(\gamma u - \mu\overline{u}) + i\kappa\{u^*, u^*, u\}]\,\mathrm{d}t, \\ u(0) &= \Pi f. \end{aligned} \tag{2.2.15}$$

By standard semigroup theory, it can be shown that (2.2.15) has a unique solution $u \in C([0,t]; L^2_x)$, which satisfies the following identities:

$$u(t) = P(t)\Pi f, \tag{2.2.16a}$$

$$u(t) = S(t)\Pi f + \int_0^t S(t-t')\big(-i\nu u - \epsilon(\gamma u - \mu\overline{u}) + i\kappa\{u^*, u^*, u\}\big)\,\mathrm{d}t'. \tag{2.2.16b}$$

Using the decay estimate (2.2.12) from Theorem 2.2.6, we first observe that

$$\|u\|_{L^1(0,T;L^2_x)} \overset{(2.2.16a)}{=} \|P(\cdot)\Pi f\|_{L^1(0,T;L^2_x)} \overset{(2.2.12)}{\leq} \|f\|_{L^2_x} \int_0^T M e^{-at}\,\mathrm{d}t \leq M a^{-1} \|f\|_{L^2_x}. \tag{2.2.17}$$

From (2.2.16b) and Theorem 2.2.3, it now follows that

$$\begin{aligned} \|u\|_{L^r(0,T;L^p_x)} &\overset{(2.2.7a),(2.2.7b)}{\leq} C\big(\|\Pi f\|_{L^2_x} + \|-i\nu u - \epsilon(\gamma u - \mu\overline{u}) + i\kappa\{u^*, u^*, u\}\|_{L^1(0,T;L^2_x)}\big) \\ &\leq C\|\Pi f\|_{L^2_x} + C'\|u\|_{L^1(0,T;L^2_x)} \overset{(2.2.17)}{\leq} C''\|f\|_{L^2_x}, \end{aligned}$$

which shows (2.2.14a). To show (2.2.14b), we use Minkowski's integral inequality and (2.2.14a):

$$\begin{aligned} \Big\|\int_0^{\cdot} P(\cdot - t')\Pi g(t')\,\mathrm{d}t'\Big\|_{L^r(0,T;L^p_x)} &= \Big\|\int_0^T \mathbb{1}_{[t',T]}(\cdot)P(\cdot - t')\Pi g(t')\,\mathrm{d}t'\Big\|_{L^r(0,T;L^p_x)} \\ &\leq \int_0^T \|\mathbb{1}_{[t',T]}(\cdot)P(\cdot - t')\Pi g(t')\|_{L^r(0,T;L^p_x)}\,\mathrm{d}t' = \int_0^T \|P(\cdot)\Pi g(t')\|_{L^r(0,T-t';L^p_x)}\,\mathrm{d}t' \\ &\overset{(2.2.14a)}{\leq} C\int_0^T \|g(t')\|_{L^2_x}\,\mathrm{d}t'. \end{aligned}$$

To obtain the stochastic estimate (2.2.14c) for $(r,p) \neq (\infty, 2)$, we simply repeat the first part of the proof of (2.2.7c) from Appendix 2.B, replacing all occurrences of $S(t)$ with $P(t)\Pi$ and using (2.2.14a) instead of (2.2.7a) in the intermediate steps. Using Hölder's inequality at the end then gives (2.2.14c).

For the case $(r,p) = (\infty, 2)$, the proof strategy in Appendix 2.B is no longer applicable, since $P(t)\Pi$ is not unitary. Instead, we estimate the stochastic convolution using the well-known factorization method (see for instance [186, Theorem 4.5] for a version applicable to our setting), which gives the result after applying (2.2.4b). □

For $P(t)\Pi^0$, there is significantly more freedom in choosing the exponents, and the requirement of admissibility can be dropped. In this case, the estimates follow not from any dispersive phenomena, but rather from the fact that the range of Π^0 has dimension one and is spanned by u^*_x.

Proposition 2.2.10. *Let $p \in [1,\infty]$. There exists a constant C, such that the estimates*

$$\|P(\cdot)\Pi^0 u_0\|_{C([0,T];L^p_x)} \le C\|u_0\|_{L^2_x}, \tag{2.2.18a}$$

$$\Big\|\int_0^\cdot P(\cdot - t')\Pi^0 f(t')\,\mathrm{d}t'\Big\|_{C([0,T];L^p_x)} \le C\|f\|_{L^1(0,T;L^2_x)}, \tag{2.2.18b}$$

$$\Big\|\int_0^\cdot P(\cdot - t')\Pi^0 g(t')\Phi\,\mathrm{d}W(t')\Big\|_{L^q_\Omega(C([0,T];L^p_x))} \le C\sqrt{q}\beta\|g\|_{L^q_\Omega(L^2(0,T;L^2_x))}, \tag{2.2.18c}$$

hold for all $q \in [2,\infty)$, $T \in (0,\infty]$, $u_0 \in L^2_x$, $f \in L^1(0,T;L^2_x)$, $g \in L^q_\Omega(L^2(0,T;L^2_x))$, and $\phi \in L^2(\mathbb{R};\mathbb{R})$ (recall (2.2.1)).

Proof. Since $\mathscr{L}u_x^* = 0$ by Theorem 2.2.6, it holds that $P(t)u_x^* = u_x^*$. After observing that the range of Π^0 is spanned by u_x^*, it follows that $P(t)\Pi^0 = \Pi^0$ for every t. Thus, we get

$$\|P(t)\Pi^0 u_0\|_{L^p_x} = \|\Pi^0 u_0\|_{L^p_x} = \frac{\|u_x^*\|_{L^p_x}}{\|u_x^*\|_{L^2_x}}\|\Pi^0 u_0\|_{L^2_x} \le C\|u_0\|_{L^2_x}, \tag{2.2.19}$$

where $\|u_x^*\|_{L^p_x} < \infty$ because u_x^* decays rapidly. Using Minkowski's inequality, we can additionally estimate

$$\Big\|\int_0^t P(t-t')\Pi^0 f(t')\,\mathrm{d}t'\Big\|_{L^p_x} \le \int_0^t \|P(t-t')\Pi^0 f(t')\|_{L^p_x}\,\mathrm{d}t' \overset{(2.2.19)}{\le} C\int_0^t \|f(t')\|_{L^2_x}\,\mathrm{d}t',$$

at which point (2.2.18b) follows by taking the supremum over $t \in [0,T]$. Finally, we estimate

$$\begin{aligned}\Big\|\int_0^\cdot P(\cdot - t')\Pi^0 g(t')\Phi\,\mathrm{d}W(t')\Big\|_{L^q_\Omega(C([0,T];L^p_x))} &= \Big\|\int_0^\cdot \Pi^0 g(t')\Phi\,\mathrm{d}W(t')\Big\|_{L^q_\Omega(C([0,T];L^p_x))}\\ &\le C\Big\|\int_0^\cdot g(t')\Phi\,\mathrm{d}W(t')\Big\|_{L^q_\Omega(C([0,T];L^2_x))}\\ &\le C'\sqrt{q}\|g\Phi\|_{L^q_\Omega(L^2(0,T;\mathscr{L}_2(L^2(\mathbb{R};\mathbb{R});L^2_x)))}\\ &\overset{(2.2.4b)}{=} C'\sqrt{q}\beta\|g\|_{L^q_\Omega(L^2(0,T;L^2_x))},\end{aligned}$$

where we have used the Burkholder–Davis–Gundy inequality for the penultimate step. □

To get appropriate Gaussian tail bounds, we need the following elementary lemma.

Lemma 2.2.11. *Let ξ be a nonnegative real-valued random variable which satisfies*

$$\|\xi\|_{L^p_\Omega} \le C\sqrt{p}$$

for all sufficiently large $p < \infty$, where C is independent of p. Then ξ satisfies the Gaussian tail bound

$$\mathbb{P}[\xi \ge \lambda] \le \exp\left(-e^{-2}C^{-2}\lambda^2\right)$$

for all sufficiently large λ.

Proof. By Markov's inequality and the assumption on ξ, we have

$$\mathbb{P}[\xi \ge \lambda] = \mathbb{P}[\xi^p \ge \lambda^p] \le \lambda^{-p}C^p\sqrt{p}^p = (\lambda^{-1}C\sqrt{p})^p$$

for p sufficiently large. Choosing $p = e^{-2}C^{-2}\lambda^2$ (which can be made sufficiently large by increasing λ) gives the result. □

2.3. MAIN RESULTS

We now state the results of the chapter. We begin by showing that (2.2.5) is well-posed in Theorem 2.3.1. Afterwards, we derive an asymptotic expansion (in terms of σ) of solutions to (2.2.5) around a solitary wave centered at the origin. This expansion is needed for the coming orbital stability results, and its validity is shown in Proposition 2.3.2.

Using the asymptotic expansion we then construct the phase processes a_1 and a_2 which describe the motion of the solitary wave to second order in σ. Our first main result in the direction of stochastic stability of the solitary wave is Theorem 2.3.6. Roughly speaking, it states that our choice of a_1 and a_2 eliminate the leading-order growth terms of the perturbations around the shifted wave. We then build on this result to prove Theorem 2.3.8, where we show an exponential estimate on the exit time from the shifted solitary wave and hence establish orbital stability on an exponential time scale.

2.3.1. WELL-POSEDNESS

Our first main result is the well-posedness of a mild formulation of (2.2.5). The proof is contained in Section 2.4.

Theorem 2.3.1. *Let $\nu,\epsilon,\gamma,\mu,\kappa>0$, let u_0 be an L^2_x-valued $\mathscr{F}_0$-measurable random variable, let $T\in(0,\infty)$ and $\phi\in L^2(\mathbb{R};\mathbb{R})$. There exists a unique $\mathbb{F}$-adapted process u in the space $C([0,T];L^2_x)\cap L^6(0,T;L^6_x)$ satisfying the mild-solution equation*

$$\begin{aligned} u(t) = S(t)u_0 + \int_0^t S(t-t')(-i\nu u(t') - \epsilon(\gamma u(t') - \mu\overline{u}(t')) - \tfrac{1}{2}\beta^2 u(t'))\,\mathrm{d}t' \\ + i\kappa\int_0^t S(t-t')|u(t')|^2 u(t')\,\mathrm{d}t - i\int_0^t S(t-t')u(t')\Phi\,\mathrm{d}W(t'), \end{aligned} \tag{2.3.1}$$

for every $t\in[0,T]$, $\mathbb{P}$-a.s. Furthermore, $u\in L^r(0,T;L^p_x)$ for any $(r,p)\neq(4,\infty)$ which satisfies (2.2.6), and we have the a priori estimate

$$\|u(t)\|_{L^2_x} \le e^{\epsilon(\mu-\gamma)t}\|u_0\|_{L^2_x}, \tag{2.3.2}$$

for every $t\in[0,T]$, $\mathbb{P}$-a.s.

If we additionally assume that $\phi\in H^s_x$ and u_0 takes values in H^s_x for some $s\in[0,\infty)$, then also $u\in C([0,T];H^s_x)\cap L^r(0,T;H^{s,p}_x)$ for any $(r,p)\neq(4,\infty)$ which satisfies (2.2.6).

2.3.2. ASYMPTOTIC EXPANSION

We now derive the asymptotic expansions which will be used in Section 2.3.3 to show orbital stability. From now on, let $\nu,\epsilon,\gamma,\mu,\kappa$, and u^* be as described in Section 2.2.4. Consider the SPFNLS equation (2.2.5), now written using our notational shorthands (cf. (2.2.1), (2.2.9), (2.2.11)), and including an additional parameter $\sigma>0$ which controls the strength of the noise:

$$\mathrm{d}u = [i\Delta u + Lu + \tfrac{1}{3}i\kappa\{u,u,u\} - \tfrac{1}{2}\beta^2\sigma^2 u]\,\mathrm{d}t - i\sigma u\Phi\,\mathrm{d}W. \tag{2.3.3}$$

The first step towards showing orbital stability of the solitary wave is to construct an asymptotic expansion to second order in σ. For this we use the following ansatz:

$$u = u^* + \sigma v_1 + \sigma^2 v_2 + z, \tag{2.3.4}$$

where z should be regarded as being $\mathcal{O}(\sigma^3)$. To match our ansatz, we supply (2.3.3) with the initial condition

$$u(0) = u^* + \sigma v_{1,0} + \sigma^2 v_{2,0}. \tag{2.3.5}$$

By using the additivity of the triple bracket, we see that (2.3.3) can be rewritten as

$$\begin{aligned}
\mathrm{d}u = {} & [(i\Delta + L)u^* + \tfrac{1}{3} i\kappa\{u^*, u^*, u^*\}]\,\mathrm{d}t \\
& + \sigma\big([(i\Delta + L)v_1 + i\kappa\{u^*, u^*, v_1\}]\,\mathrm{d}t - iu^*\Phi\,\mathrm{d}W\big) \\
& + \sigma^2\Big(\big[(i\Delta + L)v_2 + i\kappa\{u^*, u^*, v_2\} + i\kappa\{u^*, v_1, v_1\} - \tfrac{1}{2}\beta^2 u^*\big]\,\mathrm{d}t - iv_1\Phi\,\mathrm{d}W\Big) \\
& \quad + \big[(i\Delta + L)z + i\kappa\{u^*, u^*, z\} + i\kappa R - \tfrac{1}{2}\beta^2(\sigma^3 v_1 + \sigma^4 v_2 + \sigma^2 z)\big]\,\mathrm{d}t \\
& \quad - i(\sigma^3 v_2 + \sigma z)\Phi\,\mathrm{d}W,
\end{aligned} \tag{2.3.6}$$

where we have abbreviated

$$\begin{aligned}
R := {} & 2\{u^*, \sigma v_1, \sigma^2 v_2\} + \tfrac{1}{3}\{\sigma v_1, \sigma v_1, \sigma v_1\} \\
& + 2\{u^*, \sigma v_1, z\} + \{u^*, \sigma^2 v_2, \sigma^2 v_2\} + \{\sigma v_1, \sigma v_1, \sigma^2 v_2\} \\
& + 2\{u^*, \sigma^2 v_2, z\} + \{\sigma v_1, \sigma v_1, z\} + \{\sigma v_1, \sigma^2 v_2, \sigma^2 v_2\} \\
& + \{u^*, z, z\} + 2\{\sigma v_1, \sigma^2 v_2, z\} + \tfrac{1}{3}\{\sigma^2 v_2, \sigma^2 v_2, \sigma^2 v_2\} \quad . \\
& + \{\sigma v_1, z, z\} + \{\sigma^2 v_2, \sigma^2 v_2, z\} \\
& + \{\sigma^2 v_2, z, z\} \\
& + \tfrac{1}{3}\{z, z, z\}
\end{aligned} \tag{2.3.7}$$

Note that the terms in (2.3.7) are organized according to their order in σ, and all terms are $\mathcal{O}(\sigma^3)$. Taking the differential of (2.3.4) and using (2.2.11) and (2.3.6), we see that if v_1 and v_2 satisfy

$$\mathrm{d}v_1 = \mathcal{L} v_1\,\mathrm{d}t - iu^*\Phi\,\mathrm{d}W, \tag{2.3.8a}$$
$$\mathrm{d}v_2 = [\mathcal{L} v_2 + i\kappa\{u^*, v_1, v_1\} - \tfrac{1}{2}\beta^2 u^*]\,\mathrm{d}t - iv_1\Phi\,\mathrm{d}W, \tag{2.3.8b}$$
$$v_1(0) = v_{1,0}, \tag{2.3.8c}$$
$$v_2(0) = v_{2,0}, \tag{2.3.8d}$$

then z satisfies

$$\mathrm{d}z = [\mathcal{L} z + i\kappa R - \tfrac{1}{2}\beta^2(\sigma^3 v_1 + \sigma^4 v_2 + \sigma^2 z)]\,\mathrm{d}t - i(\sigma^3 v_2 + \sigma z)\Phi\,\mathrm{d}W, \tag{2.3.9a}$$
$$z(0) = 0 \tag{2.3.9b}$$

(note that $\mathrm{d}u^* = [(i\Delta + L)u^* + \frac{1}{3}\{u^*, u^*, u^*\}]\,\mathrm{d}t$ always holds, since both sides vanish). We now formulate a proposition which states that on any fixed time interval $[0, T]$, the approximation $u \approx u^* + \sigma v_1 + \sigma^2 v_2$ is accurate to second order in σ with high probability, as long as v_1 and v_2 are not too large. The proof is contained in Section 2.5.1.

Proposition 2.3.2 (Asymptotic expansion, second order)**.** *Let $v_{1,0}$ and $v_{2,0}$ be L^2_x-valued and $\mathcal{F}_0$-measurable random variables, and let u be the solution to* (2.3.3) *with initial* *condition* (2.3.5)*. The system* (2.3.8) *has a unique mild solution given by:*

$$v_1(t) = P(t)v_{1,0} - \int_0^t P(t-t')iu^*\Phi \,\mathrm{d}W(t'), \tag{2.3.10a}$$

$$\begin{aligned} v_2(t) = P(t)v_{2,0} &+ \int_0^t P(t-t')(i\kappa\{u^*, v_1, v_1\} - \tfrac{1}{2}\beta^2 u^*)\,\mathrm{d}t' \\ &- \int_0^t P(t-t')iv_1\Phi\,\mathrm{d}W(t'). \end{aligned} \tag{2.3.10b}$$

We have $v_1, v_2 \in C([0,T]; L_x^2) \cap L^r(0,T; L_x^p)$ *for every* $T \in (0,\infty)$ *and every admissible pair* $(r,p) \neq (4,\infty)$, $\mathbb{P}$*-a.s. With these* v_1 *and* v_2*, we have the asymptotic expansion*

$$u(t) = u^* + \sigma v_1(t) + \sigma^2 v_2(t) + z(t), \tag{2.3.11}$$

where z *satisfies* (2.3.9)*. Furthermore, for every* $T \in (0,\infty)$ *and every admissible pair* $(r,p) \neq (4,\infty)$*, there exist strictly positive constants* c_1*,* c_2*,* ε'*, independent of* $v_{1,0}$*,* $v_{2,0}$*, such that for the following stopping times*

$$\tau_{v_1} := \sup\{t \in [0,T] : \|v_1\|_{L^\infty(0,t;L_x^2)\cap L^6(0,t;L_x^6)} \leq \sigma^{-1}\varepsilon\}, \tag{2.3.12a}$$

$$\tau_{v_2} := \sup\{t \in [0,T] : \|v_2\|_{L^\infty(0,t;L_x^2)\cap L^6(0,t;L_x^6)} \leq \sigma^{-2}\varepsilon^2\}, \tag{2.3.12b}$$

$$\tau_z := \sup\{t \in [0,T] : \|z\|_{L^\infty(0,t;L_x^2)\cap L^r(0,t;L_x^p)} \leq c_1\varepsilon^3\}, \tag{2.3.12c}$$

we have the inequality

$$\mathbb{P}\big[\tau_z < \min\{\tau_{v_1}, \tau_{v_2}\}\big] \leq \exp(-c_2\varepsilon^2\sigma^{-2}) \tag{2.3.13}$$

for all σ*,* ε *which satisfy* $0 < \sigma \leq \varepsilon \leq \varepsilon'$.

Remark 2.3.3. It would be sufficient in (2.3.12) to control v_1 and v_2 in a slightly weaker norm. However, the choice of $L^\infty(0,t;L_x^2) \cap L^6(0,t;L_x^6)$ permits a more convenient proof, and we will be able to control v_1 and v_2 in this norm due to the Strichartz estimates.

Remark 2.3.4. The main purpose of Proposition 2.3.2 is to characterize the short-term dynamics of (2.2.5), and to serve as a building block towards the stability results in Section 2.3.3.

The following proposition is a first-order variant of Proposition 2.3.2, and will be used to show the long-term stability result (Theorem 2.3.8). The proof is a strictly simpler version of that of Proposition 2.3.2, so we choose to omit it.

Proposition 2.3.5. *Consider the setting of Proposition 2.3.2 with* $v_{2,0} = 0$ *and define* z' *via*

$$u(t) =: u^* + \sigma v_1(t) + z'(t). \tag{2.3.14}$$

For every $T \in (0,\infty)$ *and every admissible pair* $(r,p) \neq (4,\infty)$ *there exist strictly positive constants* c_1*,* c_2 *and* ε'*, independent of* $v_{1,0}$*, such that if we introduce the additional stopping time*

$$\tau_{z'} := \sup\{t \in [0,T] : \|z'\|_{L^\infty(0,t;L_x^2)\cap L^r(0,t;L_x^p)} \leq c_1\varepsilon^2\}, \tag{2.3.15}$$

we have the inequality

$$\mathbb{P}\big[\tau_{z'} < \tau_{v_1}\big] \leq \exp(-c_2\varepsilon^2\sigma^{-2}), \tag{2.3.16}$$

for all σ, ε *which satisfy* $0 < \sigma \leq \varepsilon \leq \varepsilon'$.

2.3.3. ORBITAL STABILITY

Proposition 2.3.2 implies that on any fixed time scale, we have the expansion

$$u = u^* + \sigma v_1 + \sigma^2 v_2 + \mathcal{O}(\sigma^3).$$

However, from (2.3.10) it can be seen that in general, the processes v_1 and v_2 grow with time. To show orbital stability of the solitary wave on long time scales, we need to control this growth. Therefore, we first decompose v_1 and v_2 in the following way:

$$v_1 = a_1 u^*_x + w_1, \tag{2.3.17a}$$

$$v_2 = a_2 u^*_x + \tfrac{1}{2} a_1^2 u^*_{xx} + w_2, \tag{2.3.17b}$$

where a_1 and a_2 are (real-valued) stochastic processes which we will specify later, at which point (2.3.17) determines w_1 and w_2. Substituting (2.3.17) into (2.3.4) and using Proposition 2.3.2, we get

$$u = u^* + \sigma a_1 u^*_x + \sigma^2 a_2 u^*_x + \tfrac{1}{2}\sigma^2 a_1^2 u^*_{xx} + \sigma w_1 + \sigma^2 w_2 + \mathcal{O}(\sigma^3).$$

The first four terms on the right-hand side form a Taylor expansion of $u^*(x + \sigma a_1 + \sigma^2 a_2)$ to second order in σ, and thus we have

$$u = u^*(x + \sigma a_1 + \sigma^2 a_2) + \sigma w_1 + \sigma^2 w_2 + \mathcal{O}(\sigma^3), \tag{2.3.18}$$

still on the same fixed time scale. We will see that for some particular choice of a_1 and a_2, the processes w_1 and w_2 exhibit growth behavior which is much more favorable than that of their counterparts v_1 and v_2. This is the statement of Theorem 2.3.6, which gives explicit expressions for a_1 and a_2, and characterizes the growth behavior of w_1 and w_2. This is made possible by the exponential decay of $P(t)\Pi$ (2.2.12), which is essentially the content of the deterministic stability result.

As an example, from (2.3.10a) it is clear that v_1 is expected to grow like $\sqrt{t}$ (this can be made rigorous by combining (2.3.17a), (2.3.20a), and (2.3.22a)). On the other hand, from (2.3.22a) we see that the moments of w_1 remain bounded in time. Thus, the term $a_1 u^*_x$ in (2.3.17a) fully captures the growth of v_1. Similarly, v_2 is expected to grow at a rate of t^2, whereas (2.3.22b) shows that w_2 only grows like t.

From (2.3.18) it is then clear that a_1 and a_2 have an interpretation as the first- and second-order corrections to the phase of the solitary wave. Additionally, since Φ and u^* do not depend on t and ω, it can be seen from (2.3.20a) that a_1 is a Brownian motion rescaled by $\|\mathscr{P} i u^* \Phi\|_{\mathscr{L}_2(L^2(\mathbb{R};\mathbb{R});\mathbb{R})}$ and offset by $\mathscr{P}(v_{1,0})$. The proofs of Theorem 2.3.6, Proposition 2.3.7, and Theorem 2.3.8 are contained in Section 2.5.2.

Theorem 2.3.6. *There exist predictable processes a_1, a_2, w_1, w_2, such that (2.3.17) and the condition*

$$\Pi^0 w_k = 0, \qquad k \in \{1,2\}, \tag{2.3.19}$$

both hold. The processes a_1 and a_2 are given by

$$a_1(t) = \mathscr{P}\Big[v_{1,0} - \int_0^t i u^* \Phi \,\mathrm{d}W(t')\Big], \tag{2.3.20a}$$

$$a_2(t) = \mathscr{P}\Big[v_{2,0} + \int_0^t i\kappa\{u^*, v_1, v_1\} - \tfrac{1}{2}\beta^2 u^* \,\mathrm{d}t' - \int_0^t i v_1 \Phi \,\mathrm{d}W(t') - \tfrac{1}{2} a_1(t)^2 u^*_{xx}\Big], \tag{2.3.20b}$$

and the corresponding w_1 and w_2 are given by

$$w_1 = P(t)\Pi v_{1,0} - \int_0^t P(t-t')\Pi i u^* \Phi \, \mathrm{d}W(t') \tag{2.3.21a}$$

$$\begin{aligned} w_2 = P(t)\Pi v_{2,0} &+ \int_0^t P(t-t')\Pi(i\kappa\{u^*, v_1, v_1\} - \tfrac{1}{2}\beta^2 u^*)\,\mathrm{d}t' \\ &- \int_0^t P(t-t')\Pi i v_1 \Phi \, \mathrm{d}W(t') - \tfrac{1}{2} a_1(t)^2 \Pi u^*_{xx}. \end{aligned} \tag{2.3.21b}$$

Finally, there exists a constant C, such that the estimates

$$\|w_1(t)\|_{L^q_\Omega(L^2_x)} \leq C\big(e^{-at}\|v_{1,0}\|_{L^q_\Omega(L^2_x)} + \sqrt{q}\beta \min\{t^{\frac{1}{2}}, 1\}\big), \tag{2.3.22a}$$

$$\|w_2(t)\|_{L^q_\Omega(L^2_x)} \leq C\big(e^{-at}\|v_{2,0}\|_{L^q_\Omega(L^2_x)} + \|v_{1,0}\|^2_{L^{2q}_\Omega(L^2_x)} + q\beta^2 t\big), \tag{2.3.22b}$$

hold for every $q \in [2,\infty)$, $v_{1,0} \in L^{2q}_\Omega(L^2_x)$, $v_{2,0} \in L^q_\Omega(L^2_x)$, $t \in [0,\infty)$, and $\phi \in L^2(\mathbb{R};\mathbb{R})$ (recall (2.2.1)).

Proposition 2.3.5 and Theorem 2.3.6 then allow us to show the following proposition.

Proposition 2.3.7. *Consider equation (2.3.3) with initial data $u(0) = u^* + v_0$, where v_0 is an L^2_x-valued $\mathscr{F}_0$-measurable random variable. There exist strictly positive constants T, $\tilde{c}_1$, $\tilde{c}_2$, λ, ε' such that the estimates*

$$\mathbb{P}\big[\|u(T) - u^*(x + \sigma a_1(T)\|_{L^2_x} \geq \tilde{c}_1\varepsilon\big] \leq 4\exp(-\tilde{c}_2\sigma^{-2}\varepsilon^2), \tag{2.3.23a}$$

$$\mathbb{P}\big[\|u(t) - u^*(x + \sigma a_1)\|_{L^\infty(0,T;L^2_x)} \geq \varepsilon\big] \leq 4\exp(-\tilde{c}_2\sigma^{-2}\varepsilon^2), \tag{2.3.23b}$$

hold for every $0 < \lambda\sigma \leq \varepsilon \leq \varepsilon'$, and every v_0 which satisfies $\|v_0\|_{L^2_x} \leq \tilde{c}_1\varepsilon$, $\mathbb{P}$-a.s.

From the translation invariance of the equation, it is immediate that the previous proposition also holds if we consider an initial condition of the form $u(0) = u^*(x+a) + v_0$ for any $a \in \mathbb{R}$. Thus, by (2.3.23a) we are at time T in essentially the same situation as at time 0 (with high probability). In this way, we can 'chain' Proposition 2.3.7 to finally obtain the long-term stability result.

Theorem 2.3.8. *Let v_0 be as in Proposition 2.3.7. There exist constants $c, \lambda, \varepsilon' > 0$, such that the estimate*

$$\mathbb{P}\Big[\sup_{t\in[0,T]} \inf_a \|u(t) - u^*(x+a)\|_{L^2_x} \geq \varepsilon\Big] \leq (T+1)\exp(-c\sigma^{-2}\varepsilon^2) \tag{2.3.24}$$

holds for every $T > 0$, $0 < \lambda\sigma \leq \varepsilon \leq \varepsilon'$, and every v_0 which satisfies $\|v_0\|_{L^2_x} \leq c\varepsilon$, $\mathbb{P}$-a.s.

Note that in the presence of noise, there is (generically) a nonzero probability for u to wander far from a shifted solitary wave in finite time. Hence, an estimate on the exit time is the best one could hope for.

2.4. PROOF OF WELL-POSEDNESS

2.4.1. LOCAL WELL-POSEDNESS

Following the approach of de Bouard and Debussche in [36, 38] and Hornung in [118], we first establish well-posedness of a modified version of equation (2.3.1) in which the nonlinear term $|u|^2 u$ is truncated. The truncation allows us to control the nonlinearity, which is otherwise not Lipschitz continuous.

We now fix $T_0 \in (0,\infty)$, $s \in [0,\infty)$, $\phi \in L^2(\mathbb{R};\mathbb{R}) \cap H^s_x$, and $(r,p) \neq (4,\infty)$ which satisfies (2.2.6). All of these will remain fixed throughout the proof. For $T \in (0,\infty)$, we also introduce the following spaces:

$$X^s_T := C([0,T]; H^s_x) \cap L^6(0,T; H^{s,6}_x) \cap L^r(0,T; H^{s,p}_x), \tag{2.4.1a}$$
$$X_T := C([0,T]; L^2_x) \cap L^6(0,T; L^6_x). \tag{2.4.1b}$$

Since the pairs (r,p) and $(\infty,2)$ both satisfy (2.2.6), we can freely replace the norms on the left-hand side of (2.2.7) by the X^s_T-norm, and will do so throughout.

For $R \geq 1$, let θ_R be the function which takes the value 1 on $[0,R]$, interpolates linearly between 1 and 0 on $[R,2R]$ and is identically zero on $[2R,\infty)$. Also define

$$(\Theta_R(u))(t) := \theta_R(\|u\|_{L^6(0,t;L^6_x)})\, u(t).$$

The function Θ_R will serve to truncate the nonlinearity in (2.3.1). Notice that Θ_R preserves adaptedness of u. The truncated mild equation now takes the form

$$\begin{aligned} u(t) = S(t)u_0 - \int_0^t S(t-t')(i\nu u(t') + \epsilon(\gamma u(t') - \mu\bar{u}(t')) + \tfrac{1}{2}\beta^2 u(t'))\, \mathrm{d}t' \\ + i\kappa \int_0^t S(t-t')(|\Theta_R(u)(t')|^2 \Theta_R(u)(t'))\, \mathrm{d}t' - i\int_0^t S(t-t')u(t')\Phi\, \mathrm{d}W(t'). \end{aligned} \tag{2.4.2}$$

Proposition 2.4.1 (Global well-posedness of truncated equation)**.** *For every $R \geq 1$ and every $\mathscr{F}_0$-measurable $u_0 \in L^2_\Omega(L^2_x)$, there is a unique $u \in L^2_\Omega(X_{T_0})$ which satisfies (2.4.2) for every $t \in [0,T_0]$, $\mathbb{P}$-a.s.*

Proof. Since the PFNLS equation differs from the nonlinear Schrödinger equation only by linear terms, the existence and uniqueness of a solution $u \in L^2_\Omega(X_{T_0})$ to (2.4.2) follows from the same arguments as in [36, 38, 118]. Broadly speaking, the proof consists of a fixed-point argument in $L^2_\Omega(X_T)$ and uses the fact that the truncated nonlinearity satisfies a global Lipschitz estimate in the space X_T. For detailed expositions we refer the reader to [118, Proposition 3] and [36, Proposition 3.1]. □

Let us denote by u_R the unique solution to the truncated equation (2.4.2) with radius R given by Proposition 2.4.1. We define for $R \geq 1$ the stopping time

$$\tau_R := \sup\{t \in [0,T_0] : \|u_R\|_{L^6(0,t;L^6_x)} \leq R\}, \tag{2.4.3}$$

which corresponds to the first time the norm $\|u_R\|_{L^6(0,t;L^6_x)}$ reaches size R, and before this time no truncation takes place.

We now prove additional regularity of u_R. Since no similar statement is shown in [36, 38, 118], we give the complete proof. We begin by stating a lemma relating to the regularity of the nonlinearity.

Lemma 2.4.2. *There exists a constant C, such that the estimate*

$$\||u|^2u\|_{L^1(0,T;H^s_x)} \le CT^{\frac{1}{2}}\|u\|_{L^6(0,T;H^{s,6}_x)}\|u\|^2_{L^6(0,T;L^6_x)} \tag{2.4.4}$$

holds for all $T \in (0,\infty)$ and $u \in L^6(0,T;H^{s,6}_x)$. In the case $s=0$, we can take $C=1$.

Proof. Since $|u|^2u$ can be written as $u^2\bar{u}$, the estimate (2.4.2) follows by repeated application of Hölder's inequality and the Kato–Ponce inequality (see [107, Theorem 1.4]). □

Proposition 2.4.3. *Let $u_0 \in L^2_\Omega(H^s_x)$ be $\mathscr{F}_0$-measurable and let $R \ge 1$. Then there exist $T>0$ and C, which do not depend on u_0, such that*

$$\|\mathbb{1}_{[0,\tau_R]}u_R\|_{L^2_\Omega(X^s_T)} \le C\|u_0\|_{L^2_\Omega(H^s_x)}. \tag{2.4.5}$$

Proof. For $t \le \tau_R$, u_R satisfies (2.4.2):

$$\begin{aligned}
u_R(t) &= S(t)u_0 - \int_0^t S(t-t')(i\nu u_R(t') + \epsilon(\gamma u_R(t') - \mu\bar{u}_R(t')) + \tfrac{1}{2}\beta^2 u_R(t'))\,\mathrm{d}t' \\
&\quad + i\kappa\int_0^t S(t-t')(|u_R(t')|^2u_R(t'))\,\mathrm{d}t' - i\int_0^t S(t-t')u_R(t')\Phi\,\mathrm{d}W(t') \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Using the deterministic Strichartz estimates (2.2.7a)-(2.2.7b) together with (2.4.3) and (2.4.4) we find a constant C such that

$$\begin{aligned}
\|\mathbb{1}_{[0,\tau_R]}(I_1+I_2+I_3)\|_{X^s_T} &\le C\|u_0\|_{H^s_x} + CT\|\mathbb{1}_{[0,\tau_R]}u_R\|_{C([0,T];H^s_x)} \\
&\quad + CR^2T^{1/2}\|\mathbb{1}_{[0,\tau_R]}u_R\|_{L^6(0,T;H^{s,6}_x)} \\
&\overset{(2.4.1)}{\le} C\|u_0\|_{H^s_x} + C(T+R^2T^{1/2})\|\mathbb{1}_{[0,\tau_R]}u_R\|_{X^s_T}.
\end{aligned}$$

From the stochastic Strichartz estimate (2.2.7c) there exists a constant C' such that

$$\|\mathbb{1}_{[0,\tau_R]}I_4\|_{L^2_\Omega(X^s_T)} \le C'\|\mathbb{1}_{[0,\tau_R]}u_R\|_{L^2_\Omega(L^2(0,T;H^s_x))} \overset{(2.4.1)}{\le} C'T^{1/2}\|\mathbb{1}_{[0,\tau_R]}u_R\|_{L^2_\Omega(X^s_T)}.$$

Combining the two above estimates, we find

$$\|\mathbb{1}_{[0,\tau_R]}u_R\|_{L^2_\Omega(X^s_T)} \le C\|u_0\|_{L^2_\Omega(H^s_x)} + (CT + CR^2T^{1/2} + C'T^{1/2})\|\mathbb{1}_{[0,\tau_R]}u_R\|_{L^2_\Omega(X^s_T)}.$$

Choosing T sufficiently small based on R, C, C' we can absorb the right-most term into the left-hand side, at which point the desired estimate follows. □

By iterating the above proposition on the time intervals $[T,2T]$ and so on, we find that $\mathbb{1}_{[0,\tau_R]}u(\cdot) \in L^2_\Omega(X^s_{T_0})$.

Now notice that for $t < \tau_R$, we have $\Theta_R(u_R(t)) = u_R(t)$. Hence, u_R is a genuine solution to (2.3.1) up until time τ_R. Moreover, solutions to (2.4.2) with different values of R

will be the same until the truncation time. Hence, we may patch together these solutions to obtain a maximal solution u which solves (2.3.1) up until a maximal stopping time τ^*, defined as:

$$\tau^* := \sup_{R\geq 1} \tau_R. \tag{2.4.6}$$

The following proposition summarizes the results of this procedure (see [118, Proposition 4] for more details).

Proposition 2.4.4 (Local well-posedness of SPFNLS)**.** *There exists a unique adapted process u which satisfies the following properties $\mathbb{P}$-a.s.*

1. $u \in X_t^s$ *for every* $t \in [0,\tau^*)$.
2. u *satisfies* (2.3.1) *for all* $t \in [0,\tau^*)$.
3. $\tau^* < T_0$ *implies* $\lim_{t\nearrow\tau^*} \|u(t)\|_{L^6(0,t;L_x^6)} = \infty$.

2.4.2. BLOWUP

We now show that the constructed solution can only fail to exist globally if its L_x^2-norm blows up.

Proposition 2.4.5 (Blowup criterion)**.** *The implication*

$$\sup_{t\in[0,\tau^*)} \|u\|_{C([0,t];L_x^2)} < \infty \implies \sup_{t\in[0,\tau^*)} \|u\|_{L^6(0,t;L_x^6)} < \infty$$

holds, $\mathbb{P}$*-a.s.*

Proof. Fix some $M \geq 1$, and define the stopping time

$$\tau := \sup\{\, t \in [0,\tau^*) : \|u\|_{C([0,t];L_x^2)} \leq M\,\}, \tag{2.4.7a}$$

as well as a recursive sequence of stopping times according to $\tau_0 = 0$ and

$$\tau_{N+1} := \sup\{\, t \in [\tau_N,\tau] : \|u\|_{L^6(\tau_N,t;L_x^6)} \leq 3KM\,\}, \quad N \in \mathbb{N}_0, \tag{2.4.7b}$$

where K is the constant C from the right-hand side of (2.2.7a). Additionally, we define the event

$$A := \{\omega \in \Omega : \tau_N < \tau,\ \forall N \in \mathbb{N}_0\},$$

and claim that $\mathbb{P}(A) = 0$. To see this, we start the solution from time τ_N and get the $\mathbb{P}$-a.s. equality

$$\begin{aligned}
u(t) = S(t-\tau_N)u(\tau_N) &- \int_{\tau_N}^t S(t-t')(i\nu u(t') + \epsilon(\gamma u(t') - \mu\bar{u}(t')) + \tfrac{1}{2}\beta^2 u(t'))\,\mathrm{d}t' \\
&+ i\kappa \int_{\tau_N}^t S(t-t')(|u(t')|^2 z(t'))\,\mathrm{d}t' - i\int_{\tau_N}^t S(t-t')u(t')\Phi\,\mathrm{d}W(t'). \\
=: I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{2.4.8}$$

for every $t \in [\tau_N, \tau^*)$. Since the Strichartz estimates from Theorem 2.2.3 are invariant under time translation and the pair $(6,6)$ is admissible (cf. (2.2.6)), we see that

$$\|I_1\|_{L^6(\tau_N,\tau_{N+1};L^6_x)} \overset{(2.2.7a)}{\leq} K\|u(\tau_N)\|_{L^2_x} \overset{(2.4.7a)}{\leq} KM, \tag{2.4.9a}$$

$$\|I_2\|_{L^6(\tau_N,\tau_{N+1};L^6_x)} \overset{(2.2.7b)}{\leq} C(\tau_{N+1}-\tau_N)\|u\|_{C([\tau_N,\tau_{N+1}];L^2_x)} \overset{(2.4.7a)}{\leq} C(\tau_{N+1}-\tau_N)M. \tag{2.4.9b}$$

To estimate I_3 we use Theorem 2.2.3 and Hölder's inequality:

$$\begin{aligned}\|I_3\|_{L^6(\tau_N,\tau_{N+1};L^6_x)} &\overset{(2.2.7b)}{\leq} C\||u|^2u\|_{L^1(\tau_N,\tau_{N+1};L^2_x)} \leq C(\tau_{N+1}-\tau_N)^{\frac12}\|u\|^3_{L^6(\tau_N,\tau_{N+1};L^6_x)}\\ &\overset{(2.4.7b)}{\leq} 27CK^3M^3(\tau_{N+1}-\tau_N)^{\frac12}.\end{aligned} \tag{2.4.9c}$$

Taking the $L^6(\tau_N,\tau_{N+1};L^6_x)$-norm of (2.4.8) and using the triangle inequality along with (2.4.9a)-(2.4.9c) gives

$$\begin{aligned}\|u\|_{L^6(\tau_N,\tau_{N+1};L^6_x)} &\leq KM + CM(\tau_{N+1}-\tau_N) + 27CK^3M^3(\tau_{N+1}-\tau_N)^{\frac12}\\ &\quad + \left\|\int_{\tau_N}^{\cdot} S(\cdot-t')u(t')\Phi\,\mathrm{d}W(t')\right\|_{L^6(\tau_N,\tau_{N+1};L^6_x)}.\end{aligned} \tag{2.4.10}$$

From (2.4.7b) it is clear that we must have the equality $\|u\|_{L^6(\tau_N,\tau_{N+1};L^6_x)} = 3KM$ for every N if $\omega \in A$. On the other hand, since τ_N is nondecreasing with N and bounded by T_0, the second and third term on the right-hand side of (2.4.10) converge to zero as $N \to \infty$. Combining these facts, we see that $\mathbb{P}(A)$ is bounded by the probability that the events

$$A_N := \left\{\omega \in \Omega : \left\|\int_{\tau_N}^{\cdot} S(\cdot-t')u(t')\Phi\,\mathrm{d}W(t')\right\|_{L^6(\tau_N,\tau_{N+1};L^6_x)} \geq KM\right\}$$

occur for infinitely many N. However, using Markov's inequality and Theorem 2.2.3, we can estimate

$$\begin{aligned}K^2M^2\mathbb{P}(A_N) &\leq \mathbb{E}\left[\left\|\int_{\tau_N}^{\cdot} S(\cdot-t')u(t')\Phi\,\mathrm{d}W(t')\right\|^2_{L^6(\tau_N,\tau_{N+1};L^6_x)}\right]\\ &\leq \mathbb{E}\left[\left\|\int_0^{\cdot} S(\cdot-t')\mathbb{1}_{[\tau_N,\tau_{N+1}]}(t')u(t')\Phi\,\mathrm{d}W(t')\right\|^2_{L^6(0,T_0;L^6_x)}\right]\\ &\overset{(2.2.7c)}{\leq} C^2\mathbb{E}\left[\|u\|^2_{L^2(\tau_N,\tau_{N+1};L^2_x)}\right].\end{aligned}$$

Since

$$\sum_{N=0}^{\infty}\mathbb{E}\left[\|u\|^2_{L^2(\tau_N,\tau_{N+1};L^2_x)}\right] \leq \mathbb{E}\left[\|u\|^2_{L^2(0,\tau;L^2_x)}\right] \overset{(2.4.7a)}{\leq} M^2T_0 < \infty$$

by Fubini's theorem, we see that the probabilities $\mathbb{P}(A_N)$ are summable. Thus, $\mathbb{P}(A) = 0$ by the Borel–Cantelli lemma. By definition of A, this implies $\sup_{t\in[0,\tau)}\|u\|_{L^6(0,t;L^6_x)} < \infty$, $\mathbb{P}$-a.s. Recalling that M was arbitrary, we finish the proof by choosing M larger than $\sup_{t\in[0,\tau^*)}\|u\|_{C([0,t];L^2_x)}$ (if this quantity is finite) so that $\tau = \tau^*$ by (2.4.7a). □

2.4.3. CONSERVATION

Having formulated a blowup criterion in terms of the L^2_x-norm, we now show that this norm can be controlled pathwise. This will yield global well-posedness of (2.3.1) in combination with Proposition 2.4.5.

Proposition 2.4.6. *The inequality*

$$\|u(t)\|_{L^2_x} \le e^{\epsilon(\mu-\gamma)t}\|u(0)\|_{L^2_x} \tag{2.4.11}$$

holds, $\mathbb{P}$-a.s., for every $t \in [0,\tau^*)$.

Recall that we need $\epsilon > 0$ and $\mu > \gamma$ in Section 2.2.4 in order for the solitary wave to exist. In that case, the bound (2.4.11) is exponentially growing in time.

Proof. Formally applying the Itô formula to $\|u\|^2_{L^2_x}$ and omitting terms which are identically zero gives the identity:

$$\|u(t)\|^2_{L^2_x} = \|u(0)\|^2_{L^2_x} + 2\epsilon \int_0^t \mu \operatorname{Re}\langle \overline{u}(t'), u(t')\rangle_{L^2_x} - \gamma \|u(t')\|^2_{L^2_x}\, \mathrm{d}t'.$$

for all $t \in [0,\tau^*)$. The above can be justified by applying the mild Itô formula proved by Da Prato, Jentzen and Röckner [65, Theorem 1], which in our case reduces to the standard Itô formula since $S(t)$ is unitary on L^2_x. Applying the Cauchy–Schwarz inequality allows us to deduce

$$\|u(t)\|^2_{L^2_x} \le \|u(0)\|^2_{L^2_x} + 2\epsilon \int_0^t (\mu-\gamma)\|u(t')\|^2_{L^2_x}\, \mathrm{d}t',$$

which implies (2.4.11) after using Grönwall's lemma and taking square roots. □

Proof of Theorem 2.3.1. From (2.4.11) it is immediate that $\mathbb{P}\left[\sup_{t\in[0,\tau^*)}\|u(t)\|_{L^2} = \infty\right] = 0$. Thus, by Proposition 2.4.5 the solutions constructed in Proposition 2.4.4 exist on the entire interval $[0, T_0]$, $\mathbb{P}$-a.s. It only remains to lift the assumption that $u_0 \in L^2_\Omega$. This can be done by considering the initial conditions $u_0^M = \mathbb{1}_{\|u_0\|_{L^2_x}\le M} u_0$ and taking M to infinity, using pathwise uniqueness to patch together the solutions. Since this is a well-known standard procedure, we will not elaborate. □

2.5. PROOF OF STABILITY

2.5.1. ASYMPTOTIC EXPANSION

Proof of Proposition 2.3.2. Throughout the proof, we will use the notation $A \lesssim B$ to denote that there exists a constant C, independent of v_1, v_2, ε, σ, and c_1, such that $A \le CB$.

Fix $T \in (0,\infty)$ and an admissible pair (r, p) with $p \in [6,\infty)$. If we prove the theorem for such p, it follows from an iterated application of Hölder's inequality that the theorem also holds for admissible pairs with $p \in [2,6)$, so the restriction on p does not entail any loss of generality.

The existence and uniqueness of the mild solution $v_1 \in C([0,T];L^2_x)$ to (2.3.8a) follows from standard theory (see for example [66, Theorem 5.4]. Using (2.2.14a), (2.2.14c), and (2.2.18a), (2.2.18c) of Propositions 2.2.9 and 2.2.10, we obtain from (2.3.10a) that $v_1 \in L^r(0,T;L^p_x)$, so that also $v_1 \in L^6(0,T;L^6_x)$, $\mathbb{P}$-a.s. Combining this with Hölder's inequality shows

$$\|\{u^*, v_1, v_1\}\|_{L^1(0,T;L^2_x)} \leq 3T^{\frac{1}{2}} \|u^*\|_{L^6(0,T;L^6_x)} \|v_1\|^2_{L^6(0,T;L^6_x)}.$$

By a standard localization procedure we can also get integrability in ω, so that the terms on the right-hand side of (2.3.10b) are well-defined and this is indeed the unique solution for v_2. Again, $v_2 \in L^r(0,T;L^p_x)$ by Propositions 2.2.9 and 2.2.10.

From the definition $z(t) := u(t) - u^* - \sigma v_1(t) - \sigma^2 v_2(t)$, it follows that z satisfies (2.3.9) in the mild sense, meaning for every $t \in [0,T]$ we have the $\mathbb{P}$-a.s. equality

$$\begin{aligned} z(t) = \int_0^t P(t-t') i\kappa R(t')\,\mathrm{d}t' - \tfrac{1}{2}\beta^2 \int_0^t P(t-t')(\sigma^3 v_1 + \sigma^4 v_2 + \sigma^2 z)\,\mathrm{d}t' \\ - \int_0^t P(t-t') i(\sigma^3 v_2 + \sigma z)\Phi\,\mathrm{d}W(t') =: I_1 + I_2 + I_3. \end{aligned} \tag{2.5.1}$$

To show (2.3.13) we define the stopping time $\tau := \min\{\tau_{v_1}, \tau_{v_2}, \tau_z\}$, and notice that

$$\mathbb{P}[\tau_z < \min\{\tau_{v_1}, \tau_{v_2}\}] = \mathbb{P}[\tau_z < T, \tau_z \leq \tau].$$

To estimate the latter probability, we first estimate I_1 and I_2 on the interval $[0,\tau]$. We assume $\varepsilon' \leq 1$ and $c_1 \geq 1$, so that we can estimate $\sigma^m \leq \varepsilon^m \leq 1$ for any $m \geq 0$. We will use this frequently and without further mention.

To estimate I_2, note that by Propositions 2.2.9 and 2.2.10 we have

$$\|I_2\|_{L^\infty(0,\tau;L^2_x)\cap L^r(0,\tau;L^p_x)} \overset{(2.2.14b),(2.2.18b)}{\lesssim} \|\sigma^3 v_1 + \sigma^4 v_2 + \sigma^2 z\|_{L^1(0,\tau;L^2_x)} \overset{(2.3.12)}{\lesssim} \varepsilon^3. \tag{2.5.2}$$

Using Propositions 2.2.9 and 2.2.10 again, carefully inspecting every term in (2.3.7) and using Hölder's inequality on the triple bracket, we see that we can also estimate

$$\|I_1\|_{L^\infty(0,\tau;L^2_x)\cap L^r(0,\tau;L^p_x)} \overset{(2.2.14b),(2.2.18b)}{\lesssim} \|R\|_{L^1(0,\tau;L^2_x)} \overset{(2.3.12)}{\lesssim} \varepsilon^3 + c_1^3\varepsilon^4. \tag{2.5.3}$$

Combining (2.5.1), (2.5.2), and (2.5.3) with the triangle inequality we get the estimate

$$\|z\|_{L^\infty(0,\tau;L^2_x)\cap L^r(0,\tau;L^p_x)} \leq C(\varepsilon^3 + c_1^3\varepsilon^4) + \|I_3\|_{L^\infty(0,\tau;L^2_x)\cap L^r(0,\tau;L^p_x)} \tag{2.5.4}$$

for some constant C which is independent of v_1, v_2, ε, σ, and c_1. This allows us to set $c_1 = 4C$ and $\varepsilon' = c_1^{-3}$. Suppose now that $\tau_z < T$ and $\tau_z \leq \tau$. Since $z \in C([0,T];L^2_x) \cap L^r(0,T;L^p_x)$, we then have by continuity:

$$c_1\varepsilon^3 \overset{(2.3.12c)}{=} \|z\|_{L^\infty(0,\tau_z;L^2_x)\cap L^r(0,\tau_z;L^p_x)} \overset{(2.5.4)}{\leq} \tfrac{1}{2}c_1\varepsilon^3 + \|I_3\|_{L^\infty(0,\tau;L^2_x)\cap L^r(0,\tau;L^p_x)}.$$

Since this can only happen if I_3 is sufficiently large, we can now estimate

$$\begin{aligned} \mathbb{P}[\tau_z < T, \tau_z \leq \tau] &\leq \mathbb{P}\left[\|\varepsilon^{-3} I_3\|_{L^\infty(0,\tau;L^2_x)\cap L^r(0,\tau;L^p_x)} \geq \tfrac{1}{2}c_1\right] \\ &= \mathbb{P}\left[\|\sigma^{-1}\varepsilon^{-2} I_3\|_{L^\infty(0,\tau;L^2_x)\cap L^r(0,\tau;L^p_x)} \geq \tfrac{1}{2}c_1\sigma^{-1}\varepsilon\right]. \end{aligned}$$

It only remains to estimate the latter probability. We note that for $t \le \tau$ we have the equality

$$\sigma^{-1}\varepsilon^{-2}I_3(t) \overset{(2.5.1)}{=} -\int_0^t P(t-t')(\mathbb{1}_{[0,\tau]}(t')i(\sigma^2\varepsilon^{-2}v_2(t')+\varepsilon^{-2}z(t')))\Phi\,\mathrm{d}W(t').$$

After estimating the integrand as

$$\|\mathbb{1}_{[0,\tau]}(t')(\sigma^2\varepsilon^{-2}v_2(t')+\varepsilon^{-2}z(t'))\|_{L^\infty_\Omega(L^\infty(0,T;L^2_x))} \overset{(2.3.12)}{\le} (1+c_1)\varepsilon \le 2,$$

it follows from (2.2.14c), (2.2.18c) and Lemma 2.2.11 that the Gaussian tail estimate

$$\mathbb{P}[\tau_z < T, \tau_z \le \tau] \le \exp(-c_2c_1^2\sigma^{-2}\varepsilon^2),$$

holds for some $c_2 > 0$ which is independent of ε, σ, c_1, as long as $c_1\sigma^{-1}\varepsilon$ is sufficiently large. But since $\varepsilon\sigma^{-1} \ge 1$, this can be accomplished by re-choosing c_1 to be larger than before if necessary (and also re-choosing $\varepsilon' = c_1^{-3}$). □

2.5.2. ORBITAL STABILITY

Before we prove Theorem 2.3.6, we isolate some convolution estimates which are used multiple times in the proof. These estimates essentially follow from Young's convolution inequality and the exponential decay of $P(t)\Pi$ (which we have not used before this point).

Lemma 2.5.1. *Let $r \in [1,\infty]$. There exists a constant C, such that the estimates*

$$\left\|\int_0^\cdot P(\cdot-t')\Pi f(t')\,\mathrm{d}t'\right\|_{L^\infty(0,T;L^q_\Omega(L^2_x))} \le C\min\{T^{\frac{1}{r'}},1\}\|f\|_{L^r(0,T;L^q_\Omega(L^2_x))}, \tag{2.5.5a}$$

$$\left\|\int_0^\cdot P(\cdot-t')\Pi g(t')\,\mathrm{d}t'\right\|_{L^\infty(0,T;L^q_\Omega(L^2_x))} \le C\min\{T^{\frac{1}{r'}},1\}\|g\|_{L^q_\Omega(L^r(0,T;L^2_x))}, \tag{2.5.5b}$$

$$\left\|\int_0^\cdot P(\cdot-t')\Pi h(t')\Phi\,\mathrm{d}W(t')\right\|_{L^\infty(0,T;L^q_\Omega(L^2_x))} \le C\sqrt{q}\beta\min\{T^{\frac{1}{2}},1\}\|h\|_{L^\infty(0,T;L^q_\Omega(L^2_x))}, \tag{2.5.5c}$$

hold for $q \in [2,\infty)$, $T \in (0,\infty)$, $\phi \in L^2(\mathbb{R};\mathbb{R})$ (recall (2.2.1)), $f \in L^r(0,T;L^q_\Omega(L^2_x))$, $g \in L^q_\Omega(L^r(0,T;L^2_x))$, and $h \in L^\infty(0,T;L^q_\Omega(L^2_x))$.

Proof. First we compute

$$\alpha_r(T) := \|P(\cdot)\Pi\|_{L^r(0,T;\mathcal{L}(L^2_x))} \overset{(2.2.12)}{\le} \|M\exp(-a\cdot)\|_{L^r(0,T)} \le C\min\{T^{\frac{1}{r}},1\}, \tag{2.5.6}$$

for some C which does not depend on T. It then follows from Young's convolution inequality that

$$\left\|\int_0^\cdot P(\cdot-t')\Pi f(t')\,\mathrm{d}t'\right\|_{L^\infty(0,T;L^q_\Omega(L^2_x))} \le \alpha_{r'}(T)\|f\|_{L^r(0,T;L^q_\Omega(L^2_x))},$$

and also

$$\begin{aligned}\left\|\int_0^\cdot P(\cdot-t')\Pi g(t')\,\mathrm{d}t'\right\|_{L^\infty(0,T;L^q_\Omega(L^2_x))} &\le \left\|\int_0^\cdot P(\cdot-t')\Pi g(t')\,\mathrm{d}t'\right\|_{L^q_\Omega(L^\infty(0,T;L^2_x))}\\ &\le \alpha_{r'}(T)\|g\|_{L^q_\Omega(L^r(0,T;L^2_x))},\end{aligned}$$

which in combination with (2.5.6) shows (2.5.5a) and (2.5.5b). Finally, for $t \in [0,T]$ we estimate

$$\begin{aligned}\left\|\int_0^t P(t-t')\Pi h\Phi\,\mathrm{d}W(t')\right\|_{L^q_\Omega(L^2_x)} &\leq C\sqrt{q}\|P(t-\cdot)\Pi h(\cdot)\Phi\|_{L^q_\Omega(L^2(0,t;\mathscr{L}_2(L^2(\mathbb{R};\mathbb{R});L^2_x)))}\\ &\overset{(2.2.4b)}{=} C\sqrt{q}\beta\|P(t-\cdot)\Pi h(\cdot)\|_{L^q_\Omega(L^2(0,t;L^2_x))}\\ &\leq C\sqrt{q}\beta\|P(t-\cdot)\Pi h(\cdot)\|_{L^2(0,t;L^q_\Omega(L^2_x))}\\ &\leq C\sqrt{q}\beta\|P(t-\cdot)\Pi\|_{L^2(0,t;\mathscr{L}(L^2_x))}\|h\|_{L^\infty(0,T;L^q_\Omega(L^2_x))}\\ &= C\sqrt{q}\beta\alpha_2(T)\|h\|_{L^\infty(0,T;L^q_\Omega(L^2_x))},\end{aligned}$$

where we have used [207, Theorem 1.1] for the first inequality, and the fact that $q \geq 2$ for the third inequality. Taking the supremum over $t \in [0,T]$ and using (2.5.6) gives (2.5.5c). □

Proof of Theorem 2.3.6. From Proposition 2.2.8 we obtain

$$\begin{aligned}v_1 &= \mathscr{P}[v_1]u^*_x + \Pi v_1,\\ v_2 &= \mathscr{P}\big[v_2 - \tfrac{1}{2}\mathscr{P}[v_1]^2 u^*_{xx}\big]u^*_x + \tfrac{1}{2}\mathscr{P}[v_1]^2 u^*_{xx} + \Pi\big(v_2 - \tfrac{1}{2}\mathscr{P}[v_1]^2 u^*_{xx}\big).\end{aligned}$$

If we define

$$\begin{aligned}a_1 &:= \mathscr{P}[v_1], & w_1 &:= \Pi v_1,\\ a_2 &:= \mathscr{P}\big[v_2 - \tfrac{1}{2}\mathscr{P}[v_1]^2 u^*_{xx}\big], & w_2 &:= \Pi\big(v_2 - \tfrac{1}{2}\mathscr{P}[v_1]^2 u^*_{xx}\big),\end{aligned}$$

then (2.3.17) and (2.3.19) hold. Equations (2.3.20) and (2.3.21) follow by substitution using (2.3.10) and noting that Π commutes with $P(t)$.

We will now show (2.3.22). Throughout the proof, $A \lesssim B$ means that there exists a constant C, independent of $v_{1,0}$, $v_{2,0}$, t, q and ϕ (recall (2.2.1)) such that $A \leq CB$. We first estimate w_1 as follows:

$$\begin{aligned}\|w_1(t)\|_{L^q_\Omega(L^2_x)} &\overset{(2.3.21a)}{\leq} \|P(t)\Pi v_{1,0}\|_{L^q_\Omega(L^2_x)} + \left\|\int_0^t P(t-t')\Pi i u^*\Phi\,\mathrm{d}W(t')\right\|_{L^q_\Omega(L^2_x)}\\ &\overset{(2.2.12),(2.5.5c)}{\lesssim} e^{-at}\|v_{1,0}\|_{L^q_\Omega(L^2_x)} + \sqrt{q}\beta\min\{t^{\frac{1}{2}},1\},\end{aligned}$$

which is (2.3.22a). In order to show (2.3.22b), we will need two intermediate estimates. Firstly, by Proposition 2.2.9 we have

$$\begin{aligned}\|w_1\|_{L^q_\Omega(L^6(0,t;L^6_x))} &\leq \|P(\cdot)\Pi v_{1,0}\|_{L^q_\Omega(L^6(0,t;L^6_x))} + \left\|\int_0^\cdot P(\cdot-t')\Pi i u^*\Phi\,\mathrm{d}W(t')\right\|_{L^q_\Omega(L^6(0,t;L^6_x))}\\ &\overset{(2.2.14a),(2.2.14c)}{\lesssim} \|v_{1,0}\|_{L^q_\Omega(L^2_x)} + \sqrt{q}\beta t^{\frac{1}{2}}.\end{aligned} \tag{2.5.7}$$

It also follows from [207, Theorem 1.1] that

$$\|a_1(t)\|_{L^q_\Omega} \overset{(2.3.20a)}{\lesssim} \|v_{1,0}\|_{L^q_\Omega(L^2_x)} + \sqrt{q}\beta t^{\frac{1}{2}}. \tag{2.5.8}$$

Now we have all the ingredients needed to estimate w_2. We first replace the occurrences of v_1 in (2.3.21b) by $w_1 + a_1 u_x^*$, in accordance with (2.3.17). This results in the equality

$$\begin{aligned} w_2(t) = P(t)\Pi v_{2,0} &+ \int_0^t P(t-t')\Pi i\kappa\{u^*, w_1, w_1\}\,\mathrm{d}t' \\ &+ 2\int_0^t P(t-t')\Pi i\kappa a_1\{u^*, u_x^*, w_1\}\,\mathrm{d}t' \\ &+ \int_0^t P(t-t')\Pi i\kappa a_1^2\{u^*, u_x^*, u_x^*\}\,\mathrm{d}t' \\ &- \tfrac{1}{2}\int_0^t P(t-t')\Pi\beta^2 u^*\,\mathrm{d}t' \\ &- \int_0^t P(t-t')\Pi i w_1\Phi\,\mathrm{d}W(t') \\ &- \int_0^t P(t-t')\Pi i a_1 u_x^*\Phi\,\mathrm{d}W(t') \\ &- \tfrac{1}{2} a_1^2\Pi u_{xx}^*. \end{aligned}$$

We estimate the $L^q_\Omega(L^2_x)$-norm of each term separately, which will show (2.3.22b). First, we have

$$\begin{aligned} \|P(t)\Pi v_{2,0}\|_{L^q_\Omega(L^2_x)} &\overset{(2.2.12)}{\lesssim} e^{-at}\|v_{2,0}\|_{L^q_\Omega(L^2_x)}, \\ \|a_1(t)^2\Pi u_{xx}^*\|_{L^q_\Omega(L^2_x)} \lesssim \|a_1(t)^2\|_{L^q_\Omega} = \|a_1(t)\|^2_{L^{2q}_\Omega} &\overset{(2.5.8)}{\lesssim} \|v_{1,0}\|^2_{L^{2q}_\Omega(L^2_x)} + q\beta^2 t. \end{aligned}$$

Next, we use our first intermediate estimate on the term which is quadratic in w_1.

$$\begin{aligned} \left\|\int_0^t P(t-t')\Pi i\kappa\{u^*, w_1, w_1\}\,\mathrm{d}t'\right\|_{L^q_\Omega(L^2_x)} &\overset{(2.5.5b)}{\lesssim} \|\{u^*, w_1, w_1\}\|_{L^q_\Omega(L^3(0,t;L^2_x))} \\ \lesssim \|u^*\|_{L^\infty(0,t;L^6_x)}\|w_1\|^2_{L^{2q}_\Omega(L^6(0,t;L^6_x))} &\overset{(2.5.7)}{\lesssim} \|v_{1,0}\|^2_{L^{2q}_\Omega(L^2_x)} + q\beta^2 t, \end{aligned}$$

where we have used Hölder's inequality for the second step. We also estimate

$$\begin{aligned} \left\|\int_0^t P(t-t')\Pi i\kappa a_1\{u^*, u_x^*, w_1\}\,\mathrm{d}t'\right\|_{L^q_\Omega(L^2_x)} &\overset{(2.5.5a)}{\lesssim} \|a_1\{u^*, u_x^*, w_1\}\|_{L^\infty(0,t;L^q_\Omega(L^2_x))} \\ \lesssim \|a_1\|_{L^\infty(0,t;L^{2q}_\Omega)}\|w_1\|_{L^\infty(0,t;L^{2q}_\Omega(L^2_x))} &\overset{(2.3.22a),(2.5.8)}{\lesssim} \|v_{1,0}\|^2_{L^{2q}_\Omega(L^2_x)} + q\beta^2 t, \end{aligned}$$

as well as

$$\begin{aligned} \left\|\int_0^t P(t-t')\Pi i\kappa a_1^2\{u^*, u_x^*, u_x^*\}\,\mathrm{d}t'\right\|_{L^q_\Omega(L^2_x)} &\overset{(2.5.5a)}{\lesssim} \|a_1^2\{u^*, u_x^*, u_x^*\}\|_{L^\infty(0,t;L^q_\Omega(L^2_x))} \\ \lesssim \|a_1\|^2_{L^\infty(0,t;L^{2q}_\Omega)} &\overset{(2.5.8)}{\lesssim} \|v_{1,0}\|^2_{L^{2q}_\Omega} + q\beta^2 t, \end{aligned}$$

and

$$\left\| \int_0^t P(t-t')\Pi\beta^2 u^* \,\mathrm{d}t' \right\|_{L^q_\Omega(L^2_x)} \overset{(2.2.12)}{\lesssim} \beta^2 t.$$

It only remains to estimate the stochastic integrals in (2.5.9). For the first we have

$$\begin{aligned}\left\| \int_0^t P(t-t')\Pi i w_1 \Phi \,\mathrm{d}W(t') \right\|_{L^q_\Omega(L^2_x)} &\overset{(2.5.5c)}{\lesssim} \sqrt{q}\beta t^{\frac{1}{2}} \|w_1\|_{L^\infty(0,t;L^q_\Omega(L^2_x))} \\ \le \tfrac{1}{2}\|w_1\|^2_{L^\infty(0,t;L^q_\Omega(L^2_x))} + \tfrac{1}{2} q\beta^2 t &\overset{(2.3.22a)}{\lesssim} \|v_{1,0}\|^2_{L^q_\Omega(L^2_x)} + q\beta^2 t,\end{aligned}$$

and for the second

$$\begin{aligned}\left\| \int_0^t P(t-t')\Pi i a_1 u^*_x \Phi \,\mathrm{d}W(t') \right\|_{L^q_\Omega(L^2_x)} &\overset{(2.5.5c)}{\lesssim} \sqrt{q}\beta t^{\frac{1}{2}} \|a_1\|_{L^\infty(0,t;L^q_\Omega)} \\ \le \tfrac{1}{2}\|a_1\|^2_{L^\infty(0,t;L^q_\Omega)} + \tfrac{1}{2} q\beta^2 t &\overset{(2.5.8)}{\lesssim} \|v_{1,0}\|^2_{L^q_\Omega(L^2_x)} + q\beta^2 t.\end{aligned} \qquad \square$$

Proof of Proposition 2.3.7. From our previous ansatz for u and v_1 we have the identities

$$\begin{aligned} u(t) - u^*(x+\sigma a_1(t)) &\overset{(2.3.14)}{=} u^* - u^*(x+\sigma a_1(t)) + \sigma v_1(t) + z'(t) && (2.5.10a)\\ &\overset{(2.3.17a)}{=} u^* + \sigma a_1(t) u^*_x - u^*(x+\sigma a_1(t)) + \sigma w_1(t) + z'(t). && (2.5.10b)\end{aligned}$$

From (2.5.10a) and a zeroth-order Taylor expansion we may obtain

$$\|u(t) - u^*(x+\sigma a_1(t))\|_{L^2_x} \le C_1\sigma|a_1(t)| + \sigma\|v_1(t)\|_{L^2_x} + \|z'(t)\|_{L^2_x}, \tag{2.5.11a}$$

for some constant C_1 derived from u^*. From (2.5.10b) and a first-order Taylor expansion we also get

$$\|u(t) - u^*(x+\sigma a_1(t))\|_{L^2_x} \le C_2\sigma^2|a_1(t)|^2 + \sigma\|w_1(t)\|_{L^2_x} + \|z'(t)\|_{L^2_x}, \tag{2.5.11b}$$

for some constant C_2 also derived from u^*. Now set $T = a^{-1}\log(6M)$, where a and M are the constants from (2.2.12), and fix some c_1, c_2, ε' such that Proposition 2.3.5 holds with this choice of T (note that our initial condition corresponds to setting $v_{1,0} = \sigma^{-1} v_0$). Additionally, set $\tilde{c}_1 = \frac{1}{6}\min\{M^{-1}, C_1^{-1}\|\mathscr{P}\|_{\mathscr{L}(L^2_x;\mathbb{R})}\}$. From the assumption that $\|v_0\| \le \tilde{c}_1\varepsilon$ we obtain

$$\begin{aligned}\sigma|a_1(t)| &\overset{(2.3.20a)}{\le} C_1^{-1}\frac{\varepsilon}{6} + \sigma\|\mathscr{P}\|_{\mathscr{L}(L^2_x;\mathbb{R})} \left\| \int_0^t u^*\Phi \,\mathrm{d}W(t') \right\|_{L^2_x},\\ \sigma\|v_1(t)\|_{L^2_x} &\overset{(2.3.10a)}{\le} \frac{\varepsilon}{6} + \sigma\left\| \int_0^t P(t-t')u^*\Phi \,\mathrm{d}W(t') \right\|_{L^2_x},\\ \sigma\|w_1(T)\|_{L^2_x} &\overset{(2.3.21a)}{\le} \tilde{c}_1\frac{\varepsilon}{6} + \sigma\left\| \int_0^T P(T-t')\Pi u^*\Phi \,\mathrm{d}W(t') \right\|_{L^2_x},\end{aligned}$$

where the third inequality follows from (2.2.12) since $Me^{-aT} = \frac{1}{6}$ by our choice of T. Using (2.2.14c), (2.2.18c), and Lemma 2.2.11, we can find constants λ, $c_2' > 0$, such that

$$\mathbb{P}\big[C_1\sigma|a_1|_{L^\infty(0,T)} \geq \tfrac{\varepsilon}{3}\big] \leq \exp(-c_2'\sigma^{-2}\varepsilon^2), \tag{2.5.12a}$$

$$\mathbb{P}\big[\sigma\|v_1\|_{L^\infty(0,T;L^2_x)} \geq \tfrac{\varepsilon}{3}\big] \leq \exp(-c_2'\sigma^{-2}\varepsilon^2), \tag{2.5.12b}$$

$$\mathbb{P}\big[\sigma\|w_1(T)\|_{L^2_x} \geq \tilde{c}_1\tfrac{\varepsilon}{3}\big] \leq \exp(-c_2'\sigma^{-2}\varepsilon^2), \tag{2.5.12c}$$

whenever $\sigma^{-1}\varepsilon \geq \lambda$. If we take ε' small enough such that $\tilde{c}_1\frac{\varepsilon'}{3} \geq c_1\varepsilon'^2$ (if necessary), then by Proposition 2.3.5, this also results in

$$\begin{aligned}\mathbb{P}\big[\|z'\|_{L^\infty(0,T;L^2_x)} \geq \tilde{c}_1\tfrac{\varepsilon}{3}\big] &\leq \mathbb{P}\big[\|z'\|_{L^\infty(0,T;L^2_x)} \geq c_1\varepsilon^2\big]\\ &= \mathbb{P}\big[\tau_{z'} < T\big]\\ &\leq \mathbb{P}\big[\tau_{z'} < \tau_{v_1}\big] + \mathbb{P}\big[\tau_{v_1} < T\big]\\ &\overset{(2.3.16),(2.5.12b)}{\leq} \exp(-c_2\sigma^{-2}\varepsilon^2) + \exp(-c_2'\sigma^{-2}\varepsilon^2),\end{aligned} \tag{2.5.12d}$$

for all $\varepsilon \leq \varepsilon'$. If we additionally take ε' smaller (if necessary) such that $\frac{C_1\sqrt{3\tilde{c}_1}}{\sqrt{C_2\varepsilon'}} \geq 1$, then we also get

$$\mathbb{P}\big[C_2\sigma^2|a_1|^2_{L^\infty(0,T)} \geq \tilde{c}_1\tfrac{\varepsilon}{3}\big] = \mathbb{P}\Big[C_1\sigma|a_1|_{L^\infty(0,T)} \geq \tfrac{C_1\sqrt{3\tilde{c}_1}}{\sqrt{C_2\varepsilon}}\tfrac{\varepsilon}{3}\Big] \overset{(2.5.12a)}{\leq} \exp(-c_2'\sigma^{-2}\varepsilon^2), \tag{2.5.12e}$$

for all $\varepsilon \leq \varepsilon'$. Equation (2.5.11a), a simple union bound and the fact that $\tilde{c}_1 \leq 1$ now gives

$$\begin{aligned}\mathbb{P}[\|u(\cdot) - u^*(x+\sigma a_1(\cdot))\|_{L^\infty(0,T;L^2_x)} \geq \varepsilon] &\leq \mathbb{P}\big[C_1\sigma|a_1|_{L^\infty(0,T)} \geq \tfrac{\varepsilon}{3}\big]\\ &\quad + \mathbb{P}\big[\sigma\|v_1\|_{L^\infty(0,T;L^2_x)} \geq \tfrac{\varepsilon}{3}\big]\\ &\quad + \mathbb{P}\big[\|z'\|_{L^\infty(0,T;L^2_x)} \geq \tilde{c}_1\tfrac{\varepsilon}{3}\big]\\ &\overset{(2.5.12)}{\leq} 3\exp(-c_2'\sigma^{-2}\varepsilon^2) + \exp(-c_2\sigma^{-2}\varepsilon^2).\end{aligned}$$

Similarly, from (2.5.11b) we get

$$\begin{aligned}\mathbb{P}[\|u(T) - u^*(x+\sigma a_1(T))\|_{L^2_x} \geq \tilde{c}_1\varepsilon\big] &\leq \mathbb{P}\big[C_2\sigma^2|a_1(T)|^2 \geq \tilde{c}_1\tfrac{\varepsilon}{3}\big]\\ &\quad + \mathbb{P}\big[\sigma\|w_1(T)\|_{L^2_x} \geq \tilde{c}_1\tfrac{\varepsilon}{3}\big]\\ &\quad + \mathbb{P}\big[\|z'(T)\|_{L^2_x} \geq \tilde{c}_1\tfrac{\varepsilon}{3}\big]\\ &\overset{(2.5.12)}{\leq} 3\exp(-c_2'\sigma^{-2}\varepsilon^2) + \exp(-c_2\sigma^{-2}\varepsilon^2).\end{aligned}$$

(note that although we wrote $L^\infty(0,T)$ in (2.5.12), we could have also written $C([0,T])$ so the estimate is valid). The result follows by choosing $\tilde{c}_2 = \min\{c_2, c_2'\}$. □

2.A. HILBERT–SCHMIDT OPERATORS

Proof of Proposition 2.2.1. Fix some $\phi \in L^2(\mathbb{R};\mathbb{R})$, and define for any $\psi \in L^2_x$ the following map:

$$\Phi_\psi : f \mapsto \psi * f.$$

Recall that with this notation $\Phi = \Phi_\phi$ (see (2.2.1a)). Now let e_k, $k \in \mathbb{N}$ be any orthonormal basis of $L^2(\mathbb{R};\mathbb{R})$. We see using Parseval's identity that

$$\sum_{k\in\mathbb{N}} (\Phi e_k(x))^2 = \sum_{k\in\mathbb{N}} \langle \phi(\cdot - x), e_k\rangle_{L^2_x}^2 = \|\phi(\cdot - x)\|_{L^2_x}^2 \overset{(2.2.1b)}{=} \beta^2,$$

which shows (2.2.4a). Using Fubini's theorem and Parseval's identity, we can also compute

$$\begin{aligned}
\|u\Phi\|^2_{\mathscr{L}_2(L^2(\mathbb{R};\mathbb{R});L^2_x)} &= \sum_{k\in\mathbb{N}} \|u\Phi e_k\|^2_{L^2_x} = \sum_{k\in\mathbb{N}} \int_{\mathbb{R}} |u(x)|^2 \langle \phi(\cdot - x), e_k\rangle^2_{L^2_x} \,\mathrm{d}x \\
&= \int_{\mathbb{R}} |u(x)|^2 \sum_{k\in\mathbb{N}} \langle \phi(\cdot - x), e_k\rangle^2_{L^2_x} \,\mathrm{d}x = \int_{\mathbb{R}} |u(x)|^2 \|\phi(\cdot - x)\|^2_{L^2_x} \,\mathrm{d}x \\
&= \|u\|^2_{L^2_x} \|\phi\|^2_{L^2_x},
\end{aligned}$$

which shows (2.2.4b).

To show (2.2.4c) we will make use of complex interpolation. Thus, we will now break convention and regard H^s_x and L^2_x as complex spaces for the rest of this section. We will show the complexified estimate

$$\|u\Phi\|_{\mathscr{L}_2(L^2_x;H^s_x)} \le C_s \|\phi\|_{H^s_x} \|u\|_{H^s_x}. \tag{2.A.1}$$

The result then follows after noting that an orthonormal basis of the real Hilbert space $L^2(\mathbb{R};\mathbb{R})$ is also an orthonormal basis of L^2_x when the latter is regarded as a complex Hilbert space. We first show by induction that (2.A.1) holds when $s = 2n$ for some nonnegative integer n. By repeating the previous calculation, we find again that

$$\|u\Phi\|_{\mathscr{L}_2(L^2_x;L^2_x)} = \|u\|_{L^2_x} \|\phi\|_{L^2_x},$$

which implies the base case. Therefore, we now assume that the statement holds for some n. By elementary computations, we find

$$\begin{aligned}
(1-\Delta)(u\Phi f) &= (1-\Delta)(u(\phi * f)) \\
&= u(\phi * f) - \Delta u(\phi * f) - 2\partial_x u(\partial_x \phi * f) - u(\Delta\phi * f) \\
&= u\Phi f - \Delta u \Phi f - 2\partial_x u(\Phi_{\partial_x\phi} f) - u(\Phi_{\Delta\phi} f),
\end{aligned}$$

so that

$$(1-\Delta)(u\Phi) = u\Phi - \Delta u\Phi - 2\partial_x u \Phi_{\partial_x\phi} - u\Phi_{\Delta\phi}.$$

Combining this with the triangle inequality and the induction hypothesis gives

$$\|u\Phi\|_{\mathscr{L}_2(L^2_x;H^{n+2}_x)} = \|(1-\Delta)(u\Phi)\|_{\mathscr{L}_2(L^2_x;H^n_x)} \le C\|u\|_{H^{n+2}_x} \|\phi\|_{H^{n+2}_x}.$$

Now let $s \in [0,\infty)$ be arbitrary, let n be an integer such that $2n \ge s$, let $\theta \in [0,1]$ be such that $s = 2n\theta$, and consider the bilinear map

$$B\colon (u,\phi) \mapsto u \cdot \Phi_\phi.$$

We have already shown that B is bounded from $L^2_x \times L^2_x$ to $\mathscr{L}_2(L^2_x; L^2_x)$ and from $H^{2n}_x \times H^{2n}_x$ to $\mathscr{L}_2(L^2_x, H^{2n}_x)$. Thus, by complex interpolation (using the notation $[\cdot,\cdot]_\theta$ for the intermediate space) it follows that B is also bounded from

$$[L^2_x, H^{2n}_x]_\theta \times [L^2_x, H^{2n}_x]_\theta = H^s_x \times H^s_x$$

to

$$[\mathscr{L}_2(L^2_x, L^2_x), \mathscr{L}_2(L^2_x, H^{2n}_x)]_\theta = \mathscr{L}_2(L^2_x, H^s_x). \tag{2.A.2}$$

For the interpolation of bilinear operators we have used [25, Theorem 4.4.1], and the isomorphism (2.A.2) is shown for γ-radonifying operators (which generalize Hilbert–Schmidt operators) in [120, Theorem 9.1.25]. □

2.B. STOCHASTIC STRICHARTZ ESTIMATES

To prove (2.2.7c) we distinguish between the cases $p = 2$ and $p > 2$.

Case $p > 2$. For every $t' \in [0, T]$, define the operator

$$\begin{aligned} \Psi(t') : H^s_x &\to L^r(0,T; H^{s,p}_x) \\ \psi &\mapsto 1_{[t',T]}(\cdot) S(\cdot - t')\psi, \end{aligned}$$

and observe that $\|\Psi(t')\|_{\mathscr{L}(H^s_x; L^r(0,T;H^{s,p}_x))} \le \|\Psi(0)\|_{\mathscr{L}(H^s_x;L^r(0,T;H^{s,p}_x))} \le L$ for some $L < \infty$ which is independent of T by (2.2.7a).

Since $p \in (2,\infty)$, the space L^p_x is 2-smooth [119, Proposition 3.5.30]. Using the lifting operator $(1-\Delta)^{\frac{s}{2}}$, this property immediately extends to $H^{s,p}_x$. Since $r \in (4,\infty)$, the space $L^r(0,T;H^{s,p}_x)$ has this property as well (see for instance [185, Proposition 2.2]). Thus, using our definition of Ψ we can rewrite and estimate

$$\begin{aligned} \Big\| \int_0^{\cdot} S(\cdot - t') h(t') \Phi \, \mathrm{d}W(t') \Big\|_{L^q_\Omega(L^r(0,T;H^{s,p}_x))} &= \Big\| \int_0^T \Psi(t') h(t') \Phi \, \mathrm{d}W(t') \Big\|_{L^q_\Omega(L^r(0,T;H^{s,p}_x))} \\ &\le C\sqrt{q} \|\Psi h \Phi\|_{L^q_\Omega(L^2(0,T;\gamma(L^2(\mathbb{R};\mathbb{R});L^r(0,T;H^{s,p}_x))))} \\ &\le CL\sqrt{q}\|h\Phi\|_{L^q_\Omega(L^2(0,T;\mathscr{L}_2(L^2(\mathbb{R};\mathbb{R});H^s_x)))} \\ &\overset{(2.2.4c)}{\le} C' L \sqrt{q} \|\phi\|_{H^s_x} \|h\|_{L^q_\Omega(L^2(0,T;H^s_x))}. \end{aligned}$$

The first inequality follows from [207, Theorem 1.1], and the second follows from the left-ideal property of γ-radonifying operators (which can easily be seen from the definition) and the boundedness of Ψ. □

Case $p = 2$. Since (r, p) satisfies (2.2.6) we have $r = \infty$. Using the fact that $S(t)$ is unitary on H^s_x and using [207, Theorem 1.1] again we find

$$\begin{aligned} \Big\| \int_0^{\cdot} S(\cdot - t') h(t') \Phi \, \mathrm{d}W(t') \Big\|_{L^q_\Omega(L^\infty(0,T;H^s_x))} &= \Big\| \int_0^{\cdot} S(-t') h(t') \Phi \, \mathrm{d}W(t') \Big\|_{L^q_\Omega(L^\infty(0,T;H^s_x))} \\ &\le C\sqrt{q}\|S(-\cdot)h(\cdot)\Phi\|_{L^q_\Omega(L^2(0,T;\mathscr{L}_2(L^2(\mathbb{R};\mathbb{R});H^s_x)))} \\ &= C\sqrt{q}\|h\Phi\|_{L^q_\Omega(L^2(0,T;\mathscr{L}_2(L^2(\mathbb{R};\mathbb{R});H^s_x)))} \\ &\overset{(2.2.4c)}{\le} C'\sqrt{q}\|\phi\|_{H^s_x}\|h\|_{L^q_\Omega(L^2(0,T;H^s_x))}. \end{aligned}$$

The continuity in H^s_x follows by a routine approximation argument. $\square$

3

NONCOMMUTATIVE ORBITAL STABILITY IN BANACH SPACES

This chapter is based on the preprint

[P2] J. van Winden. "Noncommutative orbital stability of stochastic patterns in Banach spaces". Preprint. 2024. [arXiv].

A preliminary investigation into patterns with noncommutative symmetry groups can be found in the author's MSc thesis [229].

Abstract. *We consider stochastic perturbations of PDEs which have special pattern solutions, such as (nonlinear) traveling waves, solitons, and spiral waves. We show orbital stability of these patterns on a time scale which is exponential in the inverse square of the noise amplitude. We systematically treat equations with noncommutative symmetry groups, and show how the noncommutativity affects the motion of the pattern. This is done by introducing a new method to track the (generalized) phase of the pattern. Furthermore, we demonstrate how orbital stability arises from a mismatch of symmetry between the pattern and the equation. Our phase tracking method does not rely on a Hilbert space structure. This allows us to show stability in general Banach spaces, and to treat noise with lower regularity than before.*

3.1. INTRODUCTION

Recently, a significant interest has developed in the stability of patterns in stochastic partial differential equations (SPDEs) [4, 33, 51, 52, 112, 113, 114, 115, 121, 147, 148, 150, 154, 157, 161]. Commonly studied patterns include traveling waves, traveling pulses, spiral waves, solitary waves, and solitons. A typical feature is that these patterns exhibit *orbital stability*, meaning that the solution to the SPDE remains close to a suitably shifted version of the pattern. The exact nature of this shift, which we will henceforth refer to as the *phase* or *phase shift*, depends on the geometry of the equation. To show orbital stability, it is often necessary to have a method to continuously track the phase of the pattern, for which various established methods are available [51, 112, 121, 148]. However, these methods typically rely on orthogonality conditions, which are only available when working in a Hilbert space. In this chapter, we make the following contribution to this field of research:

- We explain how orbital stability arises through symmetry, and show what kind of phase shift is expected for a given pattern.
- We introduce a new method of tracking the phase, which does not rely on Hilbert space geometry.
- We give explicit expressions to compute the phase, which are valid for patterns with noncommutative symmetry groups.

The main results (Theorems 3.4.7 and 3.4.8) show orbital stability of stochastically forced patterns in Banach spaces, on a time scale which is exponential in the inverse square of the noise amplitude. We also directly relate the orbital stability to the symmetry group of the equation. The main novelties of this chapter are that we treat a general noncommutative setting, and show stability without assuming an underlying Hilbert space structure. The advantage of working in a Banach space setting is that we can allow for rougher noise, as is demonstrated in Section 3.5.1.

3.1.1. ORBITAL STABILITY AND SYMMETRY

The prototypical example of an orbitally stable pattern is that of a traveling (nonlinear) wave or pulse. The literature on these waves is vast, and it is not feasible to give a comprehensive overview here. A (nonexhaustive) list of settings in which these waves have been studied consists of hydrodynamics [143], neural field equations [60], fiber optics equations [163, 164], and predator-prey models [94]. For more comprehensive treatments of this topic, we refer the reader to [129, 149, 203, 221].

The mathematical treatment of these waves can be subdivided into three somewhat separate aspects: existence, linear stability, and nonlinear stability. A prime example of each of these aspects is found in the seminal works of Evans on nonlinear waves and pulses in neural field equations [80, 81, 82, 83]. The treatment of stochastically perturbed traveling waves is much more recent (see e.g. [112, 121, 148]). However, the earliest work treating orbital stability in an SPDE that we are aware of is [75]. As our primary goal is to treat nonlinear stochastic stability, we take existence and linear stability of a pattern for granted (see Assumption 3.2 in Section 3.3).

In the case of a traveling wave, it may seem obvious that a translational correction is the 'right' way to shift the pattern. However, in higher dimensions, the situation is not as clear. A primary motivating example in this situation is a two-dimensional spiral wave, which requires a phase shift consisting of translations and rotations [28, 150]. At first glance, one may wonder why a rotational correction does not suffice to show stability. In fact, even in the case of a traveling wave, it is not immediately obvious why a phase correction is necessary in the first place. In Section 3.3, we answer these questions by stating the following principle:

> *Orbital stability arises from continuous symmetries of the equation which are not shared by the pattern.*

This immediately clarifies the origin and nature of the phase correction for the spiral wave and the traveling waves. Furthermore, it provides a guide to determine how a pattern with a more complicated symmetry group is expected to move.

Systematic treatments of patterns with bigger symmetry groups have been given before (see [129, Chapter 4.2], [161]). However, both works contain the explicit or implicit assumption that the symmetry group is commutative, which is a significant limitation. Note that the recent work [4], which does not seem to include commutativity assumptions, defers the stability proof to [161], in which commutativity is implicitly assumed. The commutativity already poses a problem when treating the two-dimensional spiral wave, as the relevant symmetry group of $\mathbb{R}^2$ (consisting of rotations and translations) is noncommutative. It can be seen in [28, 150] that this noncommutativity plays a significant role in the analysis. In the noncommutative setting, there is the work of Beyn and Thümmler [29] dealing with PDEs with continuous symmetries. Although the setting is similar to ours, [29] does not address the matter of orbital stability of patterns. Moreover, there are serious analytical challenges when one tries to adapt the 'freezing method' to a stochastic setting, as can be seen in e.g. [33].

Using the basic theory of Lie groups and Lie algebras, we give a systematic treatment of (stochastic) nonlinear stability which is valid in the noncommutative case. We show that the linearized dynamics around the pattern can be explicitly described in terms of the Lie algebra corresponding to the symmetry group of the equation (see Section 3.3.2). Moreover, from our method of phase tracking, it can directly be seen how noncommutativity affects the motion of the pattern (see Section 3.4.1).

3.1.2. PHASE TRACKING

When showing orbital stability of stochastically perturbed patterns, a crucial aspect is the issue of how to track the phase. In the recent literature, several different ways of accomplishing this have been formulated. We identify the following methods of phase tracking:

- The variational phase [121, 150, 157, 161].
- The stochastic freezing phase [33, 112, 113, 114, 115].
- The phase-lag method [74, 147, 148, 154] (also used in Chapter 2).
- The isochronal phase [1, 3, 4] (also used in Chapter 4).

Similar techniques also arise in a variety of other contexts, as can be seen in the works [12, 13, 31, 133]. For a more general introduction to center manifold reduction in the context of SPDEs, we refer the reader to [71, Chapter 6]. We also remark that any two 'valid' notions of phase must be closely related, as the pattern can only be in one location at a time.

Despite the conceptual variety of these phase tracking methods, almost all of the cited works rely in one way or another on a Hilbert space structure, which significantly limits the applicability and poses restrictions on the noise. The variational phase and stochastic freezing phase are both defined in terms of orthogonality conditions, so they are not well-defined outside of a Hilbert space. The phase-lag method and isochronal phase seem more suitable, with definitions which (partially) generalize to Banach spaces. However, the associated stability proofs still rely in a nontrivial way on the presence of an inner product.

In Section 3.4, we introduce a new method of phase tracking, which we call the *predicted phase* (3.4.3). It is defined using a decomposition of the initial profile, and can be viewed as a first-order approximation to the isochronal phase. In the case of a standing pulse solution of the form $u(t,x) = u^*(x)$, where u^* is the pulse profile, it can be described as follows. Consider an initial condition of the form $u(0,\cdot) = u^* + v_0$, where v_0 is a perturbation of the profile which is $\mathcal{O}(\varepsilon)$. If the pulse is linearly stable (see Assumption 3.4), we can find a decomposition

$$v_0 = a\partial_x u^* + w_0, \tag{3.1.1}$$

where a, w_0 are both $\mathcal{O}(\varepsilon)$, and w_0 decays exponentially under the linear dynamics. For more details on how to compute this decomposition, we refer ahead to Section 3.3.3. By (3.1.1) and a Taylor expansion, we have

$$\begin{aligned} u(0,\cdot) &= u^* + a\partial_x u^* + w_0 \\ &= u^*(\cdot + a) + w_0 + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Using the exponential decay of w_0 under the linear dynamics, and treating the nonlinear and stochastic terms perturbatively, we obtain for $t \geq 0$ the expansion

$$u(t,\cdot) = u^*(\cdot + a) + \mathcal{O}(\varepsilon e^{-t}) + \mathcal{O}(\varepsilon^2).$$

For a fixed, large enough T, and sufficiently small ε, we thus find that the difference $u(T,\cdot) - u^*(\cdot - a)$ is smaller (by a constant factor) than the initial difference $u(0,\cdot) - u^*$, when measured in suitable norms. We then repeat this procedure on the time intervals $[T, 2T]$, $[2T, 3T]$ and so on to obtain stability on long time scales.

Because of the stochastic forcing, there is on each time interval $[nT, (n+1)T]$ a probability $p > 0$ that the solution strays too far from the stable manifold. Using the subgaussian tail estimates on stochastic convolutions formulated in Section 3.2.2, we will estimate $p \lesssim \exp(-c\varepsilon^2\sigma^{-2})$ for some $c > 0$, where σ denotes the noise amplitude (see (3.4.12)). The probability $p_\varepsilon(t)$ that the solution leaves an ε-neighborhood of the stable manifold before time t can then be estimated as $p_\varepsilon(t) \lesssim t\exp(-c\varepsilon^2\sigma^{-2})$. Thus, for small noise amplitude ($\sigma \ll 1$), the pattern is stable for a long time with high probability. We refer ahead to Section 3.4.1 for the definition of the phase for patterns with more complicated (noncommutative) symmetries.

Our method has two advantages compared to the previously mentioned ones. The first is that the expression for the phase (3.4.3) is explicit and straightforward to compute (for two examples, see Sections 3.5.2 and Remark 3.5.8). The phase is directly determined from the initial condition, and there is no need to couple an SDE to the SPDE to continuously track the phase. This also bypasses analytical challenges which are present in previous works, and allows for a concise stability proof, as can be seen in Section 3.6.

Secondly, the predicted phase is defined without any reference to a Hilbert space structure. In fact, all the linear stability assumptions in Section 3.3 are formulated in a general Banach space, as are the (stochastic) nonlinear stability results in Section 3.4. To our knowledge, this is the first stochastic stability result in this setting, and the flexibility afforded by this approach allows us to treat noise of various levels of regularity in the example in Section 3.5.1. The Banach space setting is important for future applications, since L^p-theory with $p \neq 2$ has been proven to be effective in showing well-posedness of SPDEs. For parabolic equations, recent advances in this area have been made using maximal regularity techniques [7, 8, 9].

Our main stability results (Theorems 3.4.7 and 3.4.8) match the best current results obtained with other phase tracking methods [114, 161], and are expected to be optimal when stated in this generality. We pose our assumptions in a way which is mostly agnostic to the analytical properties of the PDE: we do not assume any smoothing or dispersion, and only require the nonlinearity to be locally Lipschitz. We allow for additive noise and multiplicative noise in either the Itô or Stratonovich sense. As a consequence, our results apply to parabolic equations, dispersive equations (see [228] and Chapter 2), and PDE-ODE systems such as the FitzHugh–Nagumo equation (see [74]). Being formulated without specific knowledge of the equation, our results are generally not optimal in terms of regularity when specifying to any of these settings. For example, we expect that in the parabolic setting, even lower regularity of the noise can be achieved by making use of the smoothing properties of the equation (see Remark 3.5.9).

The local Lipschitz conditions formulated in Assumptions 3.5 and 3.6 present an obstacle to showing stability with exceedingly rough noise (see Section 3.5.1), but we only formulate these assumptions to be able to deal with a very general class of equations. We expect that in many concrete situations, one can adapt our method to deal with the perturbative terms in a way which is tailored to the specific equation. For some examples of such treatments, we refer to [74, 228] and Chapter 2.

3.1.3. OUTLINE

The outline of this chapter is as follows. In Section 3.2, notation and preliminaries regarding stochastic integration are stated. Section 3.3 discusses linear stability, and motivates and introduces the main assumptions for our stability result. Afterwards, we explain in Section 3.4 how orbital stability arises from a nontrivial center space (see Definition 3.3.8). We also introduce the predicted phase (3.4.3), and formulate the main stability results (Theorems 3.4.4, 3.4.7, and 3.4.8) for deterministic and stochastic perturbations. In Section 3.5 we revisit two examples from the literature: a traveling pulse in the FitzHugh–Nagumo equation [10, 113], and a two-dimensional spiral wave in a reaction-diffusion equation [28, 150]. For the FitzHugh–Nagumo pulse, we extend the results of [74] to a wide range of noises with low regularity. In the example of the spiral wave, we compute the predicted phase explicitly, and demonstrate how the noncommutativity of the symmetry enters into the phase. Finally, Section 3.6 contains the proofs of the main stability results.

3.2. PRELIMINARIES

3.2.1. NOTATION

We use the convention that $\mathbb{N}$ does not include zero, and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout the chapter, we let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, equipped with a complete and right-continuous filtration $\{\mathscr{F}_t\}_{t\geq 0}$. When we speak of adaptedness or progressive measurability, it will be with respect to this filtration unless the contrary is explicitly stated. We write $\mathbb{E}$ for the expectation associated with $\mathbb{P}$. We write $\|\cdot\|_{\mathscr{X}}$ for the norm of a general real Banach space $\mathscr{X}$. The space of bounded linear operators between two Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ is denoted $\mathscr{L}(\mathscr{X};\mathscr{Y})$, and in the case $\mathscr{X} = \mathscr{Y}$ we write $\mathscr{L}(\mathscr{X}) := \mathscr{L}(\mathscr{X};\mathscr{X})$. For an unbounded operator A between $\mathscr{X}$ and $\mathscr{Y}$, we denote its domain by $\mathscr{D}(A)$ and the spectrum by $\sigma(A)$. We use the notation $\mathscr{X} \hookrightarrow \mathscr{Y}$ to mean that $\mathscr{X}$ embeds continuously into $\mathscr{Y}$.

A family $\{S(t,t')\}_{0\leq t'\leq t}$ consisting of bounded operators on a Banach space $\mathscr{X}$ forms a C_0-evolution family if $S(t,t) = \mathrm{I}_{\mathscr{X}}$ for all $t \geq 0$, $S(t,t')S(t',t'') = S(t,t'')$ for all $0 \leq t'' \leq t' \leq t$, and the map $(t,t') \mapsto S(t,t')$ is strongly continuous.

When M is a metric space, we write $C(M;\mathscr{X})$ (resp. $C_{\mathrm{ub}}(M;\mathscr{X})$) for the space of continuous (resp. bounded uniformly continuous) functions from M to $\mathscr{X}$. For a measure space $(S,\mathscr{G},\mu)$ and $p \in [1,\infty]$ we write $L^p(S;\mathscr{X})$ for the Lebesgue–Bochner space of strongly measurable $\mathscr{X}$-valued functions which are p-integrable (or essentially bounded if $p = \infty$). From now on, we will simply write measurable instead of strongly measurable. Note that if $\mathscr{X}$ is separable, the two notions are equivalent. We abbreviate $L^p_\Omega(\mathscr{X}) := L^p(\Omega;\mathscr{X})$ and $L^p(0,t;\mathscr{X}) := L^p([0,t];\mathscr{X})$, where the latter is equipped with the Lebesgue measure.

For $d,n \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $p \in [1,\infty]$, we denote by $W^{k,p}(\mathbb{R}^d;\mathbb{R}^n)$ the classical Sobolev space of measurable functions from $\mathbb{R}^d$ to $\mathbb{R}^n$ which have k weak derivatives which are p-integrable (or essentially bounded in the case $p = \infty$). When $p \in (1,\infty)$, $s \in [0,\infty)$, we write $H^{s,p}(\mathbb{R}^d;\mathbb{R}^n)$ for the Bessel space defined via the norm $\|(\mathrm{I}-\Delta)^{\frac{s}{2}} f\|_{L^p(\mathbb{R}^d;\mathbb{R}^n)}$, where $(\mathrm{I}-\Delta)^{\frac{s}{2}}$ is defined via the Fourier symbol $\xi \mapsto (1+|\xi|^2)^{\frac{s}{2}}$. In the case $p = 2$, we will write $H^s(\mathbb{R}^d;\mathbb{R}^n)$ instead of $H^{s,2}(\mathbb{R}^d;\mathbb{R}^n)$.

When $\mathscr{H}, \mathscr{H}'$, are Hilbert spaces, we write $\mathscr{L}_2(\mathscr{H};\mathscr{H}')$ for the subspace of $\mathscr{L}(\mathscr{H};\mathscr{H}')$ consisting of Hilbert–Schmidt operators. The space of γ-radonifying operators from $\mathscr{H}$ to $\mathscr{X}$, denoted $\gamma(\mathscr{H};\mathscr{X})$, is defined as the closure of the finite rank operators $T \in \mathscr{L}(\mathscr{H};\mathscr{X})$ with respect to the norm

$$\|T\|_{\gamma(\mathscr{H};\mathscr{X})} = \sup\Big(\tilde{\mathbb{E}}\Big[\Big\|\sum_{j=1}^n \gamma_j T h_j\Big\|_{\mathscr{X}}^2\Big]\Big)^{\frac{1}{2}},$$

where $(\gamma_j)_{j\geq 1}$ is a sequence of independent standard Gaussian random variables on some probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}})$, $\tilde{\mathbb{E}}$ denotes the expectation with respect to $\widetilde{\mathbb{P}}$, and the supremum is taken over all sets of orthonormal vectors in $\mathscr{H}$.

Finally, we write $[\cdot,\cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ for the Lie bracket of a Lie algebra $\mathfrak{g}$. For any $X \in \mathfrak{g}$ we write $\mathrm{ad}_X \in \mathscr{L}(\mathfrak{g})$ for the linear map defined by $Y \mapsto [X,Y]$, $Y \in \mathfrak{g}$. It is well-known that the map $X \mapsto \mathrm{ad}_X$ is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathscr{L}(V)$ (where the Lie bracket in $\mathscr{L}(V)$ is given by the commutator), and is commonly called the adjoint representation [137, Lemma 3.14] [122, Chapter 1.3] (note that sign conventions may differ).

3.2.2. STOCHASTIC INTEGRATION AND TAIL ESTIMATES

In this section, we let $\mathscr{H}$ be a separable Hilbert space, and let $W(t)$ be an $\mathscr{H}$-cylindrical Wiener process.

In Section 3.4.3 we will make use of the theory of stochastic integration in 2-smooth Banach spaces. For an introduction on this topic and further references, we refer the reader to [187]. The condition that $\mathscr{X}$ is 2-smooth is necessary to have a theory of stochastic integration which is satisfactory for our purposes. This condition is satisfied by many commonly used spaces, including L^p, $W^{k,p}$, and $H^{s,p}$ for $p \in [2,\infty)$. We note that 2-smoothness is generally not preserved under isomorphisms, so one must take care to use the 'right' norm for these spaces. However, the celebrated work of Pisier [196] shows that any space which has martingale type 2 admits an equivalent 2-smooth norm.

To show stability on long time scales, it will be necessary to have a subgaussian tail estimate for stochastic convolutions. Proposition 3.2.2 suffices for this purpose. The following lemma is adapted from [186, Lemma 4.3].

Lemma 3.2.1. *Let $K > 0$, and let X be a nonnegative random variable which satisfies*

$$\mathbb{E}[X^p] \le \sqrt{p}^{\,p} K^p, \quad p \in [2,\infty). \tag{3.2.1}$$

Then X satisfies the subgaussian tail estimate

$$\mathbb{P}[X > \lambda] \le e \exp\big(-(2e)^{-1}\lambda^2 K^{-2}\big), \quad \lambda \ge 0. \tag{3.2.2}$$

Proof. Let $\lambda \ge 0$, and set $q = \lambda^2 K^{-2} e^{-1}$. If $q \in [0,2]$, then

$$e \exp(-(2e)^{-1}\lambda^2 K^{-2}) = e \exp(-2^{-1} q) \ge 1,$$

so (3.2.2) is trivial. If $q \in [2,\infty)$, then we use Markov's inequality and (3.2.1) to find

$$\mathbb{P}[X > \lambda] \le \lambda^{-q}\mathbb{E}[X^q] \le \lambda^{-q}\sqrt{q}^{\,q} K^q = \exp(-(2e)^{-1}\lambda^2 K^{-2}),$$

so (3.2.2) is satisfied. $\square$

Proposition 3.2.2. *Let $\mathscr{X}$ be a 2-smooth Banach space, and let $\{S(t,t')\}_{0\le t'\le t}$ be a C_0-evolution family on $\mathscr{X}$. There exists a constant $c > 0$ such that the estimate*

$$\mathbb{P}\Big[\sup_{t\in[0,T]} \Big\| \int_0^t S(t,t') f(t')\,\mathrm{d}W(t') \Big\|_{\mathscr{X}} \ge \lambda\Big] \le e \exp\Big(\frac{-c\lambda^2}{T\|f\|^2_{L^\infty_\Omega(L^\infty(0,T;\gamma(\mathscr{H};\mathscr{X})))}}\Big) \tag{3.2.3}$$

holds for all $T > 0$, $\lambda \ge 0$, and all progressively measurable $f \in L^\infty_\Omega(L^\infty(0,T;\gamma(\mathscr{H};\mathscr{X})))$.

Proof. Fix $T > 0$ and $f \in L^\infty_\Omega(L^\infty(0,T;\gamma(\mathscr{H};\mathscr{X})))$. By [186, Theorem 4.5], there is a constant C (depending only on $\mathscr{X}, S$) such that the estimate

$$\Big\| \sup_{t\in[0,T]} \Big\| \int_0^t S(t,t') f(t')\,\mathrm{d}W(t') \Big\|_{\mathscr{X}} \Big\|_{L^p_\Omega} \le C\sqrt{p}\,T^{\frac12-\frac1p}\|f\|_{L^p_\Omega(L^p(0,T;\gamma(\mathscr{H};\mathscr{X})))}$$

holds for all $p \in [4,\infty)$. Using Hölder's inequality, we find

$$\Big\| \sup_{t\in[0,T]} \Big\| \int_0^t S(t,t') f(t')\,\mathrm{d}W(t') \Big\|_{\mathscr{X}} \Big\|_{L^p_\Omega} \le C\max\{2,\sqrt{p}\}\,T^{\frac12}\|f\|_{L^\infty_\Omega(L^\infty(0,T;\gamma(\mathscr{H};\mathscr{X})))}$$

for all $p \in [2,\infty)$. After using $2 \le \sqrt{2}\sqrt{p}$, the result follows from Lemma 3.2.1. $\square$

3.3. SYMMETRY AND LINEAR STABILITY

3.3.1. SYMMETRY

Let $\mathscr{X}$ be a Banach space, and consider the evolution equation

$$\mathrm{d}u = Au\,\mathrm{d}t + F(u)\,\mathrm{d}t, \tag{3.3.1}$$

where A is a closed (possibly unbounded) linear operator on $\mathscr{X}$ with domain $\mathscr{D}(A)$, and $F\colon \mathscr{X} \to \mathscr{X}$ is a nonlinear term. Many (semilinear) PDEs can be written in this form. A typical example is the case where A is a (possibly degenerate) second order elliptic operator with constant coefficients, and F is a Nemytskii mapping (i.e., a mapping of the form $[F(u)](x) = f(u(x))$ for some function f) on a function space with sufficient regularity.

Many interesting equations, especially physically motivated ones, are invariant under a group of continuous symmetries. For PDEs formulated on $\mathbb{R}^d$, a common symmetry is invariance under translations and rotations, and this is the principal example we have in mind. In this case, the symmetry group is the d-dimensional *special Euclidean group*, denoted $\mathrm{SE}(d)$. This group is generated by translations and rotations of $\mathbb{R}^d$, and is not commutative when $d \geq 2$. However, there are many other examples of continuous symmetries, such as scaling, dilation, or more complicated gauge transformations. This motivates the following assumption, which encodes such symmetries abstractly in the form of a Lie group. For the reader unfamiliar with this subject, we refer to [137]. However, we do not require any theory beyond the basic notions of a Lie group, Lie algebra, and their representations.

Recall that the linear operator A and the possibly nonlinear operator F originate from (3.3.1). Throughout this section, each of the upcoming Assumptions 3.1, 3.2, 3.3, and 3.4 will be in force from the moment it is introduced.

Assumption 3.1 (Symmetry of the equation)**.** *There exists a matrix Lie group* G *and a group homomorphism* $\Pi\colon \mathrm{G} \to \mathscr{L}(\mathscr{X})$ *with the following properties:*

- *For* $g \in \mathrm{G}$ *and* $\phi \in \mathscr{D}(A)$, *we have* $\Pi(g)\phi \in \mathscr{D}(A)$ *and*

$$A\phi = \Pi(g)A\Pi(g^{-1})\phi, \qquad F(\phi) = \Pi(g)F(\Pi(g^{-1})\phi). \tag{3.3.2}$$

- *For* $\phi \in \mathscr{X}$, *the map* $g \mapsto \Pi(g)\phi$ *is continuous from* G *to* $\mathscr{X}$.
- *There exists a constant* M *such that* $\|\Pi(g)\|_{\mathscr{L}(\mathscr{X})} \leq M$ *for all* $g \in \mathrm{G}$.

Remark 3.3.1. Assumption 3.1 will guarantee that $\Pi(g)u(t)$ solves (3.3.1) for any $g \in \mathrm{G}$ whenever $u(t)$ solves (3.3.1) and $u(t)$ is sufficiently regular.

Remark 3.3.2. By itself, Assumption 3.1 is trivial (as can be seen by taking G as the trivial group). However, the requirement that G is rich enough to capture all relevant symmetries of (3.3.1) will be enforced by later assumptions.

Although Assumption 3.1 is formulated in terms of the Lie group G, our following assumptions and results are formulated mostly in terms of its corresponding Lie algebra, which we denote by $\mathfrak{g}$. By a slight abuse of notation, we will write exp or e for the exponential map (which maps $\mathfrak{g}$ to G). We also fix an arbitrary norm on $\mathfrak{g}$, to be used throughout the rest of the chapter. It is not important which norm we use, as all norms on $\mathfrak{g}$ are equivalent since $\mathfrak{g}$ is finite-dimensional.

A Lie group representation Π typically gives rise to a Lie algebra representation π via differentiation at the identity:

$$\pi(Y) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Pi(\exp(tY)), \quad Y \in \mathfrak{g}. \tag{3.3.3}$$

However, if $\mathscr{X}$ is infinite dimensional, $\pi(Y)$ is generally an unbounded operator, and we must take care that the limit inherent in (3.3.3) exists. This motivates the following definition.

Definition 3.3.3. Let G, Π be as in Assumption 3.1. For $Y \in \mathfrak{g}$, we define the unbounded operator

$$\pi(Y) \colon \phi \mapsto \lim_{t\to 0} t^{-1}(\Pi(e^{tY})\phi - \phi), \tag{3.3.4}$$

where the limit is taken in the topology of $\mathscr{X}$. The domain of $\pi(Y)$ consists of exactly the elements $\phi \in \mathscr{X}$ for which this limit exists.

Remark 3.3.4. Assumption 3.1 directly implies that $t \mapsto \Pi(e^{tY})$ is a C_0-group on $\mathscr{X}$ for any $Y \in \mathfrak{g}$. As $\pi(Y)$ can be seen to be its generator, it follows that $\pi(Y)$ is closed and densely defined.

The main objects of study of this chapter are *symmetry solutions* to (3.3.1), by which we mean that the evolution of the solution is described purely by a symmetry of the equation. Typical examples which we will keep in mind throughout are traveling waves, rotating waves, and stationary solutions. It should be emphasized that existence of (nontrivial) symmetry solutions is generally a special property of a given equation. A few references where such solutions are constructed are [10, 14, 59, 84, 108, 130, 206]. Since our primary goal is to study nonlinear stability of stochastic perturbations of such solutions, we formulate their existence as an assumption.

Assumption 3.2 (Existence of a regular symmetry solution)**.** *There exist $X \in \mathfrak{g}$ and $u^* \in \mathscr{D}(A) \cap \mathscr{D}(\pi(X))$ such that*

$$\hat{u}(t) := \Pi(e^{tX})u^*, \quad t \geq 0, \tag{3.3.5}$$

is a (strong) solution to (3.3.1)*. Furthermore, we have $u^* \in \mathscr{D}(\pi(Y))$ for every $Y \in \mathfrak{g}$, and there exists a constant C such that we have the estimate*

$$\|\Pi(e^Y)u^* - u^* - \pi(Y)u^*\|_{\mathscr{X}} \leq C\|Y\|_{\mathfrak{g}}^2, \quad Y \in \mathfrak{g}. \tag{3.3.6}$$

Remark 3.3.5. Assumption 3.2 covers stationary solutions, as can be seen by taking $X = 0$.

As a concrete example, consider the case where $\mathscr{X} = L^2(\mathbb{R}^d;\mathbb{R})$, $\mathrm{G} = \mathrm{SE}(d)$, and X is the element in $\mathfrak{g}$ which generates translation by some vector $\vec{v} \in \mathbb{R}^d$. Then we have $\Pi(\exp(tX))u^*(x) = u^*(x - t\vec{v})$, so Assumption 3.2 is satisfied if the equation has a traveling wave solution with wave velocity $\vec{v}$ and (sufficiently smooth) wave profile u^*. It can also be seen that $\pi(X)$ is the directional derivative $-\partial_{\vec{v}}$.

3.3.2. DYNAMICS IN THE COMOVING FRAME

We now transfer to a coordinate frame which is comoving with $\hat{u}(t)$. Applying the transformation $\bar{u}(t) = \Pi(\exp(-tX))u(t)$, we get from (3.3.1) and Assumption 3.1:

$$d\bar{u} = [A\bar{u} - \pi(X)\bar{u}]\,dt + F(\bar{u})\,dt. \tag{3.3.7}$$

Assumption 3.2 then implies that $\bar{u}(t) \equiv u^*$ solves (3.3.7) (in the strong sense). In fact, the reverse implication also holds, so we could replace Assumption 3.2 by the assumption that (3.3.7) has a stationary solution u^*.

To study the dynamics of (3.3.7) near u^* (resp. (3.3.1) near $\hat{u}(t)$), we will look at the linearization of (3.3.7) around u^*. The next assumption is sufficient for this linearization to be meaningful.

Assumption 3.3 (Linearization)**.** *The nonlinearity $F\colon \mathscr{X} \to \mathscr{X}$ is Fréchet differentiable at u^*. Furthermore, the operator*

$$\mathscr{L}^*: \phi \mapsto A\phi - \pi(X)\phi + F'(u^*)\phi \tag{3.3.8}$$

generates a bounded C_0-semigroup $\{S^*(t)\}_{t\geq 0}$ on $\mathscr{X}$.

From now on, we will use the terms C_0-semigroup and semigroup interchangeably. We also emphasize that $\mathscr{L}^*$ is *not* an adjoint operator. Instead, the $*$ superscript indicates that $\mathscr{L}^*$ is associated with the comoving frame. The same holds for the objects $S^*(t)$, P_c^*, and P_s^*, which will be introduced later.

Remark 3.3.6. When boundedness of $F'(u^*)$ is known, it follows from [132, Chapter 9, Theorem 2.1] that $\mathscr{L}^*$ generates a C_0-semigroup on $\mathscr{X}$ whenever $A - \pi(X)$ does. Since A and $\pi(X)$ commute by Assumption 3.1, it follows using the Trotter–Kato theorem [219] that $\mathscr{L}^*$ generates a C_0-semigroup on $\mathscr{X}$ whenever A does.

Remark 3.3.7. From Assumptions 3.1 and 3.3, it follows that Assumption 3.3 also holds when u^* is replaced by $\Pi(g)u^*$ for any $g \in \mathrm{G}$. This also applies to the coming Assumptions 3.4 and 3.5, where the relevant constants can even be chosen uniformly in $g \in \mathrm{G}$.

Naively, we might hope for the semigroup $S^*(t)$ to be exponentially stable, meaning that there exist constants $M, a > 0$ such that $\|S(t)\|_{\mathscr{L}(\mathscr{X})} \leq Me^{-at}$ for all $t \geq 0$. By standard perturbative methods, this would imply that a solution which starts sufficiently close to u^* will eventually converge to u^*. However, it turns out that the presence of symmetries (in particular, nontriviality of the *center space*) can pose a significant obstacle.

Definition 3.3.8. The *center space of u^* with respect to* $\mathfrak{g}$, denoted by $\mathscr{V}$, is defined as

$$\mathscr{V} := \{\pi(Y)u^* : Y \in \mathfrak{g}\}.$$

We emphasize that the center space is determined *both* by the profile u^* and the symmetry group G. However, since we generally consider u^* and G to be fixed, we will simply speak of *the* center space. Note that $\pi(Y)u^*$ is well-defined by Assumption 3.2, and the dimension of $\mathscr{V}$ is at most that of $\mathfrak{g}$. In particular, the center space is finite-dimensional, and thus closed in $\mathscr{X}$.

It is important to note that the dimension of the center space can be strictly smaller than that of $\mathfrak{g}$. The center space can in fact be trivial, even if $\mathfrak{g}$ is very rich. In the case of PDEs on $\mathbb{R}^d$ with Euclidean symmetries, this occurs when u^* is constant throughout space.

A guiding principle is that a nontrivial center space arises not due to symmetry directly, but due to a mismatch in symmetry between the profile u^* and the equation. The more symmetries present in the equation which are not shared by the profile u^*, the richer the center space. Remarkably, the dynamics of $S^*(t)$ on the center space are entirely determined by the algebraic structure of $\mathfrak{g}$, as we will now demonstrate.

Proposition 3.3.9. *For any $Y \in \mathfrak{g}$ we have*

$$\mathscr{L}^* \pi(Y) u^* = \pi([Y,X]) u^*. \tag{3.3.9}$$

Consequently, $\mathscr{L}^*$ can be restricted to a bounded operator on $\mathscr{V}$.

Proof. Let $Y \in \mathfrak{g}$. By Assumption 3.1 and the fact that u^* is a stationary solution to (3.3.7), we have for $t \in \mathbb{R}$

$$A\Pi(\exp(tY))u^* + F(\Pi(\exp(tY))u^*) - \Pi(\exp(tY))\pi(X)u^* = 0,$$

which we rewrite as

$$\begin{aligned} A\Pi(\exp(tY))u^* + F(\Pi(\exp(tY))u^*) - \pi(X)\Pi(\exp(tY))u^* \\ = \Pi(\exp(tY))\pi(X)u^* - \pi(X)\Pi(\exp(tY))u^*. \end{aligned} \tag{3.3.10}$$

We now claim that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big(\Pi(\exp(tY))\pi(X)u^* - \pi(X)\Pi(\exp(tY))u^*\big) = \pi([Y,X])u^*.$$

If $\pi(X)u^* \in \mathscr{D}(\pi(Y))$ and $\pi(Y)u^* \in \mathscr{D}(\pi(X))$, this follows directly from Definition 3.3.3. If not, we approximate and use the fact that $\pi(X)$ and $\pi(Y)$ are closed and densely defined. Thus, the derivative at $t = 0$ of the right-hand side of (3.3.10) is well-defined and equal to $\pi(Y)\pi(X)u^* - \pi(X)\pi(Y)u^*$. Furthermore, since G is a matrix Lie group (or alternatively using Ado's theorem [122, Chapter VI]) we have the identity $\pi(Y)\pi(X)u^* - \pi(X)\pi(Y)u^* = \pi([Y,X])u^*$. Since $\pi(X)$ and A are closed and F is differentiable at u^*, we may differentiate the left-hand side of (3.3.10) at $t = 0$ to get (3.3.9). Thus, $\mathscr{L}^*$ maps $\mathscr{V}$ into $\mathscr{V}$, and the boundedness follows since $\mathscr{V}$ is finite-dimensional. □

The relation (3.3.9) prompts us to define the (bounded) linear map

$$\begin{aligned} L\colon \mathfrak{g} &\to \mathfrak{g}, \\ Y &\mapsto [Y,X], \end{aligned} \tag{3.3.11}$$

so that (3.3.9) can be formulated as

$$\mathscr{L}^* \pi(Y) u^* = \pi(LY) u^*. \tag{3.3.12}$$

Since the center space is finite-dimensional, we can define e^{tL} and $e^{t\mathscr{L}^*|_{\mathscr{V}}}$ via the usual power series, in which case $e^{t\mathscr{L}^*|_{\mathscr{V}}}$ coincides with $S^*(t)|_{\mathscr{V}}$. We now show that the relation (3.3.12) lifts to a relation between $S^*(t)$ and e^{tL}. This implies that the dynamics of $S^*(t)$ on $\mathscr{V}$ are encoded entirely in the Lie bracket.

Proposition 3.3.10. *For $t \geq 0$ and $Y \in \mathfrak{g}$ we have*

$$S^*(t)\pi(Y)u^* = \pi(e^{tL}Y)u^* = \pi(e^{-tX}Ye^{tX})u^*. \tag{3.3.13}$$

Proof. Since the power series of $e^{t\mathscr{L}^*|_{\mathscr{V}}}$ and e^{tL} both converge in the uniform topology, the first identity in (3.3.13) follows by iterating (3.3.12):

$$S^*(t)\pi(Y)u^* = \sum_{n=0}^{\infty} \frac{(t\mathscr{L}^*)^n}{n!}\pi(Y)u^* = \pi\Big(\sum_{n=0}^{\infty} \frac{(tL)^n}{n!} Y\Big)u^* = \pi(e^{tL}Y)u^*.$$

We now observe that $L = -\operatorname{ad}_X$, where ad is as in Section 3.2.1. Thus, the second identity in (3.3.13) follows from the classical identity (see [137, Lemma 3.14])

$$e^{-\operatorname{ad}_X}Y = e^{-X}Ye^X. \qquad \square$$

As a final remark, we note that (3.3.13) immediately implies

$$S^*(t)\pi(X)u^* = \pi(X)u^*, \quad t \geq 0, \tag{3.3.14}$$

from which we see that $\pi(X)u^*$ is invariant under the dynamics of $S^*(t)$.

3.3.3. LINEAR STABILITY IN THE COMOVING FRAME

From Proposition 3.3.10, it should now be clear (especially considering (3.3.14)) that we cannot expect exponential stability of $S^*(t)$ in general, unless the center space is trivial. However, in many cases a stability estimate can be recovered after 'projecting out' the center space. Essentially, this gives stability 'modulo symmetry', and this is the reason why the concept of *orbital stability* is needed. Thus, our goal is now to find a space $\mathscr{W}$ which is complementary to $\mathscr{V}$ (in the sense that $\mathscr{X}$ is the direct sum of $\mathscr{V}$ and $\mathscr{W}$) such that $S^*(t)$ leaves $\mathscr{W}$ invariant and is exponentially stable on $\mathscr{W}$. We then call $\mathscr{W}$ the *stable space*. If such a stable space exists, we can decompose the dynamics near u^* into two parts: the dynamics on $\mathscr{V}$ (which are purely determined by $\mathfrak{g}$), and the dynamics on $\mathscr{W}$ (which are exponentially stable). The next assumption guarantees that we have this decomposition.

Assumption 3.4 (Decomposition)**.** *There exist projections $P_c^*, P_s^* \in \mathscr{L}(\mathscr{X})$ with the following properties:*

- *We have the decomposition $\mathrm{I} = P_c^* + P_s^*$.*
- *The range of P_c^* coincides with the center space $\mathscr{V}$.*
- *The range of P_s^* is stable under $S^*(t)$: there exist constants $M, a > 0$ such that*

$$\|S^*(t)P_s^*\|_{\mathscr{L}(\mathscr{X})} \leq Me^{-at}, \quad t \geq 0. \tag{3.3.15}$$

The stable space $\mathscr{W}$ is given by the range of P_s^*. The subscripts in P_c^* and P_s^* stand for *center* and *stable*, motivated by the fact that they project onto the center space and the stable space, respectively. By Definition 3.3.8 and finite-dimensionality of $\mathfrak{g}$, it follows that P_c^* factorizes through a bounded linear map $\mathscr{P}\colon \mathscr{X} \to \mathfrak{g}$ as follows:

$$P_c^*\phi = \pi(\mathscr{P}\phi)u^*, \quad \phi \in \mathscr{X}. \tag{3.3.16}$$

Typically, Assumption 3.4 is verified using spectral methods and PDE techniques. If $\mathscr{X}$ is a Hilbert space, a sufficient condition on the spectrum of $\mathscr{L}^*$ may be formulated using the Gearhart–Prüss theorem [129, Theorem 4.1.5]. In this case, it suffices that the spectrum is of the form

$$\sigma(\mathscr{L}_{\mathbb{C}}^*) \subset \sigma(L_{\mathbb{C}}) \cup \{z \in \mathbb{C} : \mathrm{Re}\, z \leq -b\} \tag{3.3.17}$$

for some $b > 0$, and that the eigenspaces corresponding to $\sigma(L_{\mathbb{C}})$ are spanned by $\mathscr{V}_{\mathbb{C}}$. The assumptions on the spectrum are then verified by analyzing the linearized PDE directly.

Remark 3.3.11. The subscript $\mathbb{C}$ in (3.3.17) indicates that the condition is formulated for the complexification of the relevant spaces and operators. This is necessary, since conventional definitions of the spectrum and resolvent assume that the underlying space is complex. For a more detailed explanation on complexification (which is analogous to viewing a real matrix as a linear operator on $\mathbb{C}^n$ in the natural way) see [119, Appendix B.4].

We note that versions of Assumption 3.4 and the spectral condition (3.3.17) are common in the literature on stability of deterministic and stochastic patterns. For a (nonexhaustive) list of examples where this is explicitly assumed or proven, one can consider [113, Lemma 3.1], [121, Assumption 3.1], [150, Assumption 3.5], [161, Assumption 2.6], [74, Proposition 2.8], [10]. Moreover, Assumption 3.4 can be obtained as a corollary of many of the other cited linear stability results.

We also emphasize that (3.3.17) allows $\mathscr{L}_{\mathbb{C}}^*$ to have (point) spectrum which is located on the imaginary axis and not at the origin. This actually occurs for the rotating wave, where we have $\sigma(L_{\mathbb{C}}) = \{0, \pm i\omega\}$. One can compare (3.3.17) with [161, Assumption 2.6], which explicitly forbids this situation and only treats commutative symmetry. In the commutative case, $L \equiv 0$ and all eigenvalues on the imaginary axis must be at the origin.

3.3.4. RETURN TO THE STATIONARY FRAME

We ultimately want to solve a stochastic version of (3.3.1) in the stationary frame. To do this, we first need to formulate a solution concept. Motivated by Assumption 3.3, we first rewrite (3.3.7) as

$$\begin{aligned} \mathrm{d}\bar{u} &= [A - \pi(X) + F'(u^*)]\,\bar{u}\,\mathrm{d}t + [F(\bar{u}) - F'(u^*)\bar{u}]\,\mathrm{d}t \\ &\overset{(3.3.8)}{=} \mathscr{L}^*\,\bar{u}\,\mathrm{d}t + [F(\bar{u}) - F'(u^*)\bar{u}]\,\mathrm{d}t. \end{aligned}$$

By Duhamel's principle (variation of parameters), a solution to (3.3.7) with initial value $u_0 \in \mathscr{X}$ should then satisfy

$$\bar{u}(t) = S^*(t)u_0 + \int_0^t S^*(t-t')\big(F(\bar{u}(t')) - F'(u^*)\bar{u}(t')\big)\,\mathrm{d}t'.$$

Undoing the transformation $\bar{u}(t) = \Pi(e^{-tX})u(t)$ and using the symmetries of F from Assumption 3.1, it follows that a solution to (3.3.1) with initial value u_0 should satisfy the *mild solution formula*

$$u(t) = \Pi(e^{tX})S^*(t)u_0 + \int_0^t \Pi(e^{tX})S^*(t-t')\Pi(e^{-t'X})\big(F(u(t')) - F'(\hat{u}(t'))u(t')\big)\,\mathrm{d}t'.$$

We now define for $0 \le t' \le t$ the following bounded linear operators ($P_c(t)$ and $P_s(t)$ are intended for later purposes):

$$S(t,t') := \Pi(e^{tX}) S^*(t-t') \Pi(e^{-t'X}), \tag{3.3.18a}$$
$$P_c(t) := \Pi(e^{tX}) P_c^* \Pi(e^{-tX}), \tag{3.3.18b}$$
$$P_s(t) := \Pi(e^{tX}) P_s^* \Pi(e^{-tX}), \tag{3.3.18c}$$

so that the solution formula simplifies to its final version

$$u(t) = S(t,0)u_0 + \int_0^t S(t,t')\big(F(u(t')) - F'(\hat{u}(t'))u(t')\big)\,\mathrm{d}t'. \tag{3.3.19}$$

The following proposition translates the implications of Assumptions 3.1-3.4 back to the stationary frame.

Proposition 3.3.12. *The following statements hold:*

- $\{S(t,t')\}_{0\le t'\le t}$ *is a* C_0*-evolution family on* $\mathscr{X}$.
- *For* $t \ge 0$*, the bounded operators* $P_c(t)$ *and* $P_s(t)$ *are projections, and we have the decomposition* $\mathrm{I} = P_c(t) + P_s(t)$.
- *For* $0 \le t' \le t$ *and* $\phi \in \mathscr{X}$ *we have the identities*

$$S(t,t')P_c(t')\phi = \Pi(e^{tX})\pi(e^{(t-t')L}\mathscr{P}\Pi(e^{-t'X})\phi)u^*, \tag{3.3.20a}$$
$$S(t,t')\pi(Y)\hat{u}(t') = \Pi(e^{tX})\pi(e^{tL}Y)u^*. \tag{3.3.20b}$$

- *There exist constants* $M_1, M_2, M_3, a > 0$ *such that we have the estimates*

$$\|\Pi(g)\|_{\mathscr{L}(\mathscr{X})} \le M_1, \tag{3.3.21a}$$
$$\|S(t,t')\|_{\mathscr{L}(\mathscr{X})} \le M_2, \tag{3.3.21b}$$
$$\|S(t,t')P_s(t')\|_{\mathscr{L}(\mathscr{X})} \le M_3 e^{-a(t-t')}, \tag{3.3.21c}$$

for all $0 \le t' \le t$ *and* $g \in \mathrm{G}$.

Although (3.3.21a) was already stated in Assumption 3.1, we include it again here to have all the relevant constants in one place for later use.

Proof. Most statements follow straightforwardly from (3.3.18) and the corresponding properties of $S^*(t)$, P_c^*, and P_s^*, so we only prove (3.3.20). From Assumption 3.4 and Proposition 3.3.10 we see that

$$\begin{aligned}
S(t,t')P_c(t')\phi &\overset{(3.3.18)}{=} \Pi(e^{tX})S^*(t-t')P_c^*\Pi(e^{-t'X})\phi \\
&\overset{(3.3.16)}{=} \Pi(e^{tX})S^*(t-t')\pi(\mathscr{P}\Pi(e^{-t'X})\phi)u^* \\
&\overset{(3.3.13)}{=} \Pi(e^{tX})\pi(e^{(t-t')L}\mathscr{P}\Pi(e^{-t'X})\phi)u^*,
\end{aligned}$$

as well as

$$\begin{aligned}
S(t,t')\pi(Y)\hat{u}(t') &\overset{(3.3.5),(3.3.18a)}{=} \Pi(e^{tX})S^*(t-t')\Pi(e^{-t'X})\pi(Y)\Pi(e^{t'X})u^* \\
&\overset{(3.3.13)}{=} \Pi(e^{tX})S^*(t-t')\pi(e^{t'L}Y)u^* \\
&\overset{(3.3.13)}{=} \Pi(e^{tX})\pi(e^{(t-t')L}e^{t'L}Y)u^*,
\end{aligned}$$

which implies (3.3.20b). □

3.4. NONLINEAR STABILITY

Throughout this section, Assumptions 3.1, 3.2, 3.3 and 3.4 will be in force. Additionally, the upcoming Assumptions 3.5 and 3.6 will be in force from the moment they are introduced.

3.4.1. ORBITAL STABILITY AND THE PREDICTED PHASE

We now study the stability of $\hat{u}(t)$ in the full nonlinear equation (3.3.1). As the discussion in the previous section suggests, we generally cannot expect solutions starting close to $\hat{u}(0) = u^*$ to converge to $\hat{u}(t) = \Pi(e^{tX})u^*$. Instead, we will find that a solution which starts close to u^* will, after some positive time t, be close to $\Pi(\gamma)u^*$ for a $\gamma \in \mathrm{G}$ which is different from e^{tX}. Hence, it is the *center manifold* $\mathscr{C}$, given by the group orbit $\mathscr{C} := \{\Pi(g)u^* : g \in \mathrm{G}\}$ which is stable, instead of the solution $\hat{u}(t)$. This is precisely what we mean by *orbital stability*. We aim to answer the following questions:

1. How do we prove stability of the center manifold?
2. Which symmetries should be included in G for the center manifold to be stable?
3. How can we compute the *phase* $\gamma \in \mathrm{G}$ from the initial perturbation $u_0 - u^*$?

These questions are strongly interlinked. To prove stability, we will need to know the right symmetry group and the correct phase in advance. Conversely, a stability proof ensures that the phase shift and the symmetry group used are the correct ones.

To answer the second and third question, we look to the linear theory developed in the previous section. We consider a mild solution $u(t)$ to (3.3.1) with initial condition $u(0) = u^* + v_0$, where $v_0 = \mathscr{O}(\varepsilon)$ with $\varepsilon \ll 1$. Since the evolution family $S(t,t')$ arises from linearization around $\hat{u}(t)$, we can heuristically expect that

$$u(t) = \hat{u}(t) + S(t,0)v_0 + \mathscr{O}(\varepsilon^2). \tag{3.4.1}$$

Using the decomposition $\mathrm{I} = P_c(0) + P_s(0)$ from Proposition 3.3.12 together with (3.3.20a), we find

$$u(t) = \Pi(e^{tX})u^* + \Pi(e^{tX})\pi(e^{tL}\mathscr{P}v_0)u^* + S(t,0)P_s(0)v_0 + \mathscr{O}(\varepsilon^2).$$

In the first two terms on the right-hand side, we recognize an abstract Taylor expansion of $\Pi(e^{tX}\exp(e^{tL}\mathscr{P}v_0))u^*$ (c.f. (3.3.6)). The third term is exponentially decaying with time by (3.3.21c). Hence, we have

$$u(t) = \Pi\big(\exp(tX)\exp(e^{tL}\mathscr{P}v_0)\big)u^* + \mathscr{O}(\varepsilon e^{-at} + \varepsilon^2), \tag{3.4.2}$$

which shows that the difference between $u(t)$ and $\Pi\big(\exp(tX)\exp(e^{tL}\mathscr{P}v_0)\big)u^*$ is expected to be smaller than the difference between $u(0)$ and u^*, when t is large enough and ε is sufficiently small. Since the exponential decay from (3.3.21c) originates from Assumption 3.4, it is now clear that we must include in G enough symmetries for this assumption to hold. Recalling the discussion in Section 3.3, the orbital degrees of freedom thus originate from exactly those symmetries which are present in the equation (3.3.1) but not in the profile u^*.

Motivated by (3.4.2) we now introduce the *predicted phase at time t*, which we denote $\gamma_t\colon \mathscr{X} \to \mathrm{G}$, as

$$\gamma_t\colon v_0 \mapsto \exp(tX)\exp(e^{tL}\mathscr{P}v_0). \tag{3.4.3}$$

This is the notion of phase which we will employ to show orbital stability. In (3.4.3), v_0 should be interpreted as the deviation from the profile u^* at time $t=0$. When v_0 is clear from the context, we will simply write γ_t instead of $\gamma_t(v_0)$ to unburden the notation.

Remark 3.4.1. If $v_0 = 0$, then $\gamma_t = \exp(tX)$. This corresponds to the fact that $\hat{u}(t) = \Pi(e^{tX})u^*$ is an exact solution of (3.3.1). Hence, the predicted phase matches the exact phase in the absence of a perturbation. Moreover, this suggests that we can interpret $\exp(e^{tL}\mathscr{P}v_0)$ in (3.4.3) as the *phase correction* relative to the unperturbed evolution.

Remark 3.4.2. It is seen from (3.3.11) and (3.4.3) that any noncommutativity of $\mathfrak{g}$ (encoded in L), enters the predicted phase in a nontrivial way.

A compelling feature of the predicted phase is that it is determined directly from the initial perturbation v_0. This simplifies the stability proof, since it removes the need to couple an auxiliary SDE or ODE to the SPDE as is done in e.g. [74, 112, 121, 148, 161].

Secondly, the right-hand side of (3.4.3) can often be computed explicitly. In most concrete situations, the symmetry $\exp(tX)$ is explicitly known, so that the matrices L and e^{tL} can be calculated by hand. Furthermore, $\mathscr{P}$ can often be described in terms of the eigenfunctions of the formal adjoint of $\mathscr{L}^*$, see e.g. [129, Exercise 4.1.4]. For explicit computations of (3.4.3) in concrete situations, we refer ahead to Remark 3.5.8 and Section 3.5.2.

In [1, 3, 4], a phase description is obtained using the asymptotic behavior of the flow of (3.3.1) (see also [133] for a similar approach in an SDE setting). While this description is highly convenient for describing long-time dynamics of the phase (as will be seen in Chapter 4), it is less so for the purpose of showing stability, since the exact asymptotic behavior of a nonlinear PDE is difficult to characterize explicitly. In contrast, (3.4.3) only uses the asymptotic behavior of the linearization around u^*, and thus allows for a more explicit description.

Finally we note that the predicted phase, as well as all of our assumptions so far, is formulated without any reference to a Hilbertian structure of $\mathscr{X}$. In fact, up to this point, $\mathscr{X}$ can be an arbitrary Banach space. We believe this flexibility to be advantageous, especially if one wants to treat measure-valued equations, or (S)PDEs which are well-posed in an L^p-setting with $p \neq 2$ (see e.g. [7, 8, 9]). Note that although the phase description in [1, 4] is formulated in a Banach space as well, a stability proof is not provided.

3.4.2. STABILITY IN THE DETERMINISTIC SETTING

Our next step is to use the predicted phase to make (3.4.2) rigorous. This will answer the primary question on how to show stability in the deterministic setting, and also demonstrates that the phase shift predicted by (3.4.3) is indeed accurate.

We begin by formulating a lemma which shows how (3.4.2) follows from (3.4.1), and which will be used for the coming orbital stability proofs. The way in which we have written Lemma 3.4.3 is suggestive of how we will apply it.

Lemma 3.4.3. *Let $T > 0$, $v_0 \in \mathscr{X}$ and $f \in C([0,T];\mathscr{X})$. Let γ_t be given by (3.4.3), and set*

$$u(t) := \hat{u}(t) + S(t,0)v_0 + f(t), \quad t \in [0,T].$$

Then we have the estimate

$$\|u(t) - \Pi(\gamma_t)u^*\|_{\mathscr{X}} \le M_3 e^{-at}\|v_0\|_{\mathscr{X}} + CM_1K_T^2\|v_0\|_{\mathscr{X}}^2 + \|f(t)\|_{\mathscr{X}}, \quad t \in [0,T],$$

where C is the constant from (3.3.6), M_1, M_3, a are the constants from (3.3.21), and $K_T = \sup_{t\in[0,T]}\|e^{tL}\mathscr{P}\|_{\mathscr{L}(\mathscr{X};\mathfrak{g})}$.

Proof. For brevity, we write $Y_t = e^{tL}\mathscr{P}v_0$. By the decomposition $\mathrm{I} = P_c(0) + P_s(0)$ and (3.3.20a), we have

$$\begin{aligned} S(t,0)v_0 &= S(t,0)P_c(0)v_0 + S(t,0)P_s(0)v_0 \\ &= \Pi(e^{tX})\pi(Y_t)u^* + S(t,0)P_s(0)v_0 \end{aligned}$$

for $t \in [0,T]$. Hence, by the triangle inequality, we get

$$\begin{aligned} \|u(t) - \Pi(\gamma_t)u^*\|_{\mathscr{X}} &\le \|S(t,0)P_s(0)v_0 + f(t)\|_{\mathscr{X}} \\ &\quad + \|\Pi(\gamma_t)u^* - \hat{u}(t) - \Pi(e^{tX})\pi(Y_t)u^*\|_{\mathscr{X}}, \end{aligned}$$

so it suffices to estimate these two terms. For the first term, we observe using (3.3.21c):

$$\|S(t,0)P_s(0)v_0 + f(t)\|_{\mathscr{X}} \le M_3e^{-at}\|v_0\|_{\mathscr{X}} + \|f(t)\|_{\mathscr{X}}.$$

For the second term, we use (3.4.3) to see $\gamma_t = e^{tX}e^{Y_t}$, which together with (3.3.6) and (3.3.21a) gives:

$$\begin{aligned} \|\Pi(\gamma_t)u^* - \hat{u}(t) - \Pi(e^{tX})\pi(Y_t)u^*\|_{\mathscr{X}} &\le M_1\|\Pi(e^{Y_t})u^* - u^* - \pi(Y_t)u^*\|_{\mathscr{X}} \\ &\le CM_1\|Y_t\|_{\mathfrak{g}}^2 \le CM_1\|e^{tL}\mathscr{P}\|_{\mathscr{L}(\mathscr{X};\mathfrak{g})}^2\|v_0\|_{\mathscr{X}}^2. \end{aligned}$$ □

Before we state the nonlinear stability result, we need to assume some regularity on the nonlinearity F. Recall that we have previously assumed that F is Fréchet differentiable at u^* (see Assumption 3.3).

Assumption 3.5 (Regularity of F near u^*)**.** *For every $R > 0$, there exists a constant C such that the estimate*

$$\|F(v) - F(u^*) - F'(u^*)[v-u^*]\|_{\mathscr{X}} \le C\|v-u^*\|_{\mathscr{X}}^2 \tag{3.4.4}$$

holds for all $v \in \mathscr{X}$ satisfying $\|v-u^*\|_{\mathscr{X}} \le R$.

We can now state the deterministic orbital stability result. Theorem 3.4.4 confirms the heuristic discussion in Section 3.4.1, and rigorously shows that (3.4.3) gives an accurate prediction of the phase. The proof of Theorem 3.4.4 is contained in Section 3.6.1.

Theorem 3.4.4 (Orbital stability)**.** *Let $u(t)$ be a solution to* (3.3.19) *with initial condition $u_0 = u^* + v_0$, and let γ_t be given by* (3.4.3)*. For every $T > 0$, $\delta > 0$, there exists a constant $\varepsilon > 0$ such that we have the estimate*

$$\|u(t) - \Pi(\gamma_t)u^*\|_{\mathscr{X}} \leq (M_3 e^{-at} + \delta)\|v_0\|_{\mathscr{X}}, \quad t \in [0, T], \tag{3.4.5}$$

whenever $\|v_0\|_{\mathscr{X}} \leq \varepsilon$ (M_3 and a are the constants from (3.3.21c)*).*

Interestingly, the estimate (3.4.5) shows that the solution gets closer to the center manifold only for $t \geq a^{-1}\log(M_3)$. If $S(t, t')P_s(t')$ is not immediately contractive, the solution might move away from the center manifold between $t = 0$ and $t = a^{-1}\log(M_3)$.

Using the symmetries of (3.3.1) encoded in Assumption 3.1, it is a corollary of Theorem 3.4.4 that the center manifold is exponentially attracting for the deterministic equation. We elaborate on this point in the next section (see Theorem 3.4.8), where we treat stochastic perturbations.

3.4.3. STOCHASTIC PERTURBATIONS

In this section, we use the predicted phase (3.4.3) to show stability of stochastic perturbations of (3.3.1) on long time scales. We believe the conciseness of the proof demonstrates the strength and elegance of the phase prediction function.

For $\sigma > 0$, we consider the following SPDE:

$$\begin{aligned} \mathrm{d}u(t) &= [Au(t) + F(u(t)) + \sigma^2 H(t, u(t))]\,\mathrm{d}t + \sigma G(t, u(t))\,\mathrm{d}W(t), \\ u(0) &= u^* + v_0, \end{aligned} \tag{3.4.6}$$

where A, F are as before, $\mathscr{H}$ is a separable Hilbert space, $W(t)$ is an $\mathscr{H}$-cylindrical Wiener process, H takes values in $\mathscr{X}$, and v_0 is an $\mathscr{F}_0$-measurable, $\mathscr{X}$-valued random variable. Since we are working in an abstract Banach space, we let G take values in the space of γ-radonifying operators from $\mathscr{H}$ to $\mathscr{X}$, denoted $\gamma(\mathscr{H};\mathscr{X})$ (see Section 3.2.1). For more details and abstract results about such spaces, we refer to [119, Chapter 9]. In Section 3.5.1, we show in a concrete example what kind of noise is allowed under this condition. We allow G and H to depend on ω, but generally suppress this dependence in the notation. As discussed in Section 3.2.2, we assume from now on that $\mathscr{X}$ is 2-smooth.

The term $\sigma G\,\mathrm{d}W$ models noise present in the system, the amplitude of which is controlled by σ. We interpret (3.4.6) in the Itô sense, and include the drift term $\sigma^2 H\,\mathrm{d}t$ in (3.4.6) to account for a possible Stratonovich correction. However, we do not impose any relation between G and H. Hence, this setup allows for arbitrary deterministic perturbations which are of second order in σ.

We require some regularity assumptions on G and H for (3.4.6) to be well-posed. Since the aim of this chapter is not to treat well-posedness of stochastic PDEs, we make do with the following relatively simple local Lipschitz assumptions. For any concrete equation, more appropriate function spaces and nonlinear estimates can be formulated (see e.g. Chapter 2).

Assumption 3.6 (Regularity of G and H)**.** *For $f \in \mathscr{X}$, the processes $(\omega, t) \mapsto G(\omega, t, f)$ and $(\omega, t) \mapsto H(\omega, t, f)$ are progressively measurable. For every $R > 0$, there exist a constant C such that the estimates*

$$\|H(t, f)\|_{\mathscr{X}} \leq C, \tag{3.4.7a}$$

$$\|G(t, f)\|_{\gamma(\mathscr{H};\mathscr{X})} \leq C, \tag{3.4.7b}$$

$$\|H(t, f) - H(t, g)\|_{\mathscr{X}} \leq C\|f - g\|_{\mathscr{X}}, \tag{3.4.7c}$$

$$\|G(t, f) - G(t, g)\|_{\gamma(\mathscr{H};\mathscr{X})} \leq C\|f - g\|_{\mathscr{X}}. \tag{3.4.7d}$$

are valid for all $\omega \in \Omega$, $t \in [0, \infty)$ and $f, g \in \mathscr{X}$ which satisfy $\max\{\|f\|_{\mathscr{X}}, \|g\|_{\mathscr{X}}\} \leq R$.

It is well-known that stochastic evolution equations with globally Lipschitz coefficients and linear growth have unique local solutions. The following basic well-posedness result can be obtained from [66, Theorem 7.2] by a straightforward localization argument. Although [66] only treats the Hilbert space setting, the proof in 2-smooth Banach spaces is entirely analogous, and follows by substituting an appropriate maximal estimate for stochastic convolutions such as [186, Theorem 4.5]. We refer the reader to [47] and [139] for similar proofs in the 2-smooth setting.

Theorem 3.4.5 (Local well-posedness)**.** *There exist unique (up to indistinguishability) random variables $(\tau^*, \{u(t)\}_{t\in[0,\tau^*)})$ such that:*

- *τ^* is a stopping time.*
- *u is adapted and continuous on $[0, \tau^*)$, with values in $\mathscr{X}$.*
- *For all $t \in [0, \tau^*)$, $u(t)$ solves the following mild formulation to (3.4.6):*

$$\begin{aligned} u(t) &= S(t,0)(u^* + v_0) + \int_0^t S(t,t')\sigma G(t', u(t'))\,\mathrm{d}W(t') \\ &\quad + \int_0^t S(t,t')\big(F(u(t')) - F'(\hat{u}(t'))u(t') + \sigma^2 H(t', u(t'))\big)\,\mathrm{d}t'. \end{aligned} \tag{3.4.8}$$

- *If $\tau^* < \infty$, then $\lim_{t\nearrow\tau^*} \|u(t)\|_{\mathscr{X}} = \infty$.*

From now on, we will write u and τ^* for the solution and stopping time obtained from Theorem 3.4.5. Since the solution to (3.4.6) with $\sigma = 0$ and $v_0 \equiv 0$ is given by $u(t) \equiv \hat{u}(t)$, we propose the following asymptotic expansion for u:

$$u(t) = \hat{u}(t) + v(t) + z(t),$$

where v is $\mathscr{O}(\sigma)$ and z is $\mathscr{O}(\sigma^2)$. Substituting this ansatz into (3.4.6) and grouping the terms based on their order in σ, we get

$$\begin{aligned} \mathrm{d}u(t) = &\big(A\hat{u}(t) + F(\hat{u}(t))\big)\,\mathrm{d}t \\ &+ \big([Av(t) + F'(\hat{u}(t))v(t)]\,\mathrm{d}t + \sigma G(t, \hat{u}(t))\,\mathrm{d}W(t)\big) \\ &\quad + \big(Az(t) + F'(\hat{u}(t))z(t)\big)\,\mathrm{d}t \\ &\quad + \big(F(u(t)) - F(\hat{u}(t)) - F'(\hat{u})[u(t) - \hat{u}(t)]\big)\,\mathrm{d}t \\ &\quad + \sigma^2 H(t, u(t))\,\mathrm{d}t \\ &\quad + \sigma\big(G(t, u(t)) - G(t, \hat{u}(t))\big)\,\mathrm{d}W(t). \end{aligned}$$

Note that $\mathrm{d}\hat{u}(t) = \big(A\hat{u}(t) + F(\hat{u}(t))\big)\,\mathrm{d}t$ is satisfied, and $\hat{u}(0) = u^*$. Hence, for v we obtain the equation

$$\begin{aligned} \mathrm{d}v(t) &= [A + F'(\hat{u}(t))]\,v(t)\,\mathrm{d}t + \sigma G(t,\hat{u}(t))\,\mathrm{d}W(t), \\ v(0) &= v_0, \end{aligned} \tag{3.4.9a}$$

and for z we get

$$\begin{aligned} \mathrm{d}z(t) &= [A + F'(\hat{u}(t))]z(t)\,\mathrm{d}t + \big(F(u(t)) - F(\hat{u}(t)) - F'(\hat{u}(t))[u(t) - \hat{u}(t)]\big)\,\mathrm{d}t \\ &\quad + \sigma^2 H(t,u(t))\,\mathrm{d}t + \sigma\big(G(t,u(t)) - G(t,\hat{u}(t))\big)\,\mathrm{d}W(t), \\ z(0) &= 0. \end{aligned} \tag{3.4.9b}$$

Since we already know from Theorem 3.4.5 that there is a unique solution to (3.4.6), it is not difficult to obtain solutions for (3.4.9a) and (3.4.9b). From (3.4.9a), we should expect to obtain a solution formula for v which does not involve u or z. This is indeed possible.

Proposition 3.4.6 (Asymptotic expansion)**.** *Equation* (3.4.9a) *has a unique mild solution, explicitly given by*

$$v(t) = S(t,0)\,v_0 + \sigma \int_0^t S(t,t')G(t',\hat{u}(t'))\,\mathrm{d}W(t'), \qquad t \in [0,\infty). \tag{3.4.10}$$

With this solution, we have the following first-order asymptotic expansion:

$$u(t) = \hat{u}(t) + v(t) + z(t), \qquad t \in [0,\tau^*), \tag{3.4.11}$$

where z is the mild solution to (3.4.9b).

Proof. First note that (3.4.10) is the unique mild solution to (3.4.9a) by definition. We may now construct a process z by setting $z(t) := u(t) - \hat{u}(t) - v(t)$, so that (3.4.11) is satisfied. Substituting (3.4.11) into (3.4.8) and rearranging the terms in the same way that (3.4.9) was derived, it follows that z is indeed a mild solution to (3.4.9b). Since this amounts to repeating the calculation we just performed, we do not write this out. □

With the asymptotic expansion established, we can formulate the first stochastic stability result. Theorem 3.4.7 can be seen as a stochastic generalization of Theorem 3.4.4. The proof is contained in Section 3.6.2.

Theorem 3.4.7 (Short-term exponential stability)**.** *Let γ_t be given by* (3.4.3)*. For every $T > 0$, $\alpha > 1$, there exist constants $c, \varepsilon' > 0$ such that we have the estimate*

$$\mathbb{P}\Big[\sup_{t\in[0,T\wedge\tau^*)} \|u(t) - \Pi(\gamma_t)u^*\|_{\mathscr{X}} - M_3\alpha e^{-at}\varepsilon \geq \varepsilon,\ \|v_0\|_{\mathscr{X}} \leq \alpha\varepsilon\Big] \leq 6\exp(-c\varepsilon^2\sigma^{-2}), \tag{3.4.12}$$

for all ε, σ satisfying $0 < \sigma \leq \varepsilon \leq \varepsilon'$, and any initial condition v_0 (M_3, a are the constants from (3.3.21c)*).*

Theorem 3.4.7 does not directly imply stability on long time scales. The reason is that ε' and c both depend on T, and from the proof it can be seen that $\varepsilon' \sim T^{-1}$.

Thus, a direct application of the theorem can only give stability on a time scale $T \sim \sigma^{-1}$. However, if we choose $\alpha = 2$ and $T = a^{-1}\log(M_3\alpha)$, then (3.4.12) implies

$$\mathbb{P}\left[\|u(T) - \Pi(\gamma_T)u^*\|_{\mathscr{X}} \geq 2\varepsilon,\ \|v_0\|_{\mathscr{X}} \leq 2\varepsilon\right] \leq 6\exp(-c\varepsilon^2\sigma^{-2}).$$

Hence, if the solution at time $t = 0$ is close to the center manifold, then the solution at time $t = T$ will be equally close (with high probability). Using the symmetries of the equation, it is possible to show a similar estimate on the interval $[T, 2T]$, and so on. Combining these estimates then results in the following long-time stability result. For simplicity, we use the initial condition $v_0 \equiv 0$, but a similar result holds when $\|v_0\| \leq \varepsilon$. The proof of Theorem 3.4.8 is contained in Section 3.6.3.

Theorem 3.4.8 (Long-term stability)**.** *Set $v_0 \equiv 0$. There exist constants $c, \varepsilon' > 0$ such that we have the estimate*

$$\mathbb{P}\left[\sup_{t\in[0,T\wedge\tau^*)}\inf_{\gamma\in G}\|u(t) - \Pi(\gamma)u^*\|_{\mathscr{X}} \geq \varepsilon\right] \leq 12\,T\exp(-c\varepsilon^2\sigma^{-2}), \tag{3.4.13}$$

for all $T > 0$ and all ε, σ satisfying $0 < \sigma \leq \varepsilon \leq \varepsilon'$.

Notice that unlike in Theorem 3.4.7, the constants c, ε' in Theorem 3.4.8 do not depend on T.

LONG-TIME PHASE PROCESS

Unlike Theorem 3.4.7, where an explicit expression for the phase γ_t is given by (3.4.3), Theorem 3.4.8 only states that at each time t, there exists (with high probability) a $\gamma \in G$ such that $u(t)$ is close to $\Pi(\gamma)u^*$. In the proof, this is shown by exhibiting a process γ_t for which the bound is attained. This process is specified by the following equations:

$$\begin{aligned}\gamma_{nT+t} &= \gamma_{nT}e^{tX}\exp(e^{tL}\mathscr{P}[\Pi(\gamma_{nT}^{-1})u(nT) - u^*]), \quad n \in \mathbb{N}_0,\ t \in (0, T],\\ \gamma_0 &= e_G,\end{aligned} \tag{3.4.14}$$

where e_G denotes the identity element of G and $T > 0$ is a sufficiently large constant. Since (3.4.14) specifies $\{\gamma_t\}_{t\in(nT,(n+1)T]}$ as a function of $u(nT)$ and γ_{nT}, this results in a well-defined phase process. Note also that (3.4.14) reduces to (3.4.3) when $n = 0$.

Intuitively, (3.4.14) is obtained by tracking the pattern using (3.4.3) on each subinterval $(nT, (n+1)T]$, and 'resetting' the tracking mechanism at times nT by taking into account the full solution $u(nT)$. This resetting is necessary, since the linearization on which (3.4.3) is based gets increasingly inaccurate as the pattern wanders further from $\hat{u}(t)$. As a consequence of resetting, the process γ_t defined by (3.4.14) might have discontinuities at times nT. Note that a continuous phase process can also be obtained by smoothing γ_t afterwards.

We emphasize the remarkable phenomenon that the phase at time $(n+1)T$ is already determined by the solution at time nT. This shows that it is generally possible to 'look into the future', and accurately predict where the pattern will be after time $T + t$, solely based on the profile at time T.

Finally we note that, although (3.4.14) is an explicit recurrence relation, it does contain the solution u on the right-hand side. Hence, to compute $\gamma(nT)$ for large n, explicit information about the solution is required. However, this feature is shared by competing definitions of the phase, and we are not aware of any way to exactly compute the motion of moving patterns without explicit knowledge of the solution.

3.5. EXAMPLES

In this section, we revisit and improve on some examples from the existing literature. The first example was chosen to highlight the flexibility of working in Banach spaces, and the second example demonstrates how the phase may be concretely computed in a noncommutative scenario. The third example serves to illuminate the discussion at the beginning of Section 3.4, by showing how the dimension of the center space can vary based on symmetries of the stable solution.

3.5.1. TRAVELING PULSE IN THE FITZHUGH–NAGUMO EQUATION

We consider the FitzHugh–Nagumo equation:

$$\begin{aligned} \partial_t u(t,x) &= \partial_x^2 u(t,x) + f(u(t,x)) - v(t,x), \\ \partial_t v(t,x) &= \epsilon(u(t,x) - \gamma v(t,x)), \end{aligned} \tag{3.5.1}$$

with $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. Here $\gamma > 0$, $0 < \epsilon \ll 1$, and $f(u) = u(u-b)(1-u)$ for some $b \in (0, \frac{1}{2})$. In [60, 94], it is shown that there exists a wave speed $c > 0$ and a pulse profile $w^* = (u^*, v^*)^\top \in C^2_{\mathrm{ub}}(\mathbb{R};\mathbb{R}^2)$ such that $\hat{w}(t,x) = (u^*(x-ct), v^*(x-ct))^\top$ is a solution to (3.5.1). Moreover, $(u^*, v^*, \partial_x u^*)$ converges to $(0,0,0)$ at an exponential rate as $|x| \to \infty$, and the pulse $\hat{w}$ is orbitally stable against bounded uniformly continuous perturbations of the initial value [124, 233]. Stability of stochastic perturbations is shown in [74], using the phase-lag method introduced by Krüger and Stannat [148].

We will show how (3.5.1) fits into our framework, and establish stability of stochastic perturbations of the traveling pulse in a flexible range of spaces at a low regularity level.

There are already multiple frameworks which have treated stochastic perturbations of (3.5.1) [113, 161]. However, as previously mentioned, these works crucially rely on the Hilbert space structure of L^2 to be able to track the pulse and show stability. This poses significant restrictions on the noise, and does not allow for treatment of equations which are well-posed in L^p-spaces for $p \neq 2$. We will demonstrate that the results from Sections 3.3 and 3.4 are strong enough to directly show stability in the Bessel space $\mathscr{X} = H^{s,p}(\mathbb{R};\mathbb{R}^2)$ for $p \in [2,\infty)$, $s \in (\frac{1}{p},\infty)$, which allows us to treat noise which is much rougher than in previous results. In the rest of the section we will always assume p and s are in the previously mentioned range.

The condition $p \in [2,\infty)$ ensures that $\mathscr{X}$ is 2-smooth (when using an appropriate norm, which we assume from now on). By Sobolev embedding, the condition $s > \frac{1}{p}$ implies that $\mathscr{X} \hookrightarrow C_{\mathrm{ub}}(\mathbb{R};\mathbb{R}^2)$, and it also ensures that $\mathscr{X}$ is a Banach algebra, i.e.,

$$\|gh\|_{\mathscr{X}} \le C\|g\|_{\mathscr{X}}\|h\|_{\mathscr{X}}, \quad g,h \in \mathscr{X} \tag{3.5.2}$$

(see [200, Chapter 4.6]). We write (3.5.1) in the form (3.3.1) by taking

$$A = \begin{pmatrix} \partial_x^2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(u) - v \\ \epsilon(u - \gamma v) \end{pmatrix}.$$

We begin by establishing regularity and integrability of the pulse profile.

Proposition 3.5.1. *The profiles u^* and v^* have infinitely many bounded derivatives, all of which decay exponentially to* 0 *as* $|x| \to \infty$.

Proof. Abbreviating $z := \partial_x u^*$, we see from (3.5.1) that (u^*, v^*, z) satisfies the following system of ODEs:

$$\begin{aligned} -cz &= \partial_x z + f(u^*) - v^*, \\ -c\partial_x v^* &= \epsilon(u^* - \gamma v^*), \\ \partial_x u^* &= z. \end{aligned}$$

Since $f(0) = 0$ and (u^*, v^*, z) converges at an exponential rate to $(0,0,0)$ as $x \to \pm\infty$ this shows that $(\partial_x u^*, \partial_x v^*, \partial_x z)$ do so as well. Differentiating the ODE system and repeating this argument inductively, the claim follows. □

We now verify the assumptions formulated in Sections 3.3 and 3.4. We take $\mathrm{G} = (\mathbb{R}, +)$, and define Π according to $\Pi(a)w(x) = w(x-a)$. The corresponding Lie algebra $\mathfrak{g}$ can be identified with $(\mathbb{R}, +)$, in which case the exponential map from $\mathfrak{g}$ to G acts as the identity. The action of π is given by $\pi(a) = -a\partial_x$. Since A and F clearly commute with $\Pi(a)$ and translation is strongly continuous in $H^{s,p}$, we see that Assumption 3.1 holds true. As we have $\hat{w}(t,x) = w^*(x-ct)$, we should take $X = c$ for (3.3.5) to hold. For (3.3.6), we need to show

$$\|w^*(\cdot - a) - w^* + a\partial_x w^*\|_{\mathscr{X}} \leq C|a|^2, \quad a \in \mathbb{R},$$

which follows from a Taylor expansion and Proposition 3.5.1. The remaining statements of Assumption 3.2 can also be verified using Proposition 3.5.1.

Since f is a polynomial, it follows from (3.5.2) that F has infinitely many Fréchet derivatives on $\mathscr{X}$. Thus, Assumption 3.5 and the first part of Assumption 3.3 hold. The linearization operator in Assumption 3.3 takes the form $\mathscr{L}^* := A + B + F'$ with

$$B := c\partial_x, \qquad F' := \begin{pmatrix} f'(u^*) & -1 \\ \epsilon & -\epsilon\gamma \end{pmatrix}. \tag{3.5.3}$$

It is well-known that A generates a C_0-semigroup on $L^p(\mathbb{R};\mathbb{R}^2)$, and therefore also on $\mathscr{X}$ (since A commutes with $(I-\Delta)^{s/2}$, see Section 3.2.1). By Remark 3.3.6, it follows that $\mathscr{L}^*$ also generates a C_0-semigroup on $\mathscr{X}$, which we denote $\{S^*(t)\}_{t\geq 0}$. Hence, Assumption 3.3 is shown except for the boundedness of $S^*(t)$, which will follow from (3.3.13) and (3.3.15) once Assumption 3.4 has been established.

LINEAR STABILITY

We now show that Assumption 3.4 holds. Although it is well-known that the fast traveling pulse solution $\hat{w}$ is stable [97, 124, 233], the stability has (as far as we are aware) only been considered in the spaces $C_{\mathrm{ub}}(\mathbb{R};\mathbb{R}^2)$ and $L^2(\mathbb{R};\mathbb{R}^2)$. We now show how stability in $\mathscr{X}$ can be deduced from these existing results.

Proposition 3.5.2. *Assumption 3.4 holds when $\mathscr{X} = C_{\mathrm{ub}}(\mathbb{R};\mathbb{R}^2)$ or $\mathscr{X} = L^2(\mathbb{R};\mathbb{R}^2)$. The projections P_c^* and P_s^* do not depend on the chosen space, and are additionally bounded on $H^k(\mathbb{R};\mathbb{R}^2)$ for any $k \in \mathbb{N}$.*

Proof. The case $\mathscr{X} = C_{\mathrm{ub}}(\mathbb{R};\mathbb{R}^2)$ can be found in [124, 233]. The case $\mathscr{X} = L^2(\mathbb{R};\mathbb{R}^2)$ is treated in [97] or [74, Appendix A.2]. Now let $\mathscr{X} \in \{C_{\mathrm{ub}}(\mathbb{R};\mathbb{R}^2), L^2(\mathbb{R};\mathbb{R}^2)\}$ and let (P_c^*, P_s^*)

be the projections from Assumption 3.4 associated with the space $\mathscr{X}$. Since P_c^* maps onto constant multiples of $\partial_x w^*$, and $S^*(t)\partial_x w^* = \partial_x w^*$ for $t \geq 0$, it holds that

$$P_c^* g \overset{(3.3.15)}{=} P_c^* g + \lim_{t\to\infty} S^*(t) P_s^* g = \lim_{t\to\infty} \left(S^*(t)P_c^* g + S^*(t) P_s^* g\right) = \lim_{t\to\infty} S^*(t) g$$

for any $g \in \mathscr{X}$. Since the right-hand side is independent of the choice of $\mathscr{X}$, so is P_c^*. Using Proposition 3.5.1 and the fact that P_c^* projects onto multiples of $\partial_x w^*$, the $L^2(\mathbb{R};\mathbb{R}^2)$-boundedness of P_c^* self-improves to $H^k(\mathbb{R};\mathbb{R}^2)$-boundedness. The claims about P_s^* follow from the corresponding claims about P_c^* since $P_s^* = \mathrm{I} - P_c^*$. □

Lemma 3.5.3. *For every $k \in \mathbb{N}_0$ there exists a constant C_k such that*

$$\sup_{t\geq 0} \|S^*(t)P_s^*\|_{\mathscr{L}(H^k(\mathbb{R};\mathbb{R}^2))} \leq C_k. \tag{3.5.4}$$

Proof. Throughout the proof, we let C_k denote a constant which can depend only on k, and may change from line to line. When $k = 0$, the claim is contained in Proposition 3.5.2. Suppose now that (3.5.4) holds for some $k \in \mathbb{N}_0$. Let $g \in H^{k+1}(\mathbb{R};\mathbb{R}^2)$ and write $w(t) = S^*(t)P_s^* g$. By definition of $S^*(t)$, w then solves $\partial_t w = \mathscr{L}^* w$ with initial condition $w(0) = P_s^* g$. Differentiating in space and using (3.5.3), we find that w satisfies

$$\begin{aligned} \partial_t(\partial_x^{k+1} w) &= \partial_x^{k+1}\mathscr{L}^* w = \mathscr{L}^*(\partial_x^{k+1} w) + h, \\ (\partial_x^{k+1} w)(0) &= \partial_x^{k+1} P_s^* g, \end{aligned} \tag{3.5.5}$$

with h given by

$$h(t) := \sum_{i=0}^{k} c_{k+1,i} \begin{pmatrix} [\partial_x^{k+1-i} f'(u^*)]\partial_x^i w(t) \\ 0 \end{pmatrix}, \qquad t \geq 0, \tag{3.5.6}$$

where $c_{k,i}$ are appropriate binomial coefficients. By the Duhamel's formula we see from (3.5.5) that

$$\partial_x^{k+1} w(t) = S^*(t)\partial_x^{k+1} P_s^* g + \int_0^t S^*(t-t')h(t')\,\mathrm{d}t', \qquad t \geq 0, \tag{3.5.7}$$

and from the smoothness of f and u^* and the induction hypothesis it follows that

$$\begin{aligned} \sup_{t\geq 0}\|h(t)\|_{L^2(\mathbb{R};\mathbb{R}^2)} &\overset{(3.5.6)}{\leq} C_k \sup_{t\geq 0}\|w(t)\|_{H^k(\mathbb{R};\mathbb{R}^2)} \\ &= C_k \sup_{t\geq 0}\|S^*(t)P_s^* g\|_{H^k(\mathbb{R};\mathbb{R}^2)} \leq C_k \|g\|_{H^k(\mathbb{R};\mathbb{R}^2)}. \end{aligned} \tag{3.5.8}$$

Combining this with Proposition 3.5.2 gives

$$\begin{aligned} \|\partial_x^{k+1} w(t)\|_{L^2(\mathbb{R};\mathbb{R}^2)} &\overset{(3.3.15),(3.5.7)}{\leq} C_k e^{-at}\|\partial_x^{k+1} P_s^* g\|_{L^2(\mathbb{R};\mathbb{R}^2)} + C_k \int_0^t e^{-a(t-t')}\|h(t')\|_{L^2(\mathbb{R};\mathbb{R}^2)}\,\mathrm{d}t' \\ &\overset{(3.5.8)}{\leq} C_k \|g\|_{H^{k+1}(\mathbb{R};\mathbb{R}^2)}. \end{aligned}$$

Since the lower-order derivatives of w can be estimated using the induction hypothesis, we find $\sup_{t\geq 0}\|w(t)\|_{H^{k+1}(\mathbb{R};\mathbb{R}^2)} \leq C_k\|g\|_{H^{k+1}(\mathbb{R};\mathbb{R}^2)}$, which implies that (3.5.4) also holds for $k+1$. Since k was arbitrary, the claim follows by induction. □

The estimates required for Assumption 3.4 to hold with our choice of $\mathscr{X}$ now follow from the following interpolation argument.

Corollary 3.5.4. *Assumption 3.4 holds when $\mathscr{X} = H^{s,p}$ for any $s \geq 0$, $p \in [2,\infty)$.*

Proof. We only need to check boundedness of P_c^*, P_s^*, and (3.3.15). By applying the Marcinkiewicz interpolation theorem ([211, B.4] see also Remark 3.5.5) to P_c^*, P_s^*, and $S^*(t)P_s^*$, it follows from Proposition 3.5.2 that Assumption 3.4 holds with $\mathscr{X} = L^q(\mathbb{R};\mathbb{R}^2)$ for any $q \in [2,\infty)$. Now for fixed $s > 0$, $p \in [2,\infty)$, we choose $k \in \mathbb{N}_0$, $q \in [2,\infty)$, $\theta \in (0,1]$ such that

$$s = (1-\theta)k, \qquad \tfrac{1}{p} = (1-\theta)\tfrac{1}{2} + \theta\tfrac{1}{q}.$$

By complex interpolation [25, Chapter 4], [200, §2.5.2], there exists a constant C such that

$$\|T\|_{\mathscr{L}(H^{s,p})} \leq C\|T\|_{\mathscr{L}(H^k)}^{1-\theta}\|T\|_{\mathscr{L}(L^q)}^{\theta}$$

for any T which is bounded on H^k and L^q. For $T \in \{P_s^*, P_c^*\}$, it follows from Proposition 3.5.2 and Assumption 3.4 with $\mathscr{X} = L^q(\mathbb{R};\mathbb{R}^2)$ that $T \in \mathscr{L}(H^{s,p}(\mathbb{R};\mathbb{R}^2))$. For $T = S^*(t)P_s^*$, it follows from (3.5.3) and Assumption 3.4 applied with $\mathscr{X} = L^q(\mathbb{R};\mathbb{R}^2)$ that

$$\|S^*(t)P_s^*\|_{\mathscr{L}(H^{s,p}(\mathbb{R};\mathbb{R}^2))} \overset{(3.3.15),(3.5.3)}{\leq} CC_k^{1-\theta}(Me^{-at})^{\theta}, \qquad t \geq 0,$$

so that (3.3.15) with $\mathscr{X} = H^{s,p}(\mathbb{R};\mathbb{R}^2)$ follows since $\theta > 0$. $\square$

Remark 3.5.5. The Marcinkiewicz interpolation theorem is not usually stated for C_{ub}, so does not apply verbatim in our case. However, since C_{ub} uses the same norm as L^∞, the proof given in [211, B.4] still works after a minor modification: the only change needed is that we decompose ϕ into two continuous pieces instead of using $\phi = \mathbb{1}_{|\phi|\leq a}\phi + \mathbb{1}_{|\phi|>a}\phi$. Concretely, we define $g_a(x) = \mathbb{1}_{[0,a]}(x) + (2-x/a)\mathbb{1}_{(a,2a]}(x)$ so that we can decompose $\phi = g_a(|\phi|)\phi + (1-g_a(|\phi|))\phi$, after which the proof continues as usual.

With Assumptions 3.1, 3.2, 3.3, 3.4, 3.5 verified, we study stochastic perturbations of (3.5.1). We consider two cases.

ADDITIVE NOISE

We consider the following equation:

$$\begin{aligned} \mathrm{d}u(t,x) &= [\partial_x^2 u(t,x) + f(u(t,x)) - v(t,x)]\,\mathrm{d}t + \sigma\sum_{i\in\mathbb{N}} g_i(t,x)\,\mathrm{d}\beta_i(t), \\ \mathrm{d}v(t,x) &= \epsilon(u(t,x) - \gamma v(t,x))\,\mathrm{d}t, \end{aligned} \tag{3.5.9}$$

where $\sigma > 0$, $(\beta_i)_{i\in\mathbb{N}}$ is a sequence of independent Brownian motions, and $g_i(t,\cdot)$ is a (deterministic) sequence of functions in $H^{s,p}(\mathbb{R};\mathbb{R})$. To translate this to the language of Section 3.4.3, we take $\mathscr{H} = \ell^2(\mathbb{N})$ with the usual orthonormal basis $(e_i)_{i\in\mathbb{N}}$. For any $t \in [0,\infty)$, $\phi \in \mathscr{X}$, we define $G(t,\phi)\colon \mathscr{H} \to \mathscr{X}$ to be the linear operator which sends e_i to $(g_i(t,\cdot),0)^\top$ for every i. To verify Assumption 3.6, we only need to make sure that (3.4.7b) holds (as $H \equiv 0$ and (3.4.7d) trivially holds since G does not depend on ϕ). Since $\mathscr{X}$ has type 2, it is an immediate consequence of [119, Theorem 9.2.10] that we have

$$\|G(t,\phi)\|_{\gamma(\ell^2(\mathbb{N});\mathscr{X})}^2 = \|e_i \mapsto (g_i(t,\cdot),0)^\top\|_{\gamma(\ell^2(\mathbb{N});\mathscr{X})}^2 \leq C\sum_{i\in\mathbb{N}}\|g_i(t,\cdot)\|_{H^{s,p}(\mathbb{R};\mathbb{R})}^2,$$

(note that the first identity is the definition of G) where C does not depend on t or ϕ. Hence, to satisfy (3.4.7b) it suffices (aside from measurability) to require

$$\sup_{t\in[0,\infty]} \sum_{i\in\mathbb{N}} \|g_i(t,\cdot)\|^2_{H^{s,p}(\mathbb{R};\mathbb{R})} < \infty. \tag{3.5.10}$$

With all the assumptions verified, we obtain from Theorem 3.4.8 the following concrete result. The ε appearing in the theorem is not to be confused with ϵ appearing in (3.5.9).

Theorem 3.5.6. *Let $p \in [2,\infty)$, $s \in (\frac{1}{p},\infty)$, let g_i be such that (3.5.10) is satisfied, and let $w(t)$ be the solution to (3.5.9) with initial condition $w(0,\cdot) = w^*$. There exist constants $c, \varepsilon' > 0$ such that we have the estimate*

$$\mathbb{P}\Big[\sup_{t\in[0,T]} \inf_{a\in\mathbb{R}} \|w(t,\cdot) - w^*(\cdot - a)\|_{H^{s,p}(\mathbb{R};\mathbb{R}^2)} \geq \varepsilon\Big] \leq 12\,T\exp(-c\varepsilon^2\sigma^{-2}),$$

for all $T > 0$ and all ε, σ satisfying $0 < \sigma \leq \varepsilon \leq \varepsilon'$.

As remarked before, we can allow the regularity parameter s to be arbitrarily small (as long as p is sufficiently large).

MULTIPLICATIVE NOISE

We now consider the equation

$$\begin{aligned} \mathrm{d}u(t,x) &= [\partial_x^2 u(t,x) + f(u(t,x)) - v(t,x)]\,\mathrm{d}t + \sigma g(u)(\phi * \mathrm{d}W(t)), \\ \mathrm{d}v(t,x) &= \epsilon(u(t,x) - \gamma v(t,x))\,\mathrm{d}t, \end{aligned} \tag{3.5.11}$$

where $\sigma > 0$, $\mathrm{d}W(t)$ is space-time white noise, and $g, \phi \colon \mathbb{R} \to \mathbb{R}$ are suitable functions. The symbol $*$ denotes convolution. Observe that $\Pi(a)[\phi * \mathrm{d}W(t)] = \phi * [\Pi(a)\mathrm{d}W(t)]$, which has the same distribution as $\phi * \mathrm{d}W(t)$ since $\mathrm{d}W(t)$ is space-time white noise (recall that $\Pi(a)$ denotes a translation operator). In the literature on stochastic traveling waves, such noise is referred to as *translation-invariant noise* [115], although the invariance only holds in law.

It is well-known that an $L^2(\mathbb{R};\mathbb{R})$-cylindrical Wiener process formally corresponds to space-time white noise. Therefore, in the language of Section 3.4.3, we should take $\mathcal{H} = L^2(\mathbb{R};\mathbb{R})$, and define $G(t,w) : L^2(\mathbb{R};\mathbb{R}) \to \mathcal{X}$ via

$$G(t,w)\psi = \big(g(u(\cdot))\int_{\mathbb{R}} \phi(\cdot - y)\psi(y)\,\mathrm{d}y, 0\big)^\top, \qquad w = (u,v)^\top.$$

From Proposition 3.A.2, it follows that Assumption 3.6 is satisfied if both of the following conditions are met:

- g maps $H^{s,p}(\mathbb{R};\mathbb{R})$ into $H^{s,p}(\mathbb{R};\mathbb{R})$, and is locally Lipschitz in $H^{s,p}(\mathbb{R};\mathbb{R})$.
- $\phi \in H^s(\mathbb{R};\mathbb{R})$.

Since $H^{s,p}(\mathbb{R};\mathbb{R})$ is a Banach algebra by our choice of s, p, the first condition is satisfied by any smooth function $g\colon \mathbb{R} \to \mathbb{R}$ which satisfies $g(0) = 0$. Applying Theorem 3.4.8 thus results in the following. Again, ε in the theorem should not be confused with ϵ in (3.5.11).

Theorem 3.5.7. *Let $p \in [2,\infty)$, $s \in (\frac{1}{p},\infty)$, $\phi \in H^s(\mathbb{R};\mathbb{R})$, let g be a polynomial which satisfies $g(0) = 0$, and let $w(t)$ be the solution to (3.5.11) with initial condition $w(0,\cdot) = w^*$. There exist constants $c, \varepsilon' > 0$ such that we have the estimate*

$$\mathbb{P}\left[\sup_{t\in[0,T]} \inf_{a\in\mathbb{R}} \|w(t,\cdot) - w^*(\cdot - a)\|_{H^{s,p}(\mathbb{R};\mathbb{R}^2)} \geq \varepsilon\right] \leq 12\,T\exp(-c\varepsilon^2\sigma^{-2}),$$

for all $T > 0$ and all ε, σ satisfying $0 < \sigma \leq \varepsilon \leq \varepsilon'$.

By taking p large and s small, we see that $\phi \in H^s$ only needs to be slightly more regular than an L^2-function to have orbital stability. Since a Hilbert space setting (with $p = 2$) would necessitate $s > 1/2$, it is clear that we have truly gained something by working in the Banach space setting.

Remark 3.5.8. Since the symmetry group is commutative, in this case (3.4.3) simplifies to

$$a_t = ct + \mathscr{P} v_0$$

(note that a takes the role of γ). By the Riesz representation theorem, it also follows that $\mathscr{P} v_0 = \langle \phi^0, v_0\rangle_{L^2(\mathbb{R};\mathbb{R}^2)}$ for some $\phi^0 \in L^2(\mathbb{R};\mathbb{R}^2)$. In fact, it can be shown that ϕ^0 is an eigenfuction of the adjoint of $\mathscr{L}^*$ with eigenvalue 0 [113].

Remark 3.5.9. The proof of Theorem 3.4.7 does not make use of the smoothing effect of the Laplacian in (3.5.1). Thus, we expect that with a minor modification, we can consider noise which is even more singular in both the additive and multiplicative case. We leave this for future research.

3.5.2. ROTATING WAVES IN TWO DIMENSIONS

Our second example is a localized rotating wave in two dimensions. The existence and (non)linear stability of such waves has been studied in [28, 59, 67, 108, 150, 206]. Instead of verifying Assumptions 3.1-3.5 like in the previous example (which can be done using the results from [28]) we focus on computing the symmetry group and phase prediction function, which show how the noncommutativity enters into the phase.

We consider the reaction-diffusion equation

$$\mathrm{d}u(t,x) = [\Delta u(t,x) + f(u(t,x))]\,\mathrm{d}t, \qquad x = (x_1,x_2)^\top \in \mathbb{R}^2, \tag{3.5.12}$$

where u takes values in $\mathbb{R}^n$, Δ acts as a Laplacian in all components, and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth and satisfies $f(0) = 0$. We assume existence of a smooth and localized *rotating wave* solution of the form $\hat{u}(t,x) = u^*(R_{-\omega t}x)$, where u^* is the wave profile, $\omega > 0$ is the rotation speed, and R_θ is a rotation matrix, i.e.,

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

The natural symmetry group of (3.5.12) is the special Euclidean group SE(2), which is the semi-direct product of the group of translations and the group of rotations around the origin. SE(2) can be represented by matrices of the form

$$\mathscr{T}_{x_1,x_2,\theta} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & x_1 \\ \sin(\theta) & \cos(\theta) & x_2 \\ 0 & 0 & 1 \end{pmatrix},$$

with $x_1, x_2, \theta \in \mathbb{R}$ (notice that $(x_1, x_2, \theta) \mapsto \mathcal{T}_{x_1,x_2,\theta}$ is not injective). In this case, the group multiplication is just matrix multiplication. After identifying $(a,b)^\top$ with $(a,b,1)^\top$, we obtain an action of $\mathcal{T}_{x_1,x_2,\theta}$ on $\mathbb{R}^2$ via matrix-vector multiplication. This induces an action on the space of functions on $\mathbb{R}^2$, given by $(\Pi(\mathcal{T}_{x_1,x_2,\theta})f)(x) = f(\mathcal{T}^{-1}_{x_1,x_2,\theta}x)$. Hence, the element $\mathcal{T}_{x_1,x_2,0}$ acts as a translation by (x_1, x_2), and $\mathcal{T}_{0,0,\theta}$ acts as counterclockwise rotation around the origin by θ radians. By differentiating with respect to x_1, x_2, θ, we see that the corresponding Lie algebra (which we denote $\mathfrak{se}(2)$) can be represented as the span of the following matrices:

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.5.13}$$

Furthermore, we have

$$\begin{aligned} e^{tX_1} &= \mathcal{T}_{x_1,0,0}, & \Pi(e^{tX_1})f(x_1,x_2) &= f(x_1 - t, x_2), & \pi(X_1) &= -\partial_{x_1}, \\ e^{tX_2} &= \mathcal{T}_{0,x_2,0}, & \Pi(e^{tX_2})f(x_1,x_2) &= f(x_1, x_2 - t), & \pi(X_2) &= -\partial_{x_2}, \\ e^{tX_3} &= \mathcal{T}_{0,0,\theta}, & \Pi(e^{tX_3})f(x) &= f(R_{-t}x), & \pi(X_3) &= -x_1\partial_{x_2} + x_2\partial_{x_1}, \end{aligned}$$

where $\partial_{x_1}, \partial_{x_2}$ denote partial differentiation with respect to x_1, x_2, respectively. Using these expressions, we find that

$$\hat{u}(t) = \Pi(\mathcal{T}_{0,0,\omega t})u^* = \Pi(e^{t\omega X_3})u^*. \tag{3.5.14}$$

Thus, we need to take $X = \omega X_3$ in Assumption 3.2. From (3.5.13), we also find the following commutation relations:

$$[X_1, X_2] = 0, \quad [X_1, X_3] = -X_2, \quad [X_2, X_3] = X_1. \tag{3.5.15}$$

Abbreviating $\partial_\psi = x_1\partial_{x_2} - x_2\partial_{x_1}$, we obtain the following expression for the linearization around u^* in the comoving frame:

$$\mathcal{L}^* = \Delta + \omega\partial_\psi + f'(u^*).$$

We will now explicitly compute the predicted phase function given by (3.4.3). From (3.3.11), we find

$$LY = [Y, \omega X_3], \quad Y \in \mathfrak{se}(2).$$

Using the ordered basis (X_1, X_2, X_3), the matrix representation of L and e^{tL} (obtained from (3.5.15) and by exponentiating) are given by

$$L = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e^{tL} = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.5.16}$$

From (3.5.16) we see that $\sigma(L_\mathbb{C}) = \{0, i\omega, -i\omega\}$, so $\mathcal{L}^*$ has spectrum on the imaginary axis. The form of e^{tL} clearly shows that it is necessary to include both the translational and the rotational symmetries to have any chance of orbital stability.

It only remains to find an explicit expression for $\mathscr{P}$. To do this, we use the approach from [28, Section 2.2] (see also [150, Section 3.2]). There, it is shown that there exist functions $\phi^1, \phi^2, \phi^3 \in L^2(\mathbb{R}^2; \mathbb{R}^n)$ such that we have

$$P_c^* v_0 = -\langle \phi^1, v_0 \rangle \partial_{x_1} u^* - \langle \phi^2, v_0 \rangle \partial_{x_2} u^* - \langle \phi^3, v_0 \rangle \partial_\psi u^*, \quad v_0 \in L^2(\mathbb{R}^2; \mathbb{R}^n),$$

where the inner products are taken in $L^2(\mathbb{R}^2; \mathbb{R}^d)$. We note that such a representation of P_c^* can also be derived from the Riesz representation theorem. The functions ϕ^1, ϕ^2, ϕ^3 are suitable linear combinations of eigenfunctions of the adjoint of $\mathscr{L}^*$. Thus, using for $\mathfrak{se}(2)$ the basis (X_1, X_2, X_3), the map $\mathscr{P}$ can be written as

$$\mathscr{P} v_0 = \begin{pmatrix} \langle \phi^1, v_0 \rangle \\ \langle \phi^2, v_0 \rangle \\ \langle \phi^3, v_0 \rangle \end{pmatrix}, \quad v_0 \in L^2(\mathbb{R}^2; \mathbb{R}^n). \tag{3.5.17}$$

Combining (3.4.3), (3.5.14), (3.5.16) and (3.5.17), we arrive at the following expression for the predicted phase:

$$\gamma_t = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \exp\left(B \left(\begin{pmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \langle \phi^1, v_0 \rangle \\ \langle \phi^2, v_0 \rangle \\ \langle \phi^3, v_0 \rangle \end{pmatrix} \right) \right),$$

where B is the map sending $(a, b, c)^\top$ to $aX_1 + bX_2 + cX_3$, and exp denotes the matrix exponential.

3.5.3. ROTATION SYMMETRY

As a final example, we show how our setting can be applied to a symmetric gradient-type SDE. Consider a potential $V : \mathbb{R}^6 \to \mathbb{R}$ given by

$$V(x, y) = f(|x|^2) + f(|y|^2) + g(|x - y|^2), \quad x, y \in \mathbb{R}^3, \tag{3.5.18}$$

where f and g are smooth functions from $\mathbb{R}^+$ to $\mathbb{R}$, and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^3$. The potential V gives rise to a gradient flow, which we may perturb stochastically to obtain the following SDE:

$$\begin{aligned} \mathrm{d}X_t &= -\nabla_x V(X_t, Y_t)\,\mathrm{d}t + \sigma h_1(X_t, Y_t)\,\mathrm{d}\beta_t^1, \\ \mathrm{d}Y_t &= -\nabla_y V(X_t, Y_t)\,\mathrm{d}t + \sigma h_2(X_t, Y_t)\,\mathrm{d}\beta_t^2, \end{aligned} \tag{3.5.19}$$

where β^1 and β^2 are independent 3-dimensional Brownian motions, and h_1, h_2 are $\mathbb{R}^{3\times 3}$-valued and sufficiently smooth. The SDE (3.5.19) can be interpreted as a (perturbed) system of two interacting particles which are confined by a radially symmetric potential.

From (3.5.18), it is clear that V is invariant under the action of the special orthogonal group SO(3), where the action of $g \in \mathrm{SO}(3)$ is given by $(x, y) \mapsto (gx, gy)$. If V has a (local) minimizer (x_0, y_0), it is natural to ask whether this minimizer is (orbitally) stable for (3.5.19). We will not comment on Assumptions 3.1, 3.2, 3.3, 3.5 and 3.6, which are easily checked in this finite-dimensional setting under mild assumptions on f, g, h_1, h_2.

To check Assumption 3.4 it is necessary to first determine the dimension of the center space (Definition 3.3.8), which depends on the nature of (x_0, y_0). Since $\mathscr{V}$ is exactly the tangent space of $\mathscr{C} = \{(gx_0, gy_0) : g \in \mathrm{SO}(3)\}$ at the identity, the dimension of $\mathscr{V}$ can be inferred by examining $\mathscr{C}$. Consider the following situations:

- $x_0 = y_0 = 0$. Then $\mathscr{C} = \{(0,0)\}$ and $\dim \mathscr{V} = 0$.
- $x_0 \neq 0$, $y_0 = \alpha x_0$ for some $\alpha \in \mathbb{R}$. Then $\mathscr{C} = \{(x, \alpha x) : x \in \mathbb{R}^3, |x| = |x_0|\}$, so $\mathscr{C}$ is locally diffeomorphic to the 2-sphere and $\dim \mathscr{V} = 2$.

If neither of the above situations apply, then it follows from the axis-angle representation of 3-dimensional rotations that $\dim \mathscr{V} = 3$.

If we know $\dim \mathscr{V}$, then Assumption 3.4 can be verified by examining the Hessian matrix H of V evaluated at (x_0, y_0). Since H is symmetric and thus orthogonally diagonalizable, it is then necessary and sufficient that the kernel of H coincides with $\mathscr{V}$ and that all other eigenvalues of H are strictly positive (note the minus sign in (3.5.19)).

3.6. PROOF OF NONLINEAR STABILITY

3.6.1. DETERMINISTIC STABILITY

Proof of Theorem 3.4.4. Fix $T > 0$, $\delta > 0$. We define $v(t) := u(t) - \hat{u}(t)$. Considering that $u(t)$ and $\hat{u}(t)$ both solve (3.3.19) (with initial values $u^* + v_0$ and u^*, respectively), we see that

$$v(t) = S(t,0)v_0 + \int_0^t S(t,t')\big(F(\hat{u}(t') + v(t')) - F(\hat{u}(t')) - F'(\hat{u}(t'))v(t')\big)\,\mathrm{d}t'. \qquad (3.6.1)$$

We now claim that there exists an $\varepsilon > 0$ such that

$$\|v_0\|_{\mathscr{X}} \leq \varepsilon \implies \|v(\cdot) - S(\cdot,0)v_0\|_{C([0,T];\mathscr{X})} \leq \tfrac{1}{2}\delta\|v_0\|_{\mathscr{X}}. \qquad (3.6.2)$$

To see this, we define for $v_0 \in \mathscr{X}$ the time

$$t_{v_0} := \sup\big\{t \in [0,T] : \|v\|_{C([0,t];\mathscr{X})} \leq 3M_2\|v_0\|_{\mathscr{X}}\big\}, \qquad (3.6.3)$$

where M_2 is the constant from (3.3.21b). Using Assumption 3.5 (with $R = 3M_2$) and (3.3.21b), (3.6.1), (3.6.3), we find a constant C_1 (independent of v_0) such that we have

$$\|v(t)\|_{\mathscr{X}} \leq M_2\|v_0\|_{\mathscr{X}} + M_2TC_1(3M_2\|v_0\|_{\mathscr{X}})^2, \quad t \leq t_{v_0} \wedge T,$$

whenever $\|v_0\|_{\mathscr{X}} \leq 1$. Choosing ε small enough based on C_1, M_2, T, we get the estimate

$$\|v(t)\|_{\mathscr{X}} \leq 2M_2\|v_0\|_{\mathscr{X}}, \quad t \leq t_{v_0} \wedge T,$$

whenever $\|v_0\|_{\mathscr{X}} \leq \varepsilon$. By (3.6.3) and continuity, this implies $t_{v_0} \geq T$, which leads via (3.6.1) to the further implication that

$$\|v(t) - S(t,0)v_0\|_{\mathscr{X}} \leq M_2TC_1(3M_2\|v_0\|_{\mathscr{X}})^2, \quad t \leq T.$$

After choosing ε even smaller based on δ, M_2, T, C_1, (3.6.2) follows. Since we have the identity

$$u(t) = \hat{u}(t) + S(t,0)v_0 + (v(t) - S(t,0)v_0),$$

we can combine (3.6.2) with Lemma 3.4.3 to find

$$\|u(t) - \Pi(\gamma_t)u^*\|_{\mathscr{X}} \leq M_3e^{-at}\|v_0\|_{\mathscr{X}} + CM_1K_T^2\|v_0\|_{\mathscr{X}}^2 + \tfrac{1}{2}\delta\|v_0\|_{\mathscr{X}}, \quad t \in [0,T],$$

whenever $\|v_0\|_{\mathscr{X}} \leq \varepsilon$. Choosing ε even smaller such that $\varepsilon \leq \delta(2CM_1K_T^2)^{-1}$, we obtain (3.4.5). □

3.6.2. STOCHASTIC STABILITY, SHORT TIMES

This section contains the proof of Theorem 3.4.7. Fix $T>0$ and $\alpha>1$. For $\varepsilon,\sigma>0$, we define the events

$$A_\varepsilon := \{\omega\in\Omega : \|v_0\|_{\mathscr{X}} \le \alpha\varepsilon\}, \tag{3.6.4a}$$

$$B_{\varepsilon,\sigma} := \Big\{\omega\in\Omega : \sup_{t\in[0,T]} \Big\|\sigma\int_0^t S(t,t')G(t',\hat{u}(t'))\,\mathrm{d}W(t')\Big\|_{\mathscr{X}} \le \tfrac14\varepsilon\Big\}. \tag{3.6.4b}$$

We begin by formulating two lemmas relating to these events.

Lemma 3.6.1. *There exists a constant $\varepsilon_1'>0$, independent of v_0, such that the inequality*

$$\|u(t)-\Pi(\gamma_t)u^*\|_{\mathscr{X}} \le (M_3e^{-at}\alpha+\tfrac12)\varepsilon+\|z(t)\|_{\mathscr{X}} \tag{3.6.5}$$

holds for all $t\in[T\wedge\tau^*)$ and $\omega\in A_\varepsilon\cap B_{\varepsilon,\sigma}$, whenever $0<\sigma\le\varepsilon\le\varepsilon_1'$ (z is as in Proposition 3.4.6, and M_3 and a are the constants from (3.3.21c)).

Proof. By Lemma 3.4.3 and (3.4.10)-(3.4.11), we have for $t\in[0,\tau^*)$:

$$\begin{aligned}\|u(t)-\Pi(\gamma_t)u^*\|_{\mathscr{X}} &\le M_3e^{-at}\|v_0\|_{\mathscr{X}}+CM_1K_T^2\|v_0\|_{\mathscr{X}}^2\\ &\qquad+\Big\|\sigma\int_0^t S(t,t')G(t',\hat{u}(t'))\,\mathrm{d}W(t')\Big\|_{\mathscr{X}}+\|z(t)\|_{\mathscr{X}}.\end{aligned}$$

Therefore, for $\omega\in A_\varepsilon\cap B_{\varepsilon,\sigma}$ we have by (3.6.4)

$$\|u(t)-\Pi(\gamma_t)u^*\|_{\mathscr{X}} \le (M_3e^{-at}\alpha+\tfrac14)\varepsilon+CM_1K_T^2\alpha^2\varepsilon^2+\|z(t)\|_{\mathscr{X}},\quad t\in[0,T\wedge\tau^*).$$

The claim now follows by choosing $\varepsilon_1'=(4CM_1K_T^2\alpha^2)^{-1}$. □

Lemma 3.6.2. *There exist constants $\varepsilon_2',c>0$, independent of v_0, such that the inequality*

$$\mathbb{P}\Big[\sup_{t\in[0,T\wedge\tau^*)}\|z(t)\|_{\mathscr{X}}\ge\tfrac12\varepsilon,\,A_\varepsilon,\,B_{\varepsilon,\sigma}\Big]\le 3\exp(-c\sigma^{-2}) \tag{3.6.6}$$

holds for all $0<\sigma\le\varepsilon\le\varepsilon_2'$.

Proof. For $\varepsilon,\sigma>0$ we define

$$\tau_v := \sup\{t\in[0,T] : \|v\|_{C([0,t];\mathscr{X})}\le(\alpha M_2+\tfrac14)\varepsilon\}, \tag{3.6.7a}$$

$$\tau_z := \sup\{t\in[0,T\wedge\tau^*) : \|z\|_{C([0,t];\mathscr{X})}\le\tfrac12\varepsilon\}, \tag{3.6.7b}$$

where M_2 is the constant from (3.3.21b). By continuity and (3.4.11), we see that $\tau_v\wedge\tau_z<\tau^*$, so both $u(\tau_v\wedge\tau_z)$ and $z(\tau_v\wedge\tau_z)$ are well-defined. We also observe that the event $A_\varepsilon\cap B_{\varepsilon,\sigma}$ implies $\tau_v=T$ by (3.4.10) and (3.6.4). Therefore, by continuity of z we have

$$\begin{aligned}\mathbb{P}\Big[\sup_{t\in[0,T\wedge\tau^*)}\|z(t)\|_{\mathscr{X}}\ge\tfrac12\varepsilon,\,A_\varepsilon,\,B_{\varepsilon,\sigma}\Big]&\le\mathbb{P}\Big[\sup_{t\in[0,T\wedge\tau^*)}\|z(t)\|_{\mathscr{X}}\ge\tfrac12\varepsilon,\,\tau_v=T\Big]\\ \le\mathbb{P}\Big[\sup_{t\in[0,T\wedge\tau^*)}\|z(t)\|_{\mathscr{X}}\ge\tfrac12\varepsilon,\,\tau_z=\tau_v\wedge\tau_z<\tau^*\Big]&\le\mathbb{P}\big[\|z(\tau_v\wedge\tau_z)\|_{\mathscr{X}}=\tfrac12\varepsilon\big],\end{aligned}$$

so it suffices to estimate the latter probability.

By the mild solution formula for (3.4.9b), we have for $t \in [0, \tau_v \wedge \tau_z]$:

$$\begin{aligned}
z(t) = &\int_0^t S(t,t')\big(F(u(t')) - F(\hat{u}(t')) - F'(\hat{u}(t'))[u(t') - \hat{u}(t')]\big)\,\mathrm{d}t' \\
&+ \int_0^t S(t,t')\sigma\big(G(t',u(t')) - G(t',\hat{u}(t'))\big)\,\mathrm{d}W(t') \\
&+ \int_0^t S(t,t')\sigma^2 H(t',u(t'))\,\mathrm{d}t' \\
=:&\ T_1(t) + T_2(t) + T_3(t).
\end{aligned} \tag{3.6.8}$$

We now set $C_2 := \alpha M_2 + \frac{3}{4}$. From (3.3.5), (3.3.21a), (3.4.11) and (3.6.7), we see that for $t \in [0, \tau_v \wedge \tau_z]$ we have

$$\begin{aligned}
\|u(t)\|_{\mathscr{X}} &\le M_1 \|u^*\|_{\mathscr{X}} + C_2\varepsilon, \\
\|u(t) - \hat{u}(t)\|_{\mathscr{X}} &\le C_2\varepsilon.
\end{aligned}$$

By Assumptions 3.5 and 3.6 (applied with $R = C_2$ and $R = M_1\|u^*\|_{\mathscr{X}} + C_2$ respectively) and Remark 3.3.7, we see that there exists a constant C_3, independent of ε, σ, v_0, such that the inequalities

$$\|F(u(t)) - F(\hat{u}(t)) - F'(\hat{u}(t))[u(t) - \hat{u}(t)]\|_{\mathscr{X}} \le C_3 C_2^2 \varepsilon^2, \tag{3.6.9a}$$

$$\|G(t,u(t)) - G(t,\hat{u}(t))\|_{\gamma(\mathscr{H};\mathscr{X})} \le C_3 C_2 \varepsilon, \tag{3.6.9b}$$

$$\|H(t,u(t))\|_{\mathscr{X}} \le C_3, \tag{3.6.9c}$$

hold for all $t \in [0, \tau_v \wedge \tau_z]$ if $\varepsilon \le 1$. Therefore, by (3.3.21b) we have the estimates

$$\begin{aligned}
\|T_1(\tau_v \wedge \tau_z)\|_{\mathscr{X}} &\le M_2 T C_3 C_2^2 \varepsilon^2, \\
\|T_3(\tau_v \wedge \tau_z)\|_{\mathscr{X}} &\le M_2 T C_3 \sigma^2.
\end{aligned}$$

Choosing $\varepsilon_2' = \min\{1, (8M_2 T C_3 C_2^2)^{-1}\}$ and using (3.6.8), this gives (note that $C_2 \ge 1$)

$$\|z(\tau_v \wedge \tau_z)\|_{\mathscr{X}} \le \tfrac{1}{4}\varepsilon + \|T_2(\tau_v \wedge \tau_z)\|_{\mathscr{X}},$$

whenever $0 < \sigma \le \varepsilon \le \varepsilon_2'$. Thus, we have

$$\mathbb{P}\left[\|z(\tau_v \wedge \tau_z)\|_{\mathscr{X}} = \tfrac{1}{2}\varepsilon\right] \le \mathbb{P}\left[\|T_2(\tau_v \wedge \tau_z)\|_{\mathscr{X}} \ge \tfrac{1}{4}\varepsilon\right],$$

so it only remains to estimate this probability. Now observe that

$$\begin{aligned}
\|T_2(\tau_v \wedge \tau_z)\|_{\mathscr{X}} &\le \sup_{t \in [0,\tau_v \wedge \tau_z]} \left\| \int_0^t S(t,t')\sigma\big(G(t',u(t')) - G(t',\hat{u}(t'))\big)\,\mathrm{d}W(t') \right\|_{\mathscr{X}} \\
&\le \sup_{t \in [0,T]} \left\| \int_0^t S(t,t')\mathbb{1}_{[0,\tau_v \wedge \tau_z]}(t')\sigma\big(G(t',u(t')) - G(t',\hat{u}(t'))\big)\,\mathrm{d}W(t') \right\|_{\mathscr{X}}.
\end{aligned}$$

From (3.6.9b), we also have

$$\left\| \mathbb{1}_{[0,\tau_v \wedge \tau_z]}(\cdot)\sigma\big(G(\cdot,u(\cdot)) - G(\cdot,\hat{u}(\cdot))\big) \right\|_{L^\infty(0,T;\gamma(\mathscr{H};\mathscr{X}))} \le C_3 C_2 \sigma\varepsilon.$$

Thus, by Proposition 3.2.2, there is a constant $c > 0$, depending only on $\mathscr{X}$, S, such that

$$\mathbb{P}\left[\|T_2(\tau_v \wedge \tau_z)\|_{\mathscr{X}} \ge \tfrac{1}{4}\varepsilon\right] \le 3\exp\big(-cC_3^{-2}C_2^{-2}T^{-1}\sigma^{-2}\big). \qquad \square$$

Proof of Theorem 3.4.7. For $\varepsilon, \sigma > 0$, we introduce the additional event

$$E_{\varepsilon,\sigma} := \{\omega \in \Omega : \sup_{t\in[0,T\wedge\tau^*)} \|u(t) - \Pi(\gamma_t)u^*\|_{\mathscr{X}} - M_3\alpha e^{-at}\varepsilon \geq \varepsilon\}.$$

Let ε_1' be the constant obtained from Lemma 3.6.1, and let c_2, ε_2' be the constants obtained from Lemma 3.6.2. Set $\varepsilon' = \varepsilon_1' \wedge \varepsilon_2'$. By Lemmas 3.6.1 and 3.6.2, we have

$$\begin{aligned} \mathbb{P}[E_{\varepsilon,\sigma} \cap A_\varepsilon \cap B_{\varepsilon,\sigma}] &\overset{(3.6.5)}{\leq} \mathbb{P}\Big[\sup_{t\in[T\wedge\tau^*)} \|z(t)\|_{\mathscr{X}} \geq \tfrac{1}{2}\varepsilon,\, A_\varepsilon,\, B_{\varepsilon,\sigma}\Big] \\ &\overset{(3.6.6)}{\leq} 3\exp(-c_2\sigma^{-2}) \end{aligned} \tag{3.6.10a}$$

for $0 < \sigma \leq \varepsilon \leq \varepsilon'$. Similarly to how we treated T_2 in Lemma 3.6.2, we find using Proposition 3.2.2 a constant $c_1 > 0$ such that

$$\mathbb{P}[\Omega \setminus B_{\varepsilon,\sigma}] \leq 3\exp(-c_1\sigma^{-2}\varepsilon^2), \quad 0 < \sigma \leq \varepsilon \leq \varepsilon'. \tag{3.6.10b}$$

Thus, by a union bound and (3.6.10), we have

$$\mathbb{P}[E_{\varepsilon,\sigma} \cap A_\varepsilon] \leq \mathbb{P}[E_{\varepsilon,\sigma} \cap A_\varepsilon \cap B_{\varepsilon,\sigma}] + \mathbb{P}[\Omega \setminus B_{\varepsilon,\sigma}] \leq 3\exp(-c_1\sigma^{-2}\varepsilon^2) + 3\exp(-c_2\sigma^{-2}),$$

for all $0 < \sigma \leq \varepsilon \leq \varepsilon'$. The desired estimate (3.4.12) then follows by taking $c = c_1 \wedge c_2$. □

3.6.3. STOCHASTIC STABILITY, LONG TIMES

We now prove Theorem 3.4.8. First, we show a lemma which allows us to 'reset' the phase in Theorem 3.4.7 by performing a coordinate transformation.

Lemma 3.6.3. *Set $v_0 \equiv 0$. Let $T > 0$, and let γ_T be a G-valued $\mathscr{F}_T$-measurable random variable. There exist constants $c, \varepsilon', \widetilde{T}$, independent of T, γ_T, and an $\mathscr{F}_T$-measurable function $\gamma_{\widetilde{T}}$, such that we have the estimate*

$$\mathbb{P}\big[\|u(T+\widetilde{T}) - \Pi(\gamma_T\gamma_{\widetilde{T}})u^*\|_{\mathscr{X}} \geq \varepsilon,\, \|u(T) - \Pi(\gamma_T)u^*\|_{\mathscr{X}} \leq \varepsilon\big] \leq 6\exp(-c\varepsilon^2\sigma^{-2}),$$

for all ε, σ satisfying $0 < \sigma \leq \varepsilon \leq \varepsilon'$.

Proof. Fix $T > 0$, and set $\widetilde{T} = a^{-1}\log(2M_1^2M_3)$, where a, M_1, and M_3 are as in (3.3.21). We use the random transformation $\tilde{u}(t) = \Pi(\gamma_T^{-1})u(T+t)$. From Assumption 3.1 we find that $\tilde{u}$ solves

$$\begin{aligned} \mathrm{d}\tilde{u}(t) &= [A\tilde{u}(t) + F(\tilde{u}(t))]\,\mathrm{d}t + \sigma^2\widetilde{H}(t,\tilde{u}(t))\,\mathrm{d}t + \sigma\widetilde{G}(t,\tilde{u}(t))\,\mathrm{d}\widetilde{W}(t) \\ \tilde{u}(0) &= u^* + \tilde{v}_0, \end{aligned}$$

where we have introduced

$$\begin{aligned} \widetilde{H}(t,f) &= \Pi(\gamma_T^{-1})H(T+t,\Pi(\gamma_T)f), \\ \widetilde{G}(t,f) &= \Pi(\gamma_T^{-1})G(T+t,\Pi(\gamma_T)f), \\ \widetilde{W}(t) &= W(T+t), \\ \tilde{v}_0 &= \Pi(\gamma_T^{-1})u(T) - u^*. \end{aligned}$$

It can now be seen that Assumption 3.6 still holds when G, H are replaced by $\widetilde{G}, \widetilde{H}$ (possibly with worse constants). By choosing $\alpha = 2M_1^2$, and $T = \widetilde{T}$ in Theorem 3.4.7, we thus find $c, \varepsilon' > 0$, and $\gamma_{\widetilde{T}}$ such that the estimate

$$\mathbb{P}\left[\|\tilde{u}(\widetilde{T}) - \Pi(\gamma_{\widetilde{T}})u^*\|_{\mathscr{X}} \geq 2\varepsilon, \|\tilde{v}_0\|_{\mathscr{X}} \leq 2M_1^2\varepsilon\right] \leq 6\exp(-c\varepsilon^2\sigma^{-2}) \tag{3.6.11}$$

holds for all $0 < \sigma \leq \varepsilon \leq \varepsilon'$. Note that in Theorem 3.4.7, $\gamma_{\widetilde{T}}$ is obtained purely from $\tilde{v}_0$ (via (3.4.3)), so that $\gamma_{\widetilde{T}}$ is $\mathscr{F}_T$-measurable. Using (3.3.21a), we also find

$$\begin{aligned}\|\tilde{v}_0\|_{\mathscr{X}} &= \|\Pi(\gamma_T^{-1})(u(T) - \Pi(\gamma_T)u^*)\|_{\mathscr{X}} \\ &\leq M_1\|u(T) - \Pi(\gamma_T)u^*\|_{\mathscr{X}},\end{aligned}$$

and

$$\begin{aligned}\|u(T+\widetilde{T}) - \Pi(\gamma_T\gamma_{\widetilde{T}})u^*\|_{\mathscr{X}} &= \|\Pi(\gamma_T)(\tilde{u}(\widetilde{T}) - \Pi(\gamma_{\widetilde{T}})u^*)\|_{\mathscr{X}} \\ &\leq M_1\|\tilde{u}(\widetilde{T}) - \Pi(\gamma_{\widetilde{T}})u^*\|_{\mathscr{X}}.\end{aligned}$$

Combining this gives

$$\begin{aligned}\mathbb{P}&\left[\|u(T+\widetilde{T}) - \Pi(\gamma_T\gamma_{\widetilde{T}})u^*\|_{\mathscr{X}} \geq 2M_1\varepsilon, \|u(T) - \Pi(\gamma_T)\|_{\mathscr{X}} \leq 2M_1\varepsilon\right] \\ &\leq \mathbb{P}\left[\|\tilde{u}(\widetilde{T}) - \Pi(\gamma_{\widetilde{T}})u^*\|_{\mathscr{X}} \geq 2\varepsilon, \|\tilde{v}_0\|_{\mathscr{X}} \leq 2M_1^2\varepsilon\right] \\ &\overset{(3.6.11)}{\leq} 6\exp(-c\varepsilon^2\sigma^{-2}),\end{aligned}$$

for every $0 < \sigma \leq \varepsilon \leq \varepsilon'$. The result follows by rescaling ε', c based on M_1. □

Proof of Theorem 3.4.8. Let $c, \varepsilon', \widetilde{T}$ be as in Lemma 3.6.3. Using Lemma 3.6.3, we inductively find a sequence γ_n of $\mathscr{F}_{n\widetilde{T}}$-measurable G-valued random variables such that

$$\mathbb{P}\left[\|u((n+1)\widetilde{T}) - \Pi(\gamma_{n+1})u^*\|_{\mathscr{X}} \geq \varepsilon, \|u(n\widetilde{T}) - \Pi(\gamma_n)u^*\|_{\mathscr{X}} \leq \varepsilon\right] \leq 6\exp(-c\varepsilon^2\sigma^{-2}),$$

for all $0 < \sigma \leq \varepsilon \leq \varepsilon'$ and all $n \in \mathbb{N}_0$. Iterating this estimate and using a union bound, we find

$$\mathbb{P}\left[\max_{k\leq n}\|u(k\widetilde{T}) - \Pi(\gamma_k)u^*\|_{\mathscr{X}} \geq \varepsilon\right] \leq 6n\exp(-c\varepsilon^2\sigma^{-2}).$$

Applying Theorem 3.4.7 on the intermediate intervals $[n\widetilde{T}, (n+1)\widetilde{T}]$ then gives the result. □

3.A. RADONIFYING OPERATORS

Throughout this section, we write L^p as a shorthand for $L^p(\mathbb{R};\mathbb{R})$, and likewise for $H^{s,p}$ and $W^{k,p}$. For measurable $u, \phi, f : \mathbb{R} \to \mathbb{R}$, we define the trilinear operator G:

$$G(u,\phi,f)(x) = u(x)\int_{\mathbb{R}} \phi(x-y)f(y)\,\mathrm{d}y. \tag{3.A.1}$$

Lemma 3.A.1. *Let $p \in [1,\infty)$. There exists a constant C such that the estimate*

$$\|G(u,\phi,\cdot)\|_{\gamma(L^2;L^p)} \leq C\|u\|_{L^p}\|\phi\|_{L^2}, \tag{3.A.2}$$

holds for all $u \in L^p$, $\phi \in L^2$.

Proof. Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of L^2. By the γ-Fubini isomorphism (see [119, Theorem 9.4.8]) and Parseval's identity, we have

$$\begin{aligned}
\|G(u,\phi,\cdot)\|^p_{\gamma(L^2;L^p)} &\le C\int_{\mathbb{R}}\Big(\sum_{k\in\mathbb{N}}|G(u,\phi,e_k)(x)|^2\Big)^{\frac{p}{2}}\,\mathrm{d}x\\
&= C\int_{\mathbb{R}}\Big(\sum_{k\in\mathbb{N}}|u(x)\int_{\mathbb{R}}\phi(x-y)e_k(y)\,\mathrm{d}y|^2\Big)^{\frac{p}{2}}\,\mathrm{d}x\\
&= C\int_{\mathbb{R}}|u(x)|^p\|\phi(x-\cdot)\|^p_{L^2}\,\mathrm{d}x\\
&= C\|u\|^p_{L^p}\|\phi\|^p_{L^2}.
\end{aligned}$$
□

Proposition 3.A.2. *Let $p\in(1,\infty)$, $s\in[0,\infty)$. There exists a constant C such that the estimate*

$$\|G(u,\phi,\cdot)\|_{\gamma(L^2;H^{s,p})}\le C\|u\|_{H^{s,p}}\|\phi\|_{H^s}, \tag{3.A.3}$$

holds for all $u\in H^{s,p}$, $\phi\in H^s$.

Proof. The case $s=0$ is just (3.A.2). By differentiating (3.A.1), we find the identity

$$\partial_x G(u,\phi,f)=G(\partial_x u,\phi,f)+G(u,\partial_x\phi,f).$$

It is well-know that for $p\in(1,\infty)$, the Bessel space $H^{k,p}$ coincides with the classical Sobolev space $W^{k,p}$ for $k\in\mathbb{N}_0$, see [211, Chapter 3]. Therefore, (3.A.3) with $s=k+1$ follows from the case $s=k$ for every $k\in\mathbb{N}$. The case $s\in(0,\infty)\setminus\mathbb{N}$ follows by complex bilinear interpolation [25, Theorem 4.4.1]. Here, we use [119, Theorem 9.1.25] to interpolate between the spaces $\gamma(L^2;H^{k,p})$ for different values of k. □

4

SYNCHRONIZATION BY NOISE

This chapter is based on the article

[A1] C. Kuehn and J. van Winden. "Synchronization by noise for traveling pulses". In: *The Annals of Applied Probability* (2026). Forthcoming.

Abstract. *We consider synchronization by noise for stochastic partial differential equations which support traveling pulse solutions, such as the FitzHugh–Nagumo equation. We show that any two pulse-like solutions which start from different positions but are forced by the same realization of a multiplicative noise with amplitude* σ*, converge to each other in probability on a time scale* $\sigma^{-2} \ll t \ll \exp(\sigma^{-2})$. *The noise is assumed to be Gaussian, white in time, colored and periodic in space, and nondegenerate only in the lowest Fourier mode. The proof uses the method of phase reduction, which allows one to describe the dynamics of the stochastic pulse only in terms of its position. The position is shown to synchronize building upon existing results, and the validity of the phase reduction allows us to transfer the synchronization back to the full solution.*

4.1. INTRODUCTION

In this chapter, we study SPDEs with traveling pulse-like solutions, such as the stochastic FitzHugh–Nagumo equation:

$$\begin{aligned} \partial_t u &= \partial_{xx} u + u(u-a)(1-u) - v + \sigma g(u,v) \circ \partial_t W, \\ \partial_t v &= \epsilon(u - \gamma v), \end{aligned} \tag{4.1.1}$$

where $0 < a < 1/2$, $\epsilon \ll 1$, W is a suitable Gaussian noise and $g\colon \mathbb{R}^2 \to \mathbb{R}$ is smooth. The parameter $\sigma > 0$ controls the amplitude of the noise, and we are especially interested in the small noise regime $\sigma \ll 1$. Being a prototypical model for neural pulse propagation, it is well known that (4.1.1) with $\sigma = 0$ admits a stable traveling pulse solution, moving with a fixed speed $c > 0$; see e.g. [60, 106, 124]. In particular, the position of the pulse at time t_0 can be retroactively determined from the position at any later time t_1 simply by subtracting $c(t_1 - t_0)$.

In the presence of noise ($\sigma > 0$), the situation is significantly different. Although pulse-like solutions persist [74, 113], the noise causes a random shift of the pulse position, and in many cases this results in a change in velocity [113] (see also [112, 174, 227] for results on other equations). Moreover, in sharp contrast to the deterministic case, it has been observed that the pulse position at large times is nearly independent of the initial position [136, 181, 193, 216, 224]. Instead, the large-time position is mainly determined by stochastic forcing, a phenomenon referred to as *synchronization by noise*. However, despite strong evidence for synchronization of pulses, no mathematical proof has been given until now.

Our main result (Theorem 4.1) establishes the first rigorous proof of synchronization by noise for traveling pulse solutions to (4.1.1). Specifically, we prove that any two pulse-like solutions, starting at different positions and forced by the same noise, converge to each other in probability for $t \gg \sigma^{-2}$ as $\sigma \searrow 0$. We mention already that this theoretical synchronization result can also have concrete implications for biophysical systems such as memory processes in neuroscience [85] and cardiac arrhythmia [160]. In both applications, the FitzHugh–Nagumo equation (4.1.1) is the baseline model problem studied. We shall return to applied aspects in Section 4.5 as the proof of our main result is also very insightful from a practical perspective.

4.1.1. MAIN RESULT

Our result is formulated for an abstract semilinear PDE of the form

$$\mathrm{d}u = Au\,\mathrm{d}t + f(u)\,\mathrm{d}t, \tag{4.1.2}$$

where $u\colon \mathbb{R}^+ \times \mathbb{R} \ni (t,x) \mapsto u(t,x) \in \mathbb{R}^n$ is continuous, $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth, and A should be thought of as a (possibly degenerate) differential operator with constant coefficients.

We briefly describe our assumptions, which are formulated in more detail in Sections 4.2.2-4.2.3. The first two assumptions allow for a robust solution theory (Assumption 4.1) and ensure the existence of a *pulse profile* u^* and *pulse speed* $c \in \mathbb{R}$ such that $\hat{u}(t,x) := u^*(x - ct)$ is an orbitally stable traveling pulse solution to (4.1.2) (Assumption 4.2). It is well known that the FitzHugh–Nagumo equation (4.1.1) (with $\sigma = 0$) fits into this setting. For the stochastic equation, we introduce the noise amplitude $\sigma > 0$ and write

$$\mathrm{d}u_\sigma = Au_\sigma\,\mathrm{d}t + f(u_\sigma)\,\mathrm{d}t + \sigma g(u_\sigma) \circ \mathrm{d}W(t). \tag{4.1.3}$$

We take $g\colon \mathbb{R}^n \to \mathbb{R}^n$ sufficiently smooth and let $W(t,x)$ be a scalar[1] Gaussian noise which is white in time, colored and periodic in space, and weakly nondegenerate (Assumption 4.3).

The nondegeneracy condition on W is formulated in terms of g, u^*, the lowest Fourier modes of W, and an adjoint eigenfunction ψ which can be calculated from (4.1.2) and the pulse profile u^*. The symbol '$\circ$' in (4.1.3) indicates that the stochastic integral is interpreted in the Stratonovich sense, but this does not play a major role in the proof.

Assumptions 4.1, 4.2, and 4.3 taken together already suffice to show synchronization for the reduced SDE that describes the pulse position. However, we are only able to transfer this synchronization back to the 'full' equation (4.1.3) by additionally assuming that *either* the pulse speed c is zero, *or* that the noise W has spatially homogeneous statistics (Assumption 4.4). We refer ahead to Section 4.5.2 for further discussion of this assumption.

Let us now state the main result, which holds under Assumptions 4.1, 4.2, 4.3 and 4.4. For $x \in \mathbb{R}$ and $\sigma > 0$, we let $u_\sigma^x(\cdot)$ denote the (mild) solution of (4.1.3) with initial condition $u_\sigma^x(0,\cdot) = u^*(\cdot - x)$.

Theorem 4.1. *Suppose that the times $(t_\sigma)_{\sigma>0}$ satisfy*

$$\lim_{\sigma\to 0} t_\sigma \sigma^2 = \infty, \qquad 0 \le t_\sigma \le \exp(\sigma^{-2+q}) \tag{4.1.4}$$

for some $q \in (0,2)$. Then for any $x, y \in \mathbb{R}$, we have

$$\inf_{n\in\mathbb{Z}} \left(\| u_\sigma^x(t_\sigma, \cdot + n) - u_\sigma^y(t_\sigma,\cdot)\|_{\mathscr{X}} \right) \xrightarrow{\mathbb{P}} 0 \tag{4.1.5}$$

as $\sigma \to 0$.

In other words, *any* two solutions to (4.1.3) which start in the set of translates $\{u^*(\cdot - x) : x \in \mathbb{R}\}$ synchronize (modulo integer translations) on the time scale $\sigma^{-2} \ll t_\sigma \ll \exp(\sigma^{-2})$.

Remark 4.1.1. The choice to use initial conditions which are exact translates of u^* does not play an important role in the proof (for $t > 0$, the solution is already no longer a translate of u^* due to the noise). In fact, due to the exponential attraction towards the center manifold which consists of translates of u^*, Theorem 4.1 also holds for general initial conditions which are inside the basin of attraction.

Remark 4.1.2. The function space $\mathscr{X}$ is defined in Assumption 4.1, and always embeds into the space of continuous functions by assumption.

Remark 4.1.3. The infimum over n in (4.1.5) cannot be removed, since the periodic noise guarantees that $u_\sigma^{x+n}(t, z+n) = u_\sigma^x(t,z)$ always holds for $n \in \mathbb{N}$. Our results can be straightforwardly adapted to traveling pulses on periodic domains, in which case the infimum can be omitted. However, for the sake of readability and in view of applications we have chosen the setting where (4.1.3) is posed on the real line.

[1]There is no obstacle to considering vector-valued noise except for significant notational inconvenience.

4.1.2. SYNCHRONIZATION BY NOISE

There has been a continued mathematical interest in synchronization by noise, starting with pioneering results on stochastic flows of diffeomorphisms by Baxendale [21] and Martinelli and Scoppola [166], and continuing with seminal works by Arnold [15] and Crauel and Flandoli [62]. More recently, synchronization for SPDEs has been considered, with a particular focus on the Chafee–Infante (or Allen–Cahn) equation [30, 32, 50]. Abstract criteria for synchronization have been given in [90, 91, 188], which has led to synchronization results for porous media equations [91] and gradient type S(P)DEs [90, 96], all with additive and highly nondegenerate noise.

A common thread in these works is that there are two main strategies for showing synchronization:[2]

- Exploit an *order-preserving* structure [16, 50, 56, 62, 91, 166].
- Combine *asymptotic stability* and *irreducibility* properties [22, 90, 96, 188].

Currently, it seems that showing synchronization for (4.1.1) by treating the equation directly is totally out of reach with both methods. Firstly, since (4.1.1) is a two-component SPDE, we deem it unlikely that a useful order-preserving structure exists. Secondly, showing asymptotic stability generally requires one to verify negativity of the top Lyapunov exponent. Even with detailed information of the invariant measure (which is unavailable for (4.1.1)) this is a challenging task, see e.g. [96]. Thirdly, to have sufficient irreducibility properties, it is typically needed to impose strict nondegeneracy conditions on the noise. In contrast, we require nondegeneracy only in the lowest Fourier mode (see Assumption 4.3), a setting in which mixing (typically a necessary condition for synchronization) is already hard to prove [111].

We completely bypass these difficulties using the method of *phase reduction*. The key insight is that the dynamics of a pulse-like solution can be accurately described by tracking only the position of the pulse, and forgetting all other information. To show that the pulse synchronizes, it thus suffices to show that the position synchronizes.

In the physics literature, the possibility of showing synchronization by noise through phase reduction has been extensively discussed, especially for limit-cycle oscillators [177, 178, 179, 180, 191, 192, 194, 216, 217, 224]. However, the (temporal) roughness of the noise turns the phase reduction into a delicate procedure, which is seen in [234] and in Remark 4.4.1 ahead. This further motivates the importance of a rigorous proof, since subtle errors in the phase reduction can immediately invalidate a synchronization result.

4.1.3. PHASE REDUCTION

Consider a solution u_σ to (4.1.3), which has the deterministically stable pulse profile u^* as its initial condition. When $\sigma \ll 1$ the solution $u_\sigma(t)$ will still resemble a translate of u^* with high probability, as long as $t \ll \exp(\sigma^{-2})$. Thus, it is possible to associate to u_σ an auxiliary scalar process γ_σ, henceforth referred to as the *phase*, such that we have

$$u_\sigma(t,x) = u^*(x-\gamma_\sigma(t)) + \mathcal{O}(\sigma). \tag{4.1.6}$$

[2] A notable exception is [30], which instead relies on the noise being constant in space.

The infinite-dimensional dynamics of u_σ then effectively reduce to the one-dimensional dynamics of γ_σ, and this will be our main tool to show synchronization.

However, the condition (4.1.6) only characterizes γ_σ up to $\mathcal{O}(\sigma)$, and therefore does not lead to a canonical definition of the phase process. This is reflected in the recent literature on stochastic traveling waves/pulses, in which a variety of (nonequivalent) phase tracking methods, each with their own strengths and weaknesses, have been proposed and developed [1, 112, 121, 148] (see also Chapter 3). We will not give an overview of the different methods, but we emphasize that whichever method is preferable typically depends on the result one is trying to prove.

The phase tracking method of choice in this chapter is the *isochronal phase*, which originates in the context of nonlinear oscillators [230]. It was put on a mathematical footing by Guckenheimer [105] and has been applied to the study of transient patterns in [1, 4]. Briefly, the idea is as follows. Fix a profile $v(x)$, and let $u(t,x)$ be the solution to (4.1.2) with initial condition $u(0,x) = v(x)$. Assuming that the profile $v(\cdot)$ resembles a translate of the pulse profile u^*, it follows from the deterministic stability theory that there exists $\pi(v) \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \| u(t,\cdot) - u^*(\cdot - ct - \pi(v)) \| = 0.$$

Moreover, $\pi(v)$ is uniquely determined by v (justifying the notation), and the approximation $u(t) \approx u^*(\cdot - \pi(u(t)))$ is valid for all t. The map π is referred to as the *isochron map*. Note that π is constructed from the dynamics of (4.1.2), and thus is a purely deterministic mapping from the state space of (4.1.2) into $\mathbb{R}$.

To track the position of the stochastic pulse u_σ, we may now define the *isochronal phase*[3] process simply by setting $\gamma_\sigma := \pi(u_\sigma)$. To obtain a detailed description of the phase, we apply the Itô formula (which is shown to hold in [1]) to $\pi(u_\sigma)$ and use (4.1.6) to simplify the resulting expression. This results in an (approximate) SDE for γ_σ which no longer makes any reference to u_σ. In view of (4.1.6) the dynamics of u_σ are then characterized by this SDE, and the phase reduction is complete. We refer ahead to Section 4.3.3 for the full derivation of the phase-reduced SDE, which is given by (4.3.12).

4.1.4. PROOF STRATEGY

Phase reduction can be a powerful tool to show synchronization. We demonstrate this by means of a (surprisingly robust) analogy with the well-studied double-well SDE, given by

$$\mathrm{d}X = (X - |X|^2 X)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \qquad X \in \mathbb{R}^2, \tag{4.1.7}$$

where $W = (\beta_1, \beta_2)$ is a two-dimensional Brownian motion and $\sigma > 0$. We first observe that the set $S = \{X : |X| = 1\}$ forms a connected set of stable equilibria for the deterministic equation. Thus, it is expected that the radial component of a typical trajectory of (4.1.7) will satisfy $|X| = 1 + \mathcal{O}(\sigma)$. On the other hand, the angular component of X will diffusively wander on a one-dimensional torus. Indeed, by transforming to polar coordinates via $(X_1, X_2) = (R\cos(\Phi), R\sin(\Phi))$, we find [222]:

$$\mathrm{d}\Phi = \sigma R^{-1}(-\sin(\Phi)\,\mathrm{d}\beta_1(t) + \cos(\Phi)\,\mathrm{d}\beta_2(t)).$$

[3] This notion of isochronicity differs from the one introduced in [77], where a (time-dependent and anticipating) random isochron map is constructed directly from the stochastic dynamics.

Since we expect that $R \approx 1$ when σ is small, we can safely neglect the dynamics of R to reduce the dynamics of X to those of the angular coordinate Φ, which satisfies

$$\mathrm{d}\Phi \approx \sigma(-\sin(\Phi)\,\mathrm{d}\beta_1(t) + \cos(\Phi)\,\mathrm{d}\beta_2(t)). \tag{4.1.8}$$

This SDE is known to synchronize [22], and by rescaling time we see that the synchronization occurs on the time scale $t \sim \sigma^{-2}$. Moreover, one can verify that (4.1.8) is valid on the time scale $t \sim \sigma^{-2}\log(\sigma^{-1})$. Hence, it is expected that we can transfer the synchronization of (4.1.8) back to (4.1.7) to conclude that X synchronizes for $t \gg \sigma^{-2}$ in the limit $\sigma \searrow 0$. We remark that it is crucial that the reduced dynamics accurately capture the full dynamics on a time scale which is *strictly longer* than the typical time until synchronization. Already for (4.1.7) this is delicate, as the two time scales differ only by a logarithmic factor.

The argument sketched above consists of three main steps.

(i) Reduce the dynamics by describing (4.1.7) in terms of (4.1.8).

(ii) Show synchronization of the reduced dynamics.

(iii) Transfer synchronization back to the full dynamics.

In our actual proof, (i) is achieved using the isochronal phase as outlined in Section 4.1.3. Although the resulting SDE is not as explicit as (4.1.8), we still obtain a detailed description of the coefficients by exploiting symmetries of the isochron map π which are inherited from (4.1.2). This allows us to carry out (ii) by combining abstract results by Flandoli, Gess, and Scheutzow [90] with a calculation of the Lyapunov exponent and a control-theoretic argument. The time scales of approximation and synchronization are the same as for (4.1.8), which finally allows (iii) to succeed.

4.1.5. OUTLINE

In Section 4.2 we provide notation, specify our setting and assumptions, and formulate preliminaries regarding random dynamical systems. In Section 4.3 we derive the approximate SDE describing the pulse position and prove error estimates. We simplify and analyze the phase-reduced SDE in Section 4.4. We show that the reduced SDE synchronizes sufficiently fast, which allows us to transfer the synchronization behavior back to the full SPDE to prove Theorem 4.1.

4.2. SETTING, ASSUMPTIONS AND PRELIMINARIES

4.2.1. NOTATION

We use the convention that $\mathbb{N}$ does not include zero, and write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, as well as $\mathbb{R}^+ = [0, \infty)$. We write $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ to denote the (flat) one-dimensional torus with unit length.

We write $\|\cdot\|_{\mathscr{X}}$ for the norm of a general Banach space $\mathscr{X}$, and we write $I_{\mathscr{X}}$ for the identity function on $\mathscr{X}$. When (E,d) is a metric space and $x \in E$, $r > 0$, we write $B_r(x)$ for the open ball around x of radius r. We write $C_b(E)$ for the Banach space of bounded continuous functions $f : E \to \mathbb{R}$, equipped with the supremum norm. When $k \in \mathbb{N}$ and M is either $\mathbb{R}$, $\mathbb{T}$, or an open subset of a Banach space, we write $C^k(M; \mathscr{X})$ for the Banach space of functions $f\colon M\to\mathscr{X}$ which have k bounded continuous (Fréchet) derivatives, equipped with the usual norm. In the case $\mathscr{X}=\mathbb{R}$, we simply write $C^k(M)$. We also write $\ell^2(\mathbb{Z};\mathscr{X})$ for the space of (norm) square-integrable functions $f\colon\mathbb{Z}\to\mathscr{X}$. If $\mathscr{X}=\mathbb{R}$, we write $\ell^2(\mathbb{Z})$ or even just ℓ^2. We write $H^{s,p}(\mathbb{R};\mathbb{R}^n)$ for the Bessel potential space of functions from $\mathbb{R}$ to $\mathbb{R}^n$ with smoothness s and integrability $p\in(1,\infty)$. We write $\mathbb{E}[\cdot]$ for the expectation associated with the probability space $(\Omega,\mathscr{F},\mathbb{P})$ which is specified in Section 4.2.4. The notation $X\xrightarrow{\mathbb{P}}Y$ indicates convergence in probability with respect to $\mathbb{P}$.

We use the notation $X\lesssim Y$ to mean that there exists a constant C, possibly depending on the objects introduced in Assumptions 4.1, 4.2, 4.3, such that $X\le CY$. In the statements and proofs of Theorems 4.3.8 and 4.3.9, we will additionally use the notation $\sigma\ll_q 1$ to mean that σ is sufficiently small, based (only) on q and the objects from Assumptions 4.1, 4.2, 4.3.

4.2.2. ANALYTIC SETTING AND LINEAR STABILITY

We begin by fixing the state space for (4.1.3) by letting $p\in[2,\infty)$, $s>1/p$, $n\in\mathbb{N}$ and setting $\mathscr{X}:=H^{s,p}(\mathbb{R};\mathbb{R}^n)$. The process u is intended to be continuous in time with values in $\mathscr{X}$.

Remark 4.2.1. The condition $p\in[2,\infty)$ is used in the proof of Theorem 4.3.8, which requires Gaussian tail estimates for $\mathscr{X}$-valued stochastic integrals. Such tail estimates are available if $\mathscr{X}$ is 2-smooth [207], a property which holds for Bessel spaces iff $p\in[2,\infty)$ (this follows e.g. from [195, Proposition 2.1] and the fact that L^p is isometrically isomorphic to $H^{s,p}$). The condition $s>1/p$ is necessary for $\mathscr{X}$ to be a Banach algebra, a fact which is used at several points in the argument. By the Sobolev embedding, this restricts us to work exclusively with continuous functions.

To have a robust solution theory for (4.1.2), we make the following assumption:

Assumption 4.1 (PDE setting). *The following conditions hold:*

- *A generates a C_0-semigroup on $\mathscr{X}$ which commutes with translations.*
- *The Nemitskii map $u\mapsto f(u)$ is four times Fréchet differentiable from $\mathscr{X}$ to $\mathscr{X}$.*

The next assumption guarantees existence of a stable traveling pulse solution to (4.1.2).

Assumption 4.2 (Stable traveling pulse). *There exist $c\in\mathbb{R}$ and $u^*\in\mathscr{X}$ such that $u(t,x)=u^*(x-ct)$ solves (4.1.2) (in the mild sense) and $\partial_x^{(n)}u^*\in\mathscr{X}$ for $n\in\{1,2,3,4\}$. Moreover, the linear operator $\mathscr{L}=A+c\partial_x+f'(u^*)$ generates a C_0-semigroup $(P(t))_{t\ge0}$ on $\mathscr{X}$, and there exist projections Π^c,Π^s on $\mathscr{X}$ and $C,a>0$ such that the following hold:*

- $I_{\mathscr{X}}=\Pi^c+\Pi^s$.
- $\Pi^c\mathscr{X}=\operatorname{span}\{\partial_x u^*\}$.
- $\|P(t)\Pi^s f\|_{\mathscr{X}}\le Ce^{-at}\|f\|_{\mathscr{X}},\quad t\ge0,\ f\in\mathscr{X}$.

Remark 4.2.2. Since $\partial_x u^*\in\mathscr{X}$, it follows that we have the estimate

$$\|u^*(\cdot-x)-u^*(\cdot-y)\|_{\mathscr{X}}\lesssim|x-y|,\qquad x,y\in\mathbb{R}.$$

It follows from Assumption 4.2 that there exists a functional $\psi\colon \mathcal{X} \to \mathbb{R}$ such that

$$\Pi^c f = -\psi(f)\partial_x u^* =: -\langle \psi, f\rangle \partial_x u^*, \qquad f \in \mathcal{X}, \tag{4.2.1}$$

where the brackets denote the duality pairing. We prefer to write our equations in terms of ψ rather than Π^c, and will use the right-hand side of (4.2.1) whenever possible. In many cases it holds that ψ is a smooth function and satisfies $\mathcal{L}^*\psi = 0$, where $\mathcal{L}^*$ is the formal L^2-adjoint of $\mathcal{L}$ (see e.g. [112, (9.18)]).

4.2.3. NOISE

In order to have suitable ergodic properties of the phase-reduced SDE, we restrict our noise to be periodic in space. By rescaling the spatial variable, we may additionally assume that the period is 1 without loss of generality. This motivates the expansion of the noise in terms of the orthonormal basis $(e_k)_{k\in\mathbb{Z}}$ of $L^2(\mathbb{T})$ given by

$$e_k(x) = \begin{cases} \sqrt{2}\cos(2\pi kx), & k > 0, \\ 1, & k = 0, \\ \sqrt{2}\sin(2\pi kx), & k < 0. \end{cases} \tag{4.2.2}$$

We thus let $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ be a sequence of coefficients and let the noise be given by:

$$W(t,x) = \sum_{k\in\mathbb{Z}} \alpha_k e_k(x)\beta_k(t), \tag{4.2.3}$$

where $(\beta_k)_{k\in\mathbb{Z}}$ is a sequence of independent Brownian motions (see Section 4.2.4 for the structure of our probability space). Even though (4.2.3) is not the most general noise possible, it is a flexible formulation which is commonplace in the literature on SPDEs (see e.g. [87, 93, 111]). Furthermore, it has the benefit of allowing us to formulate the coming assumptions in terms of only the coefficients $(\alpha_k)_{k\in\mathbb{Z}}$.

Assumption 4.3 (Noise regularity and nondegeneracy)**.** *Let $\nu := s - 1/p$ (recall that $\mathcal{X} = H^{s,p}(\mathbb{R};\mathbb{R}^n)$). The Nemitskii map $u \mapsto g(u)$ is four times Fréchet differentiable from $\mathcal{X}$ to $\mathcal{X}$, and the noise coefficients satisfy*

$$\sum_{k\in\mathbb{Z}} (1+|k|^{2\nu})\alpha_k^2 < \infty. \tag{4.2.4}$$

Furthermore, both α_{-1} and α_1 are nonzero and there exists $x \in \mathbb{R}$ such that

$$\langle \psi g(u^*), \sin(2\pi \cdot + x)\rangle \neq 0, \tag{4.2.5}$$

where u^*, ψ are as in Assumption 4.2 and (4.2.1).

Remark 4.2.3. Since $\nu > 0$ by the conditions on s and p, it follows that (4.2.3) converges almost surely in the Hölder space C^α for $\alpha < \nu$, which in particular forces the noise to be continuous in space. Since our state space $\mathcal{X}$ always embeds into the space of continuous functions (see Remark 4.2.1) and we do not assume any smoothing properties, the condition that $(\alpha_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ thus forms a natural barrier for the noise regularity. In our intended applications we can however take ν arbitrarily close to zero.

Under Assumptions 4.1 and 4.3, it follows from well-known arguments (see e.g. [47, 66]) that (4.1.3) has a unique (local) mild solution. As $\mathscr{X}$ is a Banach algebra and $\|e_k\|_{\mathscr{X}} \lesssim 1+|k|^\nu$, we have

$$\sum_{k\in\mathbb{Z}} \alpha_k^2 \|g(u)e_k\|_{\mathscr{X}}^2 \lesssim \|g(u)\|_{\mathscr{X}}^2 \sum_{k\in\mathbb{Z}} \alpha_k^2 (1+|k|^{2\nu}), \qquad u \in \mathscr{X}, \tag{4.2.6}$$

which implies that the relevant stochastic integrals converge in the topology of $\mathscr{X}$. Moreover, even though the solution u_σ could potentially blow up in finite time, the stability of the pulse (see Theorem 4.3.8 ahead) guarantees that such a blowup becomes increasingly unlikely in the small noise limit $\sigma \searrow 0$. Hence, for our purposes it is harmless to think of the solution u_σ as existing for all times.

We now state our final assumption. We have formulated it separately from Assumption 4.3, since it is only needed once near the end of the proof (in Proposition 4.4.18), where it is used to transfer synchronization results from the phase-reduced SDE back to the SPDE.

Assumption 4.4. *At least one of the following conditions hold:*

- $c = 0$.
- $\alpha_k = \alpha_{-k}$ *for every* $k \in \mathbb{N}$.

Note that the first condition in Assumption 4.4 can always be satisfied by changing to a co-moving coordinate frame (in which $c = 0$) before adding noise of the form (4.2.3). The alternative condition is equivalent to the statistics of the noise being spatially homogeneous. For a detailed discussion of the implications of Assumption 4.4 and the possibility to lift it, we refer ahead to Section 4.5.2.

4.2.4. PROBABILITY, RDS, SYNCHRONIZATION

We now provide the technical setup and preliminaries for our use of random dynamical systems (RDS) in Section 4.4. We will keep the RDS theory to a minimum by introducing only the concepts which we directly use. Note that the stated definitions are standard, and robustly generalize to more involved settings. For detailed expositions, we refer the reader to [15, 63].

In our setting, the only 'source' of randomness is $W(t)$, which is constructed from independent one-dimensional Brownian motions by (4.2.3). Hence, to construct our probability space we let $\Omega = C(\mathbb{R}\times\mathbb{Z})$, let $\mathscr{F}$ be the Borel σ-field of the compact-open topology on Ω, and let $\mathbb{P}$ be the probability measure on $(\Omega, \mathscr{F})$ such that the random variables $\beta_k(t) := \omega(t,k)$ form a sequence of independent two-sided Brownian motions satisfying $\beta_k(0) = 0$. We also define the family of σ-fields $\mathbb{F} = (\mathscr{F}_{s,t})_{-\infty\le s\le t\le\infty}$ by taking $\mathscr{F}_{s,t}$ to be the σ-field generated by the increments $\{\omega(u,k) - \omega(r,k) : k \in \mathbb{Z},\ s \le r \le u \le t\}$. We abbreviate $\mathscr{F}_t = \mathscr{F}_{-\infty,t}$. Notice that $\mathscr{F}_{r,s}$ (and thus, $\mathscr{F}_s$) is independent of $\mathscr{F}_{t,u}$ when $s \le t$.

In order to 'shift time' in the probability space, we let $\theta = (\theta_t)_{t\in\mathbb{R}}$ be the family of transformations of Ω given by

$$\theta_t(\omega)(s,k) = \omega(s+t,k) - \omega(t,k), \qquad t,s \in \mathbb{R},\ k \in \mathbb{Z}.$$

Then $\theta_t \circ \theta_s = \theta_{s+t}$ and $\theta_t^* \mathbb{P} = \mathbb{P}$, so the tuple $(\Omega, \mathscr{F}, \mathbb{P}, \theta)$ forms a (continuous-time) ergodic dynamical system.

Throughout the rest of the section, let (S, d) be a separable complete metric space.

Definition 4.2.4. A *perfect cocyle* over θ is a jointly measurable map $\phi\colon \mathbb{R}^+ \times \Omega \times S \to S$, $(t, \omega, x) \mapsto \phi(t, \omega, x)$, which satisfies:

- $\phi(0, \omega, x) = x$ for all $\omega \in \Omega, x \in S$.
- $\phi(t + s, \omega, x) = \phi(s, \theta_t \omega, \phi(t, \omega, x))$ for all $t, s \in \mathbb{R}^+, \omega \in \Omega, x \in S$.

The second of these conditions is commonly referred to as the *cocycle property*. In our setting, the cocycle $\phi(t, \omega, x)$ will be the solution map of an SDE with smooth coefficients driven by the Brownian motions $\beta_k(\cdot) = \omega(\cdot, k)$. In this case, it holds that $(t, x) \mapsto \phi(t, \omega, x)$ is continuous for all $\omega \in \Omega$. Furthermore, $\phi(t, \theta_s \cdot, x)$ is $\mathscr{F}_{s,s+t}$-measurable, and from the independence between the elements of $\mathbb{F}$ it follows that $\phi(t, \theta_s \cdot, x)$ is independent of $\mathscr{F}_{-\infty,s}$. The tuple $(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F}, \theta, \phi)$ then forms a *continuous white noise random dynamical system*. Since every element of the tuple except the cocycle has already been fixed, we will only use ϕ to refer to the random dynamical system.

When ϕ is a continuous white noise RDS, we may define an associated Markovian semigroup P_t via

$$P_t f(x) = \mathbb{E}\big[f(\phi(t, \omega, x))\big], \tag{4.2.7}$$

for bounded and measurable $f\colon S \to \mathbb{R}$. We recall that a probability measure μ on S is called an *invariant measure* for P_t if the identity $\int_S P_t f \,\mathrm{d}\mu = \int_S f \,\mathrm{d}\mu$ holds for all $t \geq 0$ and all bounded measurable $f\colon S \to \mathbb{R}$. An invariant measure μ is *ergodic* if for any bounded measurable $f\colon S \to \mathbb{R}$, we have $P_t f = f$ μ-almost everywhere only if f is constant μ-almost everywhere (several different characterizations exist). Finally, we say that μ is *strongly mixing* if $\lim_{t\to\infty} P_t f(x) = \int_S f \,\mathrm{d}\mu$ for every $x \in S$ and $f \in C_b(S)$. This condition is stronger than the usual notion of strong mixing (see [63, Corollary 3.4.3]), but is chosen to remain consistent with [90].

We will use the concept of weak synchronization, introduced in [90, Definition 2.16]. Recall that a *weak point attractor* for ϕ is a random compact set $A(\omega)$ which is invariant (i.e. $A(\theta_t \omega) = \phi(t, \omega, A(\omega))$) and which satisfies

$$\lim_{t\to\infty} \mathbb{P}\left[d(\phi(t, \omega, x), A(\theta_t \omega)) > \varepsilon\right] = 0, \qquad x \in S, \varepsilon > 0.$$

A weak point attractor is *minimal* if it is contained in any other weak point attractor.

Definition 4.2.5. A white noise RDS *synchronizes weakly* if there exists a minimal weak point attractor $a(\omega)$ which consists of a single point $\mathbb{P}$-almost surely.

4.3. PHASE REDUCTION

In the study of stochastic traveling pulses, a key element is the need to define and track the position (henceforth referred to as the *phase*) of the pulse. Various phase tracking methods have been proposed [1, 112, 121, 148] (see also Chapter 3), primarily for the purpose of showing stochastic orbital stability of the pulse. As mentioned in

Section 4.1.3, the phase tracking method of choice in this chapter is the isochronal phase. This is motivated by several properties which are favorable from a theoretical perspective. Firstly, unlike phase tracking methods based on minimization or orthogonality conditions, the isochronal phase map is indifferent to the topology of the state space $\mathscr{X}$ in which (4.1.3) is solved. Hence, the phase tracking is decoupled from the solution theory, allowing for flexibility in the choice of $\mathscr{X}$. Secondly, the isochronal phase map is defined on the entire basin of attraction of the traveling pulse. Thus, the phase tracking does not break down for 'technical' reasons, and only ceases to function when the solution arguably no longer resembles a traveling pulse anyway. Thirdly, the isochron map π straightforwardly inherits all the symmetry properties of (4.1.2), a fact which we rely on to simplify the analysis of the phase-reduced SDE.

4.3.1. ISOCHRONAL PHASE

We now give a brief introduction to the idea behind the isochronal phase and prove some basic properties. Recall that Assumption 4.2 assures that (4.1.2) has a stable traveling pulse solution with profile u^*. Hence, for δ sufficiently small, the set

$$\Gamma_\delta := \{u \in \mathscr{X} : \inf_{x\in\mathbb{R}} \|u(\cdot) - u^*(\cdot - x)\|_{\mathscr{X}} < \delta\} \tag{4.3.1}$$

is contained in the basin of attraction of the traveling pulse solutions. Indeed, writing u^v for the solution to (4.1.2) with initial condition $u(0) = v$, we have the following theorem:

Theorem 4.3.1. *There exists $\delta > 0$, such that for every $v \in \Gamma_\delta$ there exists a unique $\pi(v) \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} \|u^v(t,\cdot) - u^*(\cdot - ct - \pi(v))\|_{\mathscr{X}} = 0. \tag{4.3.2}$$

Moreover, if $v \in \Gamma_{\delta'}$ with $\delta' \leq \delta$, we have

$$\|\mathscr{T}_{\pi(v)} u^* - v\|_{\mathscr{X}} \lesssim \delta'. \tag{4.3.3}$$

Remark 4.3.2. In the language of dynamical systems, the isochronal phase provides a *foliation* of $\mathscr{X}$ near the *center manifold* $\{u^*(\cdot - x) : x \in \mathbb{R}\}$, consisting of *leaves* of the form $\{u \in \mathscr{X} : \pi(u) = x\}$ for $x \in \mathbb{R}$, each of which is invariant under the flow of (4.1.2) and terminates at the manifold, see [20]. However, we will not use this terminology.

For $v \in \Gamma_\delta$, the isochronal phase is now defined to be $\pi(v)$, and the map $\pi \colon \Gamma_\delta \to \mathbb{R}$ is referred to as the isochron map. Note that the isochron map π should not be confused with the circle constant π, which has the usual meaning.

Regarding the noisy equation, it should be clear that we cannot expect to have a convergence similar to (4.3.2) for a solution u_σ to (4.1.3). However, as long as $u_\sigma(t) \in \Gamma_\delta$, the isochronal phase $\pi(u_\sigma)$ is well-defined. Moreover, from (4.3.3) it is still sensible to interpret $\pi(u_\sigma)$ as the position of the perturbed traveling pulse u_σ.

4.3.2. ISOCHRON MAP DERIVATIVES

As was done in [1, 4], we want to obtain a detailed description of $\pi(u_\sigma)$ using Itô's formula. Hence, it will be necessary to have sufficient regularity of the isochron map and to obtain as much information about the derivatives as possible. In terms of regularity, the following proposition suffices.

Proposition 4.3.3 ([1]). *The isochron map* $\pi\colon \Gamma_\delta \to \mathbb{R}$ *has four bounded Fréchet derivatives in the topology of* $\mathscr{X}$.

Since π encodes information about the long-term behavior of a nonlinear PDE, it is expected that its derivatives are in general highly nontrivial to compute. However, we benefit from transferring the symmetries of (4.1.2) to π. From now on, for $x \in \mathbb{R}$ we let $\mathscr{T}_x$ denote the right-translation operator given by $\mathscr{T}_x v = v(\cdot - x)$.

Lemma 4.3.4 (Symmetry). *For* $x \in \mathbb{R}$ *and* $v \in \Gamma_\delta$ *we have*

$$\pi(\mathscr{T}_x v) = x + \pi(v). \tag{4.3.4}$$

Proof. Since A commutes with translations, we have the identity $u^{\mathscr{T}_x v}(t) = \mathscr{T}_x[u^v(t)]$. Substituting this into (4.3.2), the claim follows. □

As a second key observation, we note that the derivative $\pi'(u^*)$ can be effectively characterized in terms of ψ from (4.2.1). Although similar statements have been shown for different notions of phase, it seems that Lemma 4.3.5 has not been observed before in the context of the isochronal phase for traveling pulses.

Lemma 4.3.5 (First derivative). *For* $v \in \mathscr{X}$ *we have the identity*

$$\pi'(u^*)[v] = \langle \psi, v \rangle, \tag{4.3.5}$$

where ψ *is as in* (4.2.1).

Proof. By transforming to a co-moving coordinate frame, we may assume without loss of generality that $c = 0$. We differentiate u^v (which was defined as the solution to (4.1.2) with $u^v(0) = v$) with respect to the initial condition. We write $u'[w]$ for the derivative of u^v with respect to v in the direction w evaluated at u^*. By (4.1.2) and the chain rule, we see that $u'[w]$ satisfies:

$$\mathrm{d}u'[w] = Au'[w]\,\mathrm{d}t + f'(u^*)u'[w]\,\mathrm{d}t = \mathscr{L}u'[w]\,\mathrm{d}t,$$

with initial condition $u'[w](0) = w$. By Assumption 4.2 it then follows that $u'[w](t) = P(t)w$ and furthermore:

$$\lim_{t\to\infty} u'[w](t) = \lim_{t\to\infty}\left(\Pi^c w + P(t)\Pi^s w\right) = \Pi^c w = -\langle \psi, w\rangle \partial_x u^*, \tag{4.3.6}$$

where we have used (4.2.1) for the final identity. On the other hand, by Theorem 4.3.1 it holds that $\lim_{t\to\infty} u^v(t) = \mathscr{T}_{\pi(v)}u^*$. Differentiating this in the same way as before we find

$$\lim_{t\to\infty} u'[w](t) = -(\mathscr{T}_{\pi(u^*)}\partial_x u^*)\pi'(u^*)[w] = -\pi'(u^*)[w]\partial_x u^*. \tag{4.3.7}$$

The claim follows by comparing (4.3.6) and (4.3.7). Although this calculation was formal, it can be made rigorous by using Assumptions 4.1 and 4.2 and the mild formulation of (4.1.2), since all involved objects have sufficient smoothness. The interchange of limit and differentiation can then be justified using the exponential orbital stability of u^*. In an ODE setting, a different proof can be found in [98, Remark 2.5]. □

The characterization (4.3.5) is highly effective from a practical standpoint, since ψ can be computed as the solution to a one-dimensional ODE. Unfortunately, expressions as nice as (4.3.5) are not available for higher derivatives of π. Although one can extend the proof method of Lemma 4.3.5 to second order to compute $\pi''(u^*)$, this results in an expression which involves a convolution with $P(t)$ similar to [113, (2.46)]. Note that calculating the second derivative is not a mere curiosity, since it has been observed that $\pi''(u^*)$ determines (to leading order) the noise-induced change in speed of the traveling pulse [98, Theorem 2.3] [113, §2.2] [4, §4.2].

However, the following symmetry properties (which are essentially inherited from the translational symmetry of (4.1.2)) suffice for our proofs.

Lemma 4.3.6 (Symmetry of derivatives)**.** *For $x \in \mathbb{R}$, $u \in \Gamma_\delta$ and $v, w \in \mathscr{X}$ we have*

$$\pi'(\mathscr{T}_x u)[v] = \pi'(u)[\mathscr{T}_{-x} v], \tag{4.3.8a}$$

$$\pi''(\mathscr{T}_x u)[v, w] = \pi''(u)[\mathscr{T}_{-x} v, \mathscr{T}_{-x} w], \tag{4.3.8b}$$

and similar for higher derivatives.

Proof. Differentiate (4.3.4) with respect to v. □

Combining (4.3.5) and (4.3.8a) we also find the identity

$$\pi'(\mathscr{T}_x u^*)[v] = \langle \psi, \mathscr{T}_{-x} v\rangle = \langle \mathscr{T}_x \psi, v\rangle, \qquad x \in \mathbb{R},\ v \in \mathscr{X}, \tag{4.3.9}$$

which will be used in the sequel.

4.3.3. REDUCED PHASE SDE

With the isochron derivatives characterized, we begin the analysis of the dynamics of the isochronal phase of the solution u_σ to (4.1.3). The aim is to derive an approximate SDE for $\pi(u_\sigma)$ which is autonomous and does not involve the 'full' solution u_σ.

Converting (4.1.3) to the equivalent Itô formulation, we see that u_σ satisfies:

$$\mathrm{d}u_\sigma = Au_\sigma\,\mathrm{d}t + f(u_\sigma)\,\mathrm{d}t + \tfrac{1}{2}\sigma^2 \sum_{k\in\mathbb{Z}} \alpha_k^2 g'(u_\sigma)g(u_\sigma)e_k^2\,\mathrm{d}t + \sigma \sum_{k\in\mathbb{Z}} \alpha_k g(u_\sigma)e_k\,\mathrm{d}\beta_k(t),$$

where the stochastic integral is interpreted in the (mild) Itô sense. Motivated by the coming application of Itô's formula, we now define $\mathfrak{a}\colon \Gamma_\delta \to \mathbb{R}$ and $\mathfrak{b}_k\colon \Gamma_\delta \to \mathbb{R}$ by

$$\mathfrak{a}(u) := \tfrac{1}{2} \sum_{k\in\mathbb{Z}} \alpha_k^2 \big(\pi'(u)[g'(u)g(u)e_k^2] + \pi''(u)[g(u)e_k, g(u)e_k]\big), \tag{4.3.10a}$$

$$\mathfrak{b}_k(u) := \alpha_k \pi'(u)[g(u)e_k], \qquad k \in \mathbb{Z}. \tag{4.3.10b}$$

Applying the Itô formula shown by Adams and MacLaurin [4, Theorem 3.16] to $\pi(u_\sigma)$, we then see that the isochronal phase satisfies

$$\mathrm{d}\pi(u_\sigma) = c\,\mathrm{d}t + \sigma^2 \mathfrak{a}(u_\sigma)\,\mathrm{d}t + \sigma \sum_{k\in\mathbb{Z}} \mathfrak{b}_k(u_\sigma)\,\mathrm{d}\beta_k(t). \tag{4.3.11}$$

At this point the right-hand side of (4.3.11) still involves the full solution u_σ. However, motivated by (4.3.3) and the expectation that $u_\sigma \in \Gamma_\delta$ with high probability, we now *postulate* the approximation $u_\sigma \approx \mathscr{T}_{\pi(u_\sigma)} u^*$. Substituting this into (4.3.11) and letting γ_σ denote the resulting approximation to $\pi(u_\sigma)$, we obtain the following SDE for γ_σ:

$$\mathrm{d}\gamma_\sigma = c\,\mathrm{d}t + \sigma^2 \mathfrak{a}(\mathscr{T}_{\gamma_\sigma} u^*)\,\mathrm{d}t + \sigma \sum_{k\in\mathbb{Z}} \mathfrak{b}_k(\mathscr{T}_{\gamma_\sigma} u^*)\,\mathrm{d}\beta_k(t). \tag{4.3.12}$$

With (4.3.12) we have achieved our goal of deriving an autonomous SDE for an approximation to $\pi(u_\sigma)$. Thus, if we can prove that the approximation $\gamma_\sigma \approx \pi(u_\sigma)$ is accurate (to a degree which will be specified shortly), we have successfully reduced the dynamics of u_σ. For this, the following regularity properties of the coefficients of (4.3.11)-(4.3.12) are needed.

Proposition 4.3.7. *We have* $\mathfrak{a} \in C^3(\Gamma_\delta)$ *and* $\mathfrak{b} \in \ell^2(\mathbb{Z}; C^4(\Gamma_\delta))$.

Proof. Recalling the definitions of e_k (4.2.2) and ν (Assumption 4.3), we first show that we have the estimate

$$\|u e_k\|_{\mathscr{X}} \lesssim (1+|k|^\nu)\|u\|_{\mathscr{X}}, \qquad u\in\mathscr{X},\ k\in\mathbb{Z}. \tag{4.3.13}$$

If $k=1$ and $\mathscr{X}$ is replaced by $W^{n,p}$ with $n\in\mathbb{N}, p\in[1,\infty]$, then (4.3.13) holds by basic calculus. By complex interpolation, (4.3.13) then also holds with $k=1$ and with $\mathscr{X}$ as in Assumption 4.1. The case $k\in\mathbb{Z}$ follows by rescaling space, since $\|u(\lambda\,\cdot)\|_{\mathscr{X}} \lesssim (1+\lambda^\nu)\|u\|_{\mathscr{X}}$.

Using (4.3.13), it follows from Proposition 4.3.3 and the chain rule that we have

$$\begin{aligned}\|u\mapsto \pi'(u)[g'(u)g(u)e_k^2] + \pi''(u)[g(u)e_k, g(u)e_k]\|_{C^3(\Gamma_\delta)} &\lesssim 1+|k|^{2\nu},\\ \|u\mapsto \pi'(u)[g(u)e_k]\|_{C^4(\Gamma_\delta)} &\lesssim 1+|k|^{\nu},\end{aligned}$$

for every $k\in\mathbb{Z}$. The regularity of $\mathfrak{a}$ and $\mathfrak{b}$ then follows from (4.2.4) and (4.3.10). □

We now refer ahead to Proposition 4.4.2 to see that Proposition 4.3.7 also guarantees that the coefficients of (4.3.12) are smooth. Hence, for $x\in\mathbb{R}$ we will write γ_σ^x for the unique solution to (4.3.12) with initial condition $\gamma_\sigma^x(0)=x$.

4.3.4. VALIDITY OF THE APPROXIMATION

We will now confirm the validity of the approximation $u_\sigma \approx \mathscr{T}_{\gamma_\sigma} u^*$, where γ_σ is defined via (4.3.12). Since our ultimate goal is to transfer the synchronization behavior of γ_σ back to u_σ, the approximation needs to be valid for longer than the time it takes γ_σ to synchronize. As the characteristic time scale of (4.3.12) is $t\sim\sigma^{-2}$ we see that if there is any hope to succeed, the approximation $\gamma_\sigma\approx\pi(u_\sigma)$ must hold *at least* until $t\sim\sigma^{-2}$ and preferably longer. We now show that this is indeed the case, and the validity holds on the (slightly) longer time scale $t\sim\sigma^{-2}\log(\sigma^{-1})$.

Before we proceed to the mathematical statement, let us remark on the initial conditions of (4.1.3) and (4.3.12). Naturally, the initial condition for γ_σ needs to be compatible with the initial condition for u_σ. Recalling that $u_\sigma^x(0)=\mathscr{T}_x u^*$ by definition and $\pi(\mathscr{T}_x u^*)=x$ by Theorem 4.3.1, we should of course use γ_σ^x to approximate $\pi(u_\sigma^x)$.

Theorem 4.3.8. *Let* $q\in(0,2/3)$ *and* $x\in\mathbb{R}$. *Then we have*

$$\mathbb{P}\Big[\sup_{t\in[0,\exp(\sigma^{-2+3q})]} \|u_\sigma^x(t) - \mathscr{T}_{\pi(u_\sigma^x(t))}u^*\|_{\mathscr{X}} \ge \sigma^q\Big] \le \exp(-\sigma^{-2+3q}) \tag{4.3.14}$$

for all $\sigma \ll_q 1$.

Theorem 4.3.9. *Let* $q \in (0, 2/9)$ *and* $x \in \mathbb{R}$. *Then we have*

$$\mathbb{P}\Big[\sup_{t\in[0,\sigma^{-2}\log(\sigma^{-1})^{1-q}]} |\pi(u_\sigma^x(t)) - \gamma_\sigma^x(t)| \geq \sigma^q\Big] \leq \sigma^q \tag{4.3.15}$$

for all $\sigma \ll_q 1$.

Recall that the notation $\sigma \ll_q 1$ means that the relevant estimates hold whenever $\sigma \leq c$ for some (small) constant c which depends on q and the objects from Assumptions 4.1, 4.2, and 4.3.

Remark 4.3.10. As q increases, the estimates (4.3.14)-(4.3.15) become increasingly suboptimal. However, since our aim with Theorems 4.3.8 and 4.3.9 is to obtain approximations which are valid on the *longest* possible time scale, we intend for q to be small. More flexible versions of the estimates can be formulated, but this complicates the presentation and does not lead to any improvement of Theorem 4.1.

Remark 4.3.11. Comparing (4.3.14) with (4.3.15), we see that the approximation $u_\sigma(t) \approx \mathcal{T}_{\pi(u_\sigma(t))}u^*$ holds on a time scale which is algebraic in σ^{-1}, whereas the approximation $\pi(u_\sigma(t)) \approx \gamma_\sigma(t)$ holds on an exponentially long time scale. Thus, the reduced SDE (4.3.12) is only accurate for a fraction of the typical lifetime of the pulse.

Proof of Theorem 4.3.8. Let C be the constant from (4.3.3). For $\sigma > 0$, choose $\varepsilon = C^{-1}\sigma^q$ and $T = \exp(\sigma^{-2+3q})$ in Theorem 3.4.8. After scaling away the constants from the theorem against appropriate powers of σ^q, we see that

$$\mathbb{P}\Big[u_\sigma^x(t) \in \Gamma_{C^{-1}\sigma^q} \text{ for all } t \in [0, \exp(\sigma^{-2+3q})]\Big] \geq 1 - \exp(-\sigma^{-2+3q}),$$

for $\sigma \ll_q 1$ (recall that Γ_δ was defined in (4.3.1)). The desired estimate (4.3.14) then follows using (4.3.3). □

Proof of Theorem 4.3.9. To keep the notation concise we write u and γ instead of u_σ^x and γ_σ^x throughout the proof. For $\sigma > 0$ we define the stopping time

$$\tau_\sigma := \sup\{t \geq 0 : \sup_{s\in[0,t]} \|u(s) - \mathcal{T}_{\pi(u(s))}u^*\|_{\mathcal{X}} \leq \sigma^{3q}\}. \tag{4.3.16}$$

When $\sigma \ll_q 1$ it then holds that $u(t) \in \Gamma_\delta$ for all $t \in [0, \tau_\sigma]$ (δ is as in Theorem 4.3.1). Thus, from Proposition 4.3.7 we see that for $t \in [0, \tau_\sigma]$ we have

$$\begin{aligned}
&|\mathfrak{a}(u(t)) - \mathfrak{a}(\mathcal{T}_{\gamma(t)}u^*)| + \|\mathfrak{b}(u(t)) - \mathfrak{b}(\mathcal{T}_{\gamma(t)}u^*)\|_{\ell^2(\mathbb{Z})} \\
&\quad\lesssim \|u(t) - \mathcal{T}_{\gamma(t)}u^*\|_{\mathcal{X}} \\
&\quad\leq \|u(t) - \mathcal{T}_{\pi(u(t))}u^*\|_{\mathcal{X}} + \|\mathcal{T}_{\pi(u(t))}u^* - \mathcal{T}_{\gamma(t)}u^*\|_{\mathcal{X}} \\
&\quad\lesssim \sigma^{3q} + |\pi(u(t)) - \gamma(t)|,
\end{aligned} \tag{4.3.17}$$

where we have used (4.3.3) and Remark 4.2.2 for the final step. Writing $X(t) = \pi(u(t)) - \gamma(t)$, we see from (4.3.11), (4.3.12), and Itô's formula that

$$\begin{aligned}
\mathrm{d}|X(t)|^2 &= 2\sigma^2 X(t)(\mathfrak{a}(u(t)) - \mathfrak{a}(\mathcal{T}_{\gamma(t)}u^*))\,\mathrm{d}t \\
&\quad + \sigma^2\|\mathfrak{b}(u(t)) - \mathfrak{b}(\mathcal{T}_{\gamma(t)}u^*)\|^2_{\ell^2(\mathbb{Z})}\,\mathrm{d}t \\
&\quad + 2\sigma X(t)\sum_{k\in\mathbb{Z}}(\mathfrak{b}_k(u(t)) - \mathfrak{b}_k(\mathcal{T}_{\gamma(t)}u^*))\,\mathrm{d}\beta_k(t).
\end{aligned}$$

Writing $X^*(t) = \sup_{s\in[0,t]} \mathbb{1}_{[0,\tau_\sigma]}(s)|X(s)|$, we combine the above with (4.3.17) and the Burkholder–Davis–Gundy inequality to obtain: (note that $X^*(0) = 0$ by our choice of initial condition)

$$\begin{aligned}
\mathbb{E}\big[X^*(t)^2\big] &\lesssim \mathbb{E}\Big[\sigma^2\int_0^{\tau_\sigma} |X(s)|(\sigma^{3q}+|X(s)|)+(\sigma^{3q}+|X(s)|)^2\,\mathrm{d}s\Big] \\
&\quad + \mathbb{E}\Big[\Big(\int_0^{\tau_\sigma} \sigma^2|X(s)|^2(\sigma^{3q}+|X(s)|)^2\,\mathrm{d}s\Big)^{1/2}\Big] \\
&\lesssim t\sigma^{2+6q}+\sigma^2\int_0^t \mathbb{E}\big[X^*(s)^2\big]\,\mathrm{d}s+\mathbb{E}\big[X^*(t)\big(\sigma^2\int_0^t \sigma^{6q}+X^*(s)^2\,\mathrm{d}s\big)^{1/2}\big] \\
&\lesssim (1+\varepsilon^{-1})\big(t\sigma^{2+6q}+\sigma^2\int_0^t \mathbb{E}\big[X^*(s)^2\big]\,\mathrm{d}s\big)+\varepsilon\mathbb{E}\big[X^*(t)^2\big],
\end{aligned}$$

where we have used the inequality $2xy \le \varepsilon x^2 + \varepsilon^{-1}y^2$ for the final step. Choosing ε sufficiently small (independent of σ) allows us to absorb the rightmost term into the left-hand side, after which we apply Grönwall's inequality to find

$$\mathbb{E}\big[X^*(t)^2\big] \lesssim t\sigma^{2+6q}e^{2C\sigma^2 t}, \qquad t\ge 0,$$

for some absolute constant $C>0$. It then follows from Markov's inequality and the definition of X^* that

$$\mathbb{P}\Big[\sup_{t\in[0,T\wedge\tau_\sigma]} |\pi(u(t))-\gamma(t)| \ge \sqrt{T}\sigma^{1+2q}e^{C\sigma^2 T}\Big] \lesssim \sigma^{2q} \le \tfrac{1}{2}\sigma^q$$

for any $T>0$ and $\sigma \ll_q 1$. With the choice $T=\sigma^{-2}\log(\sigma^{-1})^{1-q}$ we also have

$$\sqrt{T}\sigma^{1+2q}e^{C\sigma^2 T} = \sigma^{2q}\log(\sigma^{-1})^{\frac{1-q}{2}}e^{C\log(\sigma^{-1})^{1-q}} \le \sigma^q$$

for $\sigma \ll_q 1$, so that

$$\mathbb{P}\Big[\sup_{t\in[0,\sigma^{-2}\log(\sigma^{-1})^{1-q}\wedge\tau_\sigma]} |\pi(u(t))-\gamma(t)| \ge \sigma^q\Big] \le \tfrac{1}{2}\sigma^q.$$

Recalling (4.3.16) we can now combine with (4.3.14) (note that $3q<2/3$ by assumption) and a union bound to lift the restriction $t\le\tau_\sigma$, and the result follows. □

4.4. THE REDUCED SDE

With the validity of the approximation $u_\sigma \approx \mathcal{T}_{\gamma_\sigma}u^*$ established, we turn our attention towards analyzing the SDE which defines γ_σ (4.3.12). The goal of this section is to establish (weak) synchronization of γ_σ. In order to transfer the synchronization to u_σ, the synchronization moreover needs to happen on the time scale $t\sim\sigma^{-2}$, and the rate needs to be uniform in the initial condition.

Although synchronization of one-dimensional SDEs has been extensively studied (see Section 4.1.2), we were unable to find a statement in the literature which exactly fits our setting. Hence, our strategy is to verify the abstract criteria set forth in the work by Flandoli, Gess, and Scheutzow [90]. This involves verifying a mixing condition

(Section 4.4.2), an asymptotic stability condition (Section 4.4.3), and some irreducibility conditions (Section 4.4.4). The most critical part of the argument takes place in Section 4.4.5: we use Assumption 4.4 to quantify the synchronization rate in terms of σ, and use uniform mixing properties of (4.3.12) to show that the synchronization rate is uniform in the initial condition.

4.4.1. PROPERTIES OF THE COEFFICIENTS

We start our analysis by suggestively defining $\mathfrak{a}\colon \mathbb{R} \to \mathbb{R}$ and $\mathfrak{b}\colon \mathbb{R}\times\mathbb{Z} \to \mathbb{R}$ via

$$\mathfrak{a}(x) := \mathfrak{a}(\mathcal{T}_x u^*), \qquad x \in \mathbb{R}, \tag{4.4.1a}$$

$$\mathfrak{b}_k(x) := \mathfrak{b}_k(\mathcal{T}_x u^*), \qquad x \in \mathbb{R},\ k \in \mathbb{Z}, \tag{4.4.1b}$$

where the right-hand side is interpreted according to (4.3.10). Despite our abuse of notation there should not be any confusion between (4.3.10) and (4.4.1), as the latter definition will exclusively be used in the rest of this section. With the new definition, we may write (4.3.12) in the Itô formulation as

$$\mathrm{d}\gamma_\sigma = c\,\mathrm{d}t + \sigma^2 \mathfrak{a}(\gamma_\sigma)\,\mathrm{d}t + \sigma \sum_{k\in\mathbb{Z}} \mathfrak{b}_k(\gamma_\sigma)\,\mathrm{d}\beta_k(t), \tag{4.4.2}$$

or in the equivalent Stratonovich formulation as

$$\mathrm{d}\gamma_\sigma = c\,\mathrm{d}t + \sigma^2\Big(\mathfrak{a}(\gamma_\sigma) - \tfrac{1}{2}\sum_{k\in\mathbb{Z}} \mathfrak{b}_k'(\gamma_\sigma)\mathfrak{b}_k(\gamma_\sigma)\Big)\,\mathrm{d}t + \sigma \sum_{k\in\mathbb{Z}} \mathfrak{b}_k(\gamma_\sigma)\circ \mathrm{d}\beta_k(t). \tag{4.4.3}$$

Remark 4.4.1. One might expect that the extra term in (4.4.3) will cancel with the first term of $\mathfrak{a}$ in (4.3.10a), since both terms originate from an Itô–Stratonovich correction. However, this is generally not the case. For example, when $W(t,x) = \beta_0(t)$ the former term vanishes (since $\mathfrak{b}_0' \equiv 0$, see (4.4.4a) ahead) but the latter does not. This demonstrates the subtle point that naively performing a phase reduction in the Stratonovich formulation leads to an inaccurate approximation. In the physics literature, this has been observed in the context of phase reduction for nonlinear oscillators [215, 234].

Proposition 4.4.2. *We have* $\mathfrak{a} \in C^3(\mathbb{R})$ *and* $\mathfrak{b} \in \ell^2(\mathbb{Z}; C^4(\mathbb{R}))$.

Proof. This follows from the definition (4.4.1) using Proposition 4.3.7 and the chain rule, taking into account that $x \mapsto \mathcal{T}_x u^*$ is four times differentiable (with values in $\mathcal{X}$) by Assumption 4.2. □

We now gather some more facts about the coefficients aside from smoothness. We will not concern ourselves too much with the exact form of $\mathfrak{a}$, since it turns out to have no discernible effect on the synchronization properties of (4.4.2). Instead, the synchronization is facilitated mainly through the multiplicative noise coefficients $\mathfrak{b}_k$, which can be made much more explicit using Lemmas 4.3.5 and 4.3.6. Indeed, using (4.3.9) we find from (4.3.10b) and (4.4.1b) that

$$\mathfrak{b}_k(x) = \alpha_k \langle \psi g(u^*), \mathcal{T}_{-x} e_k\rangle = \alpha_k \langle \mathcal{T}_x[\psi g(u^*)], e_k\rangle, \qquad x \in \mathbb{R},\ k \in \mathbb{Z}, \tag{4.4.4a}$$

where we recall that ψ is as in (4.2.1). Recalling the definition of e_k (4.2.2) and using trigonometric addition formulas, we also find for $k \in \mathbb{N}$:

$$\begin{pmatrix} \mathfrak{b}_k(x) \\ \mathfrak{b}_{-k}(x) \end{pmatrix} = \begin{pmatrix} \alpha_k & 0 \\ 0 & \alpha_{-k} \end{pmatrix} \begin{pmatrix} \cos(2\pi kx) & -\sin(2\pi kx) \\ \sin(2\pi kx) & \cos(2\pi kx) \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k} \end{pmatrix}, \tag{4.4.4b}$$

where we have written $c_k := \langle \psi g(u^*), e_k \rangle$. This then leads to

$$\sum_{k\in\mathbb{Z}} \mathfrak{b}_k(x)\,\mathrm{d}\beta_k = \alpha_0 \langle \psi g(u^*), e_0 \rangle\,\mathrm{d}\beta_0 + \sum_{k\in\mathbb{N}} \begin{pmatrix} c_k \\ c_{-k} \end{pmatrix}^{\top} R(-2\pi kx) \begin{pmatrix} \alpha_k & 0 \\ 0 & \alpha_{-k} \end{pmatrix} \begin{pmatrix} \mathrm{d}\beta_k \\ \mathrm{d}\beta_{-k} \end{pmatrix}, \tag{4.4.4c}$$

where $R(\theta)$ denotes the usual rotation matrix which rotates $\mathbb{R}^2$ counterclockwise by θ radians. Notice that we have fully isolated the dependence on x into the rotation matrices. The nondegeneracy condition in Assumption 4.3 then gives us the following lemma.

Lemma 4.4.3. *We have $\mathfrak{b}_1^2(x) + \mathfrak{b}_{-1}^2(x) > 0$ for every $x \in \mathbb{R}$, and also*

$$\operatorname{span}\{\mathfrak{b}_{\pm 1}(\cdot)\} = \operatorname{span}\{\cos(2\pi\cdot), \sin(2\pi\cdot)\} = \{L\sin(2\pi\cdot+\eta) : L, \eta \in \mathbb{R}\}. \tag{4.4.5}$$

Proof. The second identity in (4.4.5) is well-known, and from this and (4.2.5) it follows that $c_1^2 + c_{-1}^2 > 0$. Since $\alpha_{\pm 1}$ are both nonzero by Assumption 4.3, the remaining claims can be read off from (4.4.4b). □

We also observe the following symmetry properties of $\mathfrak{a}$ and $\mathfrak{b}$, which are inherited from symmetries of the PDE and the noise.

Proposition 4.4.4. *For every $x \in \mathbb{R}$ and $k \in \mathbb{Z}$, we have*

$$\mathfrak{a}(x) = \mathfrak{a}(x+1), \qquad \mathfrak{b}_k(x) = \mathfrak{b}_k(x+1). \tag{4.4.6a}$$

If additionally $\alpha_k = \alpha_{-k}$ for every $k \in \mathbb{Z}$, then we also have

$$\mathfrak{a}(x) = \mathfrak{a}(0), \qquad \|\mathfrak{b}(x)\|_{\ell^2(\mathbb{Z})} = \|\mathfrak{b}(0)\|_{\ell^2(\mathbb{Z})}. \tag{4.4.6b}$$

Proof. The identities involving $\mathfrak{b}$ can be read off directly from (4.4.4a)-(4.4.4b), since rotation matrices are isometric. Regarding $\mathfrak{a}$, note that by Lemma 4.3.6, (4.3.10a), and (4.4.1a) we have

$$\mathfrak{a}(x) = \tfrac{1}{2} \sum_{k\in\mathbb{Z}} \alpha_k^2 \big(\pi'(u^*)[g'(u^*)g(u^*)\mathcal{T}_{-x} e_k^2] + \pi''(u^*)[g(u^*)\mathcal{T}_{-x} e_k, g(u^*)\mathcal{T}_{-x} e_k]\big).$$

The periodicity of $\mathfrak{a}$ immediately follows from periodicity of e_k, so it only remains to show the first identity of (4.4.6b) in the case where $\alpha_k = \alpha_{-k}$ for all $k \in \mathbb{N}$. For this, it suffices to note that

$$\begin{aligned}
&\sum_{k=\pm n} \alpha_k^2 \big(\pi'(u^*)[g'(u^*)g(u^*)\mathcal{T}_{-x} e_k^2] + \pi''(u^*)[g(u^*)\mathcal{T}_{-x} e_k, g(u^*)\mathcal{T}_{-x} e_k]\big) \\
&\quad = \alpha_n^2 \sum_{k=\pm n} \big(\pi'(u^*)[g'(u^*)g(u^*)\mathcal{T}_{-x} e_k^2] + \pi''(u^*)[g(u^*)\mathcal{T}_{-x} e_k, g(u^*)\mathcal{T}_{-x} e_k]\big) \\
&\quad = \alpha_n^2 \big(2\pi'(u^*)[g'(u^*)g(u^*)] + \sum_{k=\pm n} \pi''(u^*)[g(u^*)e_k, g(u^*)e_k]\big),
\end{aligned}$$

for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$, where the final identity follows (for the first term) since $\sin(x)^2 + \cos(x)^2 = 1$ and (for the second term) by trigonometric addition formulas, noting that $\pi''(u^*)$ is symmetric. □

4.4.2. RDS GENERATION AND ERGODICITY

By the periodicity of $\mathfrak{a}$ and $\mathfrak{b}$ (4.4.6a), we can now interpret (4.4.2) as an SDE on either $\mathbb{T}$ or $\mathbb{R}$. Combined with smoothness of the coefficients, this implies that (4.4.2) generates a C^2 white noise RDS on $\mathbb{T}$ as well as on $\mathbb{R}$. Recall that the maps $(\theta_t)_{t\in\mathbb{R}}$ apply a time shift to $(\beta_k)_{k\in\mathbb{Z}}$ and were defined in Section 4.2.4.

Proposition 4.4.5 (RDS generation)**.** *Let S be either $\mathbb{R}$ or $\mathbb{T}$. There exists a perfect cocycle $\phi_\sigma : \mathbb{R}^+ \times \Omega \times S \to S$ over $(\theta_t)_{t\in\mathbb{R}}$ which satisfies the following properties:*

(i) $\omega \mapsto \phi_\sigma(t, \theta_s\omega, x)$ *is $\mathscr{F}_{s,s+t}$-measurable for every $x \in S$, $s \in \mathbb{R}$, $t \in \mathbb{R}^+$.*

(ii) $t \mapsto \phi_\sigma(t,\omega,x) =: \gamma_\sigma^x(t)$ *solves* (4.4.2) *with $\gamma_\sigma^x(0) = x$, for every $x \in S$.*

(iii) $(t,x) \mapsto \phi(t,\omega,x)$ *is jointly continuous for every $\omega \in \Omega$.*

(iv) $x \mapsto \phi_\sigma(t,\omega,x) \in C^2(S;S)$ *for every $t \in \mathbb{R}^+$, $\omega \in \Omega$.*

(v) $\mathbb{E}\big[\|x \mapsto \phi_\sigma(1,\omega,x)\|_{C^2(S;S)}\big] < \infty$.

Remark 4.4.6. We will not always explicitly specify whether we consider ϕ_σ as an RDS on $\mathbb{T}$ or $\mathbb{R}$. However, the notation $d(x, y)$ will *always* refer to the distance on the torus. In the case where $x, y \in \mathbb{R}$ a priori, one should interpret $d(x, y)$ as the distance between the equivalence classes (modulo 2π) of x and y.

Proof. (i)-(iv) follow from Proposition 4.4.2 and [17, Theorem 28], and additionally (4.4.6a) in the case $S = \mathbb{T}$. For (v) it suffices by periodicity (see (4.4.6a)) to prove the case $S = \mathbb{T}$. We differentiate (4.4.2) using the chain rule to find that $\partial_x\gamma_\sigma^x$ satisfies the following SDE:

$$\mathrm{d}(\partial_x\gamma_\sigma^x) = \sigma^2\mathfrak{a}'(\gamma_\sigma^x)\partial_x\gamma_\sigma^x\,\mathrm{d}t + \sigma\sum_{k\in\mathbb{Z}}\mathfrak{b}_k'(\gamma_\sigma^x)\partial_x\gamma_\sigma^x\,\mathrm{d}\beta_k(t) \tag{4.4.7}$$

with initial condition $\partial_x\gamma_\sigma^x(0) = 1$, with similar equations being satisfied by $\partial_x^{(2)}\gamma_\sigma^x$ and $\partial_x^{(3)}\gamma_\sigma^x$. Applying well-known theory for SDEs with Lipschitz coefficients, it then follows that $\sup_{x\in\mathbb{T}}\mathbb{E}\big[|\partial_x^{(n)}\gamma_\sigma^x(1)|\big] < \infty$ for $n \in \{0,1,2,3\}$. Hence, by the Sobolev embedding and Fubini's theorem we have

$$\mathbb{E}\big[\|x \mapsto \gamma_\sigma^x(1)\|_{C^2(\mathbb{T};\mathbb{T})}\big] \lesssim \mathbb{E}\big[\|x \mapsto \gamma_\sigma^x(1)\|_{W^{3,1}(\mathbb{T};\mathbb{T})}\big] = \int_{\mathbb{T}}\mathbb{E}\Big[\sum_{k=0}^{3}|\partial_x^{(k)}\gamma_\sigma^x(1)|\Big]\,\mathrm{d}x < \infty.$$

This is exactly what we needed to show, since $\gamma_\sigma^x(1) = \phi(1,\omega,x)$ by definition. □

We now let $(P_t f)(x) := \mathbb{E}[f(\phi_\sigma(t,\omega,x))]$ be the Markov semigroup on $C_b(\mathbb{T})$ associated with ϕ_σ, and formulate the following ergodicity properties:

Proposition 4.4.7 (Ergodicity)**.** *P_t has a unique invariant measure $\mu = p\,\mathrm{d}x$, where $p \in C^2(\mathbb{T})$ is strictly positive. Moreover, μ is exponentially mixing in the sense that there exist $C, a > 0$ such that*

$$\|P_t f - \int_{\mathbb{T}} f\,\mathrm{d}\mu\|_{C_b(\mathbb{T})} \le Ce^{-at}\|f\|_{C_b(\mathbb{T})}, \qquad t \ge 0, \tag{4.4.8}$$

for every $f \in C_b(\mathbb{T})$.

Remark 4.4.8. The condition (4.4.8) is quite strong and might not be satisfied in different settings. However, (4.4.8) is only used once in the proof of Theorem 4.4.17, where the weaker condition $\lim_{t\to\infty}\|P_t f(\cdot)-\int_{\mathbb{T}} f\,\mathrm{d}\mu\|_{C_b(\mathbb{T})}=0$ (with convergence rate depending on $f\in C_b(\mathbb{T})$) would already suffice.

Proof. The generator of P_t is given by

$$\mathscr{L}f=\tfrac{1}{2}\sigma^2\sum_{k\in\mathbb{Z}}\mathfrak{b}_k^2\partial_{xx}f+(c+\sigma^2\mathfrak{a})\partial_x f,\tag{4.4.9a}$$

with formal adjoint

$$\mathscr{L}^*p=\tfrac{1}{2}\sigma^2\sum_{k\in\mathbb{Z}}\partial_{xx}(\mathfrak{b}_k^2p)-\partial_x((c+\sigma^2\mathfrak{a})p).\tag{4.4.9b}$$

We also have $\mathfrak{a}\in C^3(\mathbb{T})$ and $x\mapsto\|\mathfrak{b}(x)\|_{\ell^2(\mathbb{Z})}\in C^3(\mathbb{T})$ by Proposition 4.4.2, as well as $\inf_{x\in\mathbb{T}}\sum_{k\in\mathbb{Z}}\mathfrak{b}_k(x)^2>0$ by Lemma 4.4.3. It follows from well-known Schauder theory (see e.g. [125]) that $\mathscr{L}^*p=0$ has a solution $p\in C^2(\mathbb{T})$ with $p>0$, which means $\mu:=p\,\mathrm{d}x$ is an invariant measure. The exponential mixing (4.4.8), which also implies uniqueness of μ, can be seen from e.g. [24] or [169]. □

4.4.3. Asymptotic stability

To prove asymptotic stability, we will show (strict) negativity of the *Lyapunov exponent*, defined as

$$\lambda:=\lim_{t\to\infty}t^{-1}\log|\partial_x\phi_\sigma(t,\omega,x)|.\tag{4.4.10}$$

Although λ might a priori depend on ω and x, Oseledets' multiplicative ergodic theorem (see [15, §3.4]) ensures that λ is constant $\mathbb{P}\times\mu$-almost everywhere. Moreover, expressions for λ in terms of the invariant measure μ and the coefficients $\mathfrak{a}$ and $\mathfrak{b}$ are known. In order to have a self-contained presentation, we now prove these identities 'by hand', relying on the Birkhoff pointwise ergodic theorem. The reader who is familiar with these identities may skip the following proof. We remark that the second identity in (4.4.11) was seemingly only recently discovered by Bedrossian, Blumenthal, and Punshon-Smith [23].

Proposition 4.4.9 (Lyapunov exponent)**.** *The identity*

$$\lambda=\sigma^2\int_{\mathbb{T}}\mathfrak{a}'-\tfrac{1}{2}\sum_{k\in\mathbb{Z}}(\mathfrak{b}_k')^2\,\mathrm{d}\mu=-\tfrac{1}{2}\sigma^2\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}\frac{|\partial_x(\mathfrak{b}_kp)|^2}{p}\,\mathrm{d}x\tag{4.4.11}$$

holds for $\mathbb{P}\times\mu$*-almost all* (ω,x).

Proof. Applying Itô's formula to (4.4.7) gives

$$\mathrm{d}\log|\partial_x\phi_\sigma(t,\omega,x)|=\sigma^2\mathfrak{a}'(\gamma_\sigma^x)\,\mathrm{d}t-\tfrac{1}{2}\sigma^2\sum_{k\in\mathbb{Z}}\mathfrak{b}_k'(\gamma_\sigma^x)^2\,\mathrm{d}t+\sigma\sum_{k\in\mathbb{Z}}\mathfrak{b}_k'(\gamma_\sigma^x)\,\mathrm{d}\beta_k(t).$$

Integrating this and recalling $\gamma_\sigma^x(t)=\phi_\sigma(t,\omega,x)$, we get $\lambda=\lim_{t\to\infty}\lambda_1(t)+\lambda_2(t)$ with

$$\lambda_1(t):=\sigma^2t^{-1}\int_0^t\mathfrak{a}'(\gamma_\sigma^x(s))-\tfrac{1}{2}\sum_{k\in\mathbb{Z}}\mathfrak{b}_k'(\gamma_\sigma^x(s))^2\,\mathrm{d}s,$$

$$\lambda_2(t):=\sigma t^{-1}\sum_{k\in\mathbb{Z}}\int_0^t\mathfrak{b}_k'(\gamma_\sigma^x(s))\,\mathrm{d}\beta_k(s).$$

Since $\|b\|_{\ell^2(\mathbb{Z};C^1(\mathbb{T}))} < \infty$ by Proposition 4.4.2, it follows from the strong law of large numbers for martingales that $\lim_{t\to\infty}\lambda_2(t) = 0$ almost surely for every $x \in \mathbb{T}$. For λ_1, it follows from Proposition 4.4.7 and Birkhoff's pointwise ergodic theorem that

$$\lim_{t\to\infty}\lambda_1(t) = \sigma^2 \int_{\mathbb{T}} \mathfrak{a}'(x) - \tfrac{1}{2}\sum_{k\in\mathbb{Z}}(\mathfrak{b}_k'(x))^2\,\mathrm{d}\mu(x)$$

for $\mathbb{P}\times\mu$-almost every (ω, x), so the first identity in (4.4.11) holds.

For the second identity, we follow the proof of [23, Proposition 3.2]. Recall from Proposition 4.4.7 that $\mu = p\,\mathrm{d}x$ where p satisfies the Fokker–Planck equation $\mathscr{L}^* p = 0$. Multiplying (4.4.9b) by $\sigma^{-2}\log(p)$ and integrating, we find:

$$\tfrac{1}{2}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}\partial_{xx}(\mathfrak{b}_k^2 p)\log(p)\,\mathrm{d}x = \int_{\mathbb{T}}\partial_x((\sigma^{-2}c+\mathfrak{a})p)\log(p)\,\mathrm{d}x = \int_{\mathbb{T}}(\partial_x\mathfrak{a})p\,\mathrm{d}x, \tag{4.4.12}$$

where the second identity in (4.4.12) is obtained by integrating by parts back and forth. For the left-hand side of (4.4.12), we additionally find:

$$\begin{aligned}\tfrac{1}{2}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}\partial_{xx}(\mathfrak{b}_k^2 p)\log(p)\,\mathrm{d}x &= -\tfrac{1}{2}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}\frac{\partial_x(\mathfrak{b}_k^2 p)\partial_x p}{p}\,\mathrm{d}x\\ &= \tfrac{1}{2}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}\frac{p^2(\partial_x\mathfrak{b}_k)^2-(\partial_x(\mathfrak{b}_k p))^2}{p}\,\mathrm{d}x\\ &= \tfrac{1}{2}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}(\partial_x\mathfrak{b}_k)^2 p - \frac{(\partial_x(\mathfrak{b}_k p))^2}{p}\,\mathrm{d}x.\end{aligned} \tag{4.4.13}$$

Combining (4.4.12) and (4.4.13), the second identity in (4.4.11) follows. □

Corollary 4.4.10. *We have $\lambda < 0$, and ϕ_σ is asymptotically stable in the sense of [90, Definition 2.2].*

Proof. Since $\mu = p\,\mathrm{d}x$ is a probability measure, we see from (4.4.11) and Jensen's inequality that

$$\lambda = -\tfrac{1}{2}\sigma^2\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}\frac{|\partial_x(\mathfrak{b}_k p)|^2}{p^2}p\,\mathrm{d}x \le -\tfrac{1}{2}\sigma^2\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}|\partial_x(\mathfrak{b}_k p)|\,\mathrm{d}x \le 0.$$

Furthermore, if $\lambda = 0$ then we would necessarily have

$$\partial_x(\mathfrak{b}_1 p) \equiv \partial_x(\mathfrak{b}_{-1}p) \equiv 0,$$

which would further imply that $\mathfrak{b}_1(\cdot)$ and $\mathfrak{b}_{-1}(\cdot)$ are both multiples of p^{-1} (recall that $p > 0$) and thus linearly dependent. However, by Lemma 4.4.3 this is not the case so we have $\lambda < 0$ by contraposition. From Proposition 4.4.5-(v) and Jensen's inequality we also see that [90, (3.3)-(3.4)] are satisfied, so that asymptotic stability holds by [90, Corollary 3.4] (which extends *mutatis mutandis* to the torus). □

4.4.4. IRREDUCIBILITY AND CONTROLLABILITY

We now verify two irreducibility-type conditions needed for weak synchronization, which are formulated in terms of lower bounds on the probabilities of certain events involving ϕ_σ (see Propositions 4.4.12 and 4.4.16). Since our noise is multiplicative, this is not a trivial matter. Our strategy is to first study a control system associated with the two-point motion of (4.4.2), where $\mathrm{d}\beta_1$ and $\mathrm{d}\beta_{-1}$ are replaced with appropriate control functions. Once the relevant controllability properties are shown, the irreducibility conditions follow by applying the support theorem for diffusions by Stroock and Varadhan [212] (see also [172, Theorem 3.5] for a version which suffices for our purposes).

Throughout this section we write $\mathscr{H}$ for the space of piecewise constant compactly supported functions from $\mathbb{R}^+$ to $\mathbb{R}^2$ and write $h = (h_1, h_{-1})$ for $h \in \mathscr{H}$. The functions $h \in \mathscr{H}$ will serve as controls. For $x \in \mathbb{R}$ and $h \in \mathscr{H}$ we write, in analogy with our previous notation, $\tilde{\gamma}_h^x(t)$ and $\gamma_h^x(t)$ for the unique solutions to the following ODEs:

$$\frac{\mathrm{d}\tilde{\gamma}_h^x}{\mathrm{d}t} = \sigma\big(\mathfrak{b}_1(\tilde{\gamma}_h^x)h_1 + \mathfrak{b}_{-1}(\tilde{\gamma}_h^x)h_{-1}\big), \tag{4.4.14a}$$

$$\frac{\mathrm{d}\gamma_h^x}{\mathrm{d}t} = c + \sigma^2\big(\mathfrak{a}(\gamma_h^x) - \tfrac{1}{2}\sum_{k\in\mathbb{Z}} \mathfrak{b}_k'(\gamma_h^x)\mathfrak{b}(\gamma_h^x)\big) + \sigma\big(\mathfrak{b}_1(\gamma_h^x)h_1 + \mathfrak{b}_{-1}(\gamma_h^x)h_{-1}\big), \tag{4.4.14b}$$

with initial conditions $\tilde{\gamma}_h^x(0) = \gamma_h^x(0) = x$. Note that (4.4.14b) resembles the SDE for γ_σ in Stratonovich form (4.4.3), except we have replaced the driving white noises $\mathrm{d}\beta_k$ by control functions h_k, all of which are zero except for $k = \pm 1$.

POINTWISE STRONG SWIFT TRANSITIVITY

The first irreducibility-type condition that we show is pointwise strong swift transitivity, introduced in [90, Definition 2.22].

Lemma 4.4.11. *For all $x_1, x_2, y \in \mathbb{T}$ with $x_1 \neq x_2$, there exists $h \in \mathscr{H}$ such that*

$$\max\{d(\gamma_h^{x_1}(1), y), d(\gamma_h^{x_2}(1), y)\} \leq \tfrac{3}{2} d(x_1, x_2).$$

Proof. Since $x_1 \neq x_2$ we may write $a = d(x_1, x_2) > 0$. Now let $\eta \in \mathbb{T}$ be such that $d(\eta, y) \leq a$ and $\min\{d(x_1, \eta + \frac{1}{2}), d(x_2, \eta + \frac{1}{2})\} \geq a/4$. By Lemma 4.4.3 we can find for any $L > 0$ a control $h \in \mathscr{H}$ such that

$$\frac{\mathrm{d}\gamma_h^x}{\mathrm{d}t} = c + \sigma^2\big(\mathfrak{a}(\gamma_h^x) - \tfrac{1}{2}\sum_{k\in\mathbb{Z}} \mathfrak{b}_k'(\gamma_h^x)\mathfrak{b}(\gamma_h^x)\big) - L\sin(2\pi\gamma_h^x - \eta)$$

for $x \in \mathbb{R}$ and $t \in [0,1]$. Examining the sign of the right-hand side, we see that by choosing L sufficiently large (depending on c, σ, $\mathfrak{a}$, $\mathfrak{b}$, x_1, x_2, y) we can ensure that

$$\max\{d(\gamma_h^{x_1}(1), \eta), d(\gamma_h^{x_2}(1), \eta)\} \leq a/2,$$

since the ODE for γ_h^x is attracting towards η with a rate which increases with L. The claim then follows by the triangle inequality since $d(\eta, y) \leq a = d(x_1, x_2)$. □

Proposition 4.4.12. *The RDS ϕ_σ (on $\mathbb{T}$) is pointwise strongly swift transitive in the sense of [90, Definition 2.22].*

Proof. Fix $x_1, x_2, y \in \mathbb{T}$ with $x_1 \neq x_2$.[4] Applying the support theorem for diffusions to the two-point motion $(\phi_\sigma(t,\omega,x_1), \phi_\sigma(t,\omega,x_2))$ and using Lemma 4.4.11, the claim follows. □

MEETING CONDITION

The second irreducibility-type condition needed is that for any $x_1, x_2 \in \mathbb{T}$, we must have:

$$\mathbb{P}\left[\liminf_{t\to\infty} d(\phi_\sigma(t,\omega,x_1), \phi_\sigma(t,\omega,x_2)) = 0\right] = 1 \tag{4.4.15}$$

(see [90, (2.10)]). We refer to (4.4.15) (which is unnamed in [90]) as the *meeting condition*, since it requires that any two trajectories meet arbitrarily closely infinitely often. To show (4.4.15), our strategy is to consider for $\varepsilon, T > 0$ the events

$$B_n := \left\{ \sup_{x_1,x_2\in\mathbb{T}} \inf_{t\in[0,5T]} d(\phi_\sigma(t,\theta_{5nT}\omega,x_1), \phi_\sigma(t,\theta_{5nT}\omega,x_2)) \le 4\varepsilon \right\}, \quad n \in \mathbb{N}_0, \tag{4.4.16}$$

and make the following observations:

- B_n is $\mathscr{F}_{5nT,5(n+1)T}$-measurable by Proposition 4.4.5-(i), and thus the events B_n are independent.
- $\mathbb{P}[B_n] = \mathbb{P}[B_0]$ for all $n \in \mathbb{N}$ by invariance of $\mathbb{P}$ under θ.
- $\{B_n \text{ occurs infinitely often}\} \subset \{\liminf_{t\to\infty} d(\phi_\sigma(t,\omega,x_1), \phi_\sigma(t,\omega,x_2)) \le 4\varepsilon\}$.

Thus, if we can show that for every $\varepsilon > 0$ there exists a $T > 0$ such that $\mathbb{P}[B_0] > 0$, then (4.4.15) will follow by an application of the second Borel–Cantelli lemma.

Showing $\mathbb{P}[B_0] > 0$ is still challenging, since B_0 involves all trajectories simultaneously. Our trick to circumvent this is to show that B_0 is implied by a certain event A which involves only three trajectories. To illustrate, consider the 'squeezing' event depicted in Figure 4.1, involving trajectories of (4.4.2) starting from $z_1 = 0$, $z_2 = 1/3$, $z_3 = -1/3$. In this event, any two trajectories starting in between z_2 and z_3 on the same side as z_1 will be squeezed together near z_1 by monotonicity. By performing such a squeezing consecutively first near z_1, then z_2, and finally z_3, it follows by the pigeonhole principle that *any* two points must get squeezed at some point of the cycle, which guarantees that the event B_0 occurs.

It then only remains to show that such a 'triple squeezing cycle' occurs with positive probability. Since this only involves the three-point motion $(\phi_\sigma(t,\omega,z_i))_{i\in\{1,2,3\}}$ this can be done similarly to how we showed pointwise strong swift transitivity.

Let us now make these ideas rigorous. We begin by exhibiting a control for (4.4.14a) which witnesses the aforementioned squeezing cycle for the three-point motion.

Lemma 4.4.13. *Let $z_1 = 0$, $z_2 = 1/3$, $z_3 = -1/3$. For every $\varepsilon > 0$ there exists $T > 0$, $h \in \mathscr{H}$ such that*

$$\begin{aligned} z_1 - \varepsilon \le \tilde{\gamma}_h^{z_3}(T) < \tilde{\gamma}_h^{z_2}(T) \le z_1 + \varepsilon, \\ z_2 - \varepsilon \le \tilde{\gamma}_h^{z_1}(3T) < \tilde{\gamma}_h^{z_3}(3T) + 1 \le z_2 + \varepsilon, \\ z_3 - \varepsilon \le \tilde{\gamma}_h^{z_2}(5T) - 1 < \tilde{\gamma}_h^{z_1}(5T) \le z_3 + \varepsilon. \end{aligned} \tag{4.4.17}$$

[4] Note that [90, Definition 2.22] should contain the additional condition $x_1 \neq x_2$, as confirmed to us by the authors.

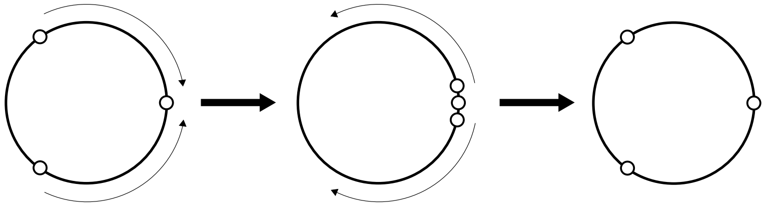


Figure 4.1: Trajectories starting at z_2 and z_3 'squeeze' near z_1 and afterwards return to their initial position.

Proof. We show that $T = 1$ suffices. Using (4.4.5) we can find for any $L > 0$ a control $h \in \mathscr{H}$ such that $\tilde{\gamma}_h^x$ (recall (4.4.14a)) satisfies

$$\frac{d\tilde{\gamma}_h^x}{dt} = -k(t)L\sin(2\pi\tilde{\gamma}_h^x(t) - \eta(t)), \qquad t \geq 0, x \in \mathbb{R}, \tag{4.4.18}$$

where

$$k(t) = \mathbb{1}_{[0,1]}(t) - \mathbb{1}_{[1,2]}(t) + \mathbb{1}_{[2,3]}(t) - \mathbb{1}_{[3,4]}(t) + \mathbb{1}_{[4,5]}(t),$$
$$\eta(t) = z_1 \cdot \mathbb{1}_{[0,2]}(t) + z_2 \cdot \mathbb{1}_{[2,4]}(t) + z_3 \cdot \mathbb{1}_{[4,5]}(t).$$

With h of this form, it follows from time reversal of (4.4.14a) that $x = \tilde{\gamma}_h^x(2) = \tilde{\gamma}_h^x(4)$ for every $x \in \mathbb{R}$. After examining (4.4.18) on the time intervals $[0,1]$, $[2,3]$, and $[4,5]$, it is seen that (4.4.17) holds when L is sufficiently large (depending on σ and $\mathfrak{b}_{\pm 1}$). □

By a perturbative argument we now show that the same statement holds for (4.4.14b).

Lemma 4.4.14. *Lemma 4.4.13 still holds when $\tilde{\gamma}$ is replaced by γ in (4.4.17).*

Proof. Fix $\varepsilon > 0$ and let $T > 0, h \in \mathscr{H}$ be such that (4.4.17) is satisfied. For $\delta > 0$ we define h_δ via $h_\delta(\cdot) = \delta^{-1}h(\delta^{-1}\cdot)$. By rescaling time in (4.4.14a) we see that

$$\tilde{\gamma}_{h_\delta}^x(\cdot) = \tilde{\gamma}_h^x(\delta^{-1}\cdot), \qquad x \in \mathbb{R}, \tag{4.4.19}$$

for any $\delta > 0$. Furthermore, subtracting (4.4.14a) from (4.4.14b) and using Proposition 4.4.2 we find that there exists $C > 0$ (depending only on σ, c, $\mathfrak{a}$, $\mathfrak{b}$) such that

$$|\gamma_{h_\delta}^x(t) - \tilde{\gamma}_{h_\delta}^x(t)| \leq Ct + C\delta^{-1}\int_0^t |\gamma_{h_\delta}^x(s) - \tilde{\gamma}_{h_\delta}^x(s)|\,ds, \qquad x \in \mathbb{R}, t \geq 0, \delta > 0.$$

It follows from Grönwall's lemma that

$$\sup_{t \in [0,\delta T]} |\gamma_{h_\delta}^x(t) - \tilde{\gamma}_{h_\delta}^x(t)| \leq \delta CT\exp(CT), \qquad \delta > 0.$$

Choosing δ sufficiently small (depending only on ε, C, T) and combining this with (4.4.17) and (4.4.19), we see that (4.4.17) is still satisfied with $\tilde{\gamma}$ replaced by γ, T replaced by δT, h replaced by h_δ, and ε replaced by 2ε. The conclusion follows since ε was arbitrary. □

Finally, we show that $\mathbb{P}[B_0]$ has positive probability using monotonicity of (4.4.2) and the Stroock–Varadhan support theorem (see [172, Theorem 3.5]).

Lemma 4.4.15. *Let $\varepsilon > 0$ and let T be as in Lemma 4.4.14. Then $\mathbb{P}[B_0] > 0$.*

Proof. Applying the support theorem to the three-point motion $(\phi_\sigma(t,\omega,z_i))_{i\in\{1,2,3\}}$ and using Lemma 4.4.14, we see that there is a strictly positive probability that (4.4.17) holds with $\tilde{\gamma}_h^{z_i}(T)$ replaced by $\phi_\sigma(T,\omega,z_i)$ everywhere. By monotonicity it then follows that the event

$$\begin{aligned} A := &\left\{\phi_\sigma(T,\omega,[z_3,z_2]) \subset B_{2\varepsilon}(z_1)\right\} \cap \left\{\phi_\sigma(3T,\omega,[z_1,z_3+1]) \subset B_{2\varepsilon}(z_2)\right\} \\ &\cap \left\{\phi_\sigma(5T,\omega,[z_2-1,z_1]) \subset B_{2\varepsilon}(z_3)\right\} \end{aligned}$$

(where ϕ_σ is interpreted as an RDS on $\mathbb{R}$) has positive probability. Now observe that for any two-point set $S \subset \mathbb{T}$ it must be the case that either $S \subset [z_3,z_2]$, $S \subset [z_1,z_3+1]$, or $S \subset [z_2-1,z_1]$ (for properly chosen representatives of S) by the pigeonhole principle. Thus, the event A implies B_0 so that $\mathbb{P}[B_0] \geq \mathbb{P}[A] > 0$. □

Proposition 4.4.16. *The meeting condition (4.4.15) holds true.*

Proof. Fix $\varepsilon > 0$. From the second Borel–Cantelli lemma, Lemma 4.4.15, and the observations directly following (4.4.16) we see that

$$\mathbb{P}\left[\liminf_{t\to\infty} d(\phi_\sigma(t,\omega,x_1),\phi_\sigma(t,\omega,x_2)) \leq 4\varepsilon\right] = \mathbb{P}[B_n \text{ infinitely often}] = 1.$$

The claim follows since ε was arbitrary. □

4.4.5. UNIFORM WEAK SYNCHRONIZATION

From this point onward we shall exclusively view ϕ_σ as an RDS on $\mathbb{T}$.

Theorem 4.4.17. *The RDS ϕ_σ synchronizes weakly, i.e., there is a minimal weak point attractor $a(\omega)$ consisting of a single point. Moreover, for any $\varepsilon > 0$ we have*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{T}} \mathbb{P}\left[d(\phi_\sigma(t,\omega,x), a(\theta_t\omega)) > \varepsilon\right] = 0, \tag{4.4.20}$$

i.e., the synchronization rate is uniform in the initial condition.

Proof. Taking Propositions 4.4.5, 4.4.7, 4.4.12 and 4.4.16 and Corollary 4.4.10 together, we see that all the conditions of [90, Theorem 2.23] are satisfied and weak synchronization holds.

To show uniform synchronization we will additionally use that the strong mixing rate of ϕ_σ is uniform in the initial condition (see Proposition 4.4.7 and Remark 4.4.8). Fix $\varepsilon, \delta > 0$ and let $z \in \mathbb{T}$ be arbitrary. Weak synchronization implies

$$\lim_{t\to\infty} \mathbb{P}[d(\phi_\sigma(t,\omega,y),\phi(t,\omega,z)) > \varepsilon] = 0, \quad y \in \mathbb{T},$$

so by dominated convergence we can find a time $t_2 > 0$ such that

$$\int_{\mathbb{T}} \mathbb{P}[d(\phi_\sigma(t_2,\omega,y),\phi_\sigma(t_2,\omega,z)) > \varepsilon]\,\mathrm{d}\mu(y) \leq \delta, \tag{4.4.21}$$

$$\mathbb{P}[d(\phi(t_2,\omega,z), a(\theta_{t_2}\omega)) > \varepsilon] \leq \delta. \tag{4.4.22}$$

Using the auxiliary function $\psi\colon x \mapsto (x-1)\mathbb{1}_{[1,2]}(x) + \mathbb{1}_{(2,\infty)}(x)$ we now define

$$f(x) = \mathbb{E}\big[\psi(\varepsilon^{-1} d(\phi_\sigma(t_2,\omega,x),\phi_\sigma(t_2,\omega,z)))\big].$$

Notice that $f \in C_b(\mathbb{T})$ by continuity of ϕ_σ, ψ and dominated convergence, and also that

$$\mathbb{P}\left[d(\phi_\sigma(t_2,\omega,x),\phi_\sigma(t_2,\omega,z)) > 2\varepsilon\right] \le f(x) \le \mathbb{P}\left[d(\phi_\sigma(t_2,\omega,x),\phi_\sigma(t_2,\omega,z)) > \varepsilon\right] \tag{4.4.23}$$

for every $x \in \mathbb{T}$, which implies $\int_{\mathbb{T}} f \,\mathrm{d}\mu \le \delta$ by (4.4.21). By (4.4.8) we then find a time $t_1 > 0$ such that

$$\sup_{x\in\mathbb{T}} |P_t f(x)| \le \delta + \int_{\mathbb{T}} f \,\mathrm{d}\mu \le 2\delta, \qquad t \ge t_1. \tag{4.4.24}$$

Now for $t = s + t_2$ with $s \ge t_1$ we find by the cocycle property and invariance of $\mathbb{P}$:

$$\begin{aligned}
&\mathbb{P}\left[d(\phi_\sigma(s+t_2,\omega,x),\phi_\sigma(t_2,\theta_s\omega,z)) > 2\varepsilon\right] \\
&\quad = \mathbb{P}\left[d(\phi_\sigma(t_2,\omega,\phi_\sigma(s,\theta_{-s}\omega,x)),\phi_\sigma(t_2,\omega,z)) > 2\varepsilon\right] \\
&\quad = P_s\Big(\mathbb{P}\left[d(\phi_\sigma(t_2,\omega,\cdot),\phi_\sigma(t_2,\omega,z)) > 2\varepsilon\right]\Big)(x) \\
&\quad \overset{(4.4.23)}{\le} \sup_{y\in\mathbb{T}} |P_s f(y)| \overset{(4.4.24)}{\le} 2\delta, \qquad x \in \mathbb{T},
\end{aligned}$$

where the second identity follows since $\phi(s,\theta_{-s}\omega,x)$ is independent of $\phi_\sigma(t_2,\omega,\cdot)$ by Proposition 4.4.5-(i). Combining this with the triangle inequality and using the invariance of $\mathbb{P}$ again yields

$$\begin{aligned}
\mathbb{P}\left[d(\phi_\sigma(s+t_2,\omega,x),a(\theta_{s+t_2}\omega)) > 3\varepsilon\right] &\le 2\delta + \mathbb{P}\left[d(\phi_\sigma(t_2,\theta_s\omega,z),a(\theta_{t_2+s}\omega)) > \varepsilon\right] \\
&= 2\delta + \mathbb{P}\left[d(\phi_\sigma(t_2,\omega,z),a(\theta_{t_2}\omega)) > \varepsilon\right] \overset{(4.4.22)}{\le} 3\delta
\end{aligned}$$

for all $s \ge t_1$ and $x \in \mathbb{T}$. Since t_1, t_2 were chosen independently of x, (4.4.20) follows. □

To prove the main result, it only remains to transfer the synchronization properties of ϕ_σ back to u_σ. However, here we encounter a limitation of Theorem 4.4.17, namely that (4.4.20) does not provide a quantitative lower bound on the synchronization time in terms of σ. This poses a problem, since Theorem 4.3.9 shows that the validity of the approximation $\pi(u_\sigma^x(t)) \approx \phi_\sigma(t,\omega,x)$ is only valid on a time scale $t \sim \sigma^{-2}\log(\sigma^{-1})$. Hence, if the time until synchronization of ϕ_σ diverges at a faster rate as $\sigma \searrow 0$, the synchronization behavior cannot be transferred to u_σ. Of course, the characteristic time scale of (4.4.2) is $t \sim \sigma^{-2}$, so it is reasonable to expect that the synchronization should take place before the approximation breaks down. However, since the term $c\,\mathrm{d}t$ in (4.4.2) breaks the time-scaling symmetry, we are currently not able to prove this without imposing Assumption 4.4 (note that this assumption has been unused until now). We believe that a quantitative version of [90, Theorem 2.23] could mitigate the need for Assumption 4.4, and leave this as a suggestion for future work.

To overcome the difficulties outlined above, we relate (4.4.2) to the following SDE in which σ no longer appears:

$$\mathrm{d}\tilde{\gamma} = \mathfrak{a}(\tilde{\gamma})\,\mathrm{d}t + \sum_{k\in\mathbb{Z}} \mathfrak{b}_k(\tilde{\gamma})\,\mathrm{d}\beta_k(t). \tag{4.4.25}$$

Note that if $c = 0$, (4.4.25) can be directly obtained from (4.4.2) by rescaling time. In the case $c \neq 0$ an additional step is needed. Throughout the following, we let $\tilde{\phi}$ denote the random dynamical system over $(\theta_t)_{t\in\mathbb{R}}$ generated by (4.4.25). The following proposition gives an exact relation between ϕ_σ and $\tilde{\phi}$.

Proposition 4.4.18. *For every $\sigma > 0$, there is a map $T\colon \Omega \to \Omega$ which satisfies $T^*\mathbb{P} = \mathbb{P}$ as well as*

$$\mathbb{P}\Big[\phi_\sigma(t,\omega,x) = ct + \tilde{\phi}(\sigma^2 t, T(\omega), x)\Big] = 1, \qquad t \in \mathbb{R}^+, x \in \mathbb{T}. \tag{4.4.26}$$

Proof. Consider first the case $c = 0$. We set $\tilde{\beta}_k(\cdot) = \sigma\beta_k(\sigma^{-2}\cdot)$ and let T_1 be the associated transformation of Ω. It is a basic property of Brownian motion that $T_1^*\mathbb{P} = \mathbb{P}$. By rescaling time in (4.4.2) it is also seen that $\tilde{\gamma}(t) := \gamma(\sigma^{-2}t)$ solves (4.4.25) with β_k replaced by $\tilde{\beta}_k$, so that (4.4.26) holds.

Consider now the case $c \neq 0$. By Assumption 4.4 we must have $\alpha_k = \alpha_{-k}$ for every $k \in \mathbb{N}$. It then follows from (4.4.6b) that $\mathfrak{a}(x)$ is constant, so we simply write $\mathfrak{a}$. Making the substitution $\tilde{\gamma}(t) := \gamma(t) - ct$, we combine (4.4.2) and (4.4.4c) to find:

$$\mathrm{d}\tilde{\gamma} = \sigma^2\mathfrak{a}\,\mathrm{d}t + \sigma\bigg(\mathfrak{b}_0\,\mathrm{d}\beta_0(t) + \sum_{k\in\mathbb{N}}\alpha_k\begin{pmatrix}c_k\\ c_{-k}\end{pmatrix}^{\top} R(-2\pi k\tilde{\gamma})R(-2\pi kct)\begin{pmatrix}\mathrm{d}\beta_k\\ \mathrm{d}\beta_{-k}\end{pmatrix}\bigg) \tag{4.4.27}$$

We now set $\tilde{\beta}_0 = \beta_0$, as well as

$$\begin{pmatrix}\tilde{\beta}_k(\cdot)\\ \tilde{\beta}_{-k}(\cdot)\end{pmatrix} = \int_0^{\cdot} R(-2\pi kcs)\begin{pmatrix}\mathrm{d}\beta_k(s)\\ \mathrm{d}\beta_{-k}(s)\end{pmatrix}, \quad k \in \mathbb{N},$$

and let T_2 be the associated map on Ω. It follows from Lévy's characterization of the Brownian motion that $T_2^*\mathbb{P} = \mathbb{P}$. Moreover, it is seen that

$$\begin{pmatrix}\mathrm{d}\tilde{\beta}_k(t)\\ \mathrm{d}\tilde{\beta}_{-k}(t)\end{pmatrix} = R(-2\pi kct)\begin{pmatrix}\mathrm{d}\beta_k(t)\\ \mathrm{d}\beta_{-k}(t)\end{pmatrix},$$

so that (4.4.27) becomes

$$\begin{aligned}\mathrm{d}\tilde{\gamma} &= \sigma^2\mathfrak{a}\,\mathrm{d}t + \sigma\bigg(\mathfrak{b}_0\,\mathrm{d}\tilde{\beta}_0(t) + \sum_{k\in\mathbb{N}}\alpha_k\begin{pmatrix}c_k\\ c_{-k}\end{pmatrix}^{\top} R(-2\pi k\tilde{\gamma})\begin{pmatrix}\mathrm{d}\tilde{\beta}_k\\ \mathrm{d}\tilde{\beta}_{-k}\end{pmatrix}\bigg)\\ &= \sigma^2\mathfrak{a}\,\mathrm{d}t + \sigma\sum_{k\in\mathbb{Z}}\mathfrak{b}_k(\tilde{\gamma})\,\mathrm{d}\tilde{\beta}_k.\end{aligned}$$

Recalling that $\gamma = \tilde{\gamma} + ct$ and using the result with $c = 0$, we find a map $T_1\colon \Omega \to \Omega$ which leaves $\mathbb{P}$ invariant and satisfies

$$\phi_\sigma(t,\omega,x) = ct + \tilde{\phi}(\sigma^2 t, T_1 \circ T_2(\omega), x).$$

We conclude by noting that $(T_1 \circ T_2)^*\mathbb{P} = T_1^*(T_2^*\mathbb{P}) = \mathbb{P}$. □

4.4.6. PROOF OF THE MAIN RESULT

Proof of Theorem 4.1. Let $(t_\sigma)_{\sigma>0}$ and q be as in the theorem statement and fix $x, y \in \mathbb{R}$. By (4.1.4) and Theorem 4.3.8 (taking into account Remark 4.2.2), it suffices to prove

$$d(\pi(u_\sigma^x(t_\sigma)), \pi(u_\sigma^y(t_\sigma))) \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad \sigma \to 0. \tag{4.4.28}$$

Also by (4.1.4), we can find a decomposition $t_\sigma = s_\sigma + c_\sigma \sigma^{-2}$ with $s_\sigma \geq 0$, and c_σ satisfying

$$0 \leq c_\sigma \leq \log(\sigma^{-1})^{1-q/9}, \qquad \lim_{\sigma\to 0} c_\sigma = \infty. \tag{4.4.29}$$

It then follows from (a time-shifted version of) Theorem 4.3.9 that

$$d(\pi(u_\sigma^x(t_\sigma)), \phi_\sigma(c_\sigma\sigma^{-2}, \theta_{s_\sigma}\omega, \pi(u_\sigma^x(s_\sigma)))) \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad \sigma \to 0, \tag{4.4.30}$$

and likewise for y. Hence, abbreviating $x_s = \pi(u_\sigma^x(s_\sigma))$ (and similarly for y) it will suffice to prove

$$d(\phi_\sigma(c_\sigma\sigma^{-2}, \theta_{s_\sigma}\omega, x_s), \phi_\sigma(c_\sigma\sigma^{-2}, \theta_{s_\sigma}\omega, y_s)) \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad \sigma \to 0, \tag{4.4.31}$$

since (4.4.28) then follows from (4.4.30)-(4.4.31) and the triangle inequality. Fix now $\varepsilon > 0$. Since x_s and y_s are both $\mathscr{F}_{s_\sigma}$-measurable, they are both independent of the map $z \mapsto \phi_\sigma(c_\sigma\sigma^{-2}, \theta_{s_\sigma}\omega, z)$ by Proposition 4.4.5-(i). Thus, we find

$$\begin{aligned}
&\mathbb{P}[d(\phi_\sigma(c_\sigma\sigma^{-2}, \theta_{s_\sigma}\omega, x_s), \phi_\sigma(c_\sigma\sigma^{-2}, \theta_{s_\sigma}\omega, y_s)) > \varepsilon] \\
&\quad \leq \sup_{x,y\in\mathbb{T}} \mathbb{P}[d(\phi_\sigma(c_\sigma\sigma^{-2}, \theta_{s_\sigma}\omega, x), \phi_\sigma(c_\sigma\sigma^{-2}, \theta_{s_\sigma}\omega, y)) > \varepsilon] \\
&\quad = \sup_{x,y\in\mathbb{T}} \mathbb{P}[d(\phi_\sigma(c_\sigma\sigma^{-2}, \omega, x), \phi_\sigma(c_\sigma\sigma^{-2}, \omega, y)) > \varepsilon] \\
&\quad = \sup_{x,y\in\mathbb{T}} \mathbb{P}[d(\tilde{\phi}(c_\sigma, \omega, x), \tilde{\phi}(c_\sigma, \omega, y)) > \varepsilon]
\end{aligned}$$

for every $\sigma > 0$, where we have used Proposition 4.4.18 for the final step. Using (4.4.29) and Theorem 4.4.17 (which applies equally well to $\tilde{\phi}$) we now conclude that (4.4.31) holds. □

4.5. OUTLOOK

We have shown that the phase reduction approach is an effective way to prove synchronization of traveling pulses. We now briefly discuss some possible extensions of the result, including the possibility to remove Assumption 4.4.

4.5.1. FIXED NOISE AMPLITUDE

Instead of considering the joint limit of small noise and long time as in Theorem 4.1, one could instead try to show synchronization for a fixed noise amplitude $\sigma > 0$. In this case, analysis of the long-time behavior is complicated by the fact that the pulse is only metastable, and typically has a limited lifetime. This may be remedied using the theory of *quasi-ergodic* (i.e., conditioned on survival) measures, which have recently been shown to exist in a setting similar to ours [2]. Furthermore, the existence of a conditioned Lyapunov exponent has been shown in [78], which was recently extended to exhibit a full conditioned Lyapunov spectrum [53]. We believe this is an interesting avenue for future research, and expect that further developments of this theory will be helpful to show transient synchronization.

4.5.2. SPATIALLY INHOMOGENEOUS NOISE

In the case where the pulse speed c is nonzero, Assumption 4.4 restricts the noise to be statistically spatially homogeneous. From a physical/symmetry perspective this assumption is not artificial or unreasonable. Moreover, spatially homogeneous noise is frequently used in this setting; see for example [115, 136, 227]. From a mathematical point of view, the extra symmetry simplifies many of the computations in Section 4.4. Most notably, it results in the following:

1. The coefficient $\mathfrak{a}$ in (4.4.2) does not depend on x. This also holds for $\|\mathfrak{b}\|_{\ell^2(\mathbb{Z})}$ and $\|\mathfrak{b}'\|_{\ell^2(\mathbb{Z})}$. From (4.4.2), we may interpret the (deterministic) quantity $c+\sigma^2\mathfrak{a}$ as the stochastically corrected pulse speed (c.f. [98, Theorem 2.3]) [113, §2.2] [4, §4.2].

2. The invariant measure μ in Proposition 4.4.7 is the Lebesgue measure on $\mathbb{T}$.

3. The Lyapunov exponent in (4.4.11) satisfies $\lambda = -\frac{1}{2}\sigma^2\|\mathfrak{b}'\|^2_{\ell^2(\mathbb{Z})}$.

We note that the validity of the phase reduction (Theorems 4.3.8 and 4.3.9), as well as synchronization of the reduced SDE (Theorem 4.4.17) are all established without use of Assumption 4.4. This leads us to conjecture that Assumption 4.4 might not be needed for Theorem 4.1 to hold. However, without Assumption 4.4 we are currently unable to get suitable quantitative control of the synchronization rate, which is otherwise provided by Proposition 4.4.18. Hence, we cannot rule out the possibility that the validity of the approximation $\gamma_\sigma \approx \pi(u_\sigma)$ breaks down before synchronization occurs. We expect that suitable estimates may be obtained by quantifying the results of [90], a problem which we believe to be of independent interest. The work [222] is a first step in this direction.

4.5.3. APPLICATIONS

This chapter also has potential implications for applications, e.g., for the biophysical systems mentioned in Section 4.1. For example, spatial synchronization by noise of nerve impulses along axons could be a natural biological robustness mechanism to avoid multiple short-time shifted pulses to arrive at a single neuron within a short time span. Furthermore, our proof also reveals two important aspects: (I) there is only an intermediate synchronization window of a time scale t_σ with $\sigma^{-2} \ll t_\sigma \ll \exp(\sigma^{-2})$, and (II) spatially homogeneous noise potentially could be beneficial for synchronization as suggested by Assumption 4.4. At first, (I)-(II) seem counter-intuitive for certain applications, e.g., for neuronal dynamics. Yet, (I) can be desirable for controlling pulse dynamics to a critical state, similarly to other self-organized criticality mechanisms, as small perturbations of the time scale can lead to different information processing outcomes. (II) might reveal a connection to chemical and electrical control of action potential propagation, i.e., changing from a heterogeneous chemical or electrical potential around the axon to a more homogeneous one may increase the propensity to synchronize slightly separated pulses along axons.

5

STOCHASTIC INTEGRALS INDEXED BY A PARAMETER

This chapter is based on the article

[A2] S. Cox and J. van Winden. "Sharp supremum and Hölder bounds for stochastic integrals indexed by a parameter". In: *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* (2026). Forthcoming.

Abstract. *We provide sharp bounds for the supremum of countably many stochastic convolutions taking values in a* 2*-smooth Banach space. As a consequence, we obtain sharp bounds on the modulus of continuity of a family of stochastic integrals indexed by a parameter* $x \in M$*, where* M *is a metric space with finite doubling dimension. In particular, we obtain a theory of stochastic integration in Hölder spaces on arbitrary bounded subsets of* $\mathbb{R}^d$*. This is done by relating the (generalized) Hölder-seminorm associated with a modulus of continuity to a supremum over countably many variables, using a Kolmogorov-type chaining argument. We provide two applications of our results: first, we show long-term bounds for Ornstein–Uhlenbeck processes, and second, we derive novel results regarding the modulus of continuity of the parabolic Anderson model.*

5.1. INTRODUCTION

It is well-known that if $(\gamma_k)_{k\in\mathbb{N}}$ is a sequence of independent standard Gaussian random variables and $(c_k)_{k\in\mathbb{N}}$ is a nonnegative decreasing sequence in $\mathbb{R}$ then we have the following equivalence:

$$\mathbb{E}\Big[\sup_{k\in\mathbb{N}}|c_k\gamma_k|\Big]<\infty \quad\Longleftrightarrow\quad \sup_{k\in\mathbb{N}}|c_k\sqrt{\log(k)}|<\infty \tag{5.1.1}$$

(see e.g. [214, Proposition 2.4.16 and Theorem 2.4.18]). In addition, we have the celebrated upper Burkholder–Davis–Gundy inequalities for stochastic integrals: let H,U be separable Hilbert spaces, $(\Omega,\mathscr{F},\mathbb{P},(\mathscr{F}_t)_{t\geq 0})$ a stochastic basis, W a U-cylindrical Wiener process with respect to $(\mathscr{F}_t)_{t\geq 0}$, and let $\mathscr{L}_2(U,H)$ denote the Hilbert–Schmidt operators from U to H. Then there exists a constant $C\in(0,\infty)$ such that

$$\Big\|\sup_{t\geq 0}\Big\|\int_0^t f(s)\,\mathrm{d}W(s)\Big\|_H\Big\|_{L^p(\Omega)}\leq C\sqrt{p}\,\Big\|\Big(\int_0^\infty \|f(s)\|^2_{\mathscr{L}_2(U,H)}\,\mathrm{d}s\Big)^{1/2}\Big\|_{L^p(\Omega)} \tag{5.1.2}$$

for all $p\in[1,\infty)$ and for every $\mathscr{L}_2(U,H)$-valued progressive process $(f(t))_{t\geq 0}$; this was proven in [69] for $H=U=\mathbb{R}$ and extends to the Hilbert space setting thanks to [126]; see [207] and [165] for detailed literature reviews on Burkholder–Davis–Gundy inequalities for stochastic integrals. Note that the asymptotic dependence on p in (5.1.2) cannot be improved [69].

Our first main result combines (5.1.1) and (5.1.2):

Theorem 5.1. *For every $p\in[1,\infty)$ and every sequence of progressive $\mathscr{L}_2(U,H)$-valued stochastic processes $(f_k(t))_{t\geq 0}$, $k\in\mathbb{N}$, it holds that*

$$\begin{aligned}&\Big\|\sup_{t\geq 0,\,k\in\mathbb{N}}\Big\|\int_0^t f_k(s)\,\mathrm{d}W(s)\Big\|_H\Big\|_{L^p(\Omega)}\\ &\qquad\leq 10\,\Big\|\sup_{k\in\mathbb{N}}\sqrt{p+\log(k)}\Big(\int_0^\infty \|f_k(s)\|^2_{\mathscr{L}_2(U,H)}\,\mathrm{d}s\Big)^{1/2}\Big\|_{L^p(\Omega)}.\end{aligned} \tag{5.1.3}$$

The actual result (Theorem 5.3.1 below) is more general: H is replaced by a 2-smooth Banach space X, and the stochastic integral is replaced by a stochastic convolution with a contractive C_0-semigroup on X. The Burkholder–Davis–Gundy inequalities (5.1.2) were proven in this setting in [207, Theorem 2.1]. Note that neither the weights $\sqrt{\log(k)}$ nor the $\sqrt{p}$-dependence in (5.1.3) can be improved.

The proof of Theorem 5.1 is obtained by using exponential tail estimates from [186, Theorem 5.6] to obtain an associated 'good-λ inequality'. This inequality is extended using a union bound, after which (5.1.3) follows from a well-known lemma of Burkholder. A similar strategy can be applied to other types of exponential tail estimates: see Theorem 5.3.6 below for a result analogous to Theorem 5.1 for the Burkholder–Rosenthal inequality for discrete-time martingales. The close relationship between inequalities of the type (5.1.2) (i.e., L^p-square-function type estimates for which the p-dependence in the inequality is known) and associated exponential tail estimates (of Azuma–Hoeffding type) is well-established in the literature, see e.g. [95, 116]. In particular, sharp estimates for the supremum of sequences of discrete-time martingales have been obtained in [95, Theorem 1.5], however, an extension to continuous time would be nontrivial.

Theorem 5.1 (or rather, its generalization Theorem 5.3.1 below) improves [185, (2.12) in Proposition 2.7], which implies that for every $p \in (0,\infty)$ there exists a constant $C_p \in (0,\infty)$ such that

$$\begin{aligned}\Big\| \sup_{t\geq 0,\, 1\leq k\leq n} \Big\| \int_0^t f_k(s)\,\mathrm{d}W(s)\Big\|_H \Big\|_{L^p(\Omega)} \\ \leq C_p \log(n) \Big\| \Big(\int_0^\infty \sup_{1\leq k\leq n} \|f_k(s)\|^2_{\mathscr{L}_2(U,H)}\,\mathrm{d}s\Big)^{1/2} \Big\|_{L^p(\Omega)}\end{aligned} \tag{5.1.4}$$

for all choices of $n\in\mathbb{N}$ and all progressive $\mathscr{L}_2(U,H)$-valued $(f_k(t))_{t\geq 0}$, $k\in\{1,\dots,n\}$ (like Theorem 5.3.1 below, [185, Proposition 2.7] actually considers case that f takes values in a 2-smooth Banach space). Note that the supremum in (5.1.4) is inside the temporal integral and the $\log(n)$ in (5.1.4) is outside of the supremum; in particular, the supremum can only be taken over a finite number of integrals. Moreover, (5.1.3) shows that the $\log(n)$ in (5.1.4) can be replaced by a $\sqrt{\log(n)}$ – in the Hilbert space setting this was also recently observed in [138, Proposition 2.3]. Finally, [185] only proves that $C_p \leq C\sqrt{p}$ for $p\in[2,\infty)$ (and not for $p\in[1,\infty)$). In this respect Theorem 5.3.1 even provides a (minor) improvement of the Burkholder–Davis–Gundy inequality [185, Theorem 4.1]. Indeed, unlike [185] and [207], we do not need to make use of a Burkholder–Rosenthal type inequality, and there is no need to discretize the stochastic integral. This results in a more straightforward proof from start to finish, allows us to cover the cases $p\in[1,2]$ and $p\in[2,\infty)$ simultaneously, and improves the final constant which is found.

Before we demonstrate the applications of Theorem 5.1 (and its generalization Theorem 5.3.1), let us mention that the related result [138, Proposition 2.3] was recently used to prove optimal pathwise convergence rates for temporal discretizations of stochastic partial differential equations [138, Theorem 1.2].

5.1.1. LONG-TIME ESTIMATES FOR ORNSTEIN–UHLENBECK PROCESSES

Our first and most straightforward application of Theorem 5.1/5.3.1 involves the derivation of long-term bounds for Ornstein–Uhlenbeck processes. The following result is a special case of Theorem 5.4.1 below:

Theorem 5.2. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on H satisfying*

$$\|S(t)\|_{\mathscr{L}(H)} \leq e^{-at}, \qquad t\geq 0, \tag{5.1.5}$$

for some $a>0$, let $p\in[1,\infty)$, and let $(f(t))_{t\in[0,T]}$ be a progressive $\mathscr{L}_2(U,H)$-valued process. Then we have the estimate

$$\begin{aligned}\Big\| \sup_{t\in[0,T]} \Big\| \int_0^t S(t-s)f(s)\,\mathrm{d}W(s)\Big\|_H \Big\|_{L^p(\Omega)} \\ \leq 18D\sqrt{p+\log(1+aT)}\,a^{-1/2} \Big\| \sup_{t\in[0,T]} \|f(t)\|_{\mathscr{L}_2(U,H)} \Big\|_{L^p(\Omega)}.\end{aligned} \tag{5.1.6}$$

Long-term estimates of Ornstein–Uhlenbeck processes of the type (5.1.6) are relevant whenever one studies the behavior of a stochastic partial differential equation near a stable manifold. In many cases, one wants to characterize a time scale on which the solutions leave a neighborhood of the manifold. It is the T-dependence of the estimate (5.1.6) which determines this time scale. In the finite-dimensional case, estimates are derived in [26, Chapter 3.1] which are foundational to the work. In the infinite-dimensional case, the development is much more recent. Key difficulties are outlined in [114], where the authors derive a tail estimate in order to show long-term stability of a traveling wave perturbed by noise. The authors use results relating to Talagrand's generic chaining and entropy bounds to derive a tail estimate with the right asymptotic dependence on p and T ([114, Proposition 3.1]). Our proof bypasses this advanced machinery, and additionally gives $L^p(\Omega)$-bounds with good constants. Moreover, we quantify the dependence of the estimate on the parameter a in (5.1.5). The inequality (5.1.6) is optimal in terms of the dependence on a, p, and T.

5.1.2. STOCHASTIC INTEGRATION IN HÖLDER SPACES

Our second, more elaborate, application of Theorem 5.1 is to construct a theory of stochastic integration in the Hölder space C^α. Recall that C^α is neither 2-smooth nor does it have the UMD property, so that 'conventional' vector-valued stochastic integration (as surveyed in [187]) fails. Instead, we take a more direct approach by transforming (5.1.3) into an estimate with C^α norms. To accomplish this, recall that for any $f \in C([0,1])$ we have

$$\sup_{\substack{x,y\in[0,1],\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \simeq_\alpha \sup_{n\in\mathbb{N},\,k\in\{1,\dots,2^n\}} 2^{\alpha n}|f(k2^{-n})-f((k-1)2^{-n})| \tag{5.1.7}$$

by Kolmogorov's chaining technique (see e.g. [214, Appendix A.2]); here '$X \simeq_\alpha Y$' means that there exist constants $c, C \in (0,\infty)$ depending only on α such that $cX \le Y \le CX$. Combining the equivalence (5.1.7) with Theorem 5.1 we obtain the following:

Theorem 5.3. *Let $T > 0$, $p \in [1,\infty)$, $\alpha \in (0,1]$, and let $((\Phi(x)(t))_{t\in[0,T]})_{x\in[0,1]}$ be a family of progressive $\mathscr{L}_2(U,\mathbb{R})$-valued processes indexed by $[0,1]$. Define $I(\Phi)\colon [0,1] \to L^p(\Omega)$ by*

$$I(\Phi)(x) = \int_0^T \Phi(x)(t)\,\mathrm{d}W(t), \quad x \in [0,1], \tag{5.1.8}$$

and suppose that

$$K(\Phi) := \left\| \sup_{x,y\in[0,1],\,x\neq y} \frac{\|\Phi(x)-\Phi(y)\|_{L^2(0,T;\mathscr{L}_2(U,\mathbb{R}))}}{(p-\log(|x-y|))^{-1/2}|x-y|^\alpha} \right\|_{L^p(\Omega)} < \infty. \tag{5.1.9}$$

Then there exists a continuous modification of $I(\Phi)$ (again denoted by $I(\Phi)$), and we have the estimate

$$\left\| \sup_{x,y\in[0,1],\,x\neq y} \frac{|I(\Phi)(x)-I(\Phi)(y)|}{|x-y|^\alpha} \right\|_{L^p(\Omega)} \le C(\alpha)K(\Phi), \tag{5.1.10}$$

for some constant $C(\alpha) \in (0,\infty)$ which depends only α.

Lévy's modulus of continuity theorem ensures that Theorem 5.3 is sharp in terms of the moduli of continuity involved in (5.1.9) and (5.1.10) (see Remark 5.9.3 below). Theorem 5.3 is a special case of Theorem 5.9.1 below. Indeed, the latter extends Theorem 5.3 in two ways:

1. Instead of only considering Hölder continuity in (5.1.10), we consider more general moduli of continuity. This gives more flexibility, and in particular allows one to move the logarithmic term back and forth between (5.1.9) and (5.1.10). However, we are restricted to moduli of continuity for which an analogue of (5.1.7) holds. In particular, we need $1 < \sup_{x\in(0,\infty)} \frac{w(x)}{w(x/2)} < \infty$ (see Definition 5.5.5 below). In Section 5.5 we provide a systematic study of such admissible moduli of continuity, and discuss relevant examples.

2. Instead of considering stochastic integrands $(\Phi(x))_{x\in[0,1]}$ indexed by $x \in [0,1]$, we consider families $(\Phi(x))_{x\in M}$ indexed by a general metric space M. This allows us to consider mixed (space-time) regularity of stochastic processes, and is crucial to prove Theorem 5.4 below. We find that the Minkowski and doubling dimensions of M play a role when seeking an appropriate analogue of (5.1.7). In Section 5.6 we recall these notions of dimension, their relation to the Kolmogorov chaining argument, and show that the dimensional requirements for chaining are satisfied for any bounded subset of Euclidean space.

As the previous discussion suggests, we need to generalize (5.1.7) as an intermediate step to obtain Theorem 5.9.1. This is done in Theorem 5.7.1, which states that for an admissible modulus of continuity w and a metric space (M, d_M) with Minkowski dimension d and finite doubling dimension, we have

$$\sup_{x,y\in M, x\neq y} \frac{\|f(x)-f(y)\|_X}{w(d_M(x,y))} \eqsim_{w,M} \sup_{n\in\mathbb{N}} \frac{\|f(x_n)-f(y_n)\|_X}{w(n^{-1/d})}$$

for all $f \in C(M,X)$, where $(x_n, y_n) \in M\times M$ is an appropriately chosen sequence (that does not depend on f or w). Theorem 5.7.1 is similar in spirit to Ciesielski's embedding for the Hölder spaces [58], but the underlying philosophy is slightly different (see Remark 5.7.4 for a more detailed discussion). As a byproduct of Theorem 5.7.1, we also obtain an elegant short proof of the Kolmogorov–Chentsov theorem, see Theorem 5.8.2 and Remark 5.8.3.

In Section 5.10 we use Theorem 5.9.1 (i.e., the generalization of Theorem 5.3) to investigate the regularity of the 1D parabolic Anderson model (i.e., the stochastic heat equation with linear multiplicative noise). More specifically, we prove the following:

Theorem 5.4. *Let $U\colon [0,T]\times[0,1]\times\Omega \to \mathbb{R}$ be the mild solution to*

$$\begin{aligned} \tfrac{\partial}{\partial t}U(t,x) &= \tfrac{\partial^2}{\partial x^2}U(t,x) + U(t,x)\xi(t,x), && (t,x)\in(0,T)\times(0,1),\\ U(t,0) &= U(t,1) = 0, && t\in(0,T],\\ U(0,x) &= U_0(x), && x\in[0,1], \end{aligned}$$

where ξ is space-time white noise and $U_0 \in L^p(\Omega; C^2([0,1]))$ for some $p\in(4,\infty)$. Then we have

$$\left\| \sup_{\substack{t,s\in[0,T];\\ x,y\in[0,1];\\ (t,x)\neq(s,y)}} \frac{|U(t,x)-U(s,y)|}{(1-\frac14\log(|t-s|))^{\frac12}|t-s|^{\frac14} + (1-\frac12\log(|x-y|))|x-y|^{\frac12}} \right\|_{L^p(\Omega)} < \infty.$$

This improves the classical regularity results for the parabolic Anderson model, which typically use Sobolev embeddings or the Kolmogorov–Chentsov theorem to prove that $U \in L^p(\Omega; C^{\alpha-\frac{1}{p},2(\alpha-\frac{1}{p})}([0,T]\times[0,1]))$ for all $\alpha \in (\frac{1}{p},\frac{1}{4})$, and then a localization argument to prove that $U \in C^{\alpha,2\alpha}([0,T]\times[0,1])$ a.s. for all $\alpha \in (0,\frac{1}{4})$ (see e.g. [183]). Our technique circumvents the use of the Kolmogorov–Chentsov theorem (whence we do not lose $\frac{1}{p}$ in regularity) *and* considers a stronger modulus of continuity (compared to [55, Theorem 2.1], we do not lose an epsilon in terms of Hölder regularity). The technique for proving Theorem 5.4 essentially combines Theorem 5.9.1 (the generalization of Theorem 5.3) with regularity results for the Dirichlet Green's function. We note that this approach can also be employed to obtain analogous results for other stochastic differential equations involving a linear second-order differential operator in the leading term.

5.2. PRELIMINARIES AND NOTATION

We use the convention that $\mathbb{N}$ is the set of strictly positive integers and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Given a finite set A, we let $\mathrm{card}(A) \in \mathbb{N}_0$ denote its cardinality. We denote the natural logarithm by log. When $(S,\mathscr{A},\mu)$ is a σ-finite measure space and $(X,\|\cdot\|_X)$ is a Banach space, we write $L^p(S;X)$ for the usual Lebesgue–Bochner space of strongly $\mathscr{A}$-measurable and p-integrable (or essentially bounded in the case $p=\infty$) X-valued functions on S. Note that if X is separable, the notions of measurability and strong measurability coincide. In the case where S is an interval (i.e., $S=(a,b)$), we write $L^p(a,b;X)$ instead of $L^p((a,b);X)$.

5.2.1. METRIC SPACES

Let (M,d_M) be a metric space. As a matter of convenience we will assume all metric spaces mentioned are nontrivial. We define the diameter $\Delta(M) \in [0,\infty]$ of M by

$$\Delta(M) = \sup_{x,y\in M} d_M(x,y),$$

and we may write $\Delta_{d_M}(M)$ to emphasize the dependence on d_M when confusion is possible. For $x \in M$ and $A \subseteq M$ we set $d_M(x,A) = \inf\{d_M(x,y) : y \in A\}$. For $x \in M$ and $r \in (0,\infty)$ we define the open ball $B_x(r)$ around x with radius r as

$$B_x(r) = \{y \in M : d_M(x,y) < r\}.$$

Similarly, the closed ball $\overline{B}_x(r)$ around x with radius r is defined as

$$\overline{B}_x(r) = \{y \in M : d_M(x,y) \le r\}.$$

We may write $B_x(d_M,r)$ (resp. $\overline{B}_x(d_M,r)$) to emphasize the dependence on d_M when confusion is possible. For $\eta \in (0,\infty)$, we let $\mathscr{N}(M,d_M,\eta)$ denote the minimal number of open d_M-balls of radius η needed to cover M, i.e.

$$\mathscr{N}(M,d_M,\eta) = \min\Big\{\mathrm{card}(A) : A \subseteq M \text{ finite}, M \subseteq \bigcup_{x\in A} B_x(\eta)\Big\},$$

where we use the convention that the minimum of the empty set is ∞.

5.2.2. FUNCTIONAL ANALYSIS

We recall the definition of a p-smooth Banach space; for details see e.g. [186, Section 2.2], [231, Chapter 3]. Note that the classical definition involving the modulus of smoothness of X (see, e.g., [231, Definition 3.1.2]) is equivalent to this definition (see [231, Proposition 3.1.2]):

Definition 5.2.1. Let $p \in [1,2]$ and $D \geq 1$. A Banach space $(X, \|\cdot\|_X)$ is called (p,D)-*smooth* if

$$\forall x, y \in X\colon \|x+y\|_X^p + \|x-y\|_X^p \leq 2\|x\|_X^p + 2D^p\|y\|_X^p,$$

and it is called p-smooth if it is (p,D)-smooth for some $D \geq 1$.

Remark 5.2.2. By the parallelogram identity every Hilbert space is $(2,1)$-smooth. Moreover, for $2 \leq p < \infty$ the Lebesgue space $L^p(S)$ is $(2, \sqrt{p-1})$-smooth, see [195, Proposition 2.1] and [185, Proposition 2.2].

Definition 5.2.3. A family $(S(t))_{t\geq 0}$ of bounded linear operators on a Banach space $(X, \|\cdot\|_X)$ is called a C_0-*semigroup* if it satisfies the following conditions:

1. $S(0) = I$.
2. $S(t+s) = S(t)S(s)$ for all $t, s \geq 0$.
3. The map $t \mapsto S(t)$ is strongly continuous on $[0,\infty)$.

A C_0-semigroup $(S(t))_{t\geq 0}$ is called *contractive* if the inequality $\|S(t)x\|_X \leq \|x\|_X$ holds for all $t \geq 0$ and all $x \in X$.

We also recall the notion of γ-radonifying operators. This class of operators generalizes Hilbert–Schmidt operators in the context of Banach-space valued stochastic integration.

Definition 5.2.4. Let H be a separable Hilbert space and let $(X, \|\cdot\|_X)$ be a Banach space. The space of γ-*radonifying operators from* H *to* X, denoted $\gamma(H,X)$, consists of the closure of the finite rank operators $T\colon H \to X$ with respect to the norm

$$\|T\|_{\gamma(H,X)} := \mathbb{E}\Big[\Big\|\sum_{i\in\mathbb{N}} \gamma_i T h_i\Big\|_X^2\Big]^{1/2}, \tag{5.2.1}$$

where $(\gamma_i)_{i\in\mathbb{N}}$ is a sequence of independent standard Gaussian variables, and $(h_i)_{i\in\mathbb{N}}$ is an orthonormal basis of H.

We remark that the definition of $\gamma(H,X)$ is independent of the choice of basis, that $\gamma(H,X) \simeq \mathscr{L}_2(H,X)$ when X is a Hilbert space, and that $\gamma(H,X) \hookrightarrow \mathscr{L}(H,X)$. See also [184] and [120, Chapter 9] for a detailed treatment of γ-radonifying operators.

5.2.3. STOCHASTIC CALCULUS

We fix once and for all a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\geq 0})$. It is implied throughout the chapter that all mentioned random variables and processes live on this space, unless the contrary is explicitly stated. Moreover, when we speak of adaptedness or progressive measurability without mentioning a filtration, it is with respect to $(\mathscr{F}_t)_{t\geq 0}$.

We also fix a separable Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$, which is to be used for constructing our Wiener processes.

We will make frequent use of the theory of stochastic integration theory in 2-smooth Banach spaces. We do not give a full introduction here (instead referring to [187]), but just recall that this allows us to define a stochastic integral of the form

$$I(f) := \int_0^T f(s)\,\mathrm{d}W(s), \tag{5.2.2}$$

when $T > 0$, $(X, \|\cdot\|_X)$ is a separable 2-smooth Banach space, W is an H-cylindrical Wiener process, and $(f(t))_{t\in[0,T]}$ is a progressive $\gamma(H,X)$-valued process (see Definition 5.2.4) with sample paths in $L^2(0,T;\gamma(H,X))$ a.s. In the finite-dimensional case, this notion of stochastic integration reduces to the usual Itô integral.

If additionally $(S(t))_{t\geq 0}$ is a contractive C_0-semigroup on X, it is well-known (see [186, Section 5] for an overview of the historical development) that the convolution process $(\Psi(t))_{t\in[0,T]}$, defined via

$$\Psi(t) := \int_0^t S(t-s)f(s)\,\mathrm{d}W(s), \quad t \in [0,T], \tag{5.2.3}$$

has a version with sample paths in $C([0,T],X)$ a.s. Throughout the chapter, when we write stochastic convolutions of the form (5.2.3), it is thus implied that we use such a continuous version.

5.3. OPTIMAL BOUNDS FOR INDEXED STOCHASTIC PROCESSES

In this section, we prove a $\sqrt{\log(n)}$-weighted bound for the moments of the supremum of countably many 'martingale-like' processes. More specifically, in Theorem 5.3.1 we consider sequences of stochastic convolutions, and Theorem 5.3.6 we consider sequences of discrete-time martingales. The general philosophy behind the proof is that an exponential tail estimate for a 'martingale-like' quantity implies a weighted $L^p(\Omega)$ estimate for sequences of this quantity. We take the tail estimates in [186, Theorem 5.6] as an input for the stochastic convolutions, and those in [195, Lemma 4.2] as an input for the discrete-time martingales. Note that this approach can be applied to other exponential tail estimates for 'martingale-like' quantities, see also Remark 5.3.7 below.

5.3.1. BDG-TYPE BOUNDS FOR INDEXED STOCHASTIC CONVOLUTIONS

Throughout the section, we let $T > 0$ and let $(X, \|\cdot\|_X)$ be a $(2,D)$-smooth Banach space. We also let $(W_j)_{j\in\mathbb{N}_0}$ be a sequence of (not necessarily independent) H-cylindrical Wiener processes on $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\geq 0})$ (recall the stochastic setup from Section 5.2), and set $W \equiv W_0$. We can now state the main result of this section.

Theorem 5.3.1. *Let $p \in [1,\infty)$, let $((\psi_j(t))_{t\in[0,T]})_{j\in\mathbb{N}}$ be a sequence of progressive $\gamma(H,X)$-valued processes, and let $((S_j(t))_{t\geq 0})_{j\in\mathbb{N}}$ be a sequence of contractive semigroups on X.*

Then we have the inequality

$$\begin{aligned}&\Big\|\sup_{j\in\mathbb{N}}\sup_{t\in[0,T]}\Big\|\int_0^t S_j(t-s)\psi_j(s)\,\mathrm{d}W_j(s)\Big\|_X\Big\|_{L^p(\Omega)}\\&\quad\le 10\,D\Big\|\sup_{j\in\mathbb{N}}\Big(\sqrt{p+\log(j)}\|\psi_j\|_{L^2(0,T;\gamma(H,X))}\Big)\Big\|_{L^p(\Omega)}.\end{aligned}\tag{5.3.1}$$

Remark 5.3.2. In Theorem 5.3.1, we could additionally let H, X, T or even the filtration or probability space depend on the index j without significantly altering the proof. However, we refrain from doing so, as it would be detrimental to the presentation.

As mentioned, the proof of Theorem 5.3.1 hinges on the exponential tail estimate [186, Theorem 5.6]. For the reader's convenience, we state a version of this theorem which is adapted to our notation and our specific use case.

Theorem 5.3.3 (Van Neerven, Veraar)**.** *Let $(S(t))_{t\ge0}$ be a contractive C_0-semigroup on X, and let $(\psi(t))_{t\in[0,T]}$ be a progressive $\gamma(H,X)$-valued process which satisfies*

$$\mathbb{P}\Big[\|\psi\|_{L^2(0,T;\gamma(H,X))}\le\sigma\Big]=1$$

for some $\sigma\in(0,\infty)$. Then the following exponential tail estimate holds:

$$\mathbb{P}\Big[\sup_{t\in[0,T]}\Big\|\int_0^t S(t-s)\psi(s)\,\mathrm{d}W(s)\Big\|_X\ge\lambda\Big]\le 3\exp\Big(-\frac{\lambda^2}{4D^2\sigma^2}\Big),\quad \lambda\ge0.\tag{5.3.2}$$

Remark 5.3.4. The original theorem in [186] is formulated for (contractive) evolution families, a setting in which our results and methods still apply. However, to avoid technical conditions and for the sake of presentation, we have restricted ourselves to the semigroup setting. Note that this setting still includes 'plain' stochastic integrals, since the identity map is trivially a C_0-semigroup.

In order to prove Theorem 5.3.1, we first derive from Theorem 5.3.3 a good-λ inequality for stochastic convolutions.

Lemma 5.3.5. *Let $(S(t))_{t\ge0}$ be a contractive C_0-semigroup on X, let $(\psi(t))_{t\in[0,T]}$ be a progressive $\gamma(H,X)$-valued process, and define*

$$\Psi(t):=\int_0^t S(t-s)\psi(s)\,\mathrm{d}W(s),\quad t\in[0,T],$$

together with $\Psi^*:=\sup_{t\in[0,T]}\|\Psi(t)\|_X$ and $s(\psi):=\|\psi\|_{L^2(0,T;\gamma(H,X))}$. Then for all $\beta>1$ and $\delta,\lambda>0$, we have the inequality

$$\mathbb{P}\big[\Psi^*>\beta\lambda,\ s(\psi)\le\delta\lambda\big]\le 3\exp\Big(-\frac{(\beta-1)^2}{4D^2\delta^2}\Big)\mathbb{P}\big[\Psi^*>\lambda\big].\tag{5.3.3}$$

Proof. We use a classical three stopping times argument. We define

$$\begin{aligned}\mu&:=\sup\{t\in[0,T]:\|\Psi(t)\|_X\le\beta\lambda\},\\ \nu&:=\sup\{t\in[0,T]:\|\Psi(t)\|_X\le\lambda\},\\ \sigma&:=\sup\{t\in[0,T]:\|\psi\|_{L^2(0,t;\gamma(H,X))}\le\delta\lambda\},\end{aligned}$$

and additionally introduce

$$\Phi(t) := \int_{\nu}^{t} S(t-s)1_{[0,\sigma]}(s)\psi(s)\,\mathrm{d}W(s), \quad t \in [\nu, T]. \tag{5.3.4}$$

together with $\Phi^* := \sup_{t\in[\nu,T]} \|\Phi(t)\|_X$. Observe that the event $\{s(\psi) \le \delta\lambda\}$ implies $\sigma = T$ so that also

$$\Psi(\mu) = S(\mu-\nu)\Psi(\nu) + \Phi(\mu).$$

Since $\|S(\mu-\nu)\Psi(\nu)\|_X \le \lambda$ by contractivity and the definition of ν, we see from the reverse triangle inequality that the event $\{\Psi^* > \beta\lambda,\, s(\psi) \le \delta\lambda\}$ implies that $\Phi^* \ge (\beta-1)\lambda$. It also trivially implies that $\nu < T$. Thus, we have

$$\begin{aligned}\mathbb{P}\left[\Psi^* > \beta\lambda,\, s(\psi) \le \delta\lambda\right] &\le \mathbb{P}\left[\Phi^* > (\beta-1)\lambda,\, \nu < T\right] \\ &= \mathbb{P}\left[\Phi^* > (\beta-1)\lambda \mid \nu < T\right]\mathbb{P}\left[\Psi^* > \lambda\right].\end{aligned}$$

It only remains to estimate the conditional probability. To do this, we note that since the integral defining Φ starts at time ν, (5.3.4) can also be interpreted as a stochastic integral with respect to the conditional probability measure $\widetilde{\mathbb{P}}(\cdot) := \mathbb{P}\left[\cdot \mid \nu < T\right]$ by the strong Markov property of a Wiener process. Thus, we can use Theorem 5.3.3 to estimate

$$\widetilde{\mathbb{P}}\big(\Phi^* > (\beta-1)\lambda\big) \le 3\exp\Big(-\frac{(\beta-1)^2}{4D^2\delta^2}\Big),$$

since $\|\mathbb{1}_{[0,\sigma]}\psi\|_{L^2(0,T;\gamma(H,X))} \le \delta\lambda$ by definition of σ. □

Proof of Theorem 5.3.1. For each $j \in \mathbb{N}$, we define

$$\Psi_j^* := \sup_{t\in[0,T]} \Big\| \int_0^t S_j(t-s)\psi_j(s)\,\mathrm{d}W(s)\Big\|_X, \quad s(\psi_j) := \|\psi_j\|_{L^2(0,T;\gamma(H,X))},$$

together with the parameters

$$\beta = 2, \quad \delta_j := \frac{1}{4D\sqrt{p+\log(j)}}, \quad \varepsilon_j := 3\exp\Big(-\frac{(\beta-1)^2}{4D^2\delta_j^2}\Big).$$

By a union bound and Lemma 5.3.5, we find the inequality

$$\begin{aligned}\mathbb{P}\Big[\sup_{j\in\mathbb{N}}\Psi_j^* > \beta\lambda,\, \sup_{j\in\mathbb{N}}\delta_j^{-1}s(\psi_j) \le \lambda\Big] &\le \sum_{j\in\mathbb{N}} \mathbb{P}\left[\Psi_j^* > \beta\lambda,\, \delta_j^{-1}s(\psi_j) \le \lambda\right] \\ &\le \sum_{j\in\mathbb{N}} \varepsilon_j \mathbb{P}\left[\Psi_j^* > \lambda\right] \le \Big(\sum_{j\in\mathbb{N}}\varepsilon_j\Big)\mathbb{P}\Big[\sup_{j\in\mathbb{N}}\Psi_j^* > \lambda\Big],\end{aligned}$$

which is valid for all $\lambda > 0$. We now observe that $\varepsilon_j = 3e^{-4p}j^{-4}$, which implies

$$\beta^p\varepsilon_j = 3(\beta e^{-4})^p j^{-4} \le 3\beta e^{-4} j^{-4}, \quad j \in \mathbb{N},$$

since $p \ge 1$ and $\beta e^{-4} = 2e^{-4} \le 1$. Therefore, we have

$$\sum_{j\in\mathbb{N}} \beta^p\varepsilon_j \le 3\beta e^{-4}\sum_{j\in\mathbb{N}} j^{-4} = \tfrac{1}{15}\pi^4 e^{-4} < \tfrac{3}{25},$$

as can be numerically verified. Thus, we may apply [48, Lemma 7.1] with $\Phi(x) = x^p$, $\gamma = 2^p$, $\varepsilon = \sum_{j\in\mathbb{N}} \varepsilon_j$, and $\delta = \eta = 1$ to find

$$\mathbb{E}\Big[\big(\sup_{j\in\mathbb{N}} \Psi_j^*\big)^p\Big] \le 2^p \tfrac{25}{22} \mathbb{E}\Big[\big(\sup_{j\in\mathbb{N}} \delta_j^{-1} s(\psi_j)\big)^p\Big].$$

Substituting the definition of δ_j and using $\frac{25}{22} \le \frac{10}{8} \le (\frac{10}{8})^p$ gives

$$\mathbb{E}\Big[\big(\sup_{j\in\mathbb{N}} \Psi_j^*\big)^p\Big] \le (10\,D)^p \mathbb{E}\Big[\big(\sup_{j\in\mathbb{N}} \sqrt{p+\log(j)}\, s(\psi_j)\big)^p\Big],$$

which results in the desired estimate upon taking p-th roots. □

5.3.2. BURKHOLDER–ROSENTHAL BOUNDS FOR INDEXED MARTINGALES

Using the same technique as in the previous section, we now extend Pinelis' Burkholder–Rosenthal inequality [195, Theorem 4.1] to sequences of martingales. We first introduce some notation. For any X-valued martingale $(f(i))_{i\in\mathbb{N}_0}$ with respect to the filtration $(\mathscr{F}_i)_{i\in\mathbb{N}_0}$, we define the quantities:

$$\begin{aligned}
f^* &:= \sup_{i\in\mathbb{N}_0} \|f(i)\|_X,\\
d^*(f) &:= \sup_{i\in\mathbb{N}} \|f(i) - f(i-1)\|_X,\\
s(f) &:= \Big(\sum_{i\in\mathbb{N}} \mathbb{E}\Big[\|f(i) - f(i-1)\|_X^2 \,\Big|\, \mathscr{F}_{i-1}\Big]\Big)^{\frac{1}{2}}.
\end{aligned}$$

The notation is chosen to remain consistent with [195] whenever possible. We also recall that a martingale $(f(i))_{i\in\mathbb{N}_0}$ with respect to a filtration $(\mathscr{F}_i)_{i\in\mathbb{N}_0}$ is said to have *conditionally symmetric increments* if $\mathbb{P}\big[f(i) - f(i-1) \in B \,|\, \mathscr{F}_{i-1}\big] = \mathbb{P}\big[f(i-1) - f(i) \in B \,|\, \mathscr{F}_{i-1}\big]$ for all Borel sets $B \subseteq X$ and all $i \in \mathbb{N}$.

Theorem 5.3.6. *Let $p \in [1,\infty)$, and let $((f_j(i))_{i\in\mathbb{N}_0})_{j\in\mathbb{N}}$ be a sequence of X-valued processes, each of which is a martingale with respect to $(\mathscr{F}_i)_{i\in\mathbb{N}_0}$, starts at zero, and has conditionally symmetric increments. Then we have the inequality*

$$\begin{aligned}
\big\|\sup_{j\in\mathbb{N}} f_j^*\big\|_{L^p(\Omega)} \le 13\,\big\|\sup_{j\in\mathbb{N}} (p+\log(j))\, d^*(f_j)\big\|_{L^p(\Omega)}&\\
+ 14\,D\,\big\|\sup_{j\in\mathbb{N}} \sqrt{p+\log(j)}\, s(f_j)\big\|_{L^p(\Omega)}&.
\end{aligned} \tag{5.3.6}$$

Proof. We set $\beta = 2 + \frac{1}{5}$, and introduce for $j \in \mathbb{N}$ the quantities

$$\delta_{j,1} := \frac{2}{11\sqrt{p+\log(j)}}, \qquad \delta_{j,2} := \frac{1}{5(p+\log(j))},$$

$$N_j := \frac{\beta - 1 - \delta_{j,2}}{\delta_{j,2}}, \qquad \varepsilon_j := 2\Big(\frac{e}{N_j}\frac{\delta_{j,1}^2}{\delta_{j,2}^2}\Big)^{N_j},$$

as well as

$$w^*(f_j) := \max\Big\{\delta_{j,2}^{-1} \cdot d^*(f_j),\ D\delta_{j,1}^{-1} \cdot s(f_j)\Big\}.$$

We begin by observing that

$$\beta - 1 - \delta_{j,2} \geq 1 > 0, \quad j \in \mathbb{N}. \tag{5.3.7}$$

Thus, by [195, Lemma 4.2], we have the inequality

$$\mathbb{P}\left[f_j^* > \beta\lambda,\, w^*(f_j) \leq \lambda\right] \leq \varepsilon_j \mathbb{P}\left[f_j^* > \lambda\right]$$

for all $\lambda > 0$ and $j \in \mathbb{N}$. By the same type of union bound used in the proof of Theorem 5.3.1, this implies

$$\mathbb{P}\left[\sup_{j\in\mathbb{N}} f_j^* > \beta\lambda,\, \sup_{j\in\mathbb{N}} w^*(f_j) \leq \lambda\right] \leq \Big(\sum_{j\in\mathbb{N}} \varepsilon_j\Big) \mathbb{P}\left[\sup_{j\in\mathbb{N}} f_j^* > \lambda\right], \quad \lambda > 0.$$

From (5.3.7), we also obtain $N_j \geq 5(p + \log(j))$ for every $j \in \mathbb{N}$, which implies

$$\varepsilon_j = 2\Big(\frac{e}{N_j}\frac{\delta_{j,1}^2}{\delta_{j,2}^2}\Big)^{N_j} \leq 2\Big(\frac{20e}{121}\Big)^{N_j} \leq 2\,e^{-4p} j^{-4}, \quad j \in \mathbb{N},$$

since $\frac{20e}{121} < e^{-\frac{4}{5}}$, as may be numerically verified. This further implies

$$\beta^p \varepsilon_j \leq 2(\beta e^{-4})^p j^{-4} \leq 2\beta e^{-4} j^{-4}, \quad j \in \mathbb{N},$$

so that

$$\sum_{j\in\mathbb{N}} \beta^p \varepsilon_j \leq 2\beta e^{-4} \sum_{j\in\mathbb{N}} j^{-4} = 2\beta e^{-4} \tfrac{\pi^4}{90} < \tfrac{1}{11}.$$

We now apply [48, Lemma 7.1] with $\Phi(x) = x^p$, $\gamma = \beta^p$, and $\delta = \eta = 1$ to obtain

$$\mathbb{E}\big[(\sup_{j\in\mathbb{N}} f_j^*)^p\big] \leq \beta^p\, \tfrac{11}{10} \mathbb{E}\big[(\sup_{j\in\mathbb{N}} w^*(f_j))^p\big].$$

Taking p-th roots and using the definition of $w^*(f_j)$ and β gives

$$\begin{aligned}\|\sup_{j\in\mathbb{N}} f_j^*\|_{L^p(\Omega)} \leq \tfrac{11}{5}\,\tfrac{11}{10}\Big(&5\|\sup_{j\in\mathbb{N}}(p+\log(j))\, d^*(f_j)\|_{L^p(\Omega)} \\ &+ \tfrac{11}{2} D \|\sup_{j\in\mathbb{N}} \sqrt{p+\log(j)}\, s(f_j)\|_{L^p(\Omega)}\Big),\end{aligned}$$

which shows the result upon estimating the fractions. □

Remark 5.3.7. As mentioned above, the proofs of Theorems 5.3.1 and 5.3.6 rely on tail bounds for martingales. In particular, analogous results could be obtained for other discrete-time martingale inequalities by employing the tail bounds provided by [195, Theorem 3.6], [182, Theorem 1.5], or [159, Theorem 1.3].

5.4. EXPONENTIALLY STABLE STOCHASTIC CONVOLUTIONS

In this section, we derive long-term estimates for the running maximum of an Ornstein–Uhlenbeck process as direct application of Theorem 5.3.1. Throughout the section, let $T>0$, let $(X,\|\cdot\|_X)$ be a $(2,D)$-smooth Banach space, and let $(S(t))_{t\geq 0}$ be a contractive C_0-semigroup on X which has generator A. We also let W be an H-cylindrical Wiener process on $(\Omega,\mathscr{F},\mathbb{P},(\mathscr{F}_t)_{t\geq 0})$ (recall the stochastic setup from Section 5.2). We consider the stochastic evolution equation

$$\mathrm{d}u(t) = Au(t)\,\mathrm{d}t + f(t)\,\mathrm{d}W(t), \quad t\in[0,\infty), \tag{5.4.1}$$

with $u(0)=0$, where $(f(t))_{t\in[0,\infty)}$ is a progressive a.s. square-integrable $\gamma(H,X)$-valued process. Recall that the (mild) solution to (5.4.1) is commonly referred to as an *Ornstein–Uhlenbeck process*; it is a continuous X-valued process satisfying the following variation-of-constants formula:

$$u(t) = \int_0^t S(t-s)f(s)\,\mathrm{d}W(s), \quad t\in[0,\infty) \tag{5.4.2}$$

(see also Section 5.2.3).

We are interested in estimating u in terms of f in the case where the dynamics of A drive the solution to zero at an exponential rate. This is captured by the following assumption:

Assumption 5.1. *The semigroup* $(S(t))_{t\geq 0}$ *is* exponentially stable, *meaning there exists a constant* $a>0$ *such that*

$$\|S(t)\|_{\mathscr{L}(X)} \leq e^{-at}, \quad t\geq 0. \tag{5.4.3}$$

Under Assumption 5.1, there is a delicate balance in (5.4.1) between exponential decay to zero due to A, and stochastic forcing away from zero due to $f\,\mathrm{d}W$. Indeed, using (5.4.2), [207, Theorem 1.1], Hölder's inequality, and Assumption 5.1, it is straightforward to derive an estimate of the form

$$\sup_{t\in[0,T]} \|u(t)\|_{L^p(\Omega;X)} \leq CD\sqrt{p}\,a^{-1/2}\Big\|\sup_{t\in[0,T]}\|f(t)\|_{\gamma(H,X)}\Big\|_{L^p(\Omega)}, \tag{5.4.4}$$

where $p\in[1,\infty)$ and C is an absolute constant. However, the case where the supremum over t is inside the expectation on the left-hand side of (5.4.4) is more delicate. There, it is no longer expected to have a constant which is independent of T. For example, the moments of a one-dimensional Ornstein–Uhlenbeck process (corresponding to $H=X=\mathbb{R}$, $A=-\lambda$ and $f\equiv\sigma$ for some $\lambda,\sigma>0$), already grow like $\sqrt{\log(T)}$, even though we have $f\in L^\infty(\Omega\times\mathbb{R}^+)$ in this case. Using the factorization method directly (see [186, Theorem 4.5] for a general version), it is possible to derive the estimate

$$\Big\|\sup_{t\in[0,T]}\|u(t)\|_X\Big\|_{L^p(\Omega)} \leq CD\sqrt{pT}\Big\|\sup_{t\in[0,T]}\|f(t)\|_{\gamma(H,X)}\Big\|_{L^p(\Omega)}, \tag{5.4.5}$$

where $p\in[1,\infty)$ and C is an absolute constant. However, this method does not incorporate the exponential stability of the semigroup, and it unclear how to do so. Consequently, (5.4.5) does not produce a constant which has the expected scaling $\sqrt{\log(T)}$, like in the one-dimensional case.

We now state Theorem 5.4.1, which shows for the first time an $L^p(\Omega)$-estimate which has the correct (joint) asymptotic dependence on a, p, and T. We hope that Theorem 5.4.1 may be of use to future authors attempting to generalize the results of [26] to an infinite-dimensional setting. Note that the proof is straightforward, and relies mostly on two direct applications of Theorem 5.3.1.

Theorem 5.4.1. *Suppose that Assumption 5.1 holds. Let $p \in [1,\infty)$, and let $(f(t))_{t\in[0,T]}$ be a progressive $\gamma(H,X)$-valued process. Then we have the estimate*

$$\begin{aligned}\Big\| \sup_{t\in[0,T]} \Big\| \int_0^t S(t-s) f(s)\,\mathrm{d}W(s) \Big\|_X \Big\|_{L^p(\Omega)} \\ \leq 18\, D \sqrt{p + \log(1+aT)}\, a^{-1/2} \Big\| \sup_{t\in[0,T]} \|f(t)\|_{\gamma(H,X)} \Big\|_{L^p(\Omega)}.\end{aligned} \tag{5.4.6}$$

Proof. By rescaling time (replace T by $a^{-1}T$, $S(\cdot)$ by $S(a^{-1}\cdot)$, f by $f(a^{-1}\cdot)$, and $W(\cdot)$ by $a^{1/2}W(a^{-1}\cdot)$), it suffices to prove the case $a = 1$. We first consider the situation $T \in \mathbb{N}$, where we set $\mathscr{I} = \{0,\dots,T-1\}$. We define

$$\begin{aligned} u(t) &= \int_0^t S(t-s) f(s)\,\mathrm{d}W(s), \qquad t \in [0,T], \\ v_n(t) &= \int_n^{n+t} S(n+t-s) f(s)\,\mathrm{d}W(s), \qquad n \in \mathscr{I},\ t \in [0,1]. \end{aligned}$$

By the semigroup property and linearity of the stochastic integral, we have the identity

$$u(n+t) = S(t)u(n) + v_n(t), \qquad n \in \mathscr{I},\ t \in [0,1].$$

Thus, by the triangle inequality and contractivity of $(S(t))_{t\geq 0}$, we get

$$\sup_{t\in[0,T]} \|u(t)\|_X \leq \sup_{n\in\mathscr{I}} \|u(n)\|_X + \sup_{n\in\mathscr{I}} \sup_{t\in[0,1]} \|v_n(t)\|_X. \tag{5.4.7a}$$

A direct application of Theorem 5.3.1 now gives

$$\Big\| \sup_{n\in\mathscr{I}} \|u(n)\|_X \Big\|_{L^p(\Omega)} \leq 10\, D \sqrt{p + \log(T)} \Big\| \sup_{n\in\mathscr{I}} \|S(n-\cdot) f(\cdot)\|_{L^2(0,n;\gamma(H,X))} \Big\|_{L^p(\Omega)},$$

at which point we apply Hölder's inequality and (5.4.3) (recall that we are treating the case $a = 1$) to find

$$\Big\| \sup_{n\in\mathscr{I}} \|u(n)\|_X \Big\|_{L^p(\Omega)} \leq 5\sqrt{2}\, D \sqrt{p + \log(T)} \Big\| \sup_{t\in[0,T]} \|f(t)\|_{\gamma(H,X)} \Big\|_{L^p(\Omega)}. \tag{5.4.7b}$$

In a similar way, Theorem 5.3.1 and Hölder's inequality also give

$$\begin{aligned} \Big\| \sup_{n\in\mathscr{I}} \sup_{t\in[0,1]} \|v_n(t)\|_X \Big\|_{L^p(\Omega)} &\leq 10\, D \sqrt{p + \log(T)} \Big\| \sup_{n\in\mathscr{I}} \|f\|_{L^2(n,n+1;\gamma(H,X))} \Big\|_{L^p(\Omega)} \\ &\leq 10\, D \sqrt{p + \log(T)} \Big\| \sup_{t\in[0,T]} \|f(t)\|_{\gamma(H,X)} \Big\|_{L^p(\Omega)}. \end{aligned} \tag{5.4.7c}$$

Combining (5.4.7a), (5.4.7b), and (5.4.7c) then yields

$$\Big\| \sup_{t\in[0,T]} \|u(t)\|_X \Big\|_{L^p(\Omega)} \le 18D\sqrt{p+\log(T)} \Big\| \sup_{t\in[0,T]} \|f(t)\|_{\gamma(H,X)} \Big\|_{L^p(\Omega)}.$$

The general result now follows by rounding T up to the nearest integer. □

Remark 5.4.2. We expect that the requirement that $S(t)$ is contractive (which is implied by Assumption 5.1) can be lifted, at the expense of having a proof which is no longer a direct application of Theorem 5.3.1. To accomplish this, one should first derive an estimate of the form

$$\mathbb{P}\Big[\sup_{t\in[0,n]} \|u(t)\|_X \ge \lambda,\ \sup_{t\in[0,n]} \|f(t)\|_{\gamma(H;X)} \le \sigma \Big] \lesssim (1+n)e^{-\frac{\lambda^2}{CD^2\sigma^2}}. \tag{5.4.8}$$

This can be done by applying a union bound to (5.4.7a) and subsequently applying tail estimates which come from the factorization method (see [186, Theorem 4.5]). With (5.4.8) in hand, the proof strategy of Lemma 5.3.5 and Theorem 5.3.1 can be repeated to obtain the result.

5.5. GENERALIZED HÖLDER SPACES

In this section we generalize the notion of a Hölder (semi)norm of functions mapping from a metric space (M,d_M) to a Banach space $(X,\|\cdot\|_X)$ to allow for more general moduli of continuity. Our results will be formulated for metric spaces (M,d_M) which satisfy $\Delta(M)\in(0,\infty)$. The condition $\Delta(M)<\infty$ is equivalent to (M,d_M) being bounded, and $\Delta(M)>0$ is equivalent to M having at least two distinct points.

Definition 5.5.1. Let (M,d_M) be a metric space satisfying $\Delta(M)\in(0,\infty)$, let $(X,\|\cdot\|_X)$ be a Banach space, and let $w\colon(0,1]\to(0,\infty)$ be a function which is nondecreasing and satisfies $\lim_{x\downarrow 0} w(x)=0$. The *generalized Hölder seminorm* $|\cdot|_{C_w(M,X)}\colon X^M\to[0,\infty]$ is defined by

$$|f|_{C_w(M,X)} = \sup_{x,y\in M, x\neq y} \frac{\|f(x)-f(y)\|_X}{w(d_M(x,y)/\Delta(M))}, \quad f\in X^M. \tag{5.5.1}$$

The associated *generalized Hölder norm* $\|\cdot\|_{C_w(M,X)}: X^M\to[0,\infty]$ is defined by

$$\|f\|_{C_w(M,X)} = \sup_{x\in M}\|f(x)\|_X + |f|_{C_w(M,X)}, \quad f\in X^M. \tag{5.5.2}$$

Finally, we define the Banach space $(C_w(M,X), \|\cdot\|_{C_w(M,X)})$ by

$$C_w(M,X) = \{f\in C(M,X) : \|f\|_{C_w(M,X)}<\infty\}. \tag{5.5.3}$$

Remark 5.5.2. Both $|\cdot|_{C_w(M,X)}$ and $\|\cdot\|_{C_w(M,X)}$ are invariant when scaling the metric d_M. We will make frequent use of this property to reduce to the case $\Delta(M)=1$ in proofs.

Remark 5.5.3. From the fact that $\lim_{x\downarrow 0} w(x)=0$ it follows that $f\in X^M$ is uniformly continuous whenever $|f|_{C_w(M,X)}<\infty$. Moreover, if $|f|_{C_w(M,X)}<\infty$ then the fact that $\Delta(M)<\infty$ implies that $\sup_{x\in M}\|f\|_X<\infty$, and therefore also $\|f\|_{C_w(M,X)}<\infty$. This also implies that if M_0 is dense in (M,d_M) and $f\in C_w(M_0,X)$, then f can be uniquely extended to a function $\tilde f\in C_w(M,X)$, which furthermore satisfies $|\tilde f|_{C_w(M,X)} = |f|_{C_w(M_0,X)}$ in the case where w is continuous.

Example 5.5.4. Let $\alpha \in (0,1]$ and let $w_\alpha \colon (0,1] \to (0,\infty)$ be given by $w_\alpha(x) = x^\alpha$, $x \in (0,1]$. Then $|\cdot|_{C_{w_\alpha}(M,X)}$ simply measures the α-Hölder continuity of a function; we set $|\cdot|_{C^\alpha(M,X)} := |\Delta(M)|^{-\alpha}|\cdot|_{C_{w_\alpha}(M,X)}$ and $C^\alpha(M,X) := C_{w_\alpha}(M,X)$. Note the somewhat unconventional definition of $\|\cdot\|_{C^1(M,X)}$: this measures the Lipschitz constant of a function and *not* the supremum norm of the derivative.

In Section 5.7 (see Theorem 5.7.1) we will prove that for certain metric spaces there exists an embedding $J\colon C(M,X) \to \ell^\infty(X)$ that defines an isomorphism between $C_w(M,X)$ and an appropriately weighted subspace of $\ell^\infty(X)$, provided w is admissible in the following sense:

Definition 5.5.5. We call $w\colon (0,1] \to (0,\infty)$ an *admissible modulus of continuity* provided that

1. w is continuous and nondecreasing.
2. There exists a constant $d_w \in (1,\infty)$ such that
$$\inf_{x\in(0,1]} \frac{w(x)}{w(x/2)} = d_w. \tag{5.5.4}$$
3. There exists a constant $c_w \in (1,\infty)$ such that
$$\sup_{x\in(0,1]} \frac{w(x)}{w(x/2)} = c_w. \tag{5.5.5}$$

We refer to c_w and d_w as the *growth constants* of w.

Remark 5.5.6. Let $w\colon (0,1] \to \mathbb{R}$ be an admissible modulus of continuity. Then it holds that $\lim_{x\downarrow 0} w(x) = 0$ by (5.5.4) and the fact that w is nondecreasing. Moreover, the fact that $d_w > 1$ implies

$$\forall m \in \mathbb{N}_0, x \in (0,1]\colon \sum_{k=m}^{\infty} w\Big(\frac{x}{2^k}\Big) \le w\Big(\frac{x}{2^m}\Big) \sum_{k=0}^{\infty} d_w^{-k} = \frac{d_w}{d_w - 1} w\Big(\frac{x}{2^m}\Big). \tag{5.5.6}$$

Example 5.5.7. We list some examples of admissible moduli of continuity:

1. The function $w_\alpha\colon (0,1] \to (0,\infty)$ given by $w_\alpha(x) = x^\alpha$ from Example 5.5.4 is an admissible modulus of continuity with growth constants $d_w = c_w = 2^\alpha$.
2. Let $w\colon (0,1] \to (0,\infty)$ be an admissible modulus of continuity with growth constants (c_w, d_w), and let $\lambda \in (0,\infty)$. Then λw is again an admissible modulus of continuity with the same growth constants (c_w, d_w).
3. Let $w\colon (0,1] \to (0,\infty)$ be an admissible modulus of continuity with growth constants (c_w, d_w), and let $\gamma \in (0,\infty)$, $\beta \in (0,\infty)$. Then $\tilde{w}\colon (0,1] \to (0,\infty)$ given by $\tilde{w}(x) = (1-\beta\log(x))^{-\gamma} w(x)$ is an admissible modulus of continuity with growth constants $(c_{\tilde{w}}, d_{\tilde{w}})$ satisfying
$$d_w \le d_{\tilde{w}} \le c_{\tilde{w}} \le (1+\beta\log(2))^\gamma c_w.$$

This follows from the fact that $x \mapsto (1-\beta\log(x))^{-\gamma}$ is an increasing positive function for $x \in (0,1]$, and that

$$\forall x \in (0,1]\colon \frac{1}{1+\beta\log(2)} \leq \frac{1-\beta\log(x)}{1-\beta\log(\frac{x}{2})} \leq 1.$$

4. Combining (1) and (3), we obtain that if $\alpha \in (0,1]$, $\gamma \in (0,\infty)$, $\beta \in (0,\infty)$, then $w\colon (0,1] \to (0,\infty)$ given by $w(x) = (1-\beta\log(x))^{-\gamma}x^{\alpha}$ is an admissible modulus of continuity, with growth constants (c_w, d_w) satisfying
$$2^{\alpha} \leq d_w \leq c_w \leq 2^{\alpha}(1+\beta\log(2))^{\gamma}.$$
5. Let $w\colon (0,1] \to (0,\infty)$ be an admissible modulus of continuity with growth constants (c_w, d_w), and let $\gamma \in (0,\infty)$, $\beta \in (0, \log(2)^{-1}(d_w^{1/\gamma}-1))$. Assume moreover that $x \mapsto (1-\beta\log(x))^{\gamma}w(x)$ is nondecreasing on $(0,1]$. Then $\tilde{w}\colon (0,1] \to (0,\infty)$ given by $\tilde{w}(x) = (1-\beta\log(x))^{\gamma}w(x)$ is an admissible modulus of continuity with growth constants $(c_{\tilde{w}}, d_{\tilde{w}})$ satisfying
$$(1+\beta\log(2))^{-\gamma}d_w \leq d_{\tilde{w}} \leq c_{\tilde{w}} \leq c_w.$$
Note that the condition on β implies $(1+\beta\log(2))^{-\gamma}d_w > 1$.
6. Combining (1) and (5) we obtain that if $\alpha \in (0,1]$, $\gamma \in (0,\infty)$, and $\beta \in (0,\frac{\alpha}{\gamma})$, then $w\colon (0,1] \to (0,\infty)$ given by $w(x) = (1-\beta\log(x))^{\gamma}x^{\alpha}$ is an admissible modulus of continuity with growth constants (c_w, d_w) satisfying
$$(1+\beta\log(2))^{-\gamma}2^{\alpha} \leq d_w \leq c_w \leq 2^{\alpha}.$$
Indeed, fact that $\beta < \frac{\alpha}{\gamma}$ guarantees that $\beta < \log(2)^{-1}(2^{\alpha/\gamma}-1)$ and insures that that w is increasing, so (5) applies.

Remark 5.5.8. Note that for $\beta > 0$ and $x \in (0,1]$ we have $\min(1,\beta) \leq \frac{1-\beta\log(x)}{1-\log(x)} \leq \max(1,\beta)$. In particular, although Example 5.5.7 (5) and (6) involve restrictions on β, the resulting seminorm $|\cdot|_{C_w(M,X)}$ is equivalent to the seminorm obtained by taking $\beta = 1$.

Generalized Hölder spaces are more natural than they may seem at first sight: indeed, when measuring the regularity of a stochastic process, one often encounters the situation that the paths of a process $X\colon [0,T]\times\Omega \to \mathbb{R}$ lie in $C^{\alpha}([0,T])$ for all $\alpha \in (0,\alpha^*)$, but not in $C^{\alpha^*}([0,T])$. The following proposition, which was pointed out by Stefan Geiss, shows that the generalized Hölder space involving the modulus of continuity $w(x) = (1-\beta\log(x))^{\gamma}x^{\alpha^*}$ measures how fast the α-Hölder constant blows up as $\alpha \uparrow \alpha^*$.

Proposition 5.5.9. *Let (M,d_M) be a metric space satisfying $\Delta(M) \in (0,\infty)$, let $(X,\|\cdot\|_X)$ be a Banach space, let $\alpha^* \in (0,1)$, $\gamma \in (0,\infty)$, $\beta \in (0,\frac{\alpha^*}{\gamma})$, and let $w\colon (0,1] \to (0,\infty)$ be given by*
$$w(x) = (1-\beta\log(x))^{\gamma}x^{\alpha^*}, \quad x \in (0,1].$$
Then for all $f \in C(M,X)$ it holds that
$$\begin{aligned}&(e^{-1}\beta\gamma)^{\gamma}|f|_{C_w(M,X)}\\ &\quad\leq \sup_{\alpha\in(0,\alpha*)} (\alpha^*-\alpha)^{\gamma}\Delta(M)^{\alpha}|f|_{C^{\alpha}(M,X)}\\ &\quad\leq (\alpha^*+e^{-1}\beta\gamma)^{\gamma}|f|_{C_w(M,X)}.\end{aligned} \tag{5.5.7}$$

Proof. By scaling the metric, we may reduce to the case $\Delta(M) = 1$. By definition of $C^\alpha(M,X)$ and $C_w(M,X)$, it then suffices to prove

$$\forall x \in (0,1]: \frac{(e^{-1}\beta\gamma)^\gamma}{x^{\alpha^*}(1-\beta\log(x))^\gamma} \le \sup_{\alpha\in(0,\alpha^*)} \frac{(\alpha^*-\alpha)^\gamma}{x^\alpha} \le \frac{(\alpha^*+e^{-1}\beta\gamma)^\gamma}{x^{\alpha^*}(1-\beta\log(x))^\gamma}.$$

This is equivalent to

$$\forall x \in (0,1]: e^{-1}\beta\gamma \le \sup_{\alpha\in(0,\alpha^*)} (\alpha^*-\alpha)x^{(\alpha^*-\alpha)/\gamma}(1-\beta\log(x)) \le \alpha^* + e^{-1}\beta\gamma.$$

Applying the substitutions $z = -\gamma^{-1}\log(x)$ and $\alpha' = z(\alpha^*-\alpha)$, this is further equivalent to showing

$$\forall z \in (0,\infty): e^{-1}\beta\gamma \le (z^{-1}+\beta\gamma) \sup_{\alpha'\in(0,z\alpha^*)} \alpha' e^{-\alpha'} \le \alpha^* + e^{-1}\beta\gamma. \tag{5.5.8}$$

We now distinguish between the cases $z\alpha^* \le 1$ and $z\alpha^* > 1$. In the first case, the supremum in (5.5.8) is attained at the endpoint $\alpha' = z\alpha^*$. Hence, the estimate follows since

$$e^{-1}\beta\gamma \le e^{-1}\alpha^* \le (z^{-1}+\beta\gamma)z\alpha^* e^{-z\alpha^*} \le \alpha^* + e^{-1}\beta\gamma.$$

In the second case, the supremum in (5.5.8) is attained at the interior point $\alpha' = 1$. Thus, the desired estimate follows because

$$e^{-1}\beta\gamma \le (z^{-1}+\beta\gamma)e^{-1} \le e^{-1}(\alpha^*+\beta\gamma). \qquad \square$$

5.6. MINKOWSKI- AND DOUBLING DIMENSIONS

In this section we recall the concepts of Minkowski- and doubling dimension and discuss their relation to chaining. More specifically, the main result (Proposition 5.6.12 below) shows that if a metric space (M, d_M) has Minkowski dimension $d \in (0,\infty)$ and finite doubling dimension, then one can construct a sequence of graphs which is in some sense 'd-dimensional', and which accurately encodes the metric structure of M. This is exactly the setup that allows for a d-dimensional Kolmogorov-type 'chaining' argument, which we will later use to construct an isomorphism between a generalized Hölder space and a subspace of an (appropriately weighted) ℓ^∞-space (see Theorem 5.7.1).

The Minkowski dimension affects the isomorphism in a more drastic way than the doubling dimension. This is convenient, since we find that it generally holds that the Minkowski dimension remains unchanged under many operations, whereas the doubling dimension might increase. In particular, we show how the Minkowski and doubling dimensions are affected by isomorphisms (Proposition 5.6.6) and by passing to a subset (Proposition 5.6.9). Finally, in view of our intended applications, we show that every bounded subset of $\mathbb{R}^d$ has Minkowski dimension d and finite doubling dimension, see Corollary 5.6.10.

Recall that if (M, d_M) is a metric space and $\eta \in (0,\infty)$, then $\mathcal{N}(M, d_M, \eta)$ denotes the minimal number of open d_M-balls of radius η needed to cover M, see also Section 5.2.1 for our notational conventions regarding metric spaces.

Definition 5.6.1. A metric space (M, d_M) which satisfies $\Delta(M) \in (0, \infty)$ has *Minkowski dimension* $d \in (0, \infty)$ with *covering constant* $c \in [1, \infty)$ if it satisfies

$$\mathcal{N}(M, d_M, \eta) \leq c(\Delta(M)/\eta)^d, \quad \eta \in (0, \Delta(M)]. \tag{5.6.1}$$

Definition 5.6.2. A metric space (M, d_M) has *doubling number* $n_2 \in \mathbb{N}$ if every open ball in M with a given radius r can be covered by n_2 open balls of radius $r/2$. In this case, we say that (M, d_M) has *doubling dimension* $d_2 := \log_2(n_2)$.

Remark 5.6.3. The condition $\Delta(M) \in (0, \infty)$ in Definition 5.6.1 is necessary for (5.6.1) to be sensible.

It is easy to see that any bounded metric space with doubling dimension d_2 also has Minkowski dimension d_2. The converse does not hold; in fact, a metric space with Minkowski dimension d might not have a finite doubling dimension at all. As an example, consider $S \subset \ell^\infty(\mathbb{N})$ given by

$$S = \bigcup_{n \in \mathbb{N}} 2^{-n}\{e_1, \ldots, e_n\}$$

(where e_n denotes the n-th element of the standard basis) equipped with the distance inherited by the $\ell^\infty(\mathbb{N})$-norm.

Example 5.6.4. Let $d \in \mathbb{N}$, $R \in (0, \infty)$, $D = [0, R]^d$, and let d_{Cheb} be the Chebyshev distance on D. Then (D, d_{Cheb}) has Minkowski dimension d with covering constant 2^d, and doubling number 3^d.

Proof. Let $r_1 > r_2 > 0$. Then any open (or closed) ball in (M, d_{Cheb}) of radius r_1 can be covered by $(r_1/r_2 + 1)^d$ open balls of radius r_2. Applying this with $r_1 = R$ shows the first claim, and applying it with $r_2 = r_1/2$ shows the second claim. □

The following example shows that the Minkowski dimension can be fractional.

Example 5.6.5. Let S be the Sierpiński triangle, and let d_{Euc} be the Euclidean metric on S. Then (S, d_{Euc}) has Minkowski dimension $\log_2(3) \approx 1.58$. This can be seen by using the self-similarity of S, which guarantees that $\mathcal{N}(S, d_{\mathrm{Euc}}, \eta/2) \leq 3\mathcal{N}(S, d_{\mathrm{Euc}}, \eta)$ for every $\eta \in (0, \Delta(S)]$.

We first show how the Minkowski dimension, covering constant, and doubling number behave under isomorphisms.

Proposition 5.6.6. *Let (M, d_M) and $(\hat{M}, d_{\hat{M}})$ be isomorphic metric spaces, i.e., there exists a bijection $I\colon M \to \hat{M}$ and constants $c_{(5.6.2)}, C_{(5.6.2)} > 0$ such that one has*

$$c_{(5.6.2)} d_M(x, y) \leq d_{\hat{M}}(I(x), I(y)) \leq C_{(5.6.2)} d_M(x, y), \quad x, y \in M. \tag{5.6.2}$$

If (M, d_M) has Minkowski dimension d with covering constant c, then $(\hat{M}, d_{\hat{M}})$ has the same Minkowski dimension d with covering constant $c \cdot c_{(5.6.2)}^{-d} C_{(5.6.2)}^d$. Moreover, if (M, d_M) has doubling number n_2, then $(\hat{M}, d_{\hat{M}})$ has doubling number $n_2^{1+\lceil \log_2(C_{(5.6.2)}/c_{(5.6.2)}) \rceil}$.

Proof. The first statement follows immediately from the fact that $\Delta(M) \le c_{(5.6.2)}^{-1}\Delta(\hat{M})$ and

$$\mathcal{N}(\hat{M}, d_{\hat{M}}, \eta) \le \mathcal{N}(M, d_M, C_{(5.6.2)}^{-1}\eta) \le c \cdot C_{(5.6.2)}^{d}(\Delta(M)/\eta)^d.$$

The second statement follows by observing that for $\hat{x} \in \hat{M}$ and $r > 0$ we have $B_{\hat{x}}(d_{\hat{M}}, r) \subseteq I(B_{I^{-1}(\hat{x})}(d_M, c_{(5.6.2)}^{-1}r))$, whereas $B_{I(y)}(d_{\hat{M}}, r/2) \subseteq B_y(d_M, C_{(5.6.2)}r/2)$ for all $y \in M$. □

Corollary 5.6.7. *Let (M, d_M) be a metric space with Minkowski dimension d and covering constant c, and doubling number n_2. Then the metric space $(M, (\Delta(M))^{-1}d_M)$ also has Minkowski dimension d with covering constant c, and doubling number n_2.*

The following elementary lemma will be used to control the behavior of Minkowski dimension, covering constant, and doubling dimension under taking subsets.

Lemma 5.6.8. *Let (M, d_M) be a metric space and let $A \subseteq M$, $k \in \mathbb{N}$, and $\eta \in (0, \infty)$. If A can be covered by k open balls with centers in M and radius η, then A can also be covered by k open balls with centers in A and radius 2η.*

Proof. Let $F \subseteq M$ be such that $\mathrm{card}(F) \le k$ and $A \subseteq \cup_{x \in F} B_x(\eta)$. Without loss of generality, we may assume $B_x(\eta) \cap A \ne \emptyset$ for any $x \in F$. For each $x \in F$, pick a point $y \in B_x(\eta) \cap A$, and denote by G the set of these points. Obviously $\mathrm{card}(G) \le k$, and by the triangle inequality we have $A \subseteq \cup_{x \in F} B_x(\eta) \subseteq \cup_{y \in G} B_y(2\eta)$. □

We can now prove that the Minkowski dimension of a metric space is preserved under taking subsets, and the doubling number is at most squared under taking subsets. Hence, the doubling dimension can increase by a factor 2.

Proposition 5.6.9. *Let (M, d_M) be a metric space with Minkowski dimension d and covering constant c, and let $A \subseteq M$ satisfy $\Delta(A) > 0$. Then $(A, d_M|_{A\times A})$ has Minkowski dimension d with covering constant $c\,(2\,\Delta(M)/\Delta(A))^d$. Moreover, if (M, d_M) has doubling number n_2 then $(A, d_M|_{A\times A})$ has doubling number n_2^2.*

Proof. It follows from Lemma 5.6.8 that $\mathcal{N}(A, d_M|_{A\times A}, \eta) \le \mathcal{N}(M, d_M, \eta/2)$. Using (5.6.1) then gives the result on the Minkowski dimension.

Next, fix $x \in A$ and $r \in (0, \infty)$. Using Definition 5.6.2 twice, we see $B_x(r)$ can be covered by n_2^2 open balls of radius $r/4$ with centers in M. Thus, by Lemma 5.6.8, $B_x(r)$ can also be covered by n_2^2 balls of radius $r/2$ with centers in A. □

From Example 5.6.4 and Propositions 5.6.6 and 5.6.9 we obtain the following result about subsets of $\mathbb{R}^d$ with the Euclidean metric.

Corollary 5.6.10. *Let $d \in \mathbb{N}$ and let $D \subseteq \mathbb{R}^d$ be a bounded set containing at least two points. Let d_{Euc} be the Euclidean metric on D. Then (D, d_{Euc}) has Minkowski dimension d with covering constant $(4d)^d$, and doubling number $(3^4 d^2)^d$.*

Proof. For $x, y \in \mathbb{R}^d$, we have

$$d_{\mathrm{Cheb}}(x, y) \le d_{\mathrm{Euc}}(x, y) \le \sqrt{d} \cdot d_{\mathrm{Cheb}}(x, y).$$

Set $R = \Delta_{d_{\mathrm{Euc}}}(D)$. From Example 5.6.4 and Proposition 5.6.6 we see that $([0, R]^d, d_{\mathrm{Euc}})$ has Minkowski dimension d, covering constant $2^d\sqrt{d}^d$, and doubling number $3^{d(1+\lceil \log_2(\sqrt{d})\rceil)}$.

By Proposition 5.6.9 and the fact that $\Delta_{\mathrm{Euc}}([0,R]^d)/\Delta_{\mathrm{Euc}}(D) \le \sqrt{d}$ we find that (D, d_{Euc}) has Minkowski dimension d with covering constant $(4d)^d$, and doubling number

$$3^{d(2+2\lceil \log_2(\sqrt{d})\rceil)} \le (3^{4+\log_2(d)})^d = (3^4 d^{\log_2(3)})^d \le (3^4 d^2)^d. \qquad \square$$

In order to establish that a metric space (M, d_M) with Minkowski dimension d and finite doubling dimension allows for the kind of chaining necessary to prove Theorem 5.7.1, we need to control the amount of 'close neighbors' that a point $x \in M$ can have. The following lemma takes care of this.

Lemma 5.6.11. *Let (M, d_M) be a metric space with doubling number n_2, and let $r \in (0,\infty)$, $k \in \mathbb{N}_0$. Let $S \subseteq M$ be contained in an open ball of radius r, and such that $d_M(x,y) \ge 2^{-k} r$ whenever $x, y \in S$ and $x \ne y$. Then we have* $\mathrm{card}(S) \le n_2^{k+1}$.

Proof. Let n be the minimum number of open balls of radius $2^{-k-1}r$ required to cover S. By iterating the doubling property, we see that $n \le n_2^{k+1}$. However, since the points of S are separated by a distance $2^{-k}r$, any open ball of radius $2^{-k-1}r$ can contain at most one point of S. Therefore, $n \ge \mathrm{card}(S)$. $\square$

Using Lemma 5.6.11 and a covering argument, we now prove the fundamental result needed for Theorem 5.7.1.

Proposition 5.6.12. *Let (M, d_M) be a metric space with Minkowski dimension d and covering constant c, and doubling number n_2. Then there exists a sequence $(V_n)_{n\in\mathbb{N}_0}$ of subsets of M such that for all $n \in \mathbb{N}_0$ we have:*

1. $V_n \subseteq V_{n+1}$.
2. $\mathrm{card}(V_n) \le c3^d 2^{dn}$.
3. $d_M(x, V_n) \le \Delta(M)\cdot 2^{-n}$ *for all* $x \in M$.
4. $\mathrm{card}(\{\{x,y\} \subseteq V_n : d(x,y) \in (0, 3\cdot 2^{-n}\Delta(M))\}) \le c3^d n_2^4 2^{dn}$.

Proof. We can assume that $\Delta(M) = 1$ by rescaling (see Corollary 5.6.7). We now claim that it suffices to construct a sequence $(V_n)_{n\in\mathbb{N}_0}$ of subsets of M such that for all $n \in \mathbb{N}_0$ we have:

(i) $V_n \subseteq V_{n+1}$.

(ii) $\mathrm{card}(V_n) \le c3^d 2^{dn}$.

(iii) $d_M(x, V_n) \le 2^{-n}$ for all $x \in M$.

(iv) $d_M(x,y) \ge \frac{2}{3} 2^{-n}$ whenever $x, y \in V_n$ and $x \ne y$.

Indeed, (iv) and Lemma 5.6.11 (with $S = B_x(3\cdot 2^{-n}) \cap V_n$ and $k = 3$) imply that for all $x \in V_n$, the number of points in V_n which are strictly within distance $3\cdot 2^{-n}$ of x is bounded by n_2^4. Combining this with (ii) results in (4).

We construct the sets V_n inductively. Pick some $x_0 \in M$ and set $V_0 = \{x_0\}$. Then (ii)-(iv) are satisfied for $n = 0$. For $n \in \mathbb{N}$, we find using Definition 5.6.1 a set $F_n \subseteq M$ such that $\operatorname{card}(F_n) \le c3^d 2^{dn}$ and $M = \cup_{x\in F_n} B_x(\frac{1}{3}2^{-n})$, and then set

$$G_n = \left\{x \in F_n : d(x, V_{n-1}) \ge \tfrac{2}{3}2^{-n}\right\}.$$

By the triangle inequality, we see that the set $\{B_x(\frac{1}{3}2^{-n}) : x \in G_n\}$ forms a cover of $M \setminus \cup_{x\in V_{n-1}} B_x(2^{-n})$. Moreover, using that $M = \cup_{x\in F_n} B_x(\frac{1}{3}2^{-n})$ and that $d_M(x, y) \ge \frac{4}{3}2^{-n}$ whenever $x, y \in V_{n-1}$ we find that $\operatorname{card}(G_n) \le c3^d 2^{dn} - \operatorname{card}(V_{n-1})$. By the (finite) Vitali covering lemma, we obtain a set $H_n \subseteq G_n$ such that the balls $\{B_x(2^{-n}) : x \in H_n\}$ again form a cover of $M \setminus \cup_{x\in V_{n-1}} B_x(2^{-n})$, and we additionally have $d(x, y) \ge \frac{2}{3}2^{-n}$ whenever $x, y \in H_n$ and $x \ne y$. We now set $V_n = V_{n-1} \cup H_n$. Note that by construction V_n satisfies (i), (ii), and (iii). From the construction of H_n it is clear that (iv) holds if $x \in H_n$ and/or $y \in H_n$, and if $x, y \in V_{n-1}$ then (iv) holds by induction. □

5.7. A CIESIELSKI-TYPE EMBEDDING BASED ON CHAINING

The goal of this section is to show that if M is a metric space with finite Minkowski and doubling dimensions (see Definitions 5.6.1 and 5.6.2), then there exists an embedding $C(M, X) \overset{J}{\hookrightarrow} \ell^\infty(X)$ such that for every admissible modulus of continuity w (see Definition 5.5.5 above) there exist constants $c_{M,w}$ and $C_{M,w}$ such that we have

$$c_{M,w}\|Jf\|_{\ell^\infty_{w_d}(X)} \le |f|_{C_w(M,X)} \le C_{M,w}\|Jf\|_{\ell^\infty_{w_d}(X)}. \tag{5.7.1}$$

for every $f \in C(M, X)$. Here, w_d is a modification of w involving the Minkowski dimension of M. The exact statement is given in Theorem 5.7.1 below (see also Corollary 5.7.3). One can think of these results as a generalization of Ciesielski's embedding [58], however, the underlying philosophy is distinctly different, see Remark 5.7.4.

Theorem 5.7.1 (Generalized Ciesielski's embedding)**.** *Let (M, d_M) be a metric space with Minkowski dimension d and covering constant c, and doubling number n_2. Let w be an admissible modulus of continuity with growth constants (c_w, d_w), and let $(X, \|\cdot\|_X)$ be a Banach space. Then there exists a sequence $(x_k, y_k)_{k\in\mathbb{N}}$ in $M \times M$ such that for every $f \in C(M, X)$ we have*

$$C^{-1}_{(5.7.3)}|f|_{C_w(M,X)} \le \sup_{k\in\mathbb{N}} \frac{\|f(x_k) - f(y_k)\|_X}{w\left(k^{-1/d}\right)} \le C_{(5.7.4)}|f|_{C_w(M,X)}, \tag{5.7.2}$$

where

$$C_{(5.7.3)} = 3c_w d_w(d_w - 1)^{-1}, \tag{5.7.3}$$

$$C_{(5.7.4)} = c_w(c3^{2d+1}d^{-1}n_2^4)^{\log_2(c_w)/d}. \tag{5.7.4}$$

Moreover, the sequence $(x_k, y_k)_{k\in\mathbb{N}}$ can be chosen independently of w.

Remark 5.7.2. If $f \notin C(M, X)$, then (5.7.2) remains valid if we replace M by $M_0 = \cup_k\{x_k, y_k\}$ on both sides of the equation (M_0 is dense in M by construction). Similarly, (5.7.9) remains valid for $f \notin C(D, X)$ if we replace D by $D_0 = \cup_k\{x_k, y_k\}$ on both sides.

Proof. By rescaling the metric (see Remark 5.5.2 and Corollary 5.6.7) we can assume without loss of generality that $\Delta(M) = 1$. Let $(V_n)_{n\in\mathbb{N}_0}$ be a sequence of subsets of M obtained from Proposition 5.6.12, and set $M_0 = \cup_{n\in\mathbb{N}_0} V_n$. It follows by Proposition 5.6.12 (2)-(3) that M_0 is countable and dense in (M, d_M). We also set

$$E_n := \{\{x,y\} \subseteq V_n : d_M(x,y) \in (0, 3\cdot 2^{-n})\}, \quad n \in \mathbb{N}_0,$$

and note that $\operatorname{card}(E_n) \le c3^d n_2^4 2^{dn}$ by Proposition 5.6.12 (4). The sequence $(x_k, y_k)_{k\in\mathbb{N}}$ is to be chosen such that $\cup_{k\in\mathbb{N}}\{x_k, y_k\} = \cup_{n\in\mathbb{N}_0} E_n$, however, we must take care in the numbering. Therefore, we define a nondecreasing sequence $(\theta(n))_{n\in\mathbb{N}_0}$ in $\mathbb{N}$ such that $\theta(0) = 0$ and, for all $n \in \mathbb{N}_0$,

$$\begin{aligned} \operatorname{card}(E_n) &= \theta(2n+1) - \theta(2n), \\ \max(2^{d(n+1)} - \operatorname{card}(E_n), 0) &\le \theta(2n+2) - \theta(2n+1). \end{aligned} \tag{5.7.5}$$

Fix some (dummy) $x^* \in M_0$. We now pick $(x_k, y_k)_{k\in\mathbb{N}}$ such that

$$E_n = \{\{x_{\theta(2n)+1}, y_{\theta(2n)+1}\}, \ldots, \{x_{\theta(2n+1)}, y_{\theta(2n+1)}\}\}, \quad n \in \mathbb{N}_0,$$

and we set $(x_k, y_k) = (x^*, x^*)$ whenever there exists an $n \in \mathbb{N}$ such that $\theta(2n+1) < k \le \theta(2n+2)$. Note that by (5.7.5) and the estimate on $\operatorname{card}(E_n)$ we have

$$2^{d(n+1)} \le \theta(2n+2) - \theta(2n) \le c3^d n_2^4 \cdot 2^{dn}, \quad n \in \mathbb{N}_0,$$

so that a telescoping series argument using $\theta(2n) = \sum_{k=0}^{n-1} \theta(2k+2) - \theta(2k)$ gives

$$2^{dn} \le \theta(2n) \le c3^d n_2^4 \frac{2^{dn} - 1}{2^d - 1} \le c3^{d+1} d^{-1} n_2^4 \cdot 2^{dn}, \quad n \in \mathbb{N}_0.$$

Since w is nondecreasing, this implies that for every $n \in \mathbb{N}_0$ and $\theta(2n) < k \le \theta(2n+2)$, we have

$$w((c3^{2d+1} d^{-1} n_2^4)^{-1/d} 3 \cdot 2^{-n}) \le w(k^{-1/d}) \le w(2^{-n}).$$

This leads via (5.5.5) to the two-sided estimate

$$C_{(5.7.4)}^{-1} w(\min(3\cdot 2^{-n}, 1)) \le w(k^{-1/d}) \le w(2^{-n}), \tag{5.7.6}$$

which holds whenever $n \in \mathbb{N}_0$ and $\theta(2n) < k \le \theta(2n+2)$. In the case $\theta(2n) < k \le \theta(2n+1)$, we have by construction $d_M(x_k, y_k) \le \min(3\cdot 2^{-n}, 1)$, so that (5.7.6) implies

$$\frac{\|f(x_k) - f(y_k)\|_X}{w(k^{-1/d})} \le C_{(5.7.4)} \frac{\|f(x_k) - f(y_k)\|_X}{w(d_M(x_k, y_k))} \le C_{(5.7.4)} |f|_{C_w(M,X)}.$$

In the case $\theta(2n+1) < k \le \theta(2n+2)$, the same inequality (without the middle term) trivially holds since $x_k = y_k = x^*$. We conclude that the second inequality in (5.7.2) is valid.

For the first inequality, fix $x, y \in M_0$ with $x \ne y$, and set

$$n_x := \min\{k \in \mathbb{N}_0 : x \in V_k\}, \qquad n_y := \min\{k \in \mathbb{N}_0 : y \in V_k\}, \tag{5.7.7a}$$

$$n_0 := \max\{k \in \mathbb{N}_0 : d_M(x,y) < 2^{-k}\}. \tag{5.7.7b}$$

To ease the notation, we also introduce for $n \in \mathbb{N}_0$ the quantities

$$K := \sup_{k\in\mathbb{N}} \frac{\|f(x_k) - f(y_k)\|_X}{w(k^{-1/d})}, \tag{5.7.7c}$$

$$K_n := \sup_{u,v\in V_n:\, d_M(u,v)<3\cdot 2^{-n}} \frac{\|f(u) - f(v)\|_X}{w(2^{-n})} \tag{5.7.7d}$$

$$= \sup_{\theta(2n)<k\le\theta(2n+1)} \frac{\|f(x_k) - f(y_k)\|_X}{w(2^{-n})}, \tag{5.7.7e}$$

where the final equality follows from the construction of $(x_k, y_k)_{k\in\mathbb{N}}$ by Proposition 5.6.12-(1). Estimating (5.7.7e) using (5.7.6) we see that $K_n \le K$, a fact which we will use multiple times.

We now proceed to 'chaining' $f(x) - f(y)$. To this end, let $\phi_n\colon M_0 \to V_n$, $n \in \mathbb{N}_0$ map every point $z \in M_0$ to (one of) its nearest point(s) in V_n. By (5.7.7a) it follows that $\phi_{n_x}(x) = x$ and $\phi_{n_y}(y) = y$. We thus have

$$\begin{aligned} f(x) - f(y) &= f(\phi_{n_x}(x)) - f(\phi_{n_y}(y)) \\ &= f(\phi_{n_0}(x)) + \sum_{j=n_0+1}^{n_x} f(\phi_j(x)) - f(\phi_{j-1}(x)) \\ &\quad - f(\phi_{n_0}(y)) - \sum_{j=n_0+1}^{n_y} f(\phi_j(y)) - f(\phi_{j-1}(y)), \end{aligned} \tag{5.7.8a}$$

where the summations are taken to be trivial if the lower index exceeds the higher index. By definition of ϕ_n and Proposition 5.6.12-(3) we have $d_M(z, \phi_n(z)) \le 2^{-n}$ for every $z \in M_0$, $n \in \mathbb{N}_0$. Using the triangle inequality and (5.7.7b), this gives

$$d_M(\phi_{n_0}(x), \phi_{n_0}(y)) \le 2\cdot 2^{-n_0} + d_M(x,y) < 3\cdot 2^{-n_0}.$$

Combining this with (5.5.5) and (5.7.7d) results in

$$\frac{\|f(\phi_{n_0}(x)) - f(\phi_{n_0}(y))\|_X}{w(2^{-(n_0+1)})} \le c_w K_{n_0} \le c_w K. \tag{5.7.8b}$$

As for the summations in (5.7.8a), notice that by the triangle inequality we have

$$d_M(\phi_j(x), \phi_{j-1}(x)) \le 2^{-j} + 2^{-(j-1)} < 3\cdot 2^{-(j-1)},$$

for any $j \in \mathbb{N}_0$. This together with (5.7.7d) and (5.5.6) implies

$$\begin{aligned} \frac{1}{w(2^{-(n_0+1)})} \Big\| \sum_{j=n_0+1}^{n_x} f(\phi_j(x)) - f(\phi_{j-1}(x)) \Big\|_X &\le \sum_{j=n_0+1}^{n_x} \frac{w(2^{-(j-1)})}{w(2^{-(n_0+1)})} K_{j-1} \\ &\le \frac{c_w d_w}{d_w - 1} K. \end{aligned} \tag{5.7.8c}$$

Clearly, (5.7.8c) also holds if we replace x by y. Finally, we know from (5.7.7b) that $d_M(x,y) \ge 2^{-(n_0+1)}$. Thus, by (5.7.8a)-(5.7.8c) and the fact that w is nondecreasing, we obtain

$$\frac{\|f(x) - f(y)\|_X}{w(d_M(x,y))} \le \frac{\|f(x) - f(y)\|_X}{w(2^{-(n_0+1)})} \le C_{(5.7.4)} K.$$

Recalling the definition of K (5.7.7c), this is exactly the estimate we wanted to show. We conclude that the first estimate of (5.7.2) holds since $x, y \in M_0$ were arbitrary and M_0 is dense in (M, d_M). □

In the Euclidean setting, we obtain from Theorem 5.7.1 and Corollary 5.6.10 the following corollary.

Corollary 5.7.3 (Generalized Ciesielski's embedding, Euclidean case)**.** *Let $d \in \mathbb{N}$ and let $D \subseteq \mathbb{R}^d$ be bounded with at least two elements. Let w be an admissible modulus of continuity with growth constants (c_w, d_w), and let $(X, \|\cdot\|_X)$ be a Banach space. Then there exists a sequence $(x_k, y_k)_{k\in\mathbb{N}}$ in $D \times D$ and constants $c_{d,w}, C_{d,w}$ such that for every $f \in C(D, X)$ we have*

$$c_{d,w}^{-1} |f|_{C_w(D,X)} \le \sup_{k\in\mathbb{N}} \frac{\|f(x_k) - f(y_k)\|_X}{w\left(k^{-1/d}\right)} \le C_{d,w} |f|_{C_w(D,X)}. \tag{5.7.9}$$

Moreover, the sequence $(x_k, y_k)_{k\in\mathbb{N}}$ can be chosen independently of w and the constants $c_{d,w}, C_{d,w}$ depend only on d and w.

Remark 5.7.4 (Relation to Ciesielski's original embedding). Ciesielski proved in [58] that there is an *isomorphism* between $C^\alpha([0,1])$ and a weighted ℓ^∞ space. Note that in Corollary 5.7.3 we only obtain an isomorphism from $C^\alpha(M)$ into a (nontrivial) subspace of a weighted ℓ^∞ space. The difference lies in the construction of the embedding: in [58], the author constructs $I\colon C([0,1]) \to \ell^\infty$ as follows

$$\begin{aligned} I(f) = \Big(& f(1) - f(0), f(\tfrac{1}{2}) - \tfrac{f(1)-f(0)}{2}, f(\tfrac{1}{4}) - \tfrac{f(\frac{1}{2})-f(0)}{2}, \\ & f(\tfrac{3}{4}) - \tfrac{f(1)-f(\frac{1}{2})}{2}, \dots \Big). \end{aligned} \tag{5.7.10}$$

The underlying philosophy is that a function $f \in C([0,1])$ can be decomposed using a spline basis $(\phi_{n,k})_{n\in\mathbb{N}_0, 1\le k\le 2^{-n}}$, and the contribution of the splines at level n scales like $n^{-\alpha}$ if and only $f \in C^\alpha([0,1])$ (this only works for $\alpha \in (0,1)$).

However, taking $M = [0,1]$ in Theorem 5.7.1 and using $V_0 = \{0\}$ and $V_n = \{k\cdot 2^{-n+1} \colon k \in \{0, 1, \dots, 2^{n-1}\}\}$, $n \in \mathbb{N}$, in the proof of Theorem 5.7.1 results in the following embedding:

$$J(f) = \left(f(1) - f(0), f(\tfrac{1}{2}) - f(0), f(1) - f(\tfrac{1}{2}), f(\tfrac{1}{4}) - f(0), f(\tfrac{1}{2}) - f(\tfrac{1}{4}), \dots\right). \tag{5.7.11}$$

In particular, $J(f)$ necessarily satisfies infinitely many constraints of the type $J(f)(2) + J(f)(3) = J(f)(1)$. This redundancy is a consequence of the chaining philosophy: for every point $x \in V_n$ one needs to control how much $f(x)$ differs from its 'neighbors' at distance at most $3 \cdot 2^{-n}\Delta(M)$.

It may be possible to extend Ciesielski's original philosophy to functions on a domain in $\mathbb{R}^d$ using wavelet techniques (wavelets being a generalization of the splines used by Ciesielski). For example, one could try to adapt the proof of [218, Theorem 2.23], where an embedding of a Triebel–Lizorkin space on an arbitrary domain in $\mathbb{R}^d$ into a weighted sequence space is shown. However, even in the Euclidean setting it is unclear to us whether this would be more efficient. Furthermore, the general (non-Euclidean) setting of Theorem 5.7.1 seems totally out of reach with these methods. For a demonstration of the power of our general setting, we refer ahead to the proof of Theorem 5.10.1, where we endow the space $[0,T]\times[0,1]$ with a metric which is specifically adapted to the situation at hand (see (5.10.7) below).

5.8. INTERMEZZO: THE KOLMOGOROV–CHENTSOV THEOREM

The Ciesielski-type embeddings of Theorem 5.7.1 and Corollary 5.7.3 allow for a quick proof of the Kolmogorov–Chentsov theorem in a setting closely related to [146, Theorem 1.1], see Theorem 5.8.2 below (for a detailed comparison see Remark 5.8.3 below). Although Theorem 5.8.2 is not a new result, we feel the proof is sufficiently elegant and straightforward to deserve attention. Indeed, the only ingredient needed aside from Theorem 5.7.1 is the following elementary lemma:

Lemma 5.8.1. *Let $p\in[1,\infty)$, $\alpha\in(\frac{1}{p},\infty)$, $\beta\in[0,\alpha-\frac{1}{p})$ and let $(\Psi_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative real-valued random variables. Then we have*

$$\Big\|\sup_{n\in\mathbb{N}} n^{\beta}\Psi_n\Big\|_{L^p(\Omega)} \le \Big(\frac{\alpha-\beta}{\alpha-1/p-\beta}\Big)^{1/p}\sup_{n\in\mathbb{N}} n^{\alpha}\|\Psi_n\|_{L^p(\Omega)}. \tag{5.8.1}$$

Proof. Note that

$$\begin{aligned}\mathbb{E}\Big[\sup_{n\in\mathbb{N}} n^{\beta p}\Psi_n^p\Big] &\le \sum_{n\in\mathbb{N}} n^{\beta p}\mathbb{E}[\Psi_n^p] \le \sum_{n\in\mathbb{N}} n^{-(\alpha-\beta)p}\sup_{k\in\mathbb{N}} k^{\alpha p}\mathbb{E}[\Psi_k^p]\\ &\le (\alpha-\beta)(\alpha-1/p-\beta)^{-1}\sup_{k\in\mathbb{N}} k^{\alpha p}\mathbb{E}[\Psi_k^p].\end{aligned}$$

The estimate then follows after taking p-th roots. □

Theorem 5.8.2. *Let (M,d_M) be a metric space with Minkowski dimension $d\in(0,\infty)$ and finite doubling dimension, and let $(X,\|\cdot\|_X)$ be a Banach space. Let $p\in(d,\infty)\cap[1,\infty)$, $\alpha\in(\frac{d}{p},1)$, $\beta\in(0,\alpha-\frac{d}{p})$ and let $Z\colon M\times\Omega\to X$ be strongly measurable with $Z\in C^{\alpha}(M,L^p(\Omega;X))$. Then there exists a continuous modification of Z (again denoted by Z) and a constant $C_M\in(0,\infty)$ depending only on the metric space M such that*

$$\big\||Z|_{C^{\beta}(M,X)}\big\|_{L^p(\Omega)} \le 24\,C_M^{\alpha}\beta^{-1}\Big(\frac{\alpha-\beta}{\alpha-d/p-\beta}\Big)^{1/p}\frac{\Delta(M)^{\alpha}}{\Delta(M)^{\beta}}|Z|_{C^{\alpha}(M,L^p(\Omega;X))}. \tag{5.8.2}$$

Proof. By rescaling the metric (see Remark 5.5.2 and Corollary 5.6.7), we can assume without loss of generality that $\Delta(M)=1$. Let w_α, w_β be as in Example 5.5.7-(1), and recall from Example 5.5.4 that in this case $|\cdot|_{C^{\alpha}}$ and $|\cdot|_{C_{w_\alpha}}$ are identical (and likewise for β). Next, let $(x_k,y_k)_{k\in\mathbb{N}}$ be a sequence obtained from Theorem 5.7.1, and set $M_0=\cup_{k\in\mathbb{N}}\{x_k,y_k\}$. Then by Theorem 5.7.1 and Remark 5.7.2 we have

$$\begin{aligned}|Z(\cdot,\omega)|_{C_{w_\beta}(M_0,X)} &\le 3\cdot 2^{2\beta}(2^{\beta}-1)^{-1}\sup_{k\in\mathbb{N}}\frac{\|Z(x_k,\omega)-Z(y_k,\omega)\|_X}{w_\beta(k^{-1/d})}\\ &\le 24\beta^{-1}\sup_{k\in\mathbb{N}} k^{\beta/d}\|Z(x_k,\omega)-Z(y_k,\omega)\|_X\end{aligned} \tag{5.8.3}$$

for all $\omega\in\Omega$, recalling that the growth constants of w_β both equal 2^{β} and using that $2^{\beta}-1\ge\beta\log(2)\ge\beta/2$. Similarly, we obtain that there exists a constant C_M depending only on the metric space M such that

$$\begin{aligned}\sup_{k\in\mathbb{N}} k^{\alpha/d}\|Z(x_k,\cdot)-Z(y_k,\cdot)\|_{L^p(\Omega;X)} &= \sup_{k\in\mathbb{N}} \frac{\|Z(x_k,\cdot)-Z(y_k,\cdot)\|_{L^p(\Omega;X)}}{w_\alpha(k^{-1/d})} \\ &\le C_M^\alpha |Z|_{C_{w_\alpha}(M,L^p(\Omega))}.\end{aligned} \tag{5.8.4}$$

Combining (5.8.3), (5.8.4), and Lemma 5.8.1 we obtain

$$\big\| |Z|_{C_{w_\beta}(M_0,X)} \big\|_{L^p(\Omega)} \le 24\, C_M^\alpha \beta^{-1} \left(\tfrac{\alpha-\beta}{\alpha-d/p-\beta}\right)^{1/p} |Z|_{C_{w_\alpha}(M,L^p(\Omega))}. \tag{5.8.5}$$

Next, define $\tilde\Omega = \{\omega\in\Omega : |Z(\cdot,\omega)|_{C_{w_\beta}(M_0,X)} < \infty\}$. By (5.8.5) we see $\mathbb{P}\big[\tilde\Omega\big] = 1$, and from Remark 5.5.3 it follows that $Z(\cdot,\omega)\in C(M_0,X)$ for $\omega\in\tilde\Omega$. Since M_0 is dense in (M,d_M), this allows us to define $\tilde Z\colon M\times\Omega\to X$ by

$$\tilde Z(z,\omega) = \begin{cases}\lim_{z_n\in M_0, z_n\to z} Z(z_n,\omega), & \omega\in\tilde\Omega;\\ 0 & \omega\in\Omega\setminus\tilde\Omega\end{cases}, \quad z\in M. \tag{5.8.6}$$

Note that $\tilde Z$ has continuous paths by construction, and that $\tilde Z$ is a modification of Z by Fatou's lemma. Finally, (5.8.2) follows from (5.8.5) and the fact that $\tilde Z$ has continuous sample paths. □

Remark 5.8.3. Theorem 1.1 in [146] considers a slightly more general setting than Theorem 5.8.2. Firstly, the authors assume Z takes values in a general metric space instead of a Banach space – however, one can easily reduce to the Banach space setting by Kuratowski's embedding. More importantly, in [146, Theorem 1.1], the metric space M is *not* required to have a finite doubling dimension, so only the assumption on the Minkowski dimension is present. We expect that [146, Lemma 6.1] (which is a modification of [214, Lemma B.2.7]) could be used to obtain a variation of Theorem 5.7.1 (and thus also Theorem 5.8.2) in this setting, although this would require a significantly more technical argument. This, together with the fact that in our cases of interest it is harmless to assume a finite doubling dimension, deterred us from pursuing this further.

5.9. HÖLDER REGULARITY FOR STOCHASTIC INTEGRALS

In this section we combine Theorems 5.3.1 and 5.7.1 to establish Hölder regularity for parameter-dependent stochastic integrals, see Theorem 5.9.1 below.

Throughout the section, we let $T>0$, let $(X,\|\cdot\|_X)$ be a $(2,D)$-smooth Banach space and let W be an H-cylindrical Wiener processes on $(\Omega,\mathscr{F},\mathbb{P},(\mathscr{F}_t)_{t\ge 0})$ (recall the stochastic setup from Section 5.2). In addition, we let (M,d_M) be a metric space with Minkowski dimension $d\in(0,\infty)$ and finite doubling dimension (see Definitions 5.6.1 and 5.6.2). Finally, let $w\colon(0,1]\to(0,\infty)$ be an admissible modulus of continuity (see Definition 5.5.5) and define

$$w_{\log,p,d}(x) = (p-d\log(x))^{-1/2} w(x), \quad x\in(0,1],\ p\in[1,\infty). \tag{5.9.1}$$

Note that by Example 5.5.7 (2)-(3), $w_{\log,p,d}$ is again an admissible modulus of continuity, with growth constants which can be bounded independently of p.

Theorem 5.9.1. *Let $p \in [1,\infty)$, and let $((\Phi(x)(t))_{t\in[0,T]})_{x\in M}$ be a family of progressive $\gamma(H,X)$-valued processes indexed by M, which additionally satisfies*

$$K(\Phi) := \big\| |\Phi|_{C_{w_{\log,p,d}}(M,L^2(0,T;\gamma(H,X)))} \big\|_{L^p(\Omega)} < \infty. \tag{5.9.2}$$

Define $I(\Phi)\colon M \to L^p(\Omega;X)$ by

$$I(\Phi)(x) = \int_0^T \Phi(x)(t)\,\mathrm{d}W(t), \quad x \in M. \tag{5.9.3}$$

Then there exists a continuous modification of $I(\Phi)$ (again denoted by $I(\Phi)$), and we have the estimate

$$\big\| |I(\Phi)|_{C_w(M,X)} \big\|_{L^p(\Omega)} \le D\,C_{(5.9.4)} K(\Phi), \tag{5.9.4}$$

for some $C_{(5.9.4)} > 0$ which depends only on (M, d_M) and w.

Remark 5.9.2. The right-hand side of (5.9.4) still depends on p via $w_{\log,p,d}$. From Definition 5.5.1 and (5.9.1), it can be seen that the right-hand side of (5.9.4) blows up at a rate $\mathcal{O}(\sqrt{p})$ as $p \to \infty$, meaning that we retain the correct scaling in p.

From (5.9.4) it is now clear how much regularity is needed in order to have a stochastic integral which is Hölder continuous. This is seen by using $w(x) = \sqrt{p - d\log(x)}\,x^\alpha$ for some $\alpha \in (0,1)$ (see Example 5.5.7-(6) and Remark 5.5.8). In this case, it follows from Proposition 5.5.9 that if Φ has regularity C^α, then $I(\Phi)$ has regularity $C^{\alpha-\varepsilon}$ for every $\varepsilon \in (0,\alpha)$, and the $C^{\alpha-\varepsilon}$-norm of $I(\Phi)$ blows up (pointwise) at a rate $\mathcal{O}(\varepsilon^{-1/2})$ as $\varepsilon \to 0$. We remark that this rate of blowup (modulo constants) does not depend the properties of the underlying metric space.

Proof of Theorem 5.9.1. By rescaling the metric (see Remark 5.5.2 and Corollary 5.6.7), we can assume without loss of generality that $\Delta(M) = 1$. We let $(x_k, y_k)_{k\in\mathbb{N}}$ be a sequence obtained from Theorem 5.7.1, and set $M_0 = \cup_{k\in\mathbb{N}}\{x_k, y_k\}$. By (5.7.2) and Remark 5.7.2, we then obtain the inequalities

$$|I(\Phi)|_{C_w(M_0,X)} \le C_{(5.7.3)} \sup_{k\in\mathbb{N}} \frac{\|I(\Phi)(x_k) - I(\Phi)(y_k)\|_X}{w(k^{-1/d})}, \tag{5.9.5a}$$

$$\sup_{k\in\mathbb{N}} \frac{\|\Phi(x_k) - \Phi(y_k)\|_{L^2(0,T;\gamma(H,X))}}{w_{\log,p,d}(k^{-1/d})} \le C_{(5.7.4)} |\Phi|_{C_{w_{\log,p,d}}(M,L^2(0,T;\gamma(H,X)))}, \tag{5.9.5b}$$

where the doubling constants of w (resp. $w_{\log,p,d}$) should be used to compute $C_{(5.7.3)}$ (resp. $C_{(5.7.4)}$). After observing the identity

$$\frac{\sqrt{p + \log(k)}}{w(k^{-1/d})} = \frac{\sqrt{p - d\log(k^{-1/d})}}{w(k^{-1/d})} = \frac{1}{w_{\log,p,d}(k^{-1/d})}, \quad k \in \mathbb{N},$$

it follows from a direct application of Theorem 5.3.1 that

$$\begin{aligned} &\left\| \sup_{k\in\mathbb{N}} \frac{\|I(\Phi)(x_k) - I(\Phi)(y_k)\|_X}{w(k^{-1/d})} \right\|_{L^p(\Omega)} \\ &\quad \le 10D \left\| \sup_{k\in\mathbb{N}} \frac{\|\Phi(x_k) - \Phi(y_k)\|_{L^2(0,T;\gamma(H,X))}}{w_{\log,p,d}(k^{-1/d})} \right\|_{L^p(\Omega)}. \end{aligned} \tag{5.9.5c}$$

Combining (5.9.5a), (5.9.5b), and (5.9.5c), we obtain

$$\big\| |I(\Phi)|_{C_w(M_0,X)} \big\|_{L^p(\Omega)} \leq 10 D C_{(5.7.3)} C_{(5.7.4)} K(\Phi). \tag{5.9.6}$$

Next, we define $\tilde{\Omega} = \{\omega \in \Omega : |I(\Phi)|_{C_w(M_0,X)} < \infty\}$ and define the modification of $I(\Phi)$ analogously to (5.8.6). The remainder of the argument is entirely analogous to the one provided in the proof of Theorem 5.8.2, noting that if $z \in M$ and $(z_k)_{k\in\mathbb{N}}$ is a sequence in M_0 converging to z, then Fatou's lemma and [207, Theorem 1.1] imply

$$\begin{aligned} \|\lim_{k\to\infty} \|I(\Phi)(z) - I(\Phi)(z_k)\|_X\|_{L^p(\Omega)} &\leq \lim_{k\to\infty} \|I(\Phi)(z) - I(\Phi)(z_k)\|_{L^p(\Omega;X)} \\ &\leq 10 D\sqrt{p} \lim_{k\to\infty} \|\Phi(z_k) - \Phi(z)\|_{L^p(\Omega;L^2(0,T;\gamma(H,X)))} = 0 \end{aligned}$$

because $K(\Phi) < \infty$. □

Remark 5.9.3. If we take $T = 1$, $H = X = \mathbb{R}$ (so $\gamma(H,X) \simeq \mathbb{R}$), $M = [0,1]$ endowed with the Euclidean metric, $w(x) = \sqrt{1 - \frac{1}{2}\log(x))x}$, and $\Phi(s)(t) = 1_{[0,s]}(t)$ $(s,t \in [0,1])$ in the setting of Theorem 5.9.1, then $I(\Phi)(s) = W(s)$ and Theorem 5.9.1 implies

$$\left\| \sup_{0\leq r<s\leq 1} \frac{|W(s) - W(r)|}{\sqrt{(s-r)(1 - \frac{1}{2}\log(s-r))}} \right\|_{L^p(\Omega)} \leq C\sqrt{p} \tag{5.9.7}$$

for all $p \in [1,\infty)$ (where $C \in (0,\infty)$ is independent of p). Recall that Lévy's modulus of continuity theorem states that

$$\lim_{h\downarrow 0} \sup_{0\leq r<s\leq 1, |r-s|<h} \frac{|W(s) - W(r)|}{\sqrt{2h|\log(h)|}} = 1 \quad \text{a.s.} \tag{5.9.8}$$

Comparing (5.9.7) and (5.9.8) we see that Theorem 5.9.1 is sharp in terms of the obtained modulus of continuity.

5.10. REGULARITY OF THE 1D PARABOLIC ANDERSON MODEL

As an application of Theorem 5.9.1, we investigate the regularity of the 1D parabolic Anderson model (PAM), which can also be viewed as a stochastic heat equation with linear multiplicative noise. To formulate the equation, we fix $\eta \in (0,\infty)$, $T \in (0,\infty)$ and let W be an $L^2(0,1)$-cylindrical Wiener process on $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\geq 0})$ (recall the stochastic setup from Section 5.2). The parabolic Anderson model is then formally given by:

$$\frac{\partial}{\partial t} U(t,x) = \frac{\partial^2}{\partial x^2} U(t,x) + \eta U(t,x) \frac{\partial}{\partial t} W(t,x), \qquad (t,x) \in (0,T) \times (0,1), \tag{5.10.1a}$$

$$U(t,0) = U(t,1) = 0, \qquad t \in (0,T], \tag{5.10.1b}$$

$$U(0,x) = U_0(x), \qquad x \in [0,1], \tag{5.10.1c}$$

where the (random) initial condition U_0 is assumed to be $\mathscr{F}_0$-measurable. To avoid complicating the presentation, we assume $U_0 \in L^p(\Omega; C^2([0,T]))$ for some $p \in (4,\infty)$, so that the regularity of U will not be limited by that of the initial value.

To obtain a solution to (5.10.1), we let $(h_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(0,1)$ and set $\beta_k(t) = W(h_k \otimes 1_{[0,t]})$ ($k \in \mathbb{N}$, $t \geq 0$), rendering $(\beta_k)_{k\in\mathbb{N}}$ a sequence of independent standard Brownian motions. We also let $G\colon [0,\infty)\times[0,1]\times[0,1] \to \mathbb{R}$ be the Green's function associated with the Dirichlet Laplacian on $[0,1]$, i.e.,

$$G(t,x,y) = \sum_{k\in\mathbb{N}} 2e^{-\pi^2 k^2 t}\sin(\pi k x)\sin(\pi k y). \tag{5.10.2}$$

It then follows from [183, Theorem 6.2] that there exists a unique (up to indistinguishability) adapted stochastic process $U \in L^p(\Omega; C([0,T]\times[0,1]))$ such that

$$\begin{aligned} U(t,x) &= \int_0^1 G(t,x,y)U_0(y)\,\mathrm{d}y \\ &\quad + \eta\sum_{k\in\mathbb{N}}\int_0^t\int_0^1 G(t-s,x,y)U(s,y)h_k(y)\,\mathrm{d}y\,\mathrm{d}\beta_k(s). \end{aligned} \tag{5.10.3}$$

The process U is conventionally called the *mild solution* to (5.10.1). From the Sobolev embedding theorem and [183, Theorem 6.3], it also follows that U takes values in the space $L^p(\Omega; C^\lambda([0,T], C^{2\gamma}([0,1])))$ for every $\lambda,\gamma \in (0,\infty)$ satisfying $\lambda+\gamma < \frac14 - \frac1p$. In particular, we have

$$\left\| \sup_{\substack{t,s\in[0,T];x,y\in[0,1];\\(t,x)\neq(s,y)}} \frac{|U(t,x)-U(s,y)|}{|t-s|^{\frac14-\frac1p-\varepsilon} + |x-y|^{\frac12-\frac2p-\varepsilon}} \right\|_{L^p(\Omega)} < \infty, \tag{5.10.4}$$

for all sufficiently small $\varepsilon > 0$. We now demonstrate how Theorem 5.9.1 can be employed to show the following refined regularity result:

Theorem 5.10.1. *Let $p \in (4,\infty)$, $U_0 \in L^p(\Omega; C^2([0,1]))$, and let U be the unique mild solution to* (5.10.1). *Then we have*

$$\left\| \sup_{\substack{t,s\in[0,T];\\x,y\in[0,1];\\(t,x)\neq(s,y)}} \frac{|U(t,x)-U(s,y)|}{(1-\frac14\log(|t-s|))^{\frac12}|t-s|^{\frac14} + (1-\frac12\log(|x-y|))|x-y|^{\frac12}} \right\|_{L^p(\Omega)} < \infty. \tag{5.10.5}$$

Proof. Throughout the proof, we write $X \lesssim Y$ if there exists a constant C depending only on p and T, such that $X \leq CY$.

Note that the first term on the right-hand side of (5.10.3) solves the heat equation with initial value U_0. Thus, by the assumed regularity of U_0, we only need to concern ourselves with the stochastic integral in (5.10.3). We begin by noting that every $h \in L^2(0,1)$ can be associated with an operator $R_h \in \gamma(L^2(0,1),\mathbb{R})$ via the relation $R_h g = \int_0^1 h(x)g(x)\,\mathrm{d}x$, in which case $\|R_h\|_{\gamma(L^2(0,1),\mathbb{R})} = \|h\|_{L^2(0,1)}$ (see [120, Proposition 9.2.9]). In particular, setting $M = [0,T]\times[0,1]$, this induces an isometric isomorphism

$$C(M, L^2(M)) \simeq C(M, L^2(0,T;\gamma(L^2(0,1),\mathbb{R}))), \tag{5.10.6}$$

so that we may define $\Phi \in L^p(\Omega; C(M, L^2(0,T;\gamma(L^2(0,1),\mathbb{R}))))$ by setting

$$\Phi(\omega)(t,x)(s,y) = G(t-s,x,y)U(s,y,\omega)1_{[0,t]}(s),$$

whence we have, with $I(\Phi)$ as in Theorem 5.9.1:

$$I(\Phi)(t,x) = \sum_{k\in\mathbb{N}} \int_0^t \int_0^1 G(t-s,x,y)U(s,y)h_k(y)\,\mathrm{d}y\,\mathrm{d}\beta_k(s), \quad (t,x)\in[0,T]\times[0,1].$$

Our intermediate goal is now to rewrite (5.10.5) in such a way that we can apply Theorem 5.9.1. To this end, we endow M with the following metric:

$$\tilde{d}((t,x),(s,y)) = |t-s|^{1/2} + (1-\tfrac{1}{2}\log(|x-y|))|x-y|. \tag{5.10.7}$$

Note that $\tilde{d}$ satisfies the triangle inequality, since $t\mapsto t^{1/2}$ is monotone and subadditive on $[0,T]$, and likewise for $x\mapsto(1-\frac{1}{2}\log(x))x$ on $[0,1]$. One can verify that $(M,\tilde{d})$ has finite doubling dimension and Minkowski dimension $3+\varepsilon$ for any $\varepsilon>0$ (we will only use $\varepsilon=1$). Also, for any $u,v\in M$ with $u=(t,x)$ and $v=(s,y)$ we have

$$\begin{aligned}
(1-\tfrac{1}{2}\log(\tilde{d}(u,v)))\tilde{d}(u,v) &= (1-\tfrac{1}{2}\log(\tilde{d}(u,v)))|t-s|^{1/2}\\
&\quad + (1-\tfrac{1}{2}\log(\tilde{d}(u,v)))(1-\tfrac{1}{2}\log(|x-y|))|x-y|\\
&\le (1-\tfrac{1}{4}\log(|t-s|))|t-s|^{1/2}\\
&\quad + (1-\tfrac{1}{2}\log(|x-y|))^2|x-y|,
\end{aligned}$$

since $\tilde{d}(u,v)\ge\max(|t-s|^{1/2},|x-y|)$. Hence, to show (5.10.5) it suffices to establish

$$K := \Big\| \sup_{u,v\in M} \frac{I(\Phi)(u)-I(\Phi)(v)}{(1-\frac{1}{2}\log(\tilde{d}(u,v)))^{1/2}\tilde{d}(u,v)^{1/2}} \Big\|_{L^p(\Omega)} < \infty.$$

We now set $w(x) = (1-\frac{1}{2}\log(x)))^{1/2}x^{1/2}$ and note that w is an admissible modulus of continuity by Example 5.5.7-(6). Recalling Definition 5.5.1 and Example 5.5.4 (both of which should be interpreted with respect to $\tilde{d}$), we apply Theorem 5.9.1 to see that

$$\begin{aligned}
K &\lesssim \big\| |I(\Phi)|_{C_w(M,\mathbb{R})} \big\|_{L^p(\Omega)} \lesssim \big\| |\Phi|_{C_{w_{\log,p,4}}(M,L^2(0,T;\gamma(L^2(0,1),\mathbb{R})))} \big\|_{L^p(\Omega)}\\
&= \big\| |\Phi|_{C_{w_{\log,p,4}}(M,L^2(M))} \big\|_{L^p(\Omega)} \lesssim \big\| |\Phi|_{C^{1/2}(M,L^2(M))} \big\|_{L^p(\Omega)},
\end{aligned} \tag{5.10.8}$$

where $w_{\log,p,4}$ is as in (5.9.1) and we have used $w_{\log,p,4}(x)\gtrsim w_{\log,1,1/2}(x)=x^{1/2}$ for the final step (see Remark 5.5.8) — also note that Hölder regularity in the space $C^{1/2}(M,L^2(M))$ is measured with respect to the metric $\tilde{d}$, whereas $L^2(M)$ is simply the usual Lebesgue space (i.e., involving the Lebesgue measure on M). In view of (5.10.8), all that remains is to show that $\big\||\Phi|_{C^{1/2}(M,L^2(M))}\big\|_{L^p(\Omega)}<\infty$. To see this, we observe that for any $t,s\in[0,T]$ and $x,y\in[0,1]$ we have

$$\begin{aligned}
\|\Phi(t,x)-\Phi(s,y)\|^2_{L^2([0,T]\times[0,1])} &\le \sup_{(\tau,\xi)\in[0,T]\times[0,1]} |U(\tau,\xi)|^2\\
&\quad\times \int_0^t\int_0^1 |G(t-r,x,z)-G(s-r,y,z)1_{[0,s]}(r)|^2\,\mathrm{d}z\,\mathrm{d}r.
\end{aligned}$$

Applying Lemma 5.A.1 and taking square roots, we thus find that for any $u,v\in M$, we have

$$\|\Phi(u)-\Phi(v)\|_{L^2(M)} \lesssim \tilde{d}(u,v)^{1/2}\|U\|_{C(M,\mathbb{R})}, \tag{5.10.9}$$

so that $\|\Phi\|_{C^{1/2}(M,L^2(M))}\lesssim\|U\|_{C(M,\mathbb{R})}$ by Example 5.5.4. We conclude by taking the $L^p(\Omega)$-norm and using $\|U\|_{L^p(\Omega;C(M,\mathbb{R}))}<\infty$. □

Remark 5.10.2. We have assumed $U_0 \in L^p(\Omega; C^2([0,1]))$ to avoid any regularity issues coming from the initial value. Similarly, the assumption $p > 4$ is only used to guarantee existence of a solution in $L^p(\Omega; C([0,T] \times [0,1]))$ from [183, Theorem 6.2], see (5.10.9). We expect that with some additional bookkeeping (which would distract from our presentation) Theorem 5.10.1 can straightforwardly be extended to rougher initial values, such as $U_0 \in L^p(\Omega; C([0,1]))$ or beyond. One could even forego the use of [183] entirely by performing a fixed-point argument in the space $L^p(\Omega; C([0,T] \times [0,1]))$ with $p \in [1,\infty)$ using the estimates outlined in the proof of Theorem 5.10.1.

5.A. REGULARITY OF THE GREEN'S FUNCTION

The proof of Theorem 5.10.1 relies on the following regularity result of the Dirichlet Green's function.

Lemma 5.A.1. *Let $G\colon [0,\infty)\times[0,1]\times[0,1] \to \mathbb{R}$ be the Dirichlet heat kernel given by (5.10.2) and let $T > 0$. Then there exists a constant $c_{(5.A.1)} \in (0,\infty)$ (possibly depending on T) such that*

$$\begin{aligned}\int_0^t \int_0^1 |G(t-r,x,z) - G(s-r,y,z)1_{[0,s]}(r)|^2 \,\mathrm{d}z\,\mathrm{d}r \\ \leq c_{(5.A.1)}\big(|t-s|^{1/2} - \log(|x-y|)|x-y|\big)\end{aligned} \tag{5.A.1}$$

for all $t,s \in [0,T]$ and $x,y \in [0,1]$ satisfying $s < t$.

Proof of Lemma 5.A.1. Fix $x,y \in [0,1]$ and $0 \leq s < t \leq T$. Throughout the proof, we write $A \lesssim B$ if there exists a constant C, independent of x,y,t,s such that $A \leq CB$.

First, using (5.10.2) and Parseval's identity (in the z variable) we can estimate

$$\begin{aligned}\int_s^t \int_0^1 |G(t-r,x,z)|^2 \,\mathrm{d}z\,\mathrm{d}r &\lesssim \int_s^t \sum_{k\in\mathbb{N}} e^{-2\pi^2k^2(t-r)}\,\mathrm{d}r \\ &= \int_0^{t-s} \sum_{k\in\mathbb{N}} e^{-2\pi^2k^2 r}\,\mathrm{d}r \leq \int_0^{t-s} r^{-1/2}\int_0^\infty e^{-2\pi^2 z^2}\,\mathrm{d}z\,\mathrm{d}r \lesssim 2(t-s)^{1/2}.\end{aligned}$$

By additionally using Minkowski's inequality, we also find

$$\begin{aligned}&\int_0^s \int_0^1 |G(t-r,x,z) - G(s-r,x,z)|^2 \,\mathrm{d}z\,\mathrm{d}r \\ &\quad\lesssim \int_0^s \sum_{k\in\mathbb{N}} (e^{-\pi^2k^2(s-r)} - e^{-\pi^2k^2(t-r)})^2\,\mathrm{d}r \\ &\quad\lesssim \int_0^s \sum_{k\in\mathbb{N}} \Big(\int_{s-r}^{t-r} k^2 e^{-\pi^2k^2\tau}\,\mathrm{d}\tau\Big)^2\,\mathrm{d}r \\ &\quad\lesssim \int_0^s \Big(\int_{s-r}^{t-r} \Big(\sum_{k\in\mathbb{N}} k^4 e^{-2\pi^2k^2\tau}\Big)^{1/2}\,\mathrm{d}\tau\Big)^2\,\mathrm{d}r \\ &\quad\lesssim \int_0^s \Big(\int_{s-r}^{t-r} \tau^{-5/4}\,\mathrm{d}\tau\Big)^2\,\mathrm{d}r \\ &\quad\lesssim \int_0^s \Big((s-r)^{-1/4} - (t-r)^{-1/4}\Big)^2\,\mathrm{d}r.\end{aligned}$$

Using the fact that $(b-a)^2 \le b^2 - a^2$ whenever $0 \le a \le b$, we obtain

$$\begin{aligned}\int_0^s \Big((s-r)^{-1/4} - (t-r)^{-1/4}\Big)^2 \,\mathrm{d}r &\le \int_0^s (s-r)^{-1/2} - (t-r)^{-1/2} \,\mathrm{d}r \\ &= 2\big(s^{1/2} - t^{1/2} + (t-s)^{1/2}\big) \le 2(t-s)^{1/2}.\end{aligned}$$

Finally, using (5.10.2), Parseval's identity, and the Hölder continuity of the sine function, we obtain for $\varepsilon \in (0,1)$:

$$\begin{aligned}&\int_0^s \int_0^1 |G(s-r,x,z) - G(s-r,y,z)|^2 \,\mathrm{d}z\,\mathrm{d}r \\ &\quad= \int_0^s \int_0^1 |G(r,x,z) - G(r,y,z)|^2 \,\mathrm{d}z\,\mathrm{d}r \\ &\quad\lesssim \int_0^s \sum_{k\in\mathbb{N}} e^{-2\pi^2 k^2 r} (\sin(k\pi x) - \sin(k\pi y))^2 \,\mathrm{d}r \\ &\quad\lesssim \sum_{k\in\mathbb{N}} k^{-2} (\sin(k\pi x) - \sin(k\pi y))^2 \\ &\quad\lesssim \sum_{k\in\mathbb{N}} k^{-(1+\varepsilon)} |x-y|^{1-\varepsilon} \\ &\quad\lesssim \varepsilon^{-1} |x-y|^{1-\varepsilon}.\end{aligned}$$

Choosing $\varepsilon^{-1} = -\log|x-y|$ and combining with the previous estimates, the proof is complete. $\square$

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[P2] J. van Winden. “Noncommutative orbital stability of stochastic patterns in Banach spaces”. Preprint. 2024. [arXiv].

[P3] M.V. Gnann, R.W.S. Westdorp, and J. van Winden. “Well-posedness of a parametrically forced nonlinear Schrödinger equation driven by translation-invariant noise”. Preprint. 2024. [arXiv]. This preprint has been superseded by [A3].

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[A1] C. Kuehn and J. van Winden. “Synchronization by noise for traveling pulses”. In: *The Annals of Applied Probability* (2026). Forthcoming.

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