The clamped elastic grid, a fourth order equation on a domain with corner

The clamped elastic grid, a fourth order equation on a domain with corner

Proefschrift

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus Prof. dr. ir. J.T. Fokkema,
voorzitter van het College voor Promoties,
in het openbaar te verdedigen
op dinsdag 29 september 2009 om 10.00 uur

door

Tymofiy GERASIMOV

Master of Science in Applied Mathematics
Zaporizhzhia Staatsuniversiteit, Oekraïne

geboren te Leninabad, Sovjet Unie

Dit proefschrift is goedgekeurd door de promotor:

Prof. dr. G.H. Sweers

Samenstelling promotiecommissie:

Rector Magnificus,	Technische Universiteit Delft, voorzitter
Prof. dr. G.H. Sweers,	Technische Universiteit Delft & Universität zu Köln, promotor
Prof. C. Davini,	Università degli Studi di Udine
Prof. dr. B. Kawohl,	Universität zu Köln
Prof. dr. ir. A.W. Heemink,	Technische Universiteit Delft
Prof. dr. J.M.A.M. van Neerven,	Technische Universiteit Delft
Prof. dr. ir. C. Vuik,	Technische Universiteit Delft
Dr. ir. E.H. van Brummelen,	Technische Universiteit Delft
Prof. ir. L. van der Sluis,	Technische Universiteit Delft, reservelid

ISBN 978-90-9024623-9

Printed in the Netherlands by: Wöhrmann Print Service

This research was carried out at the Functional Analysis Group, Delft Institute of Applied Mathematics, Department of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, the Netherlands

To Anna and Liza,

who give me the strength
in all my accomplishments

Contents

Chapter 1

Introduction

The operator $L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ can be used as a model for the vertical displacement of a two-dimensional grid that consists of two perpendicular sets of elastic fibers or rods. We are interested in the behaviour of such a grid that is clamped at the boundary and more specifically near a corner of the domain. Kondratiev supplied the appropriate setting in the sense of Sobolev type spaces tailored to find the optimal regularity. Inspired by the Laplacian and the bilaplacian models one expects, except maybe for some isolated special angles, that the optimal regularity improves when angle decreases. For the homogeneous Dirichlet problem with this special non-isotropic fourth order operator $L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ such a result does not hold true. We will prove the existence of at least one interval $\left(\frac{1}{2}\pi, \omega_\star\right)$, $\omega_\star/\pi \approx 0.528$ (in degrees $\omega_\star \approx 95.1°$), in which the optimal regularity improves with increasing opening angle.

1.1 The model

The Kirchhoff model for small deformations of a thin isotropic elastic plate is $\Delta^2 u = f$ (see e.g. the seminal paper [17]). Here f is a force density, u is the vertical displacement of a plate and $\Delta^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$ is the Bilaplace operator; the model neglects the influence of horizontal deviations.

Non-isotropic elastic plates are still modeled by fourth order differential equations but the coefficients in front of the derivatives of u may vary. The interesting extreme case is the equation

$$u_{xxxx} + u_{yyyy} = f.$$

One may think of the above equation as of the model of an elastic medium consisting of two sets of intertwined (not glued) perpendicular fibers running in Cartesian directions (Figure 1.1). We will call such medium *a grid* and the operator $L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ *a grid operator.*

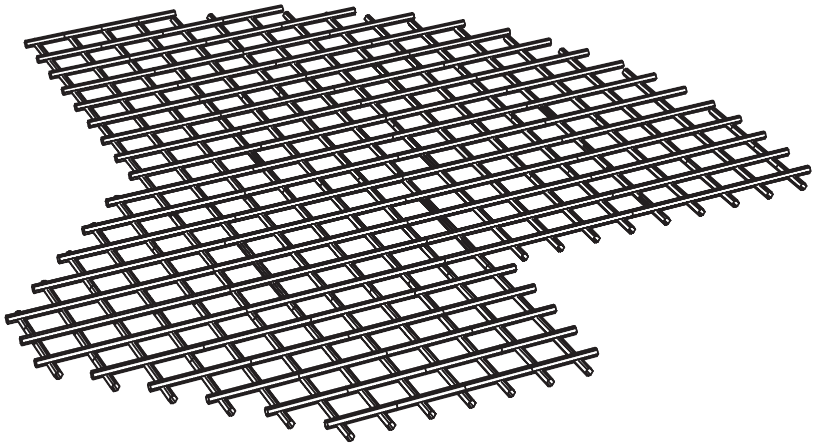

Figure 1.1: *A fragment of an elastic grid.*

The main assumption here is that sets of fibers are connected in such a way that the vertical positions coincide but there is no connection that forces a torsion in the fibers. Such torsion would occur if the fibers are glued or imbedded in a softer medium. For those models see [27]. The appropriate linearized model in that last situation would contain mixed fourth order derivatives.

A first place where operator $L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ appears is J. II. Bernoulli's paper [1]. He assumed that it was the appropriate model for an isotropic plate. It was soon dismissed as a model for such a plate, since it failed to have rotational symmetry. Indeed, the rotation of $\frac{1}{4}\pi$ transforms $\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ into $\frac{1}{2}\frac{\partial^4}{\partial x^4} + 3\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{1}{2}\frac{\partial^4}{\partial y^4}$.

1.2 The setting

We will focus on $L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ supplied with homogeneous Dirichlet boundary conditions. This problem, which we call '*a clamped grid*', is as follows:

$$\begin{cases} u_{xxxx} + u_{yyyy} = f & \text{in} \quad \Omega, \\ u = \frac{\partial}{\partial n} u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^2$ is open and bounded, and n is the unit outward normal vector on $\partial\Omega$. The boundary conditions in (1.1) correspond to the clamped situation meaning that the vertical position and the angle are fixed to be 0 at the boundary.

One verifies directly that the operator $L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ is elliptic in $\overline{\Omega}$. One may also prove, if the normal n is well-defined, that the boundary value problem (1.1) is regular elliptic. Indeed, the Dirichlet problem which fixes the zero and first order derivatives at the boundary, is regular elliptic for any fourth order uniformly elliptic operator. Hence, under the assumption that Ω is bounded and $\partial\Omega \in C^\infty$ the full classical regularity result (see e.g. [25]) for problem (1.1) can be used to find for $k \geq 0$ and $p \in (1, \infty)$:

$$\text{if} \quad f \in W^{k,p}(\Omega) \quad \text{then} \quad u \in W^{k+4,p}(\Omega). \tag{1.2}$$

If Ω in (1.1) has a piecewise smooth boundary $\partial\Omega$ with, say, one angular point, the result (1.2) in general does not apply. Instead, one may use the theory developed by Kondratiev [18]. This theory provides the appropriate treatment of problem (1.1) by employing the weighted Sobolev space $V_\beta^{k,p}(\Omega)$ (see Definition 4.1), where $k \geq 0$ is the differentiability index and $\beta \in \mathbb{R}$ characterizes the powerlike growth of the solution near the angular point of Ω. Within the framework of the Kondratiev spaces $V_\beta^{k,p}(\Omega)$ the regularity result "analogous" to (1.2) will then be as follows. There is a countable set of functions $\{u_j\}_{j\in\mathbb{N}}$ and constants $\{c_j\}_{j\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$:

$$\text{if} \quad f \in V_\beta^{k,p}(\Omega) \quad \text{then} \quad u = w + \sum_{j=1}^{J(k,p,\beta)} c_j u_j \quad \text{with} \quad w \in V_\beta^{k+4,p}(\Omega). \tag{1.3}$$

The functions $\{u_j\}_{j\in\mathbb{N}}$ in (1.3) describe the behaviour of the solution u locally in the vicinity of an angular point and are called sometimes the singular solutions to (1.1). In this thesis, we will restrict our formulations to $p = 2$.

Partial differential equations on domains with corners have obtained a lot of attention both in the mechanical and mathematical literature. For instance, in 1951 Williams in his paper [30] identified possible power singularities for a variety of homogeneous boundary conditions on the plate edges for angular elastic plates in bending treated within classical fourth-order theory. However, one may assume that the advanced qualitative theory on the subject has been developed in the seminal paper by Kondatiev [18]. Since that time many authors of which we would like to mention Kozlov, Maz'ya, Rossmann [19, 20], Grisvard [14], Dauge [7], Costabel and Dauge [4], Nazarov and Plamenevsky [26] have contributed. For applications in elasticity theory we refer to Leguillon and Sanchez-Palencia [23], Blum and Rannacher [3]. A recent paper of Kawohl and Sweers [21] concerned the positivity question for the operators $\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ and $\frac{1}{2}\frac{\partial^4}{\partial x^4} + 3\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{1}{2}\frac{\partial^4}{\partial y^4}$ in a rectangular domain for hinged boundary conditions.

1.2.1 Why grid model? Mathematical motivation

We have already mentioned above that the deformation of a thin non-isotropic elastic plate is modeled by the equation (see e.g. [24, p. 281]):

$$D_1 u_{xxxx} + D_2 u_{xxxy} + D_3 u_{xxyy} + D_4 u_{xyyy} + D_5 u_{yyyy} = f,$$

where D_j, $j = 1, ..., 5$ are elastic constants of a material a plate made of. By the standard rescaling in x and y one may turn the coefficients in front of u_{xxxx} and u_{yyyy} into 1, so that the abstract mathematical model would be

$$u_{xxxx} + b_1 u_{xxxy} + b_2 u_{xxyy} + b_3 u_{xyyy} + u_{yyyy} = f,$$

with $b_j \in \mathbb{R}$, $j = 1, 2, 3$. In Appendix A we show that provided the operator $\frac{\partial^4}{\partial x^4} + b_1 \frac{\partial^4}{\partial x^3 \partial y} + b_2 \frac{\partial^4}{\partial x^2 \partial y^2} + b_3 \frac{\partial^4}{\partial x \partial y^3} + \frac{\partial^4}{\partial y^4}$ is elliptic, there always exists an appropriate linear coordinate transformation such that in new coordinates the above equation will read as

$$u_{xxxx} + 2a u_{xxyy} + u_{yyyy} = f,$$

with $a \in [1, +\infty)$. If we set $a = 3$ in the above equation and rotate the coordinate system by $\frac{1}{4}\pi$, we will arrive (by further rescaling) at our grid model $u_{xxxx} + u_{yyyy} = f$.

1.2.2 Why corner? Mechanical motivation

A thin (non-isotropic) elastic plate is the main constructive element of almost every thin-walled engineering construction ranging from aircrafts, bridges, ships and oil rigs to storage vessels, industrial buildings and warehouses. A conventional geometry for such a plate is a polygon, that is, a planar domain with corners (both convex and concave, in general). From an engineering practice, it is well known that the presence of corners, namely, the reentrant corners in a plate may cause a significant reduction or even the loss of its load-carrying capacity. This happens due to concentration of stresses which appear near corner points of a plate and which can be extremely high (stress singularity).

Examples of such a loss are the crashes of De Havilland 106 aircrafts (see Figure 1.2) in the yearly 1950s. Also known as "The Comet" it was the first commercial airliner with jet engines and pressurized fuselage. The designers implemented cabin's pressurization in order to provide the passengers with the comfortable living conditions during the altitude flight. Within the first two years after entering service in May 1952, two of the fleet disintegrated while climbing to cruise altitude.

Figure 1.2: *May 2, 1952. "The Comet" G-ALYP departures from London's Heathrow Airport for her first scheduled flight. The picture is taken from [15].*

Extensive investigation determined the major constructive weakness of the aircraft – *square windows*. Stresses that appeared in a fuselage skin around the window corners was found to be much higher than expected. Such the stress concentration at corners resulted in a fatigue crack, which was growing rapidly due to repeated cabin's pressurizations and depressurizations, and eventually led to a sudden break-up of a fuselage. During a full scale repeated pressurization test on an aircraft removed from service, the cabin failure had been observed: a fatigue cracking that began at the lower corner of a window (see Figure 1.3). Also, the fragments collected from the scene of the crash showed that a crack had developed due to metal fatigue near direction finding aerial window (a square window situated in the front of the cabin roof).

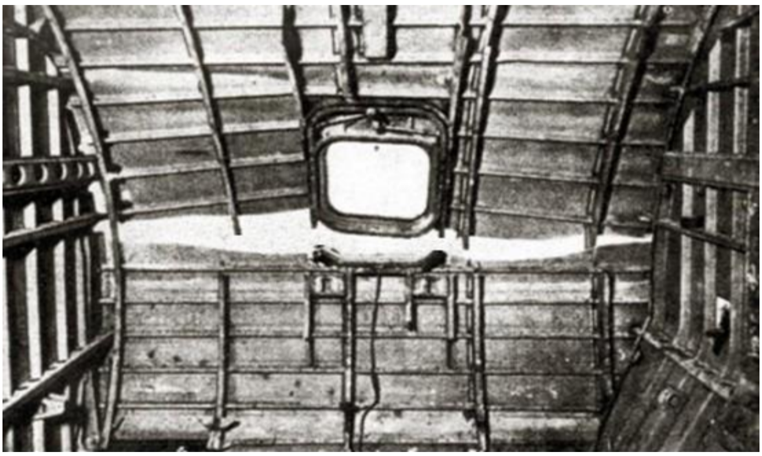

Figure 1.3: *"The Comet" fuselage cracked during the tests. The crack started at a corner of a square window. The picture is taken from [16].*

After the conclusive evidence of the reasons of crashes had been revealed, all the Comets were redesigned to have oval windows.

Remark 1.1 *Let us note that "The Comet" example is an illustrative one. Its purpose is to bring some evidence that even the smooth reentrant corner may be treated as "a weak point" of an engineering construction (here, of a fuselage panel) carrying a load, and one could expect even worse situation if the corner was sharp. The purpose of the thesis, however, is to consider the mathematical aspects of corners in the non-isotropic planar material.*

1.3 The target

In this thesis, we will focus particularly on the optimal regularity for the clamped grid problem, which depends on the opening angle of the corner. For the sake of a simple presentation, we will consider (1.1) in a domain $\Omega \subset \mathbb{R}^2$ which has one corner in $0 \in \partial\Omega$ with opening angle $\omega \in (0, 2\pi]$. Due to the Kondratiev theory a more appropriate formulation of the problem should read as:

$$\begin{cases} u_{xxxx} + u_{yyyy} = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \frac{\partial}{\partial n} u = 0 & \text{on} \quad \partial\Omega \backslash \{0\}, \end{cases} \tag{1.4}$$

"with prescribed growth behaviour near 0".

One notices that depending on the orientation of Ω in a coordinate system (x, y), our problem (1.4) will model different situations from the physical point of view. Of course, this difference seemingly will play a role locally, in the vicinity of angular point 0. It is illustrated in Figure 1.4: on the left plot a domain Ω with corner of $\frac{3}{2}\pi$ is oriented in such a way that the fibers of a grid turn out to be aligned with respect to the sides of a corner, while on the right one the fibers of a grid are arranged diagonally with respect to the corner; the mathematical model (1.4), however, remains the same for both situations.

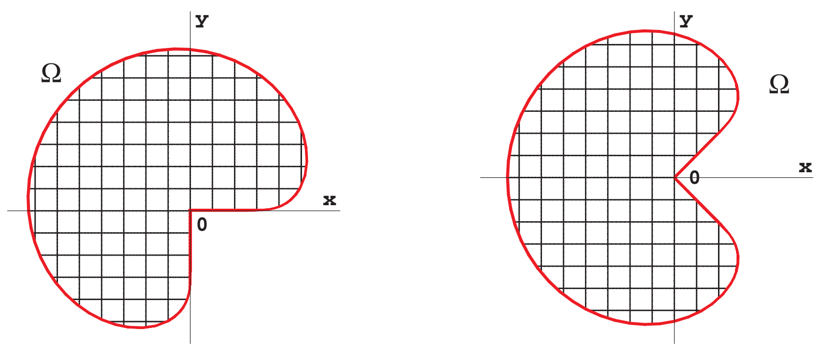

Figure 1.4: *The same Ω being oriented differently in (x, y) results in distinct physical problems.*

Hence, in order to complete the formulation of (1.4), we introduce a parameter $\alpha \in \left[0, \frac{1}{2}\pi\right)$, which defines orientation of Ω. Obviously, the cases $\alpha = 0$ and $\alpha = \frac{1}{2}\pi$ yield the identical situation.

The precise description of a domain Ω in problem (1.4) will be then as follows.

Condition 1.2 *The domain Ω has a smooth boundary except at $(x, y) = 0$, and is such that in the vicinity of 0 it locally coincides with a cone. In other words,*

1. *$\partial\Omega \backslash \{0\}$ is C^∞,*
2. *$\Omega \cap B_\varepsilon(0) = \mathcal{K}_{(\alpha,\omega)} \cap B_\varepsilon(0)$,*

where $B_\varepsilon(0) = \{(x, y) : |(x, y)| < \varepsilon\}$ is the open ball of radius $\varepsilon > 0$ centered at $(x, y) = 0$ and $\mathcal{K}_{(\alpha,\omega)}$ an infinite cone with an opening angle $\omega \in (0, 2\pi]$ and orientation angle $\alpha \in [0, \frac{1}{2}\pi)$:

$$\mathcal{K}_{(\alpha,\omega)} = \{(r\cos(\theta), r\sin(\theta)) : \ 0 < r < \infty \text{ and } \alpha < \theta < \alpha + \omega\}. \tag{1.5}$$

In Figure 1.5 a domain Ω which satisfies the condition above and corresponding cone $\mathcal{K}_{(\alpha,\omega)}$ are sketched.

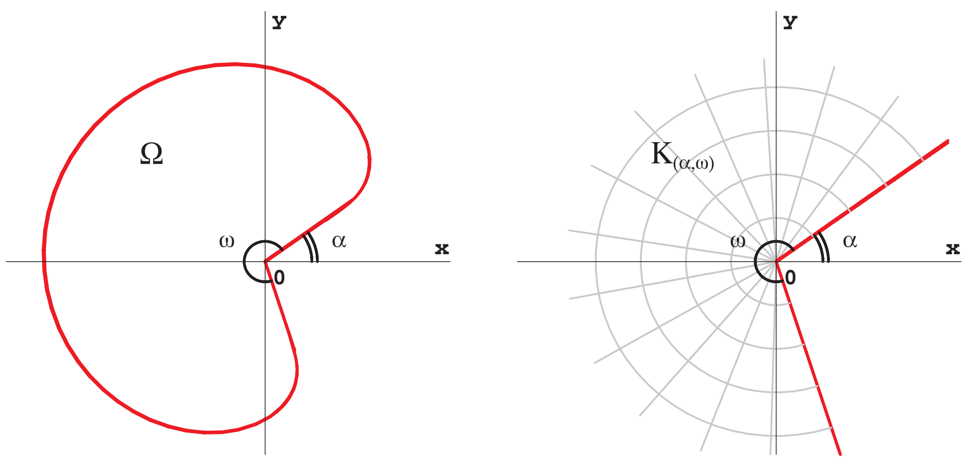

Figure 1.5: *Example for Ω and the corresponding cone $\mathcal{K}_{(\alpha,\omega)}$.*

For the elliptic problem one might roughly distinguish between papers that focus on the general theory and those papers that explicitly study in detail the results for one special model. If one chooses a special fourth order model then one usually has the biharmonic operator in the differential equation. For the biharmonic problem of the type (1.4) the optimal regularity due to the corner of Ω 'improves' when the opening angle ω decreases. In fact Kondratiev in his seminal paper [18, page 210] states that

> *"... for the number of concrete equations in § 5, it is derived that the differential properties of the solution are getting better when the cone opening decreases."*

One of the peculiar results for the present clamped grid problem which we first obtained in [12] is that this does not apply for the whole range 0 to 2π. We show that for the case $\alpha = 0$ in Condition 1.2 there is interval $\left(\frac{1}{2}\pi, \omega_\star\right)$, with $\omega_\star/\pi \approx 0.528$ (in degrees $\omega_\star \approx 95.1°$), where the optimal regularity of the solution u to problem (1.4) *increases* with increasing ω. This does not happen in the case of the clamped plate problem, i.e. when the operator in (1.4) is the bilaplacian Δ^2 (the comparison is outlined in Table 1.1 below). The actual curve that displays the connection between ω and λ, a parameter for the differential properties, is obtained numerically. The discretization is chosen fine enough such that analytical estimates show that the numerical errors are so small that they do not destroy the structure.

operator L of the problem (1.4)	opening angle ω	the regularity of the solution u to (1.4) in dependence on ω
Δ^2	$(0, 2\pi]$	decreases
$\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$	$\left(0, \frac{1}{2}\pi\right]$, $[\omega_\star, 2\pi]$	decreases
(the case $\alpha = 0$)	$\left(\frac{1}{2}\pi, \omega_\star\right)$	increases

Table 1.1: *Optimal regularity of the homogeneous Dirichlet problem for* Δ^2 *and* $\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ *when* $\alpha = 0$

For a graph displaying relation between ω and λ in this case (i.e. when $\alpha = 0$) see Figure 3.2 and in Figure 3.5 one finds a more detailed view. The lowest value of the appearing λ is a measure for the regularity. A more general result (i.e. when $\alpha \in (0, \frac{1}{2}\pi)$) is outlined in Figures 3.7 and 3.8.

1.4 Content of the thesis

This thesis is divided into six chapters and several appendices.

In Chapter 2 we recall the results for existence and uniqueness of a weak solution u to problem (1.4).

Chapter 3 is one of the key parts of this thesis. It studies the homogeneous problem in the infinite cone $\mathcal{K}_{(\alpha,\omega)}$,

$$\begin{cases} u_{xxxx} + u_{yyyy} = 0 & \text{in} \quad \mathcal{K}_{(\alpha,\omega)}, \\ u = 0 & \text{on} \quad \partial\mathcal{K}_{(\alpha,\omega)}, \\ \frac{\partial}{\partial n} u = 0 & \text{on} \quad \partial\mathcal{K}_{(\alpha,\omega)} \backslash \{0\}. \end{cases}$$

We derive (almost explicitly) a countable set of functions $\{u_j\}_{j\in\mathbb{N}}$ solving this problem. These functions describe the behaviour of the weak solution u to problem (1.4) locally in the vicinity of an angular point 0 of Ω in terms of the angle α and the opening angle ω. They will contribute in Chapter 5 to the regularity statement for u of type (1.3).

In Chapter 4 the weighted Sobolev spaces $V_\beta^{l,2}(\Omega)$ are presented and we recall the imbedding results for $W^{k,2}(\Omega)$ and $V_\beta^{l,2}(\Omega)$ based on a Hardy inequality.

Next to this, in Chapter 5 we address the Kondratiev theory and give the regularity statement for the solution u to our clamped *grid* problem (1.4) and its asymptotic representation in terms of $\{u_j\}_{j\in\mathbb{N}}$. We will also compare the results obtained with those known for the clamped *plate* problem.

Finally, in Chapter 6 we develop a system approach to our fourth order problem (1.4). It is favourized for numerical methods since one may use piecewise linear $C^{0,1}$-elements, readily available in standard programming packages. We will show that such a system approach for our clamped grid problem may fail to produce the correct solution when Ω has a reentrant (concave) corner.

The appendices contain computational and numerical results. Thus, in the first appendix we prove that every elliptic fourth order operator (which is defined by three parameters) is, in fact, equivalent to one parametric operator. This result is based on the Möbius transformation. The elaborate third appendix confirms that the errors in the numerical computations involved in order to illustrate our analytical results for Chapter 3 are small enough. This appendix also contains an explicit version of the Morse Theorem, which is necessary for an analytical error bound that confirms the numerical results. In the last appendix, we use the numerical approach developed in Chapter 6 to compare the solutions to the clamped plate and clamped grid problems when Ω has some specific geometry. We also simulate the distribution of stresses which appear in “The Comet” fuselage panel with a square and a round window under the uniformly distributed load.

Chapter 2

Existence and uniqueness

For the present so-called clamped boundary conditions existence of an appropriate weak solution can be obtained in a standard way even when the corner is not convex. We will recall the arguments for the existence of a weak solution to problem (1.4).

The function space for these weak solutions is

$$\overset{\circ}{W}{}^{2,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{2,2}(\Omega)}}. \tag{2.1}$$

where $C_c^\infty(\Omega)$ is the space of infinitely smooth functions with compact support in Ω.

Remark 2.1 *For Ω from Condition 1.2, one finds that $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ implies $u = 0$ on $\partial\Omega$ and $Du = 0$ on $\partial\Omega \backslash \{0\}$ in the sense of traces.*

Definition 2.2 *A function $\tilde{u} \in \overset{\circ}{W}{}^{2,2}(\Omega)$ is a weak solution of the boundary value problem (1.4) with $f \in L^2(\Omega)$, if*

$$\int_\Omega (\tilde{u}_{xx}\varphi_{xx} + \tilde{u}_{yy}\varphi_{yy} - f\varphi)\, dxdy = 0 \quad \textit{for all} \quad \varphi \in \overset{\circ}{W}{}^{2,2}(\Omega). \tag{2.2}$$

2.1 Approach outline

We use the direct method in the calculus of variations in order to prove the existence of a weak solution $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ to (1.4) when $f \in L^2(\Omega)$. Let us outline the method.

We consider the functional which describes the potential energy stored by the grid after it has been deformed:

$$E[u] = \int_{\Omega} \left(\tfrac{1}{2} \left(u_{xx}^2 + u_{yy}^2 \right) - fu \right) dxdy. \tag{2.3}$$

Due to the type of boundary conditions (the clamped edge), E is defined over the space $\overset{\circ}{W}{}^{2,2}(\Omega)$.

Suppose that there exists a minimizer $\tilde{u} \in \overset{\circ}{W}{}^{2,2}(\Omega)$ of E. Then the real-valued function $\tau(\varepsilon) := E[\tilde{u} + \varepsilon\varphi]$ has a minimum at $\varepsilon = 0$, meaning that $\tau'(\varepsilon)|_{\varepsilon=0} = 0$. Hence, for the minimizer $\tilde{u}$ it holds that $\frac{d}{d\varepsilon} E[\tilde{u} + \varepsilon\varphi]\big|_{\varepsilon=0} = 0$ for all $\varphi \in \overset{\circ}{W}{}^{2,2}(\Omega)$ and the expansion of the latter condition results in (2.2). For $f \in L^2(\Omega)$ and provided $\tilde{u}$ satisfying (2.2) is more regular (namely, $W^{4,2}(\Omega)$), an integration by parts of (2.2) shows that $\tilde{u}$ will fulfill the boundary value problem (1.4) in L^2-sense.

Remark 2.3 *If Ω in (1.4) is smooth enough, it is straightforward that for $f \in L^2(\Omega)$ the minimizer $\tilde{u} \in \overset{\circ}{W}{}^{2,2}(\Omega)$ of (2.3) lies in $W^{4,2}(\Omega)$.*

When Ω in (1.4) is as in Condition 1.2 we will see in Chapter 5, Theorem 5.3 that for $f \in L^2(\Omega)$ the minimizer $\tilde{u} \in \overset{\circ}{W}{}^{2,2}(\Omega)$ has the following representation $\tilde{u} = w + S$. Here w lies in $W^{4,2}(\Omega)$ and S is such that $S_{xxxx} + S_{yyyy} = 0$ in Ω. So, the integration by part in this case also yields that $\tilde{u}$ will fulfill the boundary value problem (1.4) in L^2-sense.

In the next Section we study the properties of the functional E in (2.3) over the space $\overset{\circ}{W}{}^{2,2}(\Omega)$ in order to prove that the minimizer of this E exists and is unique.

2.2 Properties of the energy functional

Due to the form of E it seems to be reasonable (and more appropriate, in fact) to endow the space $\overset{\circ}{W}{}^{2,2}(\Omega)$ with the scalar product

$$((u, v))_\star = \int_{\Omega} (u_{xx}v_{xx} + u_{yy}v_{yy})\, dxdy, \tag{2.4}$$

rather than with the standard inner product

$$(u,v) = \int\limits_{\Omega} (uv + u_x v_x + u_y v_y + u_{xx} v_{xx} + u_{xy} v_{xy} + u_{yy} v_{yy})\, dxdy.$$

With (2.4) the norm on $\overset{\circ}{W}{}^{2,2}(\Omega)$ will be given as

$$\|u\|_\star := \left(\int\limits_{\Omega} \left(u_{xx}^2 + u_{yy}^2 \right) dxdy \right)^{1/2}, \tag{2.5}$$

We show the following.

Lemma 2.4 *For $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ it holds that*

$$\left(\tfrac{1}{2} d^4 + d^2 + \tfrac{3}{2} \right)^{-\frac{1}{2}} \|u\|_{W^{2,2}(\Omega)} \le \|u\|_\star \le \|u\|_{W^{2,2}(\Omega)},$$

where d is a diameter of Ω.

Proof. The estimate from above for $\|u\|_\star$ is straightforward. Indeed, we have

$$\|u\|_\star^2 = \int\limits_{\Omega} \left(u_{xx}^2 + u_{yy}^2 \right) dxdy \le \|u\|_{W^{2,2}(\Omega)}^2 .$$

The estimate from below is obtained as follows. By the one-dimensional Poincaré inequality for all $g \in C_0^1[a,b]$ it holds that

$$\int\limits_a^b (g(x))^2\, dx \le (b-a)^2 \int\limits_a^b \left(g'(x) \right)^2 dx. \tag{2.6}$$

Hence we obtain for all $u \in C_c^\infty(\Omega)$ the following estimates:

$$\int\limits_{\Omega} u^2 dxdy \le d^2 \int\limits_{\Omega} u_x^2 dxdy, \tag{2.7}$$

alternatively,

$$\int\limits_{\Omega} u^2 dxdy \le d^2 \int\limits_{\Omega} u_y^2 dxdy, \tag{2.8}$$

and

$$\int_\Omega u_x^2 dxdy \le d^2 \int_\Omega u_{xx}^2 dxdy, \tag{2.9}$$

$$\int_\Omega u_y^2 dxdy \le d^2 \int_\Omega u_{yy}^2 dxdy, \tag{2.10}$$

where d is a diameter of Ω. Also, the integration-by-parts formula applied to $\int_\Omega u_{xy}^2 dxdy$ yields for all $u \in C_c^\infty(\Omega)$:

$$\int_\Omega u_{xy}^2 dxdy = \int_\Omega u_{xx}u_{yy} dxdy \le \tfrac{1}{2} \int_\Omega \left(u_{xx}^2 + u_{yy}^2\right) dxdy. \tag{2.11}$$

Due to (2.1), results (2.7) – (2.11) hold for $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$.

Then, combining estimates (2.7) – (2.11) we deduce that

$$\|u\|^2_{W^{2,2}(\Omega)} \le \left(\tfrac{1}{2}d^4 + d^2 + \tfrac{3}{2}\right) \int_\Omega \left(u_{xx}^2 + u_{yy}^2\right) dxdy = \left(\tfrac{1}{2}d^4 + d^2 + \tfrac{3}{2}\right) \|u\|^2_\star .$$

■

Remark 2.5 *Due to equivalence of the norms $\|\cdot\|_\star$ and $\|\cdot\|_{W^{2,2}(\Omega)}$ on $\overset{\circ}{W}{}^{2,2}(\Omega)$, (2.4) is an inner product.*

Now, our purpose is to prove that E is coercive, weakly lower semicontinuous and strictly convex on $\overset{\circ}{W}{}^{2,2}(\Omega)$ with $\|\cdot\|_\star$ as in (2.5).

For $(X, \|\cdot\|)$ a Banach space and $E : X \to \mathbb{R}$ we recall.

Definition 2.6 *A functional I is called coercive on $(X, \|\cdot\|)$ if for some function $g \in C(\mathbb{R}^+, \mathbb{R})$ with $\lim_{t\to\infty} g(t) = \infty$ it holds that*

$$I[x] \ge g(\|x\|), \quad x \in X. \tag{2.12}$$

Definition 2.7 *A functional I is called (sequentially) weakly lower semicontinuous (w.l.s-c.) on $(X, \|\cdot\|)$ if for every bounded sequence $\{x_m\} \subset X$ such that $x_m \rightharpoonup x$ in X (weak convergence), the following holds*

$$\liminf_{m\to\infty} I[x_m] \ge I[x]. \tag{2.13}$$

Definition 2.8 *Let $Y \subset X$ be a convex set. A functional E is called strictly convex on Y if for any $x, y \in Y$, $x \neq y$ and $t \in (0,1)$ it holds*

$$I\left[tx + (1-t)\, y\right] < tI\left[x\right] + (1-t)\, I[y]. \tag{2.14}$$

In three lemmas below we check that the functional E given by (2.3) satisfies these conditions.

Lemma 2.9 *E is coercive on $\overset{\circ}{W}{}^{2,2}(\Omega)$ with $\|\cdot\|_\star$ as in (2.5).*

Proof. For every $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ one straightforwardly shows that

$$\begin{aligned} E\left[u\right] &= \tfrac{1}{2}\int\limits_\Omega \left(u_{xx}^2 + u_{yy}^2\right) dxdy - \int\limits_\Omega fudxdy \geq \\ &\geq \tfrac{1}{2}\left\|u\right\|_\star^2 - \|f\|_{L^2(\Omega)} \left\|u\right\|_{L^2(\Omega)} \geq \\ &\geq \tfrac{1}{2}\left\|u\right\|_\star^2 - \left(\tfrac{1}{2}d^4 + d^2 + \tfrac{3}{2}\right)^{\frac{1}{2}} \|f\|_{L^2(\Omega)} \left\|u\right\|_\star, \end{aligned} \tag{2.15}$$

which gives the coercivity result (2.12). ■

Lemma 2.10 *E is sequentially weakly lower semicontinuous on $\overset{\circ}{W}{}^{2,2}(\Omega)$.*

Proof. Let $((u,v))_\star$ be the inner product as in (2.4). We take a bounded sequence $\{u_m\} \subset \overset{\circ}{W}{}^{2,2}(\Omega)$ such that $u_m \rightharpoonup u$ in $\overset{\circ}{W}{}^{2,2}(\Omega)$. We have

$$\begin{aligned} E\left[u_m\right] - E[u] &= \tfrac{1}{2}\int\limits_\Omega \Big(u_{m,xx}^2 - u_{xx}^2 + u_{m,yy}^2 - u_{yy}^2 - 2f\left(u_m - u\right)\Big)\, dxdy = \\ &= \tfrac{1}{2}\left\|u_m - u\right\|_\star^2 + \left((u_m - u, u)\right)_\star - \left\langle u_m - u, f\right\rangle. \end{aligned}$$

The first term is positive; the second goes to zero by the weak convergence and the third goes to zero since $u_m \rightharpoonup u$ in $\overset{\circ}{W}{}^{2,2}(\Omega)$ implies that also $\langle f, u_m\rangle \to \langle f, u\rangle$. ■

Lemma 2.11 *E is strictly convex on $\overset{\circ}{W}{}^{2,2}(\Omega)$.*

Proof. It is well known that a functional with only linear and positive quadratic terms is convex. Since, the coefficients in front of u_{xx}^2 and u_{yy}^2 in E are strictly positive and because of Lemma 2.4, we even find strict convexity of E. ■

2.3 Weak solution, existence and uniqueness result

The following statement holds.

Theorem 2.12 *Suppose $f \in L^2(\Omega)$. Then a weak solution of the boundary value problem (1.4) in the sense of Definition 2.2 exists. Moreover, this solution is unique.*

Proof. The proof basically recalls the variational approach outlined in the beginning of this Chapter. More precisely, by Lemmas 2.9 – 2.11 it follows that

$$E[u] = \int\limits_\Omega \left(\tfrac{1}{2}\left(u_{xx}^2 + u_{yy}^2\right) - fu\right) dxdy \quad \text{over} \quad \overset{\circ}{W}{}^{2,2}(\Omega),$$

is coercive, weakly lower semicontinuous and strictly convex on the space $\overset{\circ}{W}{}^{2,2}(\Omega)$. Due to the coercivity and the weak lower semicontinuity of E, the direct method in the calculus of variations (see e.g. [6]) shows us that E has a minimizer $\tilde{u} \in \overset{\circ}{W}{}^{2,2}(\Omega)$. Due to strict convexity of E the minimizer $\tilde{u}$ is unique. For this $\tilde{u}$ it holds that $\frac{d}{d\varepsilon}E[\tilde{u} + \varepsilon\varphi]\big|_{\varepsilon=0} = 0$ for all $\varphi \in \overset{\circ}{W}{}^{2,2}(\Omega)$. The expansion of the latter condition results in (2.2), meaning that $\tilde{u}$ is a weak solution of (1.4). Since a weak solution is a critical point of the given E and since the critical point is unique, so is the weak solution. ■

Remark 2.13 *For $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ we have just shown that*

$$\|u\|^2_{W^{2,2}(\Omega)} \le C \int_\Omega \left(u_{xx}^2 + u_{yy}^2\right) dxdy.$$

Let the grid be hinged, that is $u \in W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega)$. For every $u \in C^2(\overline{\Omega}) \cap C_0(\overline{\Omega})$ a Poincaré inequality still yields (2.9) and (2.10). Indeed, due to $u = 0$ on $\partial\Omega$ one find for every line $y = c$ intersecting Ω that there is x_c with $(x_c, c) \in \Omega$ such that $u_x(x_c, c) = 0$. Starting from this point one proves (2.9) and similarly (2.10). Using a density argument (see e.g [22, page 171]) results in the estimates above for every $u \in W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega)$. The real problem is $\int\limits_\Omega u_{xy}^2 dxdy$ since estimate (2.11) does not hold on domains with non-convex corners.

Chapter 3

Homogeneous problem in an infinite cone

As soon as we have the weak solution $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ to the boundary value problem (1.4) at hand we may improve its regularity.

This chapter provides all the necessary information on the local behaviour of $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ in the vicinity of angular point 0 of Ω. This behaviour is defined by the solutions of the homogeneous problem

$$\begin{cases} u_{xxxx} + u_{yyyy} = 0 & \text{in} \quad \mathcal{K}_{(\alpha,\omega)}, \\ u = 0 & \text{on} \quad \partial\mathcal{K}_{(\alpha,\omega)}, \\ \frac{\partial}{\partial n} u = 0 & \text{on} \quad \partial\mathcal{K}_{(\alpha,\omega)} \backslash \{0\}, \end{cases} \tag{3.1}$$

where $\mathcal{K}_{(\alpha,\omega)}$ is an infinite cone defined in (1.5) and sketched in Figure 3.1. We derive almost explicit formulas for power type solutions to (3.1) and this will enable us to see their contribution to the regularity of $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ in Chapter 5.

3.1 Reduced problem

The reduced problem for (3.1) is obtained in the following way. By Kondratiev [18] one should consider the power type solutions of (3.1):

$$u = r^{\lambda+1}\Phi(\theta), \tag{3.2}$$

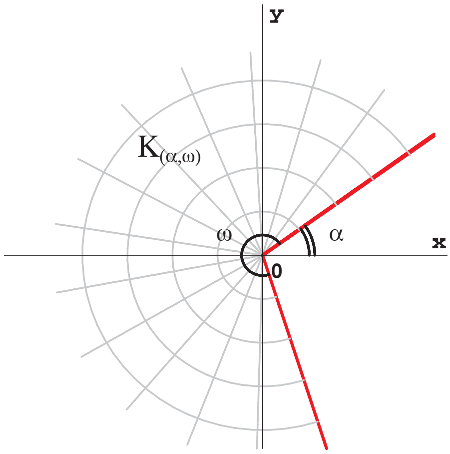

Figure 3.1: *An infinite cone* $\mathcal{K}_{(\alpha,\omega)}$.

with $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Here $\lambda \in \mathbb{C}$ and $\Phi : [\alpha, \alpha+\omega] \to \mathbb{R}$.

We insert u from (3.2) into problem (3.1) and find

$$\left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}\right)\left(r^{\lambda+1}\Phi(\theta)\right) = r^{\lambda-3}\mathcal{L}\left(\theta, \tfrac{d}{d\theta}, \lambda\right)\Phi(\theta),$$

with

$$\begin{aligned}\mathcal{L}\left(\theta, \tfrac{d}{d\theta}, \lambda\right) \quad = \quad & \tfrac{3}{4}\left(1 + \tfrac{1}{3}\cos(4\theta)\right)\tfrac{d^4}{d\theta^4} + (\lambda-2)\sin(4\theta)\tfrac{d^3}{d\theta^3} + \\ & + \tfrac{3}{2}\left(\lambda^2 - 1 - \left(\lambda^2 - 4\lambda - \tfrac{7}{3}\right)\cos(4\theta)\right)\tfrac{d^2}{d\theta^2} + \\ & + \left(-\lambda^3 + 6\lambda^2 - 7\lambda - 2\right)\sin(4\theta)\tfrac{d}{d\theta} + \\ & + \tfrac{3}{4}\left(\lambda^4 - 2\lambda^2 + 1 + \tfrac{1}{3}\left(\lambda^4 - 8\lambda^3 + 14\lambda^2 + 8\lambda - 15\right)\cos(4\theta)\right). \end{aligned} \tag{3.3}$$

Then we obtain a λ-dependent boundary value problem for Φ:

$$\begin{cases} \mathcal{L}\left(\theta, \tfrac{d}{d\theta}, \lambda\right)\Phi = 0 & \text{in} \quad (\alpha, \alpha+\omega), \\ \Phi = \tfrac{d}{d\theta}\Phi = 0 & \text{on} \quad \partial(\alpha, \alpha+\omega). \end{cases} \tag{3.4}$$

Remark 3.1 *The nonlinear eigenvalue problem (3.4) appears by a Mellin* *transformation:*

$$\Phi(\theta) = (\mathcal{M}u)(\lambda) = \int\limits_0^\infty r^{-\lambda-2} u(r,\theta)dr.$$

We call the eigenvalue problem (3.4) a reduced problem for (3.1).

Let fix some basic notions for (3.4).

Definition 3.2 *Every number $\lambda_0 \in \mathbb{C}$, such that there exists a nonzero function Φ_0 satisfying (3.4), is said to be an eigenvalue of problem (3.4), while $\Phi_0 \in C^4[\alpha, \alpha+\omega]$ is called its eigenfunction. Such pairs (λ_0, Φ_0) are called solutions to problem (3.4).*

If (λ_0, Φ_0) solves (3.4) and if Φ_1 is a nonzero function that solves

$$\begin{cases} \mathcal{L}(\lambda_0)\Phi_1 + \mathcal{L}'(\lambda_0)\Phi_0 = 0 & in \quad (\alpha, \alpha+\omega), \\ \Phi_1 = \frac{d}{d\theta}\Phi_1 = 0 & on \quad \partial(\alpha, \alpha+\omega), \end{cases} \tag{3.5}$$

where $\mathcal{L}(\lambda)$ is given by (3.3) and $\mathcal{L}'(\lambda) = \frac{d}{d\lambda}\mathcal{L}(\lambda)$, then Φ_1 is a generalized eigenfunction (of order 1) for (3.4) with eigenvalue λ_0.

Remark 3.3 *Similarly, one may define generalized eigenfunctions of higher order.*

The following holds for (3.4).

Lemma 3.4 *Let $\theta \in (\alpha, \alpha+\omega)$, with $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and $\omega \in (0, 2\pi]$. For every fixed $\lambda \notin \{\pm 1, 0\}$ in (3.4), let us set*

$$\varphi_1(\theta) = (\cos(\theta) + \tau_1 \sin(\theta))^{\lambda+1}, \quad \varphi_2(\theta) = (\cos(\theta) + \tau_2 \sin(\theta))^{\lambda+1},$$
$$\varphi_3(\theta) = (\cos(\theta) - \tau_1 \sin(\theta))^{\lambda+1}, \quad \varphi_4(\theta) = (\cos(\theta) - \tau_2 \sin(\theta))^{\lambda+1},$$

where $\tau_1 = \frac{\sqrt{2}}{2}(1+i)$, $\tau_2 = \frac{\sqrt{2}}{2}(1-i)$ and $i = \sqrt{-1}$.

The set $S_\lambda := \{\varphi_m\}_{m=1}^4$ is a fundamental system of solutions to the equation

$$\mathcal{L}\left(\theta, \tfrac{d}{d\theta}, \lambda\right)\Phi = 0 \quad on \quad (\alpha, \alpha+\omega).$$

Proof. The derivation of φ_m, $m = 1, ..., 4$ in S_λ is rather technical and we refer to Appendix B. There we also compute the Wronskian:

$$W\left(\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta), \varphi_4(\theta)\right) = \\ = 16(\lambda+1)^3 \lambda^2 (\lambda-1)\left(\cos^4(\theta) + \sin^4(\theta)\right)^{\lambda-2}.$$

It is non-zero on $\theta \in (\alpha, \alpha+\omega)$, with $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and $\omega \in (0, 2\pi]$ except for $\lambda \in \{\pm1, 0\}$. Hence, for every fixed $\lambda \notin \{\pm1, 0\}$ the set $\{\varphi_m\}_{m=1}^4$ consists of four linear independent functions on $(\alpha, \alpha+\omega)$. ■

Lemma 3.5 *In the particular cases $\lambda \in \{\pm1, 0\}$ in (3.4), one finds the following fundamental systems:*

$$S_{-1} = \left\{1,\ \arctan\left(\cos(2\theta)\right),\ \operatorname{arctanh}\left(\tfrac{\sqrt{2}}{2}\sin(2\theta)\right),\ \varphi_4(\theta)\right\},$$

$$S_0 = \left\{\sin(\theta),\ \cos(\theta),\ \varphi_3(\theta),\ \varphi_4(\theta)\right\},$$

$$S_1 = \left\{1,\ \sin(2\theta),\ \cos(2\theta),\ \varphi_4(\theta)\right\},$$

where the explicit formulas for $\varphi_4 \in S_{-1}$, $\{\varphi_3, \varphi_4\} \in S_0$ and $\varphi_4 \in S_1$ are given in Appendix B.

Proof. The fundamental systems S_{-1}, S_0, S_1 are given in Appendix B. By straightforward computations one finds that for every above S_λ, $\lambda \in \{\pm1, 0\}$ the corresponding Wronskian W is proportional to $\left(\cos^4(\theta) + \sin^4(\theta)\right)^{\lambda-2}$, $\lambda \in \{\pm1, 0\}$ and hence is nonzero on $\theta \in (\alpha, \alpha+\omega)$, with $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and $\omega \in (0, 2\pi]$. ■

In terms of the fundamental systems S we have Φ that solves $\mathcal{L}\left(\theta, \frac{\partial}{\partial\theta}, \lambda\right)\Phi = 0$ as

$$\Phi(\theta) = \sum_{m=1}^{4} b_m \varphi_m(\theta),$$

where $b_m \in \mathbb{C}$. Inserting this expression into the boundary conditions of problem (3.4), we find a homogeneous system of four equations in the unknowns $\{b_m\}_{m=1}^4$ reading as

$$Ab := \begin{pmatrix} \varphi_1(\alpha) & \varphi_2(\alpha) & \varphi_3(\alpha) & \varphi_4(\alpha) \\ \varphi_1'(\alpha) & \varphi_2'(\alpha) & \varphi_3'(\alpha) & \varphi_4'(\alpha) \\ \varphi_1(\alpha+\omega) & \varphi_2(\alpha+\omega) & \varphi_3(\alpha+\omega) & \varphi_4(\alpha+\omega) \\ \varphi_1'(\alpha+\omega) & \varphi_2'(\alpha+\omega) & \varphi_3'(\alpha+\omega) & \varphi_4'(\alpha+\omega) \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = 0,$$

where $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and $\omega \in (0, 2\pi]$. It admits non-trivial solutions for $\{b_m\}_{m=1}^{4}$ if and only if $\det(A) = 0$. Hence, the eigenvalues λ of problem (3.4) in sense of Definition 3.2 will be completely determined by the characteristic equation

$$\det(A) = 0.$$

We deduce the following four cases:

$$\det(A) := \begin{cases} \mathcal{P}(\alpha, \omega, \lambda) & \text{when} \quad \lambda \notin \{\pm 1, 0\}, \\ \mathcal{P}_{-1}(\alpha, \omega) & \text{when} \quad \lambda = -1, \\ \mathcal{P}_{0}(\alpha, \omega) & \text{when} \quad \lambda = 0, \\ \mathcal{P}_{1}(\alpha, \omega) & \text{when} \quad \lambda = 1. \end{cases} \tag{3.6}$$

The explicit formulas for $\mathcal{P}_{-1}, \mathcal{P}_0, \mathcal{P}_1$ are available in Appendix B. The formula for $\mathcal{P}$ reads as follows:

$$\begin{aligned} \mathcal{P}(\alpha, \omega, \lambda) = {} & \left(1 + \tfrac{\sqrt{2}}{2} \sin(2\alpha)\right)^{\lambda} \left(1 - \tfrac{\sqrt{2}}{2} \sin(2\alpha + 2\omega)\right)^{\lambda} + \\ & + \left(1 - \tfrac{\sqrt{2}}{2} \sin(2\alpha)\right)^{\lambda} \left(1 + \tfrac{\sqrt{2}}{2} \sin(2\alpha + 2\omega)\right)^{\lambda} + \\ & + 2\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\alpha)\right)^{\frac{1}{2}\lambda} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\alpha + 2\omega)\right)^{\frac{1}{2}\lambda} \cdot \\ & \cdot \cos\left\{\lambda\left[\arctan\left(\tfrac{\sqrt{2}}{2}\tan(2\alpha)\right) + \ell\pi - \arctan\left(\tfrac{\sqrt{2}}{2}\tan(2\alpha + 2\omega)\right) - \kappa\pi\right]\right\} - \\ & - 4\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\alpha)\right)^{\frac{1}{2}\lambda} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\alpha + 2\omega)\right)^{\frac{1}{2}\lambda} \cdot \\ & \cdot \cos\left\{\lambda\left[\arctan\left(\tan^2(\alpha)\right) - \arctan\left(\tan^2(\alpha + \omega)\right)\right]\right\}, \end{aligned} \tag{3.7}$$

where $\alpha \in \left[0, \frac{1}{2}\pi\right)$, $\omega \in (0, 2\pi]$ and

$$\begin{aligned} \ell &= 0 \quad \text{if} \quad \alpha \in \left[0, \tfrac{1}{4}\pi\right], \\ \ell &= 1 \quad \text{if} \quad \alpha \in \left(\tfrac{1}{4}\pi, \tfrac{1}{2}\pi\right), \end{aligned} \qquad \text{and} \qquad \begin{aligned} \kappa &= 0 \quad \text{if} \quad \alpha + \omega \in \left(0, \tfrac{1}{4}\pi\right], \\ \kappa &= 1 \quad \text{if} \quad \alpha + \omega \in \left(\tfrac{1}{4}\pi, \tfrac{3}{4}\pi\right], \\ \kappa &= 2 \quad \text{if} \quad \alpha + \omega \in \left(\tfrac{3}{4}\pi, \tfrac{5}{4}\pi\right], \\ \kappa &= 3 \quad \text{if} \quad \alpha + \omega \in \left(\tfrac{5}{4}\pi, \tfrac{7}{4}\pi\right], \\ \kappa &= 4 \quad \text{if} \quad \alpha + \omega \in \left(\tfrac{7}{4}\pi, \tfrac{9}{4}\pi\right], \\ \kappa &= 5 \quad \text{if} \quad \alpha + \omega \in \left(\tfrac{9}{4}\pi, \tfrac{5}{2}\pi\right). \end{aligned}$$

Our purpose is to describe the eigenvalues λ of problem (3.4) for every fixed $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and fixed $\omega \in (0, 2\pi]$. What is more important, we want to trace for a fixed $\alpha \in \left[0, \frac{1}{2}\pi\right)$ the behavior of $\omega \mapsto \lambda(\alpha, \omega)$ on $\omega \in (0, 2\pi]$.

The strategy will be as follows. First, we check for every $\alpha \in \left[0, \frac{1}{2}\pi\right)$ whether the equations $\mathcal{P}_{-1}(\alpha, \omega) = 0$, $\mathcal{P}_0(\alpha, \omega) = 0$ and $\mathcal{P}_1(\alpha, \omega) = 0$ have any solutions in ω on the interval $(0, 2\pi]$. In this way we will see whether $\lambda \in \{\pm 1, 0\}$ are eigenvalues of (3.4) or not. Next to this, we will address the transcendental equation $\mathcal{P}(\alpha, \omega, \lambda) = 0$. We will start from the basic property of the solutions λ to $\mathcal{P}(\alpha, \omega, \lambda) = 0$ for fixed $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and fixed $\omega \in (0, 2\pi]$ and then our detailed study will concern the equation $\mathcal{P}(\alpha, \omega, \lambda) = 0$ with $\alpha = 0$. In Section 3.3 we will describe the dependence of the eigenvalues λ on the opening angle ω, particularly focusing on the eigenvalue with the lowest positive real part, denoted as λ_1. For this eigenvalue we prove the existence of an interval of ω on which λ_1 as a function of ω increases with increasing ω.

3.2 General statements for the eigenvalues λ

Let $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and $\omega \in (0, 2\pi]$ in problem (3.4). It holds that:

Lemma 3.6 *For any α and ω the value $\lambda = 0$ is not an eigenvalue of (3.4).*

Proof. The proof uses the fact that the function $\mathcal{P}_0(\alpha, \omega)$ is strictly positive on $(\alpha, \omega) \in \left[0, \frac{1}{2}\pi\right) \times (0, 2\pi]$. For details see Lemma B.1 in Appendix B. ∎

Lemma 3.7 *For all $\alpha \in \left[0, \frac{1}{2}\pi\right)$ the values $\lambda \in \{\pm 1\}$ are eigenvalues of (3.4) when $\omega \in \{\pi, \omega_0, 2\pi\}$, where $\omega_0 \in (0, 2\pi)$.*

Proof. It is straightforward that for every fixed $\alpha \in \left[0, \frac{1}{2}\pi\right)$ the values $\omega \in \{\pi, 2\pi\}$ are solutions to $\mathcal{P}_{-1}(\alpha, \omega) = 0$ and $\mathcal{P}_1(\alpha, \omega) = 0$ (see Appendix B). On the other hand, due to complexity of $\mathcal{P}_{-1}$ and $\mathcal{P}_1$ one is not able to find for every fixed α the third solution $\omega_0 \in (0, 2\pi)$ of $\mathcal{P}_{-1}(\alpha, \omega) = 0$ and $\mathcal{P}_1(\alpha, \omega) = 0$ analytically. So, we have numerically assisted results. It turns out that both foregoing equations for every fixed α have identical solutions, that we denote as ω_0. For instance, fixing $\alpha = \frac{10}{180}\pi j$, $j = 0, ..., 8$ we find the following approximations for ω_0 on $(0, 2\pi)$:

α	ω_0/π	ω_0 in degrees
0	≈ 1.423	$\approx 256.25°$
$\frac{10}{180}\pi$	≈ 1.422	$\approx 256.13°$
$\frac{20}{180}\pi$	≈ 1.425	$\approx 256.61°$
$\frac{30}{180}\pi$	≈ 1.430	$\approx 257.54°$
$\frac{40}{180}\pi$	≈ 1.435	$\approx 258.47°$
$\frac{50}{180}\pi$	≈ 1.438	$\approx 258.89°$
$\frac{60}{180}\pi$	≈ 1.436	$\approx 258.57°$
$\frac{70}{180}\pi$	≈ 1.432	$\approx 257.76°$
$\frac{80}{180}\pi$	≈ 1.427	$\approx 256.87°$

We use the Maple 9.5 package for numerical computations. ■

Our next simple observation for $\mathcal{P}$ in (3.7) is that for every $\lambda \in \mathbb{C} \setminus \{\pm 1, 0\}$ it holds

$$\mathcal{P}(\alpha, \omega, -\lambda) = \frac{1}{\left(\frac{1}{2} + \frac{1}{2}\cos^2(2\alpha)\right)^{\lambda} \left(\frac{1}{2} + \frac{1}{2}\cos^2(2\alpha + 2\omega)\right)^{\lambda}} \cdot \mathcal{P}(\alpha, \omega, \lambda).$$

Hence,

Lemma 3.8 *For every fixed $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and fixed $\omega \in (0, 2\pi]$ the solutions λ of $\mathcal{P}(\alpha, \omega, \lambda) = 0$ are symmetric with respect to the ω-axis. It also holds that if λ is an eigenvalue then so is $\overline{\lambda}$.*

Hence, combining the results of Lemmas 3.6 – 3.8 it is convenient to introduce the following notation.

Notation 3.9 *For every fixed $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and fixed $\omega \in (0, 2\pi]$ we write $\{\lambda_j\}_{j=1}^{\infty}$ for the collection of the eigenvalues of problem (3.4) in the sense of Definition 3.2, which have positive real part $\mathrm{Re}\,(\lambda) > 0$ and are ordered by increasing real part.*

The complete set of eigenvalues to problem (3.4) will then read as $\{-\lambda_j, \lambda_j\}_{j=1}^{\infty}$. The following lemma describes the set $\{\lambda_j\}_{j=1}^{\infty}$.

Lemma 3.10 *Let $\mathcal{L}$ be the operator given by (3.3).*

- *For every fixed $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and fixed $\omega \in (0, 2\pi] \backslash \{\pi, \omega_0, 2\pi\}$ the set $\{\lambda_j\}_{j=1}^{\infty}$ from Notation 3.9 is given by*

$$\{\lambda_j\}_{j=1}^{\infty} = \left\{\lambda \in \mathbb{C} : \ \mathrm{Re}\,(\lambda) \in \mathbb{R}^+ \backslash \{1\}, \quad \mathcal{P}(\alpha, \omega, \lambda) = 0\right\}.$$

- *For every fixed $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and fixed $\omega \in \{\pi, \omega_0, 2\pi\}$ the set $\{\lambda_j\}_{j=1}^{\infty}$ from Notation 3.9 is given by*

$$\{\lambda_j\}_{j=1}^{\infty} = \left\{\lambda \in \mathbb{C} : \ \mathrm{Re}\,(\lambda) \in \mathbb{R}^+ \backslash \{1\}, \quad \mathcal{P}(\alpha, \omega, \lambda) = 0\right\} \cup \{1\}.$$

Here ω_0 is a solution of $\mathcal{P}_1(\alpha, \omega) = 0$ on $\omega \in (\pi, 2\pi)$ for every fixed $\alpha \in \left[0, \frac{1}{2}\pi\right)$.

Remark 3.11 *The approximations ω_0/π for some fixed α are presented in the table in the Proof of Lemma 3.7.*

The last thing we can mention in this Section is that the values $\omega \in \left\{\frac{1}{2}\pi, \pi, \frac{3}{2}\pi, 2\pi\right\}$ being set in (3.7) yield more simple formulas for $\mathcal{P}$. We find that:

$$\begin{aligned} \mathcal{P}\left(\alpha, \tfrac{1}{2}\pi, \lambda\right) = \left(1 - \tfrac{\sqrt{2}}{2}\sin(2\alpha)\right)^{2\lambda} + \left(1 + \tfrac{\sqrt{2}}{2}\sin(2\alpha)\right)^{2\lambda} + \\ +2\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\alpha)\right)^{\lambda}\cos(\lambda\pi) - \\ - 4\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\alpha)\right)^{\lambda}\cos\left\{\lambda\left[\arctan\left(\tan^2(\alpha)\right) - \arctan\left(\cot^2(\alpha)\right)\right]\right\}, \end{aligned} \tag{3.8}$$

$$\mathcal{P}(\alpha, \pi, \lambda) = -4\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\alpha)\right)^{\lambda}\sin^2(\pi\lambda), \tag{3.9}$$

$$\begin{aligned} \mathcal{P}\left(\alpha, \tfrac{3}{2}\pi, \lambda\right) = \left(1 - \tfrac{\sqrt{2}}{2}\sin(2\alpha)\right)^{2\lambda} + \left(1 + \tfrac{\sqrt{2}}{2}\sin(2\alpha)\right)^{2\lambda} + \\ +2\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\alpha)\right)^{\lambda}\cos(3\lambda\pi) - \\ - 4\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\alpha)\right)^{\lambda}\cos\left\{\lambda\left[\arctan\left(\tan^2(\alpha)\right) - \arctan\left(\cot^2(\alpha)\right)\right]\right\}, \end{aligned} \tag{3.10}$$

$$\mathcal{P}(\alpha, 2\pi, \lambda) = 16\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\alpha)\right)^{\lambda}\left[\cos^4(\pi\lambda) - \cos^2(\pi\lambda)\right]. \tag{3.11}$$

In the above formulas $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and $\lambda \notin \{\pm 1, 0\}$. Equations $\mathcal{P}(\alpha, \pi, \lambda) = 0$ and $\mathcal{P}(\alpha, 2\pi, \lambda) = 0$ admit the explicit solutions λ for every $\alpha \in \left[0, \frac{1}{2}\pi\right)$, while the equations $\mathcal{P}\left(\alpha, \frac{1}{2}\pi, \lambda\right) = 0$ and $\mathcal{P}\left(\alpha, \frac{3}{2}\pi, \lambda\right) = 0$ can be solved explicitly only for $\alpha = 0$. For details see Appendix D.

3.3 Analysis of the eigenvalues λ when $\alpha = 0$

The set $\{\lambda_j\}_{j=1}^{\infty}$ of the eigenvalues to problem for every fixed $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and fixed $\omega \in (0, 2\pi]$ has been described by Lemma 3.10. In this Section our particular study will focus on $\{\lambda_j\}_{j=1}^{\infty}$ when $\alpha = 0$. First, we will give the basic plot of some first values from $\{\lambda_j\}_{j=1}^{\infty}$ in dependence on the opening angle $\omega \in (0, 2\pi]$. This result is obtained in a numerically assisted way. Next to this, a detailed numerically-analytical analysis will be given to λ_1. We will prove that as a function of ω the first eigenvalue $\lambda_1 = \lambda_1(\omega)$ of the boundary value problem (3.4) increases on $\omega \in \left(\frac{1}{2}\pi, \omega_\star\right)$, where $\omega_\star/\pi \approx 0.528$ (in degrees $\omega_\star \approx 95.1°$).

Thus, we fix $\alpha = 0$ in (3.7) and denote

$$P(\omega, \lambda) := \mathcal{P}(\alpha, \omega, \lambda)\big|_{\alpha=0}.$$

Explicitly, P reads as follows:

$$\begin{aligned} P(\omega, \lambda) = {} & \left(1 - \tfrac{\sqrt{2}}{2}\sin(2\omega)\right)^{\lambda} + \left(1 + \tfrac{\sqrt{2}}{2}\sin(2\omega)\right)^{\lambda} + \\ & +2\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\omega)\right)^{\frac{1}{2}\lambda} \cdot \cos\left\{\lambda\left[\arctan\left(\tfrac{\sqrt{2}}{2}\tan(2\omega)\right) + \kappa\pi\right]\right\} - \\ & -4\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\omega)\right)^{\frac{1}{2}\lambda} \cdot \cos\left\{\lambda\arctan\left(\tan^2(\omega)\right)\right\}, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \kappa &= 0 \quad \text{if} \quad \omega \in \left(0, \tfrac{1}{4}\pi\right], \\ \kappa &= 1 \quad \text{if} \quad \omega \in \left(\tfrac{1}{4}\pi, \tfrac{3}{4}\pi\right], \\ \kappa &= 2 \quad \text{if} \quad \omega \in \left(\tfrac{3}{4}\pi, \tfrac{5}{4}\pi\right], \\ \kappa &= 3 \quad \text{if} \quad \omega \in \left(\tfrac{5}{4}\pi, \tfrac{7}{4}\pi\right], \\ \kappa &= 4 \quad \text{if} \quad \omega \in \left(\tfrac{7}{4}\pi, 2\pi\right]. \end{aligned}$$

Now the particular case of Lemma 3.10 for the case $\alpha = 0$ can be formulated.

Lemma 3.12 *Let $\mathcal{L}$ be the operator given by (3.3) and let $\alpha = 0$.*

- *For every fixed $\omega \in (0, 2\pi]\backslash\{\pi, \omega_0, 2\pi\}$ the set $\{\lambda_j\}_{j=1}^{\infty}$ from Notation 3.9 is given by*

$$\{\lambda_j\}_{j=1}^{\infty} = \left\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \in \mathbb{R}^+\backslash\{1\}, \quad P(\omega, \lambda) = 0\right\}.$$

- *For every fixed $\omega \in \{\pi, \omega_0, 2\pi\}$ the set $\{\lambda_j\}_{j=1}^{\infty}$ from Notation 3.9 is given by*

$$\{\lambda_j\}_{j=1}^{\infty} = \left\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \in \mathbb{R}^+\backslash\{1\}, \quad P(\omega, \lambda) = 0\right\} \cup \{1\}.$$

Here ω_0 is a solution of $\mathcal{P}_1(\alpha, \omega) = 0|_{\alpha=0}$ on $\omega \in (\pi, 2\pi)$ with the approximation $\omega_0/\pi \approx 1.424$ (in degrees $\omega_0 \approx 256.25°$).

Also, referring to the formula (3.12), we will find:

$$P\left(\tfrac{1}{2}\pi, \lambda\right) = 2 + 2\cos(\pi\lambda) - 4\cos\left(\tfrac{1}{2}\pi\lambda\right),$$

$$P(\pi, \lambda) = -4\sin^2(\pi\lambda),$$

$$P\left(\tfrac{3}{2}\pi, \lambda\right) = 8\cos^3(\pi\lambda) - 6\cos(\pi\lambda) - 4\cos\left(\tfrac{1}{2}\pi\lambda\right) + 2,$$

$$P(2\pi, \lambda) = 16\cos^4(\pi\lambda) - 16\cos^2(\pi\lambda).$$

In Appendix D we solve the above four equations explicitly.

3.3.1 Intermezzo: a comparison with Δ^2

Let the grid-operator $\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ in problems (1.4), (3.1) be replaced by the bilaplacian $\Delta^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$. That is, we have the clamped plate problem,

$$\begin{cases} \Delta^2 u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \frac{\partial}{\partial n} u = 0 & \text{on} \quad \partial\Omega\backslash\{0\}, \end{cases} \tag{3.13}$$

"with prescribed growth behaviour near 0", and the homogeneous problem in an infinite cone,

$$\begin{cases} \Delta^2 u = 0 & \text{in} \quad \mathcal{K}_{(\alpha,\omega)}, \\ u = 0 & \text{on} \quad \partial\mathcal{K}_{(\alpha,\omega)}, \\ \frac{\partial}{\partial n} u = 0 & \text{on} \quad \partial\mathcal{K}_{(\alpha,\omega)} \backslash \{0\}. \end{cases} \tag{3.14}$$

Here, Ω and $\mathcal{K}_{(\alpha,\omega)}$ are from Condition 1.2. Due to the invariance of the operator Δ^2 under rotation, the orientation angle α in problems (3.13) and (3.14) does not play a role, i.e. can be arbitrary. But in order to be consistent with the particular case $\alpha = 0$ in the grid problem we consider here, we simply assume that $\alpha = 0$ in the above bilaplacian problems too.

Here, we recall some results for the bilaplacian, namely, the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ of the corresponding reduced problem. We will compare them to those given in Lemma 3.12. For problem (3.14) the reduced problem of the type (3.4) has an operator $\mathcal{L}$ reading as (see e.g. [14, page 88]):

$$\mathcal{L}\left(\theta, \tfrac{d}{d\theta}, \lambda\right) = \tfrac{d^4}{d\theta^4} + 2\left(\lambda^2 + 1\right) \tfrac{d^2}{d\theta^2} + \left(\lambda^4 - 2\lambda^2 + 1\right). \tag{3.15}$$

The corresponding characteristic determinants are the following (see e.g. [14, page 89] or [3, page 561]):

$$\det(A) := \begin{cases} \sin^2(\lambda\omega) - \lambda^2 \sin^2(\omega) & \text{when} \quad \lambda \notin \{\pm 1, 0\}, \\ \sin^2(\omega) - \omega^2 & \text{when} \quad \lambda = 0, \\ \sin(\omega)\left(\sin(\omega) - \omega\cos(\omega)\right) & \text{when} \quad \lambda \in \{\pm 1\}. \end{cases} \tag{3.16}$$

Note that for every $\lambda \in \mathbb{C} \backslash \{\pm 1, 0\}$ the function $\sin^2(\lambda\omega) - \lambda^2 \sin^2(\omega)$ is even with respect to ω and hence the Notation 3.9 is applicable here. Analysis of $\det(A) = 0$ with $\det(A)$ as in (3.16) enables to formulate the analog of Lemma 3.12. Namely,

Lemma 3.13 *Let $\mathcal{L}$ be the operator given by (3.15).*

- *For every fixed $\omega \in (0, 2\pi] \backslash \{\pi, \omega_0, 2\pi\}$ the set $\{\lambda_j\}_{j=1}^{\infty}$ from Notation 3.9 is given by*

$$\{\lambda_j\}_{j=1}^{\infty} = \left\{\lambda \in \mathbb{C} : \ \mathrm{Re}\,(\lambda) \in \mathbb{R}^+ \backslash \{1\}, \ \sin^2(\lambda\omega) - \lambda^2 \sin^2(\omega) = 0\right\}.$$

- *For every fixed* $\omega \in \{\pi, \omega_0, 2\pi\}$ *the set* $\{\lambda_j\}_{j=1}^{\infty}$ *from Notation 3.9 is given by*

$$\{\lambda_j\}_{j=1}^{\infty} = \left\{\lambda \in \mathbb{C} : \ \mathrm{Re}\,(\lambda) \in \mathbb{R}^+ \backslash \{1\}, \ \sin^2(\lambda\omega) - \lambda^2 \sin^2(\omega) = 0\right\} \cup \{1\}.$$

Here ω_0 *is a solution of* $\tan(\omega) = \omega$ *on* $\omega \in (\pi, 2\pi)$ *with the approximation* $\omega_0/\pi \approx 1.430$ *(in degrees* $\omega_0 \approx 257.45°$*).*

3.4 Analysis of the eigenvalues λ when $\alpha = 0$ (continued)

Let (ω, λ) be the pair that solves the equations of Lemmas 3.12 and 3.13. In Figure 3.2 we plot the pairs $(\omega, \mathrm{Re}\,(\lambda))$ inside the region $(\omega, \mathrm{Re}\,(\lambda)) \in (0; 2\pi] \times [0, 7.200]$.

Remark 3.14 *The numerical computations are performed with the Maple 9.5 package in the following way: at a first cycle for every* $\omega_q = \frac{21}{180}\pi + \frac{1}{60}\pi q$, $q = 0, ..., 113$ *we compute the entries of the set* $\{\lambda_j\}_{j=1}^{N}$. *Here,* N *is determined by the condition:* $\mathrm{Re}\,(\lambda_N) \leq 7.200$ *and* $\mathrm{Re}\,(\lambda_{N+1}) > 7.200$. *The points* (ω, λ) *where* λ_j *transits from the complex plane to the real one or vice-versa are solutions to the system* $P(\omega, \lambda) = 0$ *and* $\frac{\partial P}{\partial \lambda}(\omega, \lambda) = 0$ *(the justification for the second condition will be discussed in Lemma 3.20).*

In Figure 3.2 one may observe the difference in the behavior of the eigenvalues in the corresponding cases. In particular, in the top plot (the case $L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$) there are the "loops" and the "ellipses" in the vicinities of $\omega \in \left\{\frac{1}{2}\pi, \frac{3}{2}\pi\right\}$ (we inclose them in the rectangles). The bottom plot (the case $L = \Delta^2$) looks much simpler in the same region. We will see from the regularity statements in Chapter 5 that the contribution of the first eigenvalue λ_1 to the regularity of the solution u to problem (1.4) is the most essential. So, it is important for us to know the dependence of the eigenvalues λ on the opening angle ω. In this sense, the region $(\omega, \mathrm{Re}\,(\lambda)) \in V$ (Figure 3.2, top) seems to be the most interesting part and the model one. One observes that inside V the graph of the implicit function $P(\omega,\lambda) = 0$ looks like a deformed 8-shaped curve. So, if one proves that everywhere in V, $P(\omega,\lambda) = 0$ allows its local parametrization in $\omega \mapsto \lambda = \psi(\omega)$ or $\lambda \mapsto \omega = \varphi(\lambda)$, then the bottom part of this graph is λ_1 and there is a subset of the this bottom part where λ_1 as a function of ω increases with increasing ω.

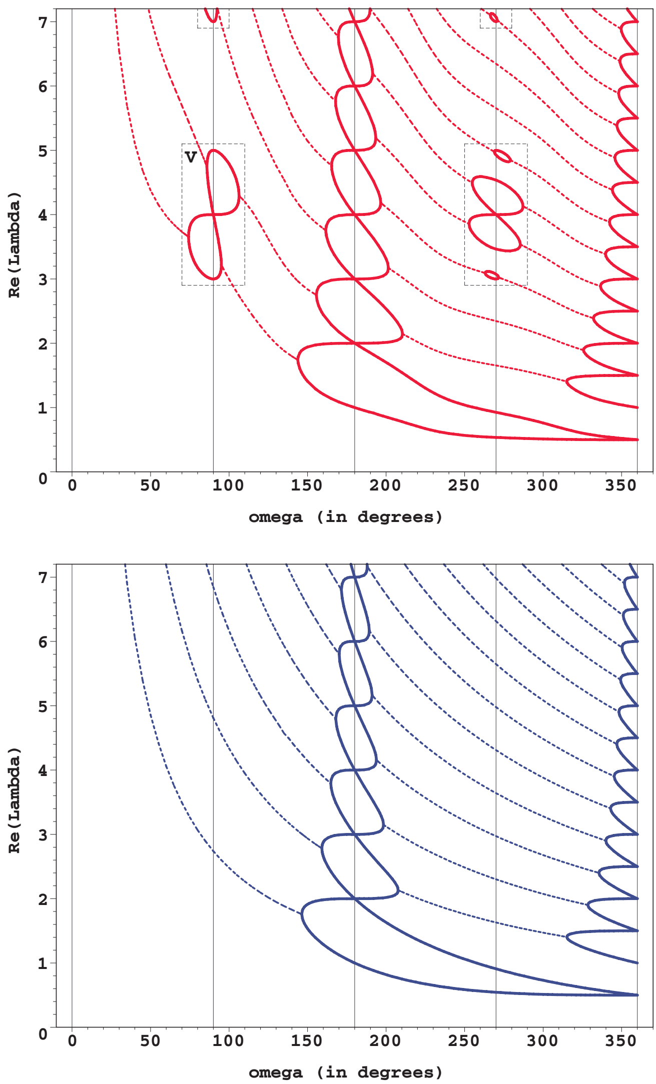

Figure 3.2: *Some first eigenvalues λ_j in $(\omega, \mathrm{Re}(\lambda)) \in (0, 2\pi] \times [0, 7.200]$ of problem (3.4), where $\mathcal{L}$ is related respectively to $\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ (on the top) and Δ^2 (on the bottom). Dashed lines depict the real part of those $\lambda_j \in \mathbb{C}$, solid lines are for purely real λ_j.*

3.4.1 Behavior of λ in V

So let us fix the open rectangular domain

$$V = \left\{(\omega, \lambda) : \left[\tfrac{70}{180}\pi, \tfrac{110}{180}\pi\right] \times [2.900, 5.100]\right\}.$$

The function $P \in C^\infty(V, \mathbb{R})$ is given by (3.12) with $\kappa = 1$:

$$\begin{aligned} P(\omega, \lambda) = & \left(1 - \tfrac{\sqrt{2}}{2}\sin(2\omega)\right)^\lambda + \left(1 + \tfrac{\sqrt{2}}{2}\sin(2\omega)\right)^\lambda + \\ & +2\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\omega)\right)^{\frac{1}{2}\lambda} \cdot \cos\left\{\lambda\left[\arctan\left(\tfrac{\sqrt{2}}{2}\tan(2\omega)\right) + \pi\right]\right\} - \\ & -4\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\omega)\right)^{\frac{1}{2}\lambda} \cdot \cos\left\{\lambda \arctan\left(\tan^2(\omega)\right)\right\}. \end{aligned} \tag{3.17}$$

We set

$$\Gamma := \{(\omega,\lambda) \in V : P(\omega,\lambda) = 0\}, \tag{3.18}$$

as a zero level set of P in V.

Remark 3.15 *In order to plot the set Γ we perform the computations to $P(\omega,\lambda) = 0$ in V in the spirit of Remark 3.14.*

In particular, for $\omega = \frac{1}{2}\pi$ being set in (3.17) we obtain $P\left(\frac{1}{2}\pi,\lambda\right) = 2 + 2\cos(\pi\lambda) - 4\cos\left(\frac{1}{2}\pi\lambda\right)$. The equation $P\left(\frac{1}{2}\pi,\lambda\right) = 0$ admits exact solutions for λ in the interval $(2.900, 5.100)$, namely, $\lambda \in \{3, 4, 5\}$. This yields the points

$$\left(\tfrac{1}{2}\pi, 3\right) =: c_1, \quad \left(\tfrac{1}{2}\pi, 4\right) =: a, \quad \left(\tfrac{1}{2}\pi, 5\right) =: c_4,$$

of Γ. It also holds straightforwardly that $\frac{\partial P}{\partial \omega}(c_1) = \frac{\partial P}{\partial \omega}(c_4) = 0$ and hence one may guess that horizontal tangents to the set Γ exist at those points (in Lemma 3.19 this situation will be discussed in details for the point c_1). For a we find directly that $\frac{\partial P}{\partial \omega}(a) = \frac{\partial P}{\partial \lambda}(a) = 0$ and hence more detailed analysis is required. Additionally to c_1, c_4, we will also specify four other points of the set Γ. Denoted as c_2, c_3, c_5, c_6, they are defined by the system $P(\omega, \lambda) = 0$ and $\frac{\partial P}{\partial \lambda}(\omega, \lambda) = 0$. The latter condition (we will justify it in Lemma 3.20 for the point c_2) gives us a hint that vertical tangents to Γ exist at those points. The approximations for the coordinates of c_i, $i = 1, ..., 6$ are listed in Table 3.1 and we plot the level set Γ in Figure 3.3.

Point c_k of Γ	Coordinates $(\omega/\pi,\ \lambda)$	ω in degrees	The property of Γ at c_k
c_1	$\left(\frac{1}{2},\ 3\right)$	$90°$	horizontal tangent
c_2	$(0.528...,\ 3.220...)$	$\approx 95.1°$	vertical tangent
c_3	$(0.591...,\ 4.291...)$	$\approx 106.4°$	vertical tangent
c_4	$\left(\frac{1}{2},\ 5\right)$	$90°$	horizontal tangent
c_5	$(0.477...,\ 4.746...)$	$\approx 85.96°$	vertical tangent
c_6	$(0.412...,\ 3.655...)$	$\approx 74.2°$	vertical tangent

Table 3.1: *Approximations for the points of the level set Γ.*

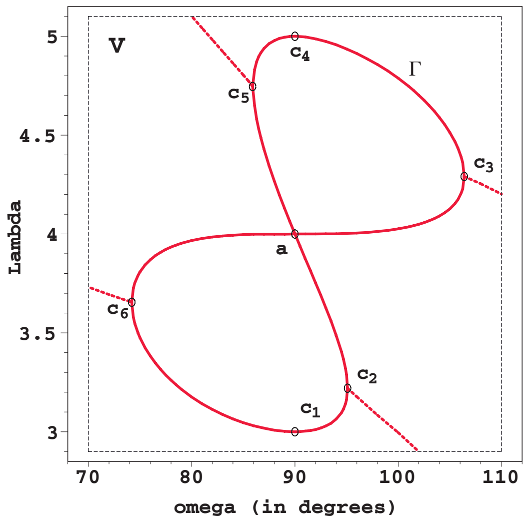

Figure 3.3: *The level set Γ (solid line) in V.*

As we mention in Remark 3.15, the set Γ as in (3.18) was found by means of numerical computations. In order to show that the plot of Γ is adequate, we study the implicit function $P(\omega,\lambda) = 0$ in V analytically. It is done in several steps.

The first lemma studies $P(\omega,\lambda)=0$ in the vicinity of the point

$$a=\left(\tfrac{1}{2}\pi,4\right)\in\Gamma. \tag{3.19}$$

Lemma 3.16 *Let $U=I\times J\subset V$ be the closed rectangle with $I=\left[\frac{88}{180}\pi,\frac{92}{180}\pi\right]$, $J=[3.940,4.060]$ and let point $a\in U$ be as in (3.19). The set Γ given by (3.18) consists of two smooth branches passing through a. Their tangents at a are $\lambda=4$ and $\lambda=-\frac{16\sqrt{2}}{\pi}\omega+4$.*

Proof. Let DP stand for the gradient vector and D^2P is the Hessian matrix.

For the given a we already know that $DP(a)=0$. We also find

$$\tfrac{\partial^2 P}{\partial\omega^2}(a)=0,\quad \tfrac{\partial^2 P}{\partial\omega\partial\lambda}(a)=-8\sqrt{2}\pi,\quad \tfrac{\partial^2 P}{\partial\lambda^2}(a)=-\pi^2.$$

That is, $\det D^2P(a)=-128\pi^2$ and by Proposition C.5 and remark C.6 (Appendix C) it holds that

$$P(\omega,\lambda)=-\tfrac{1}{2}h_2(\omega,\lambda)\left(16\sqrt{2}h_1(\omega,\lambda)+\pi h_2(\omega,\lambda)\right)\quad\text{on}\quad U, \tag{3.20}$$

where $h_1,h_2\in C^\infty(U,\mathbb{R})$ are given by almost explicit formulas in (C.13), (C.14) in the same lemma. We also have that $h_1(a)=h_2(a)=0$ and

$$\tfrac{\partial h_1}{\partial\omega}(a)=1,\quad \tfrac{\partial h_1}{\partial\lambda}(a)=0, \tag{3.21}$$

$$\tfrac{\partial h_2}{\partial\omega}(a)=0,\quad \tfrac{\partial h_2}{\partial\lambda}(a)=1. \tag{3.22}$$

Due to (3.20) we deduce that in U:

$$P(\omega,\lambda)=0\quad\Longleftrightarrow\quad h_2(\omega,\lambda)=0\quad\text{or}\quad 16\sqrt{2}h_1(\omega,\lambda)+\pi h_2(\omega,\lambda)=0. \tag{3.23}$$

By applying the Implicit Function Theorem to the functions $h_2(\omega,\lambda)=0$ and $16\sqrt{2}h_1(\omega,\lambda)+\pi h_2(\omega,\lambda)=0$ in U one finds a parametrization $\omega\mapsto\lambda=\eta(\omega)$ for each of these implicit functions. Indeed:

1) For $h_2(\omega,\lambda)=0$ it is shown in Lemma C.8 (Appendix C) that

$$\tfrac{\partial h_2}{\partial\lambda}(\omega,\lambda)>0\quad\text{on}\quad U,$$

and hence there exists $\eta_1:I\to J$, $\eta_1\in C^\infty(I)$ such that

$$h_2(\omega,\eta_1(\omega))=0,$$

and

$$\eta_1'(\omega) = -\tfrac{\partial h_2}{\partial \omega}(\omega, \eta_1(\omega)) \left[\tfrac{\partial h_2}{\partial \lambda}(\omega, \eta_1(\omega))\right]^{-1},$$

for all $\omega \in I$. We have that $\eta_1\left(\frac{1}{2}\pi\right) = 4$ and due to (3.22) we find

$$\eta_1'\left(\tfrac{1}{2}\pi\right) = 0.$$

Hence, there is a smooth branch of Γ in U passing through a, which is given by $\lambda = \eta_1(\omega)$ with the tangent $\lambda = 4$.

2) For $16\sqrt{2}h_1(\omega, \lambda) + \pi h_2(\omega, \lambda) = 0$ it is shown in Lemma C.9 (Appendix C) that

$$16\sqrt{2}\tfrac{\partial h_1}{\partial \lambda}(\omega, \lambda) + \pi \tfrac{\partial h_2}{\partial \lambda}(\omega, \lambda) > 0 \quad \text{on} \quad U,$$

and hence there exists $\eta_2 : \tilde{I} \to J$, $\eta_2 \in C^\infty(\tilde{I})$, where $\tilde{I} \subset I$, such that

$$16\sqrt{2}h_1(\omega, \eta_2(\omega)) + \pi h_2(\omega, \eta_2(\omega)) = 0,$$

and

$$\eta_2'(\omega) = -\frac{16\sqrt{2}\frac{\partial h_1}{\partial \omega}(\omega, \eta_2(\omega)) + \pi \frac{\partial h_2}{\partial \omega}(\omega, \eta_2(\omega))}{16\sqrt{2}\frac{\partial h_1}{\partial \lambda}(\omega, \eta_2(\omega)) + \pi \frac{\partial h_2}{\partial \lambda}(\omega, \eta_2(\omega))},$$

for all $\omega \in \tilde{I}$. We have that $\eta_2\left(\frac{1}{2}\pi\right) = 4$ and due to (3.21) and (3.22) we obtain

$$\eta_2'\left(\tfrac{1}{2}\pi\right) = -\tfrac{16\sqrt{2}}{\pi}.$$

Hence, there is another smooth branch of Γ in U passing through a and given by $\lambda = \eta_2(\omega)$. The tangent is $\lambda = -\frac{16\sqrt{2}}{\pi}\omega + 4$. ■

The next lemma studies $P(\omega,\lambda) = 0$ locally in V but away from the point a.

Lemma 3.17 *Let*

$$H_1 = \left\{(\omega, \lambda) : \left[\tfrac{84}{180}\pi, \tfrac{90}{180}\pi\right] \times [4.030, 4.970]\right\},$$

$$H_2 = \left\{(\omega, \lambda) : \left[\tfrac{87}{180}\pi, \tfrac{101}{180}\pi\right] \times [4.750, 5.100]\right\},$$

$$H_3 = \left\{(\omega, \lambda) : \left[\tfrac{100}{180}\pi, \tfrac{108}{180}\pi\right] \times [4.000, 4.850]\right\},$$

$$H_4 = \left\{(\omega, \lambda) : \left[\tfrac{91}{180}\pi, \tfrac{102}{180}\pi\right] \times [3.950, 4.100]\right\},$$

$$H_5 = \left\{(\omega, \lambda) : \left[\tfrac{90}{180}\pi, \tfrac{96}{180}\pi\right] \times [3.030, 3.970]\right\},$$

$$H_6 = \left\{(\omega, \lambda) : \left[\tfrac{79}{180}\pi, \tfrac{94}{180}\pi\right] \times [2.900, 3.230]\right\},$$

$$H_7 = \left\{(\omega,\lambda) : \left[\tfrac{72}{180}\pi, \tfrac{80}{180}\pi\right] \times [3.150, 4.000]\right\},$$
$$H_8 = \left\{(\omega,\lambda) : \left[\tfrac{78}{180}\pi, \tfrac{89}{180}\pi\right] \times [3.900, 4.050]\right\},$$

and U be as in Lemma 3.16. Then $\cup_{j=1}^{8} H_j$ covers the set Γ in V (see Figure 3.4) and in each H_j the following holds:

Rectangle	*Property in H_j*	*The set Γ in H_j is given by*
H_{2k-1}	$\frac{\partial P}{\partial \omega}(\omega,\lambda) \neq 0$	$\omega = \phi_{2k-1}(\lambda) : \phi_{2k-1} \in C^\infty(J_{2k-1})$
H_{2k}	$\frac{\partial P}{\partial \lambda}(\omega,\lambda) \neq 0$	$\lambda = \psi_{2k}(\omega) : \psi_{2k} \in C^\infty(I_{2k})$

Here $k = 1, ..., 4$.

Proof. In Claims C.10 – C.17 of Appendix C we constructed the rectangles $H_j \subset V$, $j = 1, ..., 8$ such that the results of the second column in a table above hold. In Figure 3.4 we sketched the covering of the set Γ in V with the rectangles H_j, $j = 1, ..., 8$.

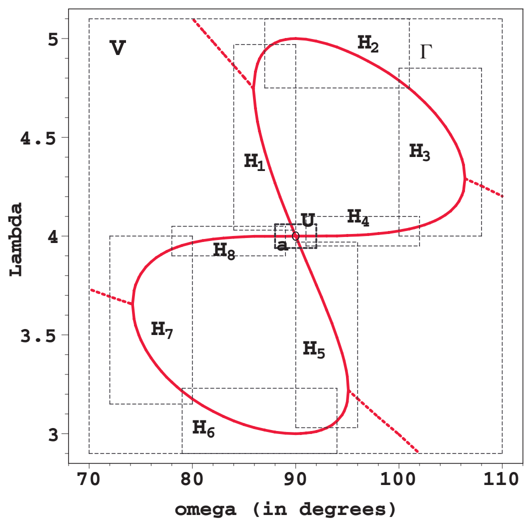

Figure 3.4: *For lemma 3.17.*

Due to result of the second column we can apply the Implicit Function Theorem to the function $P(\omega,\lambda) = 0$ in every H_j, $j = 1, ..., 8$ in order to obtain $\omega = \phi_{2k-1}(\lambda)$ or $\lambda = \psi_{2k}(\omega)$, $k = 1, ..., 4$. By assumption $P \in C^{\infty}(V, \mathbb{R})$ and hence ϕ, ψ are C^{∞} on the corresponding intervals J, I. ■

Basing on the results of the above two lemmas we arrive at the following

Proposition 3.18 *The set Γ given by (3.18) is an 8-shaped curve. That is, there exists an open set $\tilde{V} \supset [-1,1]^2$ and a C^{∞}-diffeomorphism $S : V \to \tilde{V}$ such that*

$$S(\Gamma) = \{(\sin(2t), \sin(t)),\ 0 \leq t < 2\pi\}.$$

Henceforth, we will call the set Γ a curve (having one self-intersection point) which means that every part of the set Γ is locally parametrizable in ω or λ.

3.4.2 Eigenvalue λ_1 as the bottom part of Γ

The curve Γ in a rectangle V combines the graphs of the first four eigenvalues $\lambda_1, ..., \lambda_4$ of the boundary value problem (3.4) as functions of ω as far as they are real. Here we focus on the eigenvalue λ_1 which is a bottom part of Γ (the segment $c_6c_1c_2 \subset \Gamma$ in Figure 3.3). In particular, we prove that as a function of ω the eigenvalue $\lambda_1 = \lambda_1(\omega)$ increases between the points c_1, c_2 (the approximations for their coordinates are given in Table 3.1). The situation is illustrated by Figure 3.5.

In order to prove this result, we follow the approach used in Lemmas 3.16 and 3.17. To be more precise, we fix two rectangles $\{H_0, H_\star\} \subset V$ such that $H_0 \cap H_\star = \emptyset$ and $H_0 \cup H_\star$ covers the part of Γ containing the segment c_1c_2 (see Figure 3.6). We parameterize Γ in $H_0, H_\star$ as $\omega \mapsto \lambda = \psi(\omega)$ and $\lambda \mapsto \omega = \varphi(\lambda)$, respectively, and study the properties of these parametrizations (convexity-concavity, extremum points, the intervals of increase-decrease). This will enable to gain the information about c_1c_2.

Lemma 3.19 *Let $H_0 = I_0 \times J_0 \subset V$ be the closed rectangle with $I_0 = \left[\frac{84}{180}\pi, \frac{94}{180}\pi\right]$ and $J_0 = [2.960, 3.060]$. It holds that Γ in H_0 is given by $\lambda = \psi(\omega)$, $\psi \in C^{\infty}(\omega_\alpha, \omega_\beta)$, $(\omega_\alpha, \omega_\beta) \subset I_0$ and is such that it attains its minimum on $(\omega_\alpha, \omega_\beta)$ at $\omega = \omega_0 = \frac{1}{2}\pi$ and increases monotonically on (ω_0, ω_β). Here $\omega_\alpha, \omega_\beta$ are the solutions to the equation $P(\omega, 3.060) = 0$ on $\omega \in \left(\frac{84}{180}\pi, \frac{1}{2}\pi\right)$ and on $\omega \in \left(\frac{1}{2}\pi, \frac{94}{180}\pi\right)$, respectively, with P given by (3.17).*

Proof. By Lemma 3.17 we know that

$$P(\omega, \lambda) = 0 \quad \Longleftrightarrow \quad P(\omega, \psi(\omega)) = 0 \quad \text{in } H_6, \tag{3.24}$$

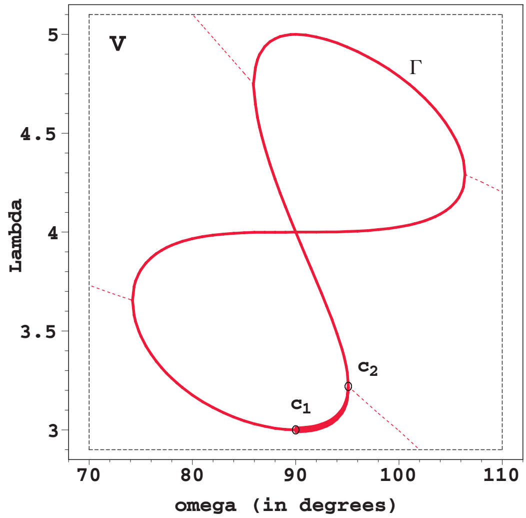

Figure 3.5: *Increase of* λ_1 *between* c_1 and c_2

and if we take the rectangle H_0 defined as in lemma above, then due to $H_0 \subset H_6$, (3.24) will also hold in H_0. Moreover, we also set H_0 in such a way that its top boundary intersects Γ at two points, meaning that we find two solutions of $P(\omega, 3.060) = 0$ with P as in (3.17). We name these two solutions $\omega_\alpha, \omega_\beta$.

Hence, we deduce that Γ in H_0 is given by $\lambda = \psi(\omega)$, $\psi \in C^\infty(\omega_\alpha, \omega_\beta)$ and satisfies $\psi(\omega_\alpha) = \psi(\omega_\beta) = 3.060$. Due to condition

$$\psi(\omega_\alpha) = \psi(\omega_\beta),$$

by Rolle's theorem there exists $\omega_0 \in (\omega_\alpha, \omega_\beta)$ such that $\psi'(\omega_0) = 0$.

Since $P\left(\omega_0, \psi\left(\omega_0\right)\right) = 0$ and due to

$$\psi'(\omega) = -\frac{\partial P}{\partial \omega}(\omega, \psi\left(\omega\right)) \left[\frac{\partial P}{\partial \lambda}(\omega, \psi\left(\omega\right))\right]^{-1},$$

we solve the system $P(\omega, \lambda) = 0$ and $\frac{\partial P}{\partial \omega}(\omega, \lambda) = 0$ in H_0 in order to find ω_0. Its solution is a point $c_1 = \left(\frac{1}{2}\pi, 3\right)$ and hence

$$\omega_0 = \tfrac{1}{2}\pi.$$

We deduce that $\lambda = \psi(\omega)$ attains its local extremum at $\omega = \omega_0$.

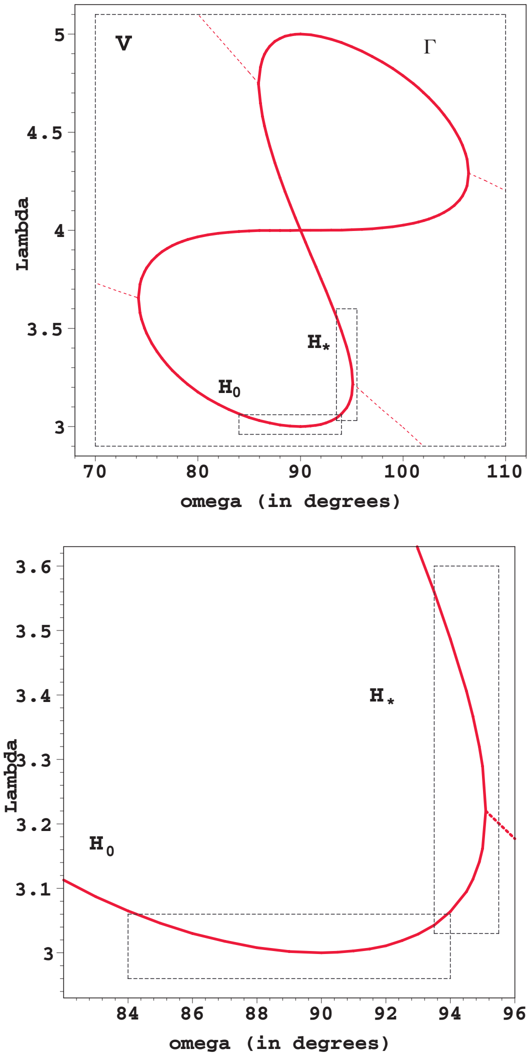

Figure 3.6: *The rectangles* $H_0, H_\star$ *from lemmas 3.19 and 3.20, respectively (on the left); the enlarged view (on the right).*

Next we show that $\lambda = \psi(\omega)$ has a minimum at $\omega = \omega_0$ on $(\omega_\alpha, \omega_\beta)$. For this purpose we consider a function $G \in C^\infty(H_0, \mathbb{R})$ such that

$$G(\omega, \psi(\omega)) = \psi''(\omega). \tag{3.25}$$

For an explicit formula for G see Appendix C. In Claim C.18 of this Appendix we show that

$$G\left(\omega,\lambda\right)>0 \quad \text{on} \quad H_0. \tag{3.26}$$

Condition (3.26) together with (3.25) yields

$$G\left(\omega,\psi(\omega)\right)=\psi''(\omega)>0 \quad \text{on} \quad (\omega_\alpha,\omega_\beta),$$

meaning that $\lambda=\psi(\omega)$ is convex on $(\omega_\alpha,\omega_\beta)$.

The result is that $\lambda=\psi(\omega)$ attains its minimum on $(\omega_\alpha,\omega_\beta)$ at $\omega=\omega_0=\frac{1}{2}\pi$ and increases monotonically on the interval $\omega\in(\omega_0,\omega_\beta)$. ■

We also have the following

Lemma 3.20 *Let $H_\star=I_\star\times J_\star\subset V$ be the closed rectangle with $I_\star=\left[\frac{93.5}{180}\pi,\frac{95.5}{180}\pi\right]$ and $J_\star=[3.030,3.600]$. It holds that Γ in $H_\star$ is given by $\omega=\varphi(\lambda)$, $\varphi\in C^\infty(\lambda_\gamma,\lambda_\delta)$, $(\lambda_\gamma,\lambda_\delta)\subset J_\star$ and is such that it attains its maximum on $(\lambda_\gamma,\lambda_\delta)$ at $\lambda=\lambda_\star\approx 3.220$ and increases monotonically on the interval $(\lambda_\gamma,\lambda_\star)$. Here $\lambda_\gamma,\lambda_\delta$ are the solutions to the equation $P\left(\frac{93.5}{180}\pi,\lambda\right)=0$ on $\lambda\in(3.030,3.100)$ and on $\lambda\in(3.500,3.600)$, respectively. Also, $\lambda_\star$ is the solution to the system $P(\omega,\lambda)=0$ and $\frac{\partial P}{\partial\lambda}(\omega,\lambda)=0$ on $\lambda\in(\lambda_\gamma,\lambda_\delta)$; P given by (3.17).*

Proof. By Lemma 3.17 we know that

$$P(\omega,\lambda)=0 \quad\Longleftrightarrow\quad P\left(\varphi(\lambda),\lambda\right)=0 \quad \text{in } H_5, \tag{3.27}$$

and if we take the rectangle $H_\star$ defined as in lemma above, then due to $H_\star\subset H_5$, (3.27) will also hold in $H_\star$. Moreover, we also set $H_\star$ in such a way that its left boundary intersects Γ at two points, meaning we find two solutions of $P\left(\frac{93.5}{180}\pi,\lambda\right)=0$ with P as in (3.17). We name these two solutions $\lambda_\gamma,\lambda_\delta$.

Hence, we deduce that Γ in $H_\star$ is given by $\omega=\varphi(\lambda)$, $\varphi\in C^\infty(\lambda_\gamma,\lambda_\delta)$ and satisfies $\varphi(\lambda_\gamma)=\varphi(\lambda_\delta)=\frac{93.5}{180}\pi$. Due to condition

$$\varphi(\lambda_\gamma)=\varphi(\lambda_\delta),$$

by Rolle's theorem there exists $\lambda_\star\in(\lambda_\gamma,\lambda_\delta)$ such that $\varphi'(\lambda_\star)=0$.

Since $P\left(\varphi(\lambda_\star),\lambda_\star\right)=0$ and due to

$$\varphi'(\lambda)=-\tfrac{\partial P}{\partial\lambda}(\varphi(\lambda),\lambda)\left[\tfrac{\partial P}{\partial\omega}(\varphi(\lambda),\lambda)\right]^{-1},$$

we solve the system $P(\omega, \lambda) = 0$ and $\frac{\partial P}{\partial \lambda}(\omega, \lambda) = 0$ in $H_\star$ in order to find $\lambda_\star$. Its solution is a point $c_2 = \left(\tilde{\omega}, \tilde{\lambda}\right)$, where $\tilde{\omega}/\pi \approx 0.528$and $\tilde{\lambda} \approx 3.220$. Hence,

$$\lambda_\star \approx 3.220.$$

We deduce that $\omega = \varphi(\lambda)$ attains its local extremum at $\lambda = \lambda_\star$.

Next we show that $\omega = \varphi(\lambda)$ has a maximum at $\lambda = \lambda_\star$ on $(\lambda_\gamma, \lambda_\delta)$. For this purpose we consider a function $F \in C^\infty(H_\star, \mathbb{R})$ such that

$$F(\varphi(\lambda), \lambda) = \varphi''(\lambda). \tag{3.28}$$

For explicit formula for F see Appendix C. In Claim C.19 of this Appendix we show that

$$F(\omega, \lambda) < 0 \quad \text{on} \quad H_\star. \tag{3.29}$$

Condition (3.29) together with (3.28) yields

$$F(\varphi(\lambda), \lambda) = \varphi''(\lambda) < 0 \quad \text{on} \quad (\lambda_\gamma, \lambda_\delta),$$

meaning that $\omega = \varphi(\lambda)$ is concave on $(\lambda_\gamma, \lambda_\delta)$.

The result is that $\omega = \varphi(\lambda)$ attains its maximum on $(\lambda_\gamma, \lambda_\delta)$ at $\lambda = \lambda_\star \approx 3.220$ and increases monotonically on the interval $\lambda \in (\lambda_\gamma, \lambda_\star)$. ■

Theorem 3.21 *As a function of ω the first eigenvalue $\lambda_1 = \lambda_1(\omega)$ of the boundary value problem (3.4) increases on $\omega \in \left(\frac{1}{2}\pi, \omega_\star\right)$. Here $\omega_\star/\pi \approx 0.528$ (in degrees $\omega_\star \approx 95.1^\circ$).*

3.5 On the behaviour of $\omega \mapsto \lambda_1(\omega)$, $\omega \in (0, 2\pi]$ when $\alpha \in \left(0, \frac{1}{2}\pi\right)$

The previous section has studied the eigenvalue λ_1 of problem (3.4) on $\omega \in (0, 2\pi]$, when $\alpha = 0$ and described $\omega \mapsto \lambda_1(\omega)$. Here we will give an impression about the behaviour of λ_1 of (3.4) on $\omega \in (0, 2\pi]$, when $\alpha \in \left(0, \frac{1}{2}\pi\right)$. In Figures 3.7 and 3.8, which are the same plots viewed from different viewpoints, we depict the eigenvalue λ_1 of (3.4) for $\alpha = \frac{10}{180}\pi j$, $j = 0, ..., 8$. Note, that although the case $\alpha = \frac{1}{2}\pi$ is identical to the case $\alpha = 0$ we, however, plot the corresponding curve (the one in green) in order to complete the row.

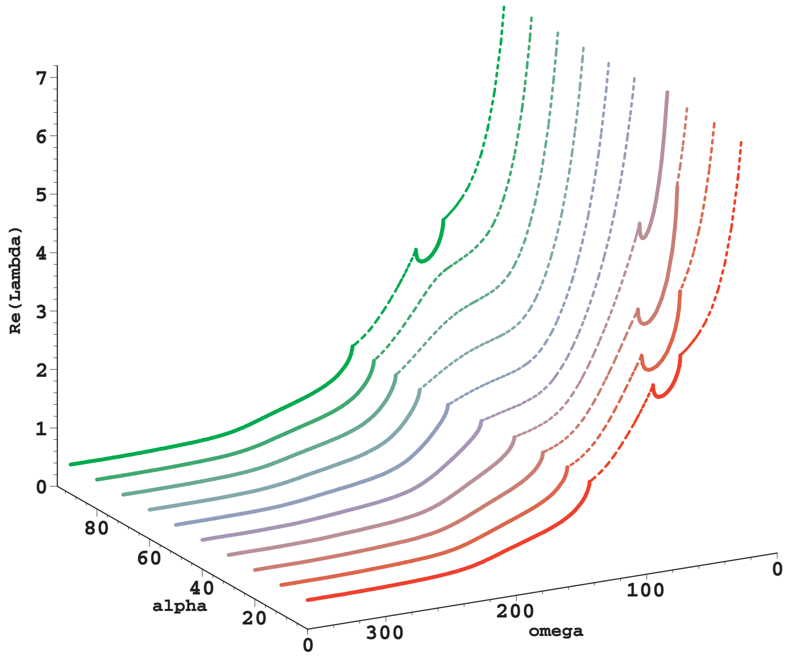

Figure 3.7: *The first eigenvalue λ_1 of the problem (3.4) when $\alpha = \frac{10}{180}\pi j$, $j = 0, ..., 9$. Dashed lines depict the real part of those $\lambda_1 \in \mathbb{C}$, solid lines are for purely real λ_1.*

3.6 Structure of solutions to (3.1) in the cone $\mathcal{K}_{(\alpha,\omega)}$

Here we proceed with the derivation of an algebraic and geometric multiplicity of the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ of problem (3.4). This will enable us to describe the precise structure of power type solutions to problem (3.1) in the cone $\mathcal{K}_{(\alpha,\omega)}$.

First we recall the following.

Definition 3.22 *Let $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and $\omega \in (0, 2\pi]$ be fixed. The eigenvalue λ_j, $j \in \mathbb{N}^+$ of problem (3.4) is said to have an algebraic multiplicity $\kappa_j \geq 1$, if for all $0 \leq k \leq \kappa_j - 1$ it holds that*

$$\frac{\partial^k}{\partial \lambda^k} \mathcal{P}(\alpha, \omega, \lambda_j) = 0,$$

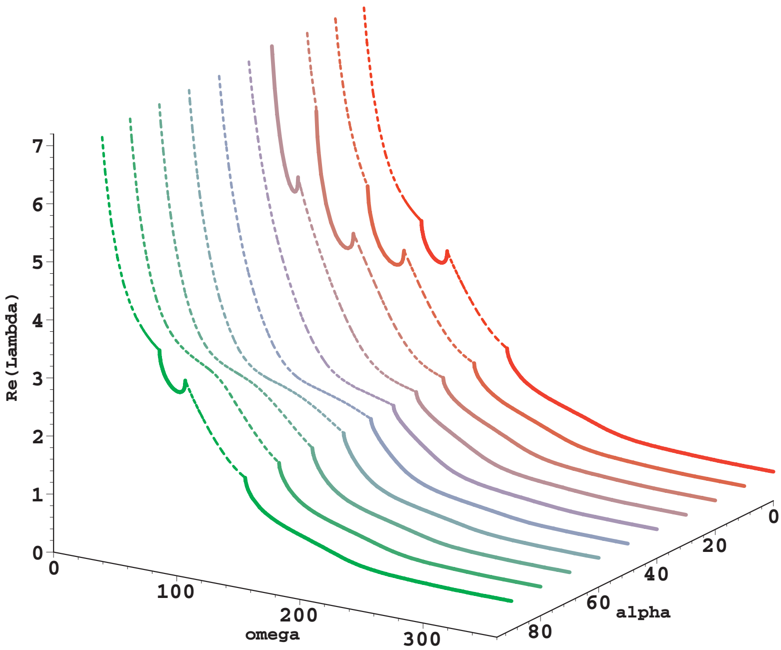

Figure 3.8: *The graph of Figure 3.7 from another viewpoint.*

and

$$\frac{d^{\kappa_j}}{d\lambda^{\kappa_j}}\mathcal{P}(\alpha,\omega,\lambda_j) \neq 0.$$

Based on the numerical approximations for some first eigenvalues λ_j, $j \in \mathbb{N}^+$ depicted in Figure 3.2 (the top one), Figures 3.7, 3.8 and partly by our derivations (namely, the existence of the solution to the system $P(\omega,\lambda) = \frac{\partial P}{\partial \lambda}(\omega,\lambda) = 0$ in Lemma 3.20) we believe that the maximal algebraic multiplicity of a certain λ_j of problem (3.4) is at most 2. Indeed, generically 3 curves never intersect at one point, meaning that geometrically the algebraic multiplicity will always be at most 2.

Definition 3.23 *The eigenvalue λ_j, $j \in \mathbb{N}^+$ of problem (3.4) is said to have a geometric multiplicity $I_j \geq 1$, if the number of the corresponding linearly independent eigenfunctions Φ equals I_j.*

For given λ_j, $j \in \mathbb{N}^+$ of problem (3.4) we distinguish the following three cases:

1. $\kappa_j = I_j = 1$ one finds a solution $(\lambda_j, \Phi_{j,0})$ of (3.4) and then the power type solution of problem (3.1) reads:

$$u_{j,0} = r^{\lambda_j+1}\Phi_{j,0}(\theta); \tag{3.30}$$

2. $\kappa_j = 2$, $I_j = 1$ one finds a solution $(\lambda_j, \Phi_{j,0})$ and a generalized solution $(\lambda_j, \Phi_{j,1})$ of (3.4). Recall that $(\lambda_j, \Phi_{j,1})$ satisfies the equation (3.5),

$$\mathcal{L}(\lambda_j)\Phi_{j,1} + \mathcal{L}'(\lambda_j)\Phi_{j,0} = 0,$$

where $\mathcal{L}(\lambda)$ is given by (3.3) and $\mathcal{L}'(\lambda) = \frac{d}{d\lambda}\mathcal{L}(\lambda)$.

Then we have two solutions of problem (3.1):

$$u_{j,0} = r^{\lambda_j+1}\Phi_{j,0}(\theta) \quad \text{and} \quad u_{j,1} = r^{\lambda_j+1}\left(\Phi_{j,1}(\theta) + \log(r)\Phi_{j,0}(\theta)\right); \tag{3.31}$$

3. $\kappa_j = I_j = 2$ one finds two solutions $(\lambda_j, \Phi_{j,0})$, $(\lambda_j, \Phi_{j,1})$ of (3.4), where $\Phi_{j,0}$ and $\Phi_{j,1}$ are linearly independent on $\theta \in (\alpha, \alpha+\omega)$. Then as in previous case, the two solutions of problem (3.1) occur:

$$u_{j,0} = r^{\lambda_j+1}\Phi_{j,0}(\theta) \quad \text{and} \quad u_{j,1} = r^{\lambda_j+1}\Phi_{j,1}(\theta). \tag{3.32}$$

For our grid operator, in some cases of α and ω one is able to find the eigenvalues $\{\lambda_j\}_j^\infty$ of problem (3.4) explicitly. This happens when

$\alpha = 0$	$\omega = \frac{1}{2}\pi$,
$\alpha \in \left[0, \frac{1}{2}\pi\right)$	$\omega = \pi$,
$\alpha = 0$	$\omega = \frac{3}{2}\pi$,
$\alpha \in \left[0, \frac{1}{2}\pi\right)$	$\omega = 2\pi$.

Moreover,

- when $\alpha = 0$ and the opening angle $\omega \in \left\{\frac{1}{2}\pi, \pi\right\}$, for every given λ_j one can compute explicitly the corresponding eigenfunctions $\Phi_{j,q}$, $q = 0, ..., \kappa_j - 1$. If $\omega \in \left\{\frac{3}{2}\pi, 2\pi\right\}$, the eigenfunctions $\Phi_{j,q}$ can be computed explicitly only for some λ_j. In Appendix D we bring the formulas of some first solutions $u_{j,q}$ to (3.1) for the corresponding cases (if computable);

- when $\alpha \in \left(0, \frac{1}{2}\pi\right)$ and the opening angle $\omega = \pi$ the first eigenvalues of problem (3.4) is $\lambda_1 = 1$. Then one computes $\Phi_{1,0}(\theta) = \sin^2(\theta - \alpha)$ which yields $u_{1,0} = r^2 \sin^2(\theta - \alpha)$. For higher values λ_j, $j \geq 2$ the solutions $u_{j,q}$ are polynomials.

These functions $r^{\lambda+1}\Phi(\theta)$ and $r^{\lambda+1}\log(r)\Phi(\theta)$ determine the bands for the regularity in Kondratiev's theory. Details are found in Chapter 5.

Chapter 4

Kondratiev's weighted Sobolev spaces

Due to Kondratiev [18], one of the appropriate functional spaces for the boundary value problems of the type (1.4) are the weighted Sobolev spaces $V_\beta^{l,2}$, where $l \in \{0, 1, 2, \dots\}$ and $\beta \in \mathbb{R}$. Such spaces can be defined in different ways: either via the set of the square-integrable weighted weak derivatives in Ω (see [18, 14]), or via the completion of the set of infinitely differentiable on Ω functions with bounded support in Ω, with respect to a certain norm (see [19, 28]).

In our case (see Condition 1.2) the domain $\Omega \subset \mathbb{R}^2$ is open, bounded, and has a corner in $0 \in \partial\Omega$. It is also assumed that $\partial\Omega\backslash\{0\}$ is smooth, and that $\Omega \cap B_\varepsilon(0) = \mathcal{K}_{(\alpha,\omega)} \cap B_\varepsilon(0)$, where $B_\varepsilon(0)$ is a ball of radius $\varepsilon > 0$ and $\mathcal{K}_{(\alpha,\omega)}$ is an infinite cone with an opening angle $\omega \in (0, 2\pi)$ and orientation angle $\alpha \in \left[0, \frac{1}{2}\pi\right)$.

These weighted spaces are as follows:

Definition 4.1 *Let*

$$C_c^\infty\left(\overline{\Omega}\backslash\{0\}\right) := \left\{u \in C_c^\infty\left(\overline{\Omega}\right) : \ \textit{support}(u) \subset \overline{\Omega}\backslash\{0\}\right\}.$$

Let $l \in \{0, 1, 2, \dots\}$ and $\beta \in \mathbb{R}$. Then $V_\beta^{l,2}(\Omega)$ is defined as a completion:

$$V_\beta^{l,2}(\Omega) = \overline{C_c^\infty\left(\overline{\Omega}\backslash\{0\}\right)}^{\|\cdot\|}, \tag{4.1}$$

with

$$\|u\| := \|u\|_{V_\beta^{l,2}(\Omega)} = \left(\sum_{|\gamma|=0}^{l} \int_\Omega r^{2(\beta-l+|\gamma|)} \left|D^\gamma u\right|^2 dxdy \right)^{\frac{1}{2}}. \tag{4.2}$$

Here $r = \left(x^2+y^2\right)^{\frac{1}{2}}$ *and* $\gamma = (\gamma_1, \gamma_2)$ *is a multi-index of order* $|\gamma| \leq l$, *so that* $D^\gamma u = \frac{\partial^{|\gamma|} u}{\partial x^{\gamma_1} \partial y^{\gamma_2}}$.

The space $V_\beta^{l,2}(\Omega)$ consists of all functions $u : \Omega \to \mathbb{R}$ such that for each multi-index $\gamma = (\gamma_1, \gamma_2)$ with $|\gamma| \leq l$, $D^\gamma u = \frac{\partial^{|\gamma|} u}{\partial x^{\gamma_1} \partial y^{\gamma_2}}$ exists in the weak sense and $r^{\beta-l+|\gamma|} D^\gamma u \in L^2(\Omega)$.

Straightforward from the definition of the norm the following continuous imbeddings hold (see [19, Section 6.2, lemma 6.2.1]):

$$V_{\beta_2}^{l_2,2}(\Omega) \subset V_{\beta_1}^{l_1,2}(\Omega) \quad \text{if} \quad l_2 \geq l_1 \geq 0 \quad \text{and} \quad \beta_2 - l_2 \leq \beta_1 - l_1. \tag{4.3}$$

In order to have the appropriate space for zero Dirichlet boundary conditions in problem (1.4) we also define the corresponding space.

Definition 4.2 *For* $l \in \{0, 1, 2, \dots\}$ *and* $\beta \in \mathbb{R}$, *set*

$$\overset{\circ}{V}{}_\beta^{l,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|}, \tag{4.4}$$

with $\|\cdot\|$ *is the norm (4.2) and* $C_c^\infty(\Omega) := \left\{ u \in C_c^\infty\left(\overline{\Omega}\right) : \text{ support}(u) \subset \Omega \right\}$.

Remark 4.3 *For* $u \in \overset{\circ}{V}{}_\beta^{l,2}(\Omega)$ *one finds that* $D^\gamma u = 0$ *on* $\partial\Omega\backslash\{0\}$ *for* $|\gamma| \leq \ell - 1$, *where* $D^\gamma u = 0$ *in the sense of traces.*

4.1 Comparing (weighted) Sobolev spaces: imbeddings

As mentioned e.g. in [18, page 240] or [19, Chapter 7, summary], the family of weighted spaces $V_\beta^{l,2}$ does not contain the ordinary Sobolev spaces without weight. More precisely: $W^{k,2} \notin \left\{ V_\beta^{l,2} \right\}_{l,\beta}$ for $k \geq 1$. We will prove the imbedding results for bounded Ω from Condition 1.2.

Our first statement is as follows.

Lemma 4.4 *Let* $\beta \in \mathbb{R}$ *and* $l \in \{0, 1, 2, \dots\}$. *Then the following holds:*

$$V_\beta^{l,2}(\Omega) \subset W^{l,2}(\Omega) \quad \Leftrightarrow \quad \beta \leq 0, \tag{4.5}$$

$$W^{l,2}(\Omega) \subset V_\beta^{l,2}(\Omega) \quad \Leftrightarrow \quad \beta \geq l, \tag{4.6}$$

Proof. Let Ω be as in Condition 1.2 and $\Omega \subset B_M(0)$, where $B_M(0)$ is an open ball of radius $M > 0$. The statement in a) goes as follows: for $(x, y) \in \Omega$ one has $0 \leq r \leq M$ and hence $r^{2(\beta-l+|\gamma|)} \geq M^{2(\beta-l+|\gamma|)}$ iff $\beta - l + |\gamma| \leq 0$. Since $0 \leq |\gamma| \leq l$, we obtain $\beta \leq 0$. This enables us to have the estimate:

$$\|u\|_{V_\beta^{l,2}(\Omega)} = \left(\sum_{|\gamma|=0}^{l} \int_\Omega r^{2(\beta-l+|\gamma|)} \left|D^\gamma u\right|^2 dxdy \right)^{\frac{1}{2}} \geq$$

$$\geq \left(\sum_{|\gamma|=0}^{l} \int_\Omega M^{2(\beta-l+|\gamma|)} \left|D^\gamma u\right|^2 dxdy \right)^{\frac{1}{2}} \geq$$

$$\geq \min(1, M^{\beta-l}) \left(\sum_{|\gamma|=0}^{l} \int_\Omega \left|D^\gamma u\right|^2 dxdy \right)^{\frac{1}{2}} = \min(1, M^{\beta-l}) \|u\|_{W^{l,2}(\Omega)},$$

which is the result in (4.5).

To prove (4.6) we notice that $r^{2(\beta-l+|\gamma|)} \leq M^{2(\beta-l+|\gamma|)}$ iff $\beta - l + |\gamma| \geq 0$. Due to $0 \leq |\gamma| \leq l$, we obtain $\beta \geq l$ and then the estimate holds:

$$\|u\|_{V_\beta^{l,2}(\Omega)} = \left(\sum_{|\gamma|=0}^{l} \int_\Omega r^{2(\beta-l+|\gamma|)} \left|D^\gamma u\right|^2 dxdy \right)^{\frac{1}{2}} \leq$$

$$\leq \left(\sum_{|\gamma|=0}^{l} \int_\Omega M^{2(\beta-l+|\gamma|)} \left|D^\gamma u\right|^2 dxdy \right)^{\frac{1}{2}} \leq$$

$$\leq \max(1, M^{\beta-l}) \left(\sum_{|\gamma|=0}^{l} \int_\Omega \left|D^\gamma u\right|^2 dxdy \right)^{\frac{1}{2}} = \max(1, M^{\beta-l}) \|u\|_{W^{l,2}(\Omega)}.$$

This is the result in (4.6). ■

For the spaces adapted to the zero-trace boundary conditions we may find the analogous result. In order to do this, let us first recall a higher-order one dimensional Hardy inequality.

Lemma 4.5 *Let $w \in C_0^\infty[x_1, x_2]$. For every $k \geq 1$ it holds that*

$$\int_{x_1}^{x_2} \left(\frac{w(x)}{(x-x_1)^k}\right)^2 dx \leq \frac{4^k}{(2k-1)^2(2k-3)^2 3^2 1^2} \int_{x_1}^{x_2} \left(w^{(k)}(x)\right)^2 dx. \tag{4.7}$$

Proof. It holds straightforwardly that

$$\int_{x_1}^{x_2} \left(\frac{w(x)}{(x-x_1)^k}\right)^2 dx = \frac{1}{1-2k} \left[(w(x))^2 (x-x_1)^{1-2k}\right]\Big|_{x_1}^{x_2} +$$

$$+\frac{2}{2k-1} \int_{x_1}^{x_2} w(x) w'(x) (x-x_1)^{1-2k} \, dx \leq$$

$$\leq \frac{2}{2k-1} \left(\int_{x_1}^{x_2} \left(\frac{w(x)}{(x-x_1)^k}\right)^2 dx\right)^{1/2} \left(\int_{x_1}^{x_2} \left(\frac{w'(x)}{(x-x_1)^{k-1}}\right)^2 dx\right)^{1/2},$$

and the first step in the proof of (4.7) follows. Repeating the argument for w' and $k-1$ etc. will give the result. ∎

Remark 4.6 *Since $\overset{\circ}{W}{}^{k,2}(x_1, x_2)$, $k \geq 1$ is the closure of $C_0^\infty[x_1, x_2]$ in the $W^{k,2}$-norm, one can use the results of Lemma 4.5 for every $w \in \overset{\circ}{W}{}^{k,2}(x_1, x_2)$, $k \geq 1$.*

The second statement about ordinary and weighted spaces follows.

Lemma 4.7 *Let $\beta \in \mathbb{R}$ and $l \in \{0, 1, 2, \dots\}$. Then the following holds:*

$$\overset{\circ}{V}{}^{l,2}_\beta(\Omega) \subset \overset{\circ}{W}{}^{l,2}(\Omega) \quad \Leftrightarrow \quad \beta \leq 0, \tag{4.8}$$

$$\overset{\circ}{W}{}^{l,2}(\Omega) \subset \overset{\circ}{V}{}^{l,2}_\beta(\Omega) \quad \Leftrightarrow \quad \beta \geq 0. \tag{4.9}$$

Corollary 4.8 *For $l \in \{0, 1, 2, \}$ one has*

$$\overset{\circ}{W}{}^{l,2}(\Omega) = \overset{\circ}{V}{}^{l,2}_0(\Omega).$$

Proof of Lemma 4.7. Let Ω be as in Condition 1.2 with $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and $\omega \in (0, 2\pi)$. Let us set $\theta = \alpha + \frac{1}{2}\omega$. We also use the fact that for our domain there exists $c > 0$ such that

$$r > c\rho\,(x, y)$$

where ρ denotes the distance from a point (x, y) on the lines

$$\ell : y = \tan(\theta)\, x + \tau,$$

with $\tau \in \mathbb{R}$ to the point $(x_1, y_1) \in \partial\Omega$. For details see Figure 4.1, where for simplicity we assume $\alpha = 0$.

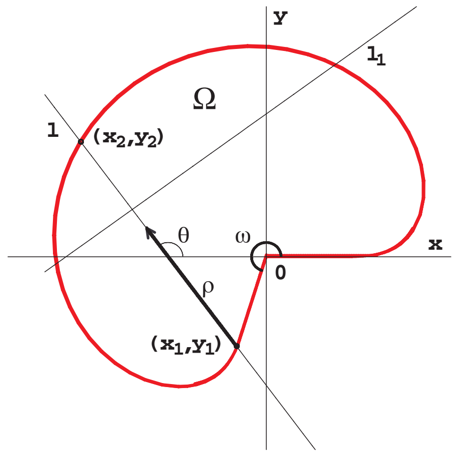

Figure 4.1: *Domain Ω with a concave corner ω intersected by ℓ. Here $\alpha = 0$.*

In particular, it holds that

$$\rho^2 = (x - x_1)^2 \left(1 + \tan^2(\theta)\right).$$

We may integrate along the lines ℓ and use the one-dimensional Hardy-inequality (4.7) to find that there exist $\tilde{a}_l \in \mathbb{R}^+$ with

$$\|u\|_{V_0^{l,2}(\Omega)} \le \tilde{a}_l \|u\|_{W^{l,2}(\Omega)} \quad \text{for all } u \in C_c^\infty(\Omega). \tag{4.10}$$

On the other hand, using the same argumentation as in proof of (4.5) in Lemma 4.4 we find $a_l \in \mathbb{R}^+$ such that

$$a_l \|u\|_{W^{l,2}(\Omega)} \le \|u\|_{V_0^{l,2}(\Omega)} \quad \text{for all } u \in C_c^\infty(\Omega). \tag{4.11}$$

Inequalities (4.10), (4.11) yield

$$\overset{\circ}{W}{}^{l,2}(\Omega) = \overset{\circ}{V}{}^{l,2}_{0}(\Omega).$$

Due to imbedding $\overset{\circ}{V}{}^{l,2}_{\beta_1}(\Omega) \subset \overset{\circ}{V}{}^{l,2}_{0}(\Omega) \subset \overset{\circ}{V}{}^{l,2}_{\beta_2}(\Omega)$ when $\beta_1 \leq 0 \leq \beta_2$ one obtains the result in (4.8) and (4.9). ■

Chapter 5

Regularity results

In this Chapter we will give the regularity result to our clamped-grid problem (1.4):

$$\begin{cases} u_{xxxx} + u_{yyyy} = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \frac{\partial}{\partial n} u = 0 & \text{on} \quad \partial\Omega \backslash \{0\}. \end{cases}$$

in the domain Ω defined as in Condition 1.2 (we sketch Ω in Figure below).

In order to do this we refer to the key theorem of the Kondratiev theory (see e.g. [19, Theorem 6.4.1]). We will also use the information from Chapter 3 on the power type solutions to the homogeneous problem (3.1) in the cone $\mathcal{K}_{(\alpha,\omega)}$.

5.1 General regularity statement

The Kondratiev theorem adapted to our problem (1.4) reads as:

Theorem 5.1 (Kondratiev) *Let $u \in V^{l_1,2}_{\beta_1}(\Omega)$ with $l_1 \in \mathbb{N}$, $\beta_1 \in \mathbb{R}$ be a solution of the elliptic boundary value problem (1.4).*

Suppose that $f \in V^{l_2,2}_{\beta_2}(\Omega)$, where $l_2 \in \mathbb{N}$, $\beta_2 \in \mathbb{R}$ and such that

$$l_1 - \beta_1 < l_2 - \beta_2 + 4.$$

If no eigenvalue λ_j of problem (3.4) lies on the lines

$$\mathrm{Re}(\lambda) = l_1 - \beta_1 - 2, \quad \mathrm{Re}(\lambda) = l_2 - \beta_2 + 2,$$

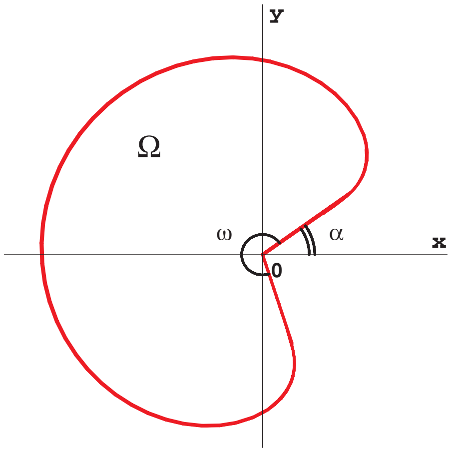

Figure 5.1: *Example for* Ω *with two governing parameters* $\alpha \in \left[0, \frac{1}{2}\pi\right)$ *and* $\omega \in (0, 2\pi]$.

while the strip

$$l_1 - \beta_1 - 2 < \mathrm{Re}(\lambda) < l_2 - \beta_2 + 2,$$

contains the eigenvalues $\lambda_s, \lambda_{s+1}, \lambda_N$, *then* u *has the representation*

$$u = w + \sum_{j=s}^{N} \sum_{q=0}^{\kappa_j - 1} c_{j,q} u_{j,q}, \tag{5.1}$$

where $w \in V_{\beta_2}^{l_2+4,2}(\Omega)$, $\kappa_j \leq 2$ *is the algebraic multiplicity of* λ_j *and* $u_{j,q}$ *are the solutions of the problem (3.1) in* $\mathcal{K}_{(\alpha,\omega)}$ *given by formulas (3.30), (3.31), (3.32).*

Remark 5.2 *Sometimes in the literature one may find instead of (5.1) the following representation*

$$u = w + \chi(r) \sum_{j=s}^{N} \sum_{q=0}^{\kappa_j - 1} c_{j,q} u_{j,q},$$

where χ *is a cut-off function such that* $\chi \equiv 1$ *in the neighborhood of the cornerpoint* $r = 0$ *and is supported sufficiently close to it.*

The statement above is too general. We specify it in the following way. Recall that by Theorem 2.12 there exists a unique weak solution $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ of (1.4) with $f \in L^2(\Omega)$. Since $L^2(\Omega) = V_0^{0,2}(\Omega)$ and by Corollary 4.8 one has $\overset{\circ}{W}{}^{2,2}(\Omega) = \overset{\circ}{V}{}_0^{2,2}(\Omega) \subset V_0^{2,2}(\Omega)$, we conclude that for $f \in V_0^{0,2}(\Omega)$ we have $u \in V_0^{2,2}(\Omega)$. Then assuming more regularity for $f \in V_0^{0,2}(\Omega)$ we apply Theorem 5.1 to obtain:

Theorem 5.3 (Weighted space) *Let $f \in L^2(\Omega)$ and let $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ be a weak solution to (1.4).*

Suppose that $f \in V_\beta^{k,2}(\Omega)$, where $k \geq 0$, $\beta \leq k$, and that no eigenvalue λ_j of problem (3.4) lies on the lines

$$\mathrm{Re}(\lambda) = 0, \quad \mathrm{Re}(\lambda) = k - \beta + 2,$$

while the strip

$$0 < \mathrm{Re}(\lambda) < k - \beta + 2,$$

contains the eigenvalues $\lambda_1, \lambda_2, \lambda_N$. Then u has the representation

$$u = w + \sum_{j=1}^{N} \sum_{q=0}^{\kappa_j - 1} c_{j,q} u_{j,q}, \tag{5.2}$$

where $w \in V_\beta^{k+4,2}(\Omega)$, $\kappa_j \leq 2$ is the algebraic multiplicity of λ_j and $u_{j,q}$ are the solutions of the problem (3.1) in $\mathcal{K}_{(\alpha,\omega)}$ given by formulas (3.30), (3.31), (3.32).

The previous theorem gives the optimal regularity in the sense of Kondratiev's spaces. Due to Lemmas 4.4 and 4.7 we are able to formulate the regularity statement in terms of commonly used Sobolev spaces, namely, when

$$f \in \begin{cases} \overset{\circ}{W}{}^{k,2}(\Omega), & k \geq 1, \\ W^{k,2}(\Omega), & k \geq 0. \end{cases}$$

The result is as follows (for the convenience we arranged it as a table):

Theorem 5.4 (Sobolev space) *Let* $f \in L^2(\Omega)$ *and let* $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ *be a weak solution to (1.4).*

Suppose also that $f \in$	$\overset{\circ}{W}{}^{k,2}(\Omega),\ k \geq 1$	$W^{k,2}(\Omega),\ k \geq 0$
and that no eigenvalue λ_j *of the problem (3.4) lies on the lines,*	$\begin{cases} \mathrm{Re}(\lambda) = 0, \\ \mathrm{Re}(\lambda) = k+2 \end{cases}$	$\begin{cases} \mathrm{Re}(\lambda) = 0, \\ \mathrm{Re}(\lambda) = 2 \end{cases}$
while the strip	$0 < \mathrm{Re}(\lambda) < k+2$	$0 < \mathrm{Re}(\lambda) < 2$
contains the eigenvalues	$\lambda_1, \lambda_2, ..., \lambda_N.$	
Then u *has the representation*	$u = w + \sum_{j=1}^{N} \sum_{q=0}^{\kappa_j - 1} c_{j,q} u_{j,q},$	
where $w \in$	$W^{k+4,2}(\Omega)$	$W^{k+4,2}(\Omega, \|x\|^{2k}\, d\mu)$

$\kappa_j \leq 2$ *is the algebraic multiplicity of* λ_j *and* $u_{j,q}$ *are the solutions of the problem (3.1) in* $\mathcal{K}_{(\alpha,\omega)}$ *given by formulas (3.30), (3.31), (3.32).*

Remark 5.5 *Away from the corner also these results are optimal. Of course the optimal regularity near a corner can not be stated using just the standard Sobolev spaces* $W^{\ell,2}(\Omega)$.

Proof. For the first column one uses the imbeddings

$$\overset{\circ}{W}{}^{k,2}(\Omega) \subset V_0^{k,2}(\Omega) \text{ and } V_0^{k+4,2}(\Omega) \subset W^{k+4,2}(\Omega),$$

and for the second

$$W^{k,2}(\Omega) \subset V_k^{k,2}(\Omega) \text{ and } V_k^{k+4,2}(\Omega) \subset W^{k+4,2}(\Omega, |x|^{2k}\, d\mu).$$

■

In Theorems 5.3 and 5.4 the function w is sometimes called in the literature a regular part of the solution $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$, as it preserves the full regularity in a classical sense: $f \in W^{k,2}(\Omega) \Rightarrow w \in W^{k+4,2}(\Omega)$. The double sum $\sum_{j=1}^{N} \sum_{q=0}^{\kappa_j - 1} c_{j,q} u_{j,q}$ in the asymptotic representation for $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ is then a so-called singular part. From formulas (3.30), (3.31), (3.32) we know that the first term of the sum, depending on the algebraic multiplicity of λ_1, reads as

$$r^{\lambda_1+1}\Phi_{1,0}(\theta) \quad \text{or} \quad r^{\lambda_1+1}\left(\Phi_{1,1}(\theta) + \log(r)\Phi_{1,0}(\theta)\right). \tag{5.3}$$

Here, $\lambda_1 \in \mathbb{C}$ is the first eigenvalue of (3.4) such that $\mathrm{Re}(\lambda_1) > 0$ and $\Phi_{1,0}(\theta), \Phi_{1,1}(\theta) \in C^\infty [\alpha, \alpha + \omega]$, $\alpha \in \left[0, \frac{1}{2}\pi\right)$, $0 < \omega < 2\pi$. Hence, in order to complete our regularity analysis we will describe the regularity in Ω of the power type terms (5.3).

5.2 Regularity for the singular part of u

We start from the following lemma

Lemma 5.6 *Let* $\Psi(\theta) \in C^\infty [\alpha, \alpha + \omega]$ *be nontrivial and* $\alpha \in \left[0, \frac{1}{2}\pi\right)$, $0 < \omega < 2\pi$. *Let also* $\lambda \in \mathbb{C} \backslash \mathbb{Z}$ *with* $\mathrm{Re}(\lambda) > 0$. *Suppose that* $k \in \{0, 1, 2, 3, \dots\}$. *Then the following are equivalent:*

1. $r^{\lambda+1}\Psi(\theta) \in W^{k,2}(\Omega)$,
2. $r^{\lambda+1}\log(r)\Psi(\theta) \in W^{k,2}(\Omega)$,
3. $\mathrm{Re}(\lambda) + 1 > k - 1$.

Proof. If λ is not an integer, one finds for $R > 0$ that $r^{\lambda+1}\Psi(\theta) \in W^{k,2}(\Omega)$ means

$$\int_0^R \left(r^{\mathrm{Re}(\lambda)+1-k}\right)^2 r dr < \infty.$$

This holds true when $2(\mathrm{Re}(\lambda) + 1 - k) + 1 > -1$ yielding the third item. Inclusion of a term with $\log(r)$ will not alter the regularity, so the same result holds. ■

Remark 5.7 *In order to restrict the already heavy technical aspects we have not considered (weighted) Sobolev spaces with non-integer coefficients* k *and Hölder spaces. A similar result will hold for* k *is non-integer. Concerning Hölder spaces:*

$$r^{\lambda+1}\Psi(\theta) \in C^{k,\gamma}(\overline{\Omega}) \quad \text{for } \mathrm{Re}(\lambda) + 1 \geq k + \gamma \quad \text{with } k \in \mathbb{N}, \gamma \in [0,1).$$

For the second function it holds that

$$r^{\lambda+1}\log(r)\Psi(\theta) \in C^{k,\gamma}(\overline{\Omega}) \quad \text{for } \mathrm{Re}(\lambda) + 1 > k + \gamma \quad \text{with } k \in \mathbb{N}, \gamma \in [0,1).$$

In order to apply Lemma 5.6 to the power type solutions (5.3),

$$r^{\lambda_1+1}\Phi_{1,0}(\theta) \quad \text{or} \quad r^{\lambda_1+1}\left(\Phi_{1,1}(\theta)+\log(r)\Phi_{1,0}(\theta)\right),$$

one has to view $\mathrm{Re}(\lambda_1)$ as a function of the opening angle ω, that is, as $(0,2\pi] \ni \omega \mapsto \mathrm{Re}(\lambda_1(\omega))$.

In Figure 5.2 we plot the curves $(0,2\pi] \ni \omega \mapsto \mathrm{Re}(\lambda_1(\omega))$ for fixed $\alpha = \alpha_j = \frac{10}{180}\pi j$, $j = 0, ..., 9$ (note that the cases $\alpha = 0$ and $\alpha = \frac{1}{2}\pi$ are identical). The horizontal black segments intersecting the corresponding curve $\mathrm{Re}(\lambda_1(\omega))$ depict the levels $\mathrm{Re}(\lambda_1) = 2$ and $\mathrm{Re}(\lambda_1) = 1$, so that one is able to trace the behaviour of $\mathrm{Re}(\lambda_1(\omega))$ on interval $\omega \in (0,2\pi]$ between these levels (recall from Appendix D that for every $\alpha \in \left[0, \frac{1}{2}\pi\right)$ one has $\lambda_1(\pi) = 1$ and $\lambda_1(2\pi) = \frac{1}{2}$).

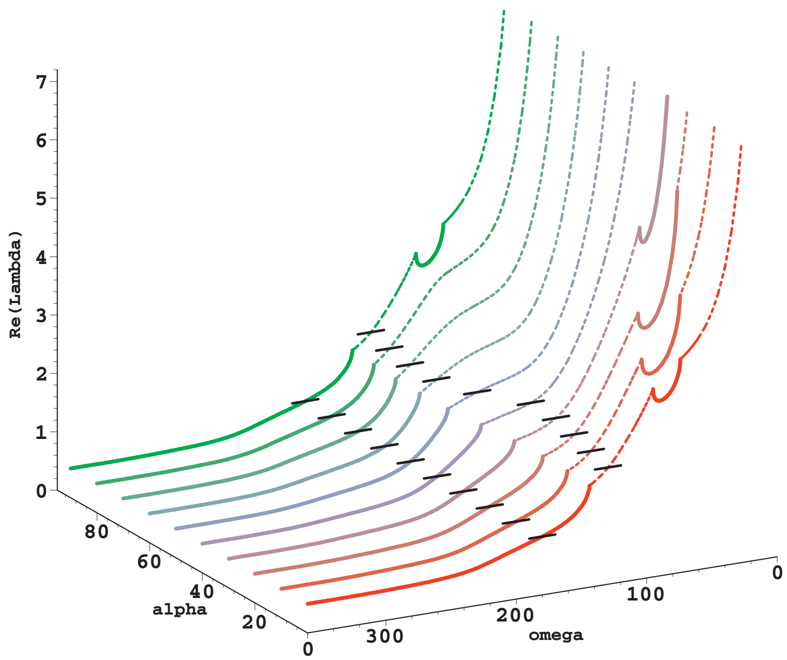

Figure 5.2: *The first eigenvalue* λ_1 *of the problem (3.4) when* $\alpha = \frac{10}{180}\pi j$, $j = 0, ..., 9$. *Dashed lines depict the real part of those* $\lambda_1 \in \mathbb{C}$, *solid lines are for purely real* λ_1. *The horizontal black segments intersecting each curve depict the levels* $\mathrm{Re}(\lambda_1) = 2$ *and* $\mathrm{Re}(\lambda_1) = 1$.

Based on the foregoing numerical approximations to λ_1, as well as, on the analytical estimates for λ_1 (obtained in Section 3.3 for the case $\alpha = 0$) we conclude that:

Claim 5.8 *For every $\alpha \in \left[0, \frac{1}{2}\pi\right)$ it holds that $(0, 2\pi] \ni \omega \mapsto \mathrm{Re}(\lambda_1(\omega))$ is a continuous function. Moreover,*

$$\begin{aligned} &\textit{for} \quad \omega \in (0, \omega_\star) &:& \quad \mathrm{Re}(\lambda_1) > 2, \\ &\textit{for} \quad \omega \in [\omega_\star, \pi) &:& \quad 2 \geq \mathrm{Re}(\lambda_1) > 1, \\ &\textit{for} \quad \omega \in [\pi, 2\pi] &:& \quad 1 \geq \lambda_1 \geq \tfrac{1}{2}. \end{aligned}$$

Here the $\omega_\star$ is the solution to the equation

$$\mathcal{P}\left(\alpha, \omega, 2 + i\xi\right) = 0,$$

on $\omega \in \left(\frac{100}{180}\pi, \frac{150}{180}\pi\right)$ for every $\alpha \in \left[0, \frac{1}{2}\pi\right)$ with $\mathcal{P}$ given by formula (3.7) and $\xi \in \mathbb{R}$.

For the corresponding fixed $\alpha = \frac{10}{180}\pi j$, $j = 0, ..., 9$ we compute the $\omega_\star$ and depict them in Table 5.1.

α	$\omega_\star/\pi$	$\omega_\star$ in degrees	α	$\omega_\star/\pi$	$\omega_\star$ in degrees
0	≈ 0.720	$\approx 129.6°$	$\frac{50}{180}\pi$	≈ 0.716	$\approx 129.0°$
$\frac{10}{180}\pi$	≈ 0.681	$\approx 122.6°$	$\frac{60}{180}\pi$	≈ 0.7786	$\approx 140.16°$
$\frac{20}{180}\pi$	≈ 0.640	$\approx 115.3°$	$\frac{70}{180}\pi$	≈ 0.7784	$\approx 140.11°$
$\frac{30}{180}\pi$	≈ 0.605	$\approx 108.9°$	$\frac{80}{180}\pi$	≈ 0.754	$\approx 135.8°$
$\frac{40}{180}\pi$	≈ 0.600	$\approx 108.1°$	$\frac{1}{2}\pi$	the same as for $\alpha = 0$	

Table 5.1: *Approximations for the $\omega_\star$ from Claim 5.8 when $\alpha = \frac{10}{180}\pi j$, $j = 0, ..., 9$.*

For the numerical results from Claim 5.8, it follows by Lemma 5.6 that:

Claim 5.9 *For every $\alpha \in \left[0, \frac{1}{2}\pi\right)$ it holds that power type term in (5.3) belongs to*

$$\begin{array}{lll} W^{4,2}(\Omega) & \text{when} & \omega \in (0, \omega_\star), \\ W^{3,2}(\Omega) & \text{when} & \omega \in [\omega_\star, \pi), \\ C^\infty\left(\overline{\Omega}\right) & \text{when} & \omega = \pi, \\ W^{2,2}(\Omega) & \text{when} & \omega \in (\pi, 2\pi]. \end{array}$$

Remark 5.10 *When the opening angle $\omega = \pi$ we know from Appendix D that every term $u_{j,q}$ of the singular part $\sum_{j=1}^{N} \sum_{q=0}^{\kappa_j - 1} c_{j,q} u_{j,q}$ is a polynomial in x, y of order $\lambda_j + 1$. That is, for every λ_j, $j \in \mathbb{N}^+$ we have*

$$r^{\lambda_j + 1} \Phi_j(\theta) = P_{\lambda_j + 1}(x, y) \in C^\infty\left(\overline{\Omega}\right). \tag{5.4}$$

For non-polynomials the result in Lemma 5.6 even holds for $\lambda \in \mathbb{N}$.

5.3 Corollary

For the numerical results from Claim 5.9, it holds by Theorem 5.4 that:

Corollary 5.11 *Let $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ be a weak solution of problem (1.4) with $f \in L^2(\Omega)$. Then*

$$\begin{array}{llll} \text{for} & \omega \in (0, \omega_\star) & : & u \in W^{4,2}(\Omega), \\ \text{for} & \omega \in (\omega_\star, \pi) & : & u \in W^{3,2}(\Omega), \\ \text{for} & \omega = \pi & : & u \in W^{4,2}(\Omega), \\ \text{for} & \omega \in (\pi, 2\pi] & : & u \in W^{2,2}(\Omega). \end{array}$$

Here $\omega_\star$ is as in Claim 5.8.

Remark 5.12 *For opening angle $\omega = \omega_\star$ we have $\operatorname{Re}(\lambda_1) = 2$ and hence Theorem 5.4 does not apply. Nevertheless, assuming $f \in L^2(\Omega)$ to be more regular, e.g. in $V_0^{1,2}(\Omega)$ or $\overset{\circ}{W}{}^{1,2}(\Omega)$, we may show that*

$$\text{for} \quad \omega = \omega_\star \quad : \quad u \in W^{3,2}(\Omega).$$

Proof. The proof is straightforward. By Theorem 5.4 if $f \in L^2(\Omega)$, the solution $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ of problem (5.3) reads as

$$u = w + \sum_{0<\mathrm{Re}(\lambda_j)<2} \sum_{q=0}^{\kappa_j - 1} c_{j,q} u_{j,q}, \tag{5.5}$$

with $w \in W^{4,2}(\Omega)$.

The summation condition is $0 < \mathrm{Re}\,(\lambda_j) < 2$ and hence the sum in (5.5) has no terms on the interval $\omega \in (0, \omega_\star)$. This yields that $u \in W^{4,2}(\Omega)$ when the opening angle $\omega \in (0, \omega_\star)$.

For $\omega \in (\omega_\star, \pi) \cup (\pi, 2\pi]$ we immediately deduce by Claim 5.9 that $u \in W^{3,2}(\Omega)$ for $\omega \in (\omega_\star, \pi)$ and $u \in W^{2,2}(\Omega)$ for $\omega \in (\pi, 2\pi]$.

Finally, for $\omega = \pi$ due to (5.4) the singular part is of $C^\infty\left(\overline{\Omega}\right)$ and hence $u \in W^{4,2}(\Omega)$ in this case. ■

5.4 Comparison with the bilaplacian case

Let us recall that in Figure 3.2 in Section 3.3 we compared the eigenvalues λ_j of problem (3.4) for $\mathcal{L}$ induced, respectively, by the operators $L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ and $L = \Delta^2$. The parameter $\alpha \in \left[0, \frac{1}{2}\pi\right)$ has been chosen to be $\alpha = 0$. In Figure 5.3 below we bring more general picture: for every $\alpha = \alpha_j = \frac{10}{180}\pi j$, $j = 0, ..., 9$ we plot $(0, 2\pi] \ni \omega \mapsto \mathrm{Re}(\lambda_1(\omega))$ for the cases $L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ and $L = \Delta^2$ simultaneously. For convenience, in the caption we denote the corresponding λ_1 as $\lambda_{1,Grid}$ and $\lambda_{1,Bilaplace}$. The horizontal black segments intersecting the curves depict the level $\mathrm{Re}(\lambda_1) = 1$, where at $\omega = \pi$ it holds that $\lambda_{1,Grid} = \lambda_{1,Bilaplace}$.

The remarkable observation is that for every $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and the opening angle $\omega > \pi$, the curves for $\lambda_{1,Grid}$ and $\lambda_{1,Bilaplace}$ behave similarly.

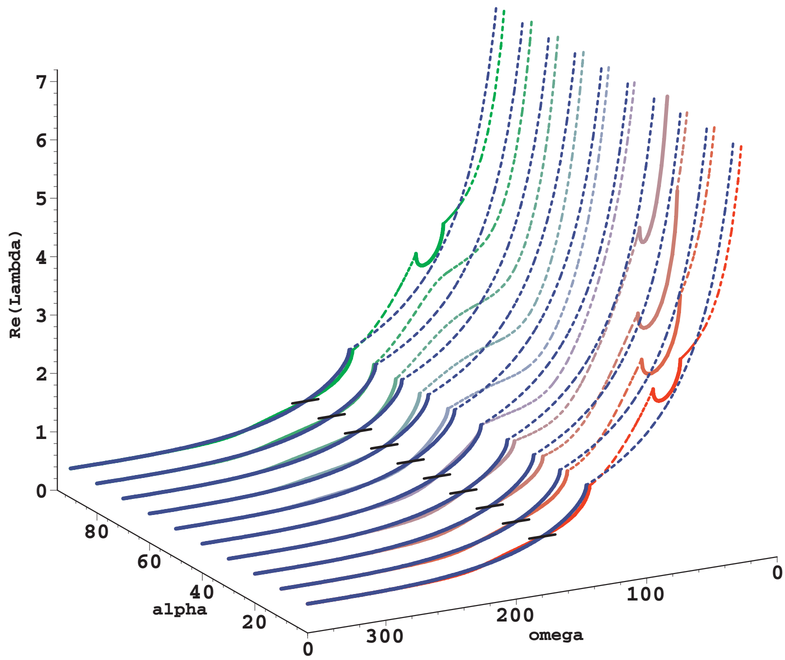

Figure 5.3: *The first eigenvalue* λ_1 *of the problem (3.4) when* $\alpha = \frac{10}{180}\pi j$*,* $j = 0, \dots, 9$*. For* $\mathcal{L}$ *in (3.4) induced, respectively, by the operators* $L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ *and* $L = \Delta^2$*, we have* $\lambda_{1,Grid}$ *(colored) and* $\lambda_{1,Bilaplace}$ *(in blue). The horizontal black segments depict the level* $Re(\lambda_1) = 1$*, where at* $\omega = \pi$ *it holds that the values* $\lambda_{1,Grid}$ *and* $\lambda_{1,Bilaplace}$ *are identical.*

Chapter 6

System approach to the clamped grid problem

A solution of a fourth order elliptic boundary value problem can be obtained either directly or by considering a system of second order equations. The system approach is usually favorized for numerical methods since one may use piecewise linear $C^{0,1}$ finite elements, instead of piecewise quadratic $C^{1,1}$ elements needed for the direct approach. Thus, in [13] the authors compare both approaches for the clamped plate problem and proved that for planar domains with reentrant (i.e. concave) corners the system approach failed to produce the correct solution. In this chapter we will study the similar question for our clamped grid problem. The material of Chapter 5 will be of use in this study.

6.1 Outline and settings

Let $\Omega \subset \mathbb{R}^2$ be open and bounded. In Chapter 2 we show that for any $f \in L^2(\Omega)$ the clamped grid problem

$$\begin{cases} u_{xxxx} + u_{yyyy} = f & \text{in} \quad \Omega, \\ u = \frac{\partial}{\partial n} u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{6.1}$$

has a unique weak solution $\overset{\circ}{W}{}^{2,2}(\Omega)$. This weak solution is a minimizer of the energy functional

$$E[u] = \int_\Omega \left(\tfrac{1}{2}\left(u_{xx}^2 + u_{yy}^2\right) - fu\right) dxdy \quad \text{over } \overset{\circ}{W}{}^{2,2}(\Omega),$$

and a solution to the weak Euler-Lagrange equation

$$\int_\Omega (u_{xx}\varphi_{xx} + u_{yy}\varphi_{yy} - f\varphi)\, dxdy = 0 \quad \text{for all} \quad \varphi \in \overset{\circ}{W}{}^{2,2}(\Omega). \tag{6.2}$$

Alternatively one may consider a system of second order equations. Indeed, due to the factorization

$$\tfrac{\partial^4}{\partial x^4} + \tfrac{\partial^4}{\partial y^4} = \left(-\Delta - \sqrt{2}\tfrac{\partial^2}{\partial x \partial y}\right)\left(-\Delta + \sqrt{2}\tfrac{\partial^2}{\partial x \partial y}\right),$$

problem (6.1) will be split into:

$$\begin{cases} -\Delta v - \sqrt{2} v_{xy} = f & \text{in} \quad \Omega, \\ -\Delta u + \sqrt{2} u_{xy} = v & \text{in} \quad \Omega, \\ u = \frac{\partial}{\partial n} u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{6.3}$$

A weak solution for the system is a pair $(v, u) \in W^{1,2}(\Omega) \times \overset{\circ}{W}{}^{1,2}(\Omega)$ such that

$$\int_\Omega \left[\left(\tfrac{\partial v}{\partial x} + \tfrac{\sqrt{2}}{2}\tfrac{\partial v}{\partial y}\right)\tfrac{\partial \psi}{\partial x} + \left(\tfrac{\partial v}{\partial y} + \tfrac{\sqrt{2}}{2}\tfrac{\partial v}{\partial x}\right)\tfrac{\partial \psi}{\partial y} + \right.$$

$$\left. + \left(\tfrac{\partial u}{\partial x} - \tfrac{\sqrt{2}}{2}\tfrac{\partial u}{\partial y}\right)\tfrac{\partial \phi}{\partial x} + \left(\tfrac{\partial u}{\partial y} - \tfrac{\sqrt{2}}{2}\tfrac{\partial u}{\partial x}\right)\tfrac{\partial \phi}{\partial y} - f\psi - v\phi\right] dxdy = 0,$$

$$\text{for all} \quad (\phi, \psi) \in W^{1,2}(\Omega) \times \overset{\circ}{W}{}^{1,2}(\Omega). \tag{6.4}$$

Remark 6.1 *Assuming for a moment (v, u) to be more regular, an integration by parts in (6.4) leads to*

$$0 = \int_\Omega \left(-\Delta v - \sqrt{2}\tfrac{\partial^2 v}{\partial x \partial y} - f\right)\psi dxdy + \int_\Omega \left(-\Delta u + \sqrt{2}\tfrac{\partial^2 u}{\partial x \partial y} - v\right)\phi dxdy +$$

$$+\int\limits_{\partial\Omega} \left(n_1\left(\frac{\partial v}{\partial x}+\frac{\sqrt{2}}{2}\frac{\partial v}{\partial y}\right)+n_2\left(\frac{\partial v}{\partial y}+\frac{\sqrt{2}}{2}\frac{\partial v}{\partial x}\right)\right)\psi dS+$$

$$+\int\limits_{\partial\Omega} \left(n_1\left(\frac{\partial u}{\partial x}-\frac{\sqrt{2}}{2}\frac{\partial u}{\partial y}\right)+n_2\left(\frac{\partial u}{\partial y}-\frac{\sqrt{2}}{2}\frac{\partial u}{\partial x}\right)\right)\phi dS,$$

where $n=(n_1,n_2)$ is the unit outward normal vector on $\partial\Omega$. The first two integrals formally results in the equations $-\Delta v-\sqrt{2}\frac{\partial^2 v}{\partial x\partial y}=f$, $-\Delta u+\sqrt{2}\frac{\partial^2 u}{\partial x\partial y}=v$ in Ω. The first boundary integral vanishes due to $\psi=0$ on $\partial\Omega$. The latter boundary integral being equal to zero results in second boundary condition $\frac{\partial u}{\partial n}=0$ on $\partial\Omega$. Indeed, in terms of the normal and tangential derivatives one has $\frac{\partial u}{\partial x}=n_1\frac{\partial u}{\partial n}-n_2\frac{\partial u}{\partial \tau}$ and $\frac{\partial u}{\partial y}=n_2\frac{\partial u}{\partial n}+n_1\frac{\partial u}{\partial \tau}$. Moreover, $u=0$ on $\partial\Omega$ and hence $\frac{\partial u}{\partial \tau}=0$ on $\partial\Omega$. As a result we deduce that

$$0=\int\limits_{\partial\Omega} \left(n_1\left(\frac{\partial u}{\partial x}-\frac{\sqrt{2}}{2}\frac{\partial u}{\partial y}\right)+n_2\left(\frac{\partial u}{\partial y}-\frac{\sqrt{2}}{2}\frac{\partial u}{\partial x}\right)\right)\phi dS=$$

$$=\int\limits_{\partial\Omega} \left(1-\sqrt{2}n_1n_2\right)\frac{\partial u}{\partial n}\phi dS. \tag{6.5}$$

Since the normal $(n_1,n_2)=(\cos(\gamma),\sin(\gamma))$ one finds that $n_1n_2=\frac{1}{2}\sin(2\gamma)\in\left[-\frac{1}{2},\frac{1}{2}\right]$ and hence that $1-\sqrt{2}n_1n_2\geq 1-\frac{\sqrt{2}}{2}>0$. The factor in (6.5) is positive and hence we obtain $\frac{\partial u}{\partial n}=0$ on $\partial\Omega$ as a second boundary condition.

The reason why we want to use the system approach (6.3) in order to solve the original fourth order problem (6.1) is that such an approach has an advantage from the computational point of view. Indeed, in order to approximate (v,u) to (6.3) one may use piecewise linear $C^{0,1}$ finite elements, which are implemented in a large number of standard programming libraries. The solution u to (6.1) has to be approximated by piecewise quadratic $C^{1,1}$ elements in order to avoid δ-type distributions behavior over the edges of adjacent elements. C. Davini in [8] follows an alternative approach by introducing a numerical way of treating distributions over mesh-sides which allowed him to stay with the piecewise linear $C^{0,1}$ elements.

The solutions u of both problems, however, may not coincide and whether it happens or not depends on the properties of the domain Ω. In particular, it holds that u of (6.1) and u in the solution (v,u) of (6.3) are identical when

Ω is smooth or has a convex corner. For the case when a reentrant (concave) corner is presented in Ω, this does not hold true. Below we will give a detailed explanation and proof to these results.

6.2 System approach

Let us set

$$\mathcal{H} = W^{1,2}(\Omega) \times \overset{\circ}{W}{}^{1,2}(\Omega).$$

First we have to find out the existence of a solution $(v, u) \in \mathcal{H}$ to the integral equation (6.4). Unfortunately, one is not able to recover a functional $H(v, u)$ such that $\partial H(v, u; \phi, \psi)$ would equal to the left-hand side of (6.4), so we can not talk about (v, u) of (6.4) in terms of a stationary point of a certain functional. Instead, we will show that if we assume some additional regularity of the corresponding solutions, the solution $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ of (6.2) and a solution $(v, u) \in \mathcal{H}$ of (6.4) will coincide. Then there will be a one to one correspondence implying that (v, u) exists and is unique.

Proposition 6.2 *Let $\Omega \subset \mathbb{R}^2$ be open and bounded and let $f \in L^2(\Omega)$.*

1. *If $(v, u) \in \mathcal{H}$ is a solution of (6.4) and $u \in W^{2,2}(\Omega)$ then u satisfies the equation (6.2).*

2. *If $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ satisfies (6.2) and $-\Delta u + \sqrt{2}u_{xy} \in W^{1,2}(\Omega)$, then $\left(-\Delta u + \sqrt{2}u_{xy}, u\right)$ is a solution of (6.4).*

Proof.

1. Let $(v, u) \in \mathcal{H}$ be a solution of of (6.4). Then we have

$$\int\limits_\Omega \left[\left(\frac{\partial v}{\partial x} + \frac{\sqrt{2}}{2}\frac{\partial v}{\partial y}\right)\frac{\partial \psi}{\partial x} + \left(\frac{\partial v}{\partial y} + \frac{\sqrt{2}}{2}\frac{\partial v}{\partial x}\right)\frac{\partial \psi}{\partial y} - f\psi\right] dxdy = 0,$$
$$\text{for all } \psi \in \overset{\circ}{W}{}^{1,2}(\Omega), \tag{6.6}$$

$$\int\limits_\Omega \left[\left(\frac{\partial u}{\partial x} - \frac{\sqrt{2}}{2}\frac{\partial u}{\partial y}\right)\frac{\partial \phi}{\partial x} + \left(\frac{\partial u}{\partial y} - \frac{\sqrt{2}}{2}\frac{\partial u}{\partial x}\right)\frac{\partial \phi}{\partial y} - v\phi\right] dxdy = 0,$$
$$\text{for all } \phi \in W^{1,2}(\Omega). \tag{6.7}$$

If $u \in W^{2,2}(\Omega)$, the for all $\phi \in W^{1,2}(\Omega)$ we derive from (6.7) that

$$0 = \int\limits_{\Omega} \left(-\Delta u + \sqrt{2}\tfrac{\partial^2 u}{\partial x \partial y} - v\right) \phi dxdy +$$

$$+ \int\limits_{\partial\Omega} \left(n_1 \left(\tfrac{\partial u}{\partial x} - \tfrac{\sqrt{2}}{2}\tfrac{\partial u}{\partial y}\right) + n_2 \left(\tfrac{\partial u}{\partial y} - \tfrac{\sqrt{2}}{2}\tfrac{\partial u}{\partial x}\right)\right) \phi dS,$$

which implies $-\Delta u + \sqrt{2}\frac{\partial^2 u}{\partial x \partial y} = v$ in Ω and $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ in a weak sense (for the latter condition see formula (6.5) in Remark 6.1). Hence, $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$. Moreover, by density,

$$\int\limits_{\Omega} \left(-\Delta u + \sqrt{2}\tfrac{\partial^2 u}{\partial x \partial y} - v\right) \phi dxdy = 0, \quad \text{for all} \quad \phi \in L^2(\Omega).$$

Taking $\psi \in \overset{\circ}{W}{}^{2,2}(\Omega)$ and using $\phi = -\Delta\psi - \sqrt{2}\frac{\partial^2 \psi}{\partial x \partial y}$ we find with (6.6) that

$$0 = \int\limits_{\Omega} \left(-\Delta u + \sqrt{2}\tfrac{\partial^2 u}{\partial x \partial y} - v\right)\left(-\Delta\psi - \sqrt{2}\tfrac{\partial^2 \psi}{\partial x \partial y}\right) dxdy =$$

$$= \int\limits_{\Omega} \left(u_{xx}\psi_{xx} + u_{yy}\psi_{yy} - f\psi\right) dxdy.$$

2. Let us define the left-hand side of equation (6.4) as $A(v, u; \phi, \psi)$. Here we will prove that if (6.2) holds true and $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$, $-\Delta u + \sqrt{2}u_{xy} \in W^{1,2}(\Omega)$ then

$$A(-\Delta u + \sqrt{2}u_{xy}, u; \phi, \psi) = 0, \quad \text{for all} \quad (\phi, \psi) \in \mathcal{H}.$$

Indeed, for $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ and $-\Delta u + \sqrt{2}u_{xy} \in W^{1,2}(\Omega)$ it holds that

$$0 = \int\limits_{\Omega} \left(u_{xx}\psi_{xx} + u_{yy}\psi_{yy} - f\psi\right) dxdy =$$

$$= \int\limits_{\Omega} \left[\left(\tfrac{\partial}{\partial x}\left(-\Delta u + \sqrt{2}u_{xy}\right) + \tfrac{\sqrt{2}}{2}\tfrac{\partial}{\partial y}\left(-\Delta u + \sqrt{2}u_{xy}\right)\right)\tfrac{\partial \psi}{\partial x} + \right.$$

$$+\left(\frac{\partial}{\partial y}\left(-\Delta u+\sqrt{2}u_{xy}\right)+\frac{\sqrt{2}}{2}\frac{\partial}{\partial x}\left(-\Delta u+\sqrt{2}u_{xy}\right)\right)\frac{\partial\psi}{\partial y}-f\psi\Big]\,dxdy,$$

$$\text{for all}\quad \psi\in \overset{\circ}{W}{}^{2,2}(\Omega).$$

Since $\overset{\circ}{W}{}^{2,2}(\Omega)$ is dense in $\overset{\circ}{W}{}^{1,2}(\Omega)$ one has for the above equation that

$$0=\int\limits_{\Omega}\Big[\left(\frac{\partial}{\partial x}\left(-\Delta u+\sqrt{2}u_{xy}\right)+\frac{\sqrt{2}}{2}\frac{\partial}{\partial y}\left(-\Delta u+\sqrt{2}u_{xy}\right)\right)\frac{\partial\psi}{\partial x}+$$

$$+\left(\frac{\partial}{\partial y}\left(-\Delta u+\sqrt{2}u_{xy}\right)+\frac{\sqrt{2}}{2}\frac{\partial}{\partial x}\left(-\Delta u+\sqrt{2}u_{xy}\right)\right)\frac{\partial\psi}{\partial y}-f\psi\Big]\,dxdy,$$

$$\text{for all}\quad \psi\in \overset{\circ}{W}{}^{1,2}(\Omega).$$

Then it follows that

$$A(-\Delta u+\sqrt{2}u_{xy},u;\phi,\psi)=$$

$$=\int\limits_{\Omega}\Big[\left(\frac{\partial}{\partial x}\left(-\Delta u+\sqrt{2}u_{xy}\right)+\frac{\sqrt{2}}{2}\frac{\partial}{\partial y}\left(-\Delta u+\sqrt{2}u_{xy}\right)\right)\frac{\partial\psi}{\partial x}+$$

$$+\left(\frac{\partial}{\partial y}\left(-\Delta u+\sqrt{2}u_{xy}\right)+\frac{\sqrt{2}}{2}\frac{\partial}{\partial x}\left(-\Delta u+\sqrt{2}u_{xy}\right)\right)\frac{\partial\psi}{\partial y}+$$

$$+\left(\frac{\partial u}{\partial x}-\frac{\sqrt{2}}{2}\frac{\partial u}{\partial y}\right)\frac{\partial\phi}{\partial x}+\left(\frac{\partial u}{\partial y}-\frac{\sqrt{2}}{2}\frac{\partial u}{\partial x}\right)\frac{\partial\phi}{\partial y}$$

$$-f\psi-\left(-\Delta u+\sqrt{2}u_{xy}\right)\phi\Big]\,dxdy=$$

$$=\int\limits_{\Omega}\Big[\left(\frac{\partial u}{\partial x}-\frac{\sqrt{2}}{2}\frac{\partial u}{\partial y}\right)\frac{\partial\phi}{\partial x}+\left(\frac{\partial u}{\partial y}-\frac{\sqrt{2}}{2}\frac{\partial u}{\partial x}\right)\frac{\partial\phi}{\partial y}-\left(-\Delta u+\sqrt{2}u_{xy}\right)\phi\Big]\,dxdy=$$

$$=\int\limits_{\partial\Omega}\left(1-\sqrt{2}n_1n_2\right)\frac{\partial u}{\partial n}\phi\,dS=0,$$

holds for all $(\phi,\psi)\in\mathcal{H}$.

■

6.3 Comparing minimizing and system approach

Let us recall that the two different types of weak solution we obtain for problem (6.1).

- Minimizer of the energy functional $E[u] = \int\limits_\Omega \left(\frac{1}{2}\left(u_{xx}^2 + u_{yy}^2\right) - fu\right) dxdy$ over $\overset{\circ}{W}{}^{2,2}(\Omega)$ yields a solution $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$.

- A solution of equation (6.4) gives a solution $u \in \overset{\circ}{W}{}^{1,2}(\Omega)$ with $-\Delta u + \sqrt{2}u_{xy} \in W^{1,2}(\Omega)$ and

$$\int\limits_\Omega \left(-\Delta u + \sqrt{2}\frac{\partial^2 u}{\partial x \partial y}\right) \phi dxdy =$$

$$= \int\limits_\Omega \left(\left(\frac{\partial u}{\partial x} - \frac{\sqrt{2}}{2}\frac{\partial u}{\partial y}\right)\frac{\partial \phi}{\partial x} + \left(\frac{\partial u}{\partial y} - \frac{\sqrt{2}}{2}\frac{\partial u}{\partial x}\right)\frac{\partial \phi}{\partial y}\right) dxdy.$$

  The last equation replaces the zero normal derivative of u.

Here we will compare these solutions by specifying the properties of the domain Ω in which problem (6.1) is posed. To be more precise, let Ω be defined as in Condition 1.2 (the sketch of Ω is depicted in Figure 6.1 below).

We claim the following.

Lemma 6.3 *The minimizer u of the clamped grid problem (6.1) satisfies $-\Delta u + \sqrt{2}u_{xy} \in W^{1,2}(\Omega)$ if and only if $\omega \leq \pi$.*

Proof. The result follows by Corollary 5.11 from Chapter 5. ∎

Corollary 6.4 *Assume that the domain Ω as in above. Then equation (6.4) possesses a solution in $\mathcal{H}$.*

Proof. It is enough to apply Lemma 6.3 with Proposition 6.2 to obtain that the unique minimizer of $E[u] = \int\limits_\Omega \left(\frac{1}{2}\left(u_{xx}^2 + u_{yy}^2\right) - fu\right) dxdy$ is a solution of (6.4). ∎

Now we proceed with the comparison of the two solutions in the case when $\omega > \pi$, i.e. when Ω has a concave (reentrant) corner. We will see that the system approach will, in general, not agree with the minimization problem.

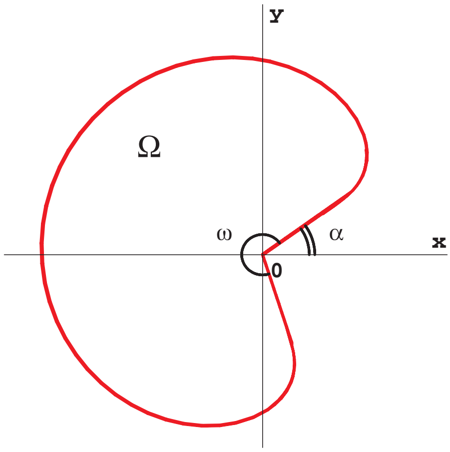

Figure 6.1: *Example for Ω with two governing parameters $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and $\omega \in (0, 2\pi]$.*

Theorem 6.5 *Let Ω be the domain as in Figure 6.1 with opening angle $\omega > \pi$. Then, there exists a right hand side f such that the unique minimizer of $E[u] = \int\limits_{\Omega} \left(\frac{1}{2}\left(u_{xx}^2 + u_{yy}^2\right) - fu\right) dxdy$ is not a solution of (6.4) in $\mathcal{H}$.*

Proof. Let $\chi \in C^{\infty}[0, \epsilon]$ be a cut-off function such that $\chi \equiv 1$ in the neighborhood of the cornerpoint $r = 0$.

Consider the singular solutions $\psi_j(r, \theta) := r^{\lambda_j + 1}\Phi_j(\theta)$ of problem (6.1). Recall that they solve the homogeneous problem (6.1) in an infinite cone with the opening angle ω. We know that

$$\int_0^1 \left(r^{\lambda+1-k}\right)^2 rdr < \infty \Leftrightarrow k < \lambda + 2$$

and similarly if $\ln r$ is included, so one finds that for $k \in \{0, 1, 2, 3, \}$,

$$\psi_j(r, \theta) := r^{\lambda_j+1}\Phi_j(\theta) \in W^{k,2}(\Omega) \Leftrightarrow k < \lambda_j + 2.$$

If for simplicity we assume that $\alpha = 0$ in Figure 6.1, then the relation between ω and λ_j be as in in Figure 6.2.

Let ψ_1 be the first positive singular eigenfunction. We set

$$u_1 = \chi\psi_1.$$

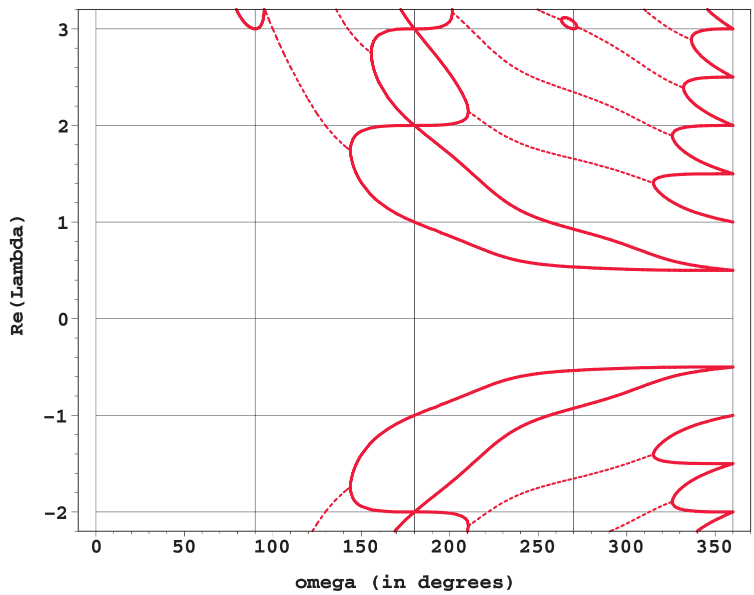

Figure 6.2: *The relation between opening angle ω and the eigenvalues λ_j in the singular solution of problem (6.1) when $\alpha = 0$.*

One finds $u_1 = 0$ on $\partial\Omega$ and $\frac{\partial u_1}{\partial n} = 0$ on $\partial\Omega \setminus \{0\}$ and since $\psi_1 \in W^{2,2}(\Omega)$ and $\psi_1 \notin W^{3,2}(\Omega)$ for $\omega > \pi$ (see Figure 6.2) we find

$$u_1 \in \overset{\circ}{W}{}^{2,2}(\Omega) \backslash W^{3,2}(\Omega).$$

Since $u_1 = \psi_1$ in the neighborhood of $r = 0$ and ψ_1 is such that $\left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}\right)\psi_1 = 0$ in Ω it follows that

$$f_1 := \left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}\right) u_1 \in C^\infty(\overline{\Omega}).$$

Taking $f = f_1$ we have u_1 as the unique minimizer of the functional $E[u] = \int_\Omega \left(\frac{1}{2}\left(u_{xx}^2 + u_{yy}^2\right) - fu\right) dxdy$.

Let us assume that $(-\Delta u_1 + \sqrt{2}\frac{\partial u_1}{\partial x \partial y}, u_1)$ is a solution of equation (6.4). Then

$$u_1 \in \overset{\circ}{W}{}^{1,2}(\Omega) \quad \text{and} \quad -\Delta u_1 + \sqrt{2}\frac{\partial u_1}{\partial x \partial y} \in W^{1,2}(\Omega). \tag{6.8}$$

Applying Theorem 5.1 from Chapter 5 adapted to the space $\overset{\circ}{W}{}^{1,2}(\Omega)$, one finds that the solution $u_1 \in \overset{\circ}{W}{}^{1,2}(\Omega)$ possesses the asymptotic representation

$$u_1 = \tilde{u}_1 + c_{-2}\psi_{-2} + c_{-1}\psi_{-1} + c_1\psi_1 + c_2\psi_2 \left(+ \text{higher order terms} \right),$$

where $\tilde{u}_1 \in W^{4,2}(\Omega)$ and

$$\psi_{-2}, \psi_{-1} \in W^{1,2}(\Omega)\backslash W^{2,2}(\Omega) \quad \text{and} \quad \psi_1, \psi_2 \in W^{2,2}(\Omega)\backslash W^{3,2}(\Omega).$$

The higher order terms lie in $W^{3,2}(\Omega)$. Moreover, since the functions ψ_i are such that

$$\left(-\Delta + \sqrt{2}\frac{\partial}{\partial x \partial y}\right)\psi_i \neq 0,$$

one obtains

$$\left(-\Delta + \sqrt{2}\frac{\partial}{\partial x \partial y}\right)\psi_{-2}, \left(-\Delta + \sqrt{2}\frac{\partial}{\partial x \partial y}\right)\psi_{-1} \notin L^2(\Omega),$$

and

$$\left(-\Delta + \sqrt{2}\frac{\partial}{\partial x \partial y}\right)\psi_1, \left(-\Delta + \sqrt{2}\frac{\partial}{\partial x \partial y}\right)\psi_2 \notin W^{1,2}(\Omega),$$

and that they do not cancel each others singularity near 0. Hence, if $-\Delta u_1 + \sqrt{2}\frac{\partial u_1}{\partial x \partial y} \in W^{1,2}(\Omega)$, then by the second condition in (6.8) one is forced to set $c_{-2} = c_{-1} = c_1 = c_2 = 0$. One finds $u_1 \in W^{3,2}(\Omega) \cap \overset{\circ}{W}{}^{2,2}(\Omega)$, a contradiction. ■

Remark 6.6 *The asymptotic analysis of the original fourth order problem and the system approach produces the same boundary value problems and imposes naturally the same boundary conditions on the angular eigenfunctions. Thus produces the exact same singular eigenfunctions ψ_j.*

Remark 6.7 *Following the argumentation of the previous theorem, one sees that when $\omega \leq \pi$, the stationary point is indeed in $W^{2,2}(\Omega)$. Thus, Proposition 6.2 implies that it is unique.*

Appendix A

Algebraic transformation

The main result of this appendix is that for every fourth order elliptic operator

$$L = \frac{\partial^4}{\partial x^4} + b_1 \frac{\partial^4}{\partial x^3 \partial y} + b_2 \frac{\partial^4}{\partial x^2 \partial y^2} + b_3 \frac{\partial^4}{\partial x \partial y^3} + \frac{\partial^4}{\partial y^4}, \tag{A.1}$$

with $b_j \in \mathbb{R}$, $j = 1, 2, 3$ there exists precisely one $a \in [1, +\infty)$ such that L is algebraically equivalent to the operator

$$L_a = \frac{\partial^4}{\partial x^4} + 2a \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}. \tag{A.2}$$

This means that one may always find an appropriate linear coordinate transformation such that due to this transformation L turns into L_a.

A.1 Ellipticity of operator L_a

Let $x \in \mathbb{R}^n$. Recall (see e.g. [25, Chapter 2, Definition 1.1]) that

Definition A.1 *The differential operator of order* $2m$*,*

$$A\left(\frac{\partial}{\partial x}\right) = \sum_{|\gamma|=2m} b_\gamma \left(\frac{\partial}{\partial x}\right)^\gamma, \tag{A.3}$$

where $b_\gamma \in \mathbb{R}$*, is said to be elliptic if its symbol satisfies*

$$A(\xi) \neq 0 \quad \textit{for all} \quad \xi \in \mathbb{R}^n \backslash \{0\}.$$

From the ellipticity it follows that $A(\xi) \geq c\,|\xi|^{2m}$ for all $\xi \in \mathbb{R}^n$, where c is a positive constant independent of ξ. Moreover, by Proposition 1.1 in [25, Chapter 2, Definition 1.1] it follows that if A is elliptic then the equation

$$\mathbb{C} \ni \tau \mapsto A(\xi + \tau\eta) = 0,$$

where $\xi, \eta \in \mathbb{R}^n$ are linearly independent vectors, does not have any real roots.

Then the following holds.

Lemma A.2 *The operator* $L_a = \frac{\partial^4}{\partial x^4} + 2a\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$ *is elliptic iff* $a > -1$.

Proof. For $a = 0$ it holds that

$$\xi_1^4 + \xi_2^4 = \tfrac{1}{2}\left(\xi_1^2 + \xi_2^2\right)^2 + \tfrac{1}{2}\left(\xi_1^2 - \xi_2^2\right)^2 \geq \tfrac{1}{2}\,|\xi|^4,$$

for all $\xi \in \mathbb{R}^2$. Then for $a > 0$ we it is straightforwardly that

$$\xi_1^4 + 2a\xi_1^2\xi_2^2 + \xi_2^4 \geq \xi_1^4 + \xi_2^4 \geq \tfrac{1}{2}\,|\xi|^4.$$

Let $a < 0$. We set $b = -a$ (that is, $b > 0$) and by Cauchy's inequality obtain

$$\xi_1^4 - 2b\xi_1^2\xi_2^2 + \xi_2^4 \geq \xi_1^4 + \xi_2^4 - b\left(\xi_1^4 + \xi_2^4\right) = (1-b)\left(\xi_1^4 + \xi_2^4\right) \geq \tfrac{1}{2}(1-b)\,|\xi|^4.$$

Hence, $1 - b > 0$ and as a result, $0 < b < 1$. In terms of a this yields $-1 < a < 0$. ∎

Let $a \in (-1, +\infty)$ in L_a. The corresponding characteristic equation

$$\tau^4 + 2a\tau^2 + 1 = 0,$$

has four solutions $\tau_1, \tau_2, \overline{\tau_1}, \overline{\tau_2}$, where

$$\begin{aligned} \tau_1 &= -\frac{\sqrt{2}}{2}\left(\sqrt{1-a} - i\sqrt{1+a}\right), \\ \tau_2 &= \frac{\sqrt{2}}{2}\left(\sqrt{1-a} + i\sqrt{1+a}\right), \quad \text{when } a \in (-1, 1], \end{aligned}$$

and

$$\begin{aligned} \tau_1 &= i\sqrt{a - \sqrt{a^2 - 1}}, \\ \tau_2 &= i\sqrt{a + \sqrt{a^2 - 1}}, \quad \text{when } a \in [1, +\infty). \end{aligned}$$

In Figure A.1 we depict the location of $\tau_1, \tau_2, \overline{\tau_1}, \overline{\tau_2}$ on the complex plane in dependence on a. In particular, for $a \in (-1, 1)$ the roots lie on the unit circle (in red) and $a \in (1, +\infty)$, they lie on the imaginary axis (in blue). The $a = 1$ corresponds to the intersection point of the circle and the imaginary axis.

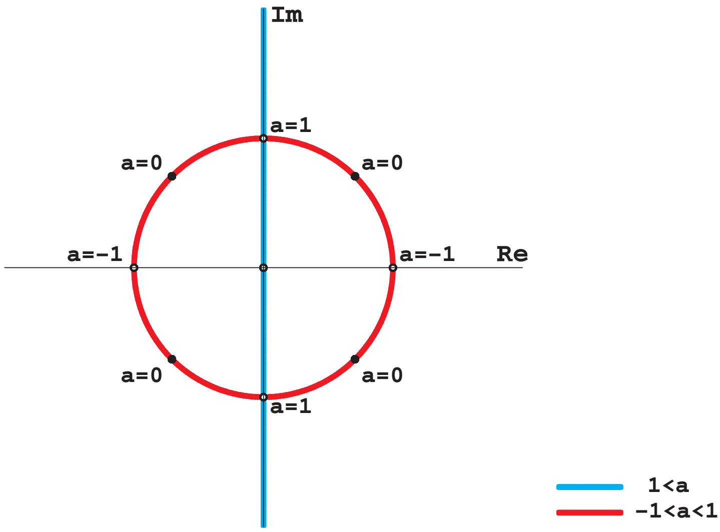

Figure A.1: *Diagram showing two possibilities in the location of the roots* $\tau \in \mathbb{C}$ *of the characteristic equation* $\tau^4 + 2a\tau^2 + 1 = 0$, $a \in (-1, +\infty)$.

A.2 Transforming the roots

Let us first prove the simple yet useful lemma.

Lemma A.3 *Let* $b_j \in \mathbb{R}$, $j = 1, 2, 3$ *be such that* $p(\tau) := \tau^4 + b_1\tau^3 + b_2\tau^2 + b_3\tau^2 + 1$ *has complex roots which come pairwise.*

There exists a Möbius mapping $f(z) = \dfrac{\alpha + \beta z}{\gamma + \delta z}$, *with real coefficients and* $\alpha\delta \neq \beta\gamma$, *such that* $p(f(\tau)) = 0$ *will have purely imaginary roots. Moreover, these roots are pairwise symmetric with respect to* 0.

Proof. Under assumption that the equation $p(\tau) = 0$ has four complex roots coming pairwise, let us denoting them as $\tau_1, \tau_2, \overline{\tau_1}, \overline{\tau_2}$ and write down

$$p(\tau) = (\tau - \tau_1)(\tau - \tau_2)(\tau - \overline{\tau_1})(\tau - \overline{\tau_2}).$$

It holds that $|\tau_1|^2 |\tau_2|^2 = 1$.

Let $f(z) = \dfrac{\alpha + \beta z}{\gamma + \delta z}$ be the mapping to apply. We obtain

$$\begin{aligned} p(f(\tau)) &= (f(\tau) - \tau_1)(f(\tau) - \tau_2)(f(\tau) - \overline{\tau_1})(f(\tau) - \overline{\tau_2}) = \\ &= \left(\frac{\alpha + \beta\tau}{\gamma + \delta\tau} - \tau_1\right)\left(\frac{\alpha + \beta\tau}{\gamma + \delta\tau} - \tau_2\right)\left(\frac{\alpha + \beta\tau}{\gamma + \delta\tau} - \overline{\tau_1}\right)\left(\frac{\alpha + \beta\tau}{\gamma + \delta\tau} - \overline{\tau_2}\right) = \\ &= A\left(\tau - \frac{-\alpha + \gamma\tau_1}{\beta - \delta\tau_1}\right)\left(\tau - \frac{-\alpha + \gamma\tau_2}{\beta - \delta\tau_2}\right)\left(\tau - \frac{-\alpha + \gamma\overline{\tau_1}}{\beta - \delta\overline{\tau_1}}\right)\left(\tau - \frac{-\alpha + \gamma\overline{\tau_2}}{\beta - \delta\overline{\tau_2}}\right) = \\ &= A\left(\tau - f^{inv}(\tau_1)\right)\left(\tau - f^{inv}(\tau_2)\right)\left(\tau - f^{inv}(\overline{\tau_1})\right)\left(\tau - f^{inv}(\overline{\tau_2})\right), \end{aligned}$$

where $A = \frac{1}{(\gamma+\delta\tau)^4}(\beta - \delta\tau_1)(\beta - \delta\tau_2)(\beta - \delta\overline{\tau_1})(\beta - \delta\overline{\tau_2})$ and f^{inv} stands for the mapping inverse to f. This inverse mapping is again a Möbius mapping: indeed, for $f(z) = \dfrac{\alpha + \beta z}{\gamma + \delta z}$, the inverse reads as $f^{inv}(w) = \dfrac{-\alpha + \gamma w}{\beta - \delta w}$.

In order to see that f^{inv} turns the roots $\tau_1, \tau_2, \overline{\tau_1}, \overline{\tau_2}$ into purely imaginary values we consider two cases.

Case 1: the roots $\tau_1, \tau_2, \overline{\tau_1}, \overline{\tau_2}$ lie one one line (see Figure A.2, top), that is, $\mathrm{Re}(\tau_1) = \mathrm{Re}(\tau_2)$ and we suppose that $0 < \mathrm{Im}(\tau_1) < \mathrm{Im}(\tau_2)$. Then in order to make $\tau_1, \tau_2, \overline{\tau_1}, \overline{\tau_2}$ purely imaginary a shift suffices, that is,

$$f^{inv}(w) = w - \mathrm{Re}(\tau_1).$$

The coefficients in f^{inv} and f will read as:

$$\alpha = \mathrm{Re}(\tau_1), \quad \beta = 1, \quad \delta = 0, \quad \gamma = 1. \tag{A.4}$$

One sees that $\alpha\delta = 0$ and $\beta\gamma = 1$. Hence $\alpha\delta \neq \beta\gamma$.

Case 2: the roots $\tau_1, \tau_2, \overline{\tau_1}, \overline{\tau_2}$ are not on line, i.e. $\mathrm{Re}(\tau_1) \neq \mathrm{Re}(\tau_2)$ and we suppose that $\mathrm{Im}(\tau_1)$ and $\mathrm{Im}(\tau_2)$ are positive. Then they lie on one circle C which is symmetric with respect to the real axis (see Figure A.2, bottom).

Let χ_1 and χ_2 denote the intersection points of this circle C with the real axis. Then we take

$$f^{inv}(w) = \frac{w - \chi_1}{w - \chi_2}.$$

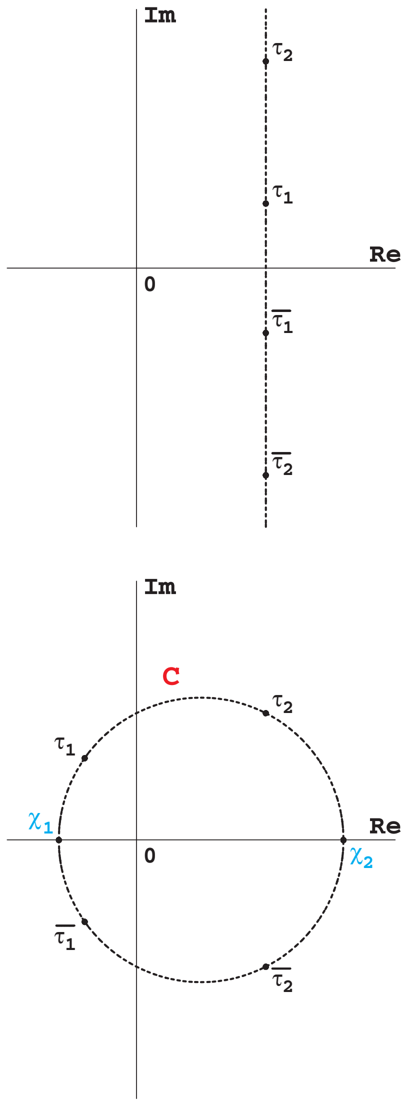

Figure A.2: *Diagram showing the location of the roots $\tau \in \mathbb{C}$ of the characteristic equation $p(\tau) = 0$.*

Möbius transformations map 'circles and lines' on 'circles and lines' and preserves the angles. That means that the circle C through $\tau_1, \tau_2, \overline{\tau_1}, \overline{\tau_2}$ and χ_1, χ_2

will be again a circle or line. In fact, due to $f^{inv}(\chi_2) = \infty$ we conclude that mapped C is a line (let us denote it as ℓ). The real axis is mapped on the real axis and since the circle C intersects the real axis perpendicularly, the line ℓ will intersect the real line perpendicularly too. Since $f^{inv}(\chi_1) = 0$, this intersection is at 0. In other words, the image ℓ of the circle C is the imaginary axis.

The coefficients in f^{inv} and f in this case are as follows as:

$$\alpha = \chi_1, \quad \beta = -\chi_2, \quad \delta = -1, \quad \gamma = 1. \tag{A.5}$$

One sees that $\alpha\delta = -\chi_1$ and $\beta\gamma = -\chi_2$. Hence $\alpha\delta \neq \beta\gamma$.

Let us note that due to the symmetry of $\tau_1, \overline{\tau_1}$ and $\tau_2, \overline{\tau_2}$ with respect to the real axis in both cases the transformation preserves this symmetry (see Formula (A.7) for details). ■

For practical reasons it is useful to have the explicit formulas for $f^{inv}(w) = \dfrac{-\alpha + \gamma w}{\beta - \delta w}$ to be applied to τ_j, $j = 1, 2$. In particular,

- When $\mathrm{Re}(\tau_1) = \mathrm{Re}(\tau_2)$, one obtains directly that

$$f^{inv}(\tau_j) = i\mathrm{Im}(\tau_j), \quad j = 1, 2. \tag{A.6}$$

  Here $i = \sqrt{-1}$.

- When $\mathrm{Re}(\tau_1) \neq \mathrm{Re}(\tau_2)$, one finds for τ_j, $j = 1, 2$ that

$$f^{inv}(\tau_j) = \frac{\tau_j - \chi_1}{\tau_j - \chi_2} = \frac{\mathrm{Re}(\tau_j) + i\mathrm{Im}(\tau_j) - \chi_1}{\mathrm{Re}(\tau_j) + i\mathrm{Im}(\tau_j) - \chi_2} =$$

$$= \frac{\chi_1\chi_2 - (\chi_1 + \chi_2)\,\mathrm{Re}(\tau_j) + |\tau_j|^2 + i\,(\chi_1 - \chi_2)\,\mathrm{Im}(\tau_j)}{\chi_2^2 - 2\chi_2\mathrm{Re}(\tau_j) + |\tau_j|^2}.$$

  We know that $f^{inv}(\tau_j)$ is purely complex, meaning that

$$\chi_1\chi_2 - (\chi_1 + \chi_2)\,\mathrm{Re}(\tau_j) + |\tau_j|^2 = 0, \quad j = 1, 2.$$

  That is, as a result

$$f^{inv}(\tau_j) = i\frac{(\chi_1 - \chi_2)\,\mathrm{Im}(\tau_j)}{\chi_2^2 - 2\chi_2\mathrm{Re}(\tau_j) + |\tau_j|^2}, \quad j = 1, 2, \tag{A.7}$$

where χ_1,χ_2 read as

$$\chi_1 = \frac{1}{2\left(\mathrm{Re}(\tau_1) - \mathrm{Re}(\tau_2)\right)}\left(|\tau_1|^2 - |\tau_2|^2 \pm \sqrt{D}\right), \tag{A.8}$$

$$\chi_2 = \frac{1}{2\left(\mathrm{Re}(\tau_1) - \mathrm{Re}(\tau_2)\right)}\left(|\tau_1|^2 - |\tau_2|^2 \mp \sqrt{D}\right), \tag{A.9}$$

and

$$D = \left(|\tau_1|^2 - |\tau_2|^2\right)^2 + 4\left(|\tau_1|^2 \mathrm{Re}^2(\tau_2) + |\tau_2|^2 \mathrm{Re}^2(\tau_1)\right) +$$
$$-4\mathrm{Re}(\tau_1)\mathrm{Re}(\tau_2)\left(|\tau_1|^2 + |\tau_2|^2\right).$$

Remark A.4 *Note that the above D is always positive. Indeed, having* $\mathrm{Re}(\tau_j) = |\tau_j|\cos(\theta_j)$, $j = 1, 2$ *and that* $|\tau_1|\,|\tau_2| = 1$ *one finds*

$$D = \left(|\tau_1|^2 - |\tau_2|^2\right)^2 + 4\left(\cos^2(\theta_1) + \cos^2(\theta_2)\right) +$$
$$-4\cos(\theta_1)\cos(\theta_2)\left(|\tau_1|^2 + |\tau_2|^2\right) \geq$$
$$\geq \left(|\tau_1|^2 - |\tau_2|^2\right)^2 + 4\left(\cos^2(\theta_1) + \cos^2(\theta_2)\right) +$$
$$-2\left(\cos^2(\theta_1) + \cos^2(\theta_2)\right)\left(|\tau_1|^2 + |\tau_2|^2\right) =$$
$$= \left(|\tau_1|^2 - |\tau_2|^2\right)^2 - 2\left(\cos^2(\theta_1) + \cos^2(\theta_2)\right)\left(|\tau_1| - |\tau_2|\right)^2 =$$
$$= \left(|\tau_1| - |\tau_2|\right)^2\left[\left(|\tau_1| + |\tau_2|\right)^2 - 2\left(\cos^2(\theta_1) + \cos^2(\theta_2)\right)\right] =$$
$$= \left(|\tau_1| - |\tau_2|\right)^2\left[|\tau_1|^2 + |\tau_2|^2 + 2 - 2\left(\cos^2(\theta_1) + \cos^2(\theta_2)\right)\right] \geq$$
$$\geq \left(|\tau_1| - |\tau_2|\right)^2\left(|\tau_1|^2 + |\tau_2|^2 - 2\right) = \left(|\tau_1| - |\tau_2|\right)^4 \geq 0.$$

Now let us address the operator L as in (A.1). Let $b_j \in \mathbb{R}$, $j = 1, 2, 3$ be such that L is elliptic, meaning that characteristic polynom

$$\tau^4 + b_1\tau^3 + b_2\tau^2 + b_3\tau^2 + 1 = 0, \tag{A.10}$$

has only complex solutions $\tau_1, \tau_2, \overline{\tau_1}, \overline{\tau_2}$. Using Lemma A.3 we may pose the following

Proposition A.5 *Let* $f^{inv}(w) = \dfrac{-\alpha + \gamma w}{\beta - \delta w}$ *be a Möbius mapping with* $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ *and* $\alpha\delta \neq \beta\gamma$ *that makes the roots* $\tau_1, \tau_2, \overline{\tau_1}, \overline{\tau_2}$ *of (A.10) purely imaginary. Then due to the linear transformation of coordinates*

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \beta & \delta \\ \alpha & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{A.11}$$

the operator L *as in (A.1) turns into the operator which contains only the terms*

$$\frac{\partial^4}{\partial \tilde{x}^4}, \quad \frac{\partial^4}{\partial \tilde{x}^2 \partial \tilde{y}^2}, \quad \frac{\partial^4}{\partial \tilde{y}^4}.$$

Proof. Indeed, for a given L the following factorization holds:

$$L = \left(\frac{\partial}{\partial x} - \tau_1 \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \tau_2 \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \overline{\tau_1} \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \overline{\tau_2} \frac{\partial}{\partial y}\right).$$

The operator L rewritten in the new coordinates $(\tilde{x}, \tilde{y})$ given by (A.11) and denoted as $\tilde{L}$ will be as follows

$$\tilde{L} = M \left(\frac{\partial}{\partial \tilde{x}} - \left(\frac{-\alpha + \gamma\tau_1}{\beta - \delta\tau_1}\right)\frac{\partial}{\partial \tilde{y}}\right)\left(\frac{\partial}{\partial \tilde{x}} - \left(\frac{-\alpha + \gamma\tau_2}{\beta - \delta\tau_2}\right)\frac{\partial}{\partial \tilde{y}}\right)$$
$$\left(\frac{\partial}{\partial \tilde{x}} - \left(\frac{-\alpha + \gamma\overline{\tau_1}}{\beta - \delta\overline{\tau_1}}\right)\frac{\partial}{\partial \tilde{y}}\right)\left(\frac{\partial}{\partial \tilde{x}} - \left(\frac{-\alpha + \gamma\overline{\tau_2}}{\beta - \delta\overline{\tau_2}}\right)\frac{\partial}{\partial \tilde{y}}\right),$$

where $M = (\beta - \delta\tau_1)(\beta - \delta\tau_2)(\beta - \delta\overline{\tau_1})(\beta - \delta\overline{\tau_2})$.

By Lemma A.3 it holds that the parameters $\alpha, \beta, \gamma, \delta$ are such that

$$\begin{cases} \dfrac{-\alpha + \gamma\tau_1}{\beta - \delta\tau_1} = ic, \\ \dfrac{-\alpha + \gamma\tau_2}{\beta - \delta\tau_2} = id, \end{cases}$$

where $i = \sqrt{-1}$ and $c, d \in \mathbb{R}$ are defined either from (A.6) or (A.7). More precisely:

When $\mathrm{Re}(\tau_1) = \mathrm{Re}(\tau_2)$, by (A.6) one has that

$$c = \mathrm{Im}(\tau_1) \quad \text{and} \quad d = \mathrm{Im}(\tau_2).$$

When $\mathrm{Re}(\tau_1) \neq \mathrm{Re}(\tau_2)$, by (A.7) it follows that

$$c = \frac{(\chi_1 - \chi_2)\,\mathrm{Im}(\tau_1)}{\chi_2^2 - 2\chi_2 \mathrm{Re}(\tau_1) + |\tau_1|^2} \quad \text{and} \quad d = \frac{(\chi_1 - \chi_2)\,\mathrm{Im}(\tau_2)}{\chi_2^2 - 2\chi_2 \mathrm{Re}(\tau_2) + |\tau_2|^2},$$

with χ_1, χ_2 as in (A.8), (A.9).

Hence the operator $\tilde{L}$ takes the form

$$\tilde{L} = M\left(\frac{\partial}{\partial\tilde{x}} - ic\frac{\partial}{\partial\tilde{y}}\right)\left(\frac{\partial}{\partial\tilde{x}} - id\frac{\partial}{\partial\tilde{y}}\right)\left(\frac{\partial}{\partial\tilde{x}} + ic\frac{\partial}{\partial\tilde{y}}\right)\left(\frac{\partial}{\partial\tilde{x}} + id\frac{\partial}{\partial\tilde{y}}\right) =$$

$$= M\left(\frac{\partial^4}{\partial\tilde{x}^4} + \left(c^2 + d^2\right)\frac{\partial^4}{\partial\tilde{x}^2\partial\tilde{y}^2} + c^2d^2\frac{\partial^4}{\partial\tilde{y}^4}\right). \tag{A.12}$$

Moreover, by setting $\tilde{x} = \mathbf{x}$ and $\tilde{y} = (cd)^{\frac{1}{2}}\,\mathbf{y}$ in (A.12), one turns $\tilde{L}$ into

$$\mathbf{L} = M\left(\frac{\partial}{\partial\mathbf{x}} - i\left(\frac{c}{d}\right)^{\frac{1}{2}}\frac{\partial}{\partial\mathbf{y}}\right)\left(\frac{\partial}{\partial\mathbf{x}} - i\left(\frac{d}{c}\right)^{\frac{1}{2}}\frac{\partial}{\partial\mathbf{y}}\right)$$

$$\left(\frac{\partial}{\partial\mathbf{x}} + i\left(\frac{c}{d}\right)^{\frac{1}{2}}\frac{\partial}{\partial\mathbf{y}}\right)\left(\frac{\partial}{\partial\mathbf{x}} + i\left(\frac{d}{c}\right)^{\frac{1}{2}}\frac{\partial}{\partial\mathbf{y}}\right) =$$

$$= M\left(\frac{\partial^4}{\partial\mathbf{x}^4} + \frac{c^2 + d^2}{cd}\frac{\partial^4}{\partial\mathbf{x}^2\partial\mathbf{y}^2} + \frac{\partial^4}{\partial\mathbf{y}^4}\right). \tag{A.13}$$

The roots $\pm i\left(\frac{c}{d}\right)^{\frac{1}{2}}, \pm i\left(\frac{d}{c}\right)^{\frac{1}{2}}$ in $\mathbf{L}$ lie on the imaginary axis and hence due to diagram in Figure A.1 one concludes that $\frac{c^2+d^2}{cd} \geq 1$. ■

Based on Proposition A.5 we conclude that

Conclusion A.6 *For every fourth order elliptic operator in* 2d *with constant coefficients there exists precisely one* $a \in [1, +\infty)$ *such that this operator is algebraically equivalent to the operator*

$$L_a = \frac{\partial^4}{\partial x^4} + 2a\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}.$$

A.3 Duality

Let us address the main result of this chapter – Conclusion A.6: every fourth order elliptic operator L as in (A.1) is algebraically equivalent to the operator L_a given by

$$L_a = \frac{\partial^4}{\partial\tilde{x}^4} + 2a\frac{\partial^4}{\partial\tilde{x}^2\partial\tilde{y}^2} + \frac{\partial^4}{\partial\tilde{y}^4}, \quad \text{with} \quad a \in [1, +\infty). \tag{A.14}$$

One may observe that the range of a in (A.14) is $[1, +\infty)$, whereas by Lemma A.2 we know that the operator L_a is elliptic on the more wide range of a, namely, on $(-1, +\infty)$. The "missing" interval $(-1, 1)$ along with Conclusion A.6 gives rise to the following

Proposition A.7 *The elliptic operator*

$$L_b = \frac{\partial^4}{\partial x^4} + 2b\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}, \quad \textit{with} \quad b \in (-1, 1]. \tag{A.15}$$

is dual to the operator L_a as in (A.14), meaning that, due to the linear transformation of coordinates

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{A.16}$$

the operator L_b turns into L_a as in (A.14), where explicitly $a = \frac{3-b}{1+b}$.

Proof. For $b \in (-1, 1]$ the roots of the characteristic equation to L_b are as follows:

$$\begin{aligned} \tau_1 &= -\tfrac{\sqrt{2}}{2}\left(\sqrt{1-b} - i\sqrt{1+b}\right), \\ \tau_2 &= \tfrac{\sqrt{2}}{2}\left(\sqrt{1-b} + i\sqrt{1+b}\right). \end{aligned}$$

Then following the same approach as in the proof of Proposition A.5 we first find by formulas (A.7) that

$$\begin{aligned} f^{inv}(\tau_1) &= -i\frac{\sqrt{1+b}}{\sqrt{2} + \sqrt{1-b}} =: ic, \\ f^{inv}(\tau_2) &= -i\frac{\sqrt{1+b}}{\sqrt{2} - \sqrt{1-b}} =: id. \end{aligned}$$

and then using (A.12) we obtain the transformed L_b:

$$\tilde{L_b} = M\left(\frac{\partial^4}{\partial \tilde{x}^4} + 2\left(\frac{3-b}{1+b}\right)\frac{\partial^4}{\partial \tilde{x}^2 \partial \tilde{y}^2} + \frac{\partial^4}{\partial \tilde{y}^4}\right), \tag{A.17}$$

where M is some non-zero constant. The coefficients in the transformation (A.16) are defined by formula (A.5), where in this case we set $\chi_1 = -1$ and $\chi_2 = 1$. ∎

This proposition outlines what may be called the *duality* between the operators L_a with the corresponding ranges of $a \in (-1, 1]$ and $a \in [1, +\infty)$.

Appendix B

A fundamental system of solutions

B.1 Derivation of system S_λ

Here we will find the fundamental set of solutions to equation

$$\mathcal{L}\left(\theta, \tfrac{d}{d\theta}, \lambda\right)\Phi = 0, \tag{B.1}$$

where $\mathcal{L}$ as in formula (3.3), namely,

$$\begin{aligned}
\mathcal{L}\left(\theta, \tfrac{d}{d\theta}, \lambda\right) \;=\; & \tfrac{3}{4}\left(1 + \tfrac{1}{3}\cos(4\theta)\right)\tfrac{d^4}{d\theta^4} + (\lambda - 2)\sin(4\theta)\tfrac{d^3}{d\theta^3} + \\
& + \tfrac{3}{2}\left(\lambda^2 - 1 - \left(\lambda^2 - 4\lambda - \tfrac{7}{3}\right)\cos(4\theta)\right)\tfrac{d^2}{d\theta^2} + \\
& + \left(-\lambda^3 + 6\lambda^2 - 7\lambda - 2\right)\sin(4\theta)\tfrac{d}{d\theta} + \\
& + \tfrac{3}{4}\left(\lambda^4 - 2\lambda^2 + 1 + \tfrac{1}{3}\left(\lambda^4 - 8\lambda^3 + 14\lambda^2 + 8\lambda - 15\right)\cos(4\theta)\right).
\end{aligned}$$

For this $\mathcal{L}$ it seems to be hard to derive a set of functions solving (B.1) explicitly. The following approach applies in this case.

For $L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ we find that $L\left(r^{\lambda+1}\Phi\right) = r^{\lambda-3}\mathcal{L}\left(\theta, \frac{d}{d\theta}, \lambda\right)\Phi$ and hence instead of $\mathcal{L}\left(\theta, \frac{d}{d\theta}, \lambda\right)\Phi = 0$ we may consider the equation

$$L\left(r^{\lambda+1}\Phi\right) = 0. \tag{B.2}$$

Operator L admits the decomposition

$$L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} = \prod_{p=1}^{2}\left(\frac{\partial}{\partial y} - \tau_p\frac{\partial}{\partial x}\right)\prod_{p=1}^{2}\left(\frac{\partial}{\partial y} + \tau_p\frac{\partial}{\partial x}\right),$$

with $\tau_1 = \frac{\sqrt{2}}{2}(1+i)$, $\tau_2 = \frac{\sqrt{2}}{2}(1-i)$ and hence every function of the form $F(x \pm \tau_p y)$ solves (B.2). Therefore, we have that

$$r^{\lambda+1}\Phi(\theta) = \sum_{p=1}^{2} c_p f_p(x+\tau_p y) + c_{p+2} f_{p+2}(x-\tau_p y),$$

and after translation $\{f_p, f_{p+2}\}_{p=1}^{2}$ into polar coordinates we set

$$f_p\left(r\cos(\theta)+\tau_p r\sin(\theta)\right) := \left(r\cos(\theta)+\tau_p r\sin(\theta)\right)^{\lambda+1},$$

$$f_{p+2}\left(r\cos(\theta)-\tau_p r\sin(\theta)\right) := \left(r\cos(\theta)-\tau_p r\sin(\theta)\right)^{\lambda+1},$$

So, the set of functions

$$\varphi_1(\theta) = \left(\cos(\theta)+\tau_1\sin(\theta)\right)^{\lambda+1}, \quad \varphi_2(\theta) = \left(\cos(\theta)+\tau_2\sin(\theta)\right)^{\lambda+1}, \tag{B.3}$$

$$\varphi_3(\theta) = \left(\cos(\theta)-\tau_1\sin(\theta)\right)^{\lambda+1}, \quad \varphi_4(\theta) = \left(\cos(\theta)-\tau_2\sin(\theta)\right)^{\lambda+1}, \tag{B.4}$$

is a set of solutions to (B.1). The Wronskian for $\{\varphi_m\}_{m=1}^{4}$ reads as

$$W\left(\varphi_1(\theta),\varphi_2(\theta),\varphi_3(\theta),\varphi_4(\theta)\right) = \det\begin{pmatrix} \varphi_1(\theta) & \varphi_2(\theta) & \varphi_3(\theta) & \varphi_4(\theta) \\ \varphi_1'(\theta) & \varphi_2'(\theta) & \varphi_3'(\theta) & \varphi_4'(\theta) \\ \varphi_1''(\theta) & \varphi_2''(\theta) & \varphi_3''(\theta) & \varphi_4''(\theta) \\ \varphi_1'''(\theta) & \varphi_2'''(\theta) & \varphi_3'''(\theta) & \varphi_4'''(\theta) \end{pmatrix}, \tag{B.5}$$

and by straightforward computations one finds

$$W = 16(\lambda+1)^3\lambda^2(\lambda-1)\left(\cos^4(\theta)+\sin^4(\theta)\right)^{\lambda-2},$$

and is non-zero on $\theta \in (\alpha, \alpha+\omega)$, with $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and $\omega \in (0, 2\pi]$ except for $\lambda \in \{\pm 1, 0\}$. Hence, except for these values $\{\varphi_m\}_{m=1}^{4}$ given in (B.3), (B.4) is a fundamental system of solutions to (B.1).

B.2 Derivation of systems S_{-1}, S_0, S_1

Here we find the fundamental systems of solutions to equation $\mathcal{L}\left(\theta, \frac{\partial}{\partial\theta}, \lambda\right)\Phi = 0$ when $\lambda \in \{\pm 1, 0\}$. We will go into details in solving the corresponding equation for every $\lambda \in \{\pm 1, 0\}$.

B.2.1 Case $\lambda = -1$

For $\lambda = -1$ the equation (B.1) reads as:

$$\tfrac{1}{4}\left(3+\cos(4\theta)\right)\Phi'''' - 3\sin(4\theta)\Phi''' + \left(3 - 11\cos(4\theta)\right)\Phi'' + 12\sin(4\theta)\Phi' = 0. \tag{B.6}$$

First we set $\Phi(\theta) = \int F(\theta)d\theta$ and obtain the equation for F:

$$\tfrac{1}{4}\left(3+\cos(4\theta)\right)F''' - 3\sin(4\theta)F'' + \left(3 - 11\cos(4\theta)\right)F' + 12\sin(4\theta)F = 0.$$

The first integral of the above equation reads as

$$\tfrac{1}{4}\left(3+\cos(4\theta)\right)F'' - 2\sin(4\theta)F' + 3(1-\cos(4\theta))F = c_0.$$

We use the change of variables $F(\theta) = \left(3+\cos(4\theta)\right)^{-1}G(\theta)$ and get

$$G'' + 4G = 4c_0.$$

Solution of the last equation reads as

$$G(\theta) = c_1\sin(2\theta) + c_2\cos(2\theta) + c_0.$$

and then

$$F(\theta) = c_1\tfrac{\sin(2\theta)}{3+\cos(4\theta)} + c_2\tfrac{\cos(2\theta)}{3+\cos(4\theta)} + c_3\tfrac{1}{3+\cos(4\theta)}.$$

As a result, Φ that solves (B.6) will read as

$$\Phi(\theta) = A_1 + A_2\int\tfrac{\sin(2\theta)}{3+\cos(4\theta)}d\theta + A_3\int\tfrac{\cos(2\theta)}{3+\cos(4\theta)}d\theta + A_4\int\tfrac{1}{3+\cos(4\theta)}d\theta,$$

and then the candidates that may form the fundamental system of solutions to (B.6) will be the following:

$$\varphi_1(\theta) = 1,$$

$$\varphi_2(\theta) = -4\int\tfrac{\sin(2\theta)}{3+\cos(4\theta)}d\theta = \arctan\left(\cos(2\theta)\right),$$

$$\varphi_3(\theta) = 4\sqrt{2}\int\tfrac{\cos(2\theta)}{3+\cos(4\theta)}d\theta = \operatorname{arctanh}\left(\tfrac{\sqrt{2}}{2}\sin(2\theta)\right),$$

$$\varphi_4(\theta) = 4\sqrt{2}\int\tfrac{1}{3+\cos(4\theta)}d\theta =$$

$$= \begin{cases} \arctan\left(\frac{\sqrt{2}}{2}\tan(2\theta)\right) + \ell\pi, & \text{for} \quad \theta \in \left(\frac{2\ell-1}{4}\pi, \frac{2\ell+1}{4}\pi\right), \\ 2\theta, & \text{for} \quad \theta = \frac{2\ell+1}{4}\pi, \end{cases}$$

with $\ell = 0, ..., 5$.

The Wronskian W computed for the given $\varphi_1, ..., \varphi_4$ reads as

$$W = -16\left(\cos^4(\theta) + \sin^4(\theta)\right)^{-3},$$

and is non-zero on $\theta \in (\alpha, \alpha+\omega)$, with $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and $\omega \in (0, 2\pi]$. Hence, $\{\varphi_m\}_{m=1}^4$ defined as above is a fundamental system of solutions to (B.6).

B.2.2 Case $\lambda = 0$

For $\lambda = 0$ the equation (B.1) reads as:

$$\frac{1}{4}\left(3+\cos(4\theta)\right)\Phi'''' - 2\sin(4\theta)\Phi''' + \frac{1}{2}\left(3 - 7\cos(4\theta)\right)\Phi'' - 2\sin(4\theta)\Phi' + \\ + \frac{3}{4}\left(1 - 5\cos(4\theta)\right)\Phi = 0, \tag{B.7}$$

and can be split as follows:

$$\left(\frac{d^2}{d\theta^2} + 1\right)\left(\frac{1}{4}\left(3+\cos(4\theta)\right)\left(\frac{d^2}{d\theta^2} + 1\right)\right)\Phi = 0.$$

So, Φ solves

$$\Phi'' + \Phi = A\frac{\sin(\theta)}{3+\cos(4\theta)} + B\frac{\cos(\theta)}{3+\cos(4\theta)},$$

and after integrating this equation we obtain

$$\Phi(\theta) = A_1\sin(\theta) + A_2\cos(\theta) +$$

$$+A_3\left(\frac{1}{2}\sin(\theta)\arctan\left(\cos(2\theta)\right) + 4\cos(\theta)\int\frac{\sin^2(\theta)}{3+\cos(4\theta)}d\theta\right) +$$

$$+A_4\left(\frac{1}{2}\cos(\theta)\arctan\left(\cos(2\theta)\right) + 4\sin(\theta)\int\frac{\cos^2(\theta)}{3+\cos(4\theta)}d\theta\right).$$

The candidates that may form the fundamental system of solutions to (B.7) will be the following:

$$\varphi_1(\theta) = \sin(\theta),$$

$$\varphi_2(\theta) = \cos(\theta),$$

$$\varphi_3(\theta) = \tfrac{1}{2}\sin(\theta)\arctan(\cos(2\theta)) + 4\cos(\theta)\int_{\alpha}^{\theta} \tfrac{\sin^2(y)}{3+\cos(4y)}dy,$$

$$\varphi_4(\theta) = \tfrac{1}{2}\cos(\theta)\arctan(\cos(2\theta)) + 4\sin(\theta)\int_{\alpha}^{\theta} \tfrac{\cos^2(y)}{3+\cos(4y)}dy.$$

Let us note that one is able to compute the integrals in the above formulas. However, the integral form is favorized for further computations when we substitute $\Phi(\theta)$ to the corresponding boundary conditions.

The Wronskian W computed for the given $\varphi_1, ..., \varphi_4$ reads as

$$W = -\left(\cos^4(\theta) + \sin^4(\theta)\right)^{-2},$$

and is non-zero on on $\theta \in (\alpha, \alpha + \omega)$, with $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and $\omega \in (0, 2\pi]$. Hence, $\{\varphi_m\}_{m=1}^4$ defined as above is a fundamental system of solutions to (B.7).

B.2.3 Case $\lambda = 1$

For $\lambda = 1$ the equation (B.1) reads as:

$$\tfrac{1}{4}\left(3 + \cos(4\theta)\right)\Phi'''' - \sin(4\theta)\Phi''' + \left(3 + \cos(4\theta)\right)\Phi'' - 4\sin(4\theta)\Phi' = 0. \quad \text{(B.8)}$$

We set $\Phi(\theta) = \int F(\theta)d\theta$ and obtain the equation for F:

$$\tfrac{1}{4}\left(3 + \cos(4\theta)\right)F''' - \sin(4\theta)F'' + \left(3 + \cos(4\theta)\right)F' - 4\sin(4\theta)F = 0.$$

It holds that

$$\begin{cases} \left(3 + \cos(4\theta)\right)F''' - 4\sin(4\theta)F'' = -2g(\theta), \\ \left(3 + \cos(4\theta)\right)F' - 4\sin(4\theta)F = \tfrac{1}{2}g(\theta). \end{cases}$$

and we obtain, respectively,

$$F''(\theta) = -4\tfrac{\int g(\theta)d\theta + C_1}{3+\cos(4\theta)}, \qquad F(\theta) = \tfrac{\int g(\theta)d\theta + C_2}{3+\cos(4\theta)}.$$

Comparing the expressions for $F''(\theta)$ and $F(\theta)$ we deduce that F solves

$$F'' + 4F = \frac{c_0}{3 + \cos(4\theta)}. \quad \text{(B.9)}$$

Solution of (B.9) reads as

$$F(\theta) = c_1 \sin(2\theta) + c_2 \cos(2\theta) +$$

$$+ c_0 \left(\tfrac{1}{4} \cos(2\theta) \arctan\left(\cos(2\theta)\right) + \tfrac{\sqrt{2}}{8} \sin(2\theta) \operatorname{arctanh}\left(\tfrac{\sqrt{2}}{2} \sin(2\theta) \right) \right),$$

and being integrated yields

$$\Phi(\theta) = A_1 + A_2 \cos(2\theta) + A_3 \sin(2\theta) + A_4 \int \Big(\cos(2\theta) \arctan\left(\cos(2\theta)\right) +$$

$$+ \tfrac{\sqrt{2}}{2} \sin(2\theta) \operatorname{arctanh}\left(\tfrac{\sqrt{2}}{2} \sin(2\theta) \right) \Big) \, d\theta$$

The candidates that may form the fundamental system of solutions to (B.8) will be the following:

$$\varphi_1(\theta) = 1,$$

$$\varphi_2(\theta) = \sin(2\theta),$$

$$\varphi_3(\theta) = \cos(2\theta),$$

$$\varphi_4(\theta) = \int_{\alpha}^{\theta} \left(\cos(2y) \arctan\left(\cos(2y)\right) + \tfrac{\sqrt{2}}{2} \sin(2y) \operatorname{arctanh}\left(\tfrac{\sqrt{2}}{2} \sin(2y) \right) \right) dy.$$

The Wronskian W computed for the given $\varphi_1, ..., \varphi_4$ reads as

$$W = -16 \left(\cos^4(\theta) + \sin^4(\theta) \right)^{-1},$$

and is non-zero on $\theta \in (\alpha, \alpha + \omega)$, with $\alpha \in \left[0, \tfrac{1}{2}\pi\right)$ and $\omega \in (0, 2\pi]$. Hence, $\{\varphi_m\}_{m=1}^{4}$ defined as above is a fundamental system of solutions to (B.8).

B.3 The explicit formulas for $\mathcal{P}_{-1}, \mathcal{P}_0, \mathcal{P}_1$

In this Section, based on the results of Section B.2, we will find the characteristic equations for the eigenvalue problem

$$\begin{cases} \mathcal{L}\left(\theta, \frac{d}{d\theta}, \lambda\right) \Phi = 0 & \text{in} \quad (\alpha, \alpha + \omega), \\ \Phi = \frac{d}{d\theta} \Phi = 0 & \text{on} \quad \partial(\alpha, \alpha + \omega). \end{cases}$$

for the cases $\lambda \in \{\pm 1, 0\}$.

B.3.1 Case $\lambda = -1$

The characteristic determinant in this case reads:

$$\mathcal{P}_{-1}(\alpha, \omega) =$$

$$= \sqrt{2}\left[\arctan\left(\cos(2\alpha + 2\omega)\right) - \arctan\left(\cos(2\alpha)\right)\right]\sin(2\alpha + \omega)\sin(\omega) +$$

$$- \left[\operatorname{arctanh}\left(\tfrac{\sqrt{2}}{2}\sin(2\alpha + 2\omega)\right) - \operatorname{arctanh}\left(\tfrac{\sqrt{2}}{2}\sin(2\alpha)\right)\right]\cos(2\alpha + \omega)\sin(\omega) +$$

$$+ \tfrac{1}{2}\left[\arctan\left(\tfrac{\sqrt{2}}{2}\tan(2\alpha + 2\omega)\right) + \ell_2\pi - \arctan\left(\tfrac{\sqrt{2}}{2}\tan(2\alpha)\right) - \ell_1\pi\right]\sin(2\omega),$$

where

$$\alpha + \omega \in \left(\tfrac{2\ell_2 - 1}{4}\pi, \tfrac{2\ell_2 + 1}{4}\pi\right) \quad \text{and} \quad \alpha \in \left(\tfrac{2\ell_1 - 1}{4}\pi, \tfrac{2\ell_1 + 1}{4}\pi\right),$$

with $\{\ell_1, \ell_2\} = 0, ..., 5$.

One immediately sees that for every $\alpha \in \left[0, \frac{1}{2}\pi\right)$ the equation $\mathcal{P}_{-1}(\alpha, \omega) = 0$ has two solutions: $\omega = \pi$ and $\omega = 2\pi$. There also exists a third solution in ω which can be found numerically.

B.3.2 Case $\lambda = 0$

The characteristic determinant in this case reads:

$$\mathcal{P}_0(\alpha, \omega) =$$

$$= 64 \int_{\alpha}^{\alpha+\omega} \tfrac{\sin^2(\theta)}{3 + \cos(4\theta)} d\theta \int_{\alpha}^{\alpha+\omega} \tfrac{\cos^2(\theta)}{3 + \cos(4\theta)} d\theta -$$

$$- \left[\arctan\left(\cos(2\alpha + 2\omega)\right) - \arctan\left(\cos(2\alpha)\right)\right]^2.$$

The following simple result holds.

Lemma B.1 *For every $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and $\omega \in (0, 2\pi]$, $\mathcal{P}_0(\alpha, \omega)$ is strictly positive.*

Proof. Let $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and $\omega \in (0, 2\pi]$. We will find the estimate for the first term in $\mathcal{P}_0$. It holds that,

$$64 \int_{\alpha}^{\alpha+\omega} \frac{\sin^2(\theta)}{3+\cos(4\theta)} d\theta \int_{\alpha}^{\alpha+\omega} \frac{\cos^2(\theta)}{3+\cos(4\theta)} d\theta =$$

$$= 64 \left[\left(\int_{\alpha}^{\alpha+\omega} \left| \frac{\sin(\theta)}{\sqrt{3+\cos(4\theta)}} \right|^2 d\theta \right)^{\frac{1}{2}} \left(\int_{\alpha}^{\alpha+\omega} \left| \frac{\cos(\theta)}{\sqrt{3+\cos(4\theta)}} \right|^2 d\theta \right)^{\frac{1}{2}} \right]^2 \overset{\text{by Hölder's}}{\geq}$$

$$\geq 16 \left[\int_{\alpha}^{\alpha+\omega} \left| \frac{\sin(2\theta)}{3+\cos(4\theta)} \right| d\theta \right]^2 \geq 16 \left[\int_{\alpha}^{\alpha+\omega} \frac{\sin(2\theta)}{3+\cos(4\theta)} d\theta \right]^2 =$$

$$= \Big[\arctan\left(\cos(2\omega + 2\alpha)\right) - \arctan\left(\cos(2\alpha)\right) \Big]. \qquad \text{(B.10)}$$

Due to (B.10) we conclude that $\mathcal{P}_0(\alpha, \omega) \geq 0$ on $\left[0, \frac{1}{2}\pi\right) \times (0, 2\pi]$. Next we will prove that $\mathcal{P}_0(\alpha, \omega) > 0$. For this purpose let us assume the opposite. Suppose there exists $(\alpha_\star, \omega_\star) \in \left[0, \frac{1}{2}\pi\right) \times (0, 2\pi]$ such that $\mathcal{P}_0(\alpha_\star, \omega_\star) = 0$. In particular, it follows from this equation that

$$\arctan\left(\cos(2\alpha_\star + 2\omega_\star)\right) - \arctan\left(\cos(2\alpha_\star)\right) =$$

$$= \pm 8 \sqrt{\int_{\alpha_\star}^{\alpha_\star+\omega_\star} \frac{\sin^2(\theta)}{3+\cos(4\theta)} d\theta} \sqrt{\int_{\alpha_\star}^{\alpha_\star+\omega_\star} \frac{\cos^2(\theta)}{3+\cos(4\theta)} d\theta}. \qquad \text{(B.11)}$$

The function $\mathcal{P}_0$ has to attain its minimum at point $(\alpha_\star, \omega_\star)$, i.e. it holds that $\frac{\partial \mathcal{P}_0}{\partial \alpha}(\alpha_\star, \omega_\star) = 0$ and $\frac{\partial \mathcal{P}_0}{\partial \omega}(\alpha_\star, \omega_\star) = 0$. We analyse these two equations and using (3.24) obtain $\tan(\alpha_\star + \omega_\star) = \tan(\alpha_\star)$. This yields $\sin(\omega_\star) = 0$. We conclude that the point $(\alpha_\star, \omega_\star) \in \left[0, \frac{1}{2}\pi\right) \times (0, 2\pi]$ where by assumption $\mathcal{P}_0(\alpha_\star, \omega_\star) = 0$ has the following coordinates: $\alpha_\star$ is arbitrary and $\omega_\star \in \{\pi, 2\pi\}$. But it is straightforward that $\mathcal{P}_0(\alpha_\star, \pi) > 0$ as well as $\mathcal{P}_0(\alpha_\star, 2\pi) > 0$, a contradiction. ∎

B.3.3 Case $\lambda = 1$

The characteristic determinant in this case reads:

$$\mathcal{P}_1(\alpha, \omega) =$$

$$= \Big[\cos(2\alpha + 2\omega)\arctan(\cos(2\alpha + 2\omega)) + \cos(2\alpha)\arctan(\cos(2\alpha))\Big]\sin^2(\omega) +$$

$$+ \tfrac{\sqrt{2}}{2}\Big[\operatorname{arctanh}\Big(\tfrac{\sqrt{2}}{2}\sin(2\alpha + 2\omega)\Big) - \operatorname{arctanh}\Big(\tfrac{\sqrt{2}}{2}\sin(2\alpha)\Big)\Big]\cos(2\alpha+\omega)\sin(\omega) +$$

$$- \tfrac{\sqrt{2}}{4}\Big[\cos(2\alpha + 2\omega)\operatorname{arctanh}\Big(\tfrac{\sqrt{2}}{2}\sin(2\alpha + 2\omega)\Big) +$$

$$- \cos(2\alpha)\operatorname{arctanh}\Big(\tfrac{\sqrt{2}}{2}\sin(2\alpha)\Big)\Big]\sin(2\omega) +$$

$$- \int_{\alpha}^{\alpha+\omega}\Big[\cos(2\theta)\arctan(\cos(2\theta)) + \tfrac{\sqrt{2}}{2}\sin(2\theta)\operatorname{arctanh}\Big(\tfrac{\sqrt{2}}{2}\sin(2\theta)\Big)\Big]\,d\theta\,\sin(2\omega).$$

One may immediately find that for every $\alpha \in \left[0, \frac{1}{2}\pi\right)$ the equation $\mathcal{P}_1(\alpha, \omega) = 0$ possesses two solutions: $\omega = \pi$ and $\omega = 2\pi$. There also exists a third solution in ω which can be derived numerically.

Appendix C

Analytical tools for the numerical computations

C.1 Implicit function and discretization

Consider a rectangle $U = [a,b] \times [c,d]$. For $n, m \in \mathbb{N}^+$ and $i = 0, ..., n$, $j = 0, ..., m$ we set

$$x_i = a + i\triangle x, \quad y_j = c + j\triangle y,$$

where

$$\triangle x = \tfrac{b-a}{n}, \quad \triangle y = \tfrac{d-c}{m}.$$

Let $F \in C^1(U, \mathbb{R})$ such that $F(x_i, y_j) > 0$ for all $i = 0, ..., n$ and $j = 0, ..., m$. The question to resolve is how fine should we take the discretization of U in order to be sure that $F > 0$ on U.

The following result holds.

Lemma C.1 *Suppose* $\min\limits_{(x_i,y_j)\in U} F(x_i, y_j) > 0$. *If*

$$\max\{\triangle x, \triangle y\} \leq \sqrt{2}\frac{\min\limits_{(x_i,y_j)\in U} F(x_i, y_j)}{\sup\limits_{U} |DF(x,y)|}, \tag{C.1}$$

then F is strictly positive on U.

Proof. For every $(x, y) \in U$ there is (x_i, y_j) with $|x - x_i| \leq \frac{1}{2}\triangle x$ and $|y - y_j| \leq \frac{1}{2}\triangle y$. By the mean value theorem there exists $(\xi_1, \xi_2)\in[(x, y), (x_i, y_j)]$ such that

$$F(x, y) = F(x_i, y_j) + DF(\xi_1, \xi_2) \cdot (x - x_i, y - y_j).$$

The following chain of estimates then holds

$$F(x, y) = F(x_i, y_j) + DF(\xi_1, \xi_2) \cdot (x - x_i, y - y_j) \geq$$

$$\geq \min_{(x_i,y_j)\in U} F(x_i, y_j) - \sup_U |DF(x, y)| \, |(x - x_i, y - y_j)| \geq$$

$$\geq \min_{(x_i,y_j)\in U} F(x_i, y_j) - \frac{\sqrt{2}}{2} \sup_U |DF(x, y)| \max\{\triangle x, \triangle y\}.$$

This last expression is positive if (C.1) holds. ■

C.2 A version of the Morse theorem

Let $V \subset \mathbb{R}^2$ be open and bounded, $F \in C^\infty(V, \mathbb{R})$.

For the gradient of F we use DF and D^2F is the Hessian matrix.

Definition C.2 *A point $a \in V$ is said to be a critical point of F if $DF(a) = 0$. Moreover, the critical point $a \in V$ is said to be non-degenerate if* $\det D^2F(a) \neq 0$.

In order to study the level set Γ defined in subsection 3.4.1 we need the Morse theorem. The original version of the theorem reads as (see [29]):

> Let V be a Banach space, O a convex neighborhood of the origin in V and $f : O \to \mathbb{R}$ a C^{k+2} function ($k \geq 1$) having the origin as a non-degenerate critical point, with $f(0) = 0$. Then there is a neighborhood U of the origin and a C^k diffeomorphism $\phi : U \to O$ with $\phi(0) = 0$ and $D\phi(0) = I$, the identity map of V, such that for $x \in U$, $f(\phi(x)) = \frac{1}{2}\left(D^2f(0)x, x\right)$.

Below we give our formulation of the theorem. This formulation is more convenient for our purposes. We will give a constructive proof that allows us to find an explicit neighborhood of a critical point where the diffeomorphism exists.

Theorem C.3 *Let $V \subset \mathbb{R}^2$ be open and bounded, $F \in C^\infty(V, \mathbb{R})$. Suppose $a = (a_1, a_2) \in V$ is a non-degenerate critical point of F. There exists a neighborhood $W_a \subset V$ of a and a C^∞-diffeomorphism $h : W_a \to U_0$, where $U_0 \subset \mathbb{R}^2$ is a neighborhood of 0, such that F in W_a is representable as:*

$$F(x) = F(a) + h(x)\left(\tfrac{1}{2}D^2F(a)\right)h(x)^T, \tag{C.2}$$

where T stands for a transposition.

Moreover, the neighborhood W_a is fixed by

$$W_a \subset \left\{x \in V : \ \det B(x) \geq 0 \quad \text{and} \quad b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x) > 0\right\}.$$

where

$$B(x) = \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix} = \left(\tfrac{1}{2}D^2F(a)\right)^{-1} \int_0^1 \int_0^1 sD^2F\left(a + ts\left(x - a\right)\right) dtds.$$

Proof. Let $a \in V$ be such that $DF(a) = 0$ and $\det D^2F(a) \neq 0$.

First, for every $x \in V$ we have

$$F(x) - F(a) = F\left(a + s(x-a)\right)\Big|_0^1 = \int_0^1 \tfrac{d}{ds}F\left(a + s(x-a)\right) ds,$$

and due to $\frac{d}{ds}F\left(a + s(x-a)\right) = DF\left(a + s(x-a)\right)(x-a)^T$ it will follow that

$$F(x) = F(a) + \int_0^1 DF\left(a + s(x-a)\right) ds(x-a)^T.$$

Analogously, we obtain

$$DF(x) = DF(a) + \int_0^1 D^2F\left(a + t(x-a)\right) dt(x-a)^T,$$

where by assumption $DF(a) = 0$.

As a result, for every $x \in V$, F is representable in terms of D^2F as:

$$F(x) = F(a) + (x-a)\int_0^1 \int_0^1 sD^2F\left(a + ts(x-a)\right) dtds(x-a)^T,$$

or shortly

$$F(x) = F(a) + (x-a)K(x)(x-a)^T. \tag{C.3}$$

Here $K(x) = \int_0^1 \int_0^1 sD^2F\left(a + ts(x-a)\right) dtds$ is a symmetric matrix. With this definition, $K(a) = \frac{1}{2}D^2F\left(a\right)$ is symmetric and invertible ($\det D^2F(a) \neq 0$ by assumption).

Let us bring some intermediate results.

1) For every $x \in V$, there exists matrix B such that

$$K(x) = K(a)B(x). \tag{C.4}$$

Indeed, since $K(a)$ is invertible, the matrix $B(x) = K(a)^{-1}K(x)$ is well-defined. We write

$$B(x) = \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix},$$

Since $F \in C^\infty(V, \mathbb{R})$, so are b_{ij}, $i, j = 1, 2$. Note that $B(a) = I$.

2) Since $x \mapsto B(x)$ is C^∞ in a neighborhood of a and $B(a) = I$, $B(x)$ is positive definite in a neighborhood of a, and hence allows a square root. In particular, it holds that

$$C(x) = \sqrt{B(x)} := \frac{1}{2\pi i} \oint_\gamma \sqrt{z}\left(Iz - B(x)\right)^{-1} dz, \tag{C.5}$$

where γ is a Jordan curve in $\mathbb{C}$ which goes around the eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ of $B(x)$ and does not intersect $\mathrm{Re}(z) \leq 0$, $\mathrm{Im}(z) = 0$.

One may check that C, defined as follows

$$C(x) = \begin{pmatrix} \frac{b_{11}(x)+\sqrt{\det B(x)}}{\sqrt{b_{11}(x)+2\sqrt{\det B(x)}+b_{22}(x)}} & \frac{b_{12}(x)}{\sqrt{b_{11}(x)+2\sqrt{\det B(x)}+b_{22}(x)}} \\ \frac{b_{21}(x)}{\sqrt{b_{11}(x)+2\sqrt{\det B(x)}+b_{22}(x)}} & \frac{b_{22}(x)+\sqrt{\det B(x)}}{\sqrt{b_{11}(x)+2\sqrt{\det B(x)}+b_{22}(x)}} \end{pmatrix},$$

is indeed such that

$$C(x)^2 = B(x). \tag{C.6}$$

With this definition, $C(a) = I$ and $C(a)^2 = I = B(a)$ as required. Also, matrix C is well defined when

$$\det B(x) \geq 0 \quad \text{and} \quad b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x) > 0. \tag{C.7}$$

3) For those x one finds

$$\begin{aligned} K(a)B(x) &= K(x) = K(x)K(a)^{-1}K(a) = K(x)^T \left(K(a)^{-1}\right)^T K(a) = \\ &= \left(K(a)^{-1}K(x)\right)^T K(a) = B^T(x)K(a). \end{aligned}$$

Due to this we deduce the following

$$(Iz - B(x))\, K(a)^{-1} = K(a)^{-1} (Iz - B(x))^T ,$$

and hence

$$K(a)\,(Iz - B(x))^{-1} = \left((Iz - B(x))\, K(a)^{-1}\right)^{-1} =$$

$$= \left(K(a)^{-1}\,(Iz - B(x))^T\right)^{-1} = \left((Iz - B(x))^T\right)^{-1} K(a) =$$

$$= \left((Iz - B(x))^{-1}\right)^T K(a).$$

Applying the integration (C.5) to the last identity we find

$$K(a)C(x) = C(x)^T K(a). \tag{C.8}$$

Combining (C.4), (C.6) and (C.8) we have

$$K(x) = K(a)B(x) = K(a)C(x)^2 = C(x)^T K(a)C(x),$$

and therefore (C.3) for those x results in

$$F(x) = F(a) + F(a) + h(x)K(a)h(x)^T, \tag{C.9}$$

where $K(a) = \frac{1}{2}D^2F(a)$ and

$$h(x)^T = C(x)(x - a)^T.$$

Note that by (C.7) the representation for F in (C.9) holds on a set $W_a \subset V$ which is star-shaped with respect to a and such that

$$W_a \subset \left\{ x \in V : \ \det B(x) \geq 0 \quad \text{and} \quad b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x) > 0 \right\}. \tag{C.10}$$

■

Remark C.4 *For each pair (F, a) one can obtain an explicit estimate for W_a in (C.10). We will do this in the next subsection for the pair we are interested in.*

C.3 The Morse Theorem applied

Let P be the function given by formula (3.17) and which is defined on

$$V = \left\{(\omega, \lambda) : \left[\tfrac{70}{180}\pi, \tfrac{110}{180}\pi\right] \times [2.900, 5.100]\right\}.$$

Let us recall it here:

$$\begin{aligned} P(\omega,\lambda) = \left(1 - \tfrac{\sqrt{2}}{2}\sin(2\omega)\right)^{\lambda} + \left(1 + \tfrac{\sqrt{2}}{2}\sin(2\omega)\right)^{\lambda} + \\ +2\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\omega)\right)^{\frac{1}{2}\lambda} \cdot \cos\left\{\lambda\left[\arctan\left(\tfrac{\sqrt{2}}{2}\tan(2\omega)\right) + \pi\right]\right\} - \\ -4\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\omega)\right)^{\frac{1}{2}\lambda} \cdot \cos\left\{\lambda \arctan\left(\tan^2(\omega)\right)\right\}, \end{aligned} \tag{C.11}$$

The point $a = \left(\frac{1}{2}\pi, 4\right)$ is such that $P(a) = 0$ and $DP(a) = 0$. Theorem C.3 gives us the tool to study P in the vicinity of a. In particular, the following holds.

Proposition C.5 *Let a be as above. There is a closed ball $W_R(a) \subset V$ of a radius R centered at a such that on $W_R(a)$ we have:*

$$P(\omega,\lambda) = -\tfrac{1}{2}h_2(\omega,\lambda)\left(16\sqrt{2}h_1(\omega,\lambda) + \pi h_2(\omega,\lambda)\right). \tag{C.12}$$

Here $h_1, h_2 \in C^\infty(W_R(a), \mathbb{R})$ are given by:

$$h_1(\omega,\lambda) = \left(\omega - \tfrac{1}{2}\pi\right)c_{11}(\omega,\lambda) + (\lambda - 4)\,c_{12}(\omega,\lambda), \tag{C.13}$$

$$h_2(\omega,\lambda) = \left(\omega - \tfrac{1}{2}\pi\right)c_{21}(\omega,\lambda) + (\lambda - 4)\,c_{22}(\omega,\lambda), \tag{C.14}$$

with $c_{ij} \in C^\infty(W_R(a), \mathbb{R})$, $i, j = 1, 2$ are the entries of matrix C:

$$C(\omega,\lambda) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}(\omega,\lambda) =$$

$$= \begin{pmatrix} \frac{b_{11}+\sqrt{\det B}}{\sqrt{b_{11}+2\sqrt{\det B}+b_{22}}} & \frac{b_{12}}{\sqrt{b_{11}+2\sqrt{\det B}+b_{22}}} \\ \frac{b_{21}}{\sqrt{b_{11}+2\sqrt{\det B}+b_{22}}} & \frac{b_{22}+\sqrt{\det B}}{\sqrt{b_{11}+2\sqrt{\det B}+b_{22}}} \end{pmatrix}(\omega,\lambda), \tag{C.15}$$

while $b_{ij} \in C^{\infty}(V,\mathbb{R})$, $i,j=1,2$ *are as follows*

$$B(\omega,\lambda) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}(\omega,\lambda) =$$

$$= \left(\tfrac{1}{2}D^2P(a)\right)^{-1} \int_0^1\int_0^1 sD^2P\left(a+ts\left((\omega,\lambda)-a\right)\right)dtds. \tag{C.16}$$

Note that $B(a)=I$ *and* $C(a)=I$.

The ball $W_R(a)$ *is fixed by*

$$W_R(a) := \left\{(\omega,\lambda)\in V : \left|\left(\omega-\tfrac{1}{2}\pi, \lambda-4\right)\right| \le R\right\}, \tag{C.17}$$

with $R = -\frac{1}{120}\pi^2 - \frac{\sqrt{2}}{18}\pi + \frac{1}{120}\pi\sqrt{\pi^2+\frac{40}{3}\sqrt{2}\pi+\frac{1568}{9}} \approx 0.078$ *(In the* ω*-direction we have that* $R\approx 4.5^{\circ}$*).*

C.3.1 Computational results I

It is straightforward for $a=\left(\frac{1}{2}\pi,4\right)$ that

$$\tfrac{\partial^2P}{\partial\omega^2}(a)=0, \quad \tfrac{\partial^2P}{\partial\omega\partial\lambda}(a)=-8\sqrt{2}\pi, \quad \tfrac{\partial^2P}{\partial\lambda^2}(a)=-\pi^2, \tag{C.18}$$

and hence

$$\det D^2P(a) = -128\pi^2. \tag{C.19}$$

To simplify the notations, we use x instead of (ω,λ) when (ω,λ) stands for a argument.

Now let us bring two alternatives representations for the entries of matrix B given by (C.16), which we will use later on.

Representation I We will need the explicit formula for the coefficients b_{ij}, $i,j=1,2$. Let us find them in a straightforward way from (C.16).

We write down the integral term $\int_0^1\int_0^1 sD^2P\left(a+ts\left(x-a\right)\right)dtds$ in (C.16) as follows

$$\int_0^1\int_0^1 sD^2P\left(a+ts\left(x-a\right)\right)dtds = \begin{pmatrix} r_1(x) & r_2(x) \\ r_2(x) & r_3(x) \end{pmatrix}, \tag{C.20}$$

where r_j, $j = 1, 2, 3$ read as:

$$r_1(x) = \int_0^1 \int_0^1 s \frac{\partial^2 P}{\partial \omega^2}(a + ts(x-a))dtds, \tag{C.21}$$

$$r_2(x) = \int_0^1 \int_0^1 s \frac{\partial^2 P}{\partial \omega \partial \lambda}(a + ts(x-a))dtds, \tag{C.22}$$

$$r_3(x) = \int_0^1 \int_0^1 s \frac{\partial^2 P}{\partial \lambda^2}(a + ts(x-a))dtds. \tag{C.23}$$

Then the entries b_{ij}, $i, j = 1, 2$ of matrix B in terms of r_j, $j = 1, 2, 3$ and due to (C.18), (C.19) will read:

$$b_{11}(x) = \frac{2}{\det D^2 P(a)} \left(\frac{\partial^2 P}{\partial \lambda^2}(a) r_1(x) - \frac{\partial^2 P}{\partial \omega \partial \lambda}(a) r_2(x) \right) = \frac{1}{64} r_1(x) - \frac{\sqrt{2}}{8\pi} r_2(x), \tag{C.24}$$

$$b_{12}(x) = \frac{2}{\det D^2 P(a)} \left(\frac{\partial^2 P}{\partial \lambda^2}(a) r_2(x) - \frac{\partial^2 P}{\partial \omega \partial \lambda}(a) r_3(x) \right) = \frac{1}{64} r_2(x) - \frac{\sqrt{2}}{8\pi} r_3(x), \tag{C.25}$$

$$b_{21}(x) = \frac{2}{\det D^2 P(a)} \left(\frac{\partial^2 P}{\partial \omega^2}(a) r_2(x) - \frac{\partial^2 P}{\partial \omega \partial \lambda}(a) r_1(x) \right) = -\frac{\sqrt{2}}{8\pi} r_1(x), \tag{C.26}$$

$$b_{22}(x) = \frac{2}{\det D^2 P(a)} \left(\frac{\partial^2 P}{\partial \omega^2}(a) r_3(x) - \frac{\partial^2 P}{\partial \omega \partial \lambda}(a) r_2(x) \right) = -\frac{\sqrt{2}}{8\pi} r_2(x). \tag{C.27}$$

Representation II On the other hand, let us obtain for r_j, $j = 1, 2, 3$ the following representations:

$$r_1(x) = \frac{1}{2} \frac{\partial^2 P}{\partial \omega^2}(a) + q_1(x),$$

$$r_2(x) = \frac{1}{2} \frac{\partial^2 P}{\partial \omega \partial \lambda}(a) + q_2(x),$$

$$r_3(x) = \frac{1}{2} \frac{\partial^2 P}{\partial \lambda^2}(a) + q_3(x),$$

where

$$q_1(x) =$$

$$= \int_0^1 \int_0^1 \int_0^1 ts^2 \left(\frac{\partial^3 P}{\partial \omega^3}(a + ts\sigma(x-a)), \frac{\partial^3 P}{\partial \omega^2 \partial \lambda}(a + ts\sigma(x-a)) \right) (x-a)^T d\sigma dtds, \tag{C.28}$$

$$q_2(x) =$$

$$= \int_0^1 \int_0^1 \int_0^1 ts^2 \left(\tfrac{\partial^3 P}{\partial \omega^2 \partial \lambda}(a + ts\sigma(x-a)), \tfrac{\partial^3 P}{\partial \omega \partial \lambda^2}(a + ts\sigma(x-a)) \right) (x-a)^T d\sigma dt ds, \tag{C.29}$$

$$q_3(x) =$$

$$= \int_0^1 \int_0^1 \int_0^1 ts^2 \left(\tfrac{\partial^3 P}{\partial \omega \partial \lambda^2}(a + ts\sigma(x-a)), \tfrac{\partial^3 P}{\partial \lambda^3}(a + ts\sigma(x-a)) \right) (x-a)^T d\sigma dt ds. \tag{C.30}$$

This will yield another representation formulas for b_{ij}, $i, j = 1, 2$ of matrix B, namely,

$$b_{11}(x) = 1 + \tfrac{2}{\det D^2 P(a)} \left(\tfrac{\partial^2 P}{\partial \lambda^2}(a) q_1(x) - \tfrac{\partial^2 P}{\partial \omega \partial \lambda}(a) q_2(x) \right) = 1 + \tfrac{1}{64} q_1(x) - \tfrac{\sqrt{2}}{8\pi} q_2(x), \tag{C.31}$$

$$b_{12}(x) = \tfrac{2}{\det D^2 P(a)} \left(\tfrac{\partial^2 P}{\partial \lambda^2}(a) q_2(x) - \tfrac{\partial^2 P}{\partial \omega \partial \lambda}(a) q_3(x) \right) = \tfrac{1}{64} q_2(x) - \tfrac{\sqrt{2}}{8\pi} q_3(x), \tag{C.32}$$

$$b_{21}(x) = \tfrac{2}{\det D^2 P(a)} \left(\tfrac{\partial^2 P}{\partial \omega^2}(a) q_2(x) - \tfrac{\partial^2 P}{\partial \omega \partial \lambda}(a) q_1(x) \right) = -\tfrac{\sqrt{2}}{8\pi} q_1(x), \tag{C.33}$$

$$b_{22}(x) = 1 + \tfrac{2}{\det D^2 P(a)} \left(\tfrac{\partial^2 P}{\partial \omega^2}(a) q_3(x) - \tfrac{\partial^2 P}{\partial \omega \partial \lambda}(a) q_2(x) \right) = 1 - \tfrac{\sqrt{2}}{8\pi} q_2(x). \tag{C.34}$$

This representation, together with the estimates from above for $|q_j|$, $j = 1, 2, 3$ on V given below, will be useful in the proof of Proposition C.5.

Estimates for $|q_j|$, $j = 1, 2, 3$ on V Let q_j, $j = 1, 2, 3$ by given by (C.28) – (C.30). We will need the estimates for $|q_j|$, $j = 1, 2, 3$ on V. Let us also use the notations $\frac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}$, $\frac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}$ for the corresponding differentiations in each q_j, $j = 1, 2, 3$. For example for q_1 it will be $\frac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}} = \frac{\partial^3 P}{\partial \omega^3}$ and $\frac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}} = \frac{\partial^3 P}{\partial \omega^2 \partial \lambda}$, etc. For every q_j, $j = 1, 2, 3$ we then in general have:

$$|q_j(x)| = \left| \int_0^1 \int_0^1 \int_0^1 ts^2 \left(\tfrac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}(a + ts\sigma(x-a)), \right. \right.$$

$$\left. \left. \tfrac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}(a + ts\sigma(x-a)) \right) (x-a)^T d\sigma dt ds \right| \le$$

$$\leq \int_0^1 \int_0^1 \int_0^1 ts^2 \sup_{x \in V} \left| \left(\frac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}(x), \frac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}(x) \right) \right| d\sigma dt ds \, |x-a| =$$

$$= \frac{1}{6} \sup_{x \in V} \left| \left(\frac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}(x), \frac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}(x) \right) \right| |x-a| \leq M_j \, |x-a| \, . \qquad \text{(C.35)}$$

where M_j is an upper bound for function $\frac{1}{6} \left| \left(\frac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}(x), \frac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}(x) \right) \right|$ on V.

In particular, one shows by explicit and tedious computations that for

$$M_1 = 180, \quad M_2 = 75, \quad M_3 = 30,$$

the estimate (C.35) for the corresponding $|q_j|$, $j = 1, 2, 3$ holds true. We depict the results in Table C.1.

$\lvert q_1(x)\rvert < 180\,\lvert x-a\rvert$	$\lvert q_2(x)\rvert < 75\,\lvert x-a\rvert$	$\lvert q_3(x)\rvert < 30\,\lvert x-a\rvert$

Table C.1: *Estimates from above for* $|q_j|$, $j = 1, 2, 3$ *on* V

Now we proceed with a proof of Proposition C.5.

C.3.2 Checking the range for Morse

Proof of Proposition C.5. Representation (C.12) is a consequence of Theorem C.3. It is straightforward for $a = \left(\frac{1}{2}\pi, 4\right)$ that $P(a) = 0$. We also find that $DP(a) = 0$, meaning a is a critical point of P. Due to (C.19) we conclude that a is a non-degenerate critical point of P.

By Theorem C.3 in a vicinity $W_a \subset V$ of a which is defined in (C.10), we obtain

$$P(x) = (h_1(x), h_2(x)) \cdot \tfrac{1}{2} D^2 P(a) \cdot (h_1(x), h_2(x))^T =$$

$$= -\tfrac{1}{2} h_2(x) \left(16\sqrt{2} h_1(x) + \pi h_2(x) \right), \qquad \text{(C.36)}$$

where $h_1, h_2 \in C^\infty(W_a, \mathbb{R})$. Their explicit formulas read as (C.13), (C.14).

We show that W_a in our case can be taken as a closed ball $W_R(a)$ centered at a of radius R and the numerical approximation for its range is given by (C.17). We will do this in two steps.

1) Let us solve the inequality $\det B(x) \geq 0$ on V.

Due to (C.31) – (C.34) we will get

$$\det B(x) = b_{11}(x) b_{22}(x) - b_{12}(x) b_{21}(x) =$$

$$= 1 - \tfrac{\sqrt{2}}{4\pi} q_2(x) + \tfrac{1}{64} q_1(x) - \tfrac{1}{32\pi^2} q_1(x) q_3(x) + \tfrac{1}{32\pi^2} q_2^2(x) \geq$$
$$\geq 1 - \tfrac{\sqrt{2}}{4\pi} |q_2(x)| - \tfrac{1}{64} |q_1(x)| - \tfrac{1}{32\pi^2} |q_1(x)| |q_3(x)| \geq \ldots$$

we use estimates for $|q_1(x)|$, $|q_2(x)|$ and $|q_3(x)|$ from Table C.1 to get

$$\ldots \geq 1 - \tfrac{75\sqrt{2}}{4\pi} |x-a| - \tfrac{180}{64} |x-a| - \tfrac{5400}{32\pi^2} |x-a|^2 .$$

The last expression is positive for all $x \in V$ such that

$$|x-a| \leq R_1,$$

with

$$R_1 = -\tfrac{1}{120}\pi^2 - \tfrac{\sqrt{2}}{18}\pi + \tfrac{1}{120}\pi\sqrt{\pi^2 + \tfrac{40}{3}\sqrt{2}\pi + \tfrac{1568}{9}}.$$

The numerical approximation is $R_1 \approx 0.078$.

Hence, the first estimate for a range of W_a is $|x-a| \leq R_1$.

2) Let us solve $b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x) > 0$ on V.

We have

$$b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x) \geq b_{11}(x) + b_{22}(x) = \ldots$$

due to formulas (C.31), (C.34) we obtain

$$\ldots = 2 - \tfrac{\sqrt{2}}{4\pi} q_2(x) + \tfrac{1}{64} q_1(x) \geq 2 - \tfrac{\sqrt{2}}{4\pi} |q_2(x)| - \tfrac{1}{64} |q_1(x)| \geq \ldots$$

we use the estimates for $|q_1(x)|$ and $|q_2(x)|$ from Table C.1 to get

$$\ldots \geq 2 - \tfrac{75\sqrt{2}}{4\pi} |x-a| - \tfrac{180}{64} |x-a| .$$

The last expression is strictly positive for all $x \in V$ such that

$$|x-a| < R_2 \quad \text{with} \quad R_2 = \tfrac{32\pi}{\left(300\sqrt{2}+45\pi\right)},$$

and this is the second estimate for W_a.

Comparing the approximations to $R_1 \approx 0.078$ and $R_2 \approx 0.178$ we set $R := R_1$ and $W_a := W_R(a) = \{x \in V : |x-a| \leq R\}$.

Result (C.17) follows. ■

Remark C.6 *Let the rectangle $U \subset W_R(a)$ containing the point $a = \left(\frac{1}{2}\pi, 4\right)$ be defined as follows:*

$$U := \left\{(\omega, \lambda) : \left[\tfrac{1}{2}\pi - \tfrac{2}{180}\pi, \tfrac{1}{2}\pi + \tfrac{2}{180}\pi\right] \times [4 - 0.060, 4 + 0.060]\right\} . \qquad \text{(C.37)}$$

Proposition C.5 holds true for the given U.

C.4 On the insecting curves from Morse

Here we consider $h_2(\omega,\lambda)=0$ and $16\sqrt{2}h_1(\omega,\lambda)+\pi h_2(\omega,\lambda)=0$ in U with h_i, $i=1,2$ as in Proposition C.5.

Consider in U given by (C.37) the two implicit functions:

$$h_2(\omega,\lambda)=0, \tag{C.38}$$

$$16\sqrt{2}h_1(\omega,\lambda)+\pi h_2(\omega,\lambda)=0. \tag{C.39}$$

At $(\omega,\lambda)=a$ it holds that $h_1(a)=h_2(a)=0$, that is,

$$h_2(a)=0,$$

$$16\sqrt{2}h_1(a)+\pi h_2(a)=0.$$

Below, by means of Lemma C.1 and some numerical computations, we will show that the following holds on U:

$$\tfrac{\partial h_2}{\partial\lambda}(\omega,\lambda)>0,$$

$$16\sqrt{2}\tfrac{\partial h_1}{\partial\lambda}(\omega,\lambda)+\pi\tfrac{\partial h_2}{\partial\lambda}(\omega,\lambda)>0,$$

and hence we can apply the Implicit Function Theorem proving that every function (C.38) and (C.39) allows its local parametrization $\omega\mapsto\lambda(\omega)$ in U. This fact is used in Lemma 3.16.

Now some preparatory technical steps are required.

C.4.1 Computational results II

Upper bounds for $|r_j|$, $j=1,2,3$ on U

Let r_j, $j=1,2,3$ be given by (C.21) – (C.23). We will find the upper bounds for $|r_j|$, $j=1,2,3$ on U.

Setting again $\frac{\partial^2 P}{\partial(\omega,\lambda)^{\alpha_j}}$ for the corresponding differentiations in each r_j, $j=1,2,3$, we in general deduce that

$$|r_j(x)|=\left|\int_0^1\int_0^1 s\tfrac{\partial^2 P}{\partial(\omega,\lambda)^{\alpha_j}}(a+ts(x-a))dtds\right|\le\int_0^1\int_0^1 s\sup_{x\in U}\left|\tfrac{\partial^2 P}{\partial(\omega,\lambda)^{\alpha_j}}(x)\right|dtds=$$

$$=\tfrac{1}{2}\sup_{x\in U}\left|\tfrac{\partial^2 P}{\partial(\omega,\lambda)^{\alpha_j}}(x)\right|\le Q_j,$$

where Q_j, $j = 1,2,3$ is an upper bound for the function $\frac{1}{2}\left|\frac{\partial^2 P}{\partial(\omega,\lambda)^{\alpha_j}}(x)\right|$ on U.

In analogous way we find the upper bounds for $\left|\frac{\partial r_j}{\partial \omega}\right|$, $\left|\frac{\partial r_j}{\partial \lambda}\right|$ and $\left|\frac{\partial^2 r_j}{\partial\omega\partial\lambda}\right|$, $\left|\frac{\partial^2 r_j}{\partial\lambda^2}\right|$, $j = 1,2,3$ on U, we will need later on. Explicit bounds are given in Table C.2. Note that we skip the derivatives $\frac{\partial^2 r_j}{\partial\omega^2}$ since there will be no need for them.

$\lvert r_1(x)\rvert < 5$	$\left\lvert\frac{\partial r_1}{\partial\omega}(x)\right\rvert < 43$ $\left\lvert\frac{\partial r_1}{\partial\lambda}(x)\right\rvert < 25$	$\left\lvert\frac{\partial^2 r_1}{\partial\omega\partial\lambda}(x)\right\rvert < 175$ $\left\lvert\frac{\partial^2 r_1}{\partial\lambda^2}(x)\right\rvert < 65$
$\lvert r_2(x)\rvert < 19$	$\left\lvert\frac{\partial r_2}{\partial\omega}(x)\right\rvert < 25$ $\left\lvert\frac{\partial r_2}{\partial\lambda}(x)\right\rvert < 6$	$\left\lvert\frac{\partial^2 r_2}{\partial\omega\partial\lambda}(x)\right\rvert < 65$ $\left\lvert\frac{\partial^2 r_2}{\partial\lambda^2}(x)\right\rvert < 33$
$\lvert r_3(x)\rvert < 6$	$\left\lvert\frac{\partial r_3}{\partial\omega}(x)\right\rvert < 6$ $\left\lvert\frac{\partial r_3}{\partial\lambda}(x)\right\rvert < 4$	$\left\lvert\frac{\partial^2 r_3}{\partial\omega\partial\lambda}(x)\right\rvert < 33$ $\left\lvert\frac{\partial^2 r_3}{\partial\lambda^2}(x)\right\rvert < 16$

Table C.2: *Estimates from above for the absolute value of* r_j, $j = 1,2,3$ *and some higher order derivatives on* U

Upper bounds for $|q_j|$, $j = 1,2,3$ **on** U

Let q_j, $j = 1,2,3$ be given by (C.28) – (C.30).

Earlier we found the estimates for $|q_j|$, $j = 1,2,3$ on V of the type $|q_j| \leq M_j |x-a|$, $j = 1,2,3$ (see Table C.1). Here, we will obtain the constants which are the upper bounds for $|q_j|$, $j = 1,2,3$ on U.

Setting $\frac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}$, $\frac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}$ for the corresponding differentiations in each q_j, $j = 1,2,3$, analogously to (C.35), we have that

$$|q_j(x)| = \left|\int_0^1\int_0^1\int_0^1 ts^2\left(\frac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}(a+ts\sigma(x-a)),\right.\right.$$

$$\left.\left.\frac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}(a+ts\sigma(x-a))\right)(x-a)^T d\sigma dt ds\right| \leq$$

$$\leq \tfrac{1}{6} \sup_{x\in U} \left| \left(\tfrac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}(x), \tfrac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}(x) \right) \right| \max_U |x-a| \leq M_j \max_U |x-a|, \quad \text{(C.40)}$$

where M_j is an upper bound for the function $\frac{1}{6}\left| \left(\frac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}(x), \frac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}(x) \right) \right|$ on U. In particular, it holds that

$$\max_U |x-a| = |x-a||_{\text{on }\partial U} =$$

$$= \sqrt{\left(\omega - \tfrac{1}{2}\pi\right)^2 + (\lambda-4)^2}\Big|_{\text{at }(\omega,\lambda)=\left(\frac{1}{2}\pi - \frac{2}{180}\pi, 4-0.060\right)} =$$

$$= \sqrt{\left(\tfrac{1}{90}\pi\right)^2 + 0.060^2}, \quad \text{(C.41)}$$

and

$$M_1 = 44, \quad M_2 = 27, \quad M_3 = 11.$$

The explicit upper bounds are given in Table C.3.

$\lvert q_1(x)\rvert < 3.1$	$\lvert q_2(x)\rvert < 1.9$	$\lvert q_3(x)\rvert < 0.8$

Table C.3: *Estimates from above for* $|q_j|$, $j = 1, 2, 3$ *on* U

Lower bounds for $F(x) = \det B(x)$ and $G(x) = b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x)$ on U

Let us set

$$\begin{cases} F(x) = \det B(x), \\ G(x) = b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x), \end{cases} \quad \text{(C.42)}$$

b_{ij}, $i, j = 1, 2$ are given in (C.31) – (C.34) and find the lower bounds for F and G on U.

By construction of the ball $W_R(a) \supset U$ from Proposition C.5 we know that $F \geq 0$ and $G > 0$ on $W_R(a)$. More precisely, for every $x \in W_R(a)$ (and hence for every $x \in U$) it holds that

$$F(x) \geq 1 - \tfrac{\sqrt{2}}{4\pi}|q_2(x)| - \tfrac{1}{64}|q_1(x)| - \tfrac{1}{32\pi^2}|q_1(x)|\,|q_3(x)|,$$

and

$$G(x) \geq 2 - \tfrac{\sqrt{2}}{4\pi}|q_2(x)| - \tfrac{1}{64}|q_1(x)|.$$

Due to results of Table C.3 we finally obtain that on U

$$\begin{cases} F(x) \geq 0.730... > 0.7, \\ G(x) \geq 1.738... > 1.7. \end{cases} \tag{C.43}$$

Upper bounds for $|b_{ij}|$, $i, j = 1, 2$ on U

Let b_{ij}, $i, j = 1, 2$ be given by (C.24) – (C.27), namely,

$$b_{11}(x) = \tfrac{1}{64} r_1(x) - \tfrac{\sqrt{2}}{8\pi} r_2(x),$$

$$b_{12}(x) = \tfrac{1}{64} r_2(x) - \tfrac{\sqrt{2}}{8\pi} r_3(x),$$

$$b_{21}(x) = -\tfrac{\sqrt{2}}{8\pi} r_1(x),$$

$$b_{22}(x) = -\tfrac{\sqrt{2}}{8\pi} r_2(x),$$

with r_j, $j = 1, 2, 3$ as in (C.21) – (C.23).

Using the results of Table C.2 we will find the following upper bounds for the absolute values of b_{ij}, $i, j = 1, 2$ and some higher order derivatives on U (see Table C.4).

Upper bounds for $F(x) = \det B(x)$ and $G(x) = b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x)$ on U

Recall that F and G are given by (C.42). They are positive functions on U with lower bounds as in (C.43). Here we find their upper bounds together with the upper bounds for some higher order derivatives.

In particular, in order to obtain the estimates for F, $\left|\frac{\partial F}{\partial \omega}\right|$, $\left|\frac{\partial F}{\partial \lambda}\right|$ and $\left|\frac{\partial^2 F}{\partial \omega \partial \lambda}\right|$, $\left|\frac{\partial^2 F}{\partial \lambda^2}\right|$ on U, we use the results of Table C.4. The estimates found are presented in the first row of Table C.5.

In order to find the estimates for $\left|\frac{\partial G}{\partial \omega}\right|$, $\left|\frac{\partial G}{\partial \lambda}\right|$ and $\left|\frac{\partial^2 G}{\partial \omega \partial \lambda}\right|$, $\left|\frac{\partial^2 G}{\partial \lambda^2}\right|$ we use

- the lower bound for Fon U, namely, $F(x) > 0.7$,
- the results of Table C.4 and
- the results from the first row of Table C.5.

E.g. for $\left|\frac{\partial G}{\partial \omega}\right|$ we will have that on U

$$\left|\frac{\partial G}{\partial \omega}(x)\right| = \left|\frac{\partial b_{11}}{\partial \omega}(x) + \frac{\frac{\partial F}{\partial \omega}(x)}{\sqrt{F(x)}} + \frac{\partial b_{22}}{\partial \omega}(x)\right| \leq$$

$\|b_{11}(x)\| < 1.2$	$\left\|\frac{\partial b_{11}}{\partial \omega}(x)\right\| < 2.1$ $\left\|\frac{\partial b_{11}}{\partial \lambda}(x)\right\| < 0.8$	$\left\|\frac{\partial^2 b_{11}}{\partial \omega \partial \lambda}(x)\right\| < 6.4$ $\left\|\frac{\partial^2 b_{11}}{\partial \lambda^2}(x)\right\| < 2.9$
$\|b_{12}(x)\| < 0.7$	$\left\|\frac{\partial b_{12}}{\partial \omega}(x)\right\| < 0.8$ $\left\|\frac{\partial b_{12}}{\partial \lambda}(x)\right\| < 0.4$	$\left\|\frac{\partial^2 b_{12}}{\partial \omega \partial \lambda}(x)\right\| < 2.9$ $\left\|\frac{\partial^2 b_{12}}{\partial \lambda^2}(x)\right\| < 1.5$
$\|b_{21}(x)\| < 0.3$	$\left\|\frac{\partial b_{21}}{\partial \omega}(x)\right\| < 2.5$ $\left\|\frac{\partial b_{21}}{\partial \lambda}(x)\right\| < 1.5$	$\left\|\frac{\partial^2 b_{21}}{\partial \omega \partial \lambda}(x)\right\| < 9.9$ $\left\|\frac{\partial^2 b_{21}}{\partial \lambda^2}(x)\right\| < 3.7$
$\|b_{22}(x)\| < 1.1$	$\left\|\frac{\partial b_{22}}{\partial \omega}(x)\right\| < 1.5$ $\left\|\frac{\partial b_{22}}{\partial \lambda}(x)\right\| < 0.4$	$\left\|\frac{\partial^2 b_{22}}{\partial \omega \partial \lambda}(x)\right\| < 3.7$ $\left\|\frac{\partial^2 b_{22}}{\partial \lambda^2}(x)\right\| < 1.9$

Table C.4: *Estimates from above for the absolute value of* b_{ij}, $i, j = 1, 2$ *and some higher order derivatives on* U

$$\leq \sup_U \left|\frac{\partial b_{11}}{\partial \omega}(x)\right| + \frac{\sup_U \left|\frac{\partial F}{\partial \omega}(x)\right|}{\inf_U \sqrt{F(x)}} + \sup_U \left|\frac{\partial b_{22}}{\partial \omega}(x)\right| <$$

$$< 2.1 + \frac{6.1}{\sqrt{0.7}} + 1.5 \approx 10.891 < 11.$$

The other estimates on U for the derivatives of G listed above are obtained in an analogous way and presented in Table C.5.

$F(x) < 1.53$	$\left\|\frac{\partial F}{\partial \omega}(x)\right\| < 6.10$ $\left\|\frac{\partial F}{\partial \lambda}(x)\right\| < 2.53$	$\left\|\frac{\partial^2 F}{\partial \omega \partial \lambda}(x)\right\| < 23.52$ $\left\|\frac{\partial^2 F}{\partial \lambda^2}(x)\right\| < 10.35$
$-$	$\left\|\frac{\partial G}{\partial \omega}(x)\right\| < 11$ $\left\|\frac{\partial G}{\partial \lambda}(x)\right\| < 4.3$	$\left\|\frac{\partial^2 G}{\partial \omega \partial \lambda}(x)\right\| < 51.4$ $\left\|\frac{\partial^2 G}{\partial \lambda^2}(x)\right\| < 22.7$

Table C.5: *Estimates from above for* F and G *on* U

Upper bounds for $|c_{ij}|$, $i,j=1,2$ on U

Let c_{ij}, $i,j=1,2$ be given by given by (C.15). It is convenient for further computations to set

$$c_{ij}(x)=\frac{b_{ij}(x)+A\sqrt{F(x)}}{\sqrt{G(x)}}, \tag{C.44}$$

where F and G are as in (C.42) and

$$A=\begin{cases}1 & \text{if} \quad (i,j)=\{(1,1),\ (2,2)\}\,,\\ 0 & \text{if} \quad (i,j)=\{(1,2),\ (2,1)\}\,.\end{cases}$$

In order to obtain the estimates for $\left|\frac{\partial c_{ij}}{\partial \omega}\right|$, $\left|\frac{\partial c_{ij}}{\partial \lambda}\right|$ and $\left|\frac{\partial^2 c_{ij}}{\partial \omega \partial \lambda}\right|$, $\left|\frac{\partial^2 c_{ij}}{\partial \lambda^2}\right|$, $i,j=1,2$ on U we use

- the lower bound for F and G on U, defined by formula (C.43),
- the results of Table C.4 and
- the results of Table C.5.

E.g. the estimate for $\left|\frac{\partial c_{ij}}{\partial \lambda}\right|$ on U is found in the following way:

$$\left|\frac{\partial c_{ij}}{\partial \lambda}(x)\right|=\left|\frac{\frac{\partial b_{ij}}{\partial \lambda}(x)}{\sqrt{G(x)}}+\frac{1}{2}A\frac{\frac{\partial F}{\partial \lambda}(x)}{\sqrt{G(x)}\sqrt{F(x)}}-\frac{1}{2}\frac{\frac{\partial G}{\partial \lambda}(x)}{G^{3/2}(x)}\left(b_{ij}(x)+A\sqrt{F(x)}\right)\right|\leq$$

$$\leq\left|\frac{\sup\limits_U\left|\frac{\partial b_{ij}}{\partial \lambda}(x)\right|}{\inf\limits_U\sqrt{G(x)}}+\frac{1}{2}A\frac{\sup\limits_U\left|\frac{\partial F}{\partial \lambda}(x)\right|}{\inf\limits_U\left(\sqrt{G(x)}\sqrt{F(x)}\right)}+\right.$$

$$\left.+\frac{1}{2}\frac{\sup\limits_U\left|\frac{\partial G}{\partial \lambda}(x)\right|}{\inf\limits_U G^{3/2}(x)}\left(\sup\limits_U|b_{ij}(x)|+A\sup\limits_U\sqrt{F(x)}\right)\right|.$$

The other estimates on U for the derivatives of c_{ij}, $i,j=1,2$ listed are obtained in an analogous way and listed in Table C.6 below.

Upper bounds for $|h_i|$, $i=1,2$ on U

Let h_i, $i=1,2$ be given by formulas (C.13), (C.14).

First we compute the following derivatives of h_i, $i=1,2$ we will need below:

$$\frac{\partial h_1}{\partial \lambda}(x)=c_{12}(x)+\left(\frac{\partial c_{11}}{\partial \lambda}(x)\,,\frac{\partial c_{12}}{\partial \lambda}(x)\right)(x-a)^T\,, \tag{C.45}$$

$\left\lvert \frac{\partial c_{11}}{\partial \lambda}(x) \right\rvert < 4.2$	$\left\lvert \frac{\partial^2 c_{11}}{\partial \omega \partial \lambda}(x) \right\rvert < 99.4$ $\left\lvert \frac{\partial^2 c_{11}}{\partial \lambda^2}(x) \right\rvert < 35$
$\left\lvert \frac{\partial c_{12}}{\partial \omega}(x) \right\rvert < 2.4$ $\left\lvert \frac{\partial c_{12}}{\partial \lambda}(x) \right\rvert < 1$	$\left\lvert \frac{\partial^2 c_{12}}{\partial \omega \partial \lambda}(x) \right\rvert < 18.7$ $\left\lvert \frac{\partial^2 c_{12}}{\partial \lambda^2}(x) \right\rvert < 8.1$
$\left\lvert \frac{\partial c_{21}}{\partial \lambda}(x) \right\rvert < 1.5$	$\left\lvert \frac{\partial^2 c_{21}}{\partial \omega \partial \lambda}(x) \right\rvert < 20.1$ $\left\lvert \frac{\partial^2 c_{21}}{\partial \lambda^2}(x) \right\rvert < 8.4$
$\left\lvert \frac{\partial c_{22}}{\partial \omega}(x) \right\rvert < 9.8$ $\left\lvert \frac{\partial c_{22}}{\partial \lambda}(x) \right\rvert < 3.8$	$\left\lvert \frac{\partial^2 c_{22}}{\partial \omega \partial \lambda}(x) \right\rvert < 94.4$ $\left\lvert \frac{\partial^2 c_{22}}{\partial \lambda^2}(x) \right\rvert < 38.7$

Table C.6: *Estimates from above for the absolute value of some higher order derivatives of c_{ij}, $i,j = 1,2$ on U*

$$\frac{\partial^2 h_1}{\partial \omega \partial \lambda}(x) = \frac{\partial c_{12}}{\partial \omega}(x) + \frac{\partial c_{11}}{\partial \lambda}(x) + \left(\frac{\partial^2 c_{11}}{\partial \omega \partial \lambda}(x), \frac{\partial^2 c_{12}}{\partial \omega \partial \lambda}(x) \right) (x-a)^T, \tag{C.46}$$

$$\frac{\partial^2 h_1}{\partial \lambda^2}(x) = 2\frac{\partial c_{12}}{\partial \lambda}(x) + \left(\frac{\partial^2 c_{11}}{\partial \lambda^2}(x), \frac{\partial^2 c_{12}}{\partial \lambda^2}(x) \right) (x-a)^T, \tag{C.47}$$

and

$$\frac{\partial h_2}{\partial \lambda}(x) = c_{22}(x) + \left(\frac{\partial c_{21}}{\partial \lambda}(x), \frac{\partial c_{22}}{\partial \lambda}(x) \right) (x-a)^T, \tag{C.48}$$

$$\frac{\partial^2 h_2}{\partial \omega \partial \lambda}(x) = \frac{\partial c_{22}}{\partial \omega}(x) + \frac{\partial c_{21}}{\partial \lambda}(x) + \left(\frac{\partial^2 c_{21}}{\partial \omega \partial \lambda}(x), \frac{\partial^2 c_{22}}{\partial \omega \partial \lambda}(x) \right) (x-a)^T, \tag{C.49}$$

$$\frac{\partial^2 h_2}{\partial \lambda^2}(x) = 2\frac{\partial c_{22}}{\partial \lambda}(x) + \left(\frac{\partial^2 c_{21}}{\partial \lambda^2}(x), \frac{\partial^2 c_{22}}{\partial \lambda^2}(x) \right) (x-a)^T, \tag{C.50}$$

where c_{ij}, $i,j = 1,2$ are as in (C.44) and $a = \left(\frac{1}{2}\pi, 4\right)$.

Using the results of Table C.6 we find the estimates for $\left\lvert \frac{\partial^2 h_i}{\partial \omega \partial \lambda} \right\rvert, \left\lvert \frac{\partial^2 h_i}{\partial \lambda^2} \right\rvert$, $i = 1,2$ on U.

E.g. for $\left\lvert \frac{\partial^2 h_1}{\partial \omega \partial \lambda} \right\rvert$ it holds that on U:

$$\left\lvert \frac{\partial^2 h_1}{\partial \omega \partial \lambda}(x) \right\rvert = \left\lvert \frac{\partial c_{12}}{\partial \omega}(x) + \frac{\partial c_{11}}{\partial \lambda}(x) + \left(\frac{\partial^2 c_{11}}{\partial \omega \partial \lambda}(x), \frac{\partial^2 c_{12}}{\partial \omega \partial \lambda}(x) \right) (x-a)^T \right\rvert \leq$$

$$\leq \sup_U \left\lvert \frac{\partial c_{12}}{\partial \omega}(x) \right\rvert + \sup_U \left\lvert \frac{\partial c_{11}}{\partial \lambda}(x) \right\rvert + \sup_U \left\lvert \left(\frac{\partial^2 c_{11}}{\partial \omega \partial \lambda}(x), \frac{\partial^2 c_{12}}{\partial \omega \partial \lambda}(x) \right) \right\rvert \max_U |x-a| < \ldots$$

for $\max_U |x-a|$ see formula (C.41) and then

$$\ldots < 2.4 + 4.2 + \sqrt{99.4^2 + 18.7^2}\sqrt{\left(\tfrac{1}{90}\pi\right)^2 + 0.060^2} \approx 13.621 < 13.7.$$

Analogously, the other estimates on U are obtained (see Table C.7).

$\left\lvert\frac{\partial^2 h_1}{\partial\omega\partial\lambda}(x)\right\rvert < 13.7$	$\left\lvert\frac{\partial^2 h_2}{\partial\omega\partial\lambda}(x)\right\rvert < 18$
$\left\lvert\frac{\partial^2 h_1}{\partial\lambda^2}(x)\right\rvert < 4.5$	$\left\lvert\frac{\partial^2 h_2}{\partial\lambda^2}(x)\right\rvert < 10.4$

Table C.7: *Estimates from above for the absolute value of some higher order derivatives of h_i, $i = 1, 2$ on U*

C.4.2 Strict positivity of the functions $\frac{\partial h_2}{\partial\lambda}(\omega,\lambda)$ and $16\sqrt{2}\frac{\partial h_1}{\partial\lambda}(\omega,\lambda) + \pi\frac{\partial h_2}{\partial\lambda}(\omega,\lambda)$ on U

Let us fix the notations which are common for two lemmas.

Notation C.7 *Let U be as in (C.37), namely,*

$$U := \left\{(\omega,\lambda) : \left[\tfrac{88}{180}\pi, \tfrac{92}{180}\pi\right] \times [3.940, 4.060]\right\}.$$

By $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\ldots,n\\ j=0,\ldots,m}}$ we mean a discretization of U which is defined as follows

$$\omega_i = \tfrac{88}{180}\pi + i\triangle\omega, \quad \lambda_j = 3.94 + j\triangle\lambda,$$

with

$$\triangle\omega = \frac{\frac{4}{180}\pi}{n}, \quad \triangle\lambda = \frac{0.12}{m}.$$

Then we deduce the following two results.

Lemma C.8 *It holds that*

$$\frac{\partial h_2}{\partial\lambda}(\omega,\lambda) > 0 \quad on \quad U. \tag{C.51}$$

Proof. Fix $n = m = 2$ and consider the discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\ldots,2\\ j=0,\ldots,2}}$ of U given by Notation C.7.

By straightforward computations it holds that

$$\tfrac{\partial h_2}{\partial \lambda}(\omega_i, \lambda_j) > 0, \tag{C.52}$$

for all $i = 0, ..., 2$, $j = 0, ..., 2$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in U} \tfrac{\partial h_2}{\partial \lambda}(\omega_i, \lambda_j) = \tfrac{\partial h_2}{\partial \lambda}(\omega_0, \lambda_0) \approx 0.952. \tag{C.53}$$

Next to this, we compute

$$a_1 = \max\{\triangle\omega, \triangle\lambda\} = \max\left\{\frac{\frac{4}{180}\pi}{n}, \frac{0.12}{m}\right\} = \frac{0.12}{2} = 0.060,$$

and, by taking into account the results of Table C.7 and (C.53), we also find

$$a_2 = \sqrt{2}\frac{\min\limits_{(\omega_i,\lambda_j)\in U} \frac{\partial h_2}{\partial \lambda}(\omega_i, \lambda_j)}{\sup\limits_U \left|\left(\frac{\partial^2 h_2}{\partial \omega \partial \lambda}(\omega, \lambda), \frac{\partial^2 h_2}{\partial \lambda^2}(\omega, \lambda)\right)\right|} \approx 0.065.$$

Since $a_1 < a_2$, by Lemma C.1 we conclude that the constructed discretization $\{(\omega_i, \lambda_j)\}_{\substack{i=0,...,2\\ j=0,...,2}}$ of U to be appropriate in the sense that condition (C.52) yields a strict positivity of $\frac{\partial h_2}{\partial \lambda}(\omega, \lambda)$ on U. ∎

Lemma C.9 *Let U be as in (C.37). It holds that*

$$16\sqrt{2}\tfrac{\partial h_1}{\partial \lambda}(\omega, \lambda) + \pi \tfrac{\partial h_2}{\partial \lambda}(\omega, \lambda) > 0 \quad on \quad U. \tag{C.54}$$

Proof. Fix $n = 7$, $m = 12$ and consider the discretization $\{(\omega_i, \lambda_j)\}_{\substack{i=0,...,7\\ j=0,...,12}}$ of U given by Notation C.7.

By straightforward computations it holds that

$$16\sqrt{2}\tfrac{\partial h_1}{\partial \lambda}(\omega_i, \lambda_j) + \pi \tfrac{\partial h_2}{\partial \lambda}(\omega_i, \lambda_j) > 0, \tag{C.55}$$

for all $i = 0, ..., 7$, $j = 0, ..., 12$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in U} \left(16\sqrt{2}\tfrac{\partial h_1}{\partial \lambda}(\omega_i, \lambda_j) + \pi \tfrac{\partial h_2}{\partial \lambda}(\omega_i, \lambda_j)\right) =$$

$$= 16\sqrt{2}\tfrac{\partial h_1}{\partial \lambda}(\omega_0, \lambda_0) + \pi \tfrac{\partial h_2}{\partial \lambda}(\omega_0, \lambda_0) \approx 2.936. \tag{C.56}$$

Next to this, we compute

$$b_1 = \max\{\triangle\omega, \triangle\lambda\} = \max\left\{\frac{\frac{4}{180}\pi}{n}, \frac{0.12}{m}\right\} = \frac{0.12}{12} = 0.010,$$

and, by taking into account the results of Table C.7 and (C.56), we also find

$$b_2 =$$

$$= \frac{\sqrt{2}\min\limits_{(\omega_i,\lambda_j)\in U}\left(16\sqrt{2}\frac{\partial h_1}{\partial\lambda}(\omega_i,\lambda_j) + \pi\frac{\partial h_2}{\partial\lambda}(\omega_i,\lambda_j)\right)}{\sup\limits_U\left|\left(16\sqrt{2}\frac{\partial^2 h_1}{\partial\omega\partial\lambda}(\omega,\lambda) + \pi\frac{\partial^2 h_2}{\partial\omega\partial\lambda}(\omega,\lambda), 16\sqrt{2}\frac{\partial^2 h_1}{\partial\lambda^2}(\omega,\lambda) + \pi\frac{\partial^2 h_2}{\partial\lambda^2}(\omega,\lambda)\right)\right|} \approx$$

$$\approx 0.011.$$

Since $b_1 < b_2$, by Lemma C.1 we conclude that the constructed discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\ldots,7\\ j=0,\ldots,12}}$ of U to be appropriate in the sense that condition (C.55) yields a strict positivity of $16\sqrt{2}\frac{\partial h_1}{\partial\lambda}(\omega,\lambda) + \pi\frac{\partial h_2}{\partial\lambda}(\omega,\lambda)$ on U. ■

C.5 On $P(\omega,\lambda) = 0$ in V away from $a = \left(\frac{1}{2}\pi, 4\right)$

Recall here function P defined on $V = \left\{(\omega,\lambda) : \left[\frac{70}{180}\pi, \frac{110}{180}\pi\right] \times [2.900, 5.100]\right\}$:

$$\begin{aligned} P(\omega,\lambda) &= \left(1 - \tfrac{\sqrt{2}}{2}\sin(2\omega)\right)^{\lambda} + \left(1 + \tfrac{\sqrt{2}}{2}\sin(2\omega)\right)^{\lambda} + \\ &+2\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\omega)\right)^{\frac{1}{2}\lambda}\cdot\cos\left\{\lambda\left[\arctan\left(\tfrac{\sqrt{2}}{2}\tan(2\omega)\right) + \pi\right]\right\} - \\ &-4\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\omega)\right)^{\frac{1}{2}\lambda}\cdot\cos\left\{\lambda\arctan\left(\tan^2(\omega)\right)\right\}, \end{aligned} \quad \text{(C.57)}$$

C.5.1 Set of Claims I

In a set of claims below we describe some properties of the first derivatives $\frac{\partial P}{\partial\omega}$ and $\frac{\partial P}{\partial\lambda}$ on V away from the point $a = \left(\frac{1}{2}\pi, 4\right)$ which are used in Lemma 3.16.

Claim C.10 *Let $H_1 \subset V$ be as follows*

$$H_1 = \left\{(\omega,\lambda) : \left[\frac{84}{180}\pi, \frac{90}{180}\pi\right] \times [4.030, 4.970]\right\}.$$

It holds that $\frac{\partial P}{\partial\omega}(\omega,\lambda) < 0$ on H_1.

Proof. First we find the following estimates:

$$\left|\frac{\partial^2 P}{\partial \omega^2}(\omega,\lambda)\right| < 162, \quad \left|\frac{\partial^2 P}{\partial \omega \partial \lambda}(\omega,\lambda)\right| < 45 \quad \text{on} \quad H_1. \tag{C.58}$$

Then we fix $n = 14$, $m = 120$ and consider the discretization $\{(\omega_i, \lambda_j)\}_{\substack{i=0,14 \\ j=0,\dots,120}}$ of H_1 such that

$$\omega_i = \tfrac{84}{180}\pi + i\triangle\omega, \quad \lambda_j = 4.030 + j\triangle\lambda,$$

with

$$\triangle\omega = \frac{\frac{6}{180}\pi}{n}, \quad \triangle\lambda = \frac{0.94}{m}.$$

By straightforward computations it holds that

$$-\tfrac{\partial P}{\partial \omega}(\omega_i, \lambda_j) > 0, \tag{C.59}$$

for all $i = 0, ..., 14$, $j = 0, ..., 120$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_1} \left(-\tfrac{\partial P}{\partial \omega}(\omega_i,\lambda_j)\right) = -\tfrac{\partial P}{\partial \omega}(\omega_9, \lambda_0) \approx 1.022. \tag{C.60}$$

Next to this, we compute

$$c_1 = \max\{\triangle\omega, \triangle\lambda\} = \max\left\{\frac{\frac{6}{180}\pi}{n}, \frac{0.94}{m}\right\} = \frac{0.94}{120} \approx 0.00783,$$

and, by taking into account (C.58) and (C.60), we also find

$$c_2 = \sqrt{2}\frac{\min\limits_{(\omega_i,\lambda_j)\in H_1} \left(-\frac{\partial P}{\partial \omega}(\omega_i,\lambda_j)\right)}{\sup\limits_{H_1}\left|\left(\frac{\partial^2 P}{\partial \omega^2}(\omega,\lambda), \frac{\partial^2 P}{\partial \omega \partial \lambda}(\omega,\lambda)\right)\right|} \approx 0.00860.$$

Since $c_1 < c_2$, by Lemma C.1 we conclude that the constructed discretization $\{(\omega_i, \lambda_j)\}_{\substack{i=0,\dots,14 \\ j=0,\dots,120}}$ of H_1 to be appropriate in the sense that condition (C.59) yields a strict positivity of $-\frac{\partial P}{\partial \omega}(\omega,\lambda)$ on H_1, or, in other words, $\frac{\partial P}{\partial \omega}(\omega,\lambda) < 0$ on H_1. ■

Claim C.11 *Let $H_2 \subset V$ be as follows*

$$H_2 = \left\{(\omega,\lambda) : \left[\tfrac{87}{180}\pi, \tfrac{101}{180}\pi\right] \times [4.750, 5.100]\right\}.$$

It holds that $\frac{\partial P}{\partial \lambda}(\omega,\lambda) > 0$ on H_2.

Proof. First we find the following estimates:

$$\left|\frac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda)\right| < 48, \quad \left|\frac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)\right| < 25 \quad \text{on} \quad H_2. \tag{C.61}$$

Then we fix $n=40$, $m=55$ and consider the discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,40\\ j=0,\dots,55}}$ of H_2 such that

$$\omega_i = \frac{87}{180}\pi + i\triangle\omega, \quad \lambda_j = 4.750 + j\triangle\lambda,$$

with

$$\triangle\omega = \frac{\frac{14}{180}\pi}{n}, \quad \triangle\lambda = \frac{0.35}{m}.$$

By straightforward computations it holds that

$$\frac{\partial P}{\partial\lambda}(\omega_i,\lambda_j) > 0, \tag{C.62}$$

for all $i=0,\dots,40$, $j=0,\dots,55$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_2}\frac{\partial P}{\partial\lambda}(\omega_i,\lambda_j) = \frac{\partial P}{\partial\lambda}(\omega_0,\lambda_0) \approx 0.245. \tag{C.63}$$

Next to this, we compute

$$c_1 = \max\{\triangle\omega,\triangle\lambda\} = \max\left\{\frac{\frac{14}{180}\pi}{n}, \frac{0.35}{m}\right\} = \frac{0.35}{55} \approx 0.00636,$$

and, by taking into account (C.61) and (C.63), we also find

$$c_2 = \sqrt{2}\frac{\min\limits_{(\omega_i,\lambda_j)\in H_2}\frac{\partial P}{\partial\lambda}(\omega_i,\lambda_j)}{\sup\limits_{H_2}\left|\left(\frac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda), \frac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)\right)\right|} \approx 0.00641.$$

Since $c_1 < c_2$, by Lemma C.1 we conclude that the constructed discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,40\\ j=0,\dots,55}}$ of H_2 to be appropriate in the sense that condition (C.62) yields a strict positivity of $\frac{\partial P}{\partial\lambda}(\omega,\lambda)$ on H_2. ■

Claim C.12 *Let $H_3 \subset V$ be as follows*

$$H_3 = \left\{(\omega,\lambda) : \left[\frac{100}{180}\pi, \frac{108}{180}\pi\right] \times [4.000, 4.850]\right\}.$$

It holds that $\frac{\partial P}{\partial\omega}(\omega,\lambda) > 0$ on H_3.

Proof. First we find the following estimates:

$$\left|\frac{\partial^2 P}{\partial \omega^2}(\omega,\lambda)\right| < 166, \quad \left|\frac{\partial^2 P}{\partial \omega \partial \lambda}(\omega,\lambda)\right| < 37 \quad \text{on} \quad H_3. \tag{C.64}$$

Then we fix $n = 10$, $m = 60$ and consider the discretization $\{(\omega_i, \lambda_j)\}_{\substack{i=0,\ldots,10 \\ j=0,\ldots,60}}$ of H_3 such that

$$\omega_i = \frac{100}{180}\pi + i\triangle\omega, \quad \lambda_j = 4.000 + j\triangle\lambda,$$

with

$$\triangle\omega = \frac{\frac{8}{180}\pi}{n}, \quad \triangle\lambda = \frac{0.85}{m}.$$

By straightforward computations it holds that

$$\frac{\partial P}{\partial \omega}(\omega_i, \lambda_j) > 0, \tag{C.65}$$

for all $i = 0, ..., 10$, $j = 0, ..., 60$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_3} \frac{\partial P}{\partial \omega}(\omega_i, \lambda_j) = \frac{\partial P}{\partial \omega}(\omega_0, \lambda_{11}) \approx 1.885. \tag{C.66}$$

Next to this, we compute

$$c_1 = \max\{\triangle\omega, \triangle\lambda\} = \max\left\{\frac{\frac{8}{180}\pi}{n}, \frac{0.85}{m}\right\} = \frac{0.85}{60} \approx 0.0142,$$

and, by taking into account (C.64) and (C.66), we also find

$$c_2 = \sqrt{2}\frac{\min\limits_{(\omega_i,\lambda_j)\in H_3} \frac{\partial P}{\partial \omega}(\omega_i, \lambda_j)}{\sup\limits_{H_3}\left|\left(\frac{\partial^2 P}{\partial \omega^2}(\omega,\lambda), \frac{\partial^2 P}{\partial \omega \partial \lambda}(\omega,\lambda)\right)\right|} \approx 0.0157.$$

Since $c_1 < c_2$, by Lemma C.1 we conclude that the constructed discretization $\{(\omega_i, \lambda_j)\}_{\substack{i=0,\ldots,10 \\ j=0,\ldots,60}}$ of H_3 to be appropriate in the sense that condition (C.65) yields a strict positivity of $\frac{\partial P}{\partial \omega}(\omega,\lambda)$ on H_3. ■

Claim C.13 *Let $H_4 \subset V$ be as follows*

$$H_4 = \left\{(\omega,\lambda) : \left[\frac{91}{180}\pi, \frac{102}{180}\pi\right] \times [3.950, 4.100]\right\}.$$

It holds that $\frac{\partial P}{\partial \lambda}(\omega,\lambda) < 0$ on H_4.

Proof. First we find the following estimates:

$$\left|\frac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda)\right| < 38, \quad \left|\frac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)\right| < 11 \quad \text{on} \quad H_4. \tag{C.67}$$

Then we fix $n=50$, $m=36$ and consider the discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,50\\ j=0,\dots,36}}$ of H_4 such that

$$\omega_i = \tfrac{91}{180}\pi + i\triangle\omega, \quad \lambda_j = 3.950 + j\triangle\lambda,$$

with

$$\triangle\omega = \frac{\frac{11}{180}\pi}{n}, \quad \triangle\lambda = \frac{0.15}{m}.$$

By straightforward computations it holds that

$$-\frac{\partial P}{\partial\lambda}(\omega_i,\lambda_j) > 0, \tag{C.68}$$

for all $i=0,\dots,50$, $j=0,\dots,36$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_4}\left(-\frac{\partial P}{\partial\lambda}(\omega_i,\lambda_j)\right) = -\frac{\partial P}{\partial\lambda}(\omega_0,\lambda_0) \approx 0.118. \tag{C.69}$$

Next to this, we compute

$$c_1 = \max\{\triangle\omega,\triangle\lambda\} = \max\left\{\frac{\frac{11}{180}\pi}{n}, \frac{0.15}{m}\right\} = \frac{0.15}{36} = 0.00416,$$

and, by taking into account (C.67) and (C.69), we also find

$$c_2 = \sqrt{2}\frac{\min\limits_{(\omega_i,\lambda_j)\in H_4}\left(-\frac{\partial P}{\partial\lambda}(\omega_i,\lambda_j)\right)}{\sup\limits_{H_4}\left|\left(\frac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda), \frac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)\right)\right|} \approx 0.00423.$$

Since $c_1 < c_2$, by Lemma C.1 we conclude that the constructed discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,50\\ j=0,\dots,36}}$ of H_4 to be appropriate in the sense that condition (C.68) yields a strict positivity of $-\frac{\partial P}{\partial\lambda}(\omega,\lambda)$ on H_4, or, in other words $\frac{\partial P}{\partial\lambda}(\omega,\lambda) < 0$ on H_4. ■

Claim C.14 *Let $H_5 \subset V$ be as follows*

$$H_5 = \left\{(\omega,\lambda) : \left[\tfrac{90}{180}\pi, \tfrac{96}{180}\pi\right] \times [3.030, 3.970]\right\}.$$

It holds that $\frac{\partial P}{\partial\omega}(\omega,\lambda) > 0$ on H_5.

Proof. First we find the following estimates:

$$\left|\frac{\partial^2 P}{\partial \omega^2}(\omega,\lambda)\right| < 105, \quad \left|\frac{\partial^2 P}{\partial \omega \partial \lambda}(\omega,\lambda)\right| < 37 \quad \text{on} \quad H_5. \tag{C.70}$$

Then we fix $n = 11$, $m = 94$ and consider the discretization $\{(\omega_i, \lambda_j)\}_{\substack{i=0,\dots,11\\ j=0,\dots,94}}$ of H_5 such that

$$\omega_i = \frac{90}{180}\pi + i\triangle\omega, \quad \lambda_j = 3.030 + j\triangle\lambda,$$

with

$$\triangle\omega = \frac{\frac{6}{180}\pi}{n}, \quad \triangle\lambda = \frac{0.94}{m}.$$

By straightforward computations it holds that

$$\frac{\partial P}{\partial \omega}(\omega_i, \lambda_j) > 0, \tag{C.71}$$

for all $i = 0, \dots, 11$, $j = 0, \dots, 94$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_5} \frac{\partial P}{\partial \omega}(\omega_i, \lambda_j) = \frac{\partial P}{\partial \omega}(\omega_0, \lambda_0) \approx 0.807. \tag{C.72}$$

Next to this, we compute

$$c_1 = \max\{\triangle\omega, \triangle\lambda\} = \max\left\{\frac{\frac{6}{180}\pi}{n}, \frac{0.94}{m}\right\} = \frac{0.94}{94} \approx 0.0100,$$

and, by taking into account (C.70) and (C.72), we also find

$$c_2 = \sqrt{2}\frac{\min\limits_{(\omega_i,\lambda_j)\in H_5} \frac{\partial P}{\partial \omega}(\omega_i, \lambda_j)}{\sup\limits_{H_5}\left|\left(\frac{\partial^2 P}{\partial \omega^2}(\omega,\lambda), \frac{\partial^2 P}{\partial \omega \partial \lambda}(\omega,\lambda)\right)\right|} \approx 0.0102.$$

Since $c_1 < c_2$, by Lemma C.1 we conclude that the constructed discretization $\{(\omega_i, \lambda_j)\}_{\substack{i=0,\dots,11\\ j=0,\dots,94}}$ of H_5 to be appropriate in the sense that condition (C.71) yields a strict positivity of $\frac{\partial P}{\partial \omega}(\omega, \lambda)$ on H_5. ■

Claim C.15 *Let $H_6 \subset V$ be as follows*

$$H_6 = \left\{(\omega,\lambda) : \left[\frac{79}{180}\pi, \frac{94}{180}\pi\right] \times [2.900, 3.230]\right\}.$$

It holds that $\frac{\partial P}{\partial \lambda}(\omega, \lambda) < 0$ on H_6.

Proof. First we find the following estimates:

$$\left|\frac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda)\right| < 33, \quad \left|\frac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)\right| < 21 \quad \text{on} \quad H_6. \tag{C.73}$$

Then we fix $n=33$, $m=41$ and consider the discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,33\\ j=0,\dots,41}}$ of H_6 such that

$$\omega_i = \tfrac{79}{180}\pi + i\triangle\omega, \quad \lambda_j = 2.900 + j\triangle\lambda,$$

with

$$\triangle\omega = \frac{\frac{15}{180}\pi}{n}, \quad \triangle\lambda = \frac{0.33}{m}.$$

By straightforward computations it holds that

$$-\tfrac{\partial P}{\partial\lambda}(\omega_i,\lambda_j) > 0, \tag{C.74}$$

for all $i=0,\dots,33$, $j=0,\dots,41$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_6}\left(-\tfrac{\partial P}{\partial\lambda}(\omega_i,\lambda_j)\right) = -\tfrac{\partial P}{\partial\lambda}(\omega_{33},\lambda_{41}) \approx 0.227. \tag{C.75}$$

Next to this, we compute

$$c_1 = \max\{\triangle\omega,\triangle\lambda\} = \max\left\{\frac{\frac{15}{180}\pi}{n}, \frac{0.33}{m}\right\} = \frac{0.33}{41} = 0.00805,$$

and, by taking into account (C.73) and (C.75), we also find

$$c_2 = \sqrt{2}\frac{\min\limits_{(\omega_i,\lambda_j)\in H_6}\left(-\frac{\partial P}{\partial\lambda}(\omega_i,\lambda_j)\right)}{\sup\limits_{H_6}\left|\left(\frac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda), \frac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)\right)\right|} \approx 0.00820.$$

Since $c_1 < c_2$, by Lemma C.1 we conclude that the constructed discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,33\\ j=0,\dots,41}}$ of H_6 to be appropriate in the sense that condition (C.74) yields a strict positivity of $-\frac{\partial P}{\partial\lambda}(\omega,\lambda)$ on H_6, or, in other words $\frac{\partial P}{\partial\lambda}(\omega,\lambda) < 0$ on H_6. ■

Claim C.16 *Let $H_7 \subset V$ be as follows*

$$H_7 = \left\{(\omega,\lambda) : \left[\tfrac{72}{180}\pi, \tfrac{80}{180}\pi\right] \times [3.150, 4.000]\right\}.$$

It holds that $\frac{\partial P}{\partial\omega}(\omega,\lambda) < 0$ on H_7.

Proof. First we find the following estimates:

$$\left|\frac{\partial^2 P}{\partial \omega^2}(\omega,\lambda)\right| < 115, \quad \left|\frac{\partial^2 P}{\partial \omega \partial \lambda}(\omega,\lambda)\right| < 21 \quad \text{on} \quad H_7. \tag{C.76}$$

Then we fix $n = 5$, $m = 30$ and consider the discretization $\{(\omega_i, \lambda_j)\}_{\substack{i=0,\dots,5 \\ j=0,\dots,30}}$ of H_7 such that

$$\omega_i = \tfrac{72}{180}\pi + i\triangle\omega, \quad \lambda_j = 3.150 + j\triangle\lambda,$$

with

$$\triangle\omega = \frac{\frac{8}{180}\pi}{n}, \quad \triangle\lambda = \frac{0.85}{m}.$$

By straightforward computations it holds that

$$-\frac{\partial P}{\partial \omega}(\omega_i, \lambda_j) > 0, \tag{C.77}$$

for all $i = 0, ..., 5$, $j = 0, ..., 30$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_7}\left(-\frac{\partial P}{\partial \omega}(\omega_i,\lambda_j)\right) = -\frac{\partial P}{\partial \omega}(\omega_5,\lambda_{27}) \approx 2.663. \tag{C.78}$$

Next to this, we compute

$$c_1 = \max\{\triangle\omega, \triangle\lambda\} = \max\left\{\frac{\frac{8}{180}\pi}{n}, \frac{0.85}{m}\right\} = \frac{0.85}{30} \approx 0.0283,$$

and, by taking into account (C.76) and (C.78), we also find

$$c_2 = \sqrt{2}\frac{\min\limits_{(\omega_i,\lambda_j)\in H_7}\left(-\frac{\partial P}{\partial \omega}(\omega_i,\lambda_j)\right)}{\sup\limits_{H_7}\left|\left(\frac{\partial^2 P}{\partial \omega^2}(\omega,\lambda), \frac{\partial^2 P}{\partial \omega \partial \lambda}(\omega,\lambda)\right)\right|} \approx 0.0322.$$

Since $c_1 < c_2$, by Lemma C.1 we conclude that the constructed discretization $\{(\omega_i, \lambda_j)\}_{\substack{i=0,\dots,5 \\ j=0,\dots,30}}$ of H_7 to be appropriate in the sense that condition (C.77) yields a strict positivity of $-\frac{\partial P}{\partial \omega}(\omega,\lambda)$ on H_7, or, in other words, $\frac{\partial P}{\partial \omega}(\omega,\lambda) < 0$ on H_7. ∎

Claim C.17 *Let $H_8 \subset V$ be as follows*

$$H_8 = \left\{(\omega,\lambda) : \left[\tfrac{78}{180}\pi, \tfrac{89}{180}\pi\right] \times [3.900, 4.050]\right\}.$$

It holds that $\frac{\partial P}{\partial \lambda}(\omega,\lambda) > 0$ on H_8.

Proof. First we find the following estimates:

$$\left|\frac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda)\right| < 36, \quad \left|\frac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)\right| < 10 \quad \text{on} \quad H_8. \tag{C.79}$$

Then we fix $n=40$, $m=30$ and consider the discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,40\\ j=0,\dots,30}}$ of H_8 such that

$$\omega_i = \frac{78}{180}\pi + i\triangle\omega, \quad \lambda_j = 3.900 + j\triangle\lambda,$$

with

$$\triangle\omega = \frac{\frac{11}{180}\pi}{n}, \quad \triangle\lambda = \frac{0.15}{m}.$$

By straightforward computations it holds that

$$\frac{\partial P}{\partial\lambda}(\omega_i,\lambda_j) > 0, \tag{C.80}$$

for all $i=0,\dots,40$, $j=0,\dots,30$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_8}\frac{\partial P}{\partial\lambda}(\omega_i,\lambda_j) = \frac{\partial P}{\partial\lambda}(\omega_{40},\lambda_{30}) \approx 0.134. \tag{C.81}$$

Next to this, we compute

$$c_1 = \max\{\triangle\omega,\triangle\lambda\} = \max\left\{\frac{\frac{11}{180}\pi}{n}, \frac{0.15}{m}\right\} = \frac{0.15}{30} = 0.00500,$$

and, by taking into account (C.79) and (C.81), we also find

$$c_2 = \sqrt{2}\frac{\min\limits_{(\omega_i,\lambda_j)\in H_8}\frac{\partial P}{\partial\lambda}(\omega_i,\lambda_j)}{\sup\limits_{H_8}\left|\left(\frac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda), \frac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)\right)\right|} \approx 0.00508.$$

Since $c_1 < c_2$, by Lemma C.1 we conclude that the constructed discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,40\\ j=0,\dots,30}}$ of H_8 to be appropriate in the sense that condition (C.80) yields a strict positivity of $\frac{\partial P}{\partial\lambda}(\omega,\lambda)$ on H_8. ∎

C.5.2 Set of Claims II

Let $H_0 \subset V$ be as follows

$$H_0 = \left\{(\omega, \lambda) : \left[\tfrac{84}{180}\pi, \tfrac{94}{180}\pi\right] \times [2.960, 3.060]\right\}.$$

Consider a function $G : H_0 \to \mathbb{R}$ given by

$$G(\omega, \lambda) = -\tfrac{\partial^2 P}{\partial \omega^2}(\omega, \lambda)\left[\tfrac{\partial P}{\partial \lambda}(\omega, \lambda)\right]^{-1} + 2\tfrac{\partial^2 P}{\partial \omega \partial \lambda}(\omega, \lambda)\tfrac{\partial P}{\partial \omega}(\omega, \lambda)\left[\tfrac{\partial P}{\partial \lambda}(\omega, \lambda)\right]^{-2} +$$

$$-\tfrac{\partial^2 P}{\partial \lambda^2}(\omega, \lambda)\left[\tfrac{\partial P}{\partial \omega}(\omega, \lambda)\right]^{2}\left[\tfrac{\partial P}{\partial \lambda}(\omega, \lambda)\right]^{-3},$$

where P is as in (C.57).

Claim C.18 *It holds that $G(\omega, \lambda) > 0$ on H_0.*

Proof. First we find the following estimates:

$$\left|\tfrac{\partial G}{\partial \omega}(\omega, \lambda)\right| < 25000, \quad \left|\tfrac{\partial G}{\partial \lambda}(\omega, \lambda)\right| < 14000 \quad \text{on} \quad H_0. \tag{C.82}$$

Then we fix $n = 600$, $m = 300$ and consider the discretization $\{(\omega_i, \lambda_j)\}_{\substack{i=0,\dots,600 \\ j=0,\dots,300}}$ of H_0 such that

$$\omega_i = \tfrac{84}{180}\pi + i\triangle\omega, \quad \lambda_j = 2.960 + j\triangle\lambda,$$

with

$$\triangle\omega = \frac{\frac{10}{180}\pi}{n}, \quad \triangle\lambda = \frac{0.1}{m}.$$

By straightforward computations it holds that

$$G(\omega_i, \lambda_j) > 0, \tag{C.83}$$

for all $i = 0, ..., 600$, $j = 0, ..., 300$ and, moreover,

$$\min_{(\omega_i, \lambda_j) \in H_0} G(\omega_i, \lambda_j) = G(\omega_0, \lambda_0) \approx 8.380. \tag{C.84}$$

Next to this, we compute

$$c_1 = \max\{\triangle\omega, \triangle\lambda\} = \max\left\{\frac{\frac{10}{180}\pi}{n}, \frac{0.1}{m}\right\} = \frac{0.1}{300} \approx 0.000(3),$$

and, by taking into account (C.82) and (C.84), we also find

$$c_2=\sqrt{2}\frac{\min\limits_{(\omega_i,\lambda_j)\in H_0} G(\omega_i,\lambda_j)}{\sup\limits_{H_0}\left|\left(\frac{\partial G}{\partial\omega}(\omega,\lambda),\,\frac{\partial G}{\partial\lambda}(\omega,\lambda)\right)\right|}\approx 0.000414.$$

Since $c_1<c_2$, by Lemma C.1 we conclude that the constructed discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,600\\ j=0,\dots,300}}$ of H_0 to be appropriate in the sense that condition (C.83) yields a strict positivity of $G(\omega,\lambda)$ on H_0. ■

Let $H_\star\subset V$ be as follows

$$H_\star=\left\{(\omega,\lambda):\left[\tfrac{93.5}{180}\pi,\,\tfrac{95.5}{180}\pi\right]\times[3.030,3.600]\right\}.$$

Consider a function $F:H_\star\to\mathbb{R}$ given by

$$F(\omega,\lambda)=-\tfrac{\partial^2P}{\partial\lambda^2}(\omega,\lambda)\left[\tfrac{\partial P}{\partial\omega}(\omega,\lambda)\right]^{-1}+2\tfrac{\partial^2P}{\partial\omega\partial\lambda}(\omega,\lambda)\tfrac{\partial P}{\partial\lambda}(\omega,\lambda)\left[\tfrac{\partial P}{\partial\omega}(\omega,\lambda)\right]^{-2}+$$
$$-\tfrac{\partial^2P}{\partial\omega^2}(\omega,\lambda)\left[\tfrac{\partial P}{\partial\lambda}(\omega,\lambda)\right]^{2}\left[\tfrac{\partial P}{\partial\omega}(\omega,\lambda)\right]^{-3},$$

where P is as in (C.57).

Claim C.19 *It holds that $F(\omega,\lambda)<0$ on $H_\star$.*

Proof. First we find the following estimates:

$$\left|\tfrac{\partial F}{\partial\omega}(\omega,\lambda)\right|<180,\quad\left|\tfrac{\partial F}{\partial\lambda}(\omega,\lambda)\right|<80\quad\text{on}\quad H_\star.\tag{C.85}$$

Then we fix $n=70$, $m=1100$ and consider the discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,70\\ j=0,\dots,1100}}$ of $H_\star$ such that

$$\omega_i=\tfrac{93.5}{180}\pi+i\triangle\omega,\quad\lambda_j=3.030+j\triangle\lambda,$$

with

$$\triangle\omega=\frac{\frac{2}{180}\pi}{n},\quad\triangle\lambda=\frac{0.57}{m}.$$

By straightforward computations it holds that

$$-F(\omega_i,\lambda_j)>0,\tag{C.86}$$

for all $i = 0, ..., 70$, $j = 0, ..., 1100$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_\star} (-F(\omega_i, \lambda_j)) = -F(\omega_{70}, \lambda_{1100}) \approx 0.0773. \tag{C.87}$$

Next to this, we compute

$$c_1 = \max\{\triangle\omega, \triangle\lambda\} = \max\left\{\frac{\frac{2}{180}\pi}{n}, \frac{0.57}{m}\right\} = \frac{0.57}{1100} = 0.000518,$$

and, by taking into account (C.85) and (C.87), we also find

$$c_2 = \sqrt{2}\frac{\min_{(\omega_i,\lambda_j)\in H_\star} (-F(\omega_i, \lambda_j))}{\sup_{H_\star} \left|\left(\frac{\partial F}{\partial\omega}(\omega,\lambda), \frac{\partial F}{\partial\lambda}(\omega,\lambda)\right)\right|} \approx 0.000555.$$

Since $c_1 < c_2$, by Lemma C.1 we conclude that the constructed discretization $\{(\omega_i, \lambda_j)\}_{\substack{i=0,...,70\\ j=0,...,1100}}$ of $H_\star$ to be appropriate in the sense that condition (C.74) yields a strict positivity of $-F(\omega, \lambda)$ on $H_\star$, or, in other words $F(\omega, \lambda) < 0$ on $H_\star$. ■

Appendix D

Explicit eigenvalues and eigenfunctions

Let us recall the λ-dependent boundary value problem (3.4):

$$\begin{cases} \mathcal{L}\left(\theta, \frac{d}{d\theta}, \lambda\right)\Phi = 0 & \text{in} \quad (\alpha, \alpha+\omega), \\ \Phi = \frac{d}{d\theta}\Phi = 0 & \text{on} \quad \partial(\alpha, \alpha+\omega), \end{cases}$$

where

$$\begin{aligned} \mathcal{L}\left(\theta, \tfrac{d}{d\theta}, \lambda\right) \;=\;& \tfrac{3}{4}\left(1+\tfrac{1}{3}\cos(4\theta)\right)\tfrac{d^4}{d\theta^4} + (\lambda-2)\sin(4\theta)\tfrac{d^3}{d\theta^3} + \\ &+\tfrac{3}{2}\left(\lambda^2-1-\left(\lambda^2-4\lambda-\tfrac{7}{3}\right)\cos(4\theta)\right)\tfrac{d^2}{d\theta^2} + \\ &+\left(-\lambda^3+6\lambda^2-7\lambda-2\right)\sin(4\theta)\tfrac{d}{d\theta} + \\ &+\tfrac{3}{4}\left(\lambda^4-2\lambda^2+1+\tfrac{1}{3}\left(\lambda^4-8\lambda^3+14\lambda^2+8\lambda-15\right)\cos(4\theta)\right). \end{aligned}$$

Here, for the cases

$\alpha = 0$	$\omega = \frac{1}{2}\pi,$
$\alpha \in \left[0, \frac{1}{2}\pi\right)$	$\omega = \pi,$
$\alpha = 0$	$\omega = \frac{3}{2}\pi,$
$\alpha \in \left[0, \frac{1}{2}\pi\right)$	$\omega = 2\pi.$

we compute the entries of the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ of problem above explicitly.

Moreover, it will be possible to give the explicit formulas for the power-type solutions $u_{j,q} = r^{\lambda_j+1}\Phi_{j,q}$, $q = 0, ..., \kappa_j - 1$, to homogeneous problem (3.1) in the $\mathcal{K}_{(\alpha,\omega)}$, namely,

$$\begin{cases} u_{xxxx} + u_{yyyy} = 0 & \text{in} \quad \mathcal{K}_{(\alpha,\omega)}, \\ u = 0 & \text{on} \quad \partial\mathcal{K}_{(\alpha,\omega)}, \\ \frac{\partial}{\partial\nu} u = 0 & \text{on} \quad \partial\mathcal{K}_{(\alpha,\omega)} \backslash \{0\}. \end{cases}$$

We will do this in the following cases

$\alpha = 0$	$\omega \in \left\{\frac{1}{2}\pi, \pi\right\}$	for all λ_j,
$\alpha = 0$	$\omega \in \left\{\frac{3}{2}\pi, 2\pi\right\}$	for some λ_j.

Remark D.1 *When $\alpha \in \left(0, \frac{1}{2}\pi\right)$ and $\omega = \pi$ one may easily compute the first solution $u_{1,0} = r^2 \sin^2(\theta - \alpha)$ to (3.1) in $\mathcal{K}_{(\alpha,\omega)}$. The higher order solutions $u_{j,q}$ will require more labor.*

D.1 Computations for $\{\lambda_j\}_{j=1}^{\infty}$

D.1.1 Case $\alpha = 0$, $\omega = \frac{1}{2}\pi$

The eigenvalues λ in more general case, namely, $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and $\omega = \frac{1}{2}\pi$ are determined by the characteristic equation:

$$\left(1 - \tfrac{\sqrt{2}}{2}\sin(2\alpha)\right)^{2\lambda} + \left(1 + \tfrac{\sqrt{2}}{2}\sin(2\alpha)\right)^{2\lambda} + 2\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\alpha)\right)^{\lambda}\cos(\lambda\pi) +$$

$$-4\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\alpha)\right)^{\lambda}\cos\left\{\lambda\left[\arctan\left(\tan^2(\alpha)\right) - \arctan\left(\cot^2(\alpha)\right)\right]\right\} = 0,$$

where $\lambda \notin \{0, \pm 1\}$. The left-hand side is obtained from formula (3.8).

In particular case $\alpha = 0$, we get

$$2 + 2\cos(\pi\lambda) - 4\cos\left(\tfrac{1}{2}\pi\lambda\right) = 0, \quad \lambda \notin \{0, \pm 1\},$$

and the set of positive solutions of the above equation reads as:

$$\{\lambda_j\}_{j=1}^{\infty} = \{\lambda_{3j-2},\ \lambda_{3j-1},\ \lambda_{3j}\}_{j=1}^{\infty} = \{-1 + 4j,\ 4j,\ 1 + 4j\}_{j=1}^{\infty}.$$

Here every values $\lambda_{3j-2}, \lambda_{3j}$ has algebraic and geometric multiplicity 1, while λ_{3j-1} has algebraic and geometric multiplicity 2.

In the table below we see that the functions that solve the homogeneous problem (3.1) in $\mathcal{K}_{(\alpha,\omega)}$ when $\alpha = 0$ and $\omega = \frac{1}{2}\pi$ are given by polynomials in x and y, which makes a difference to the case when the operator of the problem is the bilaplacian Δ^2.

D.1.2 Case $\alpha \in \left[0, \frac{1}{2}\pi\right)$, $\omega = \pi$

The eigenvalues λ in this case are determined by the characteristic equation:

$$-4\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\alpha)\right)^{\lambda}\sin^2(\pi\lambda) = 0, \quad \lambda \notin \{0, \pm 1\},$$

(the left-hand side is yielded by formula (3.9)) plus the values $\lambda = \pm 1$, which are determined by the conditions $\mathcal{P}_{-1}(\alpha,\pi) = \mathcal{P}_1(\alpha,\pi) = 0$. Hence, the set of positive solutions reads as

$$\lambda_j = j, \quad j = 1, ..., \infty,$$

where $\lambda_1 = 1$ has algebraic and geometric multiplicity 1, while λ_j for $j \geq 2$ has algebraic and geometric multiplicity 2.

In the table below we observe that in $\mathcal{K}_{(\alpha,\omega)}$ with $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and the opening angle $\omega = \pi$, the solutions (3.1) in $\mathcal{K}_{(\alpha,\omega)}$ are given by polynomials in x and y.

D.1.3 Case $\alpha = 0$, $\omega = \frac{3}{2}\pi$

In more general case, namely, $\alpha \in \left[0, \frac{1}{2}\pi\right)$ and $\omega = \frac{3}{2}\pi$, the characteristic equation is as follows:

$$\left(1 - \tfrac{\sqrt{2}}{2}\sin(2\alpha)\right)^{2\lambda} + \left(1 + \tfrac{\sqrt{2}}{2}\sin(2\alpha)\right)^{2\lambda} + 2\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\alpha)\right)^{\lambda}\cos(3\lambda\pi) +$$

$$-4\left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\alpha)\right)^{\lambda}\cos\left\{\lambda\left[\arctan\left(\tan^2(\alpha)\right) - \arctan\left(\cot^2(\alpha)\right)\right]\right\} = 0,$$

where $\lambda \notin \{0, \pm 1\}$. The left-hand side follows from formula (3.10). When $\alpha = 0$ it can be simplified and factorized, so that the eigenvalues λ are determined by the system:

$$\cos\left(\tfrac{1}{2}\pi\lambda\right) = 0, \quad \lambda \neq \pm 1,$$

$$\cos\left(\tfrac{1}{2}\pi\lambda\right) = 1, \quad \lambda \neq 0,$$

and

$$\cos^4\left(\tfrac{1}{2}\pi\lambda\right)+\cos^3\left(\tfrac{1}{2}\pi\lambda\right)-\tfrac{1}{2}\cos^2\left(\tfrac{1}{2}\pi\lambda\right)-\tfrac{1}{2}\cos\left(\tfrac{1}{2}\pi\lambda\right)+\tfrac{1}{16}=0, \quad \lambda\notin\{0,\pm1\}.$$

The solutions with positive real part of the system above read, respectively,

$$\{\lambda_{n_1}\}_{n_1=1}^{\infty}=\{1+2n_1\}_{n_1=1}^{\infty},$$
$$\{\lambda_{n_2}\}_{n_2=1}^{\infty}=\{4n_2\}_{n_2=1}^{\infty},$$

and

$$\{\lambda_{n_3}\}_{n_3=1}^{\infty}=\{-1+2n_3+(-1)^{n_3}(1-\mu_1)\}_{n_3=1}^{\infty},$$
$$\{\lambda_{n_4}\}_{n_4=1}^{\infty}=\{-1+2n_4+(-1)^{n_4}(1-\mu_2)\}_{n_4=1}^{\infty},$$
$$\{\lambda_{n_5}\}_{n_5=1}^{\infty}=\left\{-1+2n_5+(-1)^{n_5+1}(1-\gamma_1)\pm i\gamma_2\right\}_{n_5=1}^{\infty},$$

where μ_1,μ_2 are the first two positive solutions of the equation

$$s^4+s^3-\tfrac{1}{2}s^2-\tfrac{1}{2}s+\tfrac{1}{16}=0, \tag{D.1}$$

with $s=\cos\left(\frac{1}{2}\pi\mu\right)$, while (γ_1,γ_2) is the first positive solution of (D.1) with $s=-\cos\left(\frac{1}{2}\pi\gamma_1\right)\cosh\left(\frac{1}{2}\pi\gamma_2\right)+i\sin\left(\frac{1}{2}\pi\gamma_1\right)\sinh\left(\frac{1}{2}\pi\gamma_2\right)$. The numerical approximations (up to three digits) are the following: $\mu_1\approx0.536$, $\mu_2\approx0.926$and $\gamma_1\approx0.345$, $\gamma_2\approx0.179$.

Note also that every $\lambda_{n_2}=4n_2$, $n_2=1,2.3,$has algebraic and geometric multiplicity 2, while every λ_{n_k}, for each $k=1,3,4,5$, has algebraic and geometric multiplicity 1.

The set $\{\lambda_j\}_{j=1}^{\infty}$ is the combination of the found sets above.

D.1.4 Case $\alpha\in\left[0,\frac{1}{2}\pi\right)$, $\omega=2\pi$

The eigenvalues λ in this case are determined by the characteristic equation:

$$16\left(\tfrac{1}{2}+\tfrac{1}{2}\cos^2(2\alpha)\right)^{\lambda}\left[\cos^4(\pi\lambda)-\cos^2(\pi\lambda)\right]=0, \quad \lambda\notin\{0,\pm1\},$$

(the left-hand side is a simplified formula (3.11)) plus the values $\lambda=\pm1$, which are determined by the conditions $\mathcal{P}_{-1}(\alpha,2\pi)=\mathcal{P}_1(\alpha,2\pi)=0$. The positive solutions are given by the set

$$\{\lambda_j\}_{j=1}^{\infty}=\tfrac{1}{2}j, \quad j=1,...,\infty,$$

where $\lambda_2=1$ has algebraic and geometric multiplicity 1, while λ_j for $j=1$ and $j\geq3$ has algebraic and geometric multiplicity 2.

D.2 The list of tables for $u_{j,q}$

j	λ_j	κ_j	$u_{j,q} = r^{\lambda_j+1}\Phi_{j,q}(\theta)$
1	3	1	x^2y^2
$2-3$	4	2	$\begin{cases} x^3y^2 \\ x^2y^3 \end{cases}$
4	5	1	x^3y^3
5	7	1	$x^6y^2 - x^2y^6$
$6-7$	8	2	$\begin{cases} x^7y^2 - \frac{7}{3}x^3y^6 \\ x^2y^7 - \frac{7}{3}x^6y \end{cases}$
8	9	1	$x^7y^3 - x^3y^7$
9	11	1	$x^{10}y^2 - 14x^6y^6 + x^2y^{10}$
$10-11$	12	2	$\begin{cases} x^{11}y^2 - 22x^7y^6 + \frac{11}{3}x^3y^{10} \\ x^2y^{11} - 22x^6y^7 + \frac{11}{3}x^{10}y^3 \end{cases}$
12	13	1	$x^{11}y^3 - \frac{66}{7}x^7y^7 + x^3y^{11}$
etc.			

Table D.1: *The first three groups of solutions solutions* $u_{j,q} = r^{\lambda_j+1}\Phi_{j,q}(\theta)$, $q = 0, ..., \kappa_j - 1$ *to (3.1) in* $\mathcal{K}_{(\alpha,\omega)}$ *when* $\alpha = 0$ *and* $\omega = \frac{1}{2}\pi$.

j	λ_j	κ_j	$u_{j,q} = r^{\lambda_j+1}\Phi_{j,q}(\theta)$
1	1	1	y^2
$2-3$	2	2	$\begin{cases} xy^2 \\ y^3 \end{cases}$
$4-5$	3	2	$\begin{cases} x^2y^2 \\ xy^3 \end{cases}$
$6-7$	4	2	$\begin{cases} x^3y^2 \\ x^2y^3 \end{cases}$
etc.			

Table D.2: *Some first solutions* $u_{j,q} = r^{\lambda_j+1}\Phi_{j,q}(\theta)$, $q = 0, ..., \kappa_j - 1$ *to (3.1) in* $\mathcal{K}_{(\alpha,\omega)}$ *when* $\alpha = 0$ *and* $\omega = \pi$.

j	λ_j	κ_j	$u_{j,q} = r^{\lambda_j+1}\Phi_{j,q}(\theta)$
1	≈ 0.536	1	$\ldots$
2	≈ 0.926	1	$\ldots$
3	$\approx 1.655 \pm i0.179$	1	$\ldots$
4	$\approx 2.345 \pm i0.179$	1	$\ldots$
5	3	1	x^2y^2
6	≈ 3.074	1	$\ldots$
7	≈ 3.464	1	$\ldots$
$8-9$	4	2	$\begin{cases} x^3y^2 \\ x^2y^3 \end{cases}$
10	≈ 4.536	1	$\ldots$
11	≈ 4.926	1	$\ldots$
12	5	1	x^3y^3
13	$\approx 5.655 \pm i0.179$	1	$\ldots$
14	$\approx 6.345 \pm i0.179$	1	$\ldots$
15	7	1	$x^6y^2 - x^2y^6$
16	≈ 7.074	1	$\ldots$
17	≈ 7.464	1	$\ldots$
$18-19$	8	2	$\begin{cases} x^7y^2 - \frac{7}{3}x^3y^6 \\ x^2y^7 - \frac{7}{3}x^6y \end{cases}$
etc.			

Table D.3: *Some first solutions* $u_{j,q} = r^{\lambda_j+1}\Phi_{j,q}(\theta)$, $q = 0, ..., \kappa_j - 1$ *to (3.1) in* $\mathcal{K}_{(\alpha,\omega)}$ *when* $\alpha = 0$ *and* $\omega = \frac{3}{2}\pi$. *The situation without explicit formula is marked by "$\ldots$".*

j	λ_j	κ_j	$u_{j,q} = r^{\lambda_j+1}\Phi_{j,q}(\theta)$
$1-2$	$\frac{1}{2}$	2	$\ldots$
3	1	1	y^2
$4-5$	$\frac{3}{2}$	2	$\ldots$
$6-7$	2	2	$\begin{cases} xy^2 \\ y^3 \end{cases}$
$8-9$	$\frac{5}{2}$	2	$\ldots$
$10-11$	3	2	$\begin{cases} x^2y^2 \\ xy^3 \end{cases}$
$12-13$	$\frac{7}{2}$	2	$\ldots$
$14-15$	4	2	$\begin{cases} x^3y^2 \\ x^2y^3 \end{cases}$
etc.			

Table D.4: *Some first solutions* $u_{j,q} = r^{\lambda_j+1}\Phi_{j,q}(\theta)$, $q = 0, ..., \kappa_j - 1$ *to (3.1) in* $\mathcal{K}_{(\alpha,\omega)}$ *when* $\alpha = 0$ *and* $\omega = 2\pi$. *The situation when the explicit formula unavailable is marked by "$\ldots$".*

Appendix E

Application of the FreeFem++ package

Let domain $\Omega \subset \mathbb{R}^2$ be smooth or have convex corners. In this case, as it is shown in Chapter 6, one may use the system approach to find an approximate numerical solution of the clamped grid problem

$$\begin{cases} u_{xxxx} + u_{yyyy} = f & \text{in} \quad \Omega, \\ u = \frac{\partial}{\partial n} u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{E.1}$$

One of the available numerical tools to implement this approach is the FreeFem++ package [11]. In the first section of this appendix we will explain how to use it properly with respect to the package syntaxis. In the two last sections we use the package to find the solutions for the problems of our interest. More precisely, in Section E.2 we compute and compare the displacements of the clamped isotropic plate and grid under the concentrated load, and in Section E.3 we compute the distribution of stresses that occur in a fuselage panel of the DH-106 "Comet" with the rectangular and oval windows under the uniform normal load.

E.1 FreeFem++ programming

Consider the clamped grid problem (E.1). Recall from Chapter 6 that if a domain $\Omega \subset \mathbb{R}^2$ is smooth or has convex corners, the weak solution $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ to (E.1) will coincide with the solution $(v,u) \in W^{1,2}(\Omega) \times \overset{\circ}{W}{}^{1,2}(\Omega)$

of the following variational problem

$$\int_\Omega \left[\left(\frac{\partial v}{\partial x}+\frac{\sqrt{2}}{2}\frac{\partial v}{\partial y}\right)\frac{\partial \psi}{\partial x}+\left(\frac{\partial v}{\partial y}+\frac{\sqrt{2}}{2}\frac{\partial v}{\partial x}\right)\frac{\partial \psi}{\partial y}+\right.$$

$$\left.+\left(\frac{\partial u}{\partial x}-\frac{\sqrt{2}}{2}\frac{\partial u}{\partial y}\right)\frac{\partial \phi}{\partial x}+\left(\frac{\partial u}{\partial y}-\frac{\sqrt{2}}{2}\frac{\partial u}{\partial x}\right)\frac{\partial \phi}{\partial y}-f\psi-v\phi\right]dxdy=0,$$

$$\text{for all}\quad (\phi,\psi)\in W^{1,2}(\Omega)\times \overset{\circ}{W}{}^{1,2}(\Omega). \tag{E.2}$$

In this section we will bring the numerical evidence for this assertion at hand. In order to do this we will choose such the right-hand side f in problem (E.1) for which the explicit solution u is available. Then we will let the FreeFem++ package solve equation (E.2) for this given f and compare obtained u with the known original one. In this way one shows the feasibility of the system approach to problem (E.1). Also, some technical details on how to realize equation (E.2) in the FreeFem++ package will be discussed.

E.1.1 The test problem

Let us consider the functions $\varphi_i : [0,1] \to \mathbb{R}$, $i \in \mathbb{N}$ given explicitly as

$$\varphi_i(x)=\frac{\cosh(\nu_i x)-\cos(\nu_i x)}{\cosh(\nu_i)-\cos(\nu_i)}-\frac{\sinh(\nu_i x)-\sin(\nu_i x)}{\sinh(\nu_i)-\sin(\nu_i)}, \tag{E.3}$$

where ν_i is the i^{th} positive solution of the transcendental equation $\cos(\nu)\cosh(\nu)=1$.

Let $\Omega := [0,1]^2$. One directly shows that the function

$$\Phi_{ij}(x,y)=\varphi_i(x)\varphi_j(y), \quad i,j\in\mathbb{N}, \tag{E.4}$$

solves the problem

$$\begin{cases} \left(\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4}\right)\Phi_{ij}=\left(\nu_i^4+\nu_j^4\right)\Phi_{ij} & \text{in} \quad \Omega,\\ \Phi_{ij}=\frac{\partial}{\partial n}\Phi_{ij}=0 & \text{on} \quad \partial\Omega\backslash S,\end{cases}$$

where $S=\{(0,0),(0,1),(1,0),(1,1)\}$.

Consider the test boundary value problem:

$$\begin{cases} u_{xxxx}+u_{yyyy}=f:=2\nu_1^4\Phi_{11} & \text{in} \quad \Omega,\\ u=\frac{\partial}{\partial n}u=0 & \text{on} \quad \partial\Omega\backslash S,\end{cases} \tag{E.5}$$

where Φ_{11} is defined by (E.4), with ϕ_1 as in (E.3) for $\nu_1 \approx 4.73$. From the above consideration it is clear that Φ_{11} is the explicit solution to (E.5). Our purpose is to check whether u of (E.5) computed from equation (E.2) with $f := 2\nu_1^4 \Phi_{11}$ will be identical to Φ_{11}.

E.1.2 The code

One is not able to program equation (E.2) in FreeFem++ directly, that is, in the form it is. In order to do this let us one introduces two macroses

$$\ell_1(a,b) = \left(\frac{\partial a}{\partial x} - \frac{\sqrt{2}}{2}\frac{\partial a}{\partial y}\right)\frac{\partial b}{\partial x} + \left(\frac{\partial a}{\partial y} - \frac{\sqrt{2}}{2}\frac{\partial a}{\partial x}\right)\frac{\partial b}{\partial y},$$

$$\ell_2(a,b) = \left(\frac{\partial a}{\partial x} + \frac{\sqrt{2}}{2}\frac{\partial a}{\partial y}\right)\frac{\partial b}{\partial x} + \left(\frac{\partial a}{\partial y} + \frac{\sqrt{2}}{2}\frac{\partial a}{\partial x}\right)\frac{\partial b}{\partial y}.$$

The appropriate form of (E.2) which is now "readable" by the package will be as follows

$$\int_\Omega (\ell_1(w,\phi) - v\phi + \ell_2(v,\psi)) - \int_\Omega f\psi = 0 \quad \text{"with } \; w = 0 \; \text{ on } \; \partial\Omega\text{".} \qquad (E.6)$$

Below we bring the FreeFem++ program[1] to solve (E.5) through the system approach realized by (E.6):

```
border G1(t=0,1){x=t; y=0; label=1;};
border G2(t=0,1){x=1; y=t; label=1;};
border G3(t=0,1){x=1-t; y=1; label=1;};
border G4(t=0,1){x=0; y=1-t; label=1;};

mesh Th = buildmesh (G1(100)+G2(100)+G3(100)+G4(100));
plot(Th, cmm="Th", wait=true);

fespace Vh(Th,P1);
Vh w,v, psi,phi;

real k = 0.707106781186550;

macro l1(u,v) ((dx(u)-a*dy(u))*dx(v)+(dy(u)-a*dx(u))*dy(v)) // fin macro
macro l2(u,v) ((dx(u)+a*dy(u))*dx(v)+(dy(u)+a*dx(u))*dy(v)) // fin macro

real nu = 4.730040745;
```

[1]For detailed explanation of the commands in FreeFem++ see the manual [11]

```
func phi1 = (cosh(nu*x)-cos(nu*x))/(cosh(nu)-cos(nu))
          - (sinh(nu*x)-sin(nu*x))/(sinh(nu)-sin(nu)));

func phi2 = (cosh(nu*y)-cos(nu*y))/(cosh(nu)-cos(nu))
          - (sinh(nu*y)-sin(nu*y))/(sinh(nu)-sin(nu));

func f = 2*nu^4*phi1*phi2;

problem P([w,v],[psi,phi], solver=LU, eps=1e-10) =
          int2d(Th)( 1e-7*w*psi
          +l1(w,phi) - v*phi + l2(v,psi))
          -int2d(Th)(f*psi) + on(1,w=0);

P;

real[int] viso(22);
for (int i=2; i<viso.n; i++)
          viso[i] = -0.00004+i*0.00004;
          viso[0] = 0;
          viso[1] = 1e-5;

Vh Phi11 = phi1*phi2;

plot(w, cmm="u (Clamped_Grid_TEST)", viso=viso(0:viso.n-1),
     fill=1, value=1, wait=1);
plot(Phi11, cmm="Phi_11 (Clamped_Grid_TEST)", viso=viso(0:viso.n-1),
     fill=1, value=1, wait=1);
```

E.1.3 The results

In Figure E.1 we plot the Φ_{11} and the solution u to (E.5) computed from equation (E.2). They coincide as expected.

E.2 Clamped isotropic plate and grid: comparison

Here, having at hand the FreeFem++ package we will compute and compare the displacement of a clamped isotropic plate and grid (both, with the aligned and diagonal fibers) under a concentrated load. The geometry will be a rectangle and pentagon, that is, the domains with concave corners and hence the system approach to mentioned fourth order problems applies.

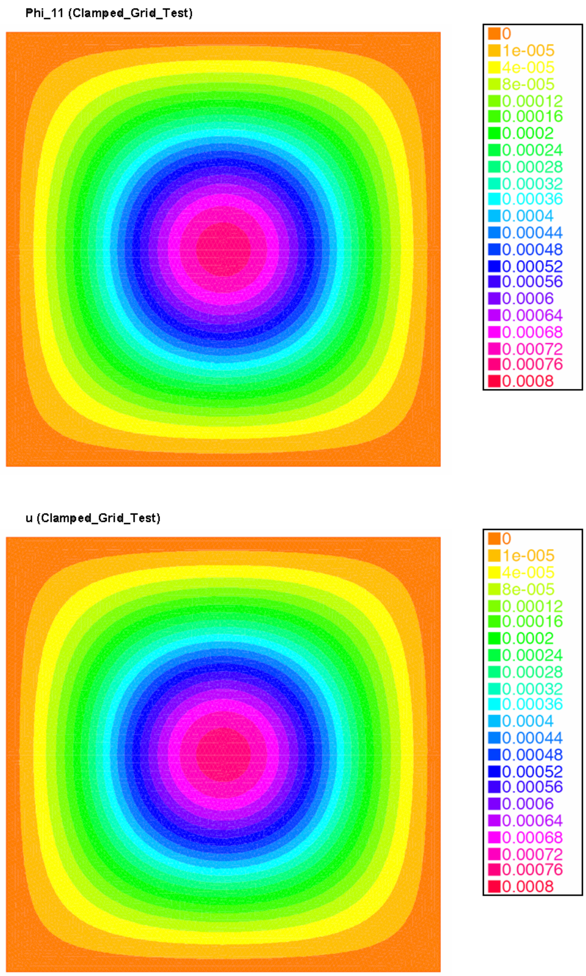

Figure E.1: *On the top the* Φ_{11}*; on the bottom the solution* u *to (E.5) computed from equation (E.2).*

More precisely, let us consider and compute the solutions of the following three boundary value problems:

$$\begin{cases} \Delta^2 u = f & \text{in} \quad \Omega, \\ u = \frac{\partial}{\partial n} u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{E.7}$$

$$\begin{cases} u_{xxxx} + u_{yyyy} = f & \text{in} \quad \Omega, \\ u = \frac{\partial}{\partial n} u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{E.8}$$

and

$$\begin{cases} u_{xxxx} + u_{yyyy} = f_1 & \text{in} \quad \Omega_1, \\ u = \frac{\partial}{\partial n} u = 0 & \text{on} \quad \partial\Omega_1. \end{cases} \tag{E.9}$$

In problems (E.7), (E.8) the Ω is a geometry of a plate and grid with the fibers aligned in Cartesian directions; f is a source term. In problem (E.9) the Ω_1 and f_1 are defined, respectively, as Ω and f rotated by $\frac{1}{4}\pi$, so that the last problem models the clamped grid with the fibers arranged diagonally in Ω_1.

E.2.1 Ω is a rectangle

Let us fix

$$\Omega := \left\{(x, y) \in \mathbb{R}^2 : (-1, 6) \times (-1, 1)\right\},$$

and

$$f := \exp\left[-100\left((x + 0.75)^2 + (y - 0.75)^2\right)\right],$$

in (E.7), (E.8) with their obvious transformation in (E.9). In Figure E.2 we plot the graph of a source term f and in Figure E.3 the solutions to (E.7) – (E.9).

Figure E.2: *Graph of a source term (concentrated load).*

One may see in Figure E.3 that the same force f applied to the clamped rectangular plate, aligned and diagonal grid induces qualitatively different displacements. One may observe for instance that the clamped diagonal grid distributes the applied force “more easily”.

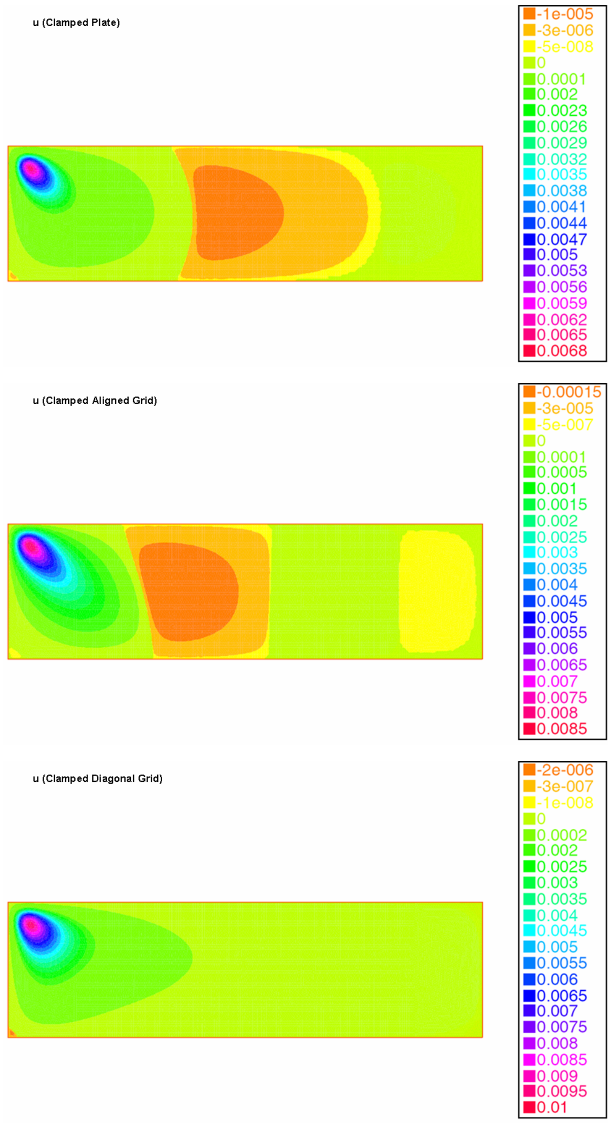

Figure E.3: *Displacement of, respectively, an isotropic rectangular plate, grid with the aligned fibers and grid with the diagonal fibers, under a concentrated load depicted in Figure E.2.*

E.2.2 Ω is a pentagon

Let us set $\Omega_{sq} := \{(x, y) \in \mathbb{R}^2 : (-1, 2) \times (-1, 1)\}$ and consider a pentagon given as follows

$$\Omega := \Omega_{sq} \setminus \{(x, y) \in \mathbb{R}^2 : y \leq -x - 1\}.$$

Also set

$$f = \exp\left[-100\left((x + 0.75)^2 + (y - 0.75)^2\right)\right].$$

We plot in Figures E.4 and E.5, respectively, the graph of a source term f and the solutions to (E.7) – (E.9).

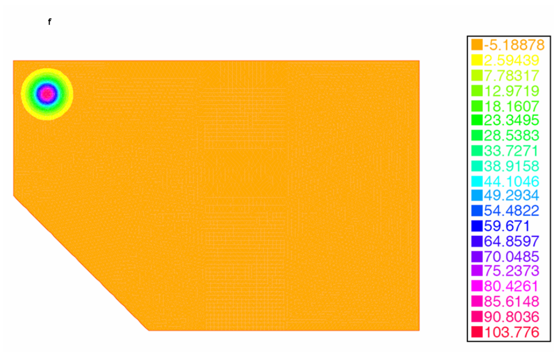

Figure E.4: *Graph of a source term (concentrated load).*

As in a previous case (rectangular domain), one may observe in Figure E.5 completely different qualitative behavior of the clamped isotropic plate, aligned and diagonal grid induced by the same force f.

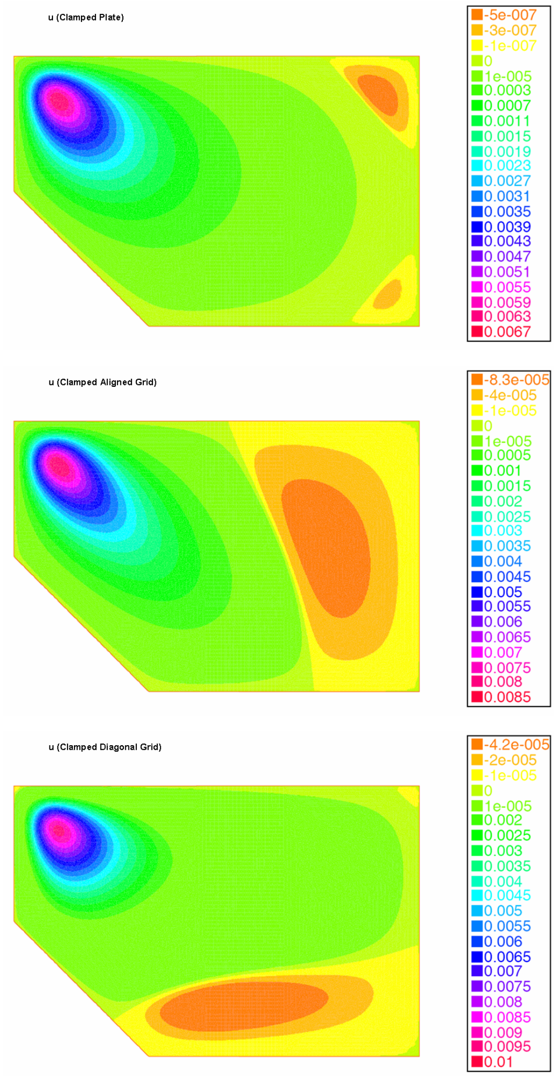

Figure E.5: *Displacement of, respectively, an isotropic rectangular plate, grid with the aligned fibers and grid with the diagonal fibers, under a concentrated load depicted in Figure E.4.*

E.3 Simulations for the DH-106 "Comet"

In this Section we will bring the numerical results on the qualitative stress distribution in a fuselage panel of "The Comet" aircraft. Here we consider the rectangular panel with two types of holes in it (which mimic the windows): the rectangular one with the smoothed corners and the oval one. We assume the panel to be an isotropic plate which is clamped on the boundary and uniformly loaded. That is, we solve the boundary value problem (E.7) in a domain Ω depicted in Figure E.6 and the source term $f = \text{const}$.

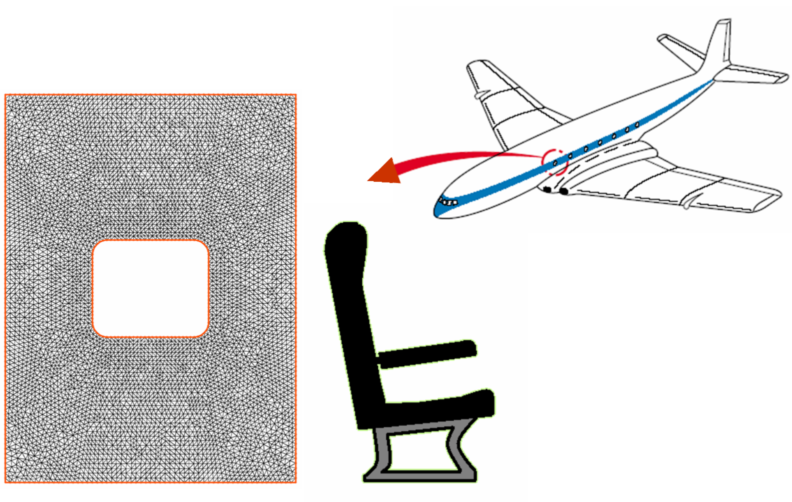

Figure E.6: *A fuselage panel with a window; a chair sketched is intended to outline a "realistic proportion " between a panel and window scale.*

The solution u is the vertical displacement of the panel. Then the bending stresses σ_x, σ_y and the twisting stress τ_{xy} (for details see [24, Chapter 9]), which play a vital role are proportional to the second order derivatives of u in the following way:

$$\sigma_x \sim u_{xx} + \nu u_{yy},$$

$$\sigma_y \sim u_{yy} + \nu u_{xx},$$

$$\tau_{xy} \sim u_{xy},$$

where ν is the Poisson ratio. In Figures E.7 we plot the quantity $u_{xx} + \nu u_{yy}$ with $\nu = 0.3$ in domains which mimic a fuselage panel with square and oval window. This quantity itself mimics the distribution of stress σ_x in the corresponding fuselage panel.

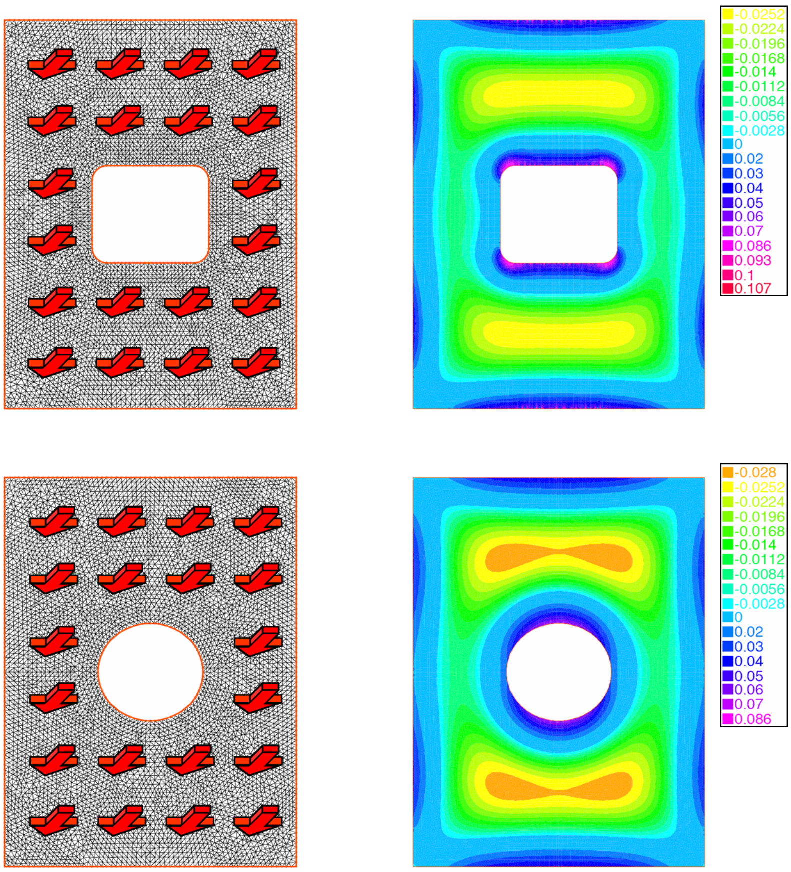

Figure E.7: *Distribution of stress in a fuselage panel with a square and oval windows under uniform load.*

The first observation that may follow from Figure E.7 (the top) is that even the smooth reentrant (i.e. concave) corner is a concentrator of stresses. By turning the "square" windows into oval we reduce the positive maximal stress occurred in a previous case by approximately 20% and, what is more important, we get rid of the concentration of stresses. Indeed, in Figure E.7 (the bottom) the maximal stress is "smashed" over the edge of an oval window.

We have to mention, that the deformation process of a real fuselage panel of an airplane has to be described by more complicated model rather than by model (E.7). Indeed, the real model is curved (see E.7), hence it is not a plate but rather a shell. Apart from normal pressure (in green in Figure E.7) applied to the panel, it also undergoes bending and stretching (in red in Figure E.7), as well as, the influence of the low and high temperatures.

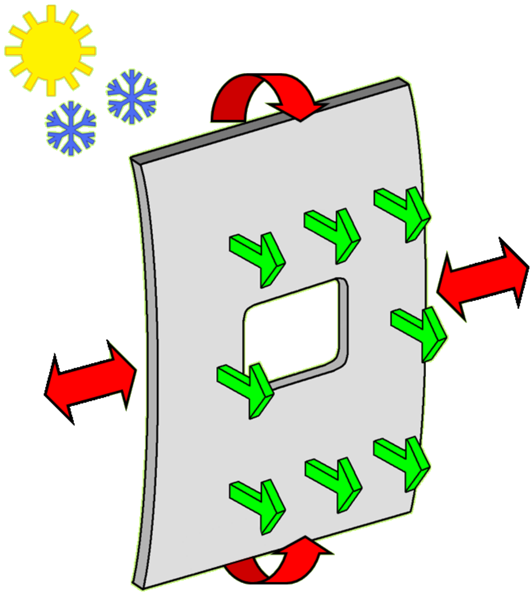

Figure E.8: *Sketch of a real fuselage panel and the factors applied to it.*

But even the results on the stresses we obtained here by considering a simplified model approve the fruitfulness of measures that the designers of "The Comet" implemented in order to remove the initial constructive weakness of the aircraft – square windows.

Bibliography

[1] Bernoulli, Jacob II., *Essai théoretique sur les vibrations des plaques élastiques, rectangulaires et libres.* Nova Acta Academiae Scientiarum Imperialis Petropolitanae V (1789), p.197.

[2] C. Bandle, M. Flucher, *Table of inequalities in elliptic boundary value problems, Recent progress in inequalities*, Series: Math. Appl., vol. 430 (1996), pp. 97–125.

[3] H. Blum, R. Rannacher, *On the boundary value problem of the biharmonic operator on domains with angular corners*, Math. Meth. in the Appl. Sci. 2 (1980), pp. 556–581.

[4] M. Costabel, M. Dauge, S. Nicaise, editors. *Boundary value problems and integral equations in nonsmooth domains*, Lecture Notes in Ppure and Apllied Mathematics, Vol. 167. Marcel Dekker, Inc., New York, 1995.

[5] Ph. Clément, G. Sweers, *Uniform anti-maximum principles*, J.Differential Equations 164 (2000), pp. 118–154.

[6] B. Dacorogna, *Introduction to the calculus of variations,* Imperial College Press, London, 2004.

[7] M. Dauge, *Elliptic boundary value problems on corner domains; smoothness and asymptotics of solutions*, Springer, Berlin, 1988.

[8] C. Davini, Γ-*convergence of External Approximations in Boundary Value Problems Involving the bi-Laplacian*, Journal of Computational and Applied Mathematics, Vol. 140 (2002), pp. 185–208.

[9] P. Drábek, J. Milota, *Methods of Nonlinear Analysis: Applications to Differential Equations,* Birkhäuser Verlag, Basel, 2007.

[10] L.C. Evans, *Partial differential equations*. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.

[11] FreeFem++, http://www.freefem.org/ff++/

[12] T. Gerasimov, G. Sweers, *The regularity for a clamped grid equation* $u_{xxxx} + u_{yyyy} = f$ *in a domain with corner*, Electronic Journal of Differential Equations, vol. 2009 (2009), No. 47, pp. 1–54. http://ejde.math.txstate.edu/

[13] T. Gerasimov, A. Stylianou, G. Sweers, *A failure in the Hamiltonian approach to the clamped plate equation*, submitted.

[14] P. Grisvard, *Singularities in boundary value problems*, Springer-Verlag, Berlin, 1992.

[15] http://www.geocities.com/CapeCanaveral/Lab/8803/comet.htm

[16] http://aviation-safety.net/database/record.php?id=19540110-1

[17] G. Kirchhoff, *Über das Gleichgewicht und die Bewegung einer elastischen Scheibe*, J. Reine Angew. Math. 40 (1850), pp. 51–88.

[18] V.A. Kondratiev, *Boundary value problems for elliptic equations in domains with conical or angular points*, Trudy Moskov. Mat. Obšč. 16 (1967), pp. 209–292 (in Russian), English transl.: Trans. Moscow Math. Soc. 16 (1967), pp. 227–313.

[19] V.A. Kozlov, V.G. Maz'ya, J. Rossmann, *Elliptic Boundary Value Problems in Domains with Point Singularities*, Mathematical Surveys and Monographs, vol. 52, American Mathematical Society, Providence, RI, 1997.

[20] V.A. Kozlov, V.G. Maz'ya, J. Rossmann, *Spectral problems associated with corner singularities of solutions to elliptic equations*, Mathematical Surveys and Monographs, vol. 85, American Mathematical Society, Providence, RI, 2001.

[21] B. Kawohl, G. Sweers, *On the differential equation* $u_{xxxx} + u_{yyyy} = f$ *for an anisotropic stiff material*, SIAM J. Math. Anal., Vol. 37, No. 6 (2006), pp. 1828–1853.

[22] O.A. Ladyzhenskaya, N.N. Ural'tseva, *Linear and quasilinear elliptic equations,* Academic Press, NY, 1968.

[23] D. Leguillon, E. Sanchez-Palencia, *Computation of singular solutions in elliptic problems and elasticity*, Masson, Paris, 1987.

[24] S.G. Lekhnitskii, *Anisotropic plates*, 2nd edition, translated from Russian, Gordon and Breach, New York, 1968.

[25] J.-L. Lions, E. Magenes, *Non-homogeneous boundary value problems and applications*, Vol. I, Springer-Verlag, Berlin, 1972.

[26] S.A. Nazarov, B.A. Plamenevsky, *Elliptic problems in domains with piecewise smooth boundaries*, De Gruyter, Berlin, 1994.

[27] S.A. Nazarov, A. Slutskij, G. Sweers, *Korn's inequalities for a reinforced plate*, submitted.

[28] S.A. Nazarov, G. Sweers, *A hinged plate equation and iterated Dirichlet Laplace operator on domains with concave corners*, J. Differential Equations 233 (2007), pp. 151–180.

[29] R.S. Palais, The Morse lemma for Banach spaces, Bull. Amer. Math. Soc. 75 (1969), pp. 968–971.

[30] M.L. Williams, *Surface Stress Singularities Resulting From Various Boundary Conditions in Angular Corners of Plates Under Bending*, Proceedings of the First U.S. National Congress of Applied Mechanics, Illinois Institute of Technology, Chicago, 1951, pp. 325–329.

Summary

The main focus of this thesis is the regularity question for the fourth order elliptic problem

$$\begin{cases} u_{xxxx} + u_{yyyy} = f & \text{in} \quad \Omega, \\ u = \frac{\partial}{\partial n} u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{S.1}$$

in an open and bounded domain $\Omega \subset \mathbb{R}^2$ which has one corner in $0 \in \partial\Omega$ with opening angle $\omega \in (0, 2\pi]$. Here n stands for the outward normal on $\partial\Omega$.

For the right-hand side $f \in L^2(\Omega)$ problem (S.1) has a unique weak solution $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ (see Chapter 2). We study the *optimal regularity*[2] of this weak solution and more specifically near a corner of the domain. Within the framework of the Kondratiev theory developed to treat the problems of type (S.1) in domains with corner singularities, it holds that for $f \in L^2(\Omega)$ the weak solution $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ to (S.1) possesses the asymptotic representation:

$$u = w + \sum_{0<\mathrm{Re}(\lambda_j)<2} \sum_{q=0}^{\kappa_j - 1} c_{j,q} r^{\lambda_j + 1} (\log(r))^q \Phi_{j,q}(\lambda_j, \theta), \tag{S.2}$$

where $w \in W^{4,2}(\Omega)$ and the double sum are, respectively, a regular and a so-called "singular" parts of u. The singular part written down in the polar coordinates (r, θ) represents the behavior of the solution u locally in the vicinity of an angular point 0. In formula (S.2), $c_{j,q}$ are the constants, λ_j are the solutions of a certain transcendental equation, κ_j is the algebraic multiplicity of each λ_j, the functions $\Phi_{j,q}$ are infinitely differentiable.

[2]In a classical sense, that is, under appropriate smoothness assumptions for $\partial\Omega$, the optimal regularity for the solution means: if $f \in W^{k,p}(\Omega)$, then $u \in W^{k+4,p}(\Omega)$, where $k \geq 0$ and $p \in (1, \infty)$. Due to the presence of a corner, this result for problem (S.1) in general does not apply.

The key parameters in (S.2) are the exponents λ_j, particularly, the λ_1 as it defines the differentiability of the whole singular part of u. For the present operator $L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ in (S.1) it holds that

$$(0, 2\pi] \times \big[0, \tfrac{1}{2}\pi\big) \ni (\omega, \alpha) \mapsto \operatorname{Re}(\lambda_1(\omega, \alpha)),$$

that is, λ_1 is a function of the opening angle ω and the parameter α. The latter defines orientation of a domain Ω in a coordinate system (x, y).

A striking consequence of the precise regularity described in this thesis is the existence of at least one interval $\big(\frac{1}{2}\pi, \omega_\star\big)$, $\omega_\star/\pi \approx 0.528$ (in degrees $\omega_\star \approx 95.1°$), on which λ_1 is the *increasing* function of ω. In terms of the optimal regularity this result means that the differential properties of solution u to (S.1) improve with increasing opening angle, what makes a drastic difference to the Laplacian and bilaplacian models. The rigorous proof concerns the case $\alpha = 0$ (Chapter 3) and for $\alpha \in \big(0, \frac{1}{2}\pi\big)$ we only give the numerical results for the behavior of λ_1 on the interval $\omega \in (0, 2\pi]$. The regularity statement for the solution u to (S.1) in terms of the weighted and standard Sobolev spaces is given in Chapter 5. It implements the embedding results for the spaces in question which is elaborated in the foregoing Chapter 4.

A secondary result of the research presented addresses the possibility to solve the fourth order problem (S.1) in a domain Ω with the opening angle $\omega \in (0, 2\pi]$ through the system approach:

$$\begin{cases} -\Delta v - \sqrt{2} v_{xy} = f & \text{in} \quad \Omega, \\ -\Delta u + \sqrt{2} u_{xy} = v & \text{in} \quad \Omega, \\ u = \frac{\partial}{\partial n} u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{S.5}$$

Thus, in Chapter 6 we show that for every $\alpha \in [0, \frac{1}{2}\pi)$ the corresponding weak solutions $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ to (S.1) and $(v, u) \in W^{1,2}(\Omega) \times \overset{\circ}{W}{}^{1,2}(\Omega)$ to problem (S.5) will coincide if the opening angle ω satisfies $\omega \leq \pi$. In general, this is not the case for all $\omega > \pi$.

Appendix A contains a result, which is although not related directly to problem (S.1) yet may be of interesting in the applications. Here we prove that for every fourth order elliptic operator

$$L = \frac{\partial^4}{\partial x^4} + b_1 \frac{\partial^4}{\partial x^3 \partial y} + b_2 \frac{\partial^4}{\partial x^2 \partial y^2} + b_3 \frac{\partial^4}{\partial x \partial y^3} + \frac{\partial^4}{\partial y^4}, \tag{S.3}$$

with $b_j \in \mathbb{R}$, $j = 1, 2, 3$, there exists precisely one $a \in [1, +\infty)$ such that L is algebraically equivalent to the operator

$$L_a = \frac{\partial^4}{\partial x^4} + 2a\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}. \tag{S.4}$$

That is, one may always find an appropriate linear coordinate transformation such that due to this transformation the operator L turns into L_a with $a \in [1, +\infty)$. This result is based on the Möbius mapping. In the mechanical context this transformation result for the operator L means that there always exists a coordinate system such that in this system every anisotropic 2d-medium is viewed as an orthotropic one.

Appendices B–D contain all supplementary computational and numerical results, while in Appendix E we use the approach developed in the last chapter of the thesis for the numerical treatment of some applied problems.

Samenvatting

Het ingeklemde elastische rooster, een vierde orde vergelijking op een gebied met hoek

van Tymofiy Gerasimov

Het centrale thema van dit proefschrift is de regulariteitsvraag voor het vierde orde elliptische probleem

$$\begin{cases} u_{xxxx} + u_{yyyy} = f & \text{in} \quad \Omega, \\ u = \frac{\partial}{\partial n} u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{S.1}$$

in een open en begrensd gebied $\Omega \subset \mathbb{R}^2$, dat een hoek bevat in $0 \in \partial\Omega$ met als opening $\omega \in (0, 2\pi]$. De letter n staat voor de naar buiten gerichte normaalvektor op $\partial\Omega$.

Voor de rechterzijde $f \in L^2(\Omega)$ heeft probleem (S.1) een eenduidige zwakke oplossing $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ (zie Hoofdstuk 2). We onderzoeken de *optimale regulariteit*[3] van deze zwakke oplossing in het bijzonder bij de hoek van dit gebied. Volgens de door Kondratiev ontwikkelde theorie voor problemen van het type (S.1), heeft de zwakke oplossing voor $f \in L^2(\Omega)$ de volgende asymptotische ontwikkeling:

[3] In de klassieke zin, onder voldoende gladheid van de rand $\partial\Omega$, geldt voor de optimale regulariteit: als $f \in W^{k,p}(\Omega)$, dan geldt $u \in W^{k+4,p}(\Omega)$, voor $k \geq 0$ en $p \in (1, \infty)$. Door de aanwezigheid van een hoek, is dit resultaat voor (S.1) in het algemeen niet van toepassing.

$$u = w + \sum_{0<\mathrm{Re}(\lambda_j)<2} \sum_{q=0}^{\kappa_j-1} c_{j,q} r^{\lambda_j+1} (\log(r))^q \Phi_{j,q}(\lambda_j,\theta), \tag{S.2}$$

waarbij $w \in W^{4,2}(\Omega)$ en de dubbele som respectievelijk het reguliere en het "singular" deel van de oplossing u vormen. Het singuliere deel is in poolcoordinaten (r,θ) beschreven en vertegenwoordigd het gedrag van de oplossing u bij het hoekpunt. In formule (S.2), zijn $c_{j,q}$ konstanten, zijn λ_j oplossingen van een bepaalde transcendentale vergelijking, zijn de κ_j de algebraïsche multipliciteiten van λ_j en vormen de $\Phi_{j,q}$ oplossingen van een bepaalde gewone differentiaalvergelijking met bijbehorende randwaarden. Deze $\Phi_{j,q}$ zijn oneindig vaak differentieerbaar.

De cruciale parameters in (S.2) zijn de λ_j, in het bijzonder λ_1 omdat deze de differentierbaarheid van het singuliere deel bepaalt. Voor de huidige operator $L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ in (S.1) geldt dat

$$(0,2\pi] \times \left[0, \tfrac{1}{2}\pi\right) \ni (\omega,\alpha) \mapsto \mathrm{Re}(\lambda_1(\omega,\alpha)),$$

dat wil zeggen, λ_1 is een functie van de openingshoek ω en de parameter α. De laatste definiert de orientatie van het gebied Ω ten opzichte van het coordinatensysteem (x,y).

Een opvallend gevolg van het regulariteitsresultaat in dit proefschrift is de existentie van minstens een interval $\left(\frac{1}{2}\pi, \omega_\star\right)$, $\omega_\star/\pi \approx 0.528$ (in graden $\omega_\star \approx 95.1°$), waar λ_1 een *stijgende* functie van ω is. Voor de optimale regulariteit betekent dit, dat de differentieerbaarheidseigenschappen van de oplossing u van (S.1) verbeteren bij toenemende hoek en dit is tegengesteld aan het gedrag bij randwaardeproblemen met de laplace- en bilaplace-operatoren. Een rigoreus bewijs betreft het geval $\alpha = 0$ (Hoofdstuk 3); voor $\alpha \in \left(0, \frac{1}{2}\pi\right)$ geven we alleen de numerieke resultaten voor λ_1 op het interval $\omega \in (0,2\pi]$. Het regulariteitsresultaat voor de oplossing u van (S.1) in de zin van gewogen en standaard Sobolev-ruimten vindt men in Hoofdstuk 5. Het gebruikt de imbeddingsresultaten die uitgewerkt zijn in het voorafgaande Hoofdstuk 4.

Een secundair resultaat van het gepresenteerde onderzoek betreft de de mogelijkheid om het vierde orde probleem (S.1) in een gebied Ω door middel van een systeem op te lossen:

$$\begin{cases} -\Delta v - \sqrt{2}v_{xy} = f & \text{in} \quad \Omega, \\ -\Delta u + \sqrt{2}u_{xy} = v & \text{in} \quad \Omega, \\ u = \frac{\partial}{\partial n}u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{S.5}$$

In Hoofdstuk 6 bewijzen we namelijk dat voor iedere $\alpha \in [0, \frac{1}{2}\pi)$ de bijbehorende zwakke oplossing $u \in \overset{\circ}{W}{}^{2,2}(\Omega)$ van (S.1) en $(v, u) \in W^{1,2}(\Omega) \times \overset{\circ}{W}{}^{1,2}(\Omega)$ van probleem (S.5) overeenkomen wanneer de openingshoek ω kleiner als π is. In het algemeen is dit niet het geval voor $\omega > \pi$.

Appendix A bevat een resultaat, dat ofschoon niet direct verwant met probleem (S.1) niettemin interessant kan zijn voor de toepassingen. Daar bewijzen we dat er voor iedere vierde orde elliptische operator

$$L = \frac{\partial^4}{\partial x^4} + b_1 \frac{\partial^4}{\partial x^3 \partial y} + b_2 \frac{\partial^4}{\partial x^2 \partial y^2} + b_3 \frac{\partial^4}{\partial x \partial y^3} + \frac{\partial^4}{\partial y^4}, \tag{S.3}$$

met $b_j \in \mathbb{R}$, $j = 1, 2, 3$, er precies één $a \in [1, +\infty)$ bestaat zodanig, dat L algebraïsch equivalent is met de operator

$$L_a = \frac{\partial^4}{\partial x^4} + 2a \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}. \tag{S.4}$$

In andere woorden, er bestaat een lineaire coordinatentransformatie zodat L in L_a verandert in dit nieuwe stelsel. Dit resultaat is gebaseerd op een geschikt gekozen Möbius-afbeelding. In de mechanische kontext betekent het, dat men altijd een coordinatensysteem zodaning kiezen kan dat het anisotrope 2d-medium als een orthotroop medium beschouwd kan worden.

Appendices B–D bevatten alle supplementaire rekentechnische en numerieke resultaten, terwijl in Appendix E de aanpak van het laatste hoofdstuk ontwikkeld wordt voor de numerieke behandeling van enkele toegepaste problemen.

Acknowledgement

At the end of this thesis I would like to thank all the people who contributed directly and indirectly in this result of mine.

First of all I would like to thank Guido Sweers, my scientific supervisor, who introduced me to this research area. I am grateful for his openness in discussions, guidance and constant willingness to explain the abstract yet sometimes obvious and simple mathematical tricks I am supposed to know, but have never heard before. I benefited a lot from his expertise on the subject, support and encouragement: I expanded my mathematical outlook, gained courage to tackle potentially tricky problems and also learned how to formulate thoughts in a precise and consistent manner.

All the other PhD-students of the Functional Analysis Group who have been my office-fellows during these four years, contributed to the nice working atmosphere: Anna Dall'Acqua, Jan Maas, Mark Veraar, Mahmoud Baroun and Sonja Cox. I am indebted to Sonja for improving my English and, especially, for her help with the translation of the Propositions into Dutch. Also, my words of gratitude go to the secretary of the Analysis Group Dori Steeneken and the computer expert Xiwei Wu for their openness and constant willingness to help.

I would like to thank Prof. dr. Victor Gristchak, the first scientific supervisor of mine in Ukraine, for introducing me to the world of science (perturbation methods and hybrid techniques). It's got obvious for me now that he "kept pushing" me in the right direction even when I did not realize that.

I am greatly indebted to Prof. dr. Igor Andrianov for his informal friendly attitude and attention he paid to my humble person in a moment when I felt confused about my scientific prospectives. He was the one who encouraged me to do PhD abroad and has been keeping his interest in the progress of my work.

Also, I would like to express my gratitude to Dr. ir. Wim van Horssen for the possibility to come to Delft and for his open and considerate manner to discuss and resolve the administrative questions.

A big thanks goes to the CICAT ladies Theda Olsder and Veronique van der Varst, for their assistance in handling many vitally important administrative matters that a foreign PhD-student may encounter in the Netherlands.

I would like to acknowledge the friends of mine in Delft, Alexey Andrianov and Alexander Pischanskiy, for a nice spirit of our conversation and help.

Finally, I thank my whole family for their love, encouragement and support.

Delft, September 2009 *Tymofiy Gerasimov*

Curriculum Vitae

Tymofiy Gerasimov was born in July 27, 1977 in Tajikistan, one of the republics of the former Soviet Union.

In 1994 he completed his high school education at the gymnasium N28 in Zaporizhzhia, Ukraine and pursued the study of Applied Mathematics at the Zaporizhzhia State University (since 2005, the Zaporizhzhia *National* University).

After graduation, between 1999 and 2004, he was involved in the teaching and research activity at the Applied Mathematics Department and the Solid Mechanics Group of the same university.

In January 2005 he started his PhD research at the Functional Analysis Group at the Delft University of Technology under the supervision of Prof. dr. G.H. Sweers.