NONLINEARITY IN GROUNDWATER FLOW

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR
IN DE TECHNISCHE WETENSCHAPPEN AAN DE TECHNISCHE
HOGESCHOOL DELFT, OP GEZAG VAN DE RECTOR MAGNIFICUS
PROF. DR. IR. F. J. KIEVITS, VOOR EEN COMMISSIE AANGEWEZEN
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FRANCISCUS BARTHOLOMEUS JOHANNES BARENDS

CIVIEL INGENIEUR

GEBOREN TE ARNHEM

I. Bij de iteratieve oplossing van een lineair stelsel vergelijkingen, waarvan de systeemmatrix positief definiet is, met behulp van de conventionele Gaus-Seidel overrelaxatie wordt de maximaal toelaatbare waarde van de relaxatie factor gewoonlijk bepaald door de grens waarboven de hoofdeigenwaarde van de iteratieve matrix complex wordt, hetgeen aanleiding geeft tot oscillaties. Niettemin zijn er toch grotere waarden voor de relaxatiefactor toelaatbaar en dit versnelt in sommige gevallen bovendien de convergentie van het rekenproces.

II. De in de literatuur gepubliceerde lijst van Mellin transformaties kan worden uitgebreid met de volgende gelijkheid:

$$\int_0^\infty \chi^{(n+1)} \zeta^{n+s} \, d\zeta = (-1)^{n+1} \tilde{\chi} \prod_{m=0}^{n} (m+s) + R_{\zeta \to \infty} - R_{\zeta \downarrow 0},$$

waarin:

$$R = \chi^{(n)} \zeta^{n+1} + \sum_{j=1}^{n} \left\{ (-1)^j \left[\prod_{k=0}^{j-1} (n+s-k) \right] \chi^{(n-j)} \zeta^{n+s-j} \right\}.$$

De grootheid $\tilde{\chi}$ stelt de Mellin transformatie van χ voor.

III. De stabiliteit van zeedijken kan nadelig worden beinvloed door refractie van drukgolven in het dijkmassief langs geometrische randen opgewekt door golfklappen op het dijktalud en wel des te meer naar mate de dijk met water verzadigd is.

IV. De methode van Drucker om een spanningsveld te beschrijven door superpositie van fictieve lijnspanningen kan worden toegepast op een belaste plaat door deze opgebouwd te denken uit meerdere orthotrope balkenlagen in verschillende richtingen, die alleen op buiging worden belast. De bekende plaatvergelijking, waarin de doorbuiging w onder invloed van de bovenbelasting p wordt beschreven volgens:

$$\nabla^2 \nabla^2 w = p/K,$$

ontstaat dan door een symmetrisch systeem van tenminste drie verschillend georiënteerde identieke balklagen te kiezen, waarbij de werkelijke plaatstijfheid K gerelateerd is aan de fictieve balkstijfheid ϱ. Zo geldt voor drie lagen (60°): $K = 9\varrho/8$, voor vier lagen (45°): $K = 3\varrho/2$, en voor oneindig veel lagen: $K = 3\pi\varrho/8$.

V. Een markant effect van het consolidatie proces is dat de fase-verschuiving bij cyclische belasting negatief kan zijn. Hierin verschilt het essentieel van het warmte diffusie proces.

F. B. J. Barends, LGM-Mededelingen XIX, 1978

VI. Van een ondergrond, die als halfruimte wordt opgevat, verloopt de bodemdaling tengevolge van continue winning van olie of gas bij aanname van perfect elastische eigenschappen van de grond aanvankelijk snel, later zeer geleidelijk, maar er is geen eindige limietwaarde.

VII. De uitdrukking voor de samendrukbaarheid van poriënwater met meegevoerde of in de poriën opgesloten vrije micro-bellen lucht kan worden uitgebreid tot de situatie, waarbij ook lucht aan de gronddeeltjes is gebonden, onder de voorwaarde dat de adhesie aan de gronddeeltjes sterker is dan de oppervlakte spanning tussen water en lucht en dat de diffusie van lucht uit vrije microbellen naar gebonden luchtvolumina niet wordt belemmerd.

F. B. J. Barends, LGM-Mededelingen XX-2, 1979

VIII. Bij de berekening van de stabiliteit van een aarden talud moet in geval zich een damwand bevindt in de beschouwde afschuivende schol een extra horizontale kracht in rekening worden gebracht, die zijn oorsprong vindt in het door de damwand opgestuwde grondwater.

F. B. J. Barends, LGM-Mededelingen XIX, 1978

IX. Een tijdelijke verhoging in de rivierstand plant zich voort in het massief van een aangrenzende dijk daarbij gedempt en vertraagd door freatische berging, zodanig dat de lengte L van het invloedsgebied gemeten langs de freatische lijn in een eenvoudige formule kan worden uitgedrukt, namelijk:

$$L = \sqrt{tT/n},$$

waarin t de tijdsduur van de verhoging voorstelt, T een gemiddelde transmissiviteit, (T = KD), en n de freatische berging is.

X. Als de freatische lijn in een massief gebieden met verschillende doorlatendheid doorsnijdt, doet zich op de grenzen van deze gebieden een singulariteit voor, die bij toepassing van de Eindige Elementen Methode met lineaire basisfuncties tot numerieke instabiliteit aanleiding kan geven, welke evenwel met een eenvoudige ingreep kan worden voorkomen.

XI. De expliciete tijdstap voor tijdsafhankelijke freatische grondwaterstromingsproblemen opgelost met een numerieke methode kan aanzienlijke worden vergroot door de fluctuaties van de randvoorwaarde zelf in de beschouwing te betrekken zonder dat daarbij de nauwkeurigheid wordt aangetast.

XII. Computerfilms kunnen het 'black box' syndroom genezen.

XIII. Zoals reeds eerder beweerd door Verruijt, maar helaas nog niet algemeen bekend, is de bergings- of consolidatie coefficient, welke een factor (1 − n) bevat (n is de porositeit), principieel onjuist. Er is in dat geval geen rekening gehouden met het feit, dat de wet van Darcy betrekking heeft op de beweging van het poriënwater ten opzichte van het bewegend poreuze medium.

A. Verruijt, proefschrift Delft, 1969, stelling 4.
Dit proefschrift, hoofdstuk 3.

XIV. Met de potentiaal:

$$\tilde{\phi} = z + \int_{p_F}^{p} (1/\varrho' g)\, d\ell \quad , \quad p_F = \alpha F/(\alpha + n\beta'),$$

kan het stromingsgedrag van samendrukbaar porienwater inclusief eventueel aanwezige micro luchtbellen in een rotatie-vrij vervormbaar poreus massief volledig worden beschreven, mits de doorlatendheid isotroop is.

Dit proefschrift, hoofdstuk 3.

XV. De algemene niet-lineaire bergingsvergelijking voor horizontaal watervoerende lagen luidt:

$$\nabla^2\tilde{\phi} + m\,(\nabla\tilde{\phi})^2 = \frac{\partial\tilde{\phi}}{c\partial t},$$

welke met de transformatie: $\chi = a \exp(m\tilde{\phi}) + b$, kan worden teruggebracht tot de conventionele lineaire bergingsvergelijking:

$$\nabla^2 \chi = \frac{\partial\chi}{c\partial t}.$$

Hierin zijn a en b vrij te bepalen, terwijl m de invloed ten gevolge van variaties in de doorlatendheid en in de dichtheid van de porien vloeistof bevat. De factor c is de consolidatie coefficient.

Dit proefschrift, hoofdstuk 4

XVI. Slechts één wijziging maakt een bestaand grondwaterstromingsprogramma geschikt voor niet-lineaire problemen.

XVII. In plaats van de effectieve porositeit behoort de porositeit zelf in de kinematische randvoorwaarde voor een bewegende freatische lijn voor te komen.

Dit proefschrift, hoofdstuk 5

XVIII. Het verminderen van sociale zekerheden verbetert sociale verhoudingen.

XIX. De Nederlandse overheidssteun voor het bedrijfsleven lijkt weinig efficient, als men bedenkt dat de regeringssteun in 1978 ter grootte van 2.5 miljard wordt overschaduwd door het verlies van 3.8 miljard dat het Nederlandse zakenleven noteerde vanwege het niet of laat betalen van geleverde goederen of diensten aan niet bonafide en valse vennootschappen.

XX. Het ongenuanceerd aaneenschakelen van feiten in menig visie-programma op de televisie bevordert eerder de vorming van een tele-visie.

DIT PROEFSCHRIFT IS GOEDGEKEURD DOOR DE PROMOTOR
PROF. DR. IR. A. VERRUIJT

TO ECHLASS

الى اخــلاص

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Curriculum vitae

Frans B. J. Barends werd geboren op 18 oktober 1946 te Arnhem en groeide op met vier zussen en vier broers. Op de middelbare school was hij naast de lessen hoofdredacteur van het schoolblad 't Snuffeltje en violist in het schoolorkest. In 1965 werd het gymnasium β voltooid. Tijdens de daarop volgende studie te Delft werd ook tijd besteed aan niet-technische bezigheden, zoals sturen bij 'n roeivereniging, zingen bij 'n zangvereniging, in de redactie van 'n annuarium, toneelspelen en de organisatie van een cultureel festijn. De praktische studie werd deels verricht in Μεγαλοπολις, een uiterst klein stadje in het midden van de Peloponnesos.
In 1971 behaalde hij het diploma van civiel ingenieur aan de Technische Hogeschool te Delft, afgestudeerd in de richting der theoretische grondmechanica. Hij bleef op de Hogeschool als wetenschappelijk medewerker zich verdiepend in consolidatie en gronddynamica onder leiding van de Josselin de Jong en Verruijt.
In 1973 viel hem een beurs ten deel om aan de Technion te Haifa een jaar van studie te besteden aan grondwaterbeheer en dispersie begeleid door Bear. In Israel werd veel aandacht besteed aan het leven daar; hij leerde hebreeuws en trouwde met Echlass Zoabi, een arabische onderwijzeres uit Haifa.
Bij zijn terugkeer in 1974 kreeg hij een aanstelling bij het Laboratorium voor Grondmechanica als speurwerk ingenieur bij de wiskunde en informatica groep, belast met het ontwikkelen van computer programmatuur. In de avonduren werd er arabisch gevolgd in Leiden. In 1978 werd hij hoofd van voornoemde groep bij het Laboratorium voor Grondmechanica.

Samenvatting

Sinds Darcy in 1856 de basis legde voor de berekening van de stroming van water door zand, heeft de grondwaterstroming de wetenschappers beziggehouden. Grondwater is essentieel voor de landbouw en de drinkwatervoorziening, maar ook speelt het een belangrijke rol, indien grond als constructie-element wordt gebruikt, zoals bij dijken, wegen en funderingen. Het mechanisch gedrag van verzadigde of droge, fijnkorrelige of grove grond maakt een groot verschil. De theorie van de grondwatermechanica moet worden gebaseerd op het systeem: water-grond-lucht. Tot nu toe heeft men zich, wat de fysische eigenschappen betreft, beperkt tot nagenoeg verzadigde en/of onvervormbare grond. In dit proefschrift wordt deze theorie uitgebreid tot die van een samendrukbare poriënvloeistof in een semi-verzadigd vervormbaar poreus medium: een water-grond-lucht mengsel, waarin de lucht voorkomt als vrije minuscule bellen en waarin het water stroomt, terwijl de grond zelf vervormt. Er wordt aangenomen, dat dit vervormingsgedrag lineair is en bovendien rotatievrij.
Na een diepgaande beschouwing wordt afgeleid, dat het mechanisch gedrag van dit systeem op een betrekkelijk eenvoudige wijze kan worden geformuleerd, waarbij rekening wordt gehouden met verschillende niet-lineaire effecten. Zo worden convectieve termen en de variatie in de doorlatendheid, als de grond vervormt, meegenomen. De geldigheid van de gevonden formulering wordt besproken. Een algemene oplosmethode met behulp van Mellin transformaties maakt het mogelijk de invloed van deze niet-lineaire termen te belichten aan de hand van analytische oplossingen van enige karakteristieke problemen.
In de grondwaterstroming kent men ook zogenaamde bewegende randen. De grondwaterspiegel, die immers varieert, is zo'n rand. Dit houdt in, dat het gebied waarin de stroming wordt bekeken, zelf verandert (geometrische niet-lineariteit). Ook op dit aspect wordt nader ingegaan. Met behulp van computerprogramma's is men in staat tijdsafhankelijke grondwaterstromingsproblemen met bewegende randen op te lossen. In de discussie wordt vaak de uitgebreide literatuur betrokken.
Tenslotte is er over niet-lineariteit in grondwaterstroming op te merken, dat in de meeste praktische gevallen de lineaire theorie goed voldoet, dat niet-lineaire effecten leiden tot een reductie van 10% tot 20% in de zogeheten invloedssfeer, en dat tijdsafhankelijke freatische problemen numeriek expliciet kunnen worden opgelost met een veel grotere tijdstap dan voorheen gebruikelijk was.

contents

1. INTRODUCTION

The subject of this thesis is the nonlinear aspects of groundwater flow.

The origine of the study of groundwater flow (seepage) goes back to Darcy (1856), who performed experiments about the flow of water in pipes filled with sand and established the steady law of this motion. Depuit (1863) and Boussinesq (1903) developed the hydraulic theory of flow of groundwater and Forchheimer (1901) gave the hydraulic theory of wells.

Later, a rigorous mathematical treatise on the theory of pore fluid flow was founded and comprehensive textbooks explaining the fundamentals of this phenomenon were written, Muskat (1937) and Polubarinova Kochina (1952). The fact that flow through porous media involves the mathematical behaviour of a deformable porous matrix was recognized by Terzaghi (1923), extended by Biot (1941) and applied in several textbooks, De Wiest (1969), Bear (1972), Gudehus (1977). Many articles covering the various topics in groundwater mechanics have been published in technical journals and in proceedings of conferences and symposia.
Some of this literature is discussed in this thesis, but no attempt is made to give a complete citation.

The theory of seepage must be based on the system water-soil-air. However, the physical properties of this system are usually restricted to saturated and/or undeformable porous media. In this thesis an attempt is made to extend the contemporary theory to compressible pore fluid flow in a semi-saturated deformable porous medium.

The contents of chapter 2 consist of an exposition of the fundamental laws describing the constitution of the substances involved as well as their interaction. The applicability of Darcy's law is investigated utilizing the principles of aimensional analysis. An empirical relation between permeability and porosity is formulated. A new formula for the compressibility of air-water mixture is derived incorporating surface tension, air solubility and water evaporation effects.
The influence of soil deformation on pore pressure generation is reconsidered from the original abstractions in soil mechanics. A rigorous restriction is introduced, since Hooke's law is assumed to govern soil deformation behaviour.

Chapter 3 is devoted to a general mathematical formulation of the equations of motion for pore fluid flow. The existence of extra terms in the continuity equation is clearified. They are induced by the compressibility of the pore fluid and the air content, and by the fact that porous matrix deformations give rise to storage effects, to convective terms and to changes in the permeability. Several approaches from literature are evaluated and their discrepancies are explained. A general potential for irrotationally deforming semi-saturated porous media is found to describe the porous flow process incorporating all the aspects concerned.

A straightforward analytical procedure to solve the general nonlinear storage equation is described in chapter 4. It involves a specific transformation, which results in a common potential equation but in terms of the so-called extensive potential. Though for the vertical direction (gravity) a nonlinear coefficient remains in the governing equation. The Mellin transformation technique permits handling this term for general boundary conditions. A proof is given and for several characteristic problems a closed form mathematical solution is presented.

A typical feature of groundwater flow is the existence of a free (phreatic) surface, which may change with pore pressure fluctuations. The flow field geometry varies and this phenomenon is discussed in chapter 5. The mathematical formulation of the behaviour of a free surface is extended to include convective terms due to soil deformation. A general view on numerical procedures to solve groundwater flow problems in inhomogeneous media for general boundary conditions is presented and some examples of realistic situations are shown. The numerical treatment of transient phreatic porous flow problems is discussed including stability and accuracy.

For the reader's convenience each chapter is preceded by a brief review of the most significant aspects, wich are then extensively outlined in the successive sections of the chapter.

In conclusion, chapter 6 elucidates once again the main lines of discussion in this thesis emphasizing the principles results.

2. CONSTITUTIVE NONLINEARITY

Summary of chapter 2

The constitution of substances involved in some thermodynamic process can approximately be described by a relation of the state variables, such as pressure and volume. Such relations are called constitutive relations. The coefficients appearing in these relations represent fundamental properties of the substances with respect to the considered process.
In this section some aspects of fundamental properties concerning groundwater flow will be discussed.

Section 2.a deals with the transport process of water through soil, which is usually governed by Darcy's law. This law expresses, that the discharge of flow is proportional to the pressure gradient. The corresponding coefficient of proportionality represents the fundamental property of this process and it comprises the specific properties of the soil and the pore fluid.
The physical background and the applicability of this law are reviewed and the fundamental property, the permeability, is extended to include nonlinear regimes of porous flow.

Section 2.b deals with the dependence of the permeability to the intrinsic pore geometry, as it changes in deformeable soil. It appears that the permeability is approximately proportional to the void ratio and consequently, a power function of the porosity itself.
Hence, the actual value of the permeability will vary strongly with changes in the porosity. This effect is essential in the nonlinear behaviour of pore water flow in deformeable soils.

Section 2.c deals with the compressibility of the pore water in cases where air is entrapped in the form of air bubbles and air pockets. For such air-water mixtures a convenient expression for the resultant compressibility is deduced, incorporating air solubility, surface tension, water vapour and the proximity of solid surfaces.
The discussion explicates the presence of air less than 15% and reveals a peculiar discontinuity in the resultant compressibility arising at a specific pressure, when free air becomes dissolved. The magnitude of this jump is significant and it can be described as a simple function of the initial state conditions.
Another advantage of the formula obtained is that the saturation degree (air in water), which is usually difficult to measure under varying pressures, does not appear. Furthermore, in the case of free air bubbles the resultant compressibility of the air-water pore content is much larger, several hundred times greater than the compressibility of pure water. It is important to include this effect in a correct description of the mechanical behaviour of semi-saturated soils.

Section 2.d deals with the constitutive relation for the soil skeleton deformation behaviour. The resistance to groundwater flow increases when soil is compacted, but storage variations causing pore pressure gradients induce porous flow and retard the soil deformation rate (consolidation). The mechanical behaviour of soil is significant to groundwater motion. Classical soil mechanics involves two materials, solid particles and water. Here, two fundamental assumptions allow for a mathematical description. The equilibrium in soil is based on Terzaghi's effective stress principle and the solid skeleton deformation is governed by Hooke's law, which represents a simple linear constitutive relationship between stresses and strains of the skeleton.
However, experiments reveal that the deformation behaviour of soil is complicated, but the obscurity of real soil parameters and the corresponding complex mathematical formulation necessitate to restrict to a simplified constitutive model. Moreover, the main attention is called to groundwater flow. Therefore, the stress-strain relation for soil is assumed to be perfectly elastic, i.e. that soil has linear, reversible, isotropic, non-retarded mechanical properties, i.e. Hooke's law. It stands to reason that due to this assumption the theory to be developed will provide an approximate description as far as the real soil deformation behaviour is concerned.

The reader who is acquainted with the subjects of chapter 2, is invited to proceed with chapter 3, which discusses nonlinearity in the continuity equation.

2.a A GENERAL POROUS FLOW LAW

The transport process of groundwater through a porous medium involves two substances: water and soil, and therefore, it will be characterised by specific properties of these two substances. The process can be described in terms of an equilibrium of forces, see De Josselin de Jong (1969). The driving force necessary to press a specific volume of pore water at a certain speed through a porous medium is in equilibrium with the resistance force generated by internal friction between the pore water and the pore structure.
In soils the driving action can be produced by a force H resulting from a pore pressure gradient ∇p and the gravity acting per unit volume ϱg. In formula the driving force per unit volume becomes:

$$H = \nabla p + \varrho g \, \nabla z \, ,$$

where z is measured vertically upwards, ϱ is the fluid density and g the gravity acceleration. The coordinate z is only significant in order to fix the direction of the gravity action and it does not matter, whether the porous medium moves or not. Moreover, it is irrelevant in the above expression if the density ϱ is a constant or a variable in space, since the unit volume considered can be conceived as a physical point.

It is sometimes possible to introduce a potential ϕ to express the volumetric driving force H, according to:

$$H = \varrho g \, \nabla \phi = \nabla p + \varrho g \, \nabla z \, . \tag{2.1}$$

In this form ϕ can be indentified with a vertical head. It is generally referred to as the piezometric head. Formula (2.1) is valid for irrotational fields, see the discussion in section 3.e.
The resistance force R generated by internal friction is characterized by Darcy's law. This law provides a relation between the filter velocity q and the corresponding volumetric resistance force R, according to:

$$R = - \varrho g W q \, . \tag{2.2}$$

W is called the coefficient of resistance to flow.
This coefficient is related to the substances being considered and it contains information regarding the properties concerning porous flow. Equation (2.2) reflects the internal constitution of these substances in the form of a volumetric flux and a generated volumetric force. In this respect it represents a constitutive relation for pore fluid flow. Such a relation can only be stated by observation and Darcy (1856) was the first to verify this behaviour by experiments. Beside the term 'filter velocity' or the 'Darcy velocity' definitions like 'specific discharge' or 'volumetric flux' are also frequently used for the same quantity q. Here, the term filter velocity is chosen, since it reflects best the vectorial character essential in this section.

The equilibrium condition is satisfied by:

$$H = R$$

Introduction of equation (2.1) and (2.2) yields:

$$\nabla \phi = - W q \, , \tag{2.3}$$

which represents a linear relation between the potential gradient and the filter velocity. Expression (2.3) is usually identified with Darcy's law and in this form it comprises two fundamental concepts: equilibrium between driving and resistance forces and a constitutive relation for porous flow.

Linear porous flow

In this section the validity of the proposed constitutive relation (2.2) is considerd. Using the principles of dimensional analysis the character of this relation can be investigated.

For linear porous flow, in which the pore fluid moves under the influence of gravity and hydrodynamical pressure, two physically relevant numbers are involved:
— the bed-Reynold number, incorporating the internal friction of flow, defined by:

$$Re = qD/\upsilon \,.$$

— the Galileo number, incorporating the gravity versus friction, defined by:

$$Ga = \upsilon^2/D^3g \,.$$

Here, υ denotes the fluid viscosity and D is some relevant length, related to a relevant pore size diameter. Contrary to expressions in hydraulics the filter velocity q is introduced for porous hydraulics instead of the real velocity. The related number is correspondingly termed a 'bed'-number, after Rumer (1966).
If it is assumed that porous fluid flow is determined by the fluid viscosity υ and the gravity g the following proportionality, consistent with dimensional requirements, can be formulated:

$$R \; (:) \; R(Ga, Re) \,. \qquad (2.4)$$

Assuming an exponential expression for the relation between the volumetric resistance force R and the filter velocity q, according to:

$$R = -aq^f, \quad f \geq 0 \,, \qquad (2.5)$$

gives with (2.4) and the definition of the Galileo, and Reynold number:

$$a \; (:) \; \upsilon^{2-f} D^{f-3} \varrho \,.$$

Considering the exponent of the viscosity υ in this formula, it becomes clear that, since for real fluids the one with a larger viscosity will generate a larger resistance at a unit filter velocity, values for f exceeding 2 will be physically unreasonable. This important conclusion, merely a consequence of simple dimensional analysis, was first suggested by Muskat (1937).

For f equal to 1, expressing linear flow, the influence of viscosity is significant and the proportionality with D^{-2} shows similarity with experimental results. In this case equation (2.5) represents Darcy's law, expression (2.2), while the quantity a is identical to the linear coefficient of resistance to flow, W. Thus, considering equation (2.3), linear porous flow is governed by:

$$\nabla\phi = -Wq \,; \quad W \, (:) \, \upsilon/D^2g \,. \qquad (2.6)$$

By introducing only effects of viscosity other eventually significant features in the flow phenomenon are excluded. From the field of hydrodynamics much is known about these effects in free flow. Dracos (1978) analyses alike effects in porous flow, such as inertia verticity, development of shear layers, propagation of surface waves and wetting fronts. He concluded that these perturbations practically have a negligible influence, since viscous forces dominate the character of flow. This does not imply, that all porous flow can be described by a linear constitutive relation.

Ante-linear porous flow; molecular linkage

For f tending to zero in equation (2.5), there exists a stationary resistance, while the filter velocity q is zero. This limit can be homologized to the so-called initial gradient, or the threshold gradient, below which there is no flow.
It occurs in clayey soils, and it is attributed to rheological non-Newtonian behaviour in saturated soils, in particular as a result of molecular linkage. It affects the porous flow and can be accounted for by introducing contact forces at the solid fluid interfaces, expressed in terms of contact tension σ, having dimensions kg/s^2, similar to the surface tension at the water-air interfaces.
For this case of very slow movements in media with very small pores a relevant dimensionless number can be introduced, i.e. the Weber number, defined by:

$$We = \varrho g\, D^2/\sigma\ ,$$

to determine the character of the constitutional relation for porous flow incorporating molecular linkage.
It can easily be verified that the non-linear expression:

$$R\ (:)\ R(Ga\ ,\ We\ ,\ Re) = Ga^f\, Re^f\, We^{f-1}\ ,$$

can be transformed into the following form, using equation **(2.6)** for W :

$$R = -\ \varrho g\, W\, (\frac{\varrho \upsilon}{\sigma})^{f-1}\, q^f\ .$$

A similar result has been mentioned by Slepicka (1961). It clearly reveals the threshold value at f tending to zero:

$$R_o = -\ \varrho g\, W\, (\frac{\varrho \upsilon}{\sigma})^{-1}\ (:)\ -\ \sigma/D^2\ . \tag{2.7}$$

The factor $(\varrho\upsilon/\sigma)$, referred to as the coefficient of molecular linkage, has the dimension of s/m and it can be conceived as a velocity damping factor, adjusting the linear flow of affinite phases in very fine pores.

Post-linear porous flow; turbulency

In the larger pores, inertial forces, such as convective and centrifugal acceleration forces, can become predominant, and the viscosity is no longer the major phenomenon characterizing the flow conditions.

These inertial forces can be represented by the bed-Froude number, defined by:

$$Fr = q^2/gD\ ,$$

resulting in a proportionality, according to:

$$R\ (:)\ R(Ga,We,Re,Fr)\ .$$

Consistent with this proportionality is the extended Forchheimer law:

$$-\nabla\phi = -\ I_o + Wq + Cq^2\ , \tag{2.8}$$

where I_o represents the initial gradient, related to the threshold value R_o in equation (2.7), according to:

$$I_o = R_o/\varrho g = -\ W(\frac{\varrho \upsilon}{\sigma})^{-1}\ .$$

Many researchers have tried to find proper expressions for the coefficients W and C in terms of fundamental material properties, see Bear (1972). Some authors even added another term: q^3, but in accordance to the previous remark that the exponent f in equation (2.5) will not exceed 2, this addition does not seem to make sense physically.
The quadratic term dominates when D or q are large. This can be understood from the resistance of a single body in a Newtonian fluid (viscosity is a constant). At low velocities the relationship becomes, according to Stokes' law:

$$\nabla\phi = -\ Wq\ \ ,\quad W = A\upsilon/gD^2\ , \tag{2.9}$$

where A denotes a dimensionless factor related to the configuration of flow around the body, including the shape of the body itself (and eventually, influences of neighbouring particles). Here, D^2/A, is identical to the intrinsic permeability (see section 2.b). The expression given for W is fully in agreement with the proportionality previously stated in relation (2.6).

The validity of Stokes' law in expressing the resistance of a sphere in a uniform flow is restricted to a certain range of the Reynold number Re. Nonlinearity is observed at $Re>5$.
Also measurement of flow in granular beds shows experimental evidence of linear porous flow until a certain value of Re, beyond which the inertia involved in laminar fluid movements in the pores can not be disregarded. Lindquist (1933) arrived at this conclusion. Karadi (1955) expounded that this type of inertia (in fact the inertia terms in the Navier-Stokes' equations describing the flow in a single pore) can only explain the deviation from the linear porous flow within a narrow range of Reynold numbers, $5 < Re < 200$. Beyond this range turbulence assumes increased significance, confirmed by numerical studies in an idealised porous medium, reported by Stark (1958).

To understand the influence of turbulent pore water motion in a porous medium, consideration of the drag force of a single sphere in a uniform flow serves to describe the constitutive behaviour of the porous flow at larger Reynolds numbers.

Following Ward (1964) and Rumer (1969), the reaction of a porous medium in a turbulent flow can be formulated in terms of a volumetric resistance force R, according to:

$$R = -\varrho g A' C_D Fr, \qquad (2.10)$$

where the factor A' is related to the configuration of turbulent flow around the particles in the porous skeleton, and Fr is the Froude number, previously defined. C_D is a dimensionless coefficient, termed the 'drag coefficient', and for incompressible flow measurements show that it is a function of the Reynold number (see Fig. I). For $Re > 600$, C_D becomes a constant, about 0.5. It is to be noted that the Froude number is proportional to q^2, so that (2.10) with C_D a constant corresponds to the conclusion drawn before, that f in equation (2.5) does not exceed 2.
A striking example of the quadratic flow law is measured by Hajdin, who investigated porous flow in karstic media, reported by Boreli (1978).

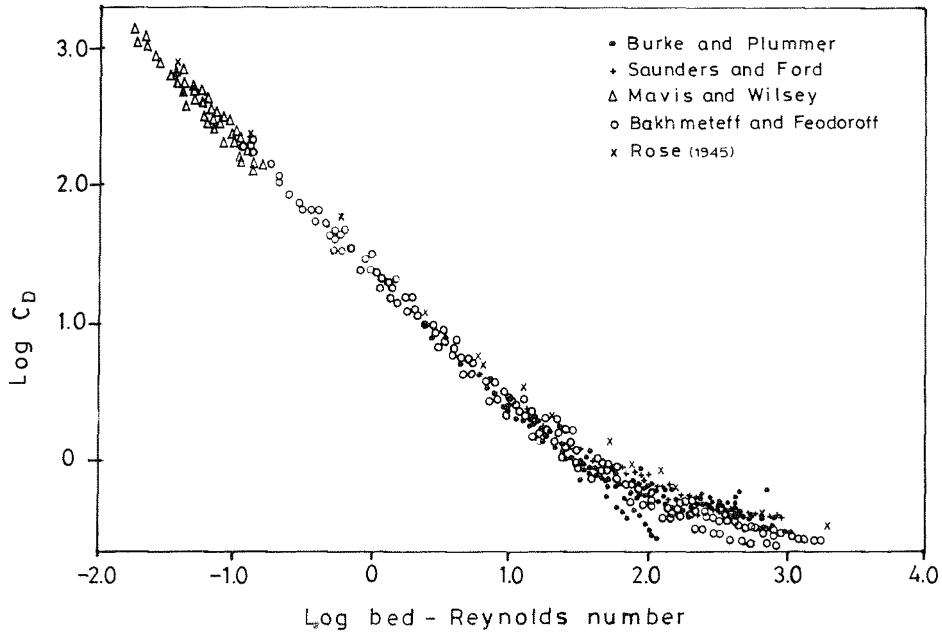


Fig. I The drag coefficient versus the bed-Reynold number.

A general porous flow law

Since C_D is a function of the Reynold number Re, it is possible to include linear porous flow in the expression (2.10). In the linear regime the drag coefficient C_D is inversely proportional to the

Reynold number, confirmed by many experiments, and consequently, equation (2.10) becomes:

$$R = -\varrho g A' Fr \left(\frac{A'}{A} Re\right)^{-1} = -\varrho g (A\upsilon / gD^2) q = -\varrho g W q \,,$$

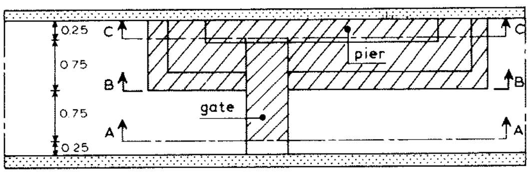
which is identical to the constitutive relation for linear porous flow, equation (2.2).
For the transitional regime the drag coefficient C_D is a non-linear function of the Reynold number, this being itself dependent on the filter velocity q. Thus, equation (2.10) covers turbulent, linear and transient flow, and subsequently, nonlinearity can be incorporated in the following linearised constitutive porous flow relation:

$$R - R_o = -\varrho q W' q \quad ; \quad W' = \sqrt{A' C_D \mid R \mid / \varrho g^2 D} \,.$$

Introducing the driving force, equation (2.1) and considering a state of equilibrium, this expression becomes:

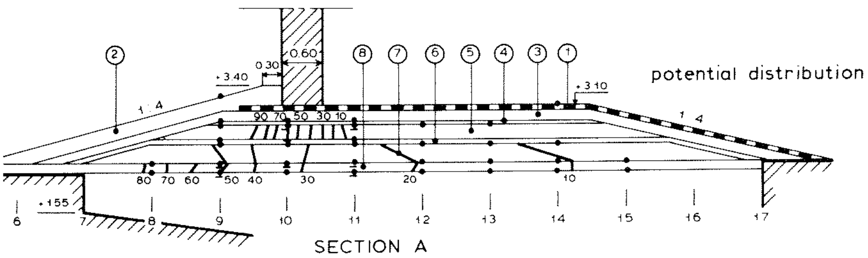
$$\nabla\phi - I_o = -W' q \quad ; \quad W' = \sqrt{A' C_D \mid \Delta\phi - I_o \mid / gD} \,. \qquad (2.11)$$

Various semi-empirical relations have been proposed for the coefficient A', respectively A. For example, De Lara (1955) suggests for coarse beds: $A' = 0.5\, n^{-5}$, where n denotes the porosity. Barends (1978) showed, that it is preferred to define the nonlinearity embedded in W' in terms of the absolute gradient $\mid \nabla\phi \mid$ rather than the filter velocity q, since the gradient is less sensitive to successive adjustments in the flow pattern during iterative computations.
Formula (2.11) has been verified in a three-dimensional, 1:10-scale model test and agreement between the pressure distribution measured and calculated was satisfactory, see Fig. II.



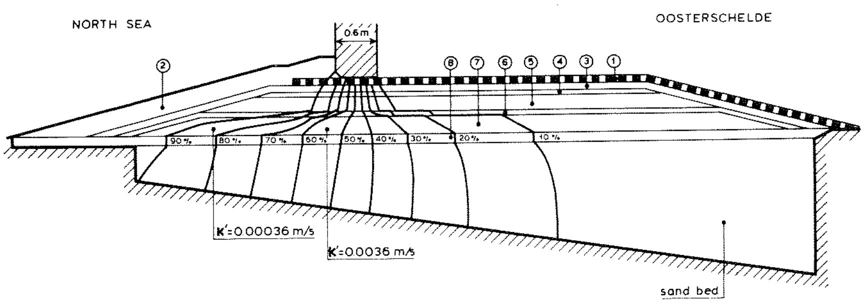
PLAN OF THE FLUME

	layer	thickness
1	concrete grid	10 cm
2	rubble 1000-3000 gr	20 cm
3	rubble 60-300 gr	12 cm
4	gravel 5-15 mm	5 cm
5	fosfor slakes	23 cm
6	sandwich	7 cm
7	fine gravel	25cm
8	fosfor slakes	15cm



SECTION A

INHOMOGENEOUS LINEAR-TURBULENT FLOW MODELTEST 1:10



LINEARIZED POROUS FLOW NUMERICAL MODEL (SEEP)

Fig. II Comparison between theory and tests.

The previous discussion about different regimes in the porous flow behaviour has been based on the assumption of a constitutional relation in the form of a power law, see expression (2.5). This approach is not generally applied, and some research workers follow the extended Forchheimer law, mentioned in (2.8).
The Forchheimer flow law has been derived theoretically using a microscopic model, in which flow is described by the Navier-Stokes' equations, by Irmay (1958), Sunada (1965) and Stark and Volker (1967), and as such it has been used to fit experimental data, frequently reported in the literature.
Cox (1977) chooses the Forchheimer relation to incorporate nonlinearity in his studies of groundwater flow. He assumes an isotropical flow resistance and suggests a 'linearized effective resistance', according to:

$$W' = W/2 + \sqrt{(W/2)^2 + C\,|\nabla\phi|}\ ,$$

in order to simulate linearized porous flow. This expression shows similarity with the coefficient of resistance to flow given in equation (2.11). The nonlinear phenomena can be incorporated in different values of the coefficient C.

Conclusion

Both formulations, the extended Forchheimer law (2.8) and formula (2.10), are equally well capable of describing porous flow over a large range of velocities. The preference for equation (2.10) is based upon the circumstance that the drag coefficient is a physical coefficient commonly applied in fluid mechanics. Measurements on this coefficient are available, and also the relation of the coefficient of porous flow configuration, i.e. A for laminair and A' for turbulent flow, with granular bed properties have been measured. For example, Benis (1968) succesfully applied a power law relation in his theory for porous flow of non-Newtonian fluids to verify various test results. Moreover, Polubarinova (1962) tried in a mathematical analysis to extend nonlinear flow behaviour in more dimensions and suggested a power relation.

Although since Darcy for more than a century the attention of many researchers has been directed to the phenomenon of pore fluid flow and much of the fundamental characteristics have been elucidated, the effect of molecular linkage in fine graded soils and the eventuation of non-coaxiality of pressure gradients and fluxes in cases of turbulent porous flow (induced anistropy) are not yet completely understood.
The previous discussion, in particular equation (2.11), covers only one-dimensional or isotropical nonlinear flow. Even then, the solution of boundary value problems with a general nonlinear porous flow law analytically is almost impossible. Numerical methods may succeed sometimes, as has been shown by Cox (1977) and Barends (1978).

In general, one might state that simplified approaches must be adopted with the obvious conse quence that the results obtained can be regarded as acceptable under the restrictions specified. In this respect the next sections will deal with an approximate, generalised porous flow law, according to expression (2.11), in which the actual nonlinearity is considered as a moderate deviation from linear flow.

2.b PERMEABILITY AND POROSITY

Since the establishment of Darcy's law basic properties of natural materials concerning porous flow received much attention. For an extensive review of these properties for rock as well as for particulate media the reader is referred to Davis (1969).

Two relevant parameters, porosity and permeability, have been the main subject of study and investigation, but despite intensive work expended in improving and adjusting measurements of these properties, reliable values representing the real character of porous flow could not be estimated. This fact is for the most part due to the complexibility of porous flow at microscopic level and to the natural inhomogeneity of soil.

Comprehension of the constitution of the permeability in terms of representative or measurable physical properties leads to the postulation that deviation in measured values is mainly

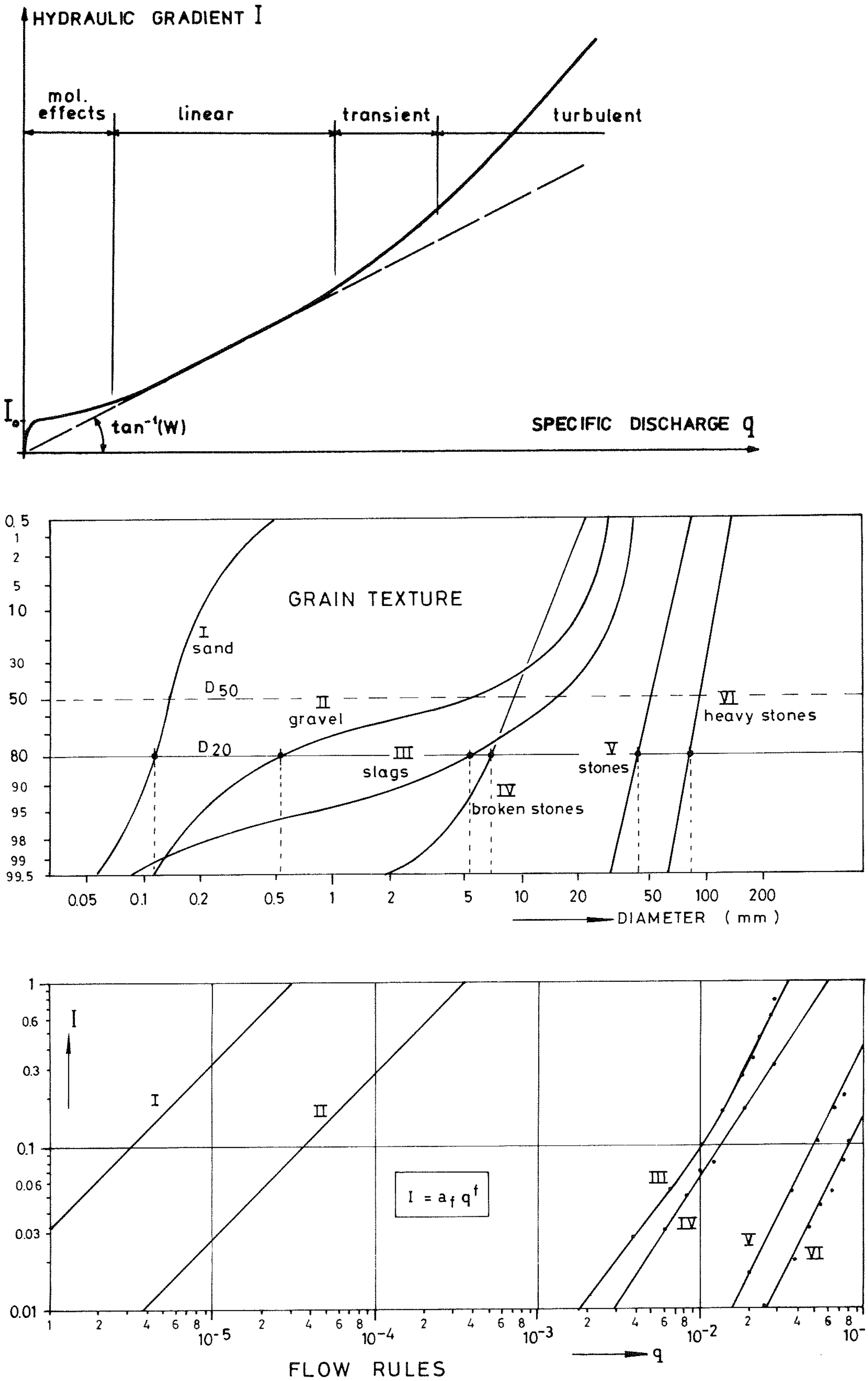


Fig. III Measured flow laws.

attributed to one particular factor: the tortuosity of the porous medium. This quantity has been introduced by Carman (1937) and is extensively discussed by Whitaker (1966) and Bear (1972). Bear states that the permeability of a porous medium depends on five basic properties.
Those of the porous matrix: porosity n, conductance D^2, tortuosity T, and those of the pore fluid: density ϱ, dynamic viscosity μ.
From section 2.a it became clear that the permeability depends on the ratio μ/ϱ, which is called the kinematic viscosity υ. It is necessary to mention this dependency, because the statement that no density effects are involved in linear porous flow, does not coincide with the actual appearance of the fluid density in a parametric expression for the hydraulic permeability.

From dimensional analysis (see section 2.a) an expression for the permeability in case of linear flow can be obtained and formula (2.9) yields for the hydraulic permeability K:

$$K = kg/\upsilon \quad ; \quad (k = D^2/A), \tag{2.12}$$

where k is referred to as the intrinsic permeability, depending on porous matrix properties only. The medium is called homogeneous, if k does not vary in space. If k varies in different spatial points, the medium is inhomogeneous, and if k varies in one point in different directions, the medium is called anisotropic, otherwise isotropic.
Anisotropy can be represented by a second-rank symmetrical tensor, already indicated by Versluijs (1915), see Vreedenburgh (1936). Ferrandon (1948) obtained a similar result from a statistical approach and De Josselin de Jong (1969) introduced an instructive method of scattering mechanics, which proves the tensor character for the case of general anisotropy. Bear and Dagan (1965) discussed the practical method of geometric scaling for anisotropic soils.

Other factors affecting the value of the intrinsic permeability are porous matrix deformation (consolidation, swelling), solution of solid parts, chemical and biological activities and saturation, see Irmay (1954) and Karadi and Nagy (1969).

For linear flow the intrinsic permeability can be represented by the relation:

$$k = CTD^2\eta, \tag{2.13}$$

where the coefficient C varies in the range of 0.005 to 0.015, see Bear (1972), and the tortuosity T, a dimensionless factor, incorporates the configuration of the intricate pore structure. In the case of anisotropy the tortuosity becomes a tensor, as has previously been mentioned. Due to non-uniformity of the complex network of interconnected channels through which flow takes place, the tortuosity varies considerably, up to 200% and more. In fact, the main cause of deviation in measured values of the permeability is embedded in this tortuosity.
The conductance D^2, appearing in equation (2.13), is related to the pore cross-section and D represents a characteristic pore width. D might be identified with a proper particle size of a granular porous medium, since the pore geometry is somehow correlated to the texture. Barends (1978) suggested D_{20} for coarse granular soil, a figure resulting from several measurements, see Fig. III.

The function η in relation (2.13) represents the general dependence of the permeability and the porosity. η is called the porosity factor. Observing several empirical expressions given in literature the following relations for the porosity factor can be deduced:

Kozeny	(1927)	$\eta = ne^2$	;
Stark	(1958)	$\eta = (ne)^{1.5}$	;
Leonards	(1962)	$\eta = n$	;
Rumer	(1969)	$\eta = ne$	,

were e represents the void ratio, defined as the volume of pores per volume of solids:

$$e = n/(1-n). \tag{2.14}$$

The above mentioned relations show sufficient similarity to conclude that the character of the porosity factor is reasonably well specified. Moreover, if the porosity itself is uniform in the porous medium being considered, the relation for the porosity factor can be useful in expressing variation of the permeability due to porous matrix deformation (consolidation). This existence of this variation is not new, see Abbot (1960), De Leeuw and Abbot (1966), Gibson et al (1967), but for soil densification the importance of this effect on groundwater flow has been outlined recently by Barends and Thabet (1978).

With the assumption that the configuration of flow and the network of interconnected pores do not essentially change during porous matrix deformation, the relative variation of the porosity factor, expressed by:

$$\eta = n^a e^b ,$$

becomes:

$$d\eta/\eta = \varkappa\, de/e = (\varkappa/n)\, dn/(1-n); \quad \varkappa = a(1-n) + b, \tag{2.15}$$

where $\varkappa$ is called the amplification factor. This relation is different from the one suggested by Lambe and Whitman (1969), who expressed the permeability k as an exponential function of the void ratio e.

Variation in the conductance D^2 can easily be accounted for, since the pore cross-sectional area will alter more or less proportionally to n^2. From the previously mentioned empirical relations for the porosity factor, relevant values for the amplification factor $\varkappa$ can be found to illustrate the effect of small alterations in the actual porosity on the permeability. For example a 20% settlement of a peat layer at an original porosity of 50% causes a formidable change in the permeability: $\eta/\eta_o = 0.20$, reducing the original permeability by 5.

Variation of the permeability due to small changes in the porosity caused by porous matrix deformation, expressed by formula (2.15), will be introduced in the complete storage equation, discussed in section 3.d.

2.c COMPRESSIBILITY OF AIR-WATER MIXTURES

Air and water comprise a large part of the soil and therefore they play a major role in soil physics. The link between the behaviour of the components and the distribution of partial stresses is determined by the individual compressibilities. It forms the basis of the effective stress principle, widely applied in soil mechanics. However, this principle is generally accepted for saturated soil without air.

Several authors, Skempton (1954), Koning (1963), Sparks (1963) and Verruijt (1969) have attempted to incorporate the presence of free air in the pore water, because it significantly affects the compressibility. Nonetheless, basic physical factors, such as surface tension, air solubility and water vapour, involved in the mechanical behaviour of an air-water mixture, are not always accounted for, and suggested formulas generally have a limited validity.

Disregarding only the water vapour, Schuurman (1964) could expound the typical behaviour of pore water pressure in partly saturated undrained clay samples under various stress conditions. His observations and conclusions include the following. As long as free air is present the resultant compressibility of the pore content depends on the pore pressure. As a consequence of air solubility and surface tension, he found that a distinct pressure exists beyond which the free air becomes dissolved quite suddenly.
In free water bubbles are unstable and shrink or grow by diffusion depending upon whether the water is under- or overconcentrated. In a porous medium the bubble growth is restricted because of non-wettability and pore size dimensions. At an increasing pressure the tendency of bubble growth will revert to shrinkage. Bubbles collapse causing an unstable increase of compressibility. Since the time necessary to dissolve small air bubbles is in the order of a second, almost instantaneously the pore volume is completely saturated, and because dissolved air in the water does not contribute, the resultant compressibility will be solely due to the water. This discontinuity is essential for a proper description of the mechanical behaviour.
In the present treatment consideration of mass balance in a fixed control volume will lead to a final expression of the resultant compressibility, which confirms Schuurman's conclusions and includes the discontinuous behaviour of the compressibility of an air-water mixture, and it takes into account to some extent the influence of solid surfaces.

The compressibility

The simplest thermodynamic system consists of a fixed mass of an isotropic fluid not influenced by external or chemical processes. Such systems are usually described by three measurable variables: pressure p, volume U and temperature T, see Abbot and van Nes (1976). Experiments show that these three are not all independent, that fixing any two determines the third. Thus, there must be a constitutive equation that interrelates these three variables for equilibrium states. When such a specific relation is known to describe the considered system, one of these variables can be solved in terms of the others. For example: $V = V(p,T)$. The mathematical description of changes which occur in the physical system leads to the exact differential of the form:

$$dU = \left(\frac{\partial U}{\partial T}\right)_p dT + \left(\frac{\partial U}{\partial p}\right)_T dp\,.$$

If T is considered to be a constant, this differential becomes:

$$dU = \left(\frac{\partial U}{\partial p}\right)_T dp\,. \qquad (2.16)$$

The partial differential quotient in this equation is directly related to a property commonly tabulated for pure substances: the isothermal compressibility β, where:

$$\beta \equiv -\frac{1}{U}\left(\frac{\partial U}{\partial p}\right)_T,$$

defined as the relative change of volume of a fixed mass subject to variations in the pressure p. Regarding equation (2.16), β can be expressed by the total derivative dU/dp at constant temperature, according to:

$$\beta = -\frac{1}{U}\frac{dU}{dp}\,. \qquad (2.17)$$

Conservation of matter of the fluid gives:

$$\varrho U = \text{constant}, \qquad (2.18)$$

in which ϱ denotes the fluid density. Since for isothermal conditions volume U and the related density ϱ are only functions of pressure p, variation of the conservation equation (2.18) yields:

$$dU/U = -d\varrho/\varrho\,.$$

Therefore, the isothermal fluid compressibility, equation (2.17), can be expressed in terms of the fluid density and by substitution it is found that:

$$\beta = \frac{1}{\varrho}\frac{d\varrho}{dp}\,. \qquad (2.19)$$

This expression states that the coefficient of compressibility can be defined as the relative change of mass per unit volume subject to variations in normal pressure at constant temperature.

It is the purpose to find an identical expression for the isothermal compressibility of an air-water mixture in terms of the density ϱ' of the mixture and the water pressure p, that is:

$$\beta' = \frac{1}{\varrho'}\frac{d\varrho'}{dp}\,. \qquad (2.20)$$

In fact two different substances are involved, water and air, partly mixed, since air dissolves in water and water evaporates in free air. With respect to the preceding discussion about one substance, it stands to reason that for a given initial air-water content the isothermal behaviour of the mixture can be described in terms of two independent variables: the water pressure and the air pressure. Equilibrium conditions provide a relation between the water and air pressure. Thus, the various states of equilibrium of the mixture can be expressed in terms of one independent variable, for example the fluid pressure. Consequently, the variations of dependent variables in the following treatment are to be conceived as exact differentials.

Simplifications and basic laws

For a real gas the constitutive equation is:

$$\varrho'' = p''/ZRT, \tag{2.21}$$

where ϱ'' denotes the gas density, p'' the gas pressure, Z the compressibility factor, R the gas constant and T the absolute temperature. In general Z depends on p'' and T, but Z is a constant for an ideal gas.
In the case: $Z = 1/M$, in which M represents the molecular weight, equation (2.21) becomes the Boyle-Mariotte law, which will be applied for the behaviour of the air in the bubbles. The absolute temperature T is a constant. Therefore, the entire term ZRT will be a constant in the proceeding analysis.

Henry's solubility law states that the mass of a slightly soluble gas, that dissolves in a definite mass of a liquid at a given temperature is very nearly proportional to the partial pressure of that gas. This holds for gases which do not unite chemically with the solvent. On the basis of this law, Hilf (1956) assumes, that all air bubbles in the air-water mixture have the same radius. The tendency to form larger bubbles is prevented in soil by small pore dimensions and the surface activity of solid particles. According to Sparks (1963) the free air will form bubbles in the pore water beyond a degree of saturation of about 85%. The bubbles are either of the same size as the pores or smaller. To simplify, the free air is supposed to be present in the form of isometric separate bubbles.

As a result of the Boyle-Mariotte law and Henry's law the weight of dissolved air is proportional to the density of the free air. This proportionality factor is referred to as the coefficient of solubility ω. It decreases slightly with increasing air pressure, but here, it will be taken as a constant. Since the compressibility of pure water is very small ($\beta \sim 5.10^{-10}$ m²/N), the weight of dissolved air is, following Henry's law, also proportional to the considered volume of water.

The mass of free and dissolved air in a fixed control volume is supposed to be a constant. As the pressure changes the saturation degree varies. Therefore, this assumption implies that only free air originally present within the control volume dissolves in the extra volume of water flowing into the control volume at a pressure increase. This approximation is justified, because the coefficient of solubility of air in water is small ($\omega \sim 0.02$).

A general accepted simplified formula for the bubble growth due to air diffusion is given by Cable (1967):

$$(r/r_i)^2 = 1 - 2\omega Dt/r_i^2 ,$$

where r_i is the initial radius, ω the air solubility in water, D the diffusivity and t the elapsed time. The usefulness of this relation has been stated by many experiments, see Krieger et al (1967). A characteristic time value can be deduced: $t_D = r_i^2/2\omega D$.
For air in cold water the diffusivity and the solubility are known: $D = 2.10^{-9}$m²/s and $\omega = 0.02$. For bubbles in the pores of a fine graded soil, $r_i \sim 10^{-5}$ m, the characteristic time becomes: $t_D \sim 1.25$ sec. The state of equilibrium will be assessed in a short period. Hence, in a relatively slowly varying pressure field the effect of air diffusion can be disregarded. The same applies to the diffusion boundary layer, and the uniformity of concentration, see Barends (1979).

The surface tension σ is a constant for isothermal conditions. The insubstantial dependence on the air pressure is disregarded conform Schuurman (1966).

Dalton's law of partial pressures states that the pressure exerted by a mixture of gases is equal to the sum of the separate pressures which each gas would exert, if it alone occupied the whole volume. According to this law, the saturation vapour pressure w is independent of the free air pressure. Hence, in the system considered here this pressure w is assumed to be a constant. In a partially saturated soil the curvature of the air-water surface and the proximity of the solid surfaces will affect the water vapour pressure, but it is rather constant for high water contents, see Raudkivi and Callander (1976).

The vapour pressure has a direct influence on the actual water pressure. For pore water containing isometric bubbles with radius r the state of equilibrium yields a relation between the water pressure p and the free air pressure p″, including the surface tension σ and the water vapour pressure w, according to:

$$p'' = p - w + 2\sigma / r . \tag{2.22}$$

Since w is a constant, variation of this equation yields:

$$dp'' = dp - (2\sigma / r)\, dr / r . \tag{2.23}$$

Recapitulation

The air-water mixture comprises water and isometric air bubbles with radius r. The saturation degree is 85% or more.
The Boyle-Mariotte law, expression (2.21) with ZRT is a constant, interrelates the air density ϱ'' and the air pressure p′. The surface tension σ, the water vapour pressure w and the air solubility ω are taken as constants. Diffusion effects are disregarded.

With the aid of previously mentioned laws (see Weast, 1976) and simplifications it is possible to find an explicit and convenient expression for the resultant compressibility of an air-water mixture in a porous medium.

The resultant compressibility

Consider a fixed volume containing an air-water mixture with a saturation degree s. In a porous medium s is defined as the volume of water per volume of pores. The remainder is occupied by N isometric bubbles with radius r containing free air. Only free air bubbles are concerned. In a porous medium surface bonded air pockets also exist. Although they can be included in the proposed analysis, they will not be considered. Their influence is discussed by Barends (1979).

Thus:

$$(1 - s) = \frac{4}{3} \pi N r^3 , \tag{2.24}$$

from which follows for the differential:

$$- ds = \frac{4}{3} \pi N\, 3 r^2 dr = (1 - s) 3 dr / r . \tag{2.25}$$

The amount of pure water in the fixed control volume equals $s\varrho$, the amount of free air equals $(1 - s)\varrho''$, and in accordance with Henry's solubility law the amount of dissolved air is equal to $s\omega\varrho''$. Hence, consider an average density of the air-water mixture, defined by:

$$\varrho' = s\varrho + (1 - s)\varrho'' + s\omega\varrho'' . \tag{2.26}$$

For stagnant bubbles, conservation of the mass of air requires that the amount of air in the bubbles and the amount of dissolved air is constant. Thus:

$$d \{ (1 - s + \omega s)\varrho'' \} = 0 . \tag{2.27}$$

Employing the Boyle-Mariotte law, equation (2.21) with ZRT constant, this becomes:

$$ds = (\frac{1}{1-\omega} - s)d\varrho''/\varrho'' = (\frac{1}{1-\omega} - s)\, dp''/p''. \tag{2.28}$$

Because of (2.27) the variation of ϱ' of the average density defined by (2.26), becomes with (2.28):

$$d\varrho' = \varrho ds + s d\varrho = \varrho\,(\frac{1}{1-\omega} - s) dp''/p'' + s d\varrho\,.$$

Substitution (2.19) and dividing by $\varrho' dp$ gives:

$$\frac{1}{\varrho'}\frac{d\varrho'}{dp} = \{(\frac{1}{1-\omega} - s)/p' + s\beta\}\,\frac{\varrho}{\varrho'}, \tag{2.29}$$

in which the auxiliary pressure p' is expressed by:

$$p' = p''\,dp/dp''. \tag{2.30}$$

The lefthand side of equation (2.29) has the character of a coefficient of compressibility in accordance with the required quantity in definition (2.20). However, the auxiliary pressure p' should be converted into an expression in terms of basic physical factors, which fortunately is possible. Eliminating dp from (2.23) and (2.30) gives for the auxiliary pressure:

$$p' = p'' + \frac{2\sigma}{r}\{\frac{p''}{r}\frac{dr}{dp''}\}. \tag{2.31}$$

Next, the term between braces will be evaluated. Reconsider the conservation of mass of air, expression (2.27), which in view of the Boyle-Mariotte law can be written in the following form:

$$d\,\{\,[\,(1-\omega)\,(1-s) + \omega]\,p''\,\} = 0\,.$$

Since ω is a constant, this becomes:

$$-(1-\omega)\,p''ds \quad + \quad [\,(1-\omega)\,(1-s) + \omega\,]\,dp'' = 0\,.$$

Eliminating ds with (2.25) and dividing by dp'' gives after some rearrangements:

$$\frac{p''}{r}\frac{dr}{dp''} = -\frac{1}{3}\{1 + \frac{\omega}{(1-\omega)\,(1-s)}\}.$$

Thus, the auxiliary pressure p', equation (2.31), becomes when employing equation (2.22):

$$p' = p - w + \frac{2\sigma}{3r}\{2 - \frac{\omega}{(1-\omega)\,(1-s)}\}. \tag{2.32}$$

Observing once again the average density, definition (2.26), and recalling the fact that the fluid density is much larger than the air density, the approximation: $\varrho' = \varrho$ can be justified. Moreover, separate bubbles are concerned, implicating a lower limit of the saturation degree of 85%.

Substitution of equation (2.32) into (2.29), and introducing the approximation: $\varrho' = s\varrho$, yields:

$$\beta' = \frac{1}{\varrho'}\frac{d\varrho'}{dp} = \beta + (\frac{1}{1-\omega} - s)\frac{1}{s}/[p - w + \frac{2\sigma}{3r}\{2 - \frac{\omega}{(1-\omega)\,(1-s)}\}]. \tag{2.33}$$

The coefficient β' represents the resultant compressibility of an air-water mixture and incorporates all the basic physical factors involved.

A discontinuity in the resultant compressibility

The new formula presented in equation (2.33), contains the constants β, w, ω, and σ, and the variable factors s and r, which are moderate functions of the fluid pressure p.
Since bubbles become stiffer at higher air pressures, it is conceivable that, when the fluid pressure rises, the surface tension increases the stiffness of the air-water mixture. In other words the resultant compressibility will decrease. In view of equation (2.33), this holds if the surface tension term is positive or:

$$\frac{2\sigma}{3r}\left\{2 - \frac{\omega}{(1-\omega)(1-s)}\right\} > 0 .$$

However, the coefficient between the braces declines at increasing fluid pressure, since the volume of air bubbles (1 – s) decreases. At a particular pressure p_s this coefficient becomes zero, corresponding to a specific saturation degree, according to:

$$s_s = 1 - \frac{\omega}{2(1-\omega)} . \qquad (2.34)$$

For fluid pressures larger than p_s, the surface tension term in equation (2.33) becomes negative, causing an assymptotic increase of the resultant compressibility. In Fig. IV the mathematical behaviour of β′ is presented as a function of the fluid pressure p.

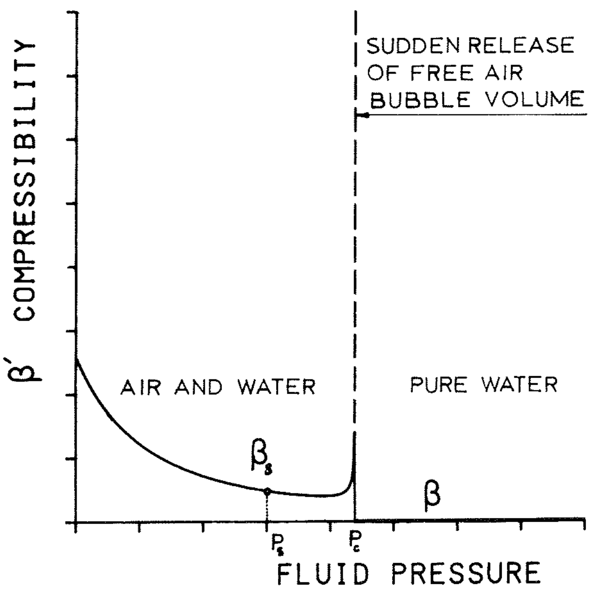


Fig. IV The air-water compressibility versus water pressure.

The physical implication of the fact that surface tension σ does not contribute anymore, is that air bubbles become unstable. The surface tension causing higher air pressures cannot prevent free air from becoming completely dissolved.
Bubbles implode releasing the volume (1 – s), creating a sudden increase of compressibility. Therefore, the resultant compressibility attains a minimum beyond the specific pressure p_s, and since the value of the compressibility is almost stationary at pressures around p_s this minimum is well approximated by the particular value at p_s.
Substitution of (2.34) into (2.33) provides this minimum value:

$$\beta'_s = \beta + \frac{3\omega}{(2-3\omega)(p_s - w)} . \qquad (2.35)$$

When all the free air is dissolved, its influence vanished and for higher pressures β′ becomes equal to β. Because the dissolving process occurs in a short period, a jump in the compressibility emerges equal to:

$$\beta_s' - \beta = \frac{3\omega}{(2-3\omega)(p_s - w)}, \tag{2.36}$$

and simultaneously the saturation degree reveals a discontinuity equal to:

$$1 - s_s = \frac{\omega}{2(1-\omega)}. \tag{2.37}$$

The actual bubble radius is involved, because it depends on the pressure. However, it varies gradually until the collapse, which can be seen from equation (2.25). For example, if the original saturation degree was 95%, the radius will have been reduced by about a factor 2 at $s_s = 99\%$. It is stressed that due to air solubility stable air bubbles cannot become infinitely small.

Verification

Skempton and Bishop (1954) disregarded surface tension, water vapour and fluid compressibility. They found:

$$\beta' = (1 - s_i + \omega s_i)\, p_i / p^2, \tag{2.38}$$

which is covered by expression (2.33) for a certain range of the saturation degree. The same applies to Koning (1963) and Verruijt (1969), who included fluid compressibility, but disregarded solubility, water vapour and partly the surface tension. They suggested:

$$\beta' = s\beta + (1-s)/p. \tag{2.39}$$

Only Koning provided a proper relation for s(p), but he did not include this in the suggested expression.
From expression (2.33) it is obvious that disregarding the air solubility ($\omega = 0$) or the surface tension ($\sigma = 0$) will consequently lead to a formula, which does not include the discontinuous behaviour, because the denominator has no real root in this case.
Therefore, the above mentioned approximate expressions (2.39) and (2.38) have a limited validity. More precisely, they are applicable for large degrees of saturation, but not far beyond the specific saturation degree s_s, given by equation (2.34), or not far beyond the specific fluid pressure p_s, which can be expressed in terms of the initial condition, see Barends (1979), according to:

$$p_s = w - \frac{2\sigma}{r_i}\sqrt[3]{2(1-\omega)(1-s)/\omega} \quad + \quad 2[1-(1-\omega)s_i][p_i - w + 2\frac{\sigma}{r_i}]/3\omega. \tag{2.40}$$

For a particular case graphs of (2.38) and (2.39) are compared to corresponding results based on formula (2.33), represented in Fig. V.

Schuurman was the first to verify the discontinuous behaviour of the resultant compressibility by triaxial tests on partly saturated undrained clay samples. He did not obtain an explicit expression for β', but could confirm the behaviour numerically. The obtained new formula, expression (2.33), is fully in accordance with the observations of Schuurman.
As a striking example one of his tests is elaborated, see Fig. V. The following values are involved:

$\omega = 0.02$
$w = 0.01\ 10^5\ N/m^2$
$\sigma/r_i = 0.14\ 10^5\ N/m^2$
$s_i = 0.95$; $p_i = 10^5\ N/m^2$
$\beta = 5.10^{-10}\ m^2/N$

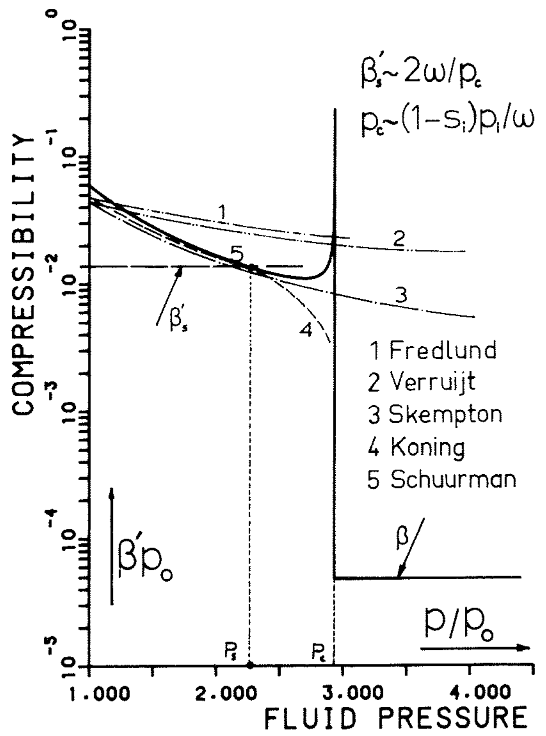


Fig. V Comparison between different formulas for the compressibility.

The jump in the saturation degree is small following equation (2.37), but the discontinuity in the compressibility is large:

$$\beta'_s = 1.40\ 10^{-7}\ m^2/N\ ,$$

which is about 300 times the compressibility of pure water. This jump starts to occur at the specific pressure: $p_s = 2.28\ 10^5\ N/m^2$. The actual collapse takes place at a higher pressure, which as a result of numerical calculation is equal to $2.93\ 10^5\ N/m^2$. The test shows a critical pressure at about $2.9\ 10^5\ N/m^2$.
In general this value depends on the initial conditions, in particular on the initial saturation and the initial bubble size, and on the loading behaviour, since the actual collapse is controlled by the diffusion process.

The abrupt behaviour along the unstable branche in Fig. V will not take place in reality. It is to be expected that this instability will spread out due to non-simultaneous collapse of the bubbles, because of slight differences in size.
According to Schuurman reduction of solubility and surface tension with increasing air pressure as well as the tendency of smaller bubbles to combine to larger ones will possibly have a retarded influence.

The actual behaviour of bubbles during collapse, i.e. cavitation, involves hydrodynamical effects and is beyond the scope of this discussion. The reader is referred to Plesset et al (1977).

Conclusions and remarks

Upper areas of groundwater might contain free air (peat sometimes produces free gases), though the saturation degree is usually large, since air bubbles tend to rise up if not entrapped by small pores. Moreover, free air can be present in the form of gas pockets attached to solid surfaces, which repulse water (hydrophobic). Although these gas pockets show a completely different behaviour with varying fluid pressures as the free (traveling) bubbles do, the presence of both kinds can be included in one formula for the compressibility, see Barends (1979).
The presence of surface bonded gas pockets seems not to essentially affect the behaviour of

the resultant compressibility. The minimum value, given by (2.35), is hardly influenced, but the value of the specific pressure and the critical pressure changes. The former pressure can be expressed in terms of the initial conditions, according to:

$$p_s = w - \frac{2\sigma}{r_1}\sqrt[3]{\frac{2(1-\omega)(1-s_i-b)}{b+\omega(1-b)} + \frac{2[1-(1-\omega)s_i][p_i - w + 2\sigma/r_i]}{3[b+\omega(1-b)]}}, \qquad (2.41)$$

where b denotes the relative volume of bonded air bubbles in the considered control volume. Correspondingly, the specific saturation degree, mentioned in equation (2.34) changes into:

$$s_s = 1 - \frac{3}{2}b - \frac{\omega}{2(1-\omega)}. \qquad (2.42)$$

Since at increasing fluid pressures causing an accelerated air pressure increase the air-water surface tension brings air becoming dissolved, the bubbles implode. This occurs in a certain pressure range, between p_s and p_c, respectively the specific pressure, given by (2.40), and the critical pressure, for which an explicit expression is suggested by Lowe and Johnson (1969): $p_c = 1.5\ p_s$. Several nummerical calculations for various situations show the validity of this formula, see Fig. VI.

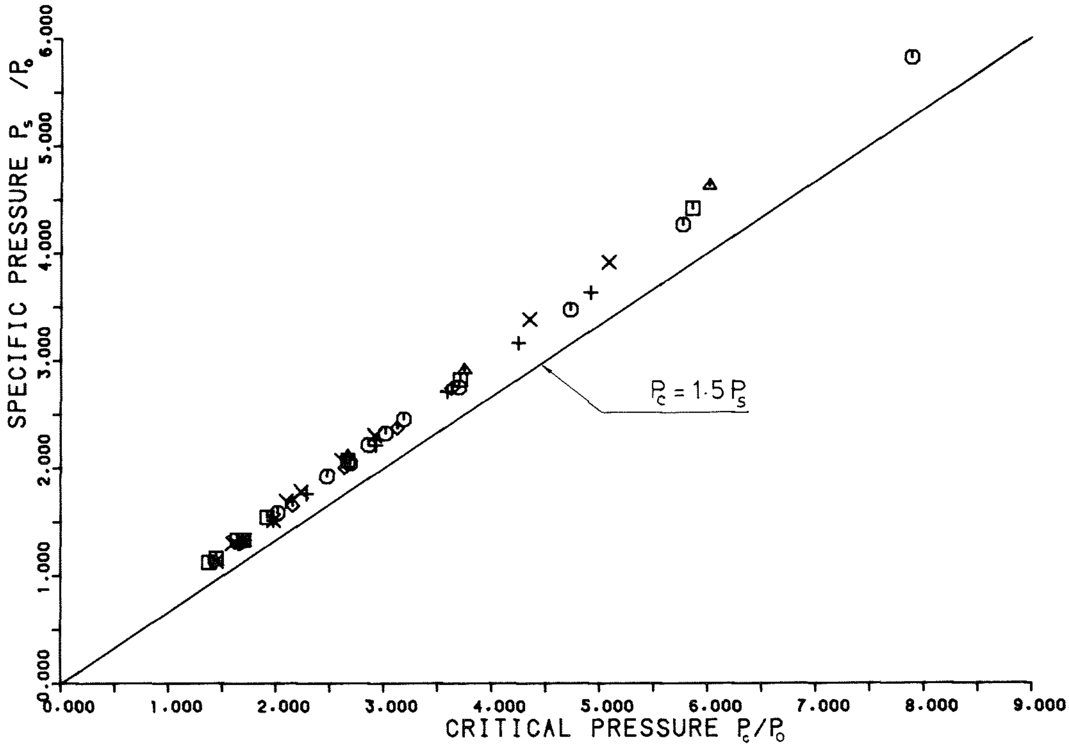


Fig. VI The relation between the specific and the critical pressure.

In this section it is shown that in the absence of surface bonded gas pockets the resultant compressibility can be expressed by two different constants, either defined by expression (2.35) in case the pore water contains free air bubbles, or similar to the compressibility of pure water in case all the free air has been dissolved.

Another advantage of the obtained formula (2.35) is that the saturation degree, which is usually difficult to measure under varying pressures, does not appear.

Furthermore, in the case of the presence of free air bubbles the resultant compressibility of the air-water mixture is much larger than the water compressibility itself. It cannot be excluded in a proper description of the mechanical behaviour of semi-saturated soils.

2.d DEFORMATION BEHAVIOUR OF SEMI-SATURATED SOILS

Soil can be visualised as a skeleton of particles encompassing interconnected voids, which in general contain an air-water mixture. For the range of stress usually encountered in soil mechanics practice the individual solid particles and the pore water can be considered to be incompressible, but air is highly compressible. The volume of the soil skeleton can change due to local deformations at the intergranular contact points and due to rearrangements of solid particles into new positions. The strength of the soil skeleton depends on the structural arrangement of the solid particles. Shear can be resisted by the skeleton itself, but compression may be resisted by the soil skeleton and by an increase in pore pressure.
In fully saturated soil a reduction of volume is possible only if some of the pore water can escape from the pores. In dry or semi-saturated soil a volume reduction is possible by compression of the air content.
Thus, changes in the pore geometry can generate pore fluid motion. An extra complication exists, because the resistance to flow increases when the porosity is reduced. On the other hand, the deformation rate is retarded due to the resistance to porous flow. This process, called consolidation, is discussed here.

Terzaghi's effective stress principle

The importance of intergranular forces in a soil skeleton was first recognised by Terzaghi (1923, 1943), who presented the principle of effective stress. This principle is shortly discussed, since it is so common in soil mechanics, that the basic assumptions have become more or less concealed.
Consider an elementary volume of saturated soil. For the motion of pore water through a deformeable porous medium the equilibrium for each of the substances, i.e. water and soil skeleton, will enable an investigation on the influence on porous flow due to deformations of the soil skeleton.
The equation of motion for the pore water occupying the pores of an elementary volume reflects the equilibrium in terms of the inertia force, the driving force H and the resistance force R per unit volume of pores, according to:

$$n\varrho\frac{\partial w}{\partial t} = n(H - R), \qquad (2.43)$$

where n represents the porosity, ϱ the water density and w the average velocity of the considered volume of pore water.
The resistance force R incorporates the internal friction, according to equation (2.2).
The driving force H combines two features, see section 2.a, to wit, pore pressure gradients ∇p and the gravity force $\varrho g\,\nabla z$. Thus,

$$H = \nabla p + \varrho g\,\nabla z. \qquad (2.44)$$

The inertia effects in porous flow can be approximately accounted for by a linearized constitutive relation, which couples the total volumetric resistance R′ comprising internal friction and inertia effects, to the volumetric flux q, according to:

$$R' = R + \varrho\frac{\partial w}{\partial t} = -\varrho g W' q. \qquad (2.45)$$

It is assumed that inertia effects in the pore fluid motion embody convective and centrifugal acceleration of laminar flow patterns, and rotating flow of turbulent patterns, dominating in the pore's inside at larger pressure gradients. Relation (2.45) was discussed in section 2.a.

The linearized coefficient of resistance to porous flow W′ contains the inertia effects and the internal friction. Therefore, the combination of equations (2.43), (2.44) and (2.45) gives the following equation of motion for the pore water:

$$n\nabla p + n\varrho g\,\nabla z + n\varrho g W' q = 0, \qquad (2.46)$$

which is fully in agreement with the (extended) law of Darcy.

The equation of motion of the soil skeleton in the considered elementary volume reflects the equilibrium in terms of the inertia force, the volumetric driving force F and the resistance force R′ per unit volume of pore water generated by the porous flow. In formula:

$$(1-n)\varrho_s \frac{\partial^2 u}{\partial t^2} = F + nR', \qquad (2.47)$$

in which ϱ_s denotes the granular density and u the average displacement of the soil skeleton. The resistance force R′ exerted by the pore fluid motion on the soil skeleton includes inertia of the pore fluid. Notice that this force refers to the pore volume and acts here as a driving force. The driving force F comprises the intergranular forces acting as concentrated forces at those parts of a solid where it makes contact with neighbouring particles, and they include also solid gravity forces.
Deformation of the soil skeleton is assumed to occur at the intergranular contact points by local deformation or by rolling and slipping, which causes rearrangements in the skeleton formation.
For the afore mentioned equation of motion referring to an elementary volume of soil, the concentrated intergranular forces can be averaged over this volume. Such an average measure is proposed by Terzaghi (1923), called the effective stress (see Fig. VII).

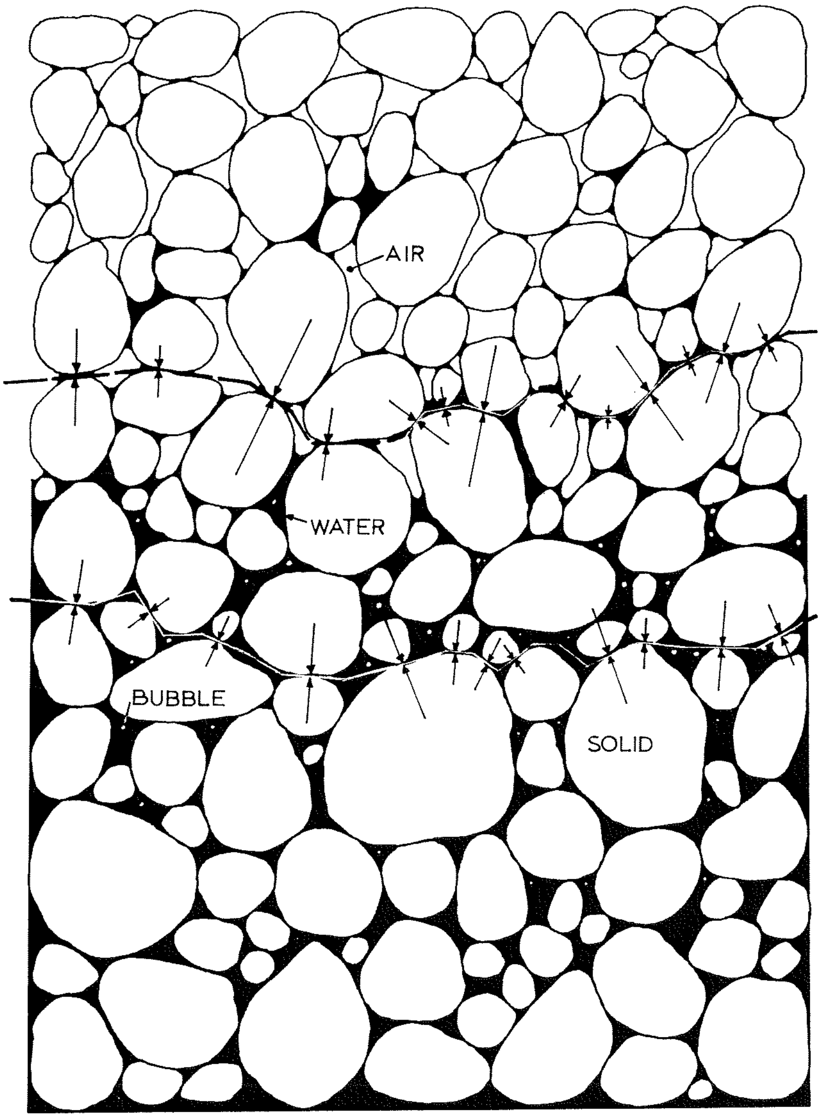


Fig. VII A visualisation of Terzaghi's effective stress principle.

In saturated soils the grains are surrounded by pore water. Therefore, also the pore pressure gradient acts as a driving force and it must be included in the equation of motion for the skeleton. Thus, the volumetric driving force F can be expressed in terms of the effective stress σ'_{ij}, which represents a tensor taken positive for tension, in terms of the gravity force $\varrho_s g \, \nabla z$, and the pore pressure gradient ∇p, according to:

$$F = -\sigma'_{ij,i} + (1-n')\,\nabla p + (1-n)\varrho_s \, g \, \nabla z \,. \qquad (2.48)$$

Here, $\sigma'_{ij,i}$ represents the resultant force acting in the i-direction on a unit volume of soil due to intergranular forces.

It must be understood that σ'_{ij} is not equal to the true contact stress between the particles, that would be a random but very much higher stress acting over the contact area between solids.

In the theory of mixtures it is common to define a stress referring to the phase volume to which it belongs, and such stresses are called phase stress or partial stress, see for example Houlsby (1979). It is to be noted that σ'_{ij} is not such a partial stress.

Clay particles may not be in direct contact due to their surrounding adhered layer of pore water, but it is assumed that the interparticle force can be transmitted through the strongly bonded water layer.

The symbol n′ denotes the effective surface area of the solid particles being in contact with the pore fluid. This area is similar to the porosity n, likewise along the cross sectional plane drawn in Fig. VII, since the contact area is usually very small. In literature there is some discussion about the equality of the volumetric porosity n, the cross-sectional porosity and the surface porosity n′, in particular in cases where very high pressures are involved. In this treatise they are considered to be equal.

The previous analysis is based on the conservation of momentum, resulting in the equations of motion including inertia of the pore fluid and solid particles.

The stated equations of motion and the subsequent derivation are not complete in a theoretical sense. It is assumed, here, that the inertia terms in the pore fluid motion contribute to a total resistance R′, according to equation (2.45), which is counterbalanced bij interfacial focus at the solid surfaces. Moreover, the real dynamic behaviour of a saturated soil might include also added mass effects due to the relative velocity of the different phases resulting in extra terms in the momentum equations, see Van der Kogel (1977, 1979). These effects are not included. In this discussion the granular inertia terms are disregarded. Only small and slow motion of the soil skeleton is considered. In this respect the equation of motion for the skeleton, equation (2.47), becomes, utilizing (2.45), (2.48) and the equality: $n = n'$,

$$-\sigma'_{ij,i} + (1-n)\,\nabla p + (1-n)\varrho_s g \,\nabla z - n\varrho g W' q = 0 \,.$$

Adding (2.46) eliminates the internal resistance force R′ and gives:

$$-\sigma'_{ij,i} + \nabla p + (n\varrho g + (1-n)\varrho_s g)\,\nabla z = 0 \,. \qquad (2.49)$$

Equilibrium in soils is governed by total stresses σ_{ij}. For a saturated soil with volumetric weight γ_s the state of equilibrium for an elementary volume of soil is expressed by:

$$-\sigma_{ij,i} + \gamma_s \,\nabla z = 0 \,. \qquad (2.50)$$

Comparison with the equilibrium condition (2.49) leads to the following conclusions:

$$-\sigma_{ij,i} = -\sigma'_{ij,i} + \nabla p \,, \qquad (2.51)$$

$$\gamma_s = n\varrho g + (1-n)\varrho_s g \,. \qquad (2.52)$$

Terzaghi noticed that a change in the pore pressure produces practically no volume change and has practically no influence on the stress condition for failure (shear). Therefore, porous materials react to pore pressure changes as if they were incompressible. Moreover, shear stresses seem independent from pore pressures. Consequently, all measurable changes in the soil skeleton deformation are exclusively due to changes in the effective stress.

In this respect the following assumptions, satisfying (2.51), seem reasonable to reflect soil behaviour:

$$\sigma = \sigma' - p \,, \qquad (2.53)$$

$$\tau = \tau' \,,$$

where σ denotes the isotropic (mean) total stress, and σ' the isotropic (mean) effective stress. The second relation expresses that total shear stress τ is equal to the effective shear stress τ', which makes sense, since water can not resist total shear stresses.
Relation (2.53) is called the effective stress equation and it constitutes the core of Terzaghi's effective stress principle.
The effective stress equation can be better understood by considering equilibrium of normal stresses acting on the wavy cross-sectional plane visualized in Fig. VII.

Concerning partly saturated soils the effective stress principle has to be adjusted. Except for high saturation degrees the pore air will form interconnected channels through the pore structure, and the pore water will be concentrated in regions around the intergranular contact areas. The pore air pressure p'' is different from the pore water pressure p because of surface tension or, in clays, because of molecular linkage. Bishop (1959) suggested for this case an adjusted effective stress equation, according to:

$$\sigma = \sigma' - \chi p - (1-\chi)p'' \quad , \quad 0 \leq \chi \leq 1 \, ,$$

where χ is somehow related tot the saturation degree, the soil structure, and to the wetting history.
For air in contact with open air ($p'' = 0$) this equation reduces to:

$$\sigma = \sigma' - \chi p \, . \tag{2.54}$$

Skempton (1960) and Henkel (1960) presented some expressions for χ incorporating phase compressibility and shear strength effects. Keveling Buisman (1939) already suggested an effective stress equation identical to (2.54) with $\chi = s$, where s denotes the saturation degree.
As can be understood from Fig. VII, a cross-sectional wave plane through the particle contact areas will pass partly through water and partly through air in the semi-saturated zone.
The effective stress principle is a rather popular subject of discussion. Leonards (1962) and Bishop and Blight (1963) considered semi-saturated soils and phase compressibilities. They reported experimental evidence for the suggested theory. Electrical forces and molecular linkage have been investigated by Lambe (1960) and measurements on this phenomenon are published by Sridharan and Venkatappa (1979). Bishop (1973) considered low compressible soils. Garg and Nur (1973) extended the effective stress principle to saturated rock. Lately, Houlsby (1979) offered in an elegant analysis a new interpretation of the principle of effective stress in terms of continuum mechanics, in that the principle reflects the independence of work input to the soil skeleton and to the pore fluid. Consequently, if the processes of skeleton deformation and seepage are uncoupled, the mechanical behaviour of the skeleton will only depend on the effective stress as suggested by Terzaghi.
If the degree of saturation of soil is such that the pore air exists in the form of free travelling bubbles or attached small air pockets, it is still possible to draw a wavy plane through the pore water only, see Fig. VII. Subsequently, all the solid particles are almost surrounded by coherent pore water. In this case Terzaghi's effective stress principle, equation (2.53), is applicable. It will be utilized in the proceeding analysis.

Consolidation in saturated soils

The previous discussion made clear that the mechanical behaviour of soil is significant to groundwater flow. The motion of pore water is influenced by changes in the pore geometry, which can occur in deformeable soils. Storage variations causing pore pressure gradients induce porous flow. Resistance to flow by internal friction between the pore water and the soil skeleton limitates the pore fluid motion and retards the deformation rate.
In conclusion, the mechanical behaviour of soil is restrained by pore water dissipation and this effect, referred to as consolidation, was first noticed by Terzaghi (1923), who mentioned the resemblance of consolidation with heat diffusion.
For example, one-dimensional linear consolidation is governed by, see Barends (1978):

$$-\frac{\partial \sigma'}{\partial t} = c \frac{\partial^2 p}{\partial z^2} \, .$$

Here, c represents the coefficient of consolidation, comprising information regarding the pore

fluid flow and the mechanical nature of the soil skeleton. p is the pore pressure and σ' the vertical effective stress.
Introducing Terzaghi's effective stress principle, equation (2.53), gives:

$$\frac{\partial p}{\partial t} - \frac{\partial \sigma}{\partial t} = c\frac{\partial^2 p}{\partial z^2}. \qquad (2.55)$$

In case the total stress σ does not vary in time, this reduces to the one-dimensional heat diffusion equation:

$$\frac{\partial p}{\partial t} = c\frac{\partial^2 p}{\partial z^2}, \qquad (2.56)$$

which is referred to as Terzaghi's consolidation. Adopting this philosophy Jacob (1940) presented a mathematical theory to account for the influence of soil elasticity to pore water dissipation in horizontal aquifers. Subsequently, several practical problems have been solved using his theory, described by:

$$\frac{\partial p}{\partial t} = c\nabla^2 p, \qquad (2.57)$$

with the restriction that the total vertical stress is a constant and horizontal displacements are negligible.

When the total vertical stress varies in time equation (2.55) and not (2.56) should be used. For a one-dimensional vertical strip of saturated soil subjected to cyclic loading, the solution shows locally a phase shift of the pore pressure response running ahead of the actual periodical loading, see Fig. VIII.

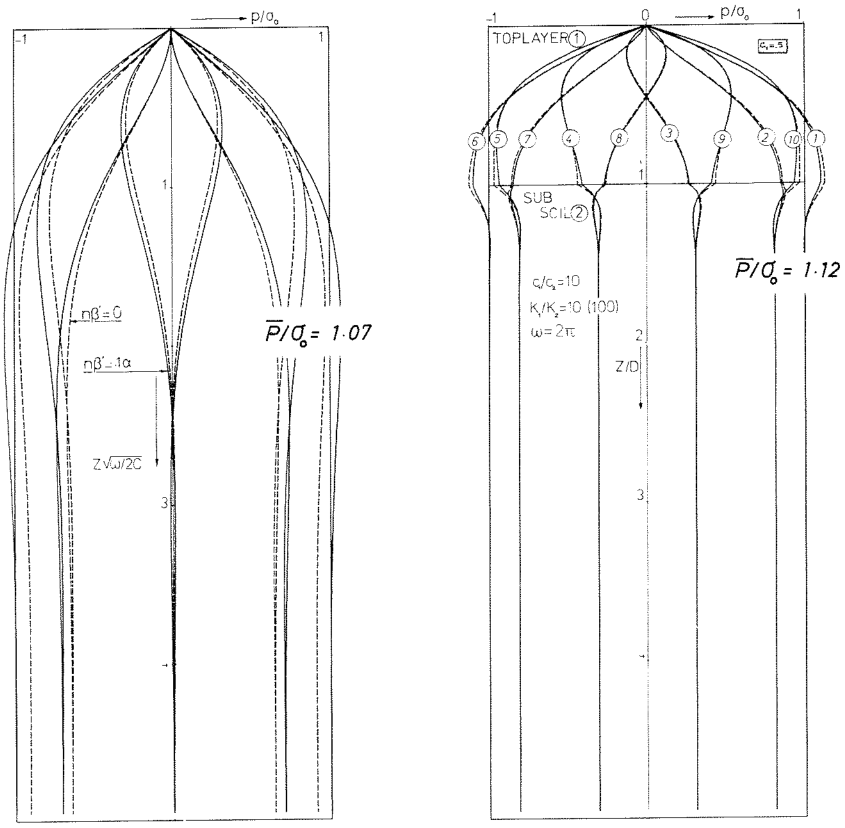


Fig. VIII Vertical cyclic consolidation.

This phenomenon has been confirmed by experiments reported by Barends and Thabet (1978), and such behaviour is characteristic for the consolidation process being different from heat diffusion. Disagreements between diffusion and consolidation for spherical symmetry have been discussed by Verruijt (1965), and by Sills (1975).

In more dimensions loading a saturated soil induces shear stresses as well, and it can be expected that this will indirectly influence the pore water dissipation. A physically consistent more-dimensional extension for a perfectly elastic soil behaviour was suggested by Biot (1941). His mathematical formulation consists of two equations derived from a state of equilibrium of the saturated soil, identical to (2.50) disregarding the gravity term:

$$-\sigma'_{ij,i} + \nabla p = 0\,, \tag{2.58}$$

and from the conservation of the mass of pore water, including the water compressibility, resulting in the so-called storage equation, after De Josselin de Jong (1963):

$$\frac{\partial \varepsilon}{\partial t} + n\beta \frac{\partial p}{\partial t} = \frac{k}{\varrho \upsilon} \nabla^2 p\,, \tag{2.59}$$

where β denotes the coefficient of compressibility of the pore water and ε is the volume strain, defined by:

$$\varepsilon = \nabla \bullet u\,. \tag{2.60}$$

The righthand side of equation (2.59) expresses the pore water dissipation, the lefthand side the volumetric storage consisting of two components, one due to soil skeleton deformation and one due to pore water compressibility. To interrelate (2.58) and (2.59) a stress-strain relation for soil is necessary, which couples the stress tensor σ_{ij} and the volumetric strain ε. Terzaghi, Jacob and Biot assumed a linear relationship.
Experiments show that the behaviour of real soils is more complicated than if they conform to a perfectly elastic material, but a fact in favour of Biot's theory is that several peculiar observations from tests could be satisfactorily explained, assuming perfect elasticity. Moreover, the obscurity of real soil parameters and the corresponding complex mathematical formulation combine to favour a simple constitutive soil model. Therefore, the stress-strain relation for soil is assumed to be perfectly elastic, i.e. Hooke's law.

Such a relationship is a rather crude approximation is regard to the microscopic nature of soil, in particular concerning the intergranular contact forces and the overall deformation of the skeleton due to local pressing, crushing, rolling and sliding of solid particles. However, the effect of most of the disturbances generating pore water dissipation is restricted to fairly moderate fluctuations of stresses in comparison to in-situ stresses. In such situations Hooke's law provides a first insight, and a description amenable to fundamental mathematical operation. Writing the equilibrium equation (2.58) in terms of the displacement vector u of the soil skeleton, see Barends (1978), gives:

$$\nabla(\nabla \bullet u) - G\alpha \nabla \times (\nabla \times u) = \alpha \nabla p\,, \tag{2.61}$$

where α represents the laterally confined compressibility, measured in the oedometer, defined by:

$$\alpha = 1/(K_s + 4G/3)\,. \tag{2.62}$$

K_s denotes the bulk modulus and G the shear modulus of the soil skeleton. Hooke's law correlates-the volumetric strain ε and the isotropic effective stress σ', according to:

$$\sigma' = K_s \varepsilon\,. \tag{2.63}$$

In conclusion, introducing (2.60) into (2.61), and (2.59) results into the following set:

$$\frac{\partial \varepsilon}{\partial t} + n\beta \frac{\partial p}{\partial t} = \frac{k}{\varrho \upsilon} \nabla^2 p\,, \tag{2.64}$$

$$\nabla \varepsilon - G\alpha \nabla \times (\nabla \times u) = \alpha \nabla p\,, \tag{2.65}$$

governing consolidation according to Biot's theory. The coupling between the soil skeleton and the pore water in this form is obviously dissimilar to the simple heat diffusion process. Such a situation is shown in Fig. IX, where measurements correspond best to Biot's theory and not to the common potential or diffusion theory (after Sellmeijer, 1978).

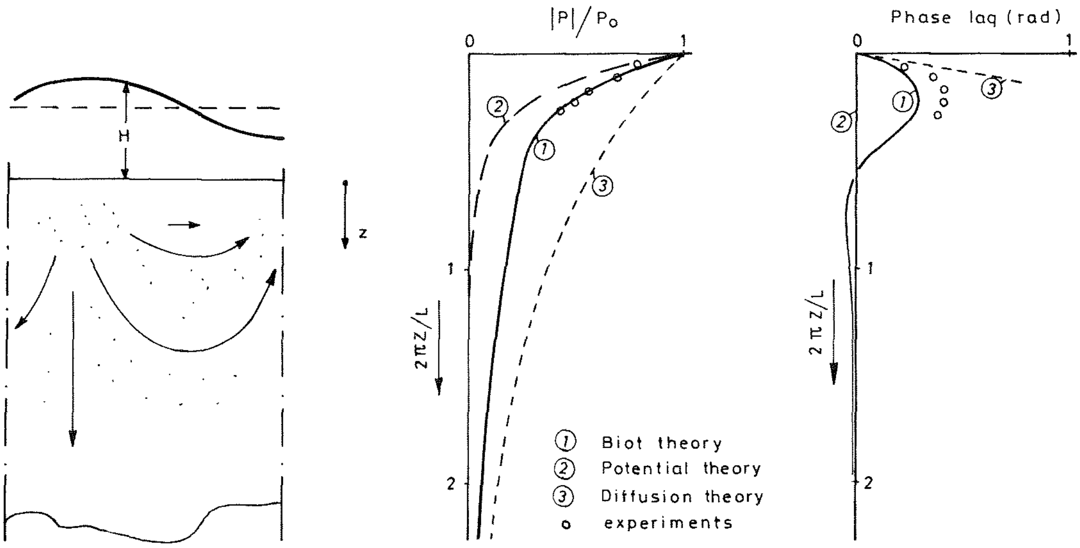


Fig. IX Consolidation due to free water waves.

This contrast becomes manifest, in a pore pressure generation, contradicting expectations, i.e. observed pore pressure increase where a decrease is expected. This feature shows some similarity with the peculiar behaviour shown in Fig. VIII. It can be explained with Biot's theory.
For example, consider the equilibrium equation (2.65). Disregard the rotational part of the displacements, i.e. $\nabla \times u = 0$, which is justified in several types of problems. Next, integration gives:

$$\varepsilon = \alpha(p - F), \tag{2.66}$$

where F is an integration constant, eventually a function of time. Inserting (2.66) into the storage equation (2.64) gives:

$$\frac{\partial p}{\partial t} - \frac{\alpha}{\alpha + n\beta}\frac{\partial F}{\partial t} = c\nabla^2 p, \tag{2.67}$$

where the coefficient of consolidation c is defined according to:

$$c = k/(\varrho\upsilon(\alpha + n\beta)). \tag{2.68}$$

Equation (2.67) is comparable to (2.55), which has been used to solve problems with varying loading.
It can be shown that the pressure function F appearing in (2.66) is related to the boundary stress. In rotational deformation fields, where $\nabla \times u \neq 0$ holds, the pressure function F depends also on the displacement vector u. This can be understood by taking the divergence of the equilibrium equation (2.65), which results in:

$$\nabla^2 \varepsilon = \alpha \nabla^2 p.$$

Integrating this relation over the considered field yields an expression similar to equation (2.66), but now F, as a result of spatial integration, is a function of space and time. This can easily be verified by taking the gradient of (2.66) and subtracting (2.65), giving:

$$\nabla F = G\nabla \times (\nabla \times u). \tag{2.69}$$

The term $\nabla \times u$ reflects the internal rotations. This particular character of the function F has been expounded by De Josselin de Jong (1963) and as such F, appearing in (2.67), symbolizes the fundamental difference between the theory of Biot and Jacob's approach, equation (2.57).

Several types of problems have been solved utilizing Biot's theory, for example, one-dimensional cyclic problems (Barends, 1978), spherically symmetrical problems (De Josselin de Jong, 1953, 1964; Cryer, 1963; Gibson, 1963, 1966; Verruijt, 1965), cylindrically symmetrical problems (De Leeuw, 1964; Calle, 1979), half plane problems (McNamee and Gibson, 1960; Koning, 1968; Barends, 1978; De Groot and Sellmeijer, 1979), axially symmetrical half space problems (Gibson and McNamee, 1962; Barends, 1971), finite layer problems (Verruijt, 1969; Gibson et al, 1970; Sellmeijer, 1976).

To demonstrate some peculiar effects of Biot's theory, consider the problem solved by Barends (1971), concerning a point well with a sudden constant production in an isolated half space. The problem was treated using the principle of mirror images and transformation techniques (Hankel- and Laplace transforms). Some results are shown in Fig. X. The calculated surface settlement above the well first increases strongly, later more moderately with time. It does not reach a final value (logarithmic), but the settlement is almost stationary at $t > 100H^2/c$, which might still represent a period of several years.
Surprisingly, the corresponding pore pressure initially increases before it decreases. This effect is due to soil deformations. It is interesting to notice the influence of the elasticity of the porous medium on this particular effect.

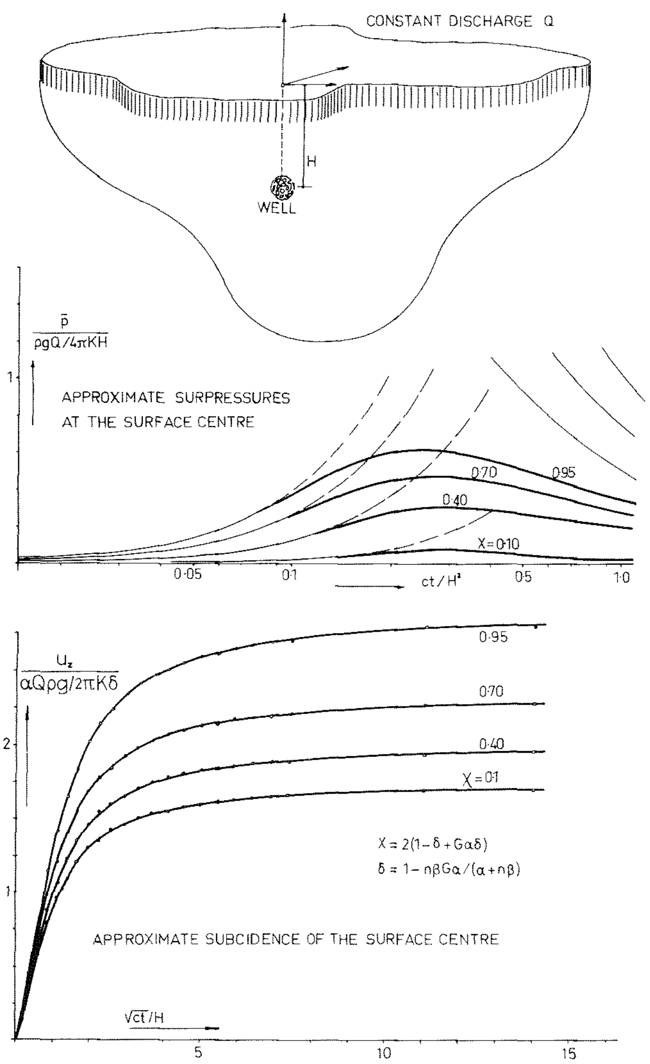


Fig. X Consolidation due to a point well in an isolated half-space.

The analyticall solution of consolidation problems using Biot's theory generally requires advanced mathematics and cumbersome calculations. Numerical techniques, i.e. finite element methods, overcome this disadvantage only partially, since numerical solutions obtained by computer programmes, do not provide parametric solutions. Good correlation and interpretation based on of computer calculations are relatively inefficient, because sometimes many sensitivity computations are needed.

It seems preferable to utilize the more simple approach according to Terzaghi's and Jacob's theory whenever possible. The conditions of applicability of these simple models are twofold. The isotropic stress must be a constant in time or the displacement field must be irrotational and loading is stationary. One can imagine a situation, which combines both conditions.
Different assumptions concerning the problem definition will lead to a specific expression for the consolidation coefficient. One such approach is already dealt with: irrotational deformation ($\nabla \times u = 0$) and constant loading ($\partial F/\partial t = 0$), rendering (2.67) into Jacob's equation (2.57), where c is defined according to expression (2.68).
Another approach, suggested by Gibson and Lumb (1953) is more straightforward. Eliminate the volume strain ε from equations (2.59) and (2.63) and next, applying Terzaghi's effective stress principle, equation (2.53), the following equation results:

$$\frac{\partial p}{\partial t} + \frac{1}{(1 + K_s n\beta)} \frac{\partial \sigma}{\partial t} = c\, \nabla^2 p \,, \tag{2.70}$$

where the coefficient of consolidation c is defined by:

$$c = kK_s/(\varrho\upsilon(1 + K_s n\beta)) \,. \tag{2.71}$$

It is obvious that, when the isotropic total stress σ is constant in time, equation (2.70) reduces to (2.57) provided that the actual expression for the coefficient of consolidation is included.
This approach is also known as pseudo-consolidation, since the pore pressure and effective stresses are decoupled, see for example Runesson (1978).
Sellmeijer (1978) considered a horizontal semi-confined aquifer. Disregarding shear stresses at the upperbound he obtained the following equation for horizontal flow:

$$\frac{\partial p}{\partial t} + \frac{2G\alpha}{\alpha + n\beta(1 - G\alpha)} \frac{\partial A}{\partial t} = c\, \nabla^2 p \,, \tag{2.72}$$

where A is some function of time, and here, the coefficient of consolidation c is defined by:

$$c = k(1 - G\alpha)/(\varrho\upsilon(\alpha + n\beta(1 - G\alpha))) \,. \tag{2.73}$$

Sellmeijer showed that for an aquifer with (semi)-infinite extent A equals zero, and (2.72) reduces to equation (2.57), corresponding to Jacob's approach, except for horizontal displacements, which are parabolic in depth.
Lately, Verruijt (1979) suggested a more practical approach. He combined the diffusion damping character with knowledge about the instantaneous behaviour and the ultimate situation based on excercises with Biot's theory. In this way the essential quantities in a consolidation problem can quickly be derived in a suitable parametric form useful to engineering practice.

Conclusion

The previous discussion showed that despite the poor description of the soil deformation behaviour by a perfectly elastic constitutive relation and the assumptions embedded in the effective stress principle, the equation of diffusion is suitable to model the consolidation process in various types of problems.
The models mentioned, in particular equations (2.67), (2.70) and (2.72), simulate the consolidation process quite well provided there are no rotations in the deformation field. Different approaches and assumptions concerning the boundary conditions resulted in specific expressions for the coefficient of consolidation, but on close examination they compare well.

Irrotational fields are completely defined by a potential (Helmholtz' theorem). Thus the entire groundwater flow phenomenon can be expressed by a second order differential equation in terms of a potential, according to, see Barends (1978):

$$\Phi = z + \int_{p_F}^{p} (1/\varrho g)\, dp\,, \quad p_F = \frac{\alpha F}{\alpha + n\beta}\,, \tag{2.74}$$

which in this form is an extension of Hubbert's potential for compressible pore water, see Hubbert (1940).
Here, the pore water density is a function of the pore water pressure only, i.e. $\varrho = \varrho(p)$. It is easily verified, that differentation with respect to time using Leibnitz' rule, which gives:

$$\varrho g \frac{\partial \Phi}{\partial t} = \frac{\partial p}{\partial t} - \frac{\alpha}{\alpha + n\beta} \frac{\partial F}{\partial t}\,,$$

renders equation (2.67) in the following form:

$$\frac{\partial \Phi}{\partial t} = c\nabla^2 \Phi\,. \tag{2.75}$$

This approach shows once more, that the diffusion type of equation is well applicable for groundwater flow in irrotationally deforming media.

In general rotations do not always vanish and therefore, they provide a concealed linkage between soil skeleton and pore water, responsible for peculiar pore pressure developments.
In section 3 the fundamental contribution of soil deformation to porous flow, in particular the volume strain ε and the characteristic function F, will be discussed including nonlinearities.
It will be explained that the equations governing the consolidation process in saturated soils are also valid in the case of semi-saturated soils, if the saturation degree is not too low.

3. NONLINEARITY IN THE CONTINUITY EQUATION

Summary of chapter 3

The concept of continuity of mass, momentum or energy is fundamental in the formulation of physical processes. However, a mathematical description utilizing continuous functions having proper derivatives suggests a material that is uniform and continuous. In case of a particulate medium such as a granular soil, and more particularly when the pore water containes entrapped air, the applicability of the continuum approach entails a thorough investigation to elucidate the restrictions of an equation of continuity formulated to characterize the physical process considered.

Some of the aspects concerning the continuity are dealt with, in that the existence of extra terms in a general continuity equation, responsible for the nonlinear character, are described.

Section 3.a discusses the scale at which the groundwater flow phenomenon is considered from a mathematical point of view. It emphasizes the averaging nature of constitutive parameters reflecting material properties suitable to describe the pore fluid motion. This scale intrinsically restricts the range of validity of the proposed mathematical description.

Section 3.b deals with the fundamental derivation of the equation of continuity, called the storage equation, governing porous flow in deformable semi-saturated soils. Due to deformations a soil element displaces and conveys the pore content. Since the interaction between the pore fluid and the soil skeleton involves friction, which is a so-called follow process (it moves-with the soil element), the description of this phenomenon must be related to the actual position of the soil element.
In physics it is common to define a process referring to a fixed coordinate system, but then application of a porous flow law, such as Darcy's law being defined in terms of the chosen

coordinate system, is likely to lead to a physically unacceptable formulation unless the conveying effect is included.
In literature several approaches are still taken for granted, though the convective terms are not accounted for. Here, they are evaluated on their deviation from the correct approach.

Section 3.c provides the theoretical background to state that the pore water motion in a semi-saturated porous medium, in which air is present in the form of micro bubbles and pockets, can equally well be represented by the storage equation valid for fully saturated soils by replacing the pore water density with the air-water mixture density and by introduction of a compressibility for the air-water pore content. An approximation of minor importance related to the volume of air pockets attached to the solid particles is the only concession to be made, while the coefficient of permeability in Darcy's law requires a small adjustment to include an increase in resistance due to presence of stagnant bubbles blocking pore necks.
The fact that the familiar storage equation also describes the motion of an air-water mixture through a deformable medium permits to conclude that the previous efforts invested in solving various saturated soil problems cover likewise semi-saturated soil problems representing more realistic situations.
It is stressed that the compressibility of the air-water pore content plays a dominent role, because the air is rather compressible.

Section 3.d discusses a trial to express the general nonlinear storage equation in terms of one variable, a flow potential. This is only possible in the case where the process is irrotational, requiring the deformation behaviour of the soil to be irrotational.
For a large class of practical problems the rotational part in the soil skeleton deformation can be disregarded. A potential function is found, by which various effects can be formulated such as variations of the permeability due to pore geometry alterations induced by soil deformations, and a pore pressure imposed by transient boundary loading. This potential resembles Hubbert's potential, but it is expressed in terms of the average density of the pore content.

Section 3.e clarifies some aspects of the nonlinear storage equation. The fact that pressure induced density variations will not give rise to rotations in the flow pattern is discussed.
It appears only to be valid in isotropic inhomogeneous media, which is an unfortunate restriction to the theory presented, since natural sediments are usually stratified and anisotropical.
The significance of convective terms is outlined. Two classes are distinguished, large strains and small strains, the latter yet generating large displacements due to an integral effect of many small strains. The first class requires a nonlinear soil deformation behaviour to be described, which is beyond the scope of the subject. In the second class convective terms should be incorporated and for two general cases the corresponding storage equation is derived approximately covering the convective effect.

3.a THE PHYSICAL CONTINUUM APPROACH

The study of groundwater flow at microscopical level is a comprehensive challenge. Representing the flow through the pores by a fictitious macroscopical model describing the average flow will stand a better chance. There are in general two methods. The first is to define or measure average properties and to compose a mathematical description of the macroscopical flow. The second way starts with the constitutive equations describing the microscopical flow and averaging these equations to a macroscopical level obtaining average parameters. The latter provides for a complete interaction between all the phenomena involved and preserves a deeper comprehension of the essence of the physical parameters, see Verruijt (1969) and Bear (1972).
However, these parameters are of little practical importance, unless they are measurable by some standard method, or unless they are related to our sense perceptions and a mathematically amenable flow model can be composed. Many such parameters have been defined already and their applicability is a fact. Sometimes a microscopical property is chosen like density, fluidity, grain shape, or a statistical characterisation, such as texture (soil particle distribution) and specific surface, sometimes a macroscopical feature like packing, porosity, saturation, or even a composed one, such as permeability, capillarity, compressibility and storativity.

The microscopical point of view represents a comparative way of considering elementary

volumes in a physical point within a fictitious continuum symbolizing the composed material, rather varied at microscopical level, and yet posessing relevant average bulk properties at macroscopical level.
This treatment of soil and pore water as continua allows to describe the flow transport mathematically. The elementary volume which might differ for any phenomenon, is precisely the physical or material point of the considered medium at the mathematical point. If such a volume exists for all the properties concerned, the obtained average flow model is physically meaningful.
The range of validity due to this averaging process has to be considered continuously, in particular for investigation of special effects like local stability of the soil particles (piping, internal migration, stability of hydraulic filters), see Bachmat and Bear (1972), Bear (1978).
An example of such an elementary volume for groundwater flow is represented in Fig. XI. The composition of a continuum out of elementary volumes is realised by mutual overlapping. In this way the average quantities and properties, defined in the centroid of an elementary volume can be represented in a continuous field. Derivatives do exist and mathematical manipulations are permitted.

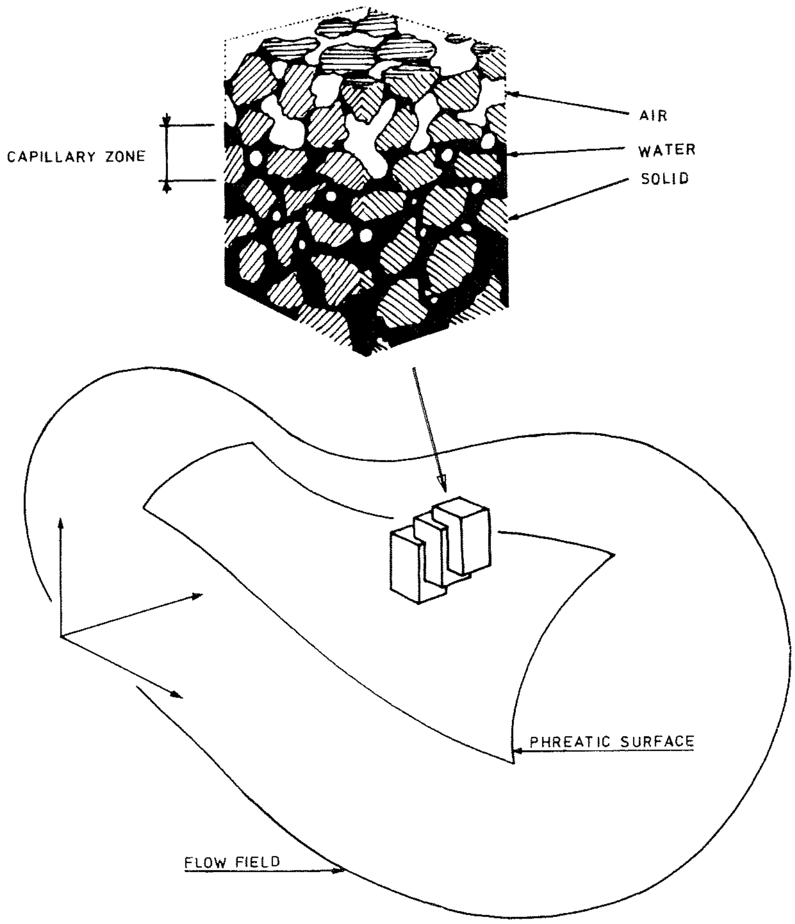


Fig. XI Elementary volumes, a physical-mathematical approach.

3.b A NONLINEAR STORAGE EQUATION FOR PORE FLUIDS

A general conservation principle valid for any fluid property related to the fluid mass, for example the extensive property g, can be expressed in the following way, see Bear (1972):

$$\int_U \{ \nabla \bullet (gw) + \partial g / \partial t \} \, dU = 0 \, ,$$

valid in an isothermal process without internal production. Here, w denotes the velocity at which the amount of property g occupying the volume U propagates. Since this expression holds for any volume U, the integrand must vanish. Therefore, the equation:

$$\nabla \bullet (gw) + \partial g / \partial t = 0 \tag{3.1}$$

represents the general conservation principle of the extensive property g without internal production.

For groundwater flow the variation of the fluid mass itself is essential to describe the quantity of stored pore water in a bulk volume of soil. Introduction of the fluid mass for property g results in the so-called storage equation. Hence, in a soil with porosity n the property g has to be identified with the relative density $n\varrho'$ of the pore fluid, where ϱ' denotes the pore fluid density including entrapped air, see section 2.c.

Since resistance to flow is directly related to the interaction between the pore fluid and the porous skeleton, the flow velocity should be formulated with respect to the skeleton, which can move due to deformations. Although this effect is sometimes assumed to be of negligible significance, it should be included in the derivation of the storage equation in order to investigate its consequences properly.

The velocity w appearing in equation (3.1) representing the absolute velocity of the pore fluid mass, is rewritten in terms of the absolute velocity v of the soil skeleton and the specific discharge q, which is measured with respect to the moving soil skeleton. Actually, the specific discharge q is related to the absolute velocity w of the pore fluid mass, to the absolute velocity v of the soil skeleton, and to the porosity n, according to:

$$q = n(w - v) . \tag{3.2}$$

Thus, equation (3.1) becomes:

$$- \nabla \bullet (\varrho' q) = \nabla \bullet (n\varrho' v) + \partial (n\varrho')/\partial t , \tag{3.3}$$

and it describes the pore fluid flow in a deformable porous medium.
Rewriting equation (3.3) in terms of substantial derivatives gives:

$$\begin{aligned} - \nabla \bullet (\varrho' q) &= n\varrho' \, \nabla \bullet v + v \bullet \nabla (n\varrho') + \partial (n\varrho')/\partial t \\ &= n\varrho' \, \nabla \bullet v + D(n\varrho')/Dt \\ &= n\varrho' \, \nabla \bullet v + \varrho' Dn/Dt + nD\varrho'/Dt . \end{aligned} \tag{3.4}$$

Velocity v varies in the field considered and is directly related to the rate of deformation of the soil. A next step to simplify equation (3.4) is to show that:

$$\nabla \bullet v = D\varepsilon / Dt , \tag{3.5}$$

where ε denotes the volume strain corresponding to the soil skeleton deformation, defined by:

$$d\varepsilon = dU/U, \tag{3.6}$$

in which $d\varepsilon$ represents an incremental volume strain.

In this form ε is referred as natural strain. Strain can be defined in various ways, see Eringen (1967), and any reasonable definition can be incorporated in a proper constitutive equation relating stresses and that particular strain. Here, definition (3.6) will be adopted. For small strains (3.6) becomes identical to (2.60), which in fact represents an incremental form of volumetric strain with respect to a fixed reference, for example, the initial state.

Consider an elementary volume U of moving soil particles of fixed identity (Lagrangian approach). Its centre position is given by $x(x_0, t)$, representing the actual position as a function of time t and of the initial position x_0 at time t_0.

The absolute velocity v of this volume is defined by:

$$v = \partial x / \partial t \Big|_{x_o} = Dx/Dt \,, \tag{3.7}$$

where the last symbol represents the substantial derivative with respect to time emphasizing the fact that the particular volume of soil particles is considered which had position x_o at time t_o.

If the volume U contains a constant amount of the extensive property g, i.e.:

$$gU = \text{constant} \,,$$

then 1/U itself represents an extensive property. The general conservation principle, according to equation (3.1), yields in this case:

$$\nabla \bullet (v/U) \quad + \quad \partial(1/U)/\partial t = 0 \,,$$

or:

$$\nabla \bullet v \quad + \quad UD(1/U)/Dt = 0 \,.$$

Elaboration of this expression results in:

$$\nabla \bullet v \quad - \quad (1/U)DU/Dt = 0 \,,$$

and introduction of definition (3.6) gives the justification of equation (3.5).

Since the volume of the individual solids remains practically constant, see section 2.d, that is:

$$d\,\{\,(1-n)U\,\} = 0 \,,$$

a variational relation between the porosity n and the volume strain ε is obtained, employing equation (3.6):

$$dn = (1-n)d\varepsilon \,. \tag{3.8}$$

From this expression the substantial derivative with respect to time is derived, resulting in:

$$\frac{Dn}{Dt} = (1-n)\frac{D\varepsilon}{Dt} \,. \tag{3.9}$$

Substitution of (3.5) and (3.9) into (3.4) gives:

$$-\,\nabla \bullet (\varrho' q) = \varrho' D\varepsilon/Dt + n\,D\varrho'/Dt \,. \tag{3.10}$$

In section 2.c the compressibility β' of the pore content consisting of pore water including entrapped air has been studied, as it was defined by equation (2.20). It was shown that β' is a very moderate function of the fluid pressure p in case air is present in the form of microbubbles. In view of equation (2.20) the rate of change of the density of the pore content within an elementary volume of soil particles of fixed identity moving at velocity v becomes:

$$\frac{D\varrho'}{Dt} = \varrho'\beta'\frac{Dp}{Dt} \,.$$

Substitution in expression (3.10) gives:

$$-\,\nabla \bullet (\varrho' q) = \varrho'\,\{\,\frac{D\varepsilon}{Dt} + n\beta'\frac{Dp}{Dt}\,\} \,, \tag{3.11}$$

which represents the storage equation for groundwater flow in a deformable porous medium including entrapped air. It is referred to as the general nonlinear storage equation, since it properly includes the nonlinear behaviour of the air-water mixture occupying the pore volume, it includes convective terms due to the soil skeleton deformations embedded in the substantial derivatives with respect to the soil movements, and it contains a specific discharge measured with respect to the soil skeleton, so that a nonlinear porous flow law, suggested in section 2.a, can be introduced in order to account for turbulent effects in the pore fluid motion. Moreover, the actual behaviour of the soil deformation is not restricted sofar, except for the assumption that the individual soil particles are relatively incompressible. A general nonlinear soil behaviour is adaptable in this storage equation.

In literature several trials have been described to define a proper storage equation governing porous flow. Many authors have contributed to a good comprehension of this process and most of the suggested approaches result in a storage equation describing the physical behaviour of groundwater flow sufficiently accurate for practical engineering purposes.
However, on the occasion when the soil deformability is considered, the common consent is lost and up to now educational books in the field of geohydrology do not contain a unique and always correct approach, which should end up with expression (3.11). Several such approaches will be briefly discussed.

Literature review

Jacob (1950) was probably the first to introduce the compressibility of the pore water and simultaneously the variation of the porosity of the soil skeleton. He considered the behaviour of aquifers and assumed that changes of lateral dimensions are by comparison negligible. His approach resulted in the following storage equation describing the rate of change of the mass ϱ of the pore water without air (soil deformation restricted, except for the vertical direction):

$$-\nabla\bullet(\varrho q) = n\varrho\,\partial\varepsilon/\partial t + \partial(n\varrho)/\partial t\,, \tag{3.12}$$

which is identical to equation (3.4) except for of the term: $v\bullet\nabla(n\varrho)$, and only if $\nabla\bullet v$ is approximated by $\partial\varepsilon/\partial t$ omitting the convective term, see equation (3.5).
Jacob based his approach on the conservation of pore mass in an elementary volume taking into account the fact that the vertical dimension varies due to changes in the porosity.
However, he did not consider the convective term which automatically arises due to soil deformations induced by these porosity alterations itself. Several authors have adopted Jacob's approach unaware of this shortcoming, see for example Raudkivi and Callander (1976).
After Biot (1941, 1956), De Josselin de Jong (1953, 1963) developed a storage equation for compressible pore water in three dimensions, according to:

$$-\nabla\bullet(\varrho q) = \varrho\left\{\frac{\partial\varepsilon}{\partial t} + n\beta\frac{\partial p}{\partial t}\right\}, \tag{3.13}$$

which differs from equation (3.11) in respect of convective terms. Although these terms usually have little effect on the process of groundwater flow, they might become significant to large strain consolidation in soft soils, see for example Kuantsai Lee and Sills (1979). Presence of entrapped air is not included in equation (3.13).

De Wiest (1966) questioned the validity of Jacob's storage equation. His comments concern the fact 'that in one side of Jacob's equation the net inward flux was calculated for a volume element without deformation while in the other side of the equation, to compute the rate of change of the mass inside the volume element, the element itself was deformed.'
This conclusion is unclear, since Jacob only did not include convective terms. To provide a better equation De Wiest suggested another approach starting from:

$$-\nabla\bullet(\varrho q) = \partial(n\varrho)/\partial t\,. \tag{3.14}$$

Sofar this equation is correct, see (3.1). It represents the mass conservation of pore water, while the specific discharge q is measured with respect to a fixed coordinate system.
Consequently, Darcy's law, which as a constitutive law relates physical quantities and subsequently is independent on a coordinate system (constitutive invariance, see Gudehus, 1969) can only be introduced in the absence of soil deformations.
Therefore, the porosity n must be a constant. Nonetheless, De Wiest introduces Darcy's law and allows at the same time the soil to deform. Disregarding convective terms he arrived at an incorrect storage equation, according to:

$$- \nabla \bullet (\varrho q) = \varrho \left\{ (1-n) \frac{\partial \varepsilon}{\partial t} + n\beta \frac{\partial p}{\partial t} \right\} . \tag{3.15}$$

Irmay (1968) utilizes the so-called fixed box analysis, which is widely applied in the field of geohydrology, in an improper way. His analysis ends up with a storage equation identical to (3.15), and this equation is essentially dissimilar to equation (3.11) by the term: $n\partial\varepsilon/\partial t$, which is merely due to an incorrect definition of the specific discharge q.
This is most easily shown for a situation where a saturated soil is considered with both the pore fluid and the individual solid grains incompressible. Consequently, the total flow of material into an arbitrary volume U fixed in space must be zero. If S is the surface bounding U and s a unit vector in the direction of the outward normal to an element dS, then this condition may be written as:

$$\int_S \{ nw + (1-n)v \} \bullet s \, dS = 0 ,$$

where w denotes the actual mean velocity of the pore fluid and v the velocity of the soil skeleton. Making use of the divergence theorem of Gauss gives:

$$\int_U \nabla \bullet \{ nw + (1-n)v \} \, dU = 0 .$$

Since U is an arbitrary volume, the integrand must always be zero. Thus:

$$\nabla \bullet \{ nw + (1-n)v \} = 0 ,$$

represents the compatibility condition for incompressible soil grains and pore fluids. Defining a specific discharge according to (3.2), and introducing (3.5) gives:

$$- \nabla \bullet q = D\varepsilon / Dt ,$$

which corresponds to expressions suggested by Jacob, Biot and De Josselin de Jong. However, defining a specific discharge according to: $q = nw$, gives:

$$- \nabla \bullet q = (1-n) \, D\varepsilon / Dt ,$$

revealing similarity to De Wiest's and Irmay's approach.

In the latter case the specific discharge q is measured with respect to fixed coordinates, so that Darcy's law can only be introduced, while considering a fixed soil skeleton. But this automatically implies that the soil skeleton must have a zero volume strain. Consequently, (3.15) reduces to the common storage equation for free compressible viscous flow. This has not properly been recognised by De Wiest, Irmay and later, Bear (1972).

It is unfortunate that still many hydrologists accept these different approaches to include soil flexibility without a deep comprehension, the more since Cooper (1966) clearly had pointed out the incompleteness of Jacob's equation and the inconsistency of De Wiest's approach.
Cooper gave a correct analysis of the storage equation of pore water in a vertically deformable aquifer and his result coincides with equation (3.4), which in turn is valid for three dimensions. Also Bear (1972) discussed the above mentioned approaches, but he did not emphasize the essential discrepancies. Bear stated that the difference between the expressions (3.13) and (3.15) stem from different assumptions in the various derivations and suggested that, since in practical cases the storativity is determined by actual field experiments, its exact dependency on the compressibility of the medium need not to enter in the definition itself.

As previously outlined the differences are essential at least for educational purposes and violating the principle of constitutive invariance can not be considered as a justifiable assumption. It should not be used to expound a physical process.

A proper judgement of existing approaches inevitable for a deep comprehension of the process of groundwater flow was missing up to now and this section is meant to give an appropriate derivation of the storage equation providing understanding of the essential differences of comparable equations suggested in literature. In the proceeding discussion equation (3.11) will be evaluated into a form amenable to analytical and numerical computation.

3.c STAGNANT AND MOVING AIR BUBBLES

Fine saturated soils are usually classified as incompressible when subjected to rapid loading, as a consequence of their low permeability, but observations reveal that for any nature of soil a first compacting operation always results in an immediate considerable settlement. This is understandable for granular soils, but traditional theories can not explain this fact for saturated impermeable soils. Subsequent research showed that most of the fine soils contain gas in the form of micro-bubbles at a content of a few percent, see Ménard and Broise (1975). The volume of these bubbles plays a fundamental role in the coefficient of compressibility of such soils, which is extensively discussed in section 2.c.

Another particular feature affected by the presence of air bubbles is the permeability and correspondingly the specific discharge. To investigate whether the storage equation (3.11) is appropriate to include the existence of micro-bubbles two different situations are considered. First case, air bubbles sticking to solid particles or blocking a pore neck will move with the soil skeleton deformations. Second case, free air bubbles in the pore water are transported by the water at the water velocity. In a real soil both possibilities simultaneously occur, but here they are dealt with separatedly.

Verruijt (1969) considered a system of three phases: a porous soil skeleton, pore water and air bubbles in the pore volume. By postulating that the individual components each as a particulate medium satisfy the conservation of mass principle and by defining a proper equation of state for each phase, sufficient and necessary conditions are available to find the correlation betweeen the partial concentrations and the substantial velocities in the form of a storage equation identical to expression (3.11). This operation will only succeed by a suitable choice of the partial fluxes, i.e. the specific discharge, as will be outlined next.
Following Verruijt the conservation of mass requires for the soil skeleton (see equation (3.1)):

$$\frac{\partial}{\partial t}((1-n)\varrho_s) + \nabla\bullet((1-n)\varrho_s v) = 0\,, \qquad (3.16)$$

for the pore water:

$$\frac{\partial}{\partial t}(sn\varrho) + \nabla\bullet(sn\varrho w) = 0\,, \qquad (3.17)$$

and for the air content:

$$\frac{\partial}{\partial t}((1-s)n\varrho'') + \nabla\bullet((1-s)n\varrho'' w'') = 0\,, \qquad (3.18)$$

in which n denotes the porosity and s the pore water saturation. The velocities v, w and w″ are absolute velocities for the soil skeleton, the pore water and the air content, respectively.
Although the air is present in the form of bubbles, it will be considered as a coherent particulate medium, so that a mean velocity and density are existent.
The following constitutive relations are defined, for the soil:

$$d\varrho_s = 0\,, \qquad (3.19)$$

corresponding to the condition of individually incompressible solid particles, for the pore water:

$$d\varrho = \varrho\beta\, dp\,, \qquad (3.20)$$

and for the air content:

$$d\varrho'' = (\varrho''/p)\, dp\,. \qquad (3.21)$$

The effect of surface tension, vapour pressure and air solubility have been disregarded in this treatment in order to make the discussion not unnecessarily complicated.
Substitution of (3.19), (3.20) and (3.21) into (3.16), (3.17) and (3.18) gives:

$$-\frac{\partial n}{\partial t} + \nabla \bullet v - \nabla \bullet (nv) = 0\,, \tag{3.22}$$

$$\frac{\partial}{\partial t}(ns) + ns\beta\frac{\partial p}{\partial t} + \nabla \bullet (nsw) + ns\beta w \bullet \nabla p = 0\,, \tag{3.23}$$

$$\frac{\partial n}{\partial t} - \frac{\partial}{\partial t}(ns) + \frac{n(1-s)}{p}\frac{\partial p}{\partial t} + \nabla \bullet (n(1-s)w'') + \frac{n(1-s)}{p}w'' \bullet \nabla p = 0\,. \tag{3.24}$$

First consider the case of stagnant bubbles. Thus:

$$v = w''\,. \tag{3.25}$$

Eliminating w″ from (3.25) and (3.24) and adding (3.22) to this result, gives:

$$-\frac{\partial}{\partial t}(ns) + \frac{n(1-s)}{p}\frac{\partial p}{\partial t} + \nabla \bullet v - \nabla \bullet (nsv) + \frac{n(1-s)}{p}v \bullet \nabla p = 0\,. \tag{3.26}$$

Next, adding (3.23) renders this into:

$$n(s\beta + \frac{1-s}{p})\frac{\partial p}{\partial t} + \nabla \bullet v + \nabla \bullet (ns(w-v)) + \frac{n(1-s)}{p}v \bullet \nabla p + ns\beta w \bullet \nabla p = 0$$

or:

$$n(s\beta + \frac{1-s}{p})\{\frac{\partial p}{\partial t} + v \bullet \nabla p\} + \nabla \bullet v + \nabla \bullet (ns(w-v)) + ns\beta(w-v) \bullet \nabla p = 0\,. \tag{3.27}$$

Using the constitutive relation (3.20) the last two terms in equation (3.27) can be combined into the form:

$$\nabla \bullet (ns(w-v)) + ns\beta(w-v) \bullet \nabla p = (1/\varrho)\nabla \bullet (\varrho ns(w-v))\,. \tag{3.28}$$

The term between braces in equation (3.27) represents the substantial derivative with respect to time following the moving soil skeleton, thus:

$$\frac{\partial p}{\partial t} + v \bullet \nabla p = \frac{Dp}{Dt}\,. \tag{3.29}$$

The second term in equation (3.27) is equal to the substantial derivative of the volume strain with respect to time, see equation (3.5). Substitution of (3.28), (3.29) and (3.5) gives finally:

$$-\nabla \bullet (\varrho ns(w-v)) = \varrho\{\frac{D\varepsilon}{Dt} + n(s\beta + \frac{1-s}{p})\frac{Dp}{Dt}\}\,. \tag{3.30}$$

Comparison with the general nonlinear storage equation (3.11) reveals that for stagnant bubbles an appropriate specific discharge exists, according to:

$$q = ns(w-v)\,, \tag{3.31}$$

and a corresponding resultant compressibility for the air-water mixture, according to:

$$\beta' = s\beta + \frac{1-s}{p}\,. \tag{3.32}$$

Since stagnant bubbles do not contribute to the specific discharge, definition (3.31) makes sense. In view of equation (3.2) it is reduced by the saturation factor s, like a pore volume reduction. Hence, also the permeability according to Darcy's law has to be refined and an adjusted porosity factor according to:

$$\eta = (sn)^3/(1-n)^2 , \tag{3.33}$$

will do as a first approximation. Actually the local air-water friction is involved and possibly the tortuosity is increased by stagnant bubbles, see section 2.b. Only measurements can provide quantative data for this phenomenon, likewise represented by Zeller (1961) and Koning (1962).

Next consider the case of moving bubbles, transported by the pore water at a similar velocity, thus:

$$w = w'' . \tag{3.34}$$

Eliminating w'' from (3.24) and (3.34) and adding (3.23) to this result gives:

$$\frac{\partial n}{\partial t} + n(s\beta + \frac{1-s}{p})\frac{\partial p}{\partial t} + \nabla \bullet (nw) + n(s\beta + \frac{1-s}{p})\, w \bullet \nabla p = 0 .$$

Addition of equation (3.22) and introduction of (3.29) and (3.5) render this equation into:

$$\frac{D\varepsilon}{Dt} + n(s\beta + \frac{1-s}{p})\frac{Dp}{Dt} + \nabla \bullet (n(w-v)) + n(s\beta + \frac{1-s}{p})(w-v) \bullet \nabla p = 0 . \tag{3.35}$$

Verruijt (1969) considered the same case of conveyed micro-bubbles and defined a specific discharge according to (3.31). Developing the third term of (3.35) by:

$$\nabla \bullet (n(w-v)) = \{\, n(w-v) \bullet \nabla s + \nabla \bullet (ns(w-v)) \,\}/s ,$$

permits to introduce this definition of the specific discharge into the storage equation (3.35), which becomes:

$$s\frac{D\varepsilon}{Dt} + ns(s\beta + \frac{1-s}{p})\frac{Dp}{Dt} + n(w-v) \bullet \nabla s + \nabla \bullet (ns(w-v))$$

$$+ ns(s\beta + \frac{1-s}{p})(w-v) \bullet \nabla p = 0 . \tag{3.36}$$

An identical expression has been obtained by Verruijt, but it can not be further evaluated. The same applies to (3.35) itself, since the last two terms can not be combined into a proper term without a constitutive relation valid for the air-water pore content as one single substance. The discussion in section 2.c, in particular equation (2.33), made clear that in the absence of effects from surface tension, water vapour, air solubility and air diffusion the following constitutive relation exists for the considered air-water mixture:

$$d\varrho' = \varrho'(\beta + \frac{1-s}{sp})\, dp , \tag{3.37}$$

in which ϱ' represents an average density according to:

$$\varrho' = s\varrho + (1-s)\varrho'' . \tag{3.38}$$

Since the resultant compressibility in the last term of equation (3.35) does not coincide with the corresponding factor in the constitutive relation (3.37), equation (3.35) is not appropriate to derive a form identical to the general nonlinear storage equation (3.11), and no definition for the specific discharge whatsoever will compensate this fact.

Another approach will do. Because the pore water conveys the micro-bubbles, the pore content behaves as one single substance. Therefore, it is preferred to start with two conservation equations, one for the soil skeleton conform (3.16) and one for the air-water pore content, according to:

$$\frac{\partial}{\partial t}(n\varrho') + \nabla \bullet (n\varrho' w) = 0 \,. \tag{3.39}$$

Employing the constitutive relations (3.19) and (3.37) yields for (3.16) and (3.39) the following set:

$$-\frac{\partial n}{\partial t} + \nabla \bullet v - \nabla \bullet (nv) = 0 \,,$$

$$\frac{\partial n}{\partial t} + n(\beta + \frac{1-s}{sp})\frac{\partial p}{\partial t} + \nabla \bullet (nw) + n(\beta + \frac{1-s}{sp}) w \bullet \nabla p = 0 \,.$$

Adding both equations and introducing (3.29) and (3.5) gives:

$$\frac{D\varepsilon}{Dt} + n(\beta + \frac{1-s}{sp})\frac{Dp}{Dt} + \nabla \bullet (n(w-v)) + n(\beta + \frac{1-s}{sp})(w-v) \bullet \nabla p = 0 \,.$$

Almost identical to the unsuitable equation (3.35), but now it is possible to combine the last two terms using the constitutive relation (3.37) valid for the pore content as one single substance. The result is:

$$-\nabla \bullet (\varrho' n(w-v)) = \varrho' \{ \frac{D\varepsilon}{Dt} + n(\beta + \frac{1-s}{sp})\frac{Dp}{Dt} \} \,.$$

At this stage disregarding the free air content in the average density ϱ', that is: $\varrho' = s\varrho$, which is justified, because the air density is practically negligible compared to the water density, renders the storage equation into:

$$-\nabla \bullet (\varrho n s(w-v)) = \varrho \{ s\frac{D\varepsilon}{Dt} + n(s\beta + \frac{1-s}{p})\frac{Dp}{Dt} \} \,. \tag{3.40}$$

Comparison with the general nonlinear storage equation (3.11) and the storage equation for stagnant bubbles (3.30) reveals that for moving micro-bubbles a storage equation exists with a similar specific discharge conform definition (3.31), and with a similar resultant compressibility conform (3.32).
In section 2.c an expression for the resultant compressibility has been obtained including effects of surface tension, water vapour and air solubility. In the above mentioned approach this compressibility can be introduced into the definition of the constitutive relation for the air-water mixture, equation (3.37), without violating the present treatment resulting in the storage equation (3.40).
Transported micro-bubbles create no specific hindrance to the process of pore water flow. Hence, the permeability needs no adjustment when using Darcy's law.
Equation (3.40) differs from the storage equation for stagnant air bubbles (3.30) in the term concerning the substantial volume strain rate of the soil skeleton by a coefficient s, representing the saturation degree of the flowing air-water mixture.
For a soil containing stagnant bubbles at a volume of bn and conveyed micro-bubbles at a volume of $(1-s-b)n$ this coefficient becomes equal to $s/(1-b)$, see Fig. XII.

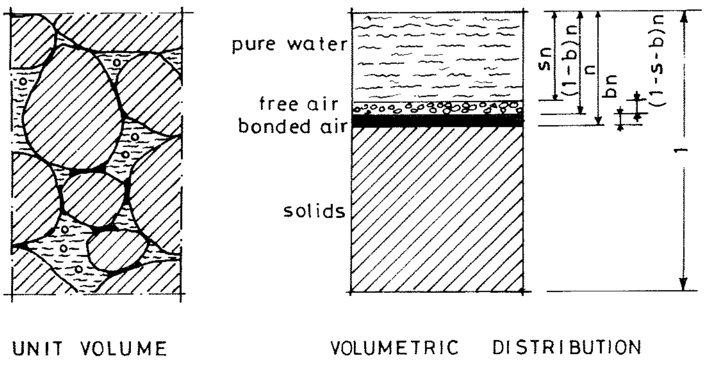


Fig. XII Volumetric distribution of solids, water, bonded and free air.

Introducing Darcy's law in this case necessitates an adjustment of the saturated permeability with respect to the relative volume of stagnant bubbles, resulting in a porosity factor according to:

$$\eta = ((1-b)n)^3/(1-n)^2 . \tag{3.41}$$

Also the tortuosity might increase by blocking bubbles, but stagnant air possibly is present on hydrophobic surfaces in micro cracks and crevices. It stands to reason that the tortuosity of the porous medium will not necessarily be increased by stagnant air.

Finally, the preceding discussion has proved that the pore water motion in a semi-saturated porous medium in which air is present in the form of micro-bubbles and pockets, can be represented by a storage equation according to:

$$-\nabla\bullet\varrho sn(w-v) = \varrho\left\{\frac{s}{1-b}\frac{D\varepsilon}{Dt} + ns\left(\beta + \frac{1-s}{sp}\right)\frac{Dp}{Dt}\right\},$$

or:

$$-\nabla\bullet(\varrho' q) = \varrho'\left\{\frac{1}{1-b}\frac{D\varepsilon}{Dt} + n\beta'\frac{Dp}{Dt}\right\}, \tag{3.42}$$

in which the specific discharge q is defined according to (3.2) and the resultant compressibility β' of the air-water mixture by (2.35) including effects of surface tension, water vapour and air solubility. Except a minor difference embedded in the coefficient $1/(1-b)$ equation (3.42) is identical to the general nonlinear storage equation (3.11), described in section 3b.
The attention is restricted to pore water with separate air bubbles, for which the saturation degree s is not less than 85%, see section 2.c. This figure is usually between 96% and 99%. Since: $1 \geq (1-b) \geq s \geq 0$, is valid, only a small error is made by approximating: $1/(1-b) = 1$.
As such the general storage equation for semi-saturated soils will be considered in the following discussion.

3.d A FLOW POTENTIAL INCLUDING IRROTATIONAL SOIL DEFORMATION

In this section the attention is directed to those situations where the groundwater flow can be fully described by a unique flow potential.

The pore water motion including air bubbles through a deformable porous medium is governed by the general storage equation (3.11):

$$-\nabla\bullet(\varrho' q) = \varrho'\left\{\frac{D\varepsilon}{Dt} + n\beta'\frac{Dp}{Dt}\right\}. \tag{3.43}$$

To arrive at a proper potential with the aid of this storage equation the specific discharge q and the volume strain ε will be expressed in terms of the fluid pressure p, which is a scalar quantity. Then, it is easy to define the potential, itself being a scalar.

After the discussion in section 2.a a linear relationship between the specific discharge q and the pressure gradient ∇p can be postulated, while the effect of different types of flow behaviour can be incorporated in the coefficient of permeability. The following relation is suggested in accordance to the previous formulas but with respect to the average density ϱ' of the air-water mixture:

$$q = -(k/\upsilon\varrho')\{\nabla p + \varrho' g\,\nabla z\}, \tag{3.44}$$

where the intrinsic permeability k is introduced according to (2.12) and z denotes the vertical coordinate. By postulating (3.44) as a flow law for porous flow including air bubbles, it is tacitly assumed that the presence of air bubbles does not affect its validity, except for a modification of the intrinsic permeability k in the porosity factor conform (3.41) as far as stagnant air bubbles are concerned. Inserting the approximation $\varrho' = s\varrho$, renders (3.44) into a flow law for an air-water pore content according to:

$$sq = -(k/\upsilon\varrho)\{\nabla p + s\varrho g\,\nabla z\}.$$

Verruijt (1969) suggested a similar expression, but he disregarded the factor s in the last term. This is not in agreement with hydrostatic equilibrium requirements, if the pressure p is regarded as an average quantity with respect to the pore cross-sectional area.
Equation (3.44) provides a relation between the pore pressure p and the specific discharge q, but at the same time the basic parameters involved are directly or indirectly dependent on the pore pressure as well. For the intrinsic permeability k depends on the porosity by means of equation (2.13), the density ϱ' changes with the pore pressure p according to (2.20) and the kinematic viscosity υ possibly is a function of the pore pressure.

Concerning the permeability an assumption is posed at this stage, in that the tortuosity will not alter, when the porosity changes at pore pressure fluctuations. In other words the flow configuration is supposed not to be affected by porous medium deformations. Therefore, the dependence of the intrinsic permeability on alterations in the flow system is limited to solely the porosity factor, conform equation (2.15), which also may include the influence of variations in the effective pore cross-sectional areas.
Concerning the viscosity υ it is known that this feature only weakly varies with pressure, but stronger with temperature.
Groundwater has a remarkably constant temperature, since the porous medium conserves it from natural heat sources, except vulcanic activities. In the case of infiltration of industrial waste and cooling water the temperature dependence of the viscosity υ might become significant (υ varies by 3% per °C approximately).
The pressure dependence of the viscosity of natural fresh water is depicted in Fig. XIII. Since the groundwater temperature is usually at about 10 °C, its kinematic viscosity υ is rather invariable to moderate pressure fluctuations induced by natural disturbances, like rain (infiltration), river and sea level changes and human activities for water supply and building purposes. Therefore, the viscosity υ of the pore water will be considered as a constant.

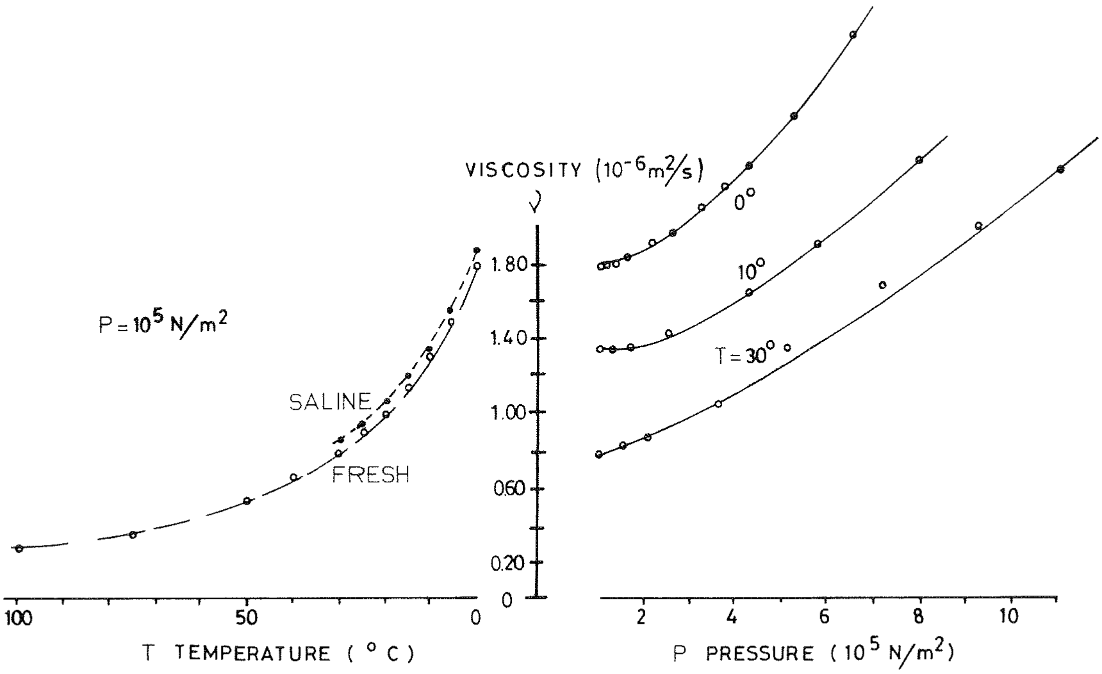


Fig. XIII Viscosity of natural water versus pressure and temperature.

Introduction of the equation of state for the linear flow motion (3.44) yields for the lefthand side of the storage equation (3.43):

$$- \nabla \bullet (\varrho' q) = \nabla \bullet (k/\upsilon) \{ \nabla p + \varrho' g \nabla z \}$$

$$= \nabla (k/\upsilon) \bullet \{ \nabla p + \varrho' g \nabla z \} + (k/\upsilon) \nabla \bullet \{ \nabla p + \varrho' g \nabla z \} . \qquad (3.45)$$

The intrinsic permeability k depends on the volume strain ε by means of the porosity factor η.

From (2.13) and (2.15) the following variational relation is derived:

$$dk = \frac{k}{\eta}\, d\eta = k \frac{\varkappa}{n} \frac{dn}{(1-n)},$$

and with equation (3.8) providing the coupling between the porosity n and the volume strain ε, this becomes:

$$dk = k(\varkappa/n)\, d\varepsilon, \tag{3.46}$$

where $\varkappa$ is defined as the amplification factor, see section 2.b. Using equation (3.46) equation(3.45) becomes:

$$-\nabla\bullet(\varrho' q) = (k/\upsilon)\left[\frac{\varkappa}{n}\nabla\varepsilon \bullet \{\nabla p + \varrho' g\, \nabla z\} + \nabla^2 p + \nabla(\varrho' g)\bullet \nabla z\right].$$

Applying the equation of state for the air-water pore content acoording to (2.20) renders this into:

$$-\nabla\bullet(\varrho' q) = (k/\upsilon)\left[\nabla^2 p + \frac{\varkappa}{n}\nabla p \bullet \nabla\varepsilon + \frac{\varrho' g}{n}\{\varkappa\nabla\varepsilon + n\beta'\nabla p\}\bullet\nabla z\right]. \tag{3.47}$$

Eliminating q from (3.43) and (3.47) gives:

$$\frac{K}{\varrho' g}\left[\nabla^2 p + \frac{\varkappa}{n}\nabla p\bullet\nabla\varepsilon + \frac{\varrho' g}{n}\{\varkappa\nabla\varepsilon + n\beta'\nabla p\}\bullet\nabla z\right] = \frac{D\varepsilon}{Dt} + n\beta'\frac{Dp}{Dt}, \tag{3.48}$$

where the hydraulic permeability K is introduced conform definition (2.12).

A similar approach is presented by Barends (1978) for a case slightly different as discussed here. The amplification factor $\varkappa$ was put equal to 2 in order to minimize the complexity of the analysis and in stead of assuming υ being a constant the treatment was based on keeping the dynamic viscosity μ a constant. The latter fact might seem essential, but straight forward analysis shows that a comparable result is obtained in case μ is kept a constant. A storage equation is obtained similar to equation (3.48), save a coefficient $2n\beta'$ in stead of $n\beta'$ in the last term on the lefthand side.

The preceding analysis explains that, if the effect of porous matrix deformations reflected in alterations in the porosity is considered, it is fundamental to include variations in the permeability as well, this itself being a function of the porosity. Altough much is written about the physical aspects of the storage equation, this fact is only sometimes mentioned, but up to now not incorporated in the basic analysis. Here, it is shown in a simple form by equation (3.48) to what extent the common disregard of permeability variations violates the generality of the storage equation. It appears to be of the same order of influence as represented by $D\varepsilon/Dt$ relatively to $n\beta' Dp/Dt$. Therefore, it makes sense to include this phenomenon in a general storage equation.

Next, the volume strain ε appearing in (3.43) has to be expressed in terms of the pore pressure p. After the discussion in section 2.d dealing with the deformation behaviour of the soil skeleton the elementary equilibrium condition for a semi-saturated, perfectly elastic porous medium provides a simple relation between the volume strain ε and the pore pressure p, see equation (2.66):

$$\varepsilon = \alpha(p - F), \tag{3.49}$$

which includes as a result of integration the pressure function F. This function is related to the rotational part of the soil deflections, following expression (2.69). In some geometries there are no such rotations due to symmetry requirements and F becomes a constant in space, which still can be a function of time to be determined from the boundary conditions. In general rotations do not vanish, as became clear from the discussion in section 2.d.

Correct solutions for groundwater flow are obtained by solving simultaneously the storage equation and satisfying the equilibrium conditions. However, the general nonlinear storage equation (3.48) is quite complicated. Although accessible with modern numerical techniques, the present treatment is restricted. Rotational effects will be disregarded, thus: $\nabla F = 0$. In that case equation (3.49) reflects the relation between ε and p, while F is only a function of time. Eliminating ε from (3.48) and (3.49) results in the following expression:

$$\frac{K}{\varrho' g}\left[\nabla^2 p + \frac{\alpha \varkappa}{n}(\nabla p)^2 + \frac{\varrho' g}{n}\{ \alpha\varkappa + n\beta' \}\frac{\partial p}{\partial z}\right] = (\alpha + n\beta')\frac{Dp}{Dt} - \alpha\frac{DF}{Dt}, \tag{3.50}$$

written in terms of p only.
In section 3.e special attention will be paid to the influence of convective terms. Here, the velocity of the soil skeleton motion is considered to be small and consequently, the substantial derivatives with respect to time can be replaced by local derivatives: $D()/Dt \sim \partial()/\partial t$. Equation (3.50) becomes:

$$\frac{K}{\varrho' g}\left[\nabla^2 p + \frac{\alpha \varkappa}{n}(\nabla p)^2 + \frac{\varrho' g}{n}\{ \alpha\varkappa + n\beta' \}\frac{\partial p}{\partial z}\right] = (\alpha + n\beta')\frac{\partial p}{\partial t} - \alpha\frac{\partial F}{\partial t}. \tag{3.51}$$

Disregarding rotational deformations indicates a restriction to the class of porous flow problems. However, horizontal flow in deformable aquifers, vertical one-dimensional flow, cylindrically and spherically symmetrical flow fields are included. In some flow fields the rotational part of the deformation behaviour can be disregarded as a second order effect.

Hemholtz' theorem states that for irrotational vector fields a scalar potential exists describing the field completely. Hence, since pore water flow is related to a deforming medium which itself does not contain rotations, a unique potential must exists. Consider the following potential Φ:

$$\Phi = z + (1/g)\int_{p_F}^{p} \frac{1}{\varrho'(\ell)}d\ell. \tag{3.52}$$

This expression is identical to (2.74) except for ϱ' in stead of ϱ. This potential resembles Hubbert's potential, but the density ϱ' to which it refers to, is the average density of the air-water mixture in the pores, and the lower boundary contains a pressure p_F, which is not necessarily the reference (atmospheric) pressure.

Because the density ϱ' is a function of the pore pressure only, see section 2.c, derivation with respect to time, using Leibnitz' rule, results in:

$$\frac{\partial \Phi}{\partial t} = \frac{1}{\varrho' g}\frac{\partial p}{\partial t} - \frac{1}{\varrho'_F g}\frac{\partial p_F}{\partial t} \cong \frac{1}{\varrho' g}\left(\frac{\partial p}{\partial t} - \frac{\partial p_F}{\partial t}\right). \tag{3.53}$$

Here, it is assumed that the difference between the density ϱ' and ϱ'_F is negligible.

In regard of equation (3.51) this potential is suitable to absorbe the function F which only is a function of time in irrotationally deforming porous media. Hence,

$$p_F = \frac{\alpha}{\alpha + n\beta'}F(t), \tag{3.54}$$

renders equation (3.53) into the following form:

$$\varrho' g(\alpha + n\beta')\frac{\partial \Phi}{\partial t} = (\alpha + n\beta')\frac{\partial p}{\partial t} - \alpha\frac{\partial F}{\partial t}, \tag{3.55}$$

which represents the right hand side of the general storage equation (3.51).

Expression (3.52) can be written in a variational form:

$$\varrho' g\, d\Phi = d(p - p_F) + \varrho' g\, dz\,.$$

Because p_F, according to (3.54), is only a function of time, this variational relation yields the following equalities, when employing (2.20):

$$\nabla^2 p = \beta' (\nabla p)^2 + \varrho' g \, \nabla^2 \phi ,$$

$$(\nabla p)^2 = (\varrho' g)^2 \{ (\nabla \phi)^2 - 2\partial\phi/\partial z + 1 \} ,$$

$$\partial p/\partial z = \varrho' g \{ \partial\phi/\partial z - 1 \} .$$

With these expressions and (3.55) the storage equation (3.51) becomes:

$$K\left[\nabla^2\phi + \frac{\varrho' g}{n}(\alpha\varkappa + n\beta')\{ (\nabla\phi)^2 - \partial\phi/\partial z \} \right] = \varrho' g(\alpha + n\beta')\partial\phi/\partial t , \qquad (3.56)$$

written in terms of a potential ϕ. The fact that pressure induced density variations will not give rise to rotations in the flow pattern is discussed in section 3.e.

Potential ϕ defined by (3.52) gives a more complete picture of the phenomenon of porous flow. It includes presence of air in the pore water and takes into account only that part of the generated pore pressure which causes porous flow, i.e. a pore fluid motion through a (deforming) skeleton. This can easily be understood by assuming $\beta' = 0$, making ϱ' a constant. Thus equation (3.52) becomes in this case, using (3.54):

$$\phi = z + (p - p_F)/\varrho' g = z + (p - F)/\varrho' g .$$

In respect of equation (3.49) the potential ϕ is directly related to the actual volume strain ε, and consequently to the boundary conditions. The above expression holds for $\alpha = 0$, valid in case of a stiff soil skeleton.
Contrary to this result the potential (the piezometric head) is commonly defined by:

$$\phi = z + p/\varrho g ,$$

which is valid for porous flow fields, where $F = 0$. This condition is not always satisfied. It is then engineering intuition to realize that ϕ covers only the pore pressure related to the pore water motion, while an instantaneous overall pressure originated from equilibrium requirements is not included.

By introducing the pressure p_F in the expression for the potential ϕ, equation (3.52), this point is clarified. Furthermore, the usual assumption applied to general aquifer flow behaviour, that is the total stress invariance principle, i.e. $F = 0$, can be extended to more general boundary conditions without violating the generality of the storage equation (3.56), as long as the rotational part of the soil deformation can be disregarded.

An example of this fact concerns barometric fluctuations causing water level changes in an open piezometer cased in a confined aquifer, see Jacob (1950).
Consider a variation dp_o in the load on the confined aquifer. Hence,

$$- d\sigma = dp_o .$$

Using expression (3.49) and Terzaghi's effective stress principle, equation (2.53), one obtains:

$$dp_o = - d\sigma = - d\sigma' + dp = - (1/\alpha)\, d\varepsilon + dp = dF .$$

Disregarding the small amount of pore water flowing into and out off the piezometer, there is no porous flow and in correspondence to (3.51) the following holds:

$$(\alpha + n\beta')\frac{dp}{dt} - \alpha\frac{dp_o}{dt} = 0 .$$

If the level fluctuations due to the atmospheric pressure variations are denoted by dh, then vertical equilibrium in the piezometer requires:

$$\frac{dp}{dt} = \frac{dp_o}{dt} + \frac{1}{\varrho' g}\frac{dh}{dt} .$$

Eliminating dp/dt from the last two equations yields the response of the water level in the piezometer to barometric variations:

$$\frac{dh}{dt} = - \frac{n\beta'}{\alpha + n\beta'} \frac{dp_o}{dt} .$$

It appears that, if the pore water is incompressible, no fluctiations are perceived. Moreover, in principal it is possible to measure the aquifer flexibility α with a piezometer.

This example clarifies that the pressure function F, or p_F, is related to that part of the pore pressure which does not give rise to porous flow. In the same manner, equation (2.70) can be covered by the same potential ϕ, defined according to (3.52). The pressure function p_F takes then the form:

$$p_F = - \frac{\alpha'}{\alpha' + n\beta'} \sigma(t) ,$$

in which α' represents the isotropic compressibility of the porous medium and σ the isotropic total stress, assumed only to be a function of time.

It would make no practical sense whatsoever to perform the extensive analysis to arrive at the nonlinear equation (3.56), if it were not amenable to a basic analytical solving procedure.
In chapter 4.a general method to solve this equation without additional simplifications is presented.

3.e ROTATIONS, CONVECTIVE TERMS AND LARGE STRAINS

Gravitational rotations

The behaviour of fluids in porous media has been traditionally described in terms of macroscopic variables, i.e. quantities averaged over microscopic distances in the order of the pore size. For instance, the specific discharge is a flow per unit macroscopic area. This view of porous materials is quite practical, because it facilitates calculation of average flow in various geometries. Furthermore, it is now well established that the porous material being complex at microscopic scale can be characterized by relatively few macroscopic parameters in many circumstances.
When considering the flow of fluid elements, i.e. the distribution of marked particles or the flow of constituents, it becomes necessary to distinguish the various constituents of the fluid, which occupies different regions of the flow system.
Not all the fluid elements move at the same velocity. Instead, the motion is diffuse as a result of molecular diffusion and of the actual variation of local fluid velocities in the pore structure. For example, the uplift of air bubbles in the pore fluid due to boyency forces induces a locally rotational flow.

In a wide range of natural situations this diffusive type of porous flow, called dispersion, may be well described macroscopically and the significant parameter is known as the dispersivity. It is observed that dispersion is not isotropic, being essentially greater in the direction of the carrying flow than perpendicular to it. The phenomenon of dispersion has been extensively studied and it is described by Bear (1972), who refers to it as hydrodynamical dispersion.

A special situation occurs when the flow parameters, such as viscosity, density and permeability, are variable. In such cases the pattern of flow may change as the flow progresses. These changes can become such that spatial variations of the velocity are amplified and protuberances or 'fingers' develop. Similar instabilities can also arise through the action of gravity in the distribution of fluids of variable density.
A review of regions of instability with respect to the flow rate, viscosity gradient and density gradient is given by Heller (1965).

The case of gravitational instabilities has been treated by De Josselin de Jong (1960, 1969), who explained the appearance of rotations due to density gradients and provided a theory of generating functions by which the fluid displacements can successively be traced. The density variation considered originates from different fluid elements at different locations in the flow field, but the fluid elements themselves are assumed to preserve their density during displacement. They are incompressible. Hence conservation of mass requires that the specific discharge q (notably a volumetric flux) obeys the continuity equation:

$$\nabla \cdot q = 0 .$$

In this form it is essentially different in comparison to the continuity equation for a compressible homogeneous fluid described in the previous sections, equation (3.11), which gives for a stationary flow in an undeformable porous medium:

$$\nabla \cdot (\varrho q) = 0 .$$

Nonetheless, the existence of pressure gradients will cause density gradients, which will be responsible to a rotational character of flow in the same manner as has been explained by De Josselin de Jong (1969, 1979).
Since Darcy's law applies to irrotational flow of fluids through porous media, it is instructive to evaluate the specific influence of density flow on the average porous flow.

A vector field q is irrotational when everywhere the curl vanishes, that is:

$$\nabla \times q = 0 . \qquad (3.57)$$

For groundwater flow the vector field is represented by the filter velocity q. Although this quantity is related to the moving porous skeleton, condition (3.57) is sufficient for the flow to be irrotational, if the soil deformation field itself is considered to be free of rotations.
The curl of the filter velocity q is elaborated. From equation (3.44) one obtains:

$$\nabla \times q = -\nabla \times \left[(k/\varrho' \upsilon) \{ \nabla p + \varrho' g \nabla z \} \right] .$$

Since the kinematic viscosity υ is considered to be a constant, see section 3.d, and because ∇z itself is obviously rotation free, this equation becomes:

$$\begin{aligned} \nabla \times q &= -(1/\upsilon) \nabla \times \left[k \nabla p / \varrho' \right] \\ &= -(1/\upsilon) \{ (1/\varrho') \nabla \times (k \nabla p) + \nabla (1/\varrho') \times (k \nabla p) \right] \\ &= -(1/\varrho' \upsilon) \{ \nabla \times (k \nabla p) - (1/\varrho') \nabla \varrho' \times (k \nabla p) \} . \end{aligned}$$

The pore pressure p is a single-valued physical scalar quantity and therefore free of rotation. The intrinsic permeability k is a scalar quantity in isotropic porous media, but in anisotropic media it represents a symmetrical tensor, see section 2.b. The product $k\nabla p$ denotes also a vector field which is irrotational. In this respect the above equation reduces to:

$$\nabla \times q = \{ \nabla \varrho' \times (k \nabla p) \} / \varrho'^2 \upsilon . \qquad (3.58)$$

In general the density ϱ' may vary in space through independent variations of pressure p, concentration c and temperature T.
Therefore:

$$d\varrho = \frac{\partial \varrho}{\partial p} dp + \frac{\partial \varrho}{\partial c} dc + \frac{\partial \varrho}{\partial T} dT .$$

The last two terms on the right hand side may cause a rotational component. Such effects due to temperature differences have already been mentioned by Hubbert (1940). Rotations in multiple fluids caused by a difference in molecular weight can be identified with variations due to concentration, and the character of this type of flow is discussed by De Josselin de Jong (1969).

Next, assume that the air-water density ϱ' is a function of the pressure p only, thus:

$$\varrho' = \varrho'(p) \tag{3.59}$$

rendering the considered equation (3.58), with:

$$\nabla\varrho' = \frac{d\varrho'}{dp}\nabla p ,$$

into:

$$\nabla \times q = \{ \nabla p \times (k\nabla p) \} \frac{d\varrho'}{dp} / \varrho'^2 \upsilon .$$

The quotient $d\varrho'/dp$ is a scalar, non-zero quantity. To satisfy condition (3.57), the following equation must hold:

$$\nabla p \times (k\nabla p) = 0 . \tag{3.60}$$

Only for isotropic porous media, where k is a scalar, the vectors ∇p and $k\nabla p$ are collinear and their outer product vanishes everwhere.
Consequently, the entire flow field is irrotational and can be represented by a single potential in isotropic porous media. In conclusion, pressure induced density variations only generate rotations in the flow pattern, if the porous medium is anisotropic.

This represents an unfortunate restriction to the presented theory, since soil sediments are anisotropic by nature.
However, the influence of this effect is not predominant, because of two reasons. First, the collinearity between the vectors $k\nabla p$ and ∇p is a fact, if one of them is in a principle direction of anisotropy; then condition (3.60) is valid. Second, the influence of density rotations is of second order importance, which can be understood from elaborating (3.58). In accordance to equation (2.20) the following holds:

$$\beta' = \frac{d\varrho'}{\varrho' dp} ,$$

and the magnitude of the rotational part becomes:

$$|\nabla \times q| = (\beta'/\varrho'\upsilon)\ |\nabla p|\ |k\nabla p|\ \sin\theta,$$

where θ is the angle between the pressure gradient ∇p **and the vector** $k\nabla p$.

Convective terms

In section 3.d the substantial derivative $D\phi/Dt$ was replaced by the local derivative $\partial\phi/\partial\phi$. This assumption is valid in the case where the convective term $v \cdot \nabla\phi$ remains relatively small. For a large displacement rate this term pretenses more significance. In order to be able to give an estimate of this effect the character of the convective term will be considered.

Recalling the definition of the potential ϕ, according to expression (3.54), the substantial derivative becomes, employing Leibnitz' rule:

$$\frac{D\phi}{Dt} = \frac{Dz}{Dt} + \frac{1}{\varrho' g}\frac{Dp}{Dt} - \frac{\alpha}{\alpha + n\beta}\frac{1}{\varrho'_F g}\frac{DF}{Dt}, \qquad p_F = \frac{\alpha F}{\alpha + n\beta'},$$

where ϱ'_F denotes the density at $p = p_F$.

In the above expression the coordinate z is measured with respect to a reference, z = 0, fixed in space. Assume that the location of an elementary volume of porous soil originally at position, z = 0, does not move. Such a situation can occur at the bottom of a deformable aquifer on a rigid base. The location of any elementary volume having an original position at $z>0$, will vary accordingly to the deformations, and consequently, the coordinate z referring to the actual position of the considered volume will be a function of time.

Next, assuming that the difference between ϱ' and ϱ'_F is negligible, the following expression can be obtained from equation (3.61):

$$(\alpha + n\beta')\frac{Dp}{Dt} - \alpha\frac{DF}{Dt} = \varrho' g(\alpha + n\beta')\{\frac{D\phi}{Dt} - \frac{Dz}{Dt}\}. \qquad (3.62)$$

Elaborating the substantial derivative at the right hand side, gives:

$$(\alpha + n\beta')\frac{Dp}{Dt} - \alpha\frac{DF}{Dt} = \varrho' g(\alpha + n\beta')\frac{\partial\phi}{\partial t} + \varrho' g(\alpha + n\beta')\, v\bullet\nabla(\phi - z), \qquad (3.63)$$

where v denotes the velocity of the porous medium. Note that Dz/Dt equals $v\bullet\nabla z$.
The form (3.63) has been introduced in the derivation of the general storage equation (3.56), except the last term, which is a mere consequence of convection caused by soil skeleton displacements.

Using (3.62) the general storage equation (3.56) can be expressed in the form:

$$K[\nabla^2\phi + \frac{\varrho' g}{n}(\alpha\varkappa + n\beta')(\nabla\phi - \nabla z)\cdot\nabla\phi] = \varrho' g(\alpha + n\beta')[\frac{\partial\phi}{\partial t} + v\cdot(\nabla\phi - \nabla z)],$$

or:

$$\nabla^2\phi + m\{(\nabla\phi)^2 - \partial\phi/\partial z\}(1 - v\cdot(mc\nabla\phi)^{-1}) = \frac{1}{c}\frac{\partial\phi}{\partial t}, \qquad (3.64)$$

where: $m = \varrho' g(\alpha\varkappa + n\beta')/n$,

$c = K/(\varrho' g(\alpha + n\beta'))$.

The last term on the left hand side represents the convective effect. It is small when the absolute pore fluid velocity w is much larger than the absolute skeleton velocity v. Since $\varkappa>1$, see section 2.b, elaboration of this term yields, employing (3.44), (3.52) and (3.2):

$$\frac{v}{mc\nabla\phi} < \frac{nv}{K\nabla\phi} = \frac{nv}{q} = \frac{nv}{n(w - v)} = \frac{1}{(\frac{w}{v} - 1)}.$$

In most practical cases $w/v >> 1$, and consequently the convective effect is negligible. In some occasions it can be included in the general storage equation without increasing its complexity.

Such a situation occurs for aquifers conveying mainly horizontal flow ($\partial\phi/\partial z \sim 0$), while horizontal deformations are precluded. Consequently, horizontal velocities of the soil skeleton vanish, and the substantial derivative of the potential ϕ becomes:

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + v\bullet\nabla\phi = \frac{\partial\phi}{\partial t}. \qquad (3.64)$$

Therefore, the storage equation, expression (3.50), using (3.52) and (3.62) becomes:

$$K[\nabla^2\phi + \frac{\varrho' g}{h}(\alpha\varkappa + n\beta')(\nabla\phi)^2] = \varrho' g(\alpha + n\beta')\{\frac{\partial\phi}{\partial t} - \frac{Dz}{Dt}\}, \qquad (3.65)$$

where the operator ∇ refers to horizontal coordinates only. Introduction of a new variable $\tilde\psi$, defined according to:

$$\tilde\psi = \phi - \int(\frac{Dz}{Dt})\,dt,$$

renders expression (3.65) into the following storage equation:

$$K\left[\nabla^2\tilde{\psi} + \frac{\varrho' g}{n}(\alpha\varkappa + n\beta')(\nabla\tilde{\psi})^2\right] = \varrho' g(\alpha + n\beta')\frac{\partial\tilde{\psi}}{\partial t}. \tag{3.66}$$

The potential $\tilde{\psi}$ is now measured with respect to the vertical convective coordinate z. An identical equation, not including $\alpha\varkappa$, has been derived by Cooper (1966). It describes horizontal porous flow in a vertically deformable aquifer including second order terms and vertical convective terms.
Since the potential $\tilde{\psi}$ corresponds to the piezometric head with respect to the substantial vertical coordinate z, the reference of the coordinate system is not essential anymore. This quantity is exactly measured with a potentiometer which moves with the soil, and therefore, automatically includes the vertical convective term.

Another particular case exists: vertical flow in a deformable porous medium. The equation of motion in terms of the potential ϕ is, according to the previous discussion:

$$K\left[\frac{\partial^2\phi}{\partial z^2} + \frac{\varrho' g}{n}(\alpha\varkappa + n\beta')\left\{\left(\frac{\partial\phi}{\partial z}\right)^2 - \frac{\partial\phi}{\partial z}\right\}\right] = \varrho' g(\alpha + n\beta')\left\{\frac{\partial\phi}{\partial t} + v\left(\frac{\partial\phi}{\partial z} - 1\right)\right\}, \tag{3.67}$$

where v represents the vertical velocity of the porous medium. Assume a reference at $z=0$, where also $v=0$ holds. In that case, if the strain ε is uniformly distributed, the following identity can be derived, see equation (3.5):

$$\frac{D\varepsilon}{Dt} = \frac{Dz}{zDt} = \frac{v}{z}. \tag{3.68}$$

In regard of equation (3.49) this implies:

$$v = \alpha z\left(\frac{Dp}{Dt} - \frac{DF}{Dt}\right). \tag{3.69}$$

Noting that Dz/Dt equals v, the vertical soil velocity, equation (3.62) becomes:

$$(\alpha + n\beta')\frac{Dp}{Dt} - \alpha\frac{DF}{Dt} = \varrho' g(\alpha + n\beta')\left\{\frac{\partial\phi}{\partial t} + v\left(\frac{\partial\phi}{\partial z} - 1\right)\right\}. \tag{3.70}$$

Eliminating Dp/Dt from equations (3.69) and (3.70) using $DF/Dt = \partial F/\partial t$, because $\nabla F = 0$, results in:

$$v = \alpha z\frac{\varrho' g\dfrac{\partial\phi}{\partial t} - \dfrac{n\beta'}{\alpha + n\beta'}\dfrac{\partial F}{\partial t}}{\left(1 - \alpha z\varrho' g\left(\dfrac{\partial\phi}{\partial z} - 1\right)\right)}. \tag{3.71}$$

Finally, substitution in the storage equation (3.67) gives:

$$K\left[\frac{\partial^2\phi}{\partial z^2} + \frac{\varrho' g}{n}(\alpha\varkappa + n\beta')\left\{\left(\frac{\partial\phi}{\partial z}\right)^2 - \frac{\partial\phi}{\partial z}\right\}\right] = \varrho' g(\alpha + n\beta')(1 + \omega)\frac{\partial\phi}{\partial t} - n\beta'\omega\frac{\partial F}{\partial t}, \tag{3.72}$$

where:

$$\omega = \frac{-1}{1 - 1/\left[\alpha z\varrho' g\left(\dfrac{\partial\phi}{\partial z} - 1\right)\right]}.$$

The coefficient ω is a smooth function of ϕ, because practical values for α, ϱ', and $\partial\phi/\partial z$ make ω small, i.e. $\omega << 1$. For example, $\alpha = 10^{-7}$ m²/N, $\varrho' = 10^4$ N/m³, $0 < z < 10$ m, $0 < \partial\phi/\partial z < 1$, give: $0 < \omega \leq 0.11$. Since soil is relatively stiff, the convective effect is usually small.

Large strain

The convective term appearing in the storage equation (3.64) is a result of the motion of a soil element caused by the deformation rate of the soil skeleton. It has been assumed that the porous medium deforms according to a perfectly elastic material. Furthermore, rotations in the deformation behaviour have categorically been disregarded. The question arises whether the deformation behaviour assumed is representative to natural soils.

Observed soil deformation behaviour is typically nonlinear. For example, based on much experimental evidence Vermeer (1978) suggested for sands a constitutive model under general stress and strain conditions, clearly revealing the nonlinear character.
In literature many such models incorporating a measured stress-strain relationship are described, but although the numerical problems are fairly well solved, the unaquaintedness of the model's relevance is still the main reason of a reserved attitude towards application in soil mechanics practice, see Roscoe (1970).

The assumption of a perfectly linear model, whereas soils exhibit a nonlinear behaviour by nature, implies a rigorous restriction to the generality of the storage equation, expression (3.56). It makes sense, however, to suppose that for small strains a linear deformation model can be regarded as reasonable, in that it gives a first approximation of the influence of the flexibility of soil on pore pressure generation, in particular when the soil properties are related to the initial stress situation. In this case the storage equation (3.56) may be useful to give some information about several second order effects, otherwise it should be applied under great reservation, because nonlinearity caused by the soil deformation assumes predominance.

Concerning convective terms in the general storage equation (3.64), which has been derived with a linear soil model, it is to be noted that in this manner these effects more likely correspond to a deformation rate related to an integral result of many small strains than to large strains. Convective terms due to large strain rates should be accounted for in combination with a realistic and therefore, nonlinear soil deformation model.

A very few trials have been published dealing with nonlinear soil deformation including the convective effects. De Leeuw and Abbot (1966) presented a numerical method for one-dimensional nonlinear consolidation of a multi-layered soil applying a time step continuously adjusted to account for the nonlinearity. The convective effect was covered by updating spatial positions every step. Gibson et al (1967) suggested a theory using a Lagrangian formulation to describe general nonlinear one-dimensional consolidation problems to trace the correct upper boundary motion of a deforming phreatic aquifer. They obtained a simple storage equation, but the flow field border is now the unknown feature (geometric nonlinearity). A discussion about geometric nonlinearity, in particular the class of transient phreatic flow problems is presented in chapter 5.

4. ELEMENTARY SOLVING METHODS

Summary of chapter 4

The previous analysis concerns the physical aspects of the process of porous flow through a deformable porous medium. The result is a general storage equation in the form of a nonlinear partial differential equation including several second order effects. To arrive at this result represents a first step. The next step, equally essential, is to solve the posed equation mathematically. Nowadays, no equation seems to be unsolvable with the aid of advanced numerical techniques and fast computers. However, to provide a deeper understanding of the consequences of nonlinearity in porous flow, it is recommendable to derive closed form solutions. Then, their parametric form will permit to evaluate immediately these effects in comparison to linear solutions. Of course, the class of problems amenable to analytical solving procedures is limited, in that they usually concern a homogeneous medium having suitable boundaries and appropriate boundary conditions. Nevertheless, the real character of the physical process being investigated can be elucidated.

In section 4.a it is shown that the nonlinear storage equation can be transformed into a linear differential equation by a simple substitution. Only the vertical contribution contains a variable coefficient. A general proof is given.
A particular result is that for horizontal flow in aquifers the general nonlinear storage equation is exactly identical to the conventional storage equation but now in terms of the so-called extensive potential, which appears to be the exponential of the piezometric head multiplied by the coefficient of nonlinearity. Thus, all the physical nonlinearity provided in the foregoing chapters is embedded in a very simple transformation. Furthermore, solutions obtained for the linear type of aquifer flow are easily extended to the nonlinear one.

In section 4.b the general solution of vertical nonlinear porous flow is investigated. A known procedure is possible, to wit, the Euler substitution. However, another method is suggested. It is shown that the Mellin transformation technique can be applied. It permits to handle general boundary conditions, while the incorporation of these conditions is realized by defining the extended potential within a finite domain. An advantage of this approach is that the inverse Mellin transform becomes identical to a Laplace inverse transform, many of which are listed in literature.

In section 4.c some characteristic cases of horizontal flow problems are discussed. The nonlinearity can be extended to phreatic aquifers as well. Special attention is payed to the area of influence related to a disturbance in the flow field of a horizontal aquifer. The nonlinearity actually provides a reduction between 10 and 20% on this area when estimated according to the linear theory. A steady state solution for semi-confined aquifers is given.

In section 4.d several solutions of nonlinear vertical porous flow are presented, applying the finite Mellin transformation technique. Cyclic vertical flow, a unit shock and a unit step disturbance are considered. In the last case the steady state solution contains a physical anomaly, which is a direct consequence of the pressure dependent density of the pore fluid, assumed to correspond to an exponential relation. The real behaviour of the density at large depths, and subsequently at large pressures, is not well described by the stated formula keeping the compressibility of the pore fluid a constant.

4.a A NONLINEAR TRANSFORMATION; THE EXTENSIVE POTENTIAL

In section 3.d the general storage equation of porous flow through an irrotationally deforming porous medium in terms of a potential ϕ, defined by:

$$\phi = z + \int_{p_F}^{p} (1/\varrho' g) d\ell , \quad p_F = \alpha F(t)/(\alpha + n\beta') ,$$

has been derived. This equation can be written in the following form, see (3.56):

$$\nabla^2 \phi + m \left\{ (\nabla \phi)^2 - \partial\phi/\partial z \right\} = (1/c)\, \partial\phi/\partial t , \tag{4.1}$$

where:

$$\begin{aligned} m &= \varrho' g\, (\alpha \varkappa + n\beta')/n , \\ c &= K/(\varrho' g\, (\alpha + n\beta')) = k/(\varrho' \upsilon\, (\alpha + n\beta')) . \end{aligned} \tag{4.2}$$

It includes the relation between the intrinsic permeability k and the porosity n, which changes due to porous matrix deformations, and also the compressibility of the pore fluid containing air bubbles.

The storage equation (4.1) is typically nonlinear and a general solution seems not available. However, a rather simple transformation exists rendering (4.1) into a linear partial differential equation with one variable coefficient. It is fortunate that such a transformation exists, and it represents a new method of solving a practicle class of flow problems governed by the nonlinear storage equation.

The transformation suggested is new in the field of geohydrology, but is based on a combination of two wellknown mathematical transformations, namely the so-called Hopf transformation, see Ames (1967), and a generalization of the Euler substitution for the vertical coordinate z. A general proof will be given.

Consider the linear partial second order differential equation:

$$\frac{\partial^2\chi}{\partial\eta^2} + g(\zeta)\frac{\partial^2\chi}{\partial\zeta^2} = \frac{\partial\chi}{\partial\tau}, \tag{4.3}$$

where η, ζ and τ are independent variables and g is some function of ζ only.
Under the transformation:

$$\chi = F(\psi) \quad , \quad \psi = \psi(\eta, \zeta, \tau), \tag{4.4}$$

the partial differential equation (4.3) becomes nonlinear in terms of ψ:

$$\frac{\partial^2\psi}{\partial\eta^2} + g(\zeta)\frac{\partial^2\psi}{\partial\zeta^2} + \left(\frac{d^2F}{d\psi^2} / \frac{dF}{d\psi}\right)\left\{\left(\frac{\partial\psi}{\partial\eta}\right)^2 + g(\zeta)\left(\frac{\partial\psi}{\partial\zeta}\right)^2\right\} = \frac{\partial\psi}{\partial\tau}. \tag{4.5}$$

Next, applying the substitution:

$$\vartheta = f(\zeta), \tag{4.6}$$

from which follows:

$$\frac{\partial\psi}{\partial\zeta} = \frac{\partial\psi}{\partial\vartheta}\,\frac{df}{d\zeta},$$

and:

$$\frac{\partial^2\psi}{\partial\zeta^2} = \frac{\partial^2\psi}{\partial\vartheta^2}\left(\frac{df}{d\zeta}\right)^2 + \frac{\partial\psi}{\partial\vartheta}\,\frac{d^2f}{d\zeta^2},$$

renders equation (4.5) into:

$$\frac{\partial^2\psi}{\partial\eta^2} + g(\zeta)\frac{\partial^2\psi}{\partial\vartheta^2}\left(\frac{df}{d\zeta}\right)^2 + g(\zeta)\frac{\partial\psi}{\partial\vartheta}\,\frac{d^2f}{d\zeta^2} +$$

$$\left(\frac{d^2F}{d\psi^2} / \frac{dF}{d\psi}\right)\left\{\left(\frac{\partial\psi}{d\eta}\right)^2 + g(\zeta)\left(\frac{\partial\psi}{d\vartheta}\right)^2\left(\frac{df}{d\zeta}\right)^2\right\} = \frac{\partial\psi}{\partial\tau}. \tag{4.7}$$

Now, let it be assumed that F is restricted so that the following identity holds:

$$\frac{d^2F}{d\psi^2} / \frac{dF}{d\psi} = 1.$$

This confines the function F to the class:

$$F = C_1 + C_2 \exp(\psi), \tag{4.8}$$

where C_1 and C_2 are arbitrary constants. Let it furthermore be assumed that the function g can be chosen in such a way that both:

$$g(\zeta)\,(df/d\zeta)^2 = 1, \tag{4.9}$$

and:

$$g(\zeta)\,d^2f/d\zeta^2 = -1. \tag{4.10}$$

Then equation (4.7) reduces to the following form:

$$\frac{\partial^2\psi}{\partial\eta^2} + \frac{\partial^2\psi}{\partial\vartheta^2} + \left(\frac{\partial\psi}{\partial\eta}\right)^2 + \left(\frac{\partial\psi}{\partial\vartheta}\right)^2 - \frac{\partial\psi}{\partial\vartheta} = \frac{\partial\psi}{\partial\tau}. \tag{4.11}$$

With a linear transformation associated to the variables used, according to:

$$\begin{aligned} \psi &= m\phi , \\ \eta &= my , \\ \vartheta &= mz , \\ \tau &= m^2ct , \end{aligned} \tag{4.12}$$

expression (4.11) becomes:

$$\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} + m\{(\frac{\partial\phi}{\partial y})^2 + (\frac{\partial\phi}{\partial z})^2 - \frac{\partial\phi}{\partial z}\} = \frac{1}{c}\frac{\partial\phi}{\partial t}. \tag{4.13}$$

Without loss of generality another space variable may be added:

$$\xi = mx , \tag{4.14}$$

having a similar character as η (ξ has been omitted to avoid lengthy formulas). This finally makes equation (4.13) identical to the general storage equation (4.1).

In the previous analysis it has been assumed that the function f and function g satisfy conditions (4.9) and (4.10). The consequences of these assumptions are still to be investigated. Eliminating $g(\zeta)$ from the two conditions gives the differential equation:

$$\frac{d^2f}{d\zeta^2} + (\frac{df}{d\zeta})^2 = 0 ,$$

which has a general solution according to:

$$f(\zeta) = \mathit{ln}\,(\zeta + C_3) + C_4 , \tag{4.15}$$

where C_3 and C_4 are arbitrary constants.
The expression for $g(\zeta)$ is determined after substitution of (4.15) into equation (4.9), and then, it is found that:

$$g(\zeta) = (\zeta + C_3)^2 , \tag{4.16}$$

satisfies both (4.9) and (4.10).

Without affecting the general storage equation (4.1) the constants C_1, C_2, C_3 and C_4 can be chosen. A convenient choice is:

$$\begin{aligned} C_1 &= -1 , \\ C_2 &= 1 , \\ C_3 &= 0 , \\ C_4 &= 0 . \end{aligned} \tag{4.17}$$

Using expressions (4.8), (4.12) and (4.15) the set of transformations becomes:

$$\begin{aligned} \chi &= \exp(m\phi) - 1 , \\ \xi &= mx , \\ \eta &= my , \\ \zeta &= \exp(mz) , \\ \tau &= m^2ct . \end{aligned} \tag{4.18}$$

Consequently, the expression for function g, equation (4.16), and the above set of transformations (4.18) inserted in (4.3), result in the transformed storage equation:

$$\frac{\partial^2\chi}{\partial\xi^2} + \frac{\partial^2\chi}{\partial\eta^2} + \zeta^2\frac{\partial^2\chi}{\partial\zeta^2} = \frac{\partial\chi}{\partial\tau} . \tag{4.19}$$

This is a linear second order partial differential equation with one variable constant, governing porous flow described by the general storage equation (3.56).

It can be conceived from the set (4.18) that general boundary conditions defined in terms of ϕ can be expressed in terms of χ. In the proceeding sections this will be shown by some general solutions of several various types of porous flow problems.

As a consequence of the form of the transformed storage equation (4.19), horizontal compressible flow in deformable porous media is fully described by the transformation set:

$$\chi = \exp(m\phi) - 1\,,$$
$$\xi = x\,,$$
$$\eta = y\,,$$
$$\tau = ct\,,$$

resulting in the storage equation:

$$\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} = \frac{1}{c}\,\frac{\partial\chi}{\partial t}\,, \tag{4.20}$$

in which reference is made to the original coordinate system. This equation is exactly similar to the conventional storage equation for horizontal flow, extensively discussed in literature. It is remarkable that all the physical nonlinearity provided in the foregoing sections is embedded in the simple transformation of the potential ϕ, according to:

$$\chi = \exp(m\phi) - 1\,. \tag{4.21}$$

For practical values encountered in groundwater, such as:

density: $\varrho' \sim 10^3\ \mathrm{N/m^3}$,
gravity: $g \sim 10\ \mathrm{m/s^2}$,
porosity: $n \sim 1$,
permeability: $\varkappa \sim 1$,
skeleton: $\alpha \sim 10^{-6} - 10^{-7}\ \mathrm{m^2/N}$,
pore water: $\beta' \sim 10^{-7} - 10^{-9}\ \mathrm{m^2/N}$,

and subsequently for the coefficient m, defined by (4.2);

$$m \sim 10^{-2} - 10^{-3}\ \mathrm{m^{-1}}\,,$$

the quantity $m\phi$ can be considered relatively small, since the fluctuations in the potential ϕ are usually in the order of a few metres. Hence,

$$m\phi < 1\,, \tag{4.22}$$

holds for practical cases, permitting a series expansion for χ, to obtain:

$$\chi = m\phi + 0(m^2\phi^2)\,.$$

Substitution in the governing storage equation (4.20) results in the ordinary storage equation for horizontal porous flow, when disregarding second order terms.
Consideration of large flow fields might violate the validity of (4.22) and nonlinearity becomes manifest. In this regard the quantity χ is called the extensive potential, since it provides information about the extent of perturbations in groundwater.

4.b APPLICATION OF INTEGRAL TRANSFORMS

In section 4.a the general nonlinear storage equation (4.1):

$$\nabla^2\phi + m\,\{(\nabla\phi)^2 - \partial\phi/\partial z\} = (1/c)\,\partial\phi/\partial t\,,$$

has been transformed into a linear partial differential equation, called the transformed storage equation (4.19):

$$\frac{\partial^2\chi}{\partial\xi^2} + \frac{\partial^2\chi}{\partial\eta^2} + \zeta^2\frac{\partial^2\chi}{\partial\zeta^2} = \frac{\partial\chi}{\partial\tau}. \tag{4.23}$$

Only the contribution along the ζ-direction, in fact the vertical direction, contains a variable coefficient preventing the application of common mathemathical techniques at the first inspection. The class of equations:

$$\zeta^2\frac{d^2\chi}{d\zeta^2} = S(\zeta) \tag{4.24}$$

is known as the Euler or Cauchy equation. As can be conceived from the result in the previous section, in particular equation (4.15) and (4.6), a simple substitution, i.e. $\zeta = \exp(\vartheta)$ called the Euler substitution, renders it into a linear differential equation with constant coefficients, but also the inhomogeneous part $S(\zeta)$ is transformed. Thus, equation (4.24) becomes:

$$\frac{d^2\chi}{d\vartheta^2} - \frac{d\chi}{d\vartheta} = S(\exp(\vartheta)),$$

for which several solving methods exist. Since for a general case the procedure to find the particular solution satisfying the inhomogeneous part, function S, might be a difficulty because of the Euler substitution, another procedure is suggested.
This procedure is found amongst integral transformation techniques. They eventually comprise an advantage above solving differential equations, in that an integral can always be majorated to give an approximate solution.
The Fourier transformation technique is known to be particularly suited to handle differential equations with variable coefficients, see Sneddon (1951). The Fourier transform and inversion theorem states the following. When the function $F(\eta)$ represents the Fourier transform of $f(y)$, given by:

$$F(\eta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(y)\exp(i\eta y)\,dy,$$

and y is a point of continuity, then the function f(y) is given by:

$$f(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(\eta)\exp(-i\eta y)\,d\eta.$$

The condition of y being a point of continuity can be relaxed to the condition that f is piecewise continuous and absolutely integrable.
In that case the last integral does not equal f(y) in a point where y is discontinuous, but:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(\eta)\exp(-i\eta y)\,dy = \frac{1}{2}\left[f(y^-) + f(y^+)\right]. \tag{4.25}$$

where $f(y^-)$ and $f(y^+)$ represent the limit value of f from the lower and upper direction, respectively, see Inselberg (1973).

Introducing new variables, according to:

$$\zeta = \exp(y),$$

and:

$$s = c + i\eta \quad ; \quad Im(c) = 0,\ Im(\eta) = 0,$$

the Fourier transform and inversion theorem becomes:

$$F\left(\frac{s-c}{i}\right) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(ln\zeta)\,\zeta^{-c}\zeta^{s-1}\,d\zeta,$$

$$f(ln\zeta) = \frac{1}{i\sqrt{2\pi}}\int_{c-i\infty}^{c+i\infty} F\left(\frac{s-c}{i}\right)\zeta^{c-s}\,ds.$$

With the definition of new functions, according to:

$$\chi(\zeta) = \frac{1}{\sqrt{2\pi}} \zeta^{-c} f(ln\, \zeta) ,$$

and:

$$\tilde{\chi}(s) = F(\frac{s-c}{i}) ,$$

the following set is obtained:

$$\tilde{\chi}(s) = \int_0^{\infty} \chi(\zeta) \zeta^{s-1} d\zeta , \tag{4.26}$$

$$\chi(\zeta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{\chi}(s) \zeta^{-s} ds , \tag{4.27}$$

which is known as the Mellin transform and inversion theorem, see Bateman (1954). The analysis holds when the integrals mentioned exist. This actually restricts the possible class of functions χ.
The inverse Mellin transform (the Mellin inversion), according to expression (4.27), is equivalent to the inverse Laplace transform, which is defined by:

$$\chi(\zeta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{\chi}(s) \exp(-s\, ln\zeta)\, ds , \tag{4.28}$$

provided that the following holds:

$$ln(\zeta) < 0 . \tag{4.29}$$

This condition suggests to confine the considered field in the domain:

$$0 < \zeta < 1 . \tag{4.30}$$

This domain is sufficient for the considered problem of nonlinear groundwater flow described by equation (4.1).
Because of the transformation of the physical field z to the field ζ, given by equation (4.18):

$$\zeta = \exp(mz) \; ; \; m > 0 ,$$

any interval of the real z-domain defined by: $z<0$ can uniquely be represented by an interval in the ζ-domain defined by (4.30). In this domain the function χ representing the physical quantity: $\exp(m\phi) - 1$, according to (4.18), can be integrated and consequently the finite integral:

$$\tilde{\chi}(s) = \int_0^1 \chi(\zeta) \zeta^{s-1} d\zeta , \tag{4.31}$$

is physically meaningful. This form is easily extended to the domain:

$$0 < \zeta < 0 ,$$

without affecting the physical meaning by defining the function χ identical to zero in the added interval, or:

$$\chi(\zeta) = 0 , \text{for } \zeta > 1 , \tag{4.32}$$

so that (4.31) becomes:

$$\tilde{\chi}(s) = \int_0^1 \chi(\zeta) \zeta^{s-1} d\zeta = \int_0^{\infty} \chi(\zeta) \zeta^{s-1} d\zeta ,$$

similar to the Mellin transform defined by (4.26). In respect of the formula (4.25) and the fact that the Fourier transform and inversion theorem hold for a piecewise continuous and absolutely integrable function, it can be conceived that also the Mellin inversion defined by (4.27) is valid.

The Mellin transformation technique is particularly suited to solve differential equations involving expressions such as (4.24). This is a mere coincidence, since the transform of (4.24) can be reduced to a linear expression in terms of the transform of χ itself. A general proof is given next.

Although not listed in literature, a special application of the Mellin transform exists. Consider the folowing integral:

$$\int_0^\infty (\zeta^{n+1}\chi^{(n+1)})\,\zeta^{s-1}\,d\zeta . \tag{4.33}$$

Performing partial integration leads to:

$$\int_0^\infty \chi^{(n+1)}\zeta^{n+s}d\zeta \;=\; \int_0^\infty \zeta^{n+s}d\,\chi^{(n)} =$$

$$-\int_0^\infty \chi^{(n)}\,d\zeta^{n+s} \;+\; \left\{\chi^{(n)}\zeta^{n+s}\right\}\Big|_0^\infty =$$

$$(-1)(n+s)\int_0^\infty \chi^{(n)}\zeta^{n+s-1}\,d\zeta \;+\; \left\{\chi^{(n)}\zeta^{n+s}\right\}\Big|_0^\infty .$$

Once again partial integration results in:

$$(-1)^2(n+s)(n+s-1)\int_0^\infty \chi^{(n-1)}\zeta^{n+s-2}d\zeta \;+\; \left\{\chi^{(n)}\zeta^{n+s} \;+\; (-1)(n+s)\,\chi^{(n-1)}\zeta^{n+s-1}\right\}\Big|_0^\infty ,$$

and after repeating this procedure j times, while $j \leq n$, the following expression is obtained:

$$(-1)^{j+1}\left[(n+s)(n+s-1)\,\ldots\ldots\,(n+s-j)\right]\int_0^\infty \chi^{(n-j)}\zeta^{n+s-1-j}\,d\zeta \quad +$$

$$\left\{\chi^{(n)}\zeta^{n+s} + (-1)(n+s)\,\chi^{(n-1)}\zeta^{n+s-1} + \ldots\ldots\right.$$

$$\left.\ldots\ldots(-1)^j\left[(n+s)(n+s-1)\,\ldots\ldots\,(n+s+1-j)\,\chi^{(n-j)}\zeta^{n+s-j}\right\}\right|_0^\infty .$$

For the special case: $j = n$, this reduces to:

$$(-1)^{n+1}\left[\prod_{m=0}^{n}(n+s-m)\right]\int_0^\infty \chi\zeta^{s-1}\,d\zeta \;+$$

$$\left\{\chi^{(n)}\zeta^{n+s} + (-1)(n+s)\,\chi^{(n-1)}\zeta^{n+s-1} + \ldots\ldots\right.$$

$$\left.\ldots\ldots(-1)^n\left[\prod_{m=0}^{n-1}(n+s-m)\right]\chi\zeta^s\right\}\Big|_0^\infty . \tag{4.34}$$

Thus, combining (4.33) and (4.34) the following identity holds for $n = 1$:

$$\int_0^\infty \zeta^2\chi^{(2)}\zeta^{s-1}d\zeta \;=\; s(s+1)\int_0^\infty \chi\zeta^{s-1}d\zeta \;+\left\{\chi^{(1)}\zeta^{s+1} - (s+1)\,\chi\zeta^s\right\}\Big|_0^\infty .$$

The integrals in this expression represent Mellin transforms, according to (4.26). It can be written in the form:

$$\widetilde{\zeta^2\chi^{(2)}} \;=\; s(s+1)\,\tilde{\chi} + \left\{\chi^{(1)}\zeta^{s+1} - (s+1)\,\chi\zeta^s\right\}\Big|_0^\infty .$$

Recalling the fact that the physical domain is restricted to $0<\zeta<1$, according to (4.30), the above formula can be reduced under the extra condition:

$$\chi^{(1)} = 0 \quad \text{for} \quad \zeta>1\,,$$

to obtain:

$$\widetilde{\zeta^2\chi^{(2)}} = s(s+1)\,\tilde{\chi} + \left\{ \chi^{(1)}\zeta^{s+1} - (s+1)\chi\zeta^s \right\} \Big|_0^1 \,,$$

or:

$$\widetilde{\zeta^2\chi^{(2)}} = s(s+1)\,\tilde{\chi} + \left\{ \chi^{(1)} - (s+1)\,\chi \right\}_{\zeta\uparrow 1} \,, \tag{4.35}$$

provided that χ and $\chi^{(1)}$ are regular in the considered domain and the following contributions vanish:

$$\mathit{Lim}_{\zeta\downarrow 0}\; \zeta^{s+1}\,\chi^{(1)} = 0\,,$$

and:

$$\mathit{Lim}_{\zeta\downarrow 0}\; \zeta^{s}\chi = 0\,.$$

Formula (4.35) is the required expression by which the differential equation (4.23) can be reduced to a linear form amenable to conventional mathematical techniques, while general boundary conditions applied at the boundary: $\zeta = 1$ can easily be introduced. Actually, the Mellin transformation is a straight forward technique, and for the inverse, being a Laplace inversion, a long list of solved integrals is available in literature, see Bateman (1954).

In the section 4.d some characteristic problems are solved using the Mellin transformation technique and the Laplace inversion technique.

4.c NONLINEAR STORAGE IN HORIZONTAL AQUIFERS

Confined and phreatic aquifers

According to the theory presented in section 4.a confined porous flow in horizontal aquifers is governed by the transformed storage equation (4.20):

$$\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} = \frac{1}{c}\,\frac{\partial\chi}{\partial t}\,, \tag{4.36}$$

in which the variabele χ, referred to as the extensive potential, is a function of the physical potential ϕ according to (4.21):

$$\chi = \exp(m\phi) - 1\,, \tag{4.37}$$

where m, defined by (4.2), incorporates nonlinear effects:

$$m = \varrho' g\,(\alpha\varkappa + n\beta')/n\,. \tag{4.38}$$

Since the amplification factor $\varkappa$, see section 2.b, is of the order unity, the nonlinear coefficient m is comparable to the coefficient of consolidation c in the following manner:

$$m \cong K/cn\,. \tag{4.39}$$

The type of equation (4.36) is well known in the field of heat diffusion, widely investigated in literature, see Carslaw and Jaeger (1951) and Bear (1972). Many solutions are available, and they are here applicable without restriction, taking into account the boundary conditions in terms of

χ and performing to the solution an additional operation, i.e. substitution of relation (4.37).

In section 4.a it has been outlined that the difference between the physical potential ϕ and the extensive potential χ becomes manifest in case $m\phi$ is of the order of unity. This is made clear for a theoretical case of flow towards a well in a rectangular aquifer. The solution both in terms of ϕ and χ is obtained by a finite element program: the SEEP code (see Barends, 1976), and results are represented in Fig. XIV.
The value of the coefficient m, in which the nonlinearity is embedded, assumes different values. From the iso-potentials it becomes clear that nonlinearity can become significant with respect to the distance from the actual location of a disturbance in the pore fluid.

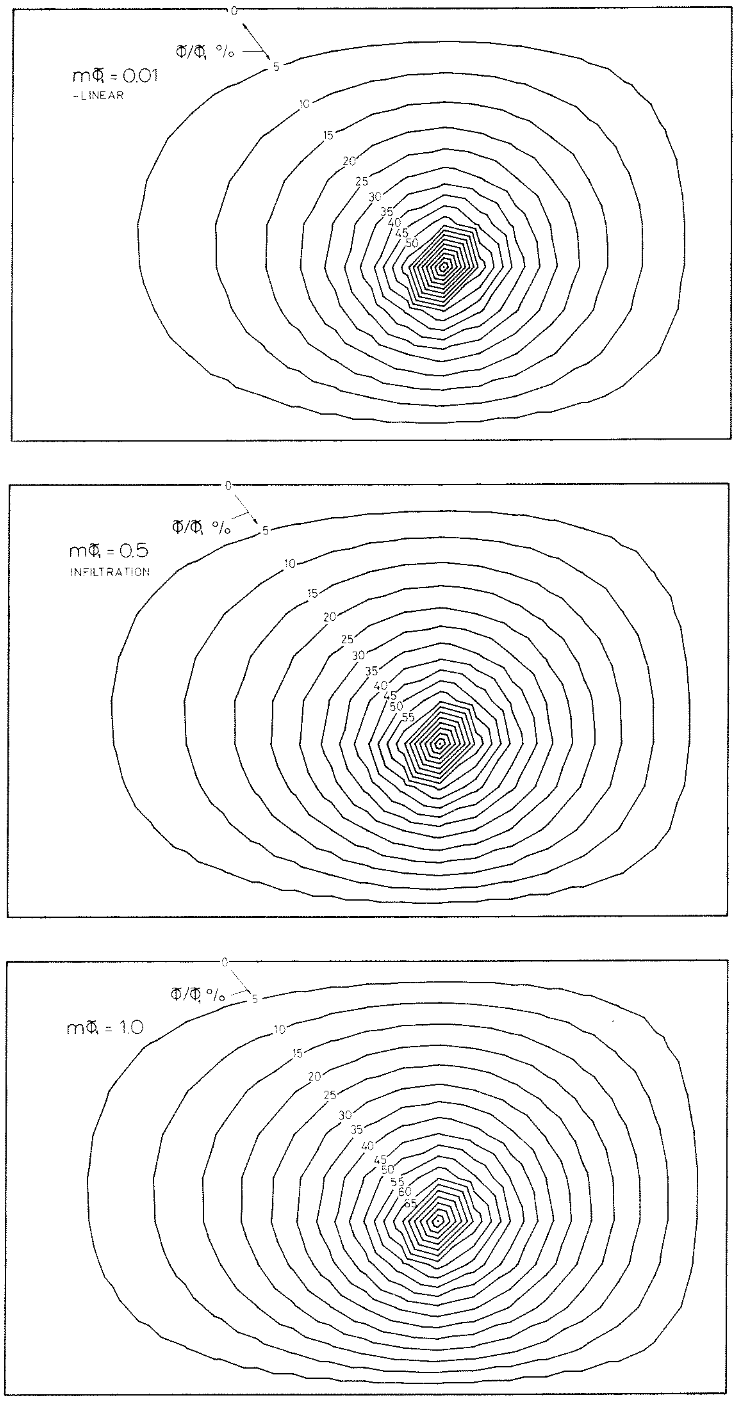


Fig. XIV Confined flow in a horizontal aquifer.

In practice the case of flow towards a well frequently occurs. For example, stationary pumping for water supply purposes or temporary, when during a period at a job site the groundwater table has been lowered to a constant level. In both cases it is important to know the area of influence due to the generated flow field, which, of course, largely depends on the geo-hydrological situation and the period of duration of pumping. In most cases the flow field has a predominantly horizontal stratification, whereas the porous flow is mainly horizontal. Therefore, the transformed storage equation (4.36) is suitable to describe this type of flow.

The character of the flow field of such horizontal situations can be either plane-symmetrical or axi-symmetrical. Moreover, the aquifer can be confined (fully saturated) or phreatic.
In the last case also the storage equation (4.36) is applicable to simulate the flow including secondary effects, while the coefficient of nonlinearity m includes another phenomenon: the phreatic storativity. This can be understood by the following explanation.

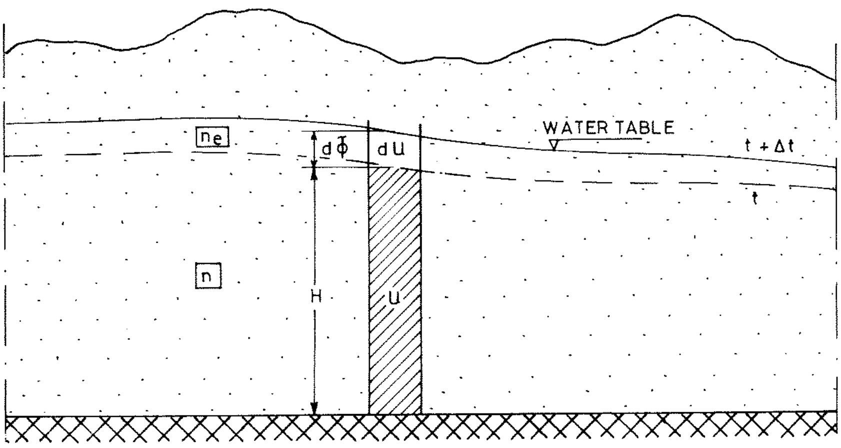


Fig. XV Phreatic fluctuations.

Consider a phreatic aquifer, see Fig. XV, having a phreatic porosity n_e denoting the volume of pores taking part in the storage of phreatic fluctuations. This quantity is also called the phreatic storativity of an aquifer, or the drainage coefficient. A fluctuation in the pressure p averaged over the aquifer height H can be expressed according to:

$$d\Phi = dp/\varrho' g\ ,$$

giving an extra storage possibility per unit surface area for a small volume dU of water:

$$dU = n_e\, dp/\varrho' g\ . \qquad (4.40)$$

The volume of water U already stored in the aquifer per unit surface area at the same location amounts:

$$U = nH\ . \qquad (4.41)$$

Conforming to the definition for the compressibility coefficient β, equation (2.17), it is possible to define a similar coefficient $\tilde{\beta}$, which includes the volume fluctuations due to phreatic storage:

$$\tilde{\beta} = -\frac{1}{U}\frac{dU}{dp}\ ,$$

After introduction of (4.40) and (4.41) this becomes:

$$\tilde{\beta} = n_e/(nH\varrho' g)\ . \qquad (4.42)$$

In respect of the discussion in section 2.c, the following expression holds for phreatic aquifers:

$$d\varrho' = \varrho'\,\tilde{\beta}\, dp$$

where $\tilde{\beta}$ is defined by (4.42), which is obviously a constant. Evidently, the preceding analysis is likely valid for phreatic horizontal flow in aquifers. This result permits another significant implication of the coefficient m. Using equation (4.38) and dropping the coefficients α and $\varkappa$, which are not considered in a phreatic aquifer, give with (4.42):

$$m = \varrho' g \tilde{\beta} = n_e/(nH). \tag{4.43}$$

In sands the phreatic porosity n_e is nearly equal to the volumetric porosity n, but in clays it might be much smaller, see Zeller (1969). Equation (4.43) reveals that m denotes a simple scale factor in the relation between the potentials ϕ and χ.

Applying equation (4.39) and (4.43) the consolidation coefficient appearing in the storage equation can be expressed by:

$$c = \frac{K}{mn} = \frac{KH}{n_e}. \tag{4.44}$$

In conclusion, phreatic horizontal flow in aquifers including geometric nonlinearity is described by the following storage equation:

$$\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} = \frac{n_e}{KH} \frac{\partial \chi}{\partial t}, \tag{4.45}$$

where:

$$\chi = \exp(m\phi) - 1, \quad m = n_e/(nH).$$

The fact that the common potential equation in terms of an exponential function of the piezometric head ϕ, describes this type of flow, is a very convenient result, since it allows to estimate nonlinear effects in a simple manner.

The value $m\phi$ appearing in the relation between ϕ and χ is bounded for physically relevant phreatic flow fields, as the drop in piezometric head ϕ can not exceed the aquifer height. Therefore, in practical cases (4.22) holds for $m\phi$.

Area of influence including nonlinear effects

Both for axi-symmetrical and plane-symmetrical flow fields in infinite aquifers, phreatic or confined, the partial differential equation (4.36) describing the nonlinear porous flow behaviour can be reduced to an ordinary differential equation with variable coefficients by definition of a new variable, known as the Boltzman transformation, see Bear (1972).

First consider the plane-symmetrical case. The new variable is:

$$\omega = x/\sqrt{ct}.$$

From the basic rules of partial differentiation, it appears that:

$$\frac{\partial \chi}{\partial t} = \frac{d\chi}{d\omega} \frac{\partial \omega}{\partial t} = \frac{d\chi}{\partial \omega} (-\omega/2t),$$

and:

$$\frac{\partial^2 \chi}{\partial x^2} = \frac{d^2 \chi}{d\omega^2} \left(\frac{\partial \omega}{\partial x}\right)^2 + \frac{d\chi}{d\omega} \left(\frac{\partial^2 \omega}{\partial x^2}\right) = \frac{d^2 \chi}{d\omega^2} \left(\frac{\omega}{x}\right)^2.$$

Substitution into equation (4.36) results in:

$$2\frac{d^2 \chi}{d\omega^2} + \omega \frac{d\chi}{d\omega} = 0. \tag{4.46}$$

This equation makes sense, when also the boundary conditions can be expressed in terms of the new variable ω.
For a sudden drawdown $\chi = 0$ at $x = 0$, $t > 0$, while $\chi = \exp(m\Phi_B) - 1 = \chi_B$ for all values of $x > 0$ at $t = 0$, the boundary conditions are:

$$\chi = 0 \quad , \quad \omega = \omega_o \downarrow 0 \, , \tag{4.47}$$

$$\chi = \chi_B \, , \quad \omega \rightarrow \infty \, .$$

Integrating equation (4.46) twice yields the final solution:

$$\chi = \chi_B \operatorname{erf}\left(\frac{\omega}{2}\right) , \tag{4.48}$$

where:

$$\operatorname{erf}(\alpha) = \frac{2}{\sqrt{\pi}} \int_0^\alpha \exp(-\ell^2)\, d\ell , \quad \text{(error function).}$$

In terms of ϕ and x this can be rewritten into:

$$\phi = \frac{1}{m} \ln\left[1 + (\exp(m\phi_B) - 1)\operatorname{erf}\left(\frac{x}{2\sqrt{ct}}\right)\right] , \tag{4.49}$$

representing the solution according to the nonlinear theory. Equation (4.48) can be expressed in the following form:

$$\frac{\chi}{\chi_B} = \left(1 - \frac{\operatorname{erfc}(\omega/2)}{\operatorname{erfc}(\omega_o/2)}\right) , \tag{4.50}$$

where ω_o tends to zero. Here, erfc denotes the complementary error function, defined according to:

$$\operatorname{erfc}(\alpha) = \frac{2}{\sqrt{\pi}} \int_\alpha^\infty \exp(-\ell^2)\, d\ell ,$$

and related to the error function erf by the relation:

$$\operatorname{erfc}(\alpha) + \operatorname{erf}(\alpha) = 1 \, .$$

The obtained result, equation (4.49), is represented in Fig. XVI for different values of m.

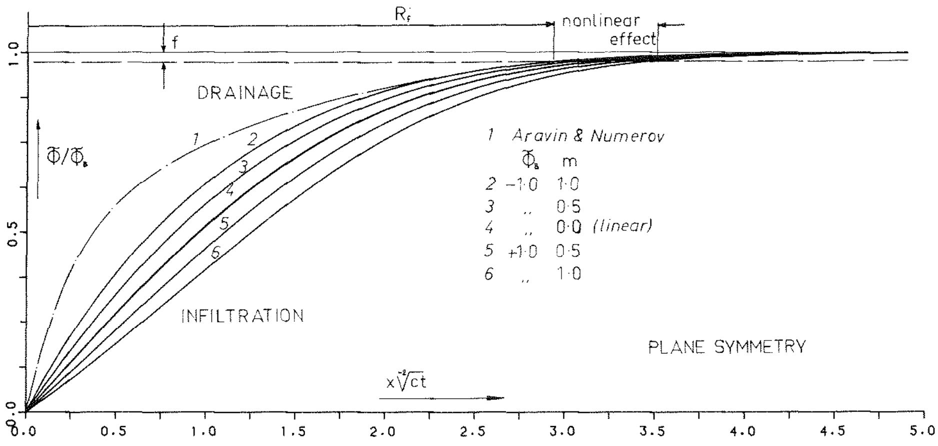


Fig. XVI Area of influence in nonlinear transient aquifer flow.

Next, consider the axi-symmetrical case. The same method is applicable. Definition of a new variable:

$$\omega = r/\sqrt{ct}\ ,$$

gives for the partial derivatives:

$$\frac{\partial \chi}{\partial t} = \frac{d\chi}{d\omega}\frac{d\omega}{dt} = \frac{d\chi}{d\omega}\left(-\frac{\omega}{2t}\right),$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \chi}{\partial r} = \frac{d^2\chi}{d\omega^2}\left(\frac{\omega}{r}\right)^2 + \frac{d\chi}{d\omega}\left(\frac{\omega}{r^2}\right).$$

Substitution into equation (4.36), which for axi-symmetry becomes:

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\chi = \frac{1}{c}\frac{\partial \chi}{\partial t}, \tag{4.51}$$

yields the differential equation:

$$\frac{d^2\chi}{d\omega^2} + \left(\frac{\omega}{2} + \frac{1}{\omega}\right)\frac{d\chi}{d\omega} = 0. \tag{4.52}$$

Integration gives:

$$\frac{d\chi}{d\omega} = \frac{\chi_o}{\omega}\exp\left(-\frac{\omega^2}{4}\right),$$

where χ_o is an integration constant related to the boundary condition. To avoid the singularity at $\omega \downarrow 0$, further integration is performed according to:

$$\chi = \int_{\omega}^{\infty}\frac{d\chi}{d\ell}d\ell = \chi_o\int_{\omega/2}^{\infty}\frac{\exp(-\ell^2)d\ell}{\ell} = \frac{1}{2}\chi_o\int_{\omega^2/4}^{\infty}\frac{\exp(-\ell)d\ell}{\ell} = \frac{1}{2}\chi_o E_1\left(\frac{\omega^2}{4}\right).$$

The function E_1 is known as the exponential integral function. The boundary conditions, in accordance to (4.47), inserted in the solution, finally ends up with:

$$\frac{\chi}{\chi_B} = \left(1 - \frac{E_1(\omega^2/4)}{E_1(\omega_o^2/4)}\right). \tag{4.53}$$

This solution is conformal to the plane-symmetrical solution, equation (4.50). The only essential difference is the function: erfc in plane-symmetry and E_1 in axi-symmetry.

Expansion of E_1 for small values of ω_o leads to the so-called 'well function', introduced by Hantush (1964), in terms of χ.
The solution in terms of ϕ and r yields:

$$\phi = \frac{1}{m}ln\left[1 + (\exp(m\phi_B) - 1)\left(1 - \frac{E_1(r^2/4ct)}{E_1(r_o^2/4ct)}\right)\right]. \tag{4.54}$$

The flow fields discussed are generated due to a sudden drawdown of ϕ_B. The area of influence R defines the area of measurable or perceivable disturbance in the form of a locally lowered groundwater level, which in comparison to ϕ_B is denoted by $\Delta\phi_B$. Only in the presence of natural infiltration, such as caused by rain, leakage or canals and rivers, a constant area of influence can be expected after certain time.
Usually one is interested in the influence of a temporary drawdown, which is chosen as the area R_f, where a fraction:

$$f = \Delta\phi_B/\phi_B\ ,$$

is found. This area R_f will grow, as the drawdown continues, but also after releasing this drawdown the recovery of the original level will start after a delay, which takes longer with greater distance.

An expression for R_f can be obtained by developing the solution (4.48) and (4.53). In both cases the fraction f can be expressed in terms of χ_B. By definition:

$$1 - \frac{\chi}{\chi_B} = \frac{\Delta\chi_B}{\chi_B}, \tag{4.55}$$

and elaborating this in terms of ϕ using (4.37) results into:

$$\frac{\Delta\chi_B}{\chi_B} = \frac{\Delta(\exp(m\phi_B) - 1)}{\exp(m\phi_B) - 1} = \Delta\phi_B(1 - \exp(-m\phi_B))^{-1}$$

$$\cong \frac{\Delta\phi_B}{\phi_B}(1 - \frac{1}{2}m\phi_B)^{-1} = f(1 - \frac{1}{2}m\phi_B)^{-1}. \tag{4.56}$$

provided that $m\phi_B < 1$ holds.

The following inequality exists for the error function, see Abramowitz and Stegun (1968):

$$(x + \sqrt{x^2 + 2}\,)^{-1} < \exp(x^2) \int_x^\infty \exp(-\ell^2)\, d\ell \le (x + \sqrt{x^2 + 4/\pi}\,)^{-1},$$

which yields for large values of x:

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-\ell^2) d\ell = \frac{2}{\sqrt{\pi}} \frac{\exp(-x^2)}{x^2} \cong \frac{2}{\sqrt{\pi}} \exp(-x^2) \ , \ x >> 1 .$$

In this respect the solution for the plane-symmetrical case, equation (4,48), can be elaborated using definition (4.55) and (4.56), to arrive at:

$$f(1 - \frac{1}{2}m\phi_B)^{-1} \cong \frac{2}{\sqrt{\pi}} \exp(-\frac{\omega^2}{4}), \quad \omega >> 1 ,$$

or:

$$\omega \cong 2\sqrt{[\,ln(1 - \tfrac{1}{2}m\phi_B) - ln f - 0.89\,]} \ , \ \omega >> 1 . \tag{4.57}$$

The following inequality exists for the exponential integral function, see Abramowitz and Stegun (1968):

$$(x + 1)^{-1} < E_1(x) \exp(x) \le x^{-1} ,$$

which yields for large values of x:

$$E_1(x) \cong \frac{\exp(-x)}{x} \cong \exp(-x) , \quad x >> 1 .$$

In regard of this result the solution for the axi-symmetrical case, expression (4.53), gives utilizing (4.55) and (4.56):

$$f(1 - \frac{1}{2}m\phi_B)^{-1} \cong \exp(-\omega^2/4)/E_1(\omega_0^2/4), \quad \omega >> 1 .$$

For small values of ω_0 the denumerator can be evaluated, see Abramowitz and Stegun (1968), and the result is:

$$f(1 - \frac{1}{2}m\phi_B)^{-1} \cong \exp(-\omega^2/4)/ln\,(2.246/\omega_0^2), \quad \omega >> 1 ,$$

or:

$$\omega \cong 2\sqrt{\left[ln(1-\tfrac{1}{2}m\Phi_B) - lnf - ln(0.809 - 2ln\omega_o)\right]}. \qquad (4.58)$$

The last term under the square root varies smoothly for $\omega_o << 1$, in fact:

$$10^{-2} > \omega_o > 10^{-8} \rightarrow \quad 2.31 < ln(0.809 - 2ln\omega_o) < 3.63\,,$$

and consequently, (4.58) can be approximated by:

$$\omega \cong 2\sqrt{\left[ln(1-\tfrac{1}{2}m\Phi_B) - ln\,f - 3.0\right]}\quad, \omega >> 1\,. \qquad (4.59)$$

Since, $\omega = R_f/\sqrt{ct}$ holds by definition at the location where the specific fraction f is considered, a formula for the area of influence R_f is obtained, both for plane-symmetrical and axi-symmetrical flow fields in horizontal aquifers:

$$R_f = \omega\sqrt{ct}\,,$$

where ω is defined by (4.57) in plane-, and by (4.59) for axi-symmetry.
For different fractions f, the following values occur for ω:

		plane-symmetry	axi-symmetry
$f = 0.01$	$m\Phi_B = 0.0$ $m\Phi_B = 1.0$	3.9 3.5	2.5 1.9
$f = 0.001$	$m\Phi_B = 0.0$ $m\Phi_B = 1.0$	4.9 4.6	4.0 3.6

Hence, nonlinearity in the storage equation, embedded in the factor $m\Phi_B$, results in a maximum reduction of about 10 to 20% in the estimated area of influence.
Since actually χ provides this information, χ is called the extensive potential.

Only if the term $m\Phi_B$ assumes large values, the influence of nonlinearity becomes manifest, even more pronounced for phreatic aquifers, but in general the nonlinearity, whether geometrical effects or density and permeability effects, is of little significance to the area of influence related to horizontal porous flow. This result is important, since it explains that the linear theory results in appropriate values to support, for example, judgement about claims on damage to agriculture or building caused by nearby pumping activities.

Semi-confined aquifers

For moderate variations of Φ in the vertical direction, it is conceivable that also χ will be almost invariable in ζ. Hence, expressing χ in terms of a mean potential $\overline{\chi}$ over the vertical, defined by:

$$\chi \cong \overline{\chi} = \frac{1}{\zeta_2 - \zeta_1}\int_{\zeta_1}^{\zeta_2} \chi d\zeta\,, \qquad (4.60)$$

will provide another possibility to consider the effect of nonlinearity.

Multiplication of the transformed storage equation (4.19) by $d\zeta/(\zeta_2 - \zeta_1)$ and performing integration according to (4.60), leads to:

$$\frac{\partial^2\overline{\chi}}{\partial\xi^2} + \frac{\partial^2\overline{\chi}}{\partial\eta^2} + \frac{1}{\zeta_2 - \zeta_1}\int_{\zeta_1}^{\zeta_2} \zeta^2\frac{\partial^2 x}{\partial\zeta^2}d\zeta = \frac{\partial\overline{\chi}}{\partial\tau}\,. \qquad (4.61)$$

Elaboration of the last term at the left hand side is possible by partial integration:

$$\zeta^2 \frac{\partial^2 \chi}{\partial \zeta^2} d\zeta = \zeta^2 d\left(\frac{\partial \chi}{\partial \zeta}\right) = d\left(\zeta^2 \frac{\partial \chi}{\partial \zeta}\right) - 2\zeta \frac{\partial \chi}{\partial \zeta} d\zeta$$

$$= d\left(\zeta^2 \frac{\partial \chi}{\partial \zeta} - 2\zeta\chi\right) + 2\chi d\zeta .$$

In respect of the definition (4.60) this renders equation (4.61) into:

$$\frac{\partial^2 \overline{\chi}}{\partial \xi^2} + \frac{\partial^2 \overline{\chi}}{\partial \eta^2} + 2\overline{\chi} = \frac{\partial \overline{\chi}}{\partial \tau} + \frac{\left(2\zeta\chi - \zeta^2 \frac{\partial \chi}{\partial \zeta}\right) \Big|_{\zeta_1}^{\zeta_2}}{(\zeta_2 - \zeta_1)} , \tag{4.62}$$

where the last term includes the boundary conditions at ζ_1 and ζ_2.

Consider an aquifer confined by a leaky layer (aquitard) at $\zeta = \zeta_1$, and an impervious layer at $\zeta = \zeta_2$. The leakage can be expressed in terms of ϕ, see Verruijt (1970):

$$\frac{\partial \phi}{\partial z} = \frac{H}{\lambda^2}(\phi - \phi_e) \quad \text{at} \quad z = z_1 , \tag{4.63a}$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{at} \quad z = z_2 . \tag{4.63b}$$

H represents the aquifer thickness, $H = z_2 - z_1$, and λ is the leakage factor, defined according to:

$$\lambda = \sqrt{KHC} ,$$

where C denotes the hydraulic resistance of the aquitard. The term ϕ_e represents the piezometric head outside the considered flow domain, see Fig. XVII.

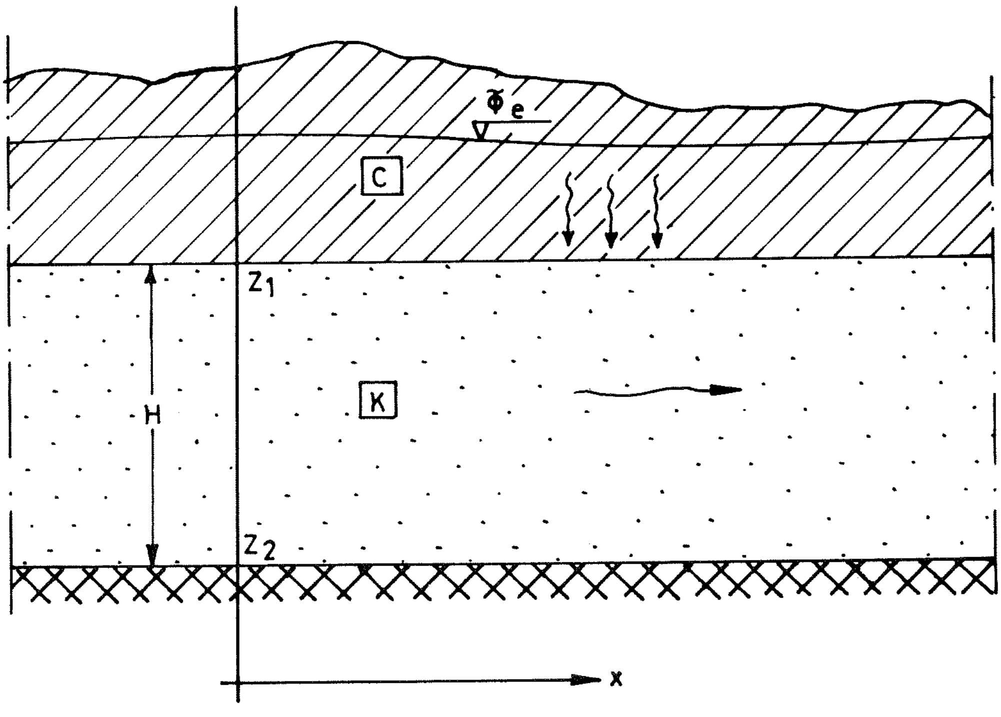


Fig. XVII A semi-confined aquifer.

Because of (4.18) χ and ϕ are interrelated, thus:

$$\zeta = \exp(mz)\,,$$

$$\chi = \exp(m\phi) - 1\,,$$

and consequently,

$$\frac{\partial \chi}{\partial \zeta} = \frac{\chi + 1}{\zeta}\ \frac{\partial \phi}{\partial z}\,.$$

Using this relation the last term in equation (4.62) can be expressed with (4.63) in the following way:

$$\frac{(2\zeta\chi - \zeta^2 \frac{\partial \chi}{\partial \zeta}) \Big|_{\zeta_1}^{\zeta_2}}{(\zeta_2 - \zeta_1)}$$

$$= \frac{2(\zeta_2\chi_2 - \zeta_1\chi_1)}{(\zeta_2 - \zeta_1)} + \frac{H}{\lambda^2}\ \frac{\zeta_1 (\chi_1 + 1)(\phi_1 - \phi_e)}{(\zeta_2 - \zeta_1)}$$

$$= \frac{2(\zeta_2\chi_2 - \zeta_1\chi_1)}{(\zeta_2 - \zeta_1)} + \frac{H}{m\lambda^2}\ \frac{\zeta_1 (\chi_1 + 1)\, ln\, [\,(\chi_1 + 1)/(\chi_e + 1)\,]}{(\zeta_2 - \zeta_1)}\,.$$

Since χ is supposed to vary smoothly with ζ this, in accordance with (4.60), reduces to:

$$2\bar{\chi} + \frac{H}{m\lambda^2}\ \frac{\zeta_1}{\zeta_2 - \zeta_1} (\bar{\chi} + 1)\, ln\, [\,(\bar{\chi} + 1)/(\chi_e + 1)\,]$$

Further, assuming that $\bar{\chi} < 1$, this can be approximated by:

$$2\bar{\chi} + (\bar{\chi} - \chi_e)/\bar{\lambda}^2\,,$$

in which the dimensionless parameter $\bar{\lambda}$ is defined according to:

$$\bar{\lambda} = \lambda \sqrt{m \frac{\exp(mz_2) - \exp(mz_1)}{\exp(mz_1)\, H}} = m\lambda \sqrt{\frac{\exp(mH) - 1}{mH}}\,. \tag{4.64}$$

Finally, equation (4.62), becomes:

$$\frac{\partial^2 \bar{\chi}}{\partial \xi^2} + \frac{\partial^2 \bar{\chi}}{\partial \eta^2} = \frac{\partial \bar{\chi}}{\partial \tau} + \frac{\bar{\chi} - \chi_e}{\bar{\lambda}^2}\,, \tag{4.65}$$

which in this form is fully in agreement with the conventional equation for semi-confined flow in regard of the linear theory, see Verruijt (1970). Therefore, all the solutions obtained in the literature are equally valid in the nonlinear theory, simply by introducing the boundary conditions in terms of $\bar{\chi}$ and replacing the leakage factor by $\bar{\lambda}$, defined in equation (4.64). Barends (1978) presented some solutions of the nonlineary theory. One example is worked out here in this section. It deals with steady axi-symmetrical flow in a semi-confined aquifer. The boundary conditions are:

$$\phi = 0 \quad \text{for } r \to \infty\,,$$

$$\phi = \phi_o \quad \text{for } r \downarrow r_o\,.$$

The governing equation for this case in terms of $\bar{\chi}$ is:

$$\frac{1}{\varrho}\ \frac{d}{d\varrho}\ \varrho \frac{d}{d\varrho}\ \bar{\chi} = \bar{\chi}/\bar{\lambda}^2 \quad , \quad \varrho = mr\,,$$

and the general solution states:

$$\bar{\chi} = \chi_o K_o (mr/\bar{\lambda}),$$

which satisfies the first of the boundary conditions, whereas the coefficient χ_o is related to the second one. Hence,

$$\chi_o = (\exp(m\Phi_o) - 1)/K_o(mr_o/\bar{\lambda}),$$

and the final solution becomes:

$$\bar{\chi} = (\exp(m\Phi_o) - 1)\, K_o(mr/\bar{\lambda})/K_o(mr_o/\bar{\lambda}), \quad r \geq r_o .$$

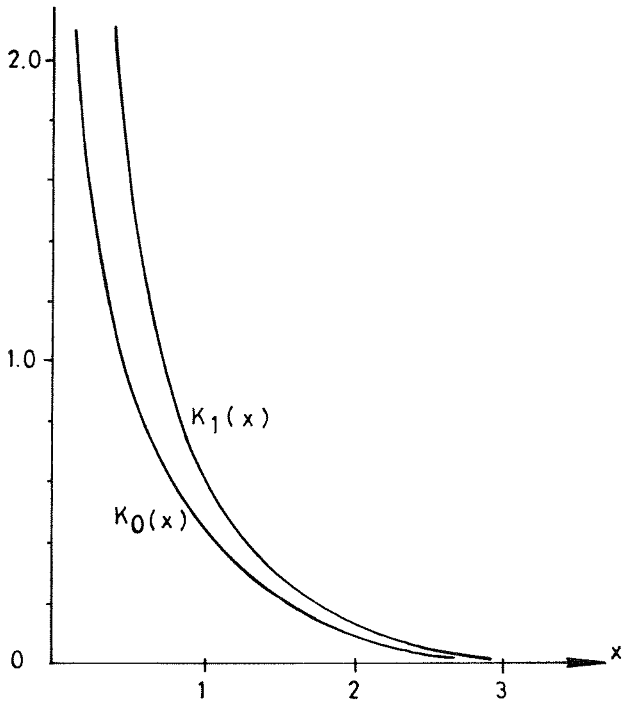


Fig. XVIII Modified Bessel functions of the second kind, order 1 and 2.

Here, K_o denotes the modified Bessel function of the second kind of order zero, a graph of which is shown in Fig. XVIII. It is verified that $\bar{\chi}<1$, which has been the only condition sofar in the derivation of the transformed storage equation (4.65). In terms of Φ the solution yields:

$$\Phi = \frac{1}{m} \ln\left[1 + (\exp(m\Phi_o) - 1)\, K_o(mr/\bar{\lambda})/K_o(mr_o/\bar{\lambda})\right], \quad r \geq r_o . \tag{4.66}$$

In this form it includes the soil flexibility and as such it reduces the drawdown at large distances in comparison to the linear theory. Some graphs of this case are presented in Fig. XIX, showing, for example, the effect of the pressure dependent density in the typical difference between pumping and infiltrating at identical discharge. In the linear theory these two disturbances would generate identical potential fields.

To clearify the effect of nonlinearity, solution (4.66) is elaborated by series expansion. In view of (4.64) $\bar{\lambda}$ can be approximated by:

$$\bar{\lambda} = m\lambda \sqrt{(mH + \tfrac{1}{2}(mH)^2 + \ldots)/mH}$$

$$\cong m\lambda (1 + mH/4) . \tag{4.67}$$

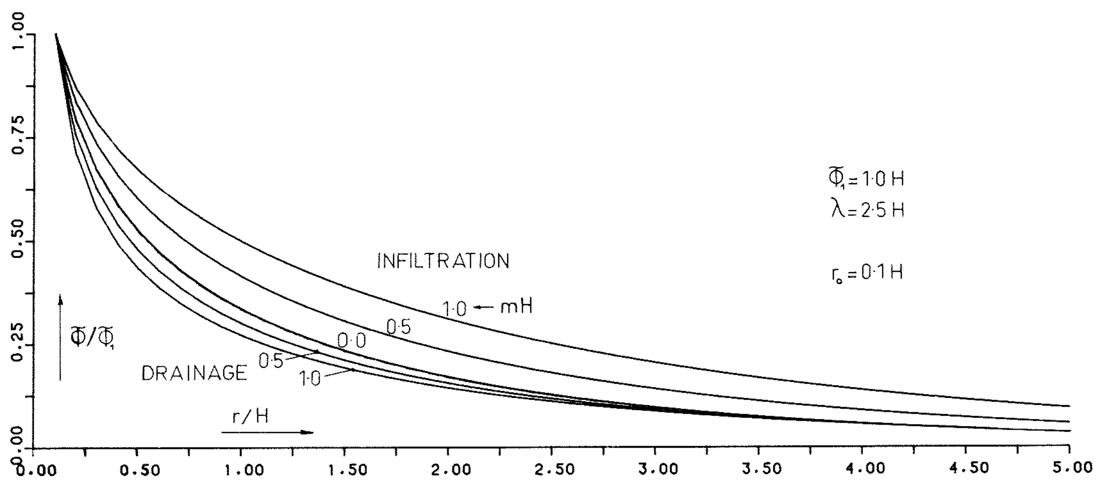


Fig. XIX Steady radial nonlinear flow in a semi-confined aquifer.

Developing the modified Bessel function in a Taylor series gives:

$$K_o(\mathrm{mr}/\bar{\lambda}) = K_o\left(\frac{r}{\lambda(1+\mathrm{mH}/4)}\right)$$

$$\cong K_o\left(\frac{r}{\lambda}\right) - \frac{\mathrm{mHr}}{4\lambda} K_o'\left(\frac{r}{\lambda}\right)$$

$$= K_o\left(\frac{r}{\lambda}\right) + \frac{\mathrm{mHr}}{4\lambda} K_1\left(\frac{r}{\lambda}\right),$$

which holds when the second term is relatively small, i.e. for values of $r<\lambda$.

In this respect the following can be deduced:

$$\frac{K_o(\mathrm{mr}/\bar{\lambda})}{K_o(\mathrm{mr_o}/\bar{\lambda})} \cong \frac{K_o(r/\lambda) + \frac{\mathrm{mHr}}{4\lambda} K_1(r/\lambda)}{K_o(r_o/\lambda) + \frac{\mathrm{mHr_o}}{4\lambda} K_1(r_o/\lambda)}$$

$$= \frac{K_o(r/\lambda)}{K_o(r_o/\lambda)}\left(1 - \frac{1}{8}(\mathrm{mH})^2 F\right), \qquad (4.68)$$

where:

$$F = \frac{r}{\lambda}\,\frac{K_1(r/\lambda)}{K_o(r/\lambda)}\;\frac{r_o}{\lambda}\,\frac{K_1(r_o/\lambda)}{K_o(r_o/\lambda)}.$$

Nothing that the area of interest is restricted by $r<\lambda$, it can be shown that $F<1.5$. The limit for $r = r_o \downarrow 0$ is of special interest, and here, F tends to zero, see Abramowitz and Stegun (1968):

$$\lim_{x\downarrow 0} \frac{xK_1(x)}{K_o(x)} = \lim_{x\downarrow 0} \frac{x^2\, ln(x/2) x^{-1} I_1(x) + 1}{-\gamma - ln(x/2)\, I_o(x)}$$

$$= \lim_{x\downarrow 0} \frac{x^2/2\; ln(x/2) + 1}{-\gamma - ln(x/2)}$$

$$= \lim_{x\downarrow 0} (x^2/2) = 0.$$

With the approximate expression:

$$\exp(m\Phi_o) - 1 \cong m\Phi_o (1 + \frac{1}{2} m\Phi_o),$$

and the obtained results (4.67) and (4.68) the solution (4.66) can be evaluated:

$$\Phi = \frac{1}{m} ln [1 + (\exp(m\Phi_o) - 1) K_o(mr/\bar{\lambda}) / K(mr_o/\bar{\lambda})]$$

$$\cong \Phi_o (1 + \frac{1}{2} m\Phi_o) \frac{K_o (r/\lambda)}{K_o (r_o/\lambda)} (1 - \frac{1}{8} (mH)^2 F)$$

$$[1 - \frac{1}{2} m\Phi_o (1 + \frac{1}{2} m\Phi_o) \frac{K_o (r/\lambda)}{K_o (r_o/\lambda)} (1 - \frac{1}{8} (mH)^2 F) + \ldots\ldots]$$

$$\cong \Phi_o \frac{K_o (r/\lambda)}{K_o (r_o/ \lambda)} [1 + \frac{1}{2} m\Phi_o (1 - \frac{K_o (r/\lambda)}{K_o (r_o/\lambda)}) - \frac{1}{8} (m\Phi_o)^2 \ldots\ldots] . \quad (4.69)$$

The nonlinearity, expressed as a converging series in terms of the coefficient of nonlinearity m, shows that it is of second order importance. It becomes manifest in case $m\Phi_o$ is large, but in practical situations $m\Phi_o < 1$. The first term represents the solution in accordance to the linear theory.

Moreover, the second order term:

$$\frac{1}{2} m\Phi_o (1 - \frac{K_o (r/\lambda)}{K_o (r_o/\lambda)}),$$

in the obtained result (4.69) reveals once again that the nonlinearity is of no importance nearby the disturbance, since there, $K_o(r/\lambda)/K_o(r_o/\lambda) \cong 1$. At greater distance the influence of nonlinearity appears, which is in agreement with the results previously mentioned about the area of influence.

4.d NONLINEAR VERTICAL POROUS FLOW

Cyclic vertical flow

Periodical loading on deformable soils is a popular subject in the field of coastal engineering and offshore construction. The marine subsoil is attacked by cyclic pressures generated by sea surface waves and a construction based on the sea bottom will move due to wave loading. On flexible soils these movements will cause deformations in the subsoil, and subsequently, pore water pressures arise. Altough not always the periodic character of this phenomenon does create serious pore water motion, forceful topical gradients may occur occasionally, which without precautionary measures can affect the stability.

The relationship of simultaneous reaction of pore water and soil skeleton to cyclic loading for a one-dimensional case in a perfectly elastic environment, according to the linear theory, is given by Barends (1978). Some peculiarities of this phenomenon have already been discussed in section 2.d, see for example Fig. VIII.

In this section some fundamental solutions will be considered in the nonlinear theory, i.e. solutions of the nonlinear storage equation (4.1).

Vertically symmetrical porous flow is governed by, see (3.56):

$$\frac{\partial^2\Phi}{\partial z^2} + m [(\frac{\partial\Phi}{\partial z})^2 - \frac{\partial\Phi}{\partial z}] = \frac{1}{c} \frac{\partial\Phi}{\partial t},$$

or by the transformed storage equation, see (4.19):

$$\zeta^2 \frac{\partial^2 \chi}{\partial \zeta^2} = \frac{\partial \chi}{\partial \tau}, \qquad (4.70)$$

while the different variables are related according to (4.18).

Mellin transformation

As has been outlined in section 4.b the Mellin transformation technique is applicable. A restriction is made to the domain: $0<\zeta<1$, corresponding to the semi-infinite vertical medium: $-\infty<z<0$, whereas the boundary conditions are acting at the boundaries: $\zeta=1$ and $\zeta=0$. It is assumed that all the contributions at $\zeta\downarrow 0$ (or $z\rightarrow -\infty$) will vanish. On the boundary: $\zeta\uparrow 1$ (or $z\uparrow 0$) cyclic vertical flow is generated and hence, a harmonic behaviour is attained. In this respect the extensive potential χ can be expressed by:

$$\chi(\zeta,\tau) = f(\zeta)\exp(i\omega\tau),$$

rendering equation (4.70) into:

$$\zeta^2 \frac{d^2 f}{d\zeta^2} = i\omega f. \qquad (4.71)$$

In accordance to formula (4.35), the Mellin transform, this can be transformed to yield:

$$s(s+1)\tilde{f} \quad + \quad \left\{\frac{df}{d\zeta} - (s+1)f\right\}_{\zeta\uparrow 1} = i\omega\tilde{f}, \qquad (4.72)$$

where the transform $\tilde{f}$ is defined by:

$$\tilde{f}(s) = \int_0^\infty f(\zeta)\,\zeta^{s-1}\,d\zeta. \qquad (4.73)$$

The solution of (4.71) in terms of $\tilde{f}$ leads to:

$$\tilde{f} = \frac{(1+s)f_1 - f'_1}{s(s+1) - i\omega}, \qquad (4.74)$$

in which the boundary conditions are related to:

$$f_1 = f(\zeta) \quad \text{at} \quad \zeta\uparrow 1,$$

$$f'_1 = \frac{df}{d\zeta} \quad \text{at} \quad \zeta\uparrow 1.$$

Inverse transformation

Next, the inverse of $\tilde{f}$ has to be determined, which can be found by performing the integration, see equation (4.27):

$$f(\zeta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\,\zeta^{-s}\,ds, \qquad (4.75)$$

where $c>k$, such that k satisfies the condition that

$$\int_0^\infty |f|\,\zeta^{k-1}\,d\zeta,$$

exists. This condition has to be fulfilled in order to ensure the existence of the actual Mellin transform (4.73).

As has been discussed in section 4.b, the function f is defined in the domain: $0<\zeta<1$, while in the remainder: $\zeta>1$, it is equal to zero. Consequently, f will be finite in the interval $0<\zeta<1$, but the behaviour at $\zeta\downarrow 0$ needs some constraint. Suppose that the following holds:

$$\lim_{\zeta\downarrow 0} f = M\zeta^{\ell} \quad , \quad \text{(M is some finite number)} .$$

Then the integral:

$$\int_0^{\infty} |f|\, \zeta^{k-1} d\zeta ,$$

is finite, if the following inequality holds:

$$\ell + k - 1 > -\frac{1}{2} ,$$

or:

$$k > \frac{1}{2} - \ell .$$

Later, for the obtained solution it will be verified whether the condition:

$$c > k > \frac{1}{2} - \ell , \tag{4.76}$$

is satisfied.

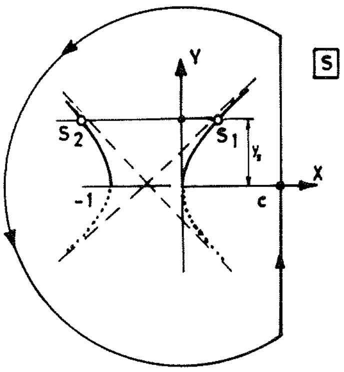


Fig. XX Contour integration in the complex s-plane.

Proceeding with the solution of the integral (4.75), in which the integrant is given by (4.74), contour integration is performed in the complex s-plane, see Fig. XX. First the poles of s are considered. They correspond to the roots of the equation:

$$s(s+1) - i\omega = 0 ,$$

or:

$$s_1 = \frac{1}{2}(s_\omega - 1) , \tag{4.77a}$$

$$s_2 = -\frac{1}{2}(s_\omega + 1) , \tag{4.77b}$$

where:

$$s_\omega = \sqrt{1 + 4i\omega} , \tag{4.77c}$$

and s_ω is chosen such that:

$$Re(s_\omega) \geq 1 . \qquad (4.77d)$$

Developing s_ω yields:

$$s_\omega = \xi + i\eta = \sqrt{1+4i\omega} = \sqrt[4]{1+16\omega^2} \; \{ \cos(\frac{1}{2} \tan^{-1}4\omega) + \sin(\frac{1}{2} \tan^{-1} 4\omega) \} .$$

This results in:

$$\xi^2 - \eta^2 = \sqrt[4]{1+16\omega^2} \cos(\tan^{-1} 4\omega) = 4 ,$$

which represents a hyberbola.

Because $\omega \geq 0$ and $Re(s_\omega) \geq 1$ is chosen, s_ω is located on a branche of a hyperbola in the positive (ξ,η) quadrant. Consequently, $\tilde{f}$ has two poles in the complex s-plane, $s = x + iy$, namely s_1 and s_2, which are points of intersection of the hyperbola:

$$(x + ½)^2 - y^2 = 1 ,$$

and the line:

$$y = 2 \sqrt{\sqrt{1+16\omega^2} \sin(\tfrac{1}{2} \tan^{-1}4\omega)} .$$

The contour along which the integration will be performed is such, that it encompasses both poles. Integration all around this contour, using Cauchy's theorem, results in:

$$\frac{1}{2\pi i} \oint \tilde{f} \exp(-s\,ln\,\zeta) ds = \frac{1}{2\pi i} \oint \tilde{f}\zeta^{-s} ds + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}\zeta^{-s} ds$$

$$= \Sigma \text{ Residuals } (\tilde{f}\zeta^{-s}) .$$

It can be shown that the contributions along the circular paths of this contour will vanish for $|s| \to \infty$, and in view of (4.75) the solution of f is:

$$f = \Sigma \text{ Residuals } (\tilde{f}\zeta^{-s}) ,$$

provided that $\tilde{f}\zeta^{-s}$ is analytical in the area in the s-plane encompassed by the contour.

The residuals with expression (4.74) give:

$$f = \frac{(1 + s_\omega) f_1 - 2f_1'}{2s_\omega} \zeta^{(1-s_\omega)/2} - \frac{(1 - s_\omega) f_1 - 2f_1'}{2s_\omega} \zeta^{(1+s_\omega)/2} . \qquad (4.78)$$

At this stage it is possible to interrelate the boundary conditions at $\zeta \uparrow 1$ and $\zeta \downarrow 0$. Corresponding to the definition of f_1 and f_1' see (4.74), the function f, expression (4.78), has to satisfy the following relations:

$$f_1 = \frac{(1 - s_\omega) f_1 - 2f_1' - (1 - s_\omega) f_1 + 2 f_1'}{2s_\omega} = f_1$$

and:

$$\lim_{\zeta \downarrow 0} f = 0 . \qquad (4.79)$$

Rewriting (4.78) results into:

$$f = \frac{1}{2s_\omega} \zeta^{(1-s_\omega)/2} \left[(1 + s_\omega) f_1 - 2f_1' - \{ (1 - s_\omega) f_1 - 2f_1' \} \zeta^{s_\omega} \right] .$$

Recalling the fact that $Re(s_\omega)\geq 1$, the limit value of this expression at ζ tending to zero yields:

$$\lim_{\zeta\downarrow 0} f = \lim_{\zeta\downarrow 0} \frac{1}{2s_\omega}\zeta^{(1-s_\omega)/2}\left[(1+s_\omega)f_1 - 2f_1'\right]$$

$$= \frac{1}{2s_\omega}\left[(1+s_\omega)f_1 - 2f_1'\right]\lim_{\zeta\downarrow 0}\zeta^{(1-s_\omega)/2}.$$

To satisfy condition (4.79) the term between square brackets has to be zero, or:

$$(1+s_\omega)f_1 = 2f_1'.$$

Therefore, the solution (4.78) becomes:

$$f = f_1\,\zeta^{(1+s_\omega)/2}. \tag{4.80}$$

Obviously the boundary conditions are satisfied. Moreover, inserting solution (4.80) into the governing differential equation (4.71) proves that it is the proper solution:

$$\frac{df}{d\zeta} = f_1\left(\frac{s_\omega+1}{2}\right)\zeta^{\frac{s_\omega+1}{2}-1}$$

$$\zeta^2\,\frac{d^2f}{d\zeta^2} = f_1\zeta^2\left(\frac{s_\omega+1}{2}\right)\left(\frac{s_\omega-1}{2}\right)\zeta^{\frac{s_\omega+1}{2}-2} = f\,(-s_2)(s_1) = i\omega f\,,$$

where the expression for the poles s_1 and s_2, according to (4.77a) and (4.77b) has been introduced.

The condition for existence, expression (4.76), is verified. In view of solution (4.80), which is valid in the domain: $0<\zeta<1$, the following holds:

$$\lim_{\zeta\downarrow 0} |f| = M\,\zeta^{\ell},$$

where ℓ is related to s_ω, and employing (4.77d) gives:

$$\ell = Re\left[(1+s_\omega)/2\right] \geq 1$$

In conclusion, the choise that c is such that $c>Re(s_\omega)$ is valid in the elaboration of the contour integration, see Fig. XX, is fully in accordance to the requirement that $c>k>\frac{1}{2}-\ell$ must be satisfied.

In terms of the extensive potential χ solution (4.80) becomes:

$$\chi = f\exp(i\omega\tau) = f_1\exp\left[i\omega\tau + \frac{1}{2}(s_\omega+1)\,ln\zeta\right]. \tag{4.81}$$

Separation of the real part leads to the response to a cyclic loading at $\zeta\uparrow 1$ conformal to:

$$\chi_1 = f_1\cos(\omega\tau).$$

Elaboration yields:

$$Re(\chi) = f_1\,Re\left[\exp\left\{i\omega\tau + \frac{1}{2}(s_\omega+1)\,ln\zeta\right\}\right]$$

$$= f_1\,Re\left[\exp\left\{i\omega\tau + \frac{1}{2}\left(\sqrt[4]{1+16\omega^2}\,\left(\cos\left(\frac{1}{2}\tan^{-1}4\omega\right) + i\sin\left(\frac{1}{2}\tan^{-1}4\omega\right)\right)+1\right)ln\zeta\right\}\right]$$

$$= f_1 \, Re\left[\exp\left\{ i(\omega\tau + \frac{1}{2}\sqrt[4]{1+16\omega^2}\,\sin(\frac{1}{2}\tan^{-1}4\omega)\, ln\zeta) + \frac{1}{2}(\sqrt[4]{1+16\omega^2}\,\cos(\frac{1}{2}\tan^{-1}4\omega)+1) ln\zeta \right\}\right]$$

$$= f_1 \cos\left[\omega\tau + \frac{1}{2}\sqrt[4]{1+16\omega^2}\,\sin(\frac{1}{2}\tan^{-1}4\omega) ln\zeta \right] \exp\left[\frac{1}{2}(\sqrt[4]{1+16\omega^2}\,\cos(\frac{1}{2}\tan^{-1}4\omega)+1) ln\zeta \right]$$

$$= f_1 \cos\left[\omega\tau + \frac{1}{2}\mu \tan\psi \, ln\zeta \right] \exp\left[\frac{1}{2}(1+\mu) ln\zeta \right],$$

where:

$$\psi = \frac{1}{2}\tan^{-1}4\omega,$$

$$\mu = \sqrt[4]{1+16\omega^2}\,\cos\psi .$$

The solution in terms of ϕ becomes, using (4.18):

$$\phi = \frac{1}{m} ln\left[1 + f_1 \cos(\omega m^2 ct + \frac{1}{2} mz\mu\tan\psi) \exp(m(1+\mu)z/2) \right]. \tag{4.82}$$

The cyclic boundary condition was expressed in terms of χ. However, the boundary condition in terms of ϕ at $z\uparrow 0$, according to:

$$\phi_o = \frac{1}{m} ln\left[1 + f_1 \cos(\omega m^2 ct) \right] = \frac{1}{m} ln\left[1 + (\exp(m\phi_1) - 1)\cos(\omega_1 t) \right],$$

can be approximated for $m\phi_1 < 1$ to:

$$\phi_o = \phi_1 \cos(\omega_1 t) \;,\; \phi_1 = \frac{1}{m} ln(f_1 + 1), \tag{4.83}$$

which shows, that the original boundary condition has a comparable cyclic character in the real domain, while the circular frequency ω_1 is related to ω by:

$$\omega_1 = \omega\, m^2 c .$$

Some graphs corresponding to this solution are represented in Fig. XXI.

Vertical flow due to a sudden disturbance

Two cases are considered, a unit shock disturbance and a unit step disturbance.
Again, the vertical flow in the nonlinear theory is described by the differential equation (4.70):

$$\zeta^2 \frac{\partial^2\chi}{\partial\zeta^2} = \frac{\partial\chi}{\partial\tau} \quad , \quad 0<\zeta<1 .$$

Consider the following boundary conditions:

$$\lim_{t\downarrow 0} \chi = 0 \quad \text{for} \quad 0<\zeta<1, \tag{4.84a}$$

$$\chi = 0 \quad \text{for} \quad \zeta\downarrow 0, \tau>o, \tag{4.84b}$$

$$\chi = c_1 n(\tau) \quad \text{for} \quad \zeta\uparrow 1, \tag{4.84c}$$

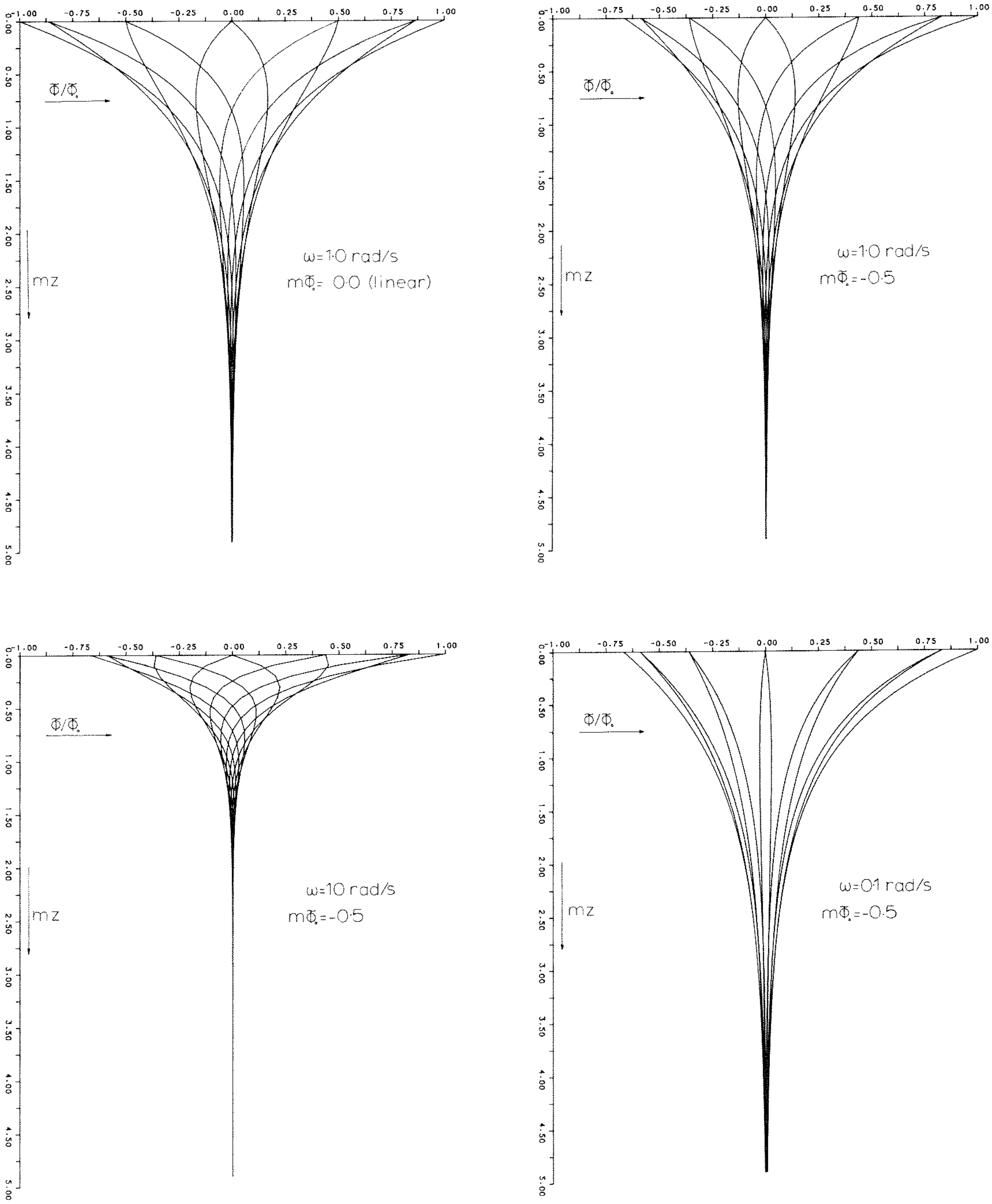


Fig. XXI Cyclic vertical nonlinear porous flow.

where $n(\tau)$, called the Dirac delta-function, is defined by:

$$n(\tau) = 0 \quad \text{for} \quad \tau > \varepsilon > 0 \;, \; (\varepsilon \text{ is small}),$$

$$\int_{-\infty}^{+\infty} n(\tau)\, d\tau \;=\; 1\,.$$

Laplace transformation, defined by:

$$\overline{\chi} \;=\; \int_0^{\infty} \chi \exp(-p\tau)\, d\tau\,,$$

yields the following differential equation for (4.70), taking into account conditions (4.84a) and (4.84b):

$$\zeta^2 \frac{d^2\bar{\chi}}{d\zeta^2} = p\bar{\chi}.$$

The general solution of this type of equation has been solved in the cyclic flow problem. The corresponding solution, equation (4.80), applies here again. Thus:

$$\bar{\chi} = c_1 \zeta^{\frac{1}{2} + \sqrt{p+1/4}} , \tag{4.85}$$

represents the solution, while c_1 is related to the boundary condition (4.84c).

In a similar manner a solution can be obtained for a slightly different boundary condition, namely in stead of (4.84c):

$$\chi = c_1 H(\tau) \quad \text{for} \quad \zeta \uparrow 1 , \tag{4.84d}$$

in which $H(\tau)$, called the Heaviside unit-step function, is defined according to:

$$H(\tau) = 0 \quad \text{for} \quad \tau < 0 ,$$

$$H(\tau) = 1 \quad \text{for} \quad \tau > 0 .$$

In this case the general solution takes the form:

$$\bar{\chi} = \frac{c_1}{p} \zeta^{\frac{1}{2} + \sqrt{p + 1/4}} . \tag{4.86}$$

Laplace inverse transformation

The solution of both transformed expressions (4.85) and (4.86) requires a Laplace inversion, which will be performed by contour integration, see Fig. XXII.

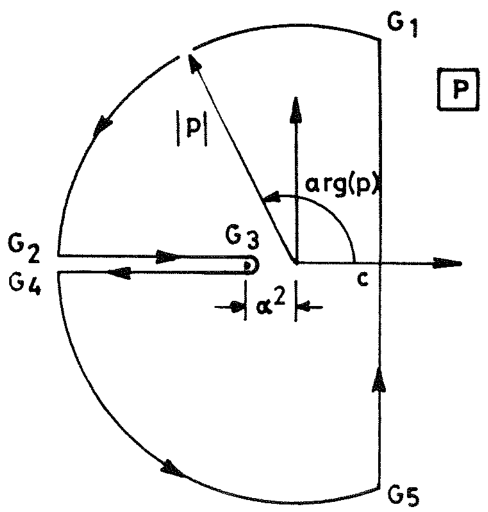


Fig. XXII Contour integration in the complex p-plane.

The inverse of the function $\bar{f}$ is sought:

$$\bar{f}(p) = \frac{A(p)}{B(p)} \exp\left[-y\sqrt{p + \alpha^2} \right], \tag{4.87}$$

with: $y<0$,

and: $Re(\alpha)>0$.

A(p) and B(p) are rational polynomials in p while A is of less or equal order compared to B.

The function $\bar{f}(p)$ is analytical in the area enclosed by the contour in the complex p-plane, see Fig. XXII, by means of a branch cut along:

$$-\infty < Re\,(p) < -\alpha^2 .$$

For the argument along this cut, the following specific choice is made.

$$\lim_{\arg(p)\uparrow\pi} \sqrt{p + \alpha^2} = iv, \tag{4.88a}$$

$$\lim_{\arg(p)\downarrow -\pi} \sqrt{p + \alpha^2} = -iv, \tag{4.88b}$$

while v is a real number, parameterizing along the branch cut:

$$p = -(v^2 + \alpha^2) .$$

$\bar{f}$ is now analytical within the enclosed area.

The inverse of $\bar{f}$ is by definition:

$$f(\tau) = \frac{1}{2\pi i} \int_{c-\infty}^{c+i\infty} \bar{f}(p) \exp(p\tau)\, dp . \tag{4.89}$$

Applying Cauchy's theorem leads to:

$$\frac{1}{2\pi i} \oint \bar{f}(p)\exp(p\tau)dp = \Sigma \text{ Residuals}\,(\bar{f}(p)\exp(p\tau)) . \tag{4.90}$$

The assumption that the contributions along the circular parts of the contour will vanish for $|p| \to \infty$, results in the solution for f:

$$f(\tau) = \Sigma \text{ Residuals}\,(\bar{f}(p)\exp(p\tau)) - \frac{1}{2\pi i} \int_{G_2}^{G_4} \bar{f}(p)\exp(p\tau)dp . \tag{4.91}$$

It will not be shown here, that for $\bar{f}$ defined by (4.87) the contributions along the circular paths vanish. A later check of the solutions by substitution will prove its completeness.

Eleboration of the integral in (4.91) yields:

$$\frac{1}{2\pi i} \int_{G_2}^{G_4} \bar{f} \exp(pt)dp = \frac{1}{2\pi i} \int_{G_2}^{G_4} \frac{A(p)}{B(p)} \exp\left[-y\sqrt{p + \alpha^2} + p\tau \right] dp$$

$$= \frac{1}{2\pi i} \int_{G_2}^{G_3} \cdots\cdots \quad + \quad \frac{1}{2\pi i} \int_{G_3}^{G_4} \cdots\cdots$$

$$= \frac{1}{2\pi i} \int_{\infty}^{0} \frac{A(-v^2-\alpha^2)}{B(-v^2-\alpha^2)} \exp(-iyv - \tau(v^2+\alpha^2))\,(-2vdv)$$

$$+\frac{1}{2\pi i}\int_0^\infty \frac{A(-v^2-\alpha^2)}{B(-v^2-\alpha^2)}\exp(+iyv-\tau(v^2+\alpha^2))\,(-2vdv)$$

$$=\frac{1}{2\pi i}\int_0^\infty \frac{A(-v^2-\alpha^2)}{B(-v^2-\alpha^2)}\exp(-\tau(v^2+\alpha^2))\left[\exp(-iyv)\; - \;\exp(iyv)\right]2vdv$$

$$=\;-\frac{2}{\pi}\int_0^\infty \frac{A(-v^2-\alpha^2)}{B(-v^2-\alpha^2)}\exp(-\tau(v^2+\alpha^2))\sin(yv)\,vdv\,. \qquad (4.92)$$

The unit shock solution

With this result it is now possible to find the inverse transform of (4.85). In this case there are no poles. Thus, according to (4.91) only the integral remains. In fact, the contribution along the branch cut represents the solution. Hence, with $A/B = c_1\,\zeta^{1/2}$:

$$\chi = c_1\zeta^{1/2}\,\frac{2}{\pi}\exp(-\tau/4)\int_0^\infty \exp(-\tau v^2)\;\sin(yv)\,vdv$$

$$= c_1\zeta^{1/2}\exp(-\tau/4)\int_0^\infty \exp(-\tau\theta)\sin(y\sqrt{\theta}\,)\,d\theta\,.$$

The integral in this expression itself represents a Laplace transform, of which the closed form solution is given in literature, see Bateman (1954):

$$\frac{1}{\sqrt{\pi k}}\int_0^\infty \sin(2\sqrt{k\theta}\,)\exp(-\tau\theta)d\theta = \frac{\exp(-k/\tau)}{\tau\sqrt{\tau}}\,.$$

This leads to the final solution, with $y = -ln\zeta$:

$$\chi = c_1\zeta^{1/2}\,y(2\tau\sqrt{\pi\tau}\,)^{-1}\exp(-\tau/4\; - \;y^2/4\tau)$$

$$= c_1 y\,(2\tau\sqrt{\pi\tau}\,)^{-1}\exp(-y/2)\exp(-\tau/4\; - \;y^2/4\tau)$$

$$= c_1 y\,(2\tau\sqrt{\pi\tau}\,)^{-1}\exp(-(\tau+y)^2/4\tau)$$

$$= -c_1\,ln\zeta\,(2\tau\sqrt{\pi\tau}\,)^{-1}\exp(-\;(\tau-ln\zeta)^2/4\tau)\,. \qquad (4.93)$$

In terms of ϕ this gives:

$$\phi = \frac{1}{m}\,ln\left[1-c_1\frac{mz}{2\tau\sqrt{\pi\tau}}\exp(-(\tau-mz)^2/4\tau)\right],\quad \tau=m^2ct\,,\quad z<0\,, \qquad (4.94)$$

denoting the response of porous flow to a unit shock load in the nonlinear theory. Some graphs of this solution are given in Fig. XXIII.

The unit step solution

Next, the inverse transform of (4.86) is considered. There exists a pole in $p=0$. The residual is: $c_1\zeta$. The contribution along the branch cut, according to expression (4.92) with $A/B = c_1\zeta^{1/2}/p$ gives the solution in terms of χ:

$$\chi = c_1\zeta(1-\frac{2}{\pi}\zeta^{-1/2}\exp(-\tau/4)\int_0^\infty \frac{\sin(yv)}{v^2+1/4}\exp(-\tau v^2)vdv)\,. \qquad (4.95)$$

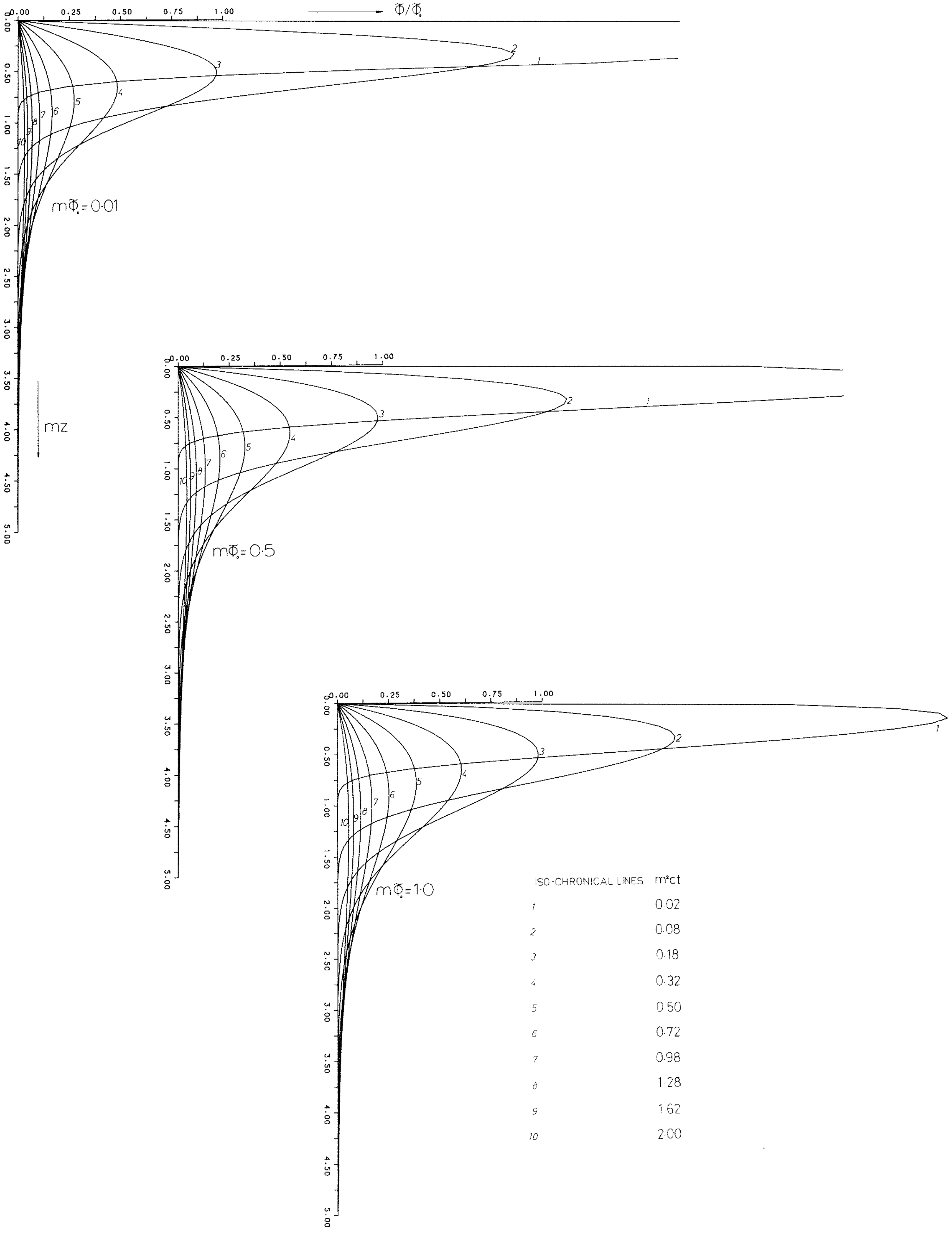


Fig. XXIII The unit shock solution for vertical nonlinear porous flow.

The integral in this formula is a Fourier transform and the closed form solution is mentioned in literature, see Bateman (1954):

$$\int_0^\infty \frac{v}{v^2+\alpha^2} \exp(-\tau v^2)\sin(yv)\,dv =$$

$$\frac{\pi}{4}\exp(\tau\alpha^2)\{\exp(-\alpha y)\,\mathrm{erfc}(\alpha\sqrt{\tau} - y/2\sqrt{\tau}) - \exp(\alpha y)\,\mathrm{erfc}(\alpha\sqrt{\tau} + y/2\sqrt{\tau})\}.$$

This renders (4.95) into its final form, with $y = -\mathit{ln}\zeta$:

$$\chi = c_1\zeta - \frac{1}{2}c_1\zeta^{1/2}\{\zeta^{1/2}\,\mathrm{erfc}(\sqrt{\tau}/2 + \mathit{ln}\zeta/2\sqrt{\tau}) - \zeta^{-1/2}\mathrm{erfc}(\sqrt{\tau}/2 - \mathit{ln}\zeta/2\sqrt{\tau})\}$$

$$= c_1\{\zeta(1 - \frac{1}{2}\mathrm{erfc}(\sqrt{\tau}/2 + \mathit{ln}\zeta/2\sqrt{\tau})) + \frac{1}{2}\mathrm{erfc}(\sqrt{\tau}/2 - \mathit{ln}\zeta/2\sqrt{\tau})\} \qquad (4.96)$$

In terms of ϕ the solution becomes:

$$\phi = \frac{1}{m}\mathit{ln}[1 + c_1\{\exp(mz)(1 - \frac{1}{2}\mathrm{erfc}(\frac{1}{2}m\sqrt{ct} + z/2\sqrt{ct})) + \frac{1}{2}\mathrm{erfc}(\frac{1}{2}m\sqrt{ct} - z/2\sqrt{ct})\}], \quad z<0,\ t>0, \qquad (4.97)$$

representing the response of porous flow to a unit step load in the nonlinear theory. Some graphs of (4.97) are represented in Fig. XXIV.

It can be shown that expression (4.96) satisfies the differential equation (4.70) and the stated boundary conditions (4.84a), (484b) and (4.84d). The proof is quite laborious. It is presented here, since in an earlier stage of this discussion the contribution along the circular paths of the contour have arbitrarily been assumed identical to zero.

Check on the completeness of the unit step solution

First some limit values are considered.

$$\lim_{\tau\to\infty} \mathrm{erfc}(\sqrt{\tau}/2 \pm \mathit{ln}\zeta/2\sqrt{\tau}) = \lim_{\tau\to\infty}\mathrm{erfc}(\sqrt{\tau}/2) = 0, \quad \zeta>0$$

$$\left.\begin{array}{l}\lim_{\tau\downarrow 0}\mathrm{erf}(y/2\sqrt{\tau}) = 1, \quad y>0 \\ -\mathrm{erf}(y) = \mathrm{erf}(-y)\end{array}\right\} \quad \lim_{\tau\downarrow 0}\mathrm{erf}(\mathit{ln}\zeta/2\sqrt{\tau}) = -1, \quad 0<\zeta<1.$$

Thus,

$$\lim_{\tau\downarrow 0}\mathrm{erfc}(\sqrt{\tau}/2 \pm \mathit{ln}\zeta/2\sqrt{\tau}) = \lim_{\tau\downarrow 0}\mathrm{erfc}(\pm\mathit{ln}\zeta/2\sqrt{\tau})$$

$$= \lim_{\tau\downarrow 0}(1 - \mathrm{erf}(\pm\mathit{ln}\zeta/2\sqrt{\tau})) = \lim_{\tau\downarrow 0}(1 \mp \mathrm{erf}(\mathit{ln}\zeta/2\sqrt{\tau})) = 1 \pm 1, \quad 0<\zeta<1.$$

Obviously the boundary conditions (4.84a), (4.84b) and (4.84d) are satisfied, since:

$$\lim_{\tau\to\infty}\chi = c_1\zeta, \quad 0<\zeta<1, \qquad (4.98)$$

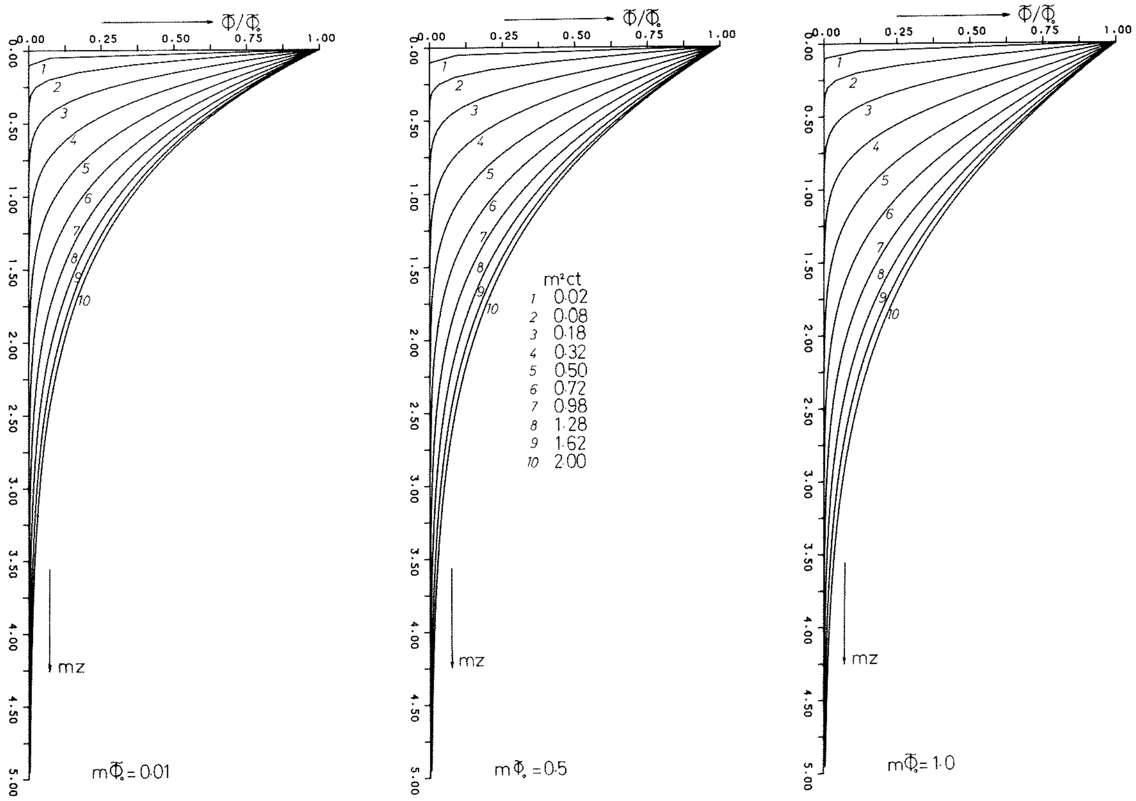


Fig. XXIV The unit step solution for vertical nonlinear porous flow.

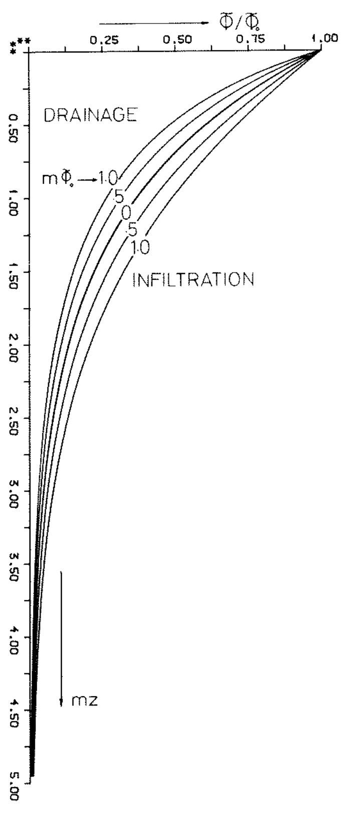


Fig. XXV Anomalous steady vertical nonlinear flow.

which represents the steady state solution, see Barends (1978), and:

$$\lim_{\tau \downarrow 0} \chi = c_1 \{ \zeta(1-\frac{1}{2}2) + 0 \} = 0 \quad , \quad 0<\zeta<1 \, ,$$

and also:

$$\lim_{\zeta \uparrow 1} ln\zeta/2\sqrt{\tau} = 0 \quad \rightarrow \quad \chi(1,\tau) = c_1 \quad , \quad \tau>0 \, .$$

Finally, the question whether the solution satisfies the governing field equation, is considered.

The derivatives:

$$\frac{d}{dx} \text{erfc}\,(f) = -\frac{2}{\sqrt{\pi}} e^{-f^2} \frac{df}{dx} \, ,$$

$$\frac{d^2}{dx^2} \text{erfc}\,(f) = -\frac{2}{\sqrt{\pi}} e^{-f^2} \{ \frac{d^2f}{dx^2} - 2f\,(\frac{df}{dx})^2 \} \, ,$$

with:

$$f^{\pm} = f^{\pm}(\zeta, \tau) = \sqrt{\tau}/2 \pm ln\zeta/2\sqrt{\tau} \, ,$$

$$\frac{\partial f^{\pm}}{\partial t} = \frac{1}{4\sqrt{\tau}} (1 \mp ln\zeta/\tau) \, ,$$

$$\frac{\partial f^{\pm}}{\partial \zeta} = \pm \, 1/(2\zeta\sqrt{\tau}\,) \, ,$$

$$\frac{\partial^2 f^{\pm}}{\partial \zeta^2} = \mp \, 1/(2\zeta^2\sqrt{\tau}\,) \, ,$$

yield:

$$\zeta^2 \frac{\partial^2}{\partial \zeta^2} \text{erfc}(f^{\pm}) = \frac{1}{2\sqrt{\pi\tau}} e^{-f^2} \{ \pm 2 + 1 \pm ln\zeta/\tau \} \, ,$$

$$\frac{\partial}{\partial \tau} \text{erfc}(f^{\pm}) = \frac{-1}{2\sqrt{\pi\tau}} e^{-f^2} \{ 1 \mp ln\zeta/\tau \} \, .$$

Moreover, the following holds:

$$\zeta^2 \frac{d^2}{d\zeta^2} [\, \zeta(1-g(\zeta)) \,] = -2\zeta^2 \frac{dg}{d\zeta} - \zeta^3 \frac{d^2g}{d\zeta^2} \, .$$

The governing field equation is:

$$\zeta^2 \frac{\partial^2 \chi}{\partial \chi^2} = \frac{\partial \chi}{\partial \tau} \, . \tag{4.99}$$

Elaboration of the left hand side with solution (4.96) using the expressions above mentioned gives:

$$\zeta^2 \frac{\partial^2 \chi}{\partial \zeta^2} = \zeta^2 \frac{\partial^2}{\partial \zeta^2} [\, c_1 \{ \zeta(1-\frac{1}{2}\text{erfc}(f^+)) + \frac{1}{2}\text{erfc}(f^-) \} \,]$$

$$= c_1 [\, -\zeta^2 \frac{\partial}{\partial \zeta} \text{erfc}(f^+) - \frac{1}{2} \zeta^3 \frac{\partial^2}{\partial \zeta^2} \text{erfc}(f^+) + \frac{1}{2} \zeta^2 \frac{\partial^2}{\partial \zeta^2} \text{erfc}(f^-) \,]$$

$$= c_1 \left[\zeta^2 \frac{2}{\sqrt{\pi}} e^{-f^2}/(2\zeta\sqrt{\tau}\,) - \frac{1}{2} \zeta e^{-f^2} (\zeta + ln\zeta/\tau)/(2\sqrt{\pi\tau}\,) \right.$$

$$\left. + \frac{1}{2} e^{-f^2} (-1 - ln\zeta/\tau)/(2\sqrt{\pi\tau}\,) \right]$$

$$= c_1 \, e^{-f^2} \left[\zeta(1 - ln\zeta/\tau) - (1 + ln\zeta/\tau) \right] / (4\sqrt{\pi\tau}\,) .$$

Elaboration of the right hand side of equation (4.99) yields:

$$\frac{\partial \chi}{\partial \tau} = \frac{\partial}{\partial \tau} c_1 \left[\zeta(1 - \frac{1}{2} \operatorname{erfc}(f^+)) + \frac{1}{2} \operatorname{erfc}(f^-) \right]$$

$$= c_1 \left[\frac{1}{2} \zeta e^{-f^2} (1 - ln\zeta/\tau)/(2\sqrt{\pi\tau}\,) - \frac{1}{2} e^{-f^2} (1 + ln\zeta/\tau)/(2\sqrt{\pi\tau}\,) \right]$$

$$= c_1 e^{-f^2} \left[\zeta (1 - ln\zeta/\tau) - (1 + ln\zeta/\tau) \right] / (4\sqrt{\pi\tau}\,) .$$

And the completeness of solution (4.96) is proved.

A physical anomaly

In this section a general solution procedure has been applied to derive complete solutions of some characteristic problems in the fields of porous flow in deformable soils including non-linear effects. One remark has to be made on the steady state solution. In the last example steady state is attained at $t \to \infty$, see expression (4.98), of which the form in terms of ϕ can be described by, see Barends (1978):

$$\phi = \frac{1}{m} ln \left[1 + (\exp(m\phi_1) - 1)\exp(mz) \right] , \quad -\infty < z < 0 ,$$

where ϕ_1 denotes the value at $z \uparrow 0$. Some graphs of this particular case are represented in Fig. XXV.

This result implies that a vertical steady flow exists according to the nonlinear theory, while the boundary condition at $z \to -\infty$ is not affected. In other words the medium shows an infinite capacity of storage. This physical anomaly is a direct consequence of the pressure dependent density, $\varrho'(p)$, defined according to (2.20) which assumes:

$$\varrho' = \varrho_o' \exp(\beta'(p - p_o)) \; , \; \beta' > 0 .$$

In an infinite vertical strip it can be expected that the density will tend to infinite values for $z \to \infty$. Thus, the 'deep' boundary at $z \to -\infty$ contains such a density, that vertical transport of pore water is possible, without actually changing the situation at this boundary. The steady state vertical flow solution owes its character to this peculiarity, which is physically not correct, because the real behaviour of the density at large pressures is not described well by the stated formula keeping the compressibility β' a constant.

The presented solutions in the semi-infinite domain which do not affect the mean state of pressure, like the case of the unit shock response and the cyclic agitation, do not include this anomaly, and they are valid in the entire domain.

5. GEOMETRIC NONLINEARITY

Summary of chapter 5

A remarkable feature of groundwater flow is the existence of a free surface, or a phreatic surface. It constitutes the separation between the flow domain and the exterior, where the eventually present pore water does not take part in the gravitational flow process. In nature such a sharp interface does not exist. The transition between the flow domain and the exterior appears in a zone. However, for the mathematical formulation of groundwater flow a sharp border is assumed.
A similar situation occurs when the flow domain contains two immiscible fluids each of them occupying a different region. At the interface of both fluids a separation surface is assumed, which behaves somehow like a free surface.

The free surface represents a part of the flow field border along which specific conditions are imposed: a pressure condition, because there must be always an equilibrium, and a storage condition, which controls the flux across the surface. In case a boundary flux occurs at a free surface, pore water is transported to the exterior. Since all the pore water taking part in the flow process considered must be encompassed by the flow field border, the free surface must move with the efflux. This implies, that the position of the free surface is determined by the boundary flux and precisely the phreatic storage condition governs this process. It is also referred to as the moving or kinematic boundary condition. Because the geometry of the flow field varies, this class of problems is typified by geometric nonlinearity.

In section 5.a the background of the moving boundary condition is outlined. It is shown that the compressibility of the pore water and volumetric storage variations due to soil deformations have no influence on the formulation of the free surface storage condition. Moreover, the convective terms caused by infiltration at the free surface are included. The moving boundary condition is essentially nonlinear.

In section 5.b several aspects of a numerical procedure to solve general porous flow problems including presence of a free boundary are considered. The calculus of variations provides a mean to discretize. The fundamentals of the calculus of variations are described, whereas general boundary conditions are included in the governing functional. The basis of a finite element formulation is given and some different methods are briefly discussed. Application of a user-orientied computer program, the SEEP code, is presented for a case concerning different types of flow behaviour, for an axi-symmetric free surface problem, for a drainage and infiltration problem in an inhomogeneous flow field and for a case of a discontinuous phreatic surface. About the contribution of capillary water some remarks are mentioned.

In section 5.c the numerical treatment of time dependent phreatic flow problems is considered in detail. Several methods are discussed. Most of them are incomplete in case of infiltration. For an explicit method the stability of the numerical procedure is expressed in the form of a time step criterion, which includes the coherence of the flow field and the rate of change of prescribed time dependent boundary values. A reduction is advised to improve accuracy. An example of a sand dyke subjected to tides and a storm surge is included to prove the reliability of the time step criterion suggested.

5.a NONLINEAR TERMS IN THE MOVING BOUNDARY CONDITION

A phreatic surface separates the saturated flow field from the unsaturated area. In nature there is no such sharp border, i.e. a surface of discontinuity, between the saturated and unsaturated parts in soils. In this section the phreatic surface is defined as the imaginary surface where the pressure equals the atmospheric pressure p_o, which is assumed to be a constant. Thus, if the vertical position of an element of the surface is denoted by z, the piezometric head ϕ at that position is described by:

$$\phi = z + p_o/\varrho g \,. \tag{5.1}$$

The pore fluid present in the porous area above the phreatic surface also will contribute to

porous flow. Here, this contribution is disregarded and the phreatic surface is considered as a material-of-fluid surface. In other words, when there is no accretion (infiltration) or evaporation, this surface contains the same fluid elements. Once on a phreatic surface such an elementary fluid volume will remain at this surface, but not necessarily at the same position.
Since in time dependent flow the separation surface is moving, it can be represented by an equation, of which the general form is:

$$S(\ell,t) = 0\,, \quad \ell = (x,y,z)\,, \tag{5.2}$$

where ℓ denotes the position vector. Another interpretation is, that property S, which according to (5.1) can be defined by:

$$S = z + p_o/\varrho g - \Phi = 0\,, \tag{5.3}$$

is assigned to any elementary fluid volume at the phreatic surface.

Consider such a volume at initial position ℓ_o at time t_o.
The position at any later time t is defined by $\ell(\ell_o,t)$. The rate of change of this position represents the absolute velocity:

$$w = \frac{D\ell}{Dt}\,, \quad \ell_o \text{ is constant}. \tag{5.4}$$

Here, D/Dt denotes the (fluid) substantial derivative, indicating partial differentation with respect to time, while following a fluid volume of fixed identity, as it is displaced in the flow field.

The substantial derivative of property S is defined in a similar manner. Since S is a function of ℓ and t, for ℓ_o is a constant, the chain rule for differentiation applies:

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial \ell} \cdot \frac{D\ell}{Dt}\,,$$

which becomes in virtue of (5.4):

$$\frac{DS}{Dt} = \frac{\partial s}{\partial t} + w \cdot \nabla S\,. \tag{5.5}$$

Note that $\partial/\partial\ell$ is identical to the spatial gradient.

The general conservation principle, as it has been derived in section 3.a, equation (3.1), holds also for property S, while propagating at velocity w. Thus:

$$\frac{\partial S}{\partial t} + \nabla \cdot (wS) = J\,, \tag{5.6}$$

where J denotes the free internal production of property in the considered fluid element.

In view of the discussion in section 3.b the conservation of matter of a compressible pore fluid, propagating at velocity w in a deforming soil skeleton moving at velocity v, can be expressed by:

$$-\nabla \cdot \{\varrho' n(w - v)\} = \varrho' \frac{D\varepsilon}{Dt} + n\varrho'\beta' \frac{Dp}{Dt}\,,$$

Here, D/Dt, which is called the (skeleton) substantial derivative, is defined by:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla\,,$$

and it denotes the substantial derivative, while following a soil particle moving at velocity v.
The above equation can be rewritten into the form:

$$\nabla \cdot w = -F, \tag{5.7}$$

where F is given by:

$$F = \frac{1}{n}\left\{ (1-n)\frac{D\varepsilon}{Dt} + n\beta' \frac{Dp}{Dt} + (w-v)\cdot\nabla n \right\}.$$

Eliminating $\nabla \cdot w$ from equations (5.6) and (5.7) results in:

$$\frac{\partial S}{\partial t} + w \cdot \nabla S = \frac{DS}{Dt} = J + SF. \tag{5.8}$$

Property S incorporates the equilibrium condition (5.3) at any time. Thus, the last term in expression (5.8) vanishes:

$$\frac{\partial S}{\partial t} + w \cdot \nabla S = \frac{DS}{Dt} = J. \tag{5.9}$$

The (fluid) substantial derivative of property S equals the internal production of property S, independently from the mechanical behaviour of the pore fluid and the porous medium in the flow field.

Assume that the internal production only takes place at the phreatic surface. It represents for example accretion (infiltration) or evaporation. Essentially, it has to produce a change in phreatic position in accordance to the definition of the property S, equation (5.3).
Let the internal production J, being a contribution of fluid material at the phreatic surface, be stored in the porous area immediately above the phreatic surface. Hence, it causes a change of position of the phreatic surface.
The motion of the elementary fluid volume considered at the surface is affected by this contribution. In this respect, the convective velocity of property S in equation (5.9) has to be adjusted. It must be enlarged with the velocity produced by the storage of internal production. Therefore, equation (5.9) becomes:

$$\frac{\partial S}{\partial t} + (w + d) \cdot \nabla S = 0,$$

where d represents the velocity due to internal production J. This expression can be written in the form:

$$\frac{DS}{Dt} + (w - v + d) \cdot \nabla S = 0,$$

where the (skeleton) substantial derivative is introduced. Inserting the specific discharge q, given by (3.2), yields:

$$n\frac{DS}{Dt} + (q + B) \cdot \nabla S = 0. \tag{5.10}$$

Here, B denotes the fluid volume added per unit total surface per unit time:

$$B = nd. \tag{5.11}$$

Note that B represents in fact a kind of specific discharge along velocity d. It can be conceived as the time rate of change of the position of the phreatic surface due to accretion.

The specific discharge q is related to the potential ϕ, according to:

$$q = -K\nabla\phi,$$

and equation (5.10) becomes, employing (5.3):

$$nv\cdot\nabla z - n\frac{D\phi}{Dt} + (-K\nabla\phi + B)\cdot\nabla(z-\phi) = 0\,,$$

$$-n\frac{\partial\phi}{\partial t} + (-K\nabla\phi + B + nv)\cdot\nabla(z-\phi) = 0\,,$$

$$n\frac{\partial\phi}{\partial t} = K(\nabla\phi)^2 - (B+nv)\cdot\nabla\phi + (-K\nabla\phi + B+nv)\cdot\nabla z\,. \tag{5.12}$$

This expression gives a complete picture of the boundary condition at a moving phreatic surface. It is referred to as the moving or kinematic boundary condition, and in this discussion the influence of second order terms, free contribution and convective terms due to soil motion have been included.
Particularly the last effect has not been considered in literature up to now. In view of equation (5.12) the convective effect by soil motion can be incorporated in another definition of accretion, to wit, by considering the rate of infiltration including the soil skeleton velocity.
In the absence of skeleton deformations, i.e. $v=0$, equation (5.12) becomes identical to a form given by Todsen (1971). Assuming, that the accretion is only vertical:

$$B = (0, 0, N)\,,$$

and disregarding the soil motion renders (5.12) into:

$$n\frac{\partial\phi}{\partial t} = K(\nabla\phi)^2 - (K+N)\frac{\partial\phi}{\partial t} + N\,, \tag{5.13}$$

which is described by Bear (1972).

By purpose the quantity 'effective porosity' (n_e) has been avoided in the previous discussion, since, according to the definition of the specific discharge, equation (3.2), and to the general storage equation (3.42) valid for air-water mixtures with stagnant air, the mass transport in the pores is considered, whether it contributes partly or completely to the flow motion. In particular in the phreatic zone effects of stagnant air play an important role and a reduction of the saturation degree up to 20% and more is observed, caused by entrapped air. The approach outlined in section 3.c shows here that it is not necessary to distinguish between effective porosity (or phreatic storativity) and the volumetric porosity. The influence of stagnant air must be embedded in a reduction of the permeability, a formula of which is given by expression (3.41). The moving boundary condition should not contain any other quantity than the volumetric porosity. The existence of stagnant pore water (not taking part yet in the gravitational flow) can be incorporated in the infiltration term.

5.b A NUMERICAL MODEL ; THE COMPUTER CODE SEEP

Numerical methods

The analytical method draws attention to general trends in physical processes and helps to distinguish between which factors are of primary significance and which are of secondary importance. It is surely a useful supplement to engineering intuition, but not always the best procedure for quantifying results. Geometric complexity and natural inhomogeneity limit the application of analytical methods. It is therefore evident that, as soon as the generality of numerical methods of analysis were recognized having fast digital computers at one's disposal, emphasis was directed to develop practical methods to treat complex geometries and material properties.

In the field of seepage flow through porous media observed in various disciplines of engineering, the application of computers has become a common practice. Several numerical techniques have been designed amongst which the finite difference method is mentioned, resulted from the early experience with discrete nets of electric resistances (analogue models), and later the finite element method which proved to be a more flexible procedure. Both classical methods are well described by Verruijt (1970). Later, Verruijt (1975) developed a method to incorporate the time dependent storage term by introducing an average value per time step.

Also more sophisticated methods are applied, such as a boundary integral method using a complex potential suggested by Van der Veer (1976). It shows much similarity with the method of collocation, where the approach requires that the chosen field equation is exactly satisfied at a limited number of points per subdomain. Another intricate method is proposed by Baiocchi (1973). He introduced a special function. This method is particularly suited to free surface problems.

The various numerical methods are all attempts to find trial functions, which approximately satisfy the field equations and the boundary conditions. Characteristic differences between these methods are in general of second order importance, but for specific classes of problems a certain method can be preferred.
In this section some aspects of the finite element method are reviewed.

Calculus of variations

Boundary value problems for differential equations governing seepage flow are equivalent to problems of the calculus of variations. Recognition of this fact goes back to Euler and Bernouilli, and the general three-dimensional approach was first considered by Green, in 1837. The calculus of variations can be conceived as an extension of the concept of differentiation. In this manner it is very useful to obtain numerical values for boundary value problems.
The method is based on a minimum principle of a functional. A functional is a variable that assumes a specific numerical value for each function which is substituted into it, for example:

$$J(\phi) = \int_a^b \phi(x)\,dx\,. \tag{5.14}$$

Here, J represents a functional for a class of functions ϕ defined in the domain $a \leq x \leq b$.
The fundamental problem in the calculus of variations is to find a function ϕ, such that increments $\delta\phi$ in this function yield only a second order increment in the functional J.
This condition gives an equation, called the Euler equation, which determines ϕ. If the Euler equation is equivalent to the field equation considered, then the condition that the first order variation of the functional, δJ, vanishes for arbitrary variations in the trial function ϕ, is sufficient to solve the problem.

The variation of a functional is sometimes defined as a derivative, see Nozicka (1969):

$$\delta J(\phi) \equiv \lim_{c \to 0} \frac{J\{\phi(x) + c\delta\phi(x)\} - J\{\phi\}}{c}, \tag{5.15}$$

where c is independent of the space variable x.
Consider a function:

$$L = L\{x, \phi(x), d\phi/dx\}\,.$$

Allowing ϕ to vary by $\delta\phi$ yields for the first order in J, employing the mean value theorem:

$$\begin{aligned}\delta J &= \delta \int_a^b L\,dx = \int_a^b (\delta L)\,dx \\ &= \int_a^b \lim_{c\to 0} \left[\frac{L\{x, \phi + c\delta\phi, \frac{d}{dx}(\phi + c\delta\phi)\} - L\{x, \phi, d\phi/dx\}}{c} \right] dx \\ &= \int_a^b \left\{ \frac{\partial L}{\partial\phi}\delta\phi + \frac{\partial L}{\partial(d\phi/dx)} \frac{d\delta\phi}{dx} \right\} dx\,. \end{aligned} \tag{5.16}$$

Integrating by parts results in:

$$\delta J = \int_a^b \{ \frac{\partial L}{\partial \phi} \delta\phi + \frac{d}{dx}(\frac{\partial L}{\partial(d\phi/(dx)}\delta\phi) - \delta\phi \frac{d}{dx}(\frac{\partial L}{\partial(d\phi/dx)}) \} dx$$

$$= \int_a^b \{ \frac{\partial L}{\partial \phi} - \frac{d}{dx}(\frac{\partial L}{\partial(d\phi/dx)}) \} \delta\phi dx \quad + \quad \frac{\partial L}{\partial(d\phi/dx)} \delta\phi \Big|_a^b . \tag{5.17}$$

The variation $\delta\phi$ is arbitrary in the considered domain. Therefore, the first variation of the functional J is equal to zero, if the differential equation:

$$\frac{\partial L}{\partial \phi} - \frac{d}{dx}(\frac{\partial L}{\partial(d\phi/dx)}) = 0 , \tag{5.18}$$

is satisfied and the boundary contribution:

$$\frac{\partial L}{\partial(d\phi/dx)} \delta\phi \Big|_a^b \tag{5.19}$$

vanishes. Equation (5.18) is called the Euler equation. The function L is referred to as the Lagrange function.
The second order variation of J determines the character of the 'stationary value' of J where $\delta J = 0$. Is $\delta(\delta J)$ positive definite, negative definite or of undeterminate sign, then the value of J is an absolute minimum, an absolute maximum, or a saddle point, respectively.

The approach is easily extended to more dimensions. In order to incorporate also the boundary contributions (5.19) the functional is extended to, see Edelen (1969):

$$J(\phi) = \int_D L\, dV \quad - \quad \int_{\partial D} L' \cdot s\, dA . \tag{5.20}$$

The domain D is the three-dimensional space, where dV denotes a volume element. ∂D is the border of the domain D and dA represents a surface element of this border. The symbol s in expression (5.20) is a unit vector outwardly directed on the surface element dA. The Lagrange function L is a function of space, of the function ϕ and its first derivatives, thus:

$$L = L \{ x, y, z, \phi(x, y, z), \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \} .$$

The function L' is defined on the border as a function of space and of the function $\phi(x,y,z)$, thus:

$$L' = L' \{ x, y, z, \phi(x,y,z) \} .$$

The first variation of J can be expressed as follows:

$$\delta J = \int_D (\delta L)\, dV \quad - \quad \int_{\partial D} (\delta L') \cdot s\, dA$$

$$= \int_D \{ \frac{\partial L}{\partial \phi} \delta\phi + \frac{\partial L}{\partial \phi_{,i}} (\delta\phi)_{,i} \} dV \quad - \quad \int_{\partial D} \frac{\partial L'}{\partial \phi} \delta\phi \cdot s\, dA$$

$$= \int_D \{ \frac{\partial L}{\partial \phi} - (\frac{\partial L}{\partial \phi_{,i}})_{,i} \} \delta\phi\, dV \quad + \quad \int_D (\frac{\partial L}{\partial \phi_{,i}} \phi)_{,i}\, dV \quad - \quad \int_{\partial D} \frac{\partial L'}{\partial \phi} \delta\phi \cdot s\, dA$$

$$= \int_D \{ \frac{\partial L}{\partial \phi} - (\frac{\partial L}{\partial \phi_{,i}})_{,i} \} \delta\phi dV \quad + \quad \int_{\partial D} \{ \frac{\partial L}{\partial \phi_{,m}} - \frac{\partial L'}{\partial \phi} \} \delta\phi \cdot s\, dA . \tag{5.21}$$

The suffix preceded by a comma denotes partial differentiation. Repeated suffixes denote summation (Einstein convention).

In respect of the above result disappearance of δJ for arbitrary $\delta\phi$ is equivalent to the following conditions:

$$\frac{\partial L}{\partial \phi} - \left(\frac{\partial L}{\partial \phi_{,i}}\right)_{,i} = 0 \quad , \quad \text{in D} \tag{5.22}$$

$$\partial \phi = 0 \quad , \quad \text{on} \quad \partial D_1 \tag{5.23}$$

$$\left(\frac{\partial L}{\partial \phi_{,m}} - \frac{\partial L'}{\partial \phi}\right) \cdot s = 0 \quad , \quad \text{on} \quad \partial D_2 \tag{5.24}$$

where the border ∂D is split up into two parts: $\partial D = \partial D_1 + \partial D_2$.

The addition of L′ to the functional J can be considered as a compensation for the contribution arising from constraints on the gradient of the function ϕ at the boundary ∂D_2.

Stationary groundwater flow is described by the set:

$$(K\phi_{,i})_{,i} = 0 \quad , \quad \text{in} \quad D \tag{5.25}$$

$$\phi = f \quad , \quad \text{on} \quad \partial D_1 \tag{5.26}$$

$$K \frac{\partial \phi}{\partial s} = g \quad , \quad \text{on} \quad \partial D_2 \tag{5.27}$$

where:
- ϕ : potential,
- f : a constant,
- s : the outer normal on ∂D_2,
- K : the permeability,
- D : flow field,
- $\partial D_1 + \partial D_2$: the flow field border.

For the choice:

$$L = \frac{1}{2} K (\phi_{,i})^2 \quad , \quad L' = g\phi, \tag{5.28}$$

the set (5.25), (5.26) and (5.27) becomes equal to the conditions (5.22), (5.23) and (5.24). Therefore, the functions (5.28) are the Lagrange functions of which the Euler equations correspond to the field equations of porous flow, while general boundary conditions can be incorporated. The functional J accordingly defined represents a suitable quantity from which the numerical solution can be obtained.

Finite element formulation

The integral representation of the functional is appropriate to spatial approximations. The domain D in which flow is considered is subdived into many small subdomains. In such a subdomain the local potential distribution can be approximated by a given spatial function, such as:

$$\phi \cong \phi(x, y, z, \alpha_1, \alpha_2, \ldots \alpha_m), \tag{5.29}$$

where α_i are yet undetermined parameters. Substitution into the functional J for all subdomains results in an expression in terms of the parameters α_i. For any arbitrary variation in each α_i the functional J must be a constant, in that the first variation of J vanishes. Hence, for each α_i a condition can be formulated when employing the identity:

$$\frac{\partial J}{\partial \alpha_i} \delta\alpha_i = 0,$$

so that a complete set of equations is found from which all the parameters α_i can be determined. Subsequently, the approximate solution (5.29) is defined.

The set of equations can become large, but nowadays fast computers with large memories and sophisticated methods to efficiently solve many equations, allow for almost any flow problem to determine an approximate (weak) solution in a straightforward manner taking into account arbitrary boundary conditions.
The most common approximation per subdomain is a simple linear function parameterized in nodal values at the nodal points suspending the subdomain in space. The derivation of the corresponding set of discrete equations in this case is extensively described in many textbooks, see for example Verruijt (1970). It will not be further outlined.

Also the boundary integral method makes use of the calculus of variation. In this case exact solutions are parameterized in a finite number of boundary nodes defining the border of a subdomain. The final solution is therefore exact inside the subdomains, but fluctuation between selected points can violate the imposed boundary conditions and the compatibility of the solution at the subdomain separations.

The previous analysis shows that, since the functional J is not affected by a limited number of finite jumps in the Lagrange function, different values for the permeability can be assigned per subdomain. There is also no restriction to the form or distribution of subdomains, except that they must cover the flow domain completely.
With the calculus of variations and the availability of large computers the limitations of analytical solving methods, i.e. complex geometry and inhomogeneous material properties, are mastered.

The free boundary condition

As has been the subject of discussion in section 5.a a specific boundary condition exists in the field of seepage flow, to wit, the free surface. This surface coincides with the border of the considered flow domain and it is determined by pressure conditions (atmospheric pressure). The position is usually unknown prior to the calculation, unless the border is fixed, for example a seepage surface.
According to the preceding analysis the application of the calculus of variations upon a functional defined as an integral over the entire flow domain does not automatically include variation in the flow domain itself. A fundamental reconsideration of the analysis is possible to incorporate variation of the domain (defining a kind of Leibnitz' rule for variations).
Another approach was designed by Baiocchi (1972). The idea is to extent the domain over the unsaturated area by definition of a suitable new variable related to the potential Φ.
Consequently, the flow domain can be defined as a constant area in space for this new variable. The field equation in terms of the new variable is appropriate to apply a solution procedure based on calculus of variations.
However, not all boundary conditions can be formulated, and the procedure to obtain the solution in terms of the potential (by derivation) causes loss of accuracy, see Engering (1976).

Recently, another method was suggested and worked out by Bathe and Khoshgoftaar (1979). Their solution algorithm employs a nonlinear permeability description to account for the a priori unknown free boundary position. In fact, an iterative procedure determines the unsaturated area, where corresponding to negative values of the potential head the permeability is assigned fictituous small values (numerically zero).

Time dependency causing a contineously changing position of the free boundary is not covered by the method of Baiocchi or Bathe and Khoshgoftaar. Although their methods do not seem unpractical to treat free surface porous flow problems, no preference is yet found above applying an explicit method, simply using a steady state solving procedure every time step.
The flow domain is allowed to change after every time step in accordance to the moving boundary condition, equation (5.12).
Aware of the fact that the character of the moving boundary condition is nonlinear, whereas the field equation and other boundary conditions are linear, it seems profound to design an explicit procedure. Such a method has been proposed by Verruijt (1970). It involves iteration of the finite element mesh, i.e. an updating of free surface node positions every time step. It requires

therefore a thorough organisation of numerical data and a time step which does not give rise to dominant numerical noise or divergence. In section 5.c a semi-empirical formula for the maximum admissible time step is derived.

The computer code SEEP

The purpose of analysis is to solve practical problems or better, to provide a practical tool to solve problems. Such a user-oriented tool governing general seepage flow is the computer program SEEP, which has been composed at the Delft Soil Mechanics Laboratory by Barends (1976). It uses the finite element technique.

The emphasis for this program was directed towards:

— general application; more dimensional, free surface flow in heterogeneous flow fields of arbitrary geometry under varying boundary conditions;

— automatic accuracy; the user is not hindered by mathematical details and no specific knowledge concerning coherence of matrices, and convergence of iterative solving procedures is required;

— flexibility; different kinds of problems or various aspects of a certain problem can be simultaneously evaluated at little extra effort;

— communication facilities; a simple user-language to input a problem is composed and various possibilities to visualize results are available.

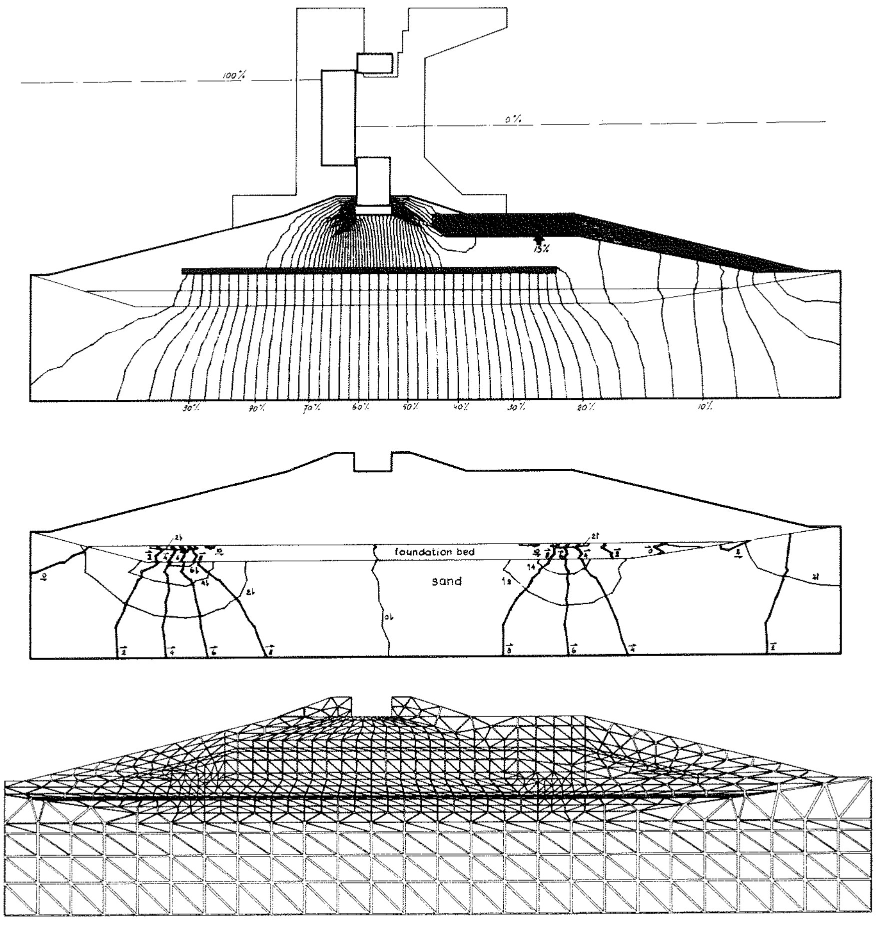


Fig. XXVI Subsurface, nonlinear flow through a foundation of a sea barrier.

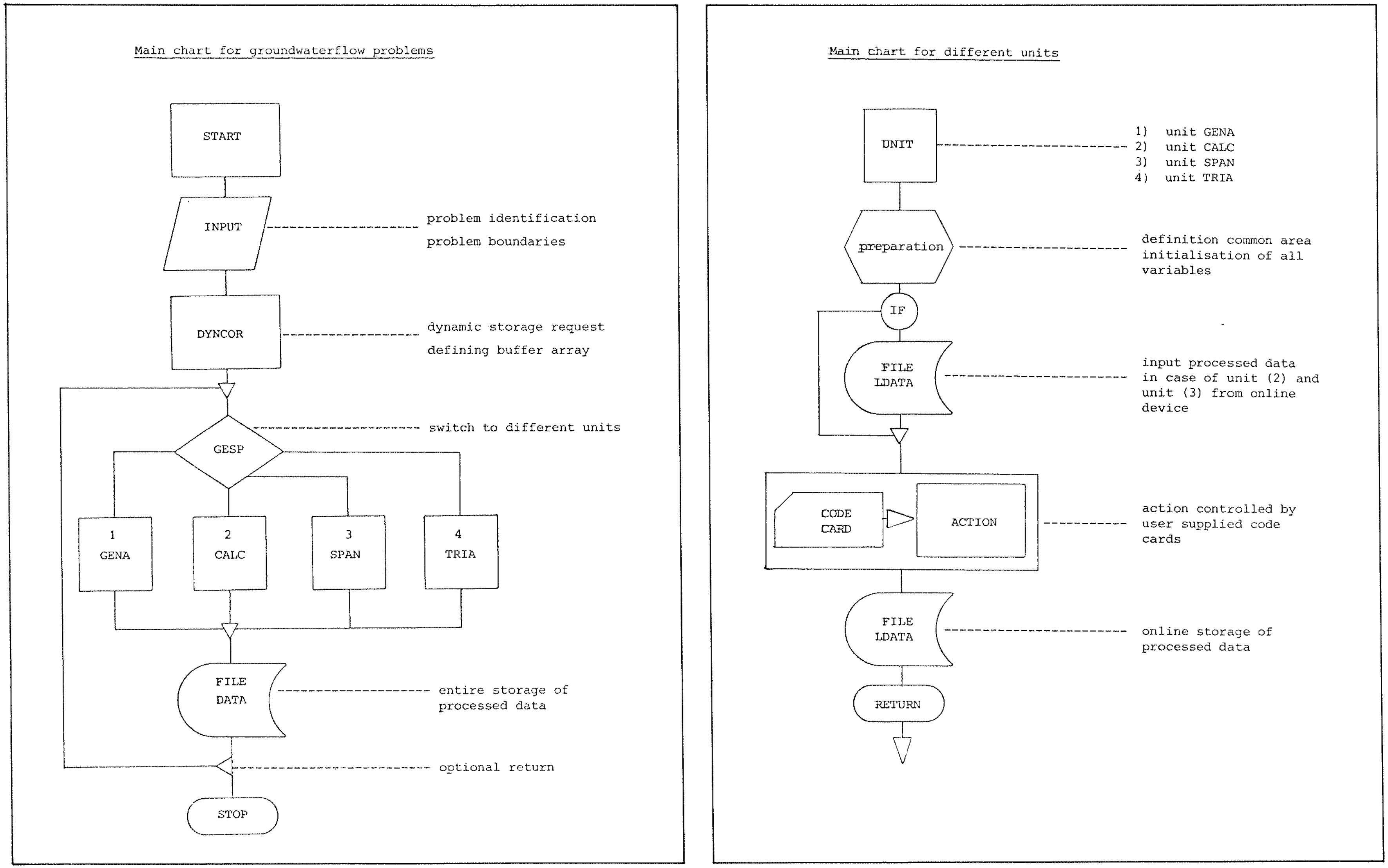


Fig. XXVII The phase structure of SEEP, the block scheme of a phase.

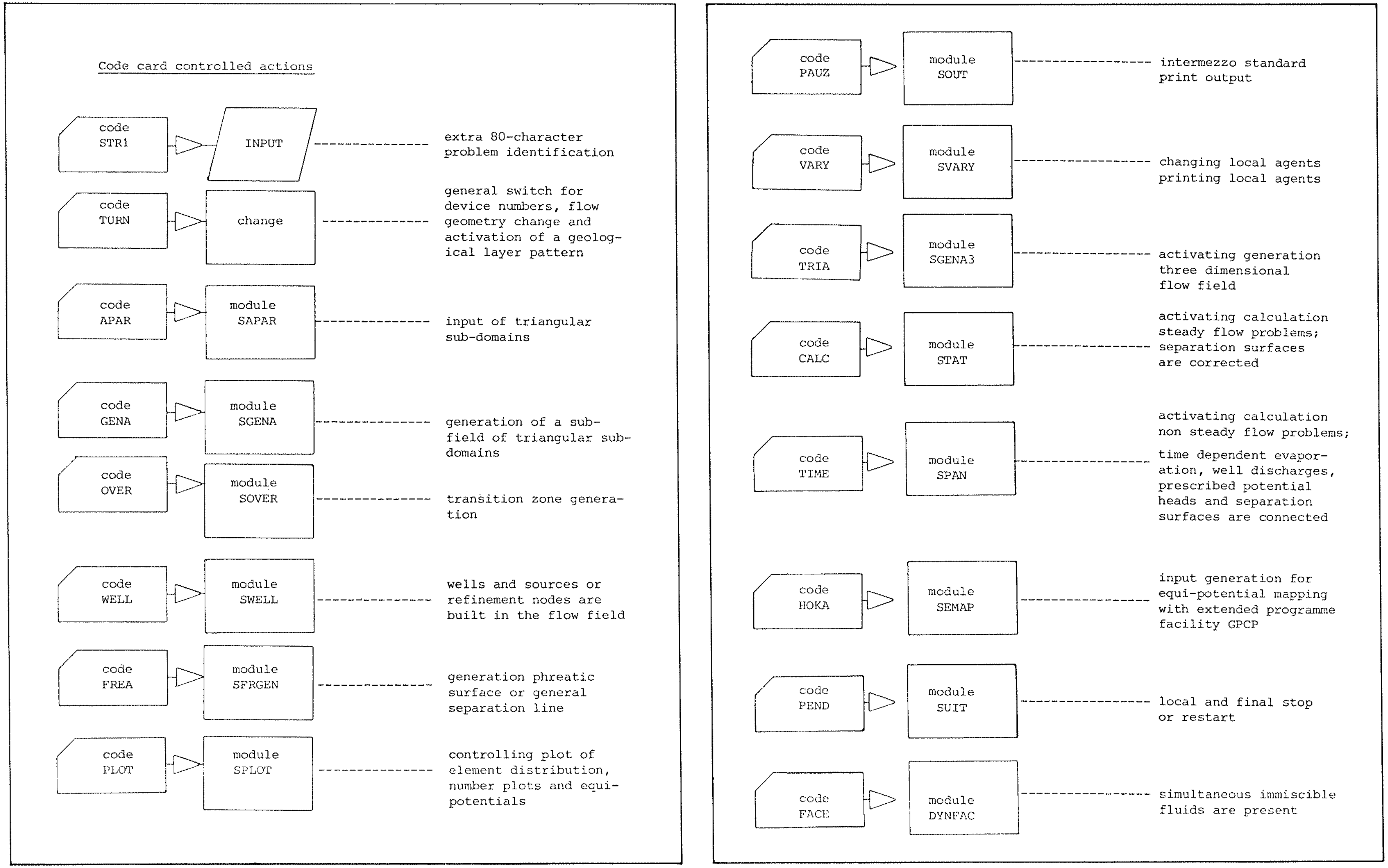


Fig. XXVIII User supplied actions in the SEEP code.

By realizing these conditions a method became available to optimize the design process and to increase insight in the aspects of seepage flow in all kinds of earth works and foundations. In Fig. XXVI a recent example of a complex problem is shown.
It concerns the subsurface flow through the filter construction of the sill in the foundation of the Oosterschelde storm surge barrier, see Barends and Thabet (1978). The constitution of the flow varied from linear to transitional and turbulent types. The finite element mesh, iso-potential lines for a steady state flow and corresponding iso-gradient lines in the sand bottom and the first sea-gravel layer are represented.

The concepts of the SEEP code are outlined in a bulky system documentation, and a compact user's manual is written. The structure of the program is such, that the user can model the required numerical actions. Four phases (units) are distinguished:
— unit GENA for mesh generation and composition of the system matrix,
— unit CALC for steady state flow calculation,
— unit TRIA for three-dimensional mesh generation,
— unit TIME for transient free surface flow calculation.
These different units can be combined into one jobstream, see Fig. XXVII. In every phase the user can direct individual operations, and over twenty user controlled actions can be started, see Fig. XXVIII. The code contains about 7000 fortran statements and the additional plotting library another 4000.

Next, three examples of utilizing the code are given. The first example, mentioned in Fig. XXIX, concerns steady, free surface, axi-symmetrical flow towards a well including a seepage surface.

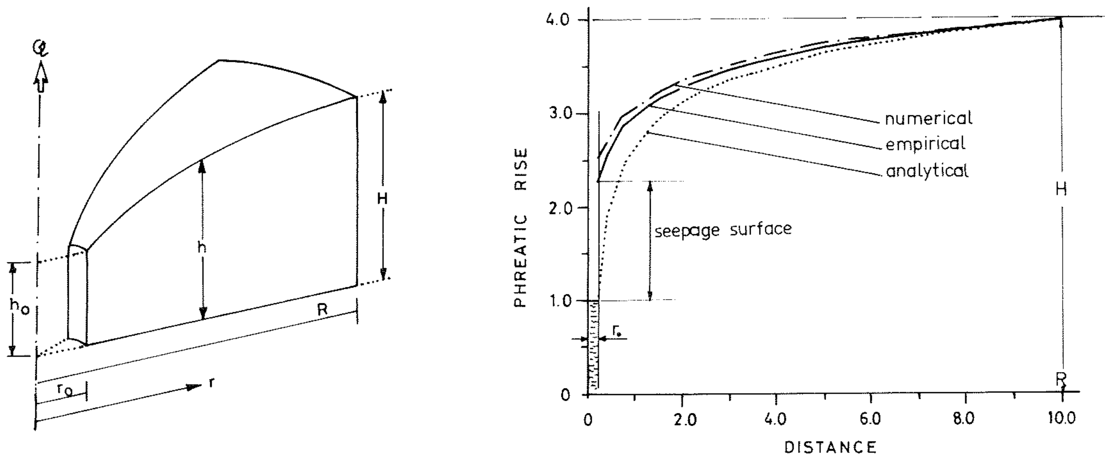


Fig. XXIX Steady, free surface, axi-symmetrical flow towards a well.

A second example in Fig. XXX shows the case of locally lowering of the groundwater table by horizontal drains required for a deep situated highway. The contribution due to infiltration (rain) is taken into account. The discussion in section 5.a made clear that the effect of internal production along a free surface is equivalent to infiltration at the free surface. In the finite element approach a nodal discharge at the free surface nodes simulates infiltration. The program SEEP determines the correct steady state free surface position including a constant infiltration. Also a sheet piling on either side of the highway has been installed to separate the groundwater regime in the highway section from the environment, a polder. A sheet piling construction is not impermeable, see Brauns (1978). The purpose of calculations was to optimise the location and dimensions of the drainage systems.
A third case presented in Fig. XXXI deals with a free boundary porous flow problem, where the phreatic surface is discontinuous. Along some parts of the flow field border the free boundary conditions are converted to seepage conditions or artesian conditions. The position of such parts is unknown. Though in some situations an approximate analytical solution exists, see Moench and Prickett (1972) even for inhomogeneous multi-phase flow fields see Strack (1979), a numerical iterative procedure can be applied as well. Furthermore, a numerical method is

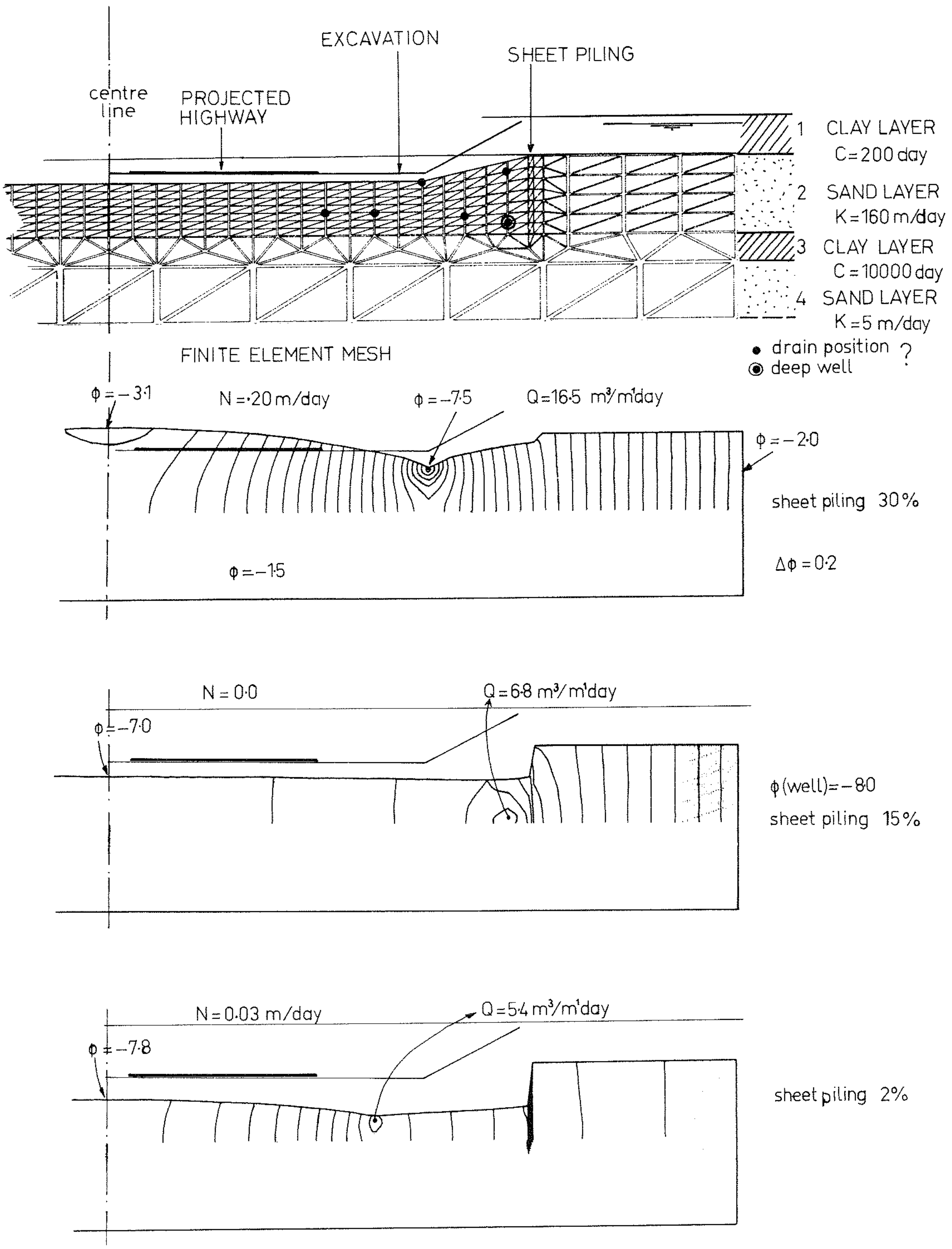


Fig. XXX Drainage of groundwater under a deep situated highway.

appropriate to general inhomogeneous flow fields with arbitrary geometry, see Fig. XXXI. Some intermediate approximations of the free surface position during the iterative process is shown as it has been determined by the program SEEP. A few iterations provide a sufficiently accurate result, and a corresponding iso-potential plot is composed.

In the previous examples the capillary zone has been disregarded. It may sometimes have a pronounced contribution. This has already been indicated by Terzaghi (1943), and an experimental study by Kerr (1954) clearly reveals the effect in extreme cases.

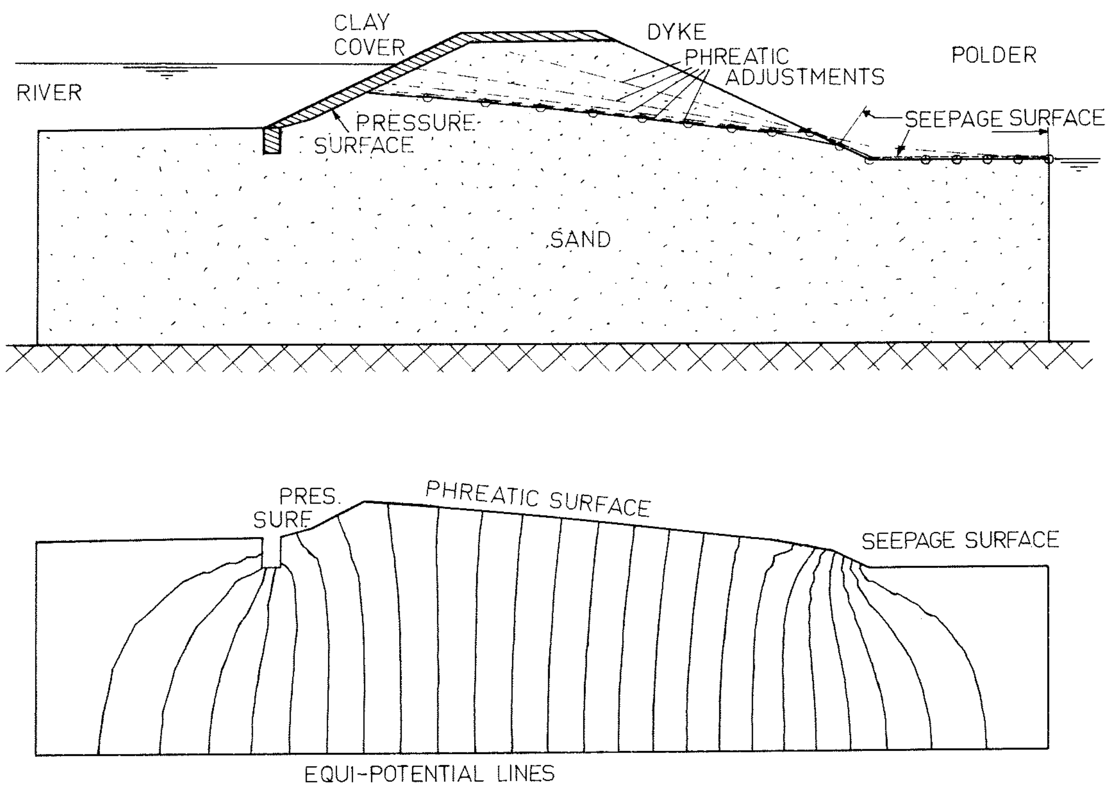


Fig. XXXI A discontinuous phreatic porous flow problem.

For a limited range of boundary geometry Chapman (1959) gave some mathematical formulations of the contribution of the capillary zone. Also a numerical procedure will do. Another constraint at the free surface, in fact introducing a capillary pressure head in stead of the atmospheric pressure, will automatically include this zone within the flow field.
However, at the fixed border of the flow field pore water can not seep out at pressures under the atmospheric pressure, and consequently some special conditions have to be imposed. For the finite element approach this does not represent a serious problem.

The program SEEP contains a hundred error checks in order to try to defend the user against his own fantasy and to show the plausibility of the calculated results. The various plotting facility has proved to be an important feature to scan the program's reliability.

5.c NUMERICAL AND ACCURACY OF PHREATIC FLOW PROBLEMS

In this section the case of non-steady flow will be considered. Storage of water due to the compressibility of pore water or due to volumetric deformations of the soil can be included by solving the field equation in terms of the extensive potential, equation (4.19), for the steady case, and explicitely solving the free boundary condition in terms of the potential ϕ. According to the discussion in section 5.a this condition is independent from the volumetric storage variations in the flow field.
However, in most situations the effect of volumetric storage within the field is small in order compared to phreatic storage effects. This becomes evident from the corresponding values of the coefficient of nonlinearity, described in section 4.c, equations (4.39) and (4.43).

The time dependent character of flow is introduced through the presence of a free moving boundary. The position of the boundary is a function of time. The field equation however is the same as for steady flow.

In section 5.a the kinematic boundary condition along such a free surface is derived. It has been shown that equation (5.13) governs phreatic storage with vertical infiltration N:

$$n\frac{\partial\Phi}{\partial t} = K(\nabla\Phi)^2 - (K+N)\frac{\partial\Phi}{\partial z} + N\,. \tag{5.30}$$

In several textbooks a similar expression is adopted, but not always the second term on the right hand side is complete, as it misses then the infiltration N. This particular term covers the convective contribution due to infiltration, and it should be included in the phreatic storage equation .

At the free boundary a second condition is imposed, in fact a pressure condition, see equation (5.1):

$$\Phi = z + p_o/\varrho g \tag{5.31}$$

where p_o denotes the atmospheric pressure. When a capillary zone is considered this pressure has to be replaced by the capillary pressure p_c. Both conditions, (5.30) and (5.31), control the equilibrium and the position at the free surface.

Steady free surface flow

Steady free surface porous flow is solved by imposing (5.30), infact defining an impermeable boundary along the free surface, the position of which is yet undetermined. Lacher (1975) gave a method to determine a steady phreatic surface for a turbulent type of flow. He derived an analytical solution for a simple geometry and verified this result with an experiment. In a finite element formulation an impermeable boundary condition is a natural condition, and no special arrangements are required. Steady infiltration is taken into account by nodal discharges. To satisfy the second condition (5.31) the position has to be adjusted. The SEEP code calculates the correct location of the free surface applying an iterative method, according to:

$$H_i^{j+1} = (1-\omega)\,H_i^j + \omega(\Phi_i^j - p_{o_i}/\varrho g)\,, \tag{5.32}$$

where H_i represents the vertical position of the free surface at different free surface nodes i. The factor ω is an interpolation coefficient, the value of which is usually somewhere in the range: $0<\omega<1$. The flow fields presented in Fig. XXIX, Fig. XXX and Fig. XXXI have been determined using this method.

Transient free surface flow

Also in non-steady surface flow condition (5.30) and (5.31) are imposed at the free boundary. In this case the numerical procedure is different. It starts from a steady state situation, generated at initial time. Upon this steady state solution the time dependency is superimposed, step by step.

The phreatic position is in pressure equilibrium. Hence, condition (5.31) is imposed along the free surface by a guess (fixed nodal potentials). The efflux (nodal discharge or potential gradient normal to the surface) at the free boundary due to a time step change in the prescribed time dependent agents determines the phreatic surface adjustments through the storage equation (5.30) being solved explicitly. However, the adjustment of the free surface will not take place linearly in time. They cause a slightly different potential distribution and a different efflux at the phreatic surface within the same time step (without altering the actual values of the time dependent agents). A second refinement in the same time step is recommendable.

Numerical methods

There are several methods to determine the transient free surface position, see Perrels (1975). For example, Verruijt (1970) utilizes nodal discharges caused by time step alterations. The storage equation (5.13) becomes by introduction of the filter velocity, $q = -K\nabla\Phi$:

$$n\frac{\partial\Phi}{\partial t} = N - q\cdot\nabla(\Phi - (1+\frac{N}{K})\,z)\,. \tag{5.33}$$

Verruijt solves this equation for plane vertical flow in a linearized form, but he disregarded the term N/K:

$$n\frac{\partial H}{\partial t} = q_z - q_x\frac{\partial H}{\partial x} + N\,, \tag{5.34}$$

$$\Phi = H\,.$$

Another method is suggested by Todsen (1971). He makes use of the potential gradients, which are available in a finite element formulation in the free surface elements. The difficulty which arises is that the position and the potential are nodal agents, whereas the gradients are average values in elements.
Todsen utilizes an interpolation scheme which is solved explicitely. Also equation (5.34) is used which misses the term N/K of (5.33). Cheng and Li (1973) designed a modified method, also based however on (5.34). The pressure boundary condition (5.31) is written in the following form:

$$\Phi\,(x, H(x, t), t) = H(x, t)\,,$$

where H is related to the position of the free surface.
Differentiation with respect to x gives:

$$\frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial z}\,\frac{\partial H}{\partial x} = \frac{\partial H}{\partial x}\,,$$

or:

$$\frac{\partial\Phi}{\partial x} = (1 - \frac{\partial\Phi}{\partial z})\frac{\partial H}{\partial z}\,. \tag{5.35}$$

In a similar manner differentiation with respect to t leads to:

$$\frac{\partial\Phi}{\partial t} = (1 - \frac{\partial\Phi}{\partial z})\frac{\partial H}{\partial t}\,. \tag{5.36}$$

Substitution of (5.35) and (5.36) into the phreatic storage equation (5.33) yields:

$$n\frac{\partial H}{\partial t} = K(1 - \frac{\partial\Phi}{\partial z})(\frac{\partial H}{\partial z})^2 - ((K+N)\frac{\partial\Phi}{\partial z} - N)/(1 - \frac{\partial\Phi}{\partial z})\,. \tag{5.37}$$

Cheng and Li derived a slightly different formula, namely:

$$n\frac{\partial H}{\partial t} = K(1 - \frac{\partial\Phi}{\partial z})(\frac{\partial H}{\partial x})^2 - (K\frac{\partial\Phi}{\partial z} - N)\,. \tag{5.38}$$

They did not include the time dependence of H according to (5.36), and also the convective contribution $N\partial\Phi/\partial z$ is not considered.
Next, equation (5.38) is linearized by introducing the approximation:

$$(\frac{\partial H}{\partial x})^2 \cong (\frac{\partial \tilde{H}}{\partial x})^2 + 2\frac{\partial \tilde{H}}{\partial x}(\frac{\partial H}{\partial x} - \frac{\partial \tilde{H}}{\partial x})\,.$$

The calculation starts with an initial guess for $\tilde{H}$, which is then adjusted until a certain accuracy.
Barends (1976) applies the method suggested by Verruijt.

Stability and accuracy

Solution of the phreatic storage equation involves an equation of the type:

$$\frac{\partial F}{\partial t} = a\frac{\partial F}{\partial x} + b\,.$$

The corresponding difference scheme is:

$$\frac{F_i^{j+i} - F_i^j}{\Delta t} = a\left\{ \omega \frac{F_i^{j+1} - F_i^{j+1}}{\Delta x} + (1-\omega)\frac{F_i^j - F_{i-1}^j}{\Delta x} \right\}, \tag{5.39}$$

in which ω represents an interpolation factor between time level j and j+1. The stability criterion for this equation yields:

$$\Delta t < \frac{\Delta x}{a(1-2\omega)}, \quad 0 \leq \omega < \frac{1}{2}. \tag{5.40}$$

Equation (5.39) is unconditionally stable for $\frac{1}{2} \leq \omega \leq 1$.

In respect of equation (5.34), keeping in mind that the filter velocity q is determined and kept a constant in the considered time step, this criterion becomes:

$$\Delta t < n\Delta x \,/\, (K \frac{\partial \phi}{\partial x} (1-2\omega)),$$

or:

$$\Delta t < \frac{n(\Delta x)^2}{K\Delta\phi\,(1-2\omega)}. \tag{5.41}$$

The variation $\Delta\phi$ is related to the variation in the time dependent boundary value, which is denoted by $\Delta\Phi$, during the same time step. Thus:

$$\Delta\phi \leq A\Delta\Phi. \tag{5.42}$$

The factor A is related to the configuration of the flow pattern, i.e. to the coherence of the considered boundary node and the actual prescribed time dependent boundary value. For groundwater flow $A<1$ holds (the extreme values for ϕ are situated on the border of the flow field). Since the time dependent behaviour of Φ is known, the value of $\Delta\Phi$ can be determined by:

$$\Delta\Phi \leq \lambda\Delta t, \tag{5.43}$$

where λ denotes the extreme value of $|d\Phi/dt|$, see Fig. XXXII.

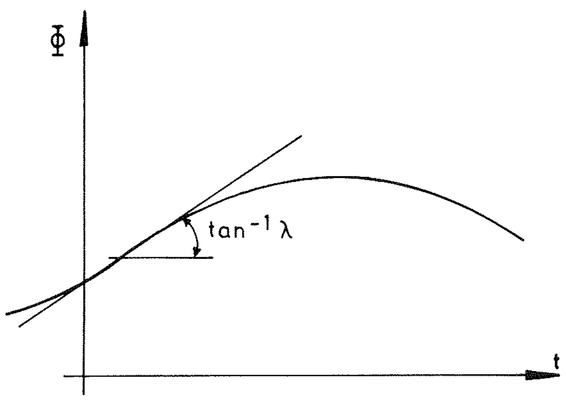


Fig. XXXII Variations in the boundary value.

Substitution of (5.42) and (5.43) into the stability criterion (5.41) gives:

$$\Delta t < \Delta x\sqrt{n/(K\lambda A(1-2\omega))}. \tag{5.44}$$

Stability does not imply accuracy. A reduction on the time step determined by this criterion will improve the accuracy of the results. Barends (1977) checked the validity of (5.44) for different

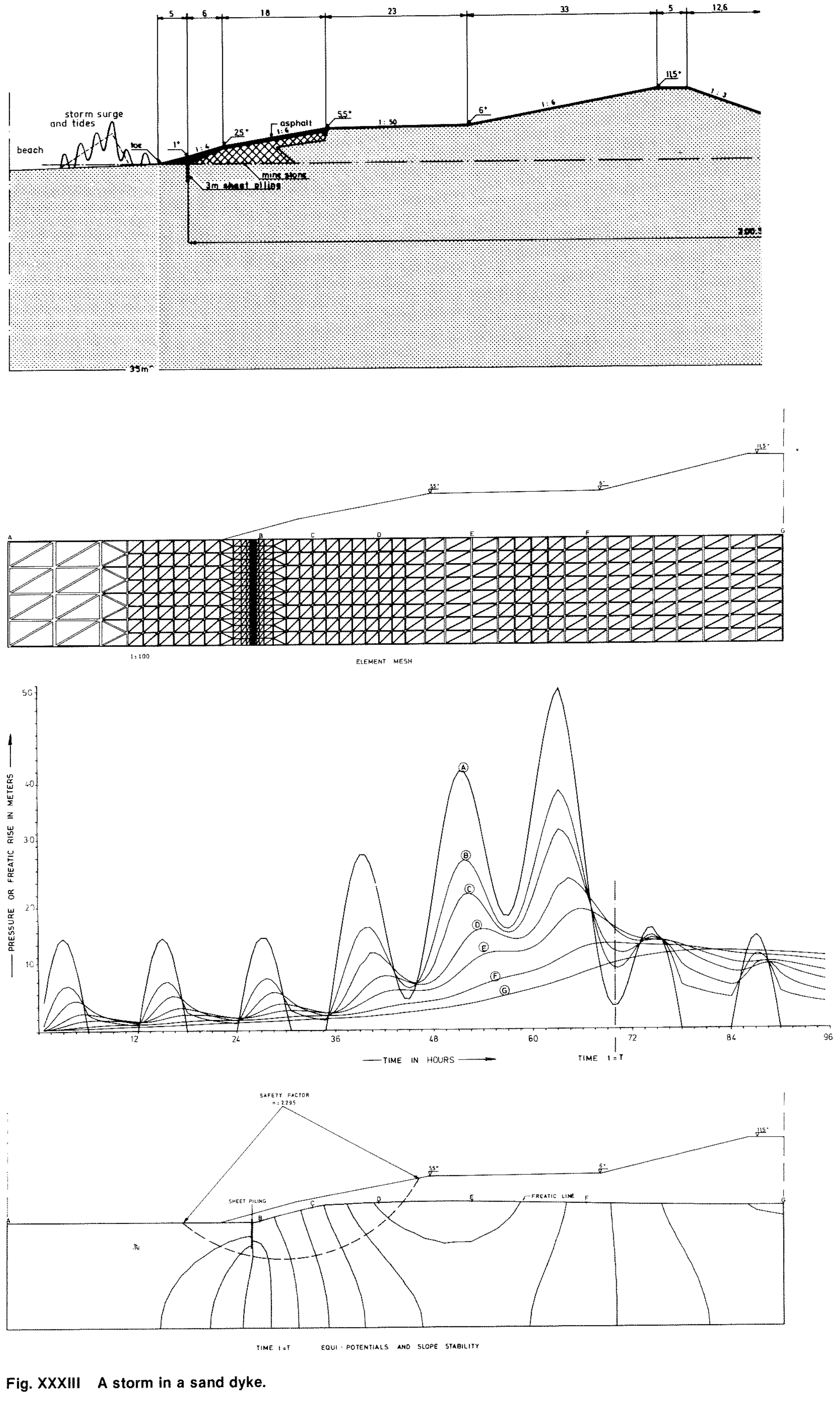


Fig. XXXIII A storm in a sand dyke.

configurations using a fully explicite scheme ($\omega = 0$). Sufficient accuracy is found for a time step according to:

$$\Delta t < \frac{1}{3} \Delta x \sqrt{n/(K\lambda A)} \,. \tag{5.45}$$

It must be noted that a secondary of the influence of boundary motion on the stationary gradients at the free surface has been taken into account by a so-called multi-time-step method, which is not further discussed.

An example of the application of the computer program SEEP for complex transient free surface flow problems is represented in Fig. XXXIII. The case concerns a sand dyke subjected to tides and a storm surge. The slope of the sand dyke is partly covered by an asphaltic layer and at the toe a sheet piling construction is installed. The attention is drawn to the effect of accumulation of phreatic pore water during the storm and to the pressure at the asphaltic layer immediately after the storm fall off. Fortunately, due to the time step criterion, equation (5.45), a large time step (about one hour) could be used without affecting the stability and the accuracy.
A similar problem has been simulated by an electric analogon, see Pereboom (1978). Though difference in discretization between a digital and an analogous model exist, a good agreement was found. Hence, the accuracy of the calculation results was sufficient. Digital computers can efficiently be used for large transient free surface porous flow problems.

6. SURVEY OF NONLINEARITY IN GROUNDWATER FLOW

To summerize the principical results of this thesis a survey of the various aspects about the process of groundwater flow, discussed in the preceding chapters, is presented emphasizing the deviations from to the linear theory.

Groundwater flow is described by Darcy's law according to:

$$q = -\frac{k}{\mu}(\nabla p + \varrho g \nabla z) = -\frac{kg}{\upsilon}\nabla\phi \,, \tag{6.1}$$

which comprises two fundamental concepts: a constitutive relation for groundwater flow, and equilibrium between driving and resistance forces. A dimensional analysis reveals that with four physical numbers (the Galileo, the Weber, the Reynold, and the Froude number) the constitutive relation for porous flow can be formulated to cover general ante-linear, linear and post-linear flow behaviour. On physical grounds the nonlinearity is bounded by a parabolic behaviour. Turbulence in a porous medium is geometrically restricted to pore size dimensions. An approximate linearization technique is suggested, tested and evaluated experimentally. The linearization involves an iterative procedure about the intrinsic permeability k, which is defined by:

$$k = \upsilon (A' C_D g |\nabla\phi| / D)^{-1/2} \tag{6.2}$$

In coarse beds a post-linear porous flow behaviour is more significant to the specific discharge (q) distribution than to the potential (ϕ) distribution.

The permeability k, a fundamental property of porous flow, is related to the void ratio e (or porosity n) in the following way:

$$k/k_i = (e/e_i)^{\varkappa} \,, \quad e = n/(1-n) \,. \tag{6.3}$$

The power $\varkappa$ is generally larger than unity ($3 < \varkappa < 5$) causing an amplified effect in the permeability when the porosity varies (soil deformations). $\varkappa$ is determined by experiments.

The resultant compressibility β' of an air-water mixture can be expressed by two different constants, defining either the case where small air bubbles are present, or the case where all free air has been dissolved. The corresponding discontinuity occurs at a critical pore pressure p_c, which is fully determined by the air solubility coefficient ω and the initial saturation degree s_i measured at pore pressure p_i. Disregarding surface tension it is found that:

$$\beta' \cong 2\omega / p_c \quad \text{for} \quad p < p_c ,$$

$$\beta' = \beta \quad \text{for} \quad p > p_c , \tag{6.4}$$

where:

$$p_c \cong (1 - s_i)\, p_i / \omega . \tag{6.5}$$

and where β is the compressibility of pure water.
Since in the presence of free air bubbles the compressibility β' is much larger that in the absence of free air in the pore water, this phenomenon cannot be ignored in a proper description of the mechanical behaviour of semi-saturated soils.

The interaction between pore water and the porous medium is reconsidered in the form of continuity of momentum for both substances. Under very general conditions this formulation reduces to Terzaghi's effective stress principle:

$$\sigma = \sigma' - p , \tag{6.6}$$

which states a relation between the total isotropical stress σ, the effective isotropical stress σ' and the pore pressure p. For semi-saturated soils this principle holds when the pore water remains coherent. Therefore, the present theory is restricted to circumstances where the saturation degree s of the pores is above 85% ($s > 0.85$).

The mechanical behaviour of soil is significant to groundwater flow. The motion of pore water is influenced by changes in the pore geometry, which can occur in deformable soils. It affects the permeability and induces storage variations causing pore pressure gradients, which give rise to porous flow. Resistance to porous flow by internal friction between the pore fluid and the porous skeleton limits the pore fluid motion and retards the deformation rate of the skeleton. Hence, the mechanical behaviour of soil is restrained by pore water dissipation (consolidation) and for specific circumstances (irrotational elastic deformations) this effect can be described by a storage equation according to:

$$\frac{\partial (p - p_F)}{\partial t} = c \nabla^2 p , \tag{6.7}$$

where p_F is related to the boundary conditions (or to the total isotropic stress σ). The presence of this function p_F is the reason that the storage equation is not always similar to the familiar heat diffusion equation, even in the simple case of irrotational deformations.

In deriving equation (6.7) a perfectly elastic soil deformation behaviour has been assumed. Such an assumption in a rather crude approximation in regard to the microscopic nature of soil, in particular concerning the intergranular contact forces and the overall deformation of the porous skeleton due to local pressing, crushing, rolling and sliding of solid particles. However, the proposed linear behaviour provides a first estimation, and a description amenable to fundamental mathematical operations, contributing to a deeper comprehension of the fundamental character of the process.

Due to deformations a soil element displaces and conveyes the pore content. Hence, it should be obvious that the interaction phenomenon between the pore fluid and the porous skeleton must be formulated with respect to the moving soil.

This implies that Darcy's law must be introduced relative to the soil motion, thus:

$$q = n(w - v), \tag{6.8}$$

where w denotes the absolute pore fluid velocity and v is the absolute porous skeleton velocity. Several authors have violated this fundamental concept and they obtain a storage equation with an erroneous storage coefficient (consolidation coefficient) when including soil deformations.

It has been shown that the familiar storage equation for a fully saturated porous medium according to:

$$-\nabla\cdot(\varrho q) = \varrho\left\{\frac{D\varepsilon}{Dt} + n\beta\frac{Dp}{Dt}\right\}, \tag{6.9}$$

where D/Dt represents the substantial derivative with respect to a deforming (moving) soil ($D/Dt = \partial/\partial t + v\cdot\nabla$), is also valid for semi-saturated porous media. In that particular case, in a slightly modified form, the governing storage equation becomes:

$$-\nabla\cdot(\varrho' q) = \varrho'\left\{\frac{1}{1-b}\frac{D\varepsilon}{Dt} + n\beta'\frac{Dp}{Dt}\right\}. \tag{6.10}$$

Here, ϱ' constitutes the pore fluid density including the air content ($\varrho' = s\varrho + (1-s)\varrho''$) and b is the specific volume of stagnant air, which is considered to be small ($b < 1 - s$). ε denotes the volumetric strain of the skeleton.

Darcy's law is defined by:

$$q = -\frac{k}{\varrho'\upsilon}(\nabla p + \varrho' g\nabla z) = -\frac{kg}{\upsilon}\nabla\phi = -K\nabla\phi, \tag{6.11}$$

for semi-saturated soils (provided that $s > 0.85$). The permeability is affected by the presence of stagnant air. A modification of the permeability is suggested in the form of a reduction by a factor $(1-b)^3$.

For an irrotationally deforming porous media the porous flow field itself is irrotational. Therefore, the process can be formulated in terms of a potential. It is found that the following potential:

$$\phi = z + \int_{p_F}^{p}(1/\varrho' g)d\ell, \tag{6.12}$$

transforms the governing storage equation (6.10) into:

$$\nabla^2\phi + m\left((\nabla\phi)^2 - \frac{\partial\phi}{\partial z}\right) = \frac{D\phi}{cDt} \tag{6.13}$$

where c is the common consolidation coefficient and m denotes the coefficient of nonlinearity, defined by:

$$m = \varrho' g(\alpha\varkappa + n\beta')/n. \tag{6.14}$$

It incorporates the dependence of permeability and porosity, expression (6.3), the compressibility of the air-water mixture β' and a soil deformation parameter α. Its dimension is the inverse of a lenght.

In anisotropic porous media the pressure induced density variations (ϱ' is a function of p only) will give rise to density rotations in the flow pattern. This effect is small of second order.

In general the significance of convective terms (the second term in the right hand side of equation (6.13), $D\phi/Dt = \partial\phi/\partial t + v \cdot \nabla\phi$) is negligible. In some particular cases it can be taken into account in an approximate way.

For horizontal (confined or phreatic) aquifers equation (6.13) can be transformed into a common form, namely:

$$\nabla^2\chi = \frac{D\chi}{cDt} . \tag{6.15}$$

The potential χ appearing in this equation is called the extensive potential. It is related to the potential ϕ according to:

$$\chi = \exp(m\phi) - 1 . \tag{6.16}$$

Since (6.15) itself represents an ordinary storage equation and because general boundary conditions can be formulated in terms of χ, the nonlinear effects in the original storage equation (6.13) can be included in a very simple manner.

The nonlinearity becomes manifest in a reduction of the area of influence by 10% tot 20%.

For vertical flow the storage equation (6.13) can be transformed into:

$$\zeta^2 \frac{\partial^2\chi}{\partial\zeta^2} = \frac{D\chi}{D\tau} , \tag{6.17}$$

where:

$$\zeta = \exp(mz) ,$$
$$\tau = m^2 ct .$$

This equation can be solved for general boundary conditions using a finite Mellin transformation technique. Several characteristic solutions (a unit shock, a unit step, and a cyclic boundary condition) reveal a specific influence of nonlinearity.

A numerical procedure to solve (6.17) is a simple matter.

Since in most practicle cases the coefficient of nonlinearity m is relatively fairly small, the linear theory of groundwater flow provides sufficiently accurate results (provided that the flow behaviour can be described by Darcy's law). At far distances from the disturbance considered the nonlinearity becomes relatively more predominent. The real nonlinear soil deformation behaviour, which is not discussed in this thesis, may contribute to this effect.

Moving boundaries (phreatic surfaces) form another type of nonlinearity encountered in groundwater flow (geometric nonlinearity). The general phreatic storage equation, or the kinematic boundary condition is derived, including infiltration B and convective effects due to soil deformations:

$$n\frac{\partial\phi}{\partial t} = K(\nabla\phi)^2 - (B + nv)\cdot\nabla\phi + (B + nv - K\nabla\phi)\cdot\nabla z . \tag{6.18}$$

In a linearized form this condition states:

$$n\frac{\partial\phi}{\partial t} = N - q \cdot \nabla(\phi - (1 + \frac{N}{K})z) . \tag{6.19}$$

including only vertical infiltration N measured with respect to the soil skeleton velocity v. A survey of numerical methods to treat (6.19) is given. The term N/K is usually disregarded, but this seems to be unnecessary ($K = kg/\upsilon$: hydraulic permeability).

Volumetric storage variations due to soil deformation and to pore fluid compressibility do not influence the kinematic boundary condition (6.19). General nonlinear groundwater flow can be solved in the flow field in terms of the extensive potential χ, whilst the phreatic boundary condition is solved in terms of ϕ, or employing in (6.19) the identity:

$$\phi = \frac{1}{m} ln(\chi + 1) . \qquad (6.20)$$

A stability criterion for the time step Δt to solve (6.19) is derived according to:

$$\Delta t < \Delta x \, (K \lambda A(1 - 2\omega)/n)^{-1/2} , \quad 0 \leq \omega < 1/2 . \qquad (6.21)$$

Here, Δx denotes a space discretization, λ is the maximum time rate in a given boundary value and A is related to the configuration of the flow pattern. ω is an interpolation factor in the time domain. Introduction of λ and A in the criterion makes an explicit procedure to solve transient porous flow problems rather efficient.

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Symbol index

a	$[Ns^f/m^{3+f}]$	coefficient of nonlinear porous flow
b	[1]	relative volume of bonded air pockets
c	$[m^2/s]$	coefficient of consolidation: $K/(\varrho' g(\alpha + n\beta')) = c$
d	[m/s]	convection velocity by internal production
e	[1]	void ratio: $n/(1-n) = e$
f	[1]	exponent of nonlinear flow
g	$[m/s^2]$	gravity acceleration: $9.81\ m/s^2$
i	—	complex unity: $i^2 = -1$
k	$[m^2]$	intrinsic permeability: $D^2/A = CTD^2\eta = k$
m	[1/m]	coefficient of nonlinearity: $\varrho' g(\alpha\varkappa + n\beta')/n = m$
n	[1]	porosity
o	—	unit vector in the direction of the outward normal
p	$[N/m^2]$	pore pressure
p	[1/s]	Laplace transform variable
p_o	$[N/m^2]$	atmospheric pressure
p_c	$[N/m^2]$	capillary pressure
p''	$[N/m^2]$	air pressure
p_F	$[N/m^2]$	boundary pressure: $\alpha F/(\alpha + n\beta') = p_F$
q	[m/s]	filter velocity, specific discharge
r	[m]	bubble radius; radial coordinate
s	[1]	saturation degree
s	[1]	Mellin transform variable
t	[s]	time variable
u	[m]	displacement vector
v	[m/s]	soil skeleton velocity
w	$[N/m^2]$	water vapour pressure
w	[m/s]	pore fluid velocity
w''	[m/s]	air bubble velocity
x	[m]	coordinate, position vector
y	[m]	coordinate
z	[m]	vertical coordinate
A,A′	[1]	coefficient of porous flow configuration (linear, turbulent)
B	[m/s]	specific infiltration: $nd = B$
C	[s]	hydraulic resistance of confining layer: $d/K = C$
C_D	[1]	drag coefficient
D	[m]	particle size
D	$[m^2/s]$	diffusion coefficient
F	$[N/m^2]$	pressure function
F	$[N/m^3]$	volumetric driving force
G	$[N/m^2]$	shear modulus
H	$[N/m^3]$	volumetric driving force
H	[m]	aquifer thickness
I_o	[1]	initial gradient: $\lvert \nabla\phi \rvert_{q \downarrow 0} = I_o$
J	$[kg/sm^3]$	rate of volumetric mass production
K	[m/s]	hydraulic permeability: $kg/\upsilon = K$
K_s	$[N/m^2]$	bulk modulus
M	[kg/mol]	molecular weight
N	[1]	specific number of bubbles
Q	$[m^3/s]$	total volumetric discharge
R	[Nm/molK]	gas constant
R	$[N/m^3]$	volumetric resistance force
R	[m]	area of radial extent, area of influence
S	[1]	aquifer storativity: $HK/c = S$
S	[1/m]	specific volumetric storativity: $K/c = S$
T	[K]	absolute temperature
T	[1]	tortuosity
U	$[m^3]$	specific volume
W	[s/m]	linear flow resistance: $A\upsilon/gD^2 = W$
W′	[s/m]	nonlinear flow resistance: $\sqrt{A'C_D \lvert \nabla\phi \rvert /gD} = W'$
Z	[mol/kg]	gas compressibility factor

α	[m²/N]	laterally confined compressibility: $1/(K_s + 4G/3) = \alpha$
α'	[m²/N]	isotropic soil skeleton compressibility: $1/K_s = \alpha'$
β	[m²/N]	pure water compressibility
β'	[m²/N]	compressibility of an air-water mixture
ε	[1]	volume strain
η	[1]	porosity factor
η	[1]	transformation variable: $my = \eta$
ζ	[1]	transformation variable: $\exp[\theta] = \zeta$
θ	[1]	transformation variable: $mz = \theta$
$\varkappa$	[1]	permeability amplification factor
λ	[m]	leakage factor: $\sqrt{KHC} = \lambda$
$\bar{\lambda}$	[s]	relative leakage factor
μ	[kg/ms]	dynamic viscosity: $\varrho\upsilon = \mu$
υ	[m²/s]	kinematic viscosity: $\mu/\varrho = \upsilon$
ξ	[1]	transformation variable: $mx = \xi$
π	[1]	3.141529
ϱ	[kg/m³]	fluid density
ϱ''	[kg/m³]	air density
ϱ'	[kg/m³]	air-water mixture density
ϱ_s	[kg/m³]	solid particle density
σ	[N/m]	surface tension
σ	[N/m²]	total stress
σ'	[N/m²]	effective stress: $\sigma + p = \sigma'$
σ_{ij}	[N/m²]	stress tensor
τ	[1]	transformation variable: $m^2ct = \tau$
ϕ	[m]	piezometric head; potential: $z + p/\varrho g = \phi$
Φ	[m]	potential: $z + \int_{p_F}^{p} (1/\varrho' g)dp = \Phi$
χ	[1]	extensive potential: $\exp(m\Phi) - 1 = \chi$
ψ	[1]	phase shift
ω	[1]	air solubility coefficient (in water)
ω	[1]	non-dimensional coordinate: $r/\sqrt{ct} = \omega$
ω	[rad]	circular frequency
∇	–	gradient operator
$\nabla \bullet$	–	divergence operator
$\nabla \times$	–	curl operator
∇^2	–	Laplacian operator: $\nabla \bullet \nabla = \nabla^2$
∂/∂	–	partial derivative symbol
D/Dt	–	substantial derivative following the soil skeleton
D/Dt	–	substantial derivative following the pore fluid
δ	–	variation
Σ	–	summation symbol
Π	–	multiplication symbol
Fr	[1]	bed-Froude number: $q^2/gD = Fr$
Ga	[1]	Galileo number: $\upsilon^2/gD^3 = Ga$
Re	[1]	bed-Reynold number: $qD/\upsilon = Re$
We	[1]	Weber number: $\varrho g D^2/\sigma = We$

Figure index

Summary

Since 1856 when Darcy laid the basis for the calculation of the flow of water through sands, researchers have been interested in groundwater flow. Groundwater is essential for agriculture and water supply, but it also plays an important role when soil is used as a construction element, such as for dykes, roads and foundations. The mechanical behaviour of saturated or dry, fine graded or coarse soils are quite different.

The theory of groundwater mechanics must be based on the system: water-soil-air. Up to now study has been restricted to mainly saturated and/or undeformable soil. In this thesis the contemporary theory is extended to compressible fluid flow in a semi-saturated deformable porous medium; a water-soil-air mixture, the air in which appears in the form of micro-bubbles. The pore water moves, whereas the soil itself deforms. It is assumed that this deformation behaviour is linear and free of rotations.

From a fundamental reconsideration it is shown that the mechanical behaviour of this system can be formulated in a rather simple way taking into account various nonlinear effects. Convective terms and the variation of the permeability related to soil deformation are included. The validity of the formulation derived is discussed. A general solving procedure applying the Mellin transformation technique allows elucidation of the influence of these nonlinear terms on the basis of analytical solutions of some characteristic problems.

In the phenomenon of groundwater flow so-called moving boundaries also occur. The free surface of natural groundwater, which actually varies, is such a boundary. This implies that the domain in which the process of porous flow is considered, changes (geometric nonlinearity). This aspect is explained. Transient phreatic porous flow problems can be solved by applying numerical models. In the discussion reference is made to the extensive literature.

In conclusion, the following statements hold for nonlinear groundwater flow. In most practical cases the linear theory is sufficiently accurate, nonlinearity becomes manifest in a reduction of the area of influence, and time dependent porous flow problems can be explicitely solved applying a time step much larger than formerly assumed.

الخلاصة

منذ أرسى " دارسي " سنة ١٨٥٦ الأسس لحساب تغلغل الماء في الطبقات الرملية ، ازداد اهتمام الباحثين في المياه الجوفية لكونها ضرورية في الزراعة والامداد المائي ولدورها الهام البيّن عند كون التربة عنصرا من عناصر اقامة السدود ،شق الطرق أو انشاء الاساسات. ويظهر الفرق جليّا في السلوك الميكانيكي للتربة المشبعة بالماء ، الجافة ، الخشنة أو الناعمة .

يجب الاعتماد في نظرية السلوك الميكانيكي للمياه الجوفية على النظام التالي: ماء – تربه – هواء . وقد تركّزت الابحاث حتى الان حول الميزات الفيزيائيّة للتربة المشبعة أو لكلتيهما. وهذه الاطروحه تلقي الضوء على جريان السوائل الضغيطة عبر وسط مسامي نصف مشبع وغير قابل للتشكل ؛ خليط من الماء – التربة – والهواء . حيث يظهر الهواء على شكل فقاقيع مجهريّة ،أما الماء فدائم الحركة حيث تتشكل التربة . والمفروض ان هذا السلوك العدم التشكلي – متناسق وليس دوراني .

اثر اعادة نظر اساسية تبيّن أنه يمكن صياغة هذا السلوك الميكانيكي في معادلة سهلة ، مع الاخذ بعين الاعتبار مختلف التاثيرات عديمة التناسق والتغيّر في قابليّة النفاذ للتربة عديمة التناسق. وقد مكّنت نظرية " ميلين " من توضيح تاثير هذه الظروف عديمةالتناسق على أسس تحليلية لبعض المشاكل المتميزة .

في ظاهرة جريان المياه الجوفية يحدث ما يدعى بالتحركات النطاقية ، وهذه تختلف تبعا لسطح المياه الجوفية الطبيعية الدائمة الحركة ، وهذا يعني حدوث تغيّر في الوسط الذي يحدث فيه الجريان (هندسة عدم التناسق) وقد شرحت هذه النقطة في الاطروحة . يمكن حل مشاكل الجريان النفيذ على أساس " المعامل العددي " وقد احتوى البحث على مراجع توضّح هذه النقطة .

وختاما ، فان هذا البحث يبرهن صحة الفرضية حول الجريان الغير المتناسق للمياه الجوفيّة . وفي معظم الحالات العملية اتضحت صحة نظرية التناسق ، كما واصبحت نظرية عدم التناسق جليّة وعلى جانب كاف من الدقّة في حدود تأثير أصغر . أما مشاكل الجريان النافذ المتغيّر خلال الوقت فيمكن حلها خلال فترات زمنية اطول مما افترض سابقا .