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Synchronization of impacting mechanical systems with a single constraint

Michael Baumann [a], J.J. Benjamin Biemond [b,c], Remco I. Leine [d,*], Nathan van de Wouw [e,f,g]

[a] Institute of Mechanical Systems, ETH Zurich, Switzerland
[b] Department of Computer Science, KU Leuven, Belgium
[c] Department of Optomechatronics, Netherlands Organization for Applied Scientific Research, TNO, Delft, The Netherlands
[d] Institute for Nonlinear Mechanics, University of Stuttgart, Germany
[e] Department of Mechanical Engineering, Eindhoven University of Technology, The Netherlands
[f] Department of Civil, Environmental & Geo-Engineering, University of Minnesota, USA
[g] Delft Center for Systems and Control, Delft University of Technology, The Netherlands

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ABSTRACT

This paper addresses the synchronization problem of mechanical systems subjected to a single geometric unilateral constraint. The impacts of the individual systems, induced by the unilateral constraint, generally do not coincide even if the solutions are arbitrarily 'close' to each other. The mismatch in the impact time instants demands a careful choice of the distance function to allow for an intuitively correct comparison of the discontinuous solutions resulting from the impacts. We propose a distance function induced by the quotient metric, which is based on an equivalence relation using the impact map. The distance function obtained in this way is continuous in time when evaluated along jumping solutions.

The property of maximal monotonicity, which is fulfilled by most commonly used impact laws, is used to significantly reduce the complexity of the distance function. Based on the simplified distance function, a Lyapunov function is constructed to investigate the synchronization problem for two identical one-dimensional mechanical systems. Sufficient conditions for the uncoupled individual systems are provided under which local synchronization is guaranteed. Furthermore, we present an interaction law which ensures global synchronization, also in the presence of grazing trajectories and accumulation points (Zeno behavior). The results are illustrated using numerical examples of a 1-DOF mechanical impact oscillator which serves as stepping stone in the direction of more general systems.

1. Introduction

Synchronization of coupled dynamical systems leads to motion in unison which is a fundamental phenomenon appearing in, for example, biological and engineering systems. The synchronization of chaotic oscillators, neural systems and mechanical systems has been studied extensively, see [1–5] and references therein. For both diffusively coupled differential equations and impulsively coupled maps, synchronization properties are generally studied through the analysis of the error dynamics which describes the difference between the states of the systems [1,3,6]. The error dynamics is typically characterized by a smooth differential equation or map and, consequently, linearization techniques and bifurcation theory have allowed to describe the convergence properties of the error dynamics and to study the effect of the interaction network. In this manner, the effect of the network topology, coupling strengths and delays of the network interaction on (master–slave, partial or global) synchronization is relatively well-understood for smooth systems [4,7,8] as well as for phase-coupled oscillators, which are naturally analyzed using Poincaré sections [9]. In contrast, synchronization of nonsmooth systems has received significantly less attention and, to the best of the authors' knowledge, the problem of synchronization for unilaterally constrained mechanical systems has not yet been addressed.

In this paper, synchronization is analyzed for mechanical systems with a single geometric unilateral constraint, which occur generally if mechanical systems (such as, e.g., robots) interact with a rigid environment. The dynamics of these systems comprises impacts which induce velocity jumps, rendering the system dynamics of an impulsive, hybrid nature [10–13]. Accumulation of infinitely many impacts in a finite time interval, which is known as Zeno-behavior, is a natural feature of unilaterally constrained mechanical systems. To describe the dynamics which includes such accumulation events, system models in terms of Measure Differential Inclusions (MDIs) are employed in [10,14–16].

* Corresponding author.
E-mail address: remco.leine@inm.uni-stuttgart.de (R.I. Leine).

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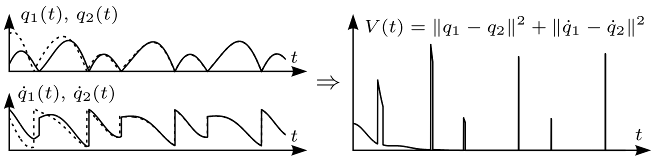


Fig. 1. 'Peaking behavior' of the Euclidean synchronization error.

Because impacts of unilaterally constrained mechanical systems are a consequence of collisions and therefore are state-triggered events (i.e., occur at a certain position), they generally do not occur at the same time instants for nearby trajectories. Therefore, one expects a small time-mismatch of the impact time instants even for arbitrarily close initial conditions. A small Euclidean synchronization error prior to the first impact therefore does not imply that the Euclidean error is small during the time in between the impacts of two neighboring trajectories. As an illustrative example, Fig. 1 shows the time evolution of $V(t) = \|q_1-q_2\|^2 + \|\dot{q}_1 - \dot{q}_2\|^2$ evaluated along the solutions of two synchronizing 1-DOF impacting systems with the generalized coordinates $q_1(t)$ and $q_2(t)$ and discontinuous generalized velocities $\dot{q}_1(t)$ and $\dot{q}_2(t)$, respectively. This 'peaking phenomenon' of the Euclidean synchronization error has also been observed in [10,17–21] and implies that the Euclidean synchronization error dynamics is generally unstable in the sense of Lyapunov. Consequently, the existing synchronization results are not applicable to mechanical systems with unilateral position constraints. An exception is the case of synchronization between a mechanical system and an observer, in which the impacts of the observer state can be made to coincide with the impacts of the mechanical system, as exploited in [22].

Recently, focusing on the stability of jumping trajectories, the 'peaking phenomenon' has been addressed for hybrid systems in the framework of [11] by considering stability in terms of a novel distance function which takes the jump characteristics into account [17,23]. This approach has been extended in [24] towards incremental stability. These approaches, however, are not applicable if the time between state jumps can be arbitrarily small, as occurs generally in unilaterally constrained mechanical systems (especially in Zeno events) since the hybrid system framework employed is not suitable to investigate solutions after the occurrence of Zeno behavior.

In this paper, we establish the generic synchronization problem for mechanical systems subjected to a geometric unilateral constraint. Furthermore, the synchronization problem of mechanical systems with one degree of freedom, possibly featuring accumulation of impacts, is investigated in more detail. Synchronization between the systems is induced either by an intrinsic incremental stability property or by an interaction law.

We distinguish three main contributions of this paper. First, we propose a distance function, inspired by [25], applicable to mechanical systems with multiple degrees of freedom and a single geometric unilateral constraint. This distance function defines when solutions are considered close to synchronization or when they are synchronized. The synchronization problem formulation, which we establish based on the presented distance function, is applicable to generic mechanical systems with a unilateral constraint. Synchronization defined in this manner corresponds to the intuitive notion of synchrony. The synchronization problem does not suffer from the 'peaking phenomenon' and can deal with accumulation of impacts. Second, Lyapunov arguments are used to investigate this synchronization problem for the one-dimensional case and provide conditions on the individual systems and their interaction law which guarantee that synchronization occurs. The non-expansive nature of most commonly used instantaneous impact laws, including the generalized Poisson's and Newton's impact law with global coefficient of restitution, is exploited in the Lyapunov function design to guarantee non-increasing behavior of the Lyapunov function over the impacts. Third, we design a interaction law to enforce synchronization using finite forces generated by the interaction network. Finally, the results are illustrated with numerical examples.

A forerunner of this work has been published as extended abstract in [26]. In the current paper, the proofs of the main and auxiliary statements are given, which are both original and constructive and form an essential part of the scientific work. The current paper extends [26] additionally by the analysis of uncoupled individual systems for which sufficient conditions for local synchronization are provided. Furthermore, many of the ideas presented in [26] are deepened in the current work. In particular, the choice of the distance function and the Lyapunov function design, which are at the core of defining and solving the synchronization problem, are discussed in more detail. Additional numerical simulations are presented which illustrate the performance and robustness of the interaction law.

The outline of this paper is as follows. In Section 2, the model of the mechanical system under study is presented in terms of a measure differential inclusion, and the non-expansivity of the impact law is discussed. Subsequently, in Section 3, a distance function is presented which is suitable to compare discontinuous solutions and which we use to define the synchronization problem for multi-dimensional mechanical systems subjected to single unilateral constraint. Section 4 presents sufficient conditions for coupled and uncoupled synchronization using Lyapunov stability analysis and some illustrative numerical examples are presented in Section 5. Conclusions are formulated in Section 6.

2. Dynamics of mechanical systems with a single unilateral constraint

We consider an n-DOF (degrees of freedom) mechanical system subjected to a single frictionless geometric unilateral constraint. The state of the system is described by the generalized coordinates $\boldsymbol{q}(t) \in \mathbb{R}^n$ and velocities $\boldsymbol{u}(t) \in \mathbb{R}^n$. The non-impulsive dynamics is described by the kinematic equation and the equation of motion given by

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{u}, \\ \boldsymbol{M}\dot{\boldsymbol{u}} - \boldsymbol{h}(\boldsymbol{q}, \boldsymbol{u}, \boldsymbol{\tau}, t) &= \boldsymbol{w}\lambda, \end{aligned} \tag{1}$$

where $\boldsymbol{h}(\boldsymbol{q}, \boldsymbol{u}, \boldsymbol{\tau}, t)$ is a function of the state $(\boldsymbol{q}; \boldsymbol{u})$, the coupling forces $\boldsymbol{\tau}$ and the time t explicitly. Throughout the document, we will use the notation $(\boldsymbol{x}; \boldsymbol{y}) = \begin{pmatrix}\boldsymbol{x}^\top & \boldsymbol{y}^\top\end{pmatrix}^\top$, where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. The mass matrix $\boldsymbol{M} = \boldsymbol{M}^\top \succ \boldsymbol{0}$ is symmetric and assumed to be constant and positive definite. The motion of the system is restricted by a single scleronomic geometric unilateral constraint $g(\boldsymbol{q}) \geq 0$, where $g : \mathbb{R}^n \to \mathbb{R}$ is an affine function of $\boldsymbol{q}$. The constraint velocity $\gamma(\boldsymbol{q}, \boldsymbol{u}) = \frac{\mathrm{d}g(\boldsymbol{q}(t))}{\mathrm{d}t} = \boldsymbol{w}^\top \boldsymbol{u}$ is the time derivative of the constraint distance g, where $\boldsymbol{w} = \left(\frac{\partial g}{\partial \boldsymbol{q}}\right)^\top$ is the generalized force direction.

The force law for the constraint force λ is described by the inequality complementarity condition [27] (also referred to as Signorini's law):

$$-\lambda \in \mathcal{N}_{\mathbb{R}_0^+}(g) := \begin{cases} 0 & \text{if } g > 0, \\ (-\infty, 0] & \text{if } g = 0, \end{cases}$$

where the normal cone $\mathcal{N}_{\mathbb{R}_0^+}$ is a set-valued operator. The admissible set of states is $\mathcal{A} := \{(\boldsymbol{q}; \boldsymbol{u}) \in \mathbb{R}^{2n} \,|\, g(\boldsymbol{q}) \geq 0\}$. The boundary of $\mathcal{A}$ is partitioned as $\partial\mathcal{A} = \partial\mathcal{A}^+ \cup \partial\mathcal{A}^-$ with $\partial\mathcal{A}^+ := \{(\boldsymbol{q}; \boldsymbol{u}) \in \mathbb{R}^{2n} \,|\, g(\boldsymbol{q}) = 0,\ \gamma(\boldsymbol{q}, \boldsymbol{u}) \geq 0\}$ and $\partial\mathcal{A}^- = \{(\boldsymbol{q}; \boldsymbol{u}) \in \mathbb{R}^{2n} \,|\, g(\boldsymbol{q}) = 0,\ \gamma(\boldsymbol{q}, \boldsymbol{u}) < 0\}$. An impact is imminent if the state is in $\partial\mathcal{A}^-$ because an impact is required for the state to remain in the admissible set $\mathcal{A}$. The impulsive dynamics is described by the impact equation

$$\boldsymbol{M}(\boldsymbol{u}^+ - \boldsymbol{u}^-) = \boldsymbol{w}\Lambda, \tag{2}$$

where $\boldsymbol{u}^-(t) = \lim_{\tau\uparrow 0} \boldsymbol{u}(t+\tau)$ and $\boldsymbol{u}^+(t) = \lim_{\tau\downarrow 0} \boldsymbol{u}(t+\tau)$ are the pre- and post-impact velocities, respectively. The impact law for the constraint impulse Λ is written in the form

$$-\Lambda \in \mathcal{H}_g(\bar{\gamma}), \tag{3}$$

where $\bar{\gamma} = \frac{1}{2}(\gamma^+ + \gamma^-)$ is the local kinematic quantity dual to Λ and $\gamma^-(t) = \lim_{\tau\uparrow 0} \gamma(\boldsymbol{q}(t+\tau), \boldsymbol{u}(t+\tau))$ and $\gamma^+(t) = \lim_{\tau\downarrow 0} \gamma(\boldsymbol{q}(t+\tau), \boldsymbol{u}(t+\tau))$ denote pre- and post-impact constraint velocities, respectively. The representation (2)–(3) allows to describe well-known impact laws including generalized Newton's and Poisson's impact law [28,29].

In unilaterally constrained mechanical systems, infinitely many impacts can naturally occur in a finite time interval, known as Zeno behavior or the accumulation of impact time instants. Hence, a complete description of the dynamics cannot be given in the hybrid dynamical systems framework [11]. This motivates the measure differential inclusion formulation (1)–(3), which can be written in the compact form (see [10,14–16])

$$\begin{aligned}
&\mathrm{d}\boldsymbol{q} = \boldsymbol{u}\mathrm{d}t,\\
&\boldsymbol{M}\mathrm{d}\boldsymbol{u} - \boldsymbol{h}(\boldsymbol{q}, \boldsymbol{u}, \boldsymbol{\tau}, t)\mathrm{d}t = \boldsymbol{w}\mathrm{d}P,\\
&\mathrm{d}P \in \left\{\lambda\mathrm{d}t + \Lambda\mathrm{d}\eta \mid -\lambda \in \mathcal{N}_{\mathbb{R}_0^+}(g),\ -\Lambda \in \mathcal{H}_g(\bar{\gamma})\right\}.
\end{aligned}$$

The generalized coordinates $\boldsymbol{q} : \mathbb{R} \to \mathbb{R}^n$ are absolutely continuous in time and their measure $\mathrm{d}\boldsymbol{q}$ has the density $\boldsymbol{u}$ with respect to the Lebesgue measure $\mathrm{d}t$. The generalized velocities $\boldsymbol{u} : \mathbb{R} \to \mathbb{R}^n$ are discontinuous due to the impulsive dynamics, but they are assumed to be functions of special locally bounded variation [30]. The pre- and post-impact velocities $\boldsymbol{u}^-(t)$ and $\boldsymbol{u}^+(t)$, respectively, are therefore defined for every point in time. The measure $\mathrm{d}\boldsymbol{u}$ has a density $\dot{\boldsymbol{u}}$ with respect to the Lebesgue measure $\mathrm{d}t$ and a density $(\boldsymbol{u}^+ - \boldsymbol{u}^-)$ with respect to the atomic measure $\mathrm{d}\eta$, i.e., $\mathrm{d}\boldsymbol{u} = \dot{\boldsymbol{u}}\mathrm{d}t + (\boldsymbol{u}^+ - \boldsymbol{u}^-)\mathrm{d}\eta$. The atomic measure $\mathrm{d}\eta = \sum_i \mathrm{d}\delta_{t_i}$ is the sum of Dirac point measures $\mathrm{d}\delta_{t_i}$ at the discontinuity points t_i [27].

To formulate the synchronization problem in the next section, we will employ an explicit impact map from pre- to post-impact states. Namely, the impact equation (2) together with the impact law (3) results in an explicit impact map $\bar{Z} : (\boldsymbol{q}; \boldsymbol{u}^-) \mapsto (\boldsymbol{q}; \boldsymbol{u}^+) = \bar{Z}(\boldsymbol{q}; \boldsymbol{u}^-)$, where we note that the generalized coordinates $\boldsymbol{q}$ are not altered by the impact map.

We adopt the following assumption on the impact map $\bar{Z}$.

Assumption 1. Consider the impact map $\bar{Z} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$.

- If the domain of $\bar{Z}$ is restricted to $\partial\mathcal{A}$, then the map $\bar{Z}$ is non-expansive in the metric $\boldsymbol{A} = \mathrm{diag}(\boldsymbol{K}, \boldsymbol{M})$, where $\boldsymbol{M} \in \mathbb{R}^{n\times n}$ is the mass matrix and $\boldsymbol{K} \in \mathbb{R}^{n\times n}$ is an arbitrary symmetric positive definite matrix. This property can be written as $\|\bar{Z}(\boldsymbol{q}_1; \boldsymbol{u}_1^-) - \bar{Z}(\boldsymbol{q}_2; \boldsymbol{u}_2^-)\|_{\boldsymbol{A}} \le \|(\boldsymbol{q}_1; \boldsymbol{u}_1^-) - (\boldsymbol{q}_2; \boldsymbol{u}_2^-)\|_{\boldsymbol{A}}$ $\forall (\boldsymbol{q}_1; \boldsymbol{u}_1^-), (\boldsymbol{q}_2; \boldsymbol{u}_2^-) \in \partial\mathcal{A}$, where $\|\cdot\|_{\boldsymbol{A}} : \boldsymbol{x} \mapsto \sqrt{\boldsymbol{x}^\mathsf{T}\boldsymbol{A}\boldsymbol{x}}$ is the norm induced by the inner product with the metric $\boldsymbol{A}$.
- Admissible velocities are unchanged, that is, $(\boldsymbol{q}; \boldsymbol{u}^-) \in \mathcal{A} \setminus \partial\mathcal{A}^- \Rightarrow \bar{Z}(\boldsymbol{q}; \boldsymbol{u}^-) = (\boldsymbol{q}; \boldsymbol{u}^-)$.
- Post-impact velocities are admissible, that is, $(\boldsymbol{q}; \boldsymbol{u}^-) \in \partial\mathcal{A}^- \Rightarrow \bar{Z}(\boldsymbol{q}; \boldsymbol{u}^-) \in \partial\mathcal{A}^+$.

The condition of non-expansivity of $\bar{Z}$ is equivalent to Lipschitz continuity of $\bar{Z}$ (in the metric $\boldsymbol{A}$) with Lipschitz constant equal to one [31].

Assumption 1 is a natural assumption, which is fulfilled for the commonly used generalized Newton's impact law for a single constraint. The impact law of the generalized Newton's impact law for a closed constraint is given by [32]

$$-\Lambda \in \mathcal{H}_g(\bar{\gamma}) = \begin{cases} \dfrac{2(1+e)}{e-1}\mathrm{prox}_{\mathbb{R}_0^+}\!\left(-\dfrac{1}{G}\bar{\gamma}\right) & \text{if } 0 \le e < 1,\\ \mathcal{N}_{\mathbb{R}_0^+}(\bar{\gamma}) & \text{if } e = 1, \end{cases} \tag{4}$$

and the impulsive force is zero if the contact is open. The operator $\mathrm{prox}_{\mathcal{C}}(\boldsymbol{v}) := \mathrm{argmin}_{\boldsymbol{v}^*\in\mathcal{C}}\|\boldsymbol{v} - \boldsymbol{v}^*\|$ denotes the proximal point to the set $\mathcal{C}$. Here, the Delassus-operator $G := \boldsymbol{w}^\mathsf{T}\boldsymbol{M}^{-1}\boldsymbol{w}$ is scalar and not a function of $\boldsymbol{q}$. The parameter $e \in [0, 1]$ is the coefficient of restitution and captures the energy dissipation of the impact. According to [32], the explicit expression for the impact map $\bar{Z}$ for the generalized Newton's impact law (4) is

$$\begin{aligned}
&\bar{Z}\begin{pmatrix}\boldsymbol{q}\\ \boldsymbol{u}^-\end{pmatrix} = \begin{pmatrix}\boldsymbol{q}\\ Z_q(\boldsymbol{u}^-)\end{pmatrix}\\
&\text{with } Z_q(\boldsymbol{u}^-) = (1+e)\mathrm{prox}^{\boldsymbol{M}}_{\mathcal{T}_{\mathcal{C}}(\boldsymbol{q})}(\boldsymbol{u}^-) - e\boldsymbol{u}^-,\\
&\text{where } \mathcal{T}_{\mathcal{C}}(\boldsymbol{q}) = \begin{cases}\left\{\boldsymbol{u} | \boldsymbol{w}^\mathsf{T}\boldsymbol{u} \ge 0\right\} & \text{if } g(\boldsymbol{q}) = 0,\\ \mathbb{R}^n & \text{if } g(\boldsymbol{q}) > 0.\end{cases}
\end{aligned} \tag{5}$$

It is shown in [32] that the impact map Z_q in (5) from pre- to post-impact velocities is non-expansive in the metric $\boldsymbol{M}$ for closed constraints. Hence, Assumption 1 is indeed fulfilled for Newton's impact law.

In the following section, we consider the synchronization problem for mechanical systems of the form (1)–(3). The 'peaking phenomenon', which appears when the Euclidean synchronization error is considered, is induced by the nature of the underlying system. In order to obtain a distance function which is continuous when evaluated along solutions, the impact map $\bar{Z}$ is explicitly used in the construction. The property of non-expansivity of $\bar{Z}$ leads to a great simplification in the construction of the distance function.

3. Synchronization problem

For smooth systems, where solutions are continuous functions in time, we consider two solutions as synchronized if they are equal. In the case of nonsmooth systems, we need to take the discontinuous behavior of the solutions into account. We do so by introducing the following definition.

Definition 1. Two solutions $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ are called *synchronized at time t* if $\boldsymbol{x}^+(t) = \boldsymbol{y}^+(t)$.

Using the right-limit in Definition 1 is a small but considerable change compared to the smooth case. At impact time instants, two solutions are considered synchronized if they are mapped to the same post-impact state and, for continuous motion, it agrees with the notion of synchronization for smooth systems. We note that the right-hand limits in Definition 1 can be replaced by left-hand limits without altering the defined property of the solutions if the impact map $\bar{Z}$ is invertible (e.g., Newton's impact law with $e > 0$).

In Section 3.1, we define the synchronization set as a subset of the admissible state space such that it corresponds to synchronization in the sense of Definition 1. Furthermore, solutions are considered *close to synchronization* if they are close to the synchronization set. This notion of distance is introduced in Section 3.2 and it is used to formulate the synchronization problem for systems of the form (1)–(3) in Section 3.3.

3.1. Synchronization set

Following Definition 1, we say that two states $\boldsymbol{x} = (\boldsymbol{q}_x; \boldsymbol{u}_x)$ and $\boldsymbol{y} = (\boldsymbol{q}_y; \boldsymbol{u}_y)$ are synchronized if they are identical or if they are mapped to the same point in the state space by the impact map $\bar{Z}$, which yields the equivalence relation

$$\boldsymbol{x} \sim \boldsymbol{y} \;\Leftrightarrow\; \bar{Z}(\boldsymbol{x}) = \bar{Z}(\boldsymbol{y}). \tag{6}$$

Similarly to the synchronization manifold defined for smooth systems (see, e.g., [3]), we define the synchronization set $\mathcal{S} := \{(\boldsymbol{x}; \boldsymbol{y}) \in \mathcal{A}^2 \mid \boldsymbol{x} \sim \boldsymbol{y}\}$. The synchronization set $\mathcal{S}$ can be partitioned as

$$\mathcal{S} = \mathcal{S}_{00} \cup \mathcal{S}_{01} \cup \mathcal{S}_{10} \cup \mathcal{S}_{11} \tag{7}$$

with the four subsets defined by

$$\mathcal{S}_{00} = \left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in \mathcal{S} \,\middle|\, \boldsymbol{x}, \boldsymbol{y} \in \operatorname{int}\mathcal{A} \vee \boldsymbol{x}, \boldsymbol{y} \in \partial\mathcal{A}^+ \right\}, \tag{8}$$

$$\mathcal{S}_{01} = \left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in \mathcal{S} \,\middle|\, \boldsymbol{x} \in \partial\mathcal{A}^+ \wedge \boldsymbol{y} \in \partial\mathcal{A}^- \right\}, \tag{9}$$

$$\mathcal{S}_{10} = \left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in \mathcal{S} \,\middle|\, \boldsymbol{x} \in \partial\mathcal{A}^- \wedge \boldsymbol{y} \in \partial\mathcal{A}^+ \right\}, \tag{10}$$

$$\mathcal{S}_{11} = \left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in \mathcal{S} \,\middle|\, \boldsymbol{x}, \boldsymbol{y} \in \partial\mathcal{A}^- \right\}. \tag{11}$$

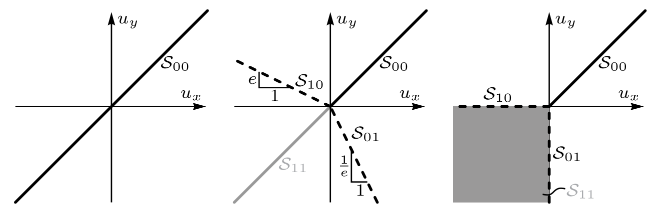
If two states are equivalent, then either both states are in the interior $\operatorname{int}\mathcal{A}$ *or* both are on the boundary $\partial\mathcal{A}$ of $\mathcal{A}$. The partitioning (7)–(11) distinguishes whether the states $\boldsymbol{x}$ or $\boldsymbol{y}$ are immediately prior to an impact or not. More precisely, $\boldsymbol{x}$ has an imminent impact if $(\boldsymbol{x}; \boldsymbol{y}) \in \mathcal{S}_{10} \cup \mathcal{S}_{11}$ and $\boldsymbol{y}$ has an imminent impact if $(\boldsymbol{x}; \boldsymbol{y}) \in \mathcal{S}_{01} \cup \mathcal{S}_{11}$. Let us illustrate the synchronization set $\mathcal{S}$ and its partitioning (7)–(11) by an example.

Example 1. The equivalence relation (6) and the partitioning (7)–(11) are illustrated using a 1-DOF mechanical system with the state vector $(q; u) \in \mathbb{R}^2$. We consider a single constraint $g(q) = q \geq 0$ with a generalized Newton's impact law. The impact map (5) simplifies to

$$\begin{pmatrix} q \\ u^+ \end{pmatrix} = \bar{Z}\begin{pmatrix} q \\ u^- \end{pmatrix} = \begin{pmatrix} q \\ Z_q(u^-) \end{pmatrix}$$
$$\text{with } Z_q(u^-) = \begin{cases} -e\,u^- & \text{if } q = 0 \wedge u^- < 0, \\ u^- & \text{otherwise.} \end{cases} \tag{12}$$

A necessary condition for the equivalence of two points in the state space $\boldsymbol{x} = (q_x; u_x)$ and $\boldsymbol{y} = (q_y; u_y)$ is $q_x = q_y$ because the impact map $\bar{Z}$ does not alter the generalized coordinate. In the case of open constraints (here: $q_x = q_y > 0$), two states $\boldsymbol{x}$ and $\boldsymbol{y}$ are equivalent if and only if the velocities are identical and the synchronization set consists only of the region $(\boldsymbol{x}; \boldsymbol{y}) \in \mathcal{S}_{00}$ as depicted in Fig. 2(a). The case of closed constraints (here: $q_x = q_y = 0$) is depicted in Fig. 2(b) for a partially elastic impact and in Fig. 2(c) for a completely inelastic impact. The region $(\boldsymbol{x}; \boldsymbol{y}) \in \mathcal{S}_{01}$ captures the case where $\boldsymbol{y}$ is mapped to $\boldsymbol{x}$ by the impact (i.e., $u_x = -eu_y$) and vice versa for $(\boldsymbol{x}; \boldsymbol{y}) \in \mathcal{S}_{10}$. The region $\mathcal{S}_{11}$ fills the entire quadrant $u_x < 0, u_y < 0$ in the case of a completely inelastic impact.

We have now defined equivalence of points in the state space. However, we are interested in the synchronization of *trajectories* and therefore it is necessary to compare trajectories point-wise in time. Since the velocities are discontinuous at impact time instants, the state is undefined as it jumps from one point in the state space to another. To deal with this problem, we introduce $[\boldsymbol{x}] := \{\tilde{\boldsymbol{x}} | \tilde{\boldsymbol{x}} \sim \boldsymbol{x}\}$ as the equivalence class of $\boldsymbol{x}$ using the equivalence relation (6). Furthermore, let $\mathcal{A}/\!\sim := \{[\boldsymbol{x}] \mid \boldsymbol{x} \in \mathcal{A}\}$ denote the quotient space, which is the set of all equivalence classes. Since the quotient map is constructed using the impact map of the dynamics, the solutions $\boldsymbol{x}(t)$ on the quotient map are continuous in time, that is, $[\boldsymbol{x}^-(t)] = [\boldsymbol{x}^+(t)]$, and we can define $[\boldsymbol{x}(t)] := [\boldsymbol{x}^+(t)]$ for all t. Therefore, two solutions $\boldsymbol{x}(t)$, $\boldsymbol{y}(t)$ are synchronized at time t



(a) $q_x = q_y > 0$, $e \in [0, 1]$. (b) $q_x = q_y = 0$, $e \in (0,1]$. (c) $q_x = q_y > 0$, $e = 0$.

Fig. 2. Partitioning of the synchronization set $\mathcal{S}$ for open constraints (a), closed constraints with $e \in (0, 1]$ (b), and closed constraints with $e = 0$ (c).

according to Definition 1 if they belong to the same equivalence class at time t, that is, if $[\boldsymbol{x}(t)] = [\boldsymbol{y}(t)]$.

3.2. Distance function

We will now introduce a notion of distance between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ in the state space in order to measure how far two solutions are away from synchrony at a certain time t. In contrast to the similar distance notions introduced in [23] and [33], the distance function introduced here exploits the physical properties of the impact map $\bar{Z}$. In order to avoid the 'peaking phenomenon' when evaluated along solutions, two states should also be considered close if one state has just experienced an impact and the other state is still on the verge of an impact. Using the equivalence relation (6), this can be achieved by defining the distance function $d(\boldsymbol{x}, \boldsymbol{y})$ as

$$d(\boldsymbol{x}, \boldsymbol{y}) = \inf \left\{ \sum_{j=0}^{N} \left\| \boldsymbol{x}^j - \boldsymbol{y}^j \right\| \,\middle|\, N \in \mathbb{N}_0, \boldsymbol{x} = \boldsymbol{x}^0, \right.$$
$$\left. \boldsymbol{y}^j \sim \boldsymbol{x}^{j+1} \text{ for } 0 \leq j < N, \boldsymbol{y}^N = \boldsymbol{y} \right\}, \tag{13}$$

where $\|\cdot\| : \boldsymbol{x} \mapsto \sqrt{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}$ denotes the Euclidean norm. The distance function d is the quotient metric on the quotient space $\mathcal{A}/\!\sim$ obtained by the equivalence relation (6), cf. [34]. The quotient metric is indeed a metric on the quotient space if Assumption 1 on the impact map $\bar{Z}$ is fulfilled, which is shown in the following proposition.

Proposition 1. *Let Assumption 1 be fulfilled. Then the quotient distance function $d(\boldsymbol{x}, \boldsymbol{y})$ in (13) is a metric on the quotient space, that is, $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{A}$*

(i) $d(\boldsymbol{x}, \boldsymbol{y}) = 0 \;\Leftrightarrow\; (\boldsymbol{x}; \boldsymbol{y}) \in \mathcal{S}$,
(ii) $d(\boldsymbol{x}, \boldsymbol{y}) = d(\boldsymbol{y}, \boldsymbol{x})$,
(iii) $d(\boldsymbol{x}, \boldsymbol{y}) \leq d(\boldsymbol{x}, \boldsymbol{z}) + d(\boldsymbol{z}, \boldsymbol{y})$.

Proof. See Appendix. □

Conditions (ii) and (iii) in Proposition 1 are also fulfilled if the distance function d is considered on the original space $\mathcal{A}$. Additionally, $\boldsymbol{x} = \boldsymbol{y}$ implies $d(\boldsymbol{x}, \boldsymbol{y}) = 0$, however, the converse is not true as $d(\boldsymbol{x}, \boldsymbol{y}) = 0$ merely implies $\boldsymbol{x} \sim \boldsymbol{y}$. The distance function d is therefore only a pseudometric on the space $\mathcal{A}$.

The distance function d is continuous in time when evaluated along solution $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$, since it fulfills $d(\boldsymbol{x}, \boldsymbol{y}) = d(\boldsymbol{x}^*, \boldsymbol{y}^*)$ $\forall \boldsymbol{x} \in [\boldsymbol{x}^*]$, $\boldsymbol{y} \in [\boldsymbol{y}^*]$ by construction and solutions of (1)–(3) are continuous in time on the quotient space. The distance function at impact time instants can thus be defined as $d(\boldsymbol{x}(t), \boldsymbol{y}(t)) := d(\boldsymbol{x}^+(t), \boldsymbol{y}^+(t))$, as the result is not altered if the left limit for $\boldsymbol{x}(t)$ or $\boldsymbol{y}(t)$ would be taken.

In the next step, we construct a simpler (quotient) distance function $d^A(\boldsymbol{x},\boldsymbol{y})$ which is equivalent[1] to the distance function d if the impact map is non-expansive. In the definition of the distance function d in (13), the points $\boldsymbol{x}^{j+1} \sim \boldsymbol{y}^j$ can be seen as intermediate points. The number of these points, denoted by N, is generally unbounded. The new distance function $d^A(\boldsymbol{x},\boldsymbol{y})$ is simpler in the sense that at most two intermediate points are necessary as shown in the following theorem.

Theorem 1. *Let Assumption 1 be fulfilled and let $d(\boldsymbol{x},\boldsymbol{y})$ be the quotient distance function in (13) with the equivalence relation (6). Let $d^A(\boldsymbol{x},\boldsymbol{y})$ be defined by*

$$d^A(\boldsymbol{x},\boldsymbol{y}) := \min\left\{d^A_{00}, d^A_{01}, d^A_{10}, d^A_{11}\right\}, \tag{14}$$

where

$$d^A_{00} = \|\boldsymbol{x}-\boldsymbol{y}\|_{\boldsymbol{A}}, \tag{15}$$

$$d^A_{01} = \inf\left\{\left\|\boldsymbol{x}-\boldsymbol{y}^0\right\|_{\boldsymbol{A}} + \left\|\boldsymbol{x}^1-\boldsymbol{y}\right\|_{\boldsymbol{A}} \;\middle|\; \begin{pmatrix}\boldsymbol{x}^1\\ \boldsymbol{y}^0\end{pmatrix} \in \mathcal{S}_{10}\right\}, \tag{16}$$

$$d^A_{10} = \inf\left\{\left\|\boldsymbol{x}-\boldsymbol{y}^0\right\|_{\boldsymbol{A}} + \left\|\boldsymbol{x}^1-\boldsymbol{y}\right\|_{\boldsymbol{A}} \;\middle|\; \begin{pmatrix}\boldsymbol{x}^1\\ \boldsymbol{y}^0\end{pmatrix} \in \mathcal{S}_{01}\right\}, \tag{17}$$

$$d^A_{11} = \inf\left\{\left\|\boldsymbol{x}-\boldsymbol{y}^0\right\|_{\boldsymbol{A}} + \left\|\boldsymbol{x}^1-\boldsymbol{y}^1\right\|_{\boldsymbol{A}} + \left\|\boldsymbol{x}^2-\boldsymbol{y}\right\|_{\boldsymbol{A}} \;\middle|\; \begin{pmatrix}\boldsymbol{x}^1\\ \boldsymbol{y}^0\end{pmatrix} \in \mathcal{S}_{01} \wedge \begin{pmatrix}\boldsymbol{x}^2\\ \boldsymbol{y}^1\end{pmatrix} \in \mathcal{S}_{10}\right\} \tag{18}$$

with $\boldsymbol{A} = \mathrm{diag}(\boldsymbol{K},\boldsymbol{M})$, where $\boldsymbol{M}$ is the mass matrix and $\boldsymbol{K}$ is a symmetric positive definite matrix. Then the distance functions $d(\boldsymbol{x},\boldsymbol{y})$ and $d^A(\boldsymbol{x},\boldsymbol{y})$ are equivalent. Furthermore, if $\boldsymbol{A}$ is the identity matrix $\boldsymbol{I}$, then $d^A(\boldsymbol{x},\boldsymbol{y}) = d^I(\boldsymbol{x},\boldsymbol{y}) = d(\boldsymbol{x},\boldsymbol{y})$.

Proof. See Appendix. □

Theorem 1 bears similarity with [25, Lemma 2.1], which considers one-dimensional systems together with Newton's impact law. In contrast to [25], Theorem 1 is more generically applicable to multi-dimensional systems and the impact law is only assumed to be non-expansive (which includes Newton's impact law, but allows for more generic impact laws).

Remark 1. If $\boldsymbol{K}$ in $\boldsymbol{A} = \mathrm{diag}(\boldsymbol{K},\boldsymbol{M})$ is the stiffness matrix of the considered mechanical system, then $\|\cdot\|_{\boldsymbol{A}}$ corresponds to the metric of the mechanical energy of the system.

Example 1 (*Revisited*). We revisit Example 1 in order to illustrate the distance function d^A in (14)–(18). We write an intermediate point $(\boldsymbol{x}^1;\boldsymbol{y}^0) \in \mathcal{S}_{10}$ as $\boldsymbol{x}^1 = (0;-z)$, $\boldsymbol{y}^0 = (0;ez)$ with $z > 0$. The distance function d^A can therefore be written as (14), where

$$d^A_{00} = \left\|\begin{pmatrix}q_x\\u_x\end{pmatrix} - \begin{pmatrix}q_y\\u_y\end{pmatrix}\right\|_{\boldsymbol{A}},$$

$$d^A_{01} = \inf_{z>0}\left\{\left\|\begin{pmatrix}q_x\\u_x\end{pmatrix} - \begin{pmatrix}0\\ez\end{pmatrix}\right\|_{\boldsymbol{A}} + \left\|\begin{pmatrix}0\\-z\end{pmatrix} - \begin{pmatrix}q_y\\u_y\end{pmatrix}\right\|_{\boldsymbol{A}}\right\},$$

$$d^A_{10} = \inf_{z>0}\left\{\left\|\begin{pmatrix}q_x\\u_x\end{pmatrix} - \begin{pmatrix}0\\-z\end{pmatrix}\right\|_{\boldsymbol{A}} + \left\|\begin{pmatrix}0\\ez\end{pmatrix} - \begin{pmatrix}q_y\\u_y\end{pmatrix}\right\|_{\boldsymbol{A}}\right\},$$

$$d^A_{11} = \inf_{\substack{z_1>0\\z_2>0}}\left\{\left\|\begin{pmatrix}q_x\\u_x\end{pmatrix} - \begin{pmatrix}0\\-z_1\end{pmatrix}\right\|_{\boldsymbol{A}} + \left\|\begin{pmatrix}0\\ez_1\end{pmatrix} - \begin{pmatrix}0\\ez_2\end{pmatrix}\right\|_{\boldsymbol{A}} + \left\|\begin{pmatrix}0\\-z_2\end{pmatrix} - \begin{pmatrix}q_y\\u_y\end{pmatrix}\right\|_{\boldsymbol{A}}\right\}$$

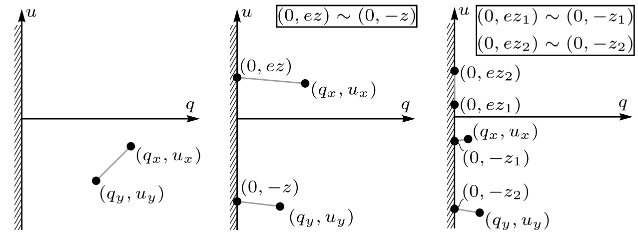


Fig. 3. Distance function $d^A(\boldsymbol{x},\boldsymbol{y})$ for Example 1 for (a) d^A_{00}, (b) d^A_{01} and (c) d^A_{11}.

with $\boldsymbol{A} = \mathrm{diag}(k,m)$ positive definite. The term d^A_{00} corresponds to the Euclidean distance in the metric $\boldsymbol{A}$, which is depicted in Fig. 3(a). The term d^A_{01} consists of the sum of two Euclidean distances, where the intermediate point $(0;-z;0;ez) \in \mathcal{S}_{10}$ is chosen as depicted in Fig. 3(b). Note that the term d^A_{01} vanishes for $\boldsymbol{x} = \bar{Z}(\boldsymbol{y}) \neq \boldsymbol{y}$. The term d^A_{10} is the symmetric case to d^A_{01}, that is, $d^A_{10}(\boldsymbol{x},\boldsymbol{y}) = d^A_{01}(\boldsymbol{y},\boldsymbol{x})$. The term d^A_{11} is the sum of three Euclidean distances using the intermediate points $(0;ez_1;0;-z_1) \in \mathcal{S}_{01}$ and $(0;-z_2;0;ez_2) \in \mathcal{S}_{10}$ as depicted in Fig. 3(c).

Remark 2. In [23], which focuses on hybrid systems formulated in the framework of [11] with invertible jump laws, a related distance function $\tilde{d}(\boldsymbol{x},\boldsymbol{y})$ has been introduced as

$$\tilde{d}(\boldsymbol{x},\boldsymbol{y}) = \min_{\boldsymbol{z}\in\mathcal{S}}\left\|\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix} - \boldsymbol{z}\right\|. \tag{19}$$

The distance function $\tilde{d}$ and the quotient distance function d are equivalent under certain assumptions on the impact map Z. More specifically, if Assumption 1 is fulfilled and if there exists a constant $\alpha \in (0\ 1]$ such that $\frac{1}{\alpha}\bar{Z}$ restricted to $\partial\mathcal{A}^-$ is non-contractive (i.e., strictly expansive) in the metric $\boldsymbol{A}$, then the distance functions defined by (13) and (19) are equivalent. This statement is proven in Proposition 2 in Appendix. Let us consider, for example, the generalized Newton's impact law for a single geometric unilateral constraint for which the impact map $\bar{Z}$ is given by (5). If the coefficient of restitution e is sufficiently larger than 0 (i.e., if $1 > (1-e^2)\,\mathrm{cond}(\boldsymbol{M})$, where $\mathrm{cond}(\boldsymbol{M})$ is the condition number of the mass matrix), then there exists a constant $\alpha \in (0,\ 1]$ such that $\frac{1}{\alpha}\bar{Z}$ restricted to $\partial\mathcal{A}^-$ is non-contractive in the metric $\boldsymbol{A}$ as shown in Lemma 1 in Appendix. Together with Assumption 1, this implies that the distance function $\tilde{d}$ in (19) and the quotient distance function d in (13) are equivalent for this example.

If the distance functions d and $\tilde{d}$ are equivalent, then any statement using stability in the sense of Lyapunov does not depend on the choice of either distance function. In the case of a completely inelastic collision, the equivalence cannot hold as $\mathrm{cond}(\boldsymbol{M}) \geq 1$ by definition. This is due to the fact that the definition of $\tilde{d}$ in (19) uses only one intermediate point, whereas the definition of d allows for more intermediate points. In any case, both distance functions vanish if and only if the considered states are equivalent according to Proposition 1. The present formulation of the distance function d is preferred as it is the most natural choice of a metric on the quotient space. Furthermore, in contrast to (19), the distance function $d(\boldsymbol{x}(t),\boldsymbol{y}(t))$ evaluated along solutions $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ is guaranteed to be a continuous function in time. Consequently, $d(\boldsymbol{x}(t),\boldsymbol{y}(t))$ is also defined at impact time instants.

3.3. Synchronization problem

We define the synchronization problem for mechanical systems of the form (1)–(3) using the quotient distance function d

[1] Let $d_\alpha, d_\beta : E \times E \to \mathbb{R}$ be two distance functions on a set E. Analogously to equivalence of norms, the distance functions are called equivalent if there exist two numbers $c_1 > 0$, $c_2 > 0$ such that $c_1 d_\alpha(\boldsymbol{x},\boldsymbol{y}) \leq d_\beta(\boldsymbol{x},\boldsymbol{y}) \leq c_2 d_\alpha(\boldsymbol{x},\boldsymbol{y})\ \forall \boldsymbol{x},\boldsymbol{y} \in E$.

given by (13). Given two trajectories $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$, the error signal $e(t) = d(\boldsymbol{x}^+(t), \boldsymbol{y}^+(t))$ is a continuous function in time since $d(\boldsymbol{x}^-(t), \boldsymbol{y}^-(t)) = d(\boldsymbol{x}^+(t), \boldsymbol{y}^+(t))$. This observation allows us to formulate the synchronization problem as follows (confer [35] for definitions of synchronization for smooth systems).

Definition 2 (*Synchronization Problem*)**.** Consider two mechanical systems described by (1)–(3) with solutions $\boldsymbol{x}(t) = (\boldsymbol{q}_x(t); \boldsymbol{u}_x(t))$ and $\boldsymbol{y}(t) = (\boldsymbol{q}_y(t); \boldsymbol{u}_y(t))$ for the initial conditions $\boldsymbol{x}^-(t_0), \boldsymbol{y}^-(t_0) \in \mathcal{A}$. Let the coupling forces $\boldsymbol{\tau}_x$ and $\boldsymbol{\tau}_y$ acting on the first and second system, respectively, be defined by a static interaction law $(\boldsymbol{\tau}_x(t), \boldsymbol{\tau}_y(t)) = (\boldsymbol{\kappa}_x(\boldsymbol{x}(t), \boldsymbol{y}(t), t), \boldsymbol{\kappa}_y(\boldsymbol{x}(t), \boldsymbol{y}(t), t))$ and let the distance function d be defined by (13). The coupled systems are said to achieve *local synchronization* if for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$d(\boldsymbol{x}(t_0), \boldsymbol{y}(t_0)) < \delta(\varepsilon) \Rightarrow d(\boldsymbol{x}(t), \boldsymbol{y}(t)) < \varepsilon, \ \forall t \geq t_0 \tag{20}$$

and there exists a $\delta_0 > 0$ such that

$$d(\boldsymbol{x}(t_0), \boldsymbol{y}(t_0)) < \delta_0 \Rightarrow \lim_{t\to\infty} d(\boldsymbol{x}(t), \boldsymbol{y}(t)) = 0. \tag{21}$$

Furthermore, the coupled systems are said to achieve *global synchronization* if (20) and (21) are fulfilled and δ_0 in (21) can be chosen arbitrarily large.

The quotient distance function d gives a natural notion of distance when comparing solutions and it is therefore appropriate in the definition of the synchronization problem. If two solutions $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ are close at a certain point in time (i.e., $d(\boldsymbol{x}(t), \boldsymbol{y}(t))$ is small) and if the solutions are far away from the constraint, then the Euclidean distance $\|\boldsymbol{x}(t) - \boldsymbol{y}(t)\|$ is small as well. The Euclidean distance might be large in the vicinity of the constraints even if the solutions are arbitrarily close to each other w.r.t. d. However, generally for unilaterally constrained mechanical systems, the width of the 'peaks' of the Euclidean distance tends to zero as the solutions approach each other, see [17,23].

Remark 3. In the next section, the two systems in Definition 2 are coupled by the forces $\boldsymbol{\tau}_x$ and $\boldsymbol{\tau}_y$, which are used to achieve (local/global) synchronization. For $\boldsymbol{\tau}_x = 0$, Definition 2 describes a master–slave synchronization problem where $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ are the solutions of the master and slave system, respectively. In this case, synchronization requires the solution of the master system to be an asymptotically stable solution of the slave system with respect to the distance function d, as defined in [17]. Here, we consider the generic case of mutual synchronization.

4. 1-DOF mechanical impact oscillators

In this section, we consider the synchronization problem for a 1-DOF mechanical impact oscillator as depicted in Fig. 4. Even though this is a canonical representative of the class of mechanical systems presented in Section 2, the synchronization problem including accumulation points has remained unsolved for this case. This shows that the theory of synchronization of impacting systems including accumulation points is still in its infancy. The one-dimensional case is a challenging first step and serves as stepping stone in the direction of more general systems.

In the following, we design a Lyapunov function for the 1-DOF system which enables us to construct sufficient conditions for *local* synchronization in the sense of Definition 2 without any interaction. Furthermore, we design a synchronizing interaction law and construct sufficient conditions for *global* synchronization for the coupled case.

The states of the two coupled systems are denoted by $\boldsymbol{x} = (q_x; u_x)$ and $\boldsymbol{y} = (q_y; u_y)$. The equation of motion is described by (1)

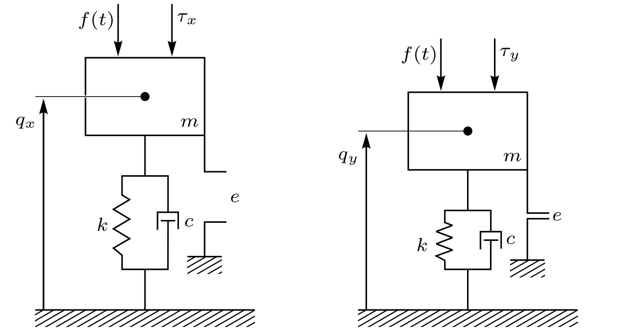


Fig. 4. Two identical unilaterally constrained 1-DOF mechanical systems subjected to an external forcing $f(t)$ and coupling forces τ_x and τ_y.

with $h(q, u, \tau, t) = -cu - kq - f(t) - \tau$. The impact equation is given by (2). We assume that the stiffness k and the damping c are strictly positive. Without loss of generality, we choose $m = k = 1$ as well as $w = 1$ and $g = q$. This can always be achieved by rescaling the states and time. The equations of motion of the coupled system are obtained as

$$\begin{aligned} &\dot{u}_x + cu_x + q_x = \lambda_x - f(t) - \tau_x \quad \text{with } \dot{q}_x = u_x \text{ a.e.,} \\ &\dot{u}_y + cu_y + q_y = \lambda_y - f(t) - \tau_y \quad \text{with } \dot{q}_y = u_y \text{ a.e.} \end{aligned} \tag{22}$$

The external forcing $f(t)$ in (22) is identical for both systems, whereas the coupling forces τ_x and τ_y are generally unequal. The two systems are coupled if the force τ_x depends on the state $\boldsymbol{y}$ and/or vice versa for τ_y and $\boldsymbol{x}$. The unilateral constraints are closed if $q_x = 0$ or $q_y = 0$, respectively, and the constraint forces obey the Signorini conditions

$$-\lambda_x \in \mathcal{N}_{\mathbb{R}_0^+}(q_x), \quad -\lambda_y \in \mathcal{N}_{\mathbb{R}_0^+}(q_y). \tag{23}$$

The generalized Newton's impact law is chosen and completely inelastic collisions are excluded, that is, the coefficient of restitution fulfills $e \in (0 \ \ 1]$. The corresponding impact map $\bar{Z}$ fulfills Assumption 1 and is given by (12) as

$$\begin{aligned} &\bar{Z}\begin{pmatrix} q_x \\ u_x^- \end{pmatrix} = \begin{pmatrix} q_x \\ Z_{q_x}(u_x^-) \end{pmatrix}, \quad \bar{Z}\begin{pmatrix} q_y \\ u_y^- \end{pmatrix} = \begin{pmatrix} q_y \\ Z_{q_y}(u_y^-) \end{pmatrix} \\ &\text{with } Z_q(u^-) = \begin{cases} -e\,u^- & \text{if } q = 0 \wedge u^- < 0, \\ u^- & \text{otherwise.} \end{cases} \end{aligned} \tag{24}$$

Remark 4. Collisions with $e = 0$ dissipate the entire energy of a 1-DOF system, which typically leads to finite-time synchronization after a few impacts, also in the absence of any coupling.

As we are interested in the synchronization problem for the system described by (22)–(24), we aim to study the evolution of the quotient distance function d defined in (13) along solutions. Since the mass is normalized to be equal to one, the matrix $\boldsymbol{A}$ in (14)–(18) can be chosen as the identity matrix. Additionally, Theorem 1 implies that the distance function d^l is identical to d. Therefore, we can reduce the complexity of the problem by considering the simpler distance function d^l.

In order to design the interaction laws

$$\tau_x(t) = \kappa_x(\boldsymbol{x}(t), \boldsymbol{y}(t), t), \quad \tau_y(t) = \kappa_y(\boldsymbol{x}(t), \boldsymbol{y}(t), t) \tag{25}$$

and to study synchronization of the coupled system (22)–(24), we will now present a Lyapunov function suitable for investigating synchronization according to Definition 2. A naive approach would be to choose the candidate Lyapunov function $\frac{1}{2}(d^l)^2 = \frac{1}{2}\min\{{d^l_{00}}^2, {d^l_{01}}^2, {d^l_{10}}^2, {d^l_{11}}^2\}$ and differentiate this function with respect to time. However, this approach requires explicit knowledge of the intermediate points which have to be obtained by

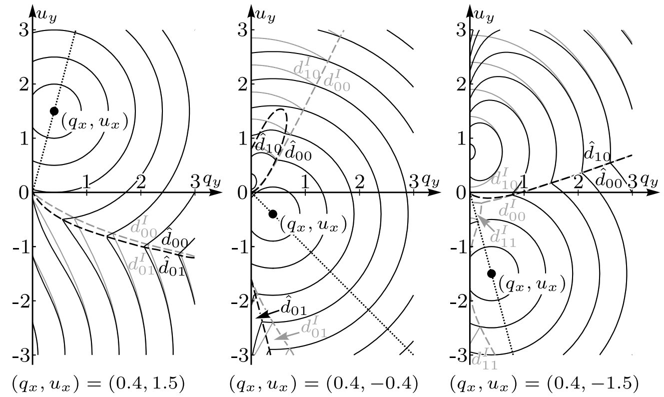


Fig. 5. Level sets of the distance functions $d^I(\boldsymbol{x}, \boldsymbol{y}) = d(\boldsymbol{x}, \boldsymbol{y})$ (gray) and $\hat{d}(\boldsymbol{x}, \boldsymbol{y})$ (black) for $e = 0.5$ and three different choices of $\boldsymbol{x}$.

solving the minimization problem in the definition of d^I_{01}, d^I_{10} and d^I_{11}, see (16)–(18). In order to avoid this complication and to obtain an explicit definition for a Lyapunov function, we approximate the minimizers in (16)–(18) and obtain the following candidate Lyapunov function:

$$V(\boldsymbol{x}, \boldsymbol{y}) := \min\{V_{00}(\boldsymbol{x}, \boldsymbol{y}), V_{01}(\boldsymbol{x}, \boldsymbol{y}), V_{10}(\boldsymbol{x}, \boldsymbol{y})\}, \tag{26}$$

where $V_{00} := \frac{1}{2}\hat{d}_{00}^2$, $V_{01} := \frac{1}{2}\hat{d}_{01}^2$, $V_{10} := \frac{1}{2}\hat{d}_{10}^2$,

$$\hat{d}_{00} := \sqrt{(q_x - q_y)^2 + (u_x - u_y)^2}, \tag{28}$$

$$\hat{d}_{01} := \begin{cases} \sqrt{(q_x + q_y)^2 + \left(\dfrac{q_x + q_y}{q_x + e q_y}\right)^2 (u_x + e u_y)^2} \\ \qquad \text{if } u_x q_y - u_y q_x > 0, \\ \sqrt{q_x^2 + u_x^2} + \sqrt{q_y^2 + u_y^2} \text{ if } u_x q_y - u_y q_x \le 0, \end{cases} \tag{29}$$

$$\hat{d}_{10}(\boldsymbol{x}, \boldsymbol{y}) := \hat{d}_{01}(\boldsymbol{y}, \boldsymbol{x}). \tag{30}$$

Remark 5. The Lyapunov function (26)–(30) can be written as $V = \frac{1}{2}\hat{d}^2$, where $\hat{d}(\boldsymbol{x}, \boldsymbol{y}) := \min\{\hat{d}_{00}(\boldsymbol{x}, \boldsymbol{y}), \hat{d}_{01}(\boldsymbol{x}, \boldsymbol{y}), \hat{d}_{10}(\boldsymbol{x}, \boldsymbol{y})\}$ is an explicit approximation of the distance function d^I in (14)–(18). First, we note that $\hat{d}_{00}(\boldsymbol{x}, \boldsymbol{y}) = d_{00}(\boldsymbol{x}, \boldsymbol{y})$. Furthermore, the function d_{11} in (18) is neglected, since d_{11} can be bounded from below[2] using d_{00}. The function $\hat{d}_{01}(\boldsymbol{x}, \boldsymbol{y})$ is obtained by selecting in (16) the intermediate point $(\tilde{\boldsymbol{x}}^1; \tilde{\boldsymbol{y}}^0) \in \mathcal{S}_{10}$ with $-\tilde{\boldsymbol{x}}^1 = \frac{1}{e}\tilde{\boldsymbol{y}}^0 = (0; z_{01})$ and $z_{01} = \begin{cases} \frac{u_x q_y - u_y q_x}{q_x + e q_y} & \text{if } u_x q_y - u_y q_x > 0, \\ 0 & \text{if } u_x q_y - u_y q_x \le 0. \end{cases}$. Therefore, $(\tilde{\boldsymbol{x}}^1, \tilde{\boldsymbol{y}}^0)$ acts as an approximation of the minimizer in (16), thereby generating an upper bound for $d_{01}(\boldsymbol{x}, \boldsymbol{y})$. Analogously, the function $\hat{d}_{10}(\boldsymbol{x}, \boldsymbol{y})$ is an approximation of $d_{10}(\boldsymbol{x}, \boldsymbol{y})$. An interpretation of the approximation of the intermediate point will be given later in Lemma 2.

Fig. 5 depicts level sets of the distance function $d^I = d$ (gray) and of the approximation $\hat{d} = \min\{\hat{d}_{00}, \hat{d}_{01}, \hat{d}_{10}\}$ (black) in the q_y-u_y-plane for three different choices of $\boldsymbol{x}$. Depending on the intermediate points, we can distinguish between four regions for $d^I = \min\{d^I_{00}, d^I_{01}, d^I_{10}, d^I_{11}\}$ and three regions for $\hat{d} = \min\{\hat{d}_{00}, \hat{d}_{01}, \hat{d}_{10}\}$. These regions are separated by dashed lines in Fig. 5. If the points $\boldsymbol{x}$ and $\boldsymbol{y}$ are close, then the distance functions are given by $d^I = d^I_{00}$ and $\hat{d} = \hat{d}_{00}$, respectively, and the level sets are concentric circles.

If the points $\boldsymbol{x}$ and $\boldsymbol{y}$ are close to the constraint but not close to each other, then the minimum is no longer attained using the Euclidean distance. The dotted line running through the origin and $\boldsymbol{x}$ is given by $u_x q_y - u_y q_x = 0$ and it is used in the definition of $\hat{d}_{01}$ and $\hat{d}_{10}$, see (29). We note that $\hat{d} = \hat{d}_{00}$ on the line $u_x q_y - u_y q_x = 0$. Furthermore, we have $\hat{d} < \hat{d}_{10}$ in the region $u_x q_y - u_y q_x > 0$ and, analogously, $\hat{d} < \hat{d}_{01}$ in the region $u_x q_y - u_y q_x < 0$. Consequently, the distance function $\hat{d}$ can be written as

$$\hat{d} = \begin{cases} \min\{\hat{d}_{00}, \hat{d}_{01}\} & \text{if } u_x q_y - u_y q_x > 0, \\ \hat{d}_{00} & \text{if } u_x q_y - u_y q_x = 0, \\ \min\{\hat{d}_{00}, \hat{d}_{10}\} & \text{if } u_x q_y - u_y q_x < 0. \end{cases} \tag{31}$$

The candidate Lyapunov function V is positive definite in the distance function d^I, which is shown in the following lemma.

Lemma 2. *Let V be the function defined in (26)–(30) and let the quotient distance function d in (13) be defined using the impact map $\bar{Z}$ in (24). Then,*

$$\frac{1}{2}d^2(\boldsymbol{x}, \boldsymbol{y}) \le V(\boldsymbol{x}, \boldsymbol{y}) \le \frac{1}{2}\left(\frac{d(\boldsymbol{x}, \boldsymbol{y})}{e}\right)^2.$$

Proof. See Appendix. □

Note that Lemma 2 implies $V(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}d^2(\boldsymbol{x}, \boldsymbol{y})$ for $e = 1$.

The candidate Lyapunov function (26)–(30) measures the distance between two states. Evaluated along solutions $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ of (22)–(24), a scalar function $V(\boldsymbol{x}(t), \boldsymbol{y}(t))$ is obtained. The differential measure $\mathrm{d}V$ has a density $\dot{V}$ with respect to the Lebesgue measure $\mathrm{d}t$ and a density $V^+ - V^-$ with respect to the atomic measure $\mathrm{d}\eta$, i.e., $\mathrm{d}V = \dot{V}\mathrm{d}t + (V^+ - V^-)\mathrm{d}\eta$. This is the case because the jumps in the Lyapunov function $V^+ - V^-$ are well defined for all points in time in the Lebesgue-negligible set $\{t \in \mathbb{R} \,|\, q_x = q_y = 0 \wedge |u_x| + |u_y| \neq 0\}$ (see later in Lemma 3). For every other point in time not in the set $\{t \in \mathbb{R} \,|\, q_x = q_y = 0 \wedge |u_x| + |u_y| \neq 0\}$, the function V is locally Lipschitz in both arguments and the considered solutions are functions of special locally bounded variation. From [10, Prop. 6.3] it follows that the candidate Lyapunov function is of special locally bounded variation, which directly implies $\mathrm{d}V = \dot{V}\mathrm{d}t + (V^+ - V^-)\mathrm{d}\eta$. In the following, the densities $\dot{V}$ and $(V^+ - V^-)$ are evaluated for system (22)–(24), see Lemmas 4 and 3, respectively, which is used later for the Lyapunov-based stability analysis.

Lemma 3. *The Lyapunov function (26)–(30) evaluated along solutions $\boldsymbol{x}(t), \boldsymbol{y}(t)$ of (22)–(24) satisfies*

$$V(\boldsymbol{x}^+(t), \boldsymbol{y}^+(t)) - V(\boldsymbol{x}^-(t), \boldsymbol{y}^-(t)) \le 0 \; \forall t.$$

Proof. See Appendix. □

The density $\dot{V}$ is generally given by $\dot{V} = \boldsymbol{\xi}^{\mathrm{T}}(\dot{\boldsymbol{x}}; \dot{\boldsymbol{y}})$ with $\boldsymbol{\xi} \in \partial V(\boldsymbol{x}, \boldsymbol{y})$, where $\partial V(\boldsymbol{x}, \boldsymbol{y})$ denotes the Clarke's generalized gradient of $V(\boldsymbol{x}, \boldsymbol{y})$ [36]. In the following, we consider the three cases (i) $V_{00} < \min\{V_{01}, V_{10}\}$, (ii) $V_{01} < \min\{V_{00}, V_{10}\}$ and (iii) $V_{10} < \min\{V_{00}, V_{01}\}$. In these cases, the generalized gradient consists of a single element, that is, the gradient in the classical sense. The case for which the generalized gradient is set-valued is considered later separately.

Lemma 4. *Let the Lyapunov function V in (26)–(30) be evaluated along solutions $\boldsymbol{x}(t)$, $\boldsymbol{y}(t)$ of (22)–(24). Consider the cases (i) $V_{00} < \min\{V_{01}, V_{10}\}$, (ii) $V_{01} < \min\{V_{00}, V_{10}\}$ and (iii) V_{10}*

[2] The lower bound is explicitly given in the proof of Proposition 2. The necessary conditions are fulfilled for the impact map (24) with $e \in (0, 1]$ as shown in Lemma 1 given in Appendix.

$< \min\{V_{00}, V_{01}\}$. *Depending on the case, the density* $\dot{V}$ *is equal to*

$$\text{(i) } \dot{V}_{00} = -c(u_x - u_y)^2 + (u_x - u_y)(-\tau_x + \lambda_x + \tau_y - \lambda_y), \tag{32}$$

$$\begin{aligned}\text{(ii) } \dot{V}_{01} = &-c\frac{(q_x+q_y)^2}{(q_x+eq_y)^2}(u_x+eu_y)^2 \\ &-(1-e)\frac{q_x+q_y}{q_x+eq_y}\left(1+\frac{(u_x+eu_y)^2}{(q_x+eq_y)^2}\right)(u_xq_y-q_xu_y) \\ &+\frac{(q_x+q_y)^2}{(q_x+eq_y)^2}(u_x+eu_y)((\lambda_x+e\lambda_y)-(1+e)f-(\tau_x+e\tau_y)),\end{aligned} \tag{33}$$

$$\text{(iii) } \dot{V}_{10}\text{: symmetric to case (ii).} \tag{34}$$

Proof. See Appendix. □

Lemma 2–4 show that the Lyapunov function (26)–(30) is positive definite in the quotient distance function, it is non-increasing at discontinuities and the density $\dot{V}$ in the different cases takes a simple form. Therefore, the proposed Lyapunov function is suitable to study the synchronization problem of the mechanical system (22)–(24).

Let us first consider two solutions which both keep a minimal distance from the origin, that is, stay away from accumulation points and grazing trajectories. Then, the constraint forces are zero almost everywhere since there are no persistent contacts. If the external forcing f satisfies a certain bound, then *local* synchronization can be achieved without any coupling, which is stated in the following theorem.

Theorem 2. *Let* $\boldsymbol{x}(t)$ *and* $\boldsymbol{y}(t)$ *be the solution of* (22)–(24) *with* $\tau_x = \tau_y = 0$ *for the initial conditions* $\boldsymbol{x}^-(t_0)$, $\boldsymbol{y}^-(t_0) \in \mathcal{A}$ *and let* $V(\boldsymbol{x}(t), \boldsymbol{y}(t))$ *be the Lyapunov function defined by* (26)–(30) *evaluated along the solutions. Let the external forcing be bounded by* $|f(t)| < f_{\max} < \infty$. *If there exists a constant* $r_1 > \frac{1+e}{1-e}f_{\max}$ *such that* $q_x(t) + |u_x(t)| > r_1$ *and* $q_y(t) + |u_y(t)| > r_1$ *for all* $t \geq t_0$, *then* $\mathrm{d}V \leq 0$ *and* $\lim_{t\to\infty} V(\boldsymbol{x}(t), \boldsymbol{y}(t)) = 0$ *for all initial conditions with* $d(\boldsymbol{x}^-(t_0), \boldsymbol{y}^-(t_0)) < \frac{e}{2}r_1$. *Therefore, local synchronization in the sense of Definition 2 is achieved.*

Proof. See Appendix. □

The region of attraction in Theorem 2 shrinks to zero for $e \to 0$. However, the condition $d(\boldsymbol{x}^-(t_0), \boldsymbol{y}^-(t_0)) < \frac{e}{2}r_1$ can be replaced by $\hat{d}(\boldsymbol{x}^-(t_0), \boldsymbol{y}^-(t_0)) < \frac{1}{2}r_1$, where $\hat{d}$ is the approximated distance function in (26)–(30), and Theorem 2 still holds.

In the following, we will design an interaction law for τ_x and τ_y such that *global* synchronization is achieved without a boundedness assumption of the external forcing f and also in the presence of accumulation points and grazing trajectories, that is, the conditions $q_x(t)+|u_x(t)| > r_1$ and $q_y(t)+|u_y(t)| > r_1$ for all $t \geq t_0$ are not required anymore. The proposed interaction law for the coupling forces τ_x and τ_y is given by (25), where

$$\kappa_x = \begin{cases} -f & \text{if } q_x > 0 \wedge q_y > 0 \wedge \min\{V_{01}, V_{10}\} < V_{00}, \\ -f & \text{if } q_x > 0 \wedge q_y = u_y = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{35}$$

$$\kappa_y = \begin{cases} -f & \text{if } q_x > 0 \wedge q_y > 0 \wedge \min\{V_{01}, V_{10}\} < V_{00}, \\ -f & \text{if } q_x = u_x = 0 \wedge q_y > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

Using the interaction law (35)–(36), the right-hand sides in (22) (without impacts) become discontinuous. Therefore, we will consider Filippov-type solutions of system (22)–(24) together with (35)–(36) [37].

We note that the coupling forces vanish if the solutions are synchronized. The proposed interaction law compensates the external forcing $f(t)$ whenever necessary such that the density $\dot{V}$ of the Lyapunov function (26)–(30) evaluated along solutions is non-positive. Using this interaction law, the global synchronization problem is solved as shown in the following theorem.

Theorem 3. *Let* $\boldsymbol{x}(t)$, $\boldsymbol{y}(t)$ *be the Filippov-type solutions of system* (22)–(24), *where the coupling forces* τ_x, τ_y *are given by the interaction law* (35)–(36) *and let* $V(\boldsymbol{x}(t), \boldsymbol{y}(t))$ *be the Lyapunov function defined by* (26)–(30) *evaluated along the solutions. Then* $\mathrm{d}V \leq 0$ *and* $\lim_{t\to\infty} V(\boldsymbol{x}(t), \boldsymbol{y}(t)) = 0$ *for all initial conditions* $\boldsymbol{x}^-(t_0)$, $\boldsymbol{y}^-(t_0) \in \mathcal{A}$. *Therefore, global synchronization is achieved in the sense of Definition 2.*

Proof. See Appendix. □

Corollary 1. *Let* $\boldsymbol{x}(t)$, $\boldsymbol{y}(t)$ *be the Filippov-type solutions of system* (22)–(24) *for the initial conditions* $\boldsymbol{x}^-(t_0)$, $\boldsymbol{y}^-(t_0) \in \mathcal{A}$ *using the interaction law*

$$\begin{aligned}\tau_x &= \kappa_x(\boldsymbol{x}, \boldsymbol{y}) + k_\mathrm{d}u_x + k_\mathrm{p}q_x, \\ \tau_y &= \kappa_y(\boldsymbol{x}, \boldsymbol{y}) + k_\mathrm{d}u_y + k_\mathrm{p}q_y\end{aligned} \tag{37}$$

with $k_\mathrm{p}, k_\mathrm{d} \geq 0$ *and* $\kappa_x(\boldsymbol{x}, \boldsymbol{y})$ *and* $\kappa_y(\boldsymbol{x}, \boldsymbol{y})$ *given by* (35)–(36). *Then, global synchronization is achieved in the sense of Definition 2.*

Proof. The differential gain k_d simply increases the damping constant c. The proportional gain k_p increases the stiffness and another rescaling of the states and time is necessary to obtain the normalized equations of motion (22). Theorem 3 is applicable to the rescaled systems, which implies global synchronization in the sense of Definition 2. □

The interaction law (37) extends the interaction law (35)–(36) with a PD-type coupling and generally increases the synchronization speed. The coupling forces do not vanish if the solutions are synchronized and the limit sets of the uncoupled systems are therefore not preserved in the coupled case. Furthermore, the stiffness and damping constants of the impact oscillators considered in this section (see (22)) are assumed to be strictly positive. This assumption can be dropped when the extended interaction law (37) is used with $k_\mathrm{p}, k_\mathrm{d} > 0$.

5. Illustrative examples

We illustrate the synchronization problem of a 1-DOF mechanical impact oscillator with several numerical examples. We consider system (22)–(24) with a damping constant $c = 0.01$ and a coefficient of restitution $e = 0.8$. The external forcing is chosen as $f(t) = 1+2\cos t+\cos 3t$. During the first part of the simulation, the two mechanical systems are uncoupled. The interaction law (35)–(36) is switched on at $t = 20$.

The solutions $\boldsymbol{x}(t) = (q_x(t); u_x(t))$ and $\boldsymbol{y}(t) = (q_y(t); u_y(t))$ for the initial conditions $\boldsymbol{x}(t_0) = (1; -0.2)$ and $\boldsymbol{y}(t_0) = (1.1; 0.1)$ are depicted in Fig. 6. No synchronization is achieved in the uncoupled case even though they are initialized close to each other. This does not contradict Theorem 2. Namely, the solutions diverge quickly at $t \approx 3$, where both solutions are close to the origin and the assumptions of Theorem 2 are violated. After the interaction law is switched on at $t = 20$, the distance between the solutions decreases and synchronization is achieved in accordance with Theorem 3.

Fig. 7 shows the Lyapunov function $V(\boldsymbol{x}(t), \boldsymbol{y}(t)) = \frac{1}{2}\hat{d}(\boldsymbol{x}, \boldsymbol{y})$ defined by (26)–(30) (solid black line). It is continuous in time except when both constraints are closed at the same time, that is, when one solution has an impact and the other is in persistent contact. The Lyapunov function is an approximation of $\frac{1}{2}(d^I(\boldsymbol{x}, \boldsymbol{y}))^2$ as discussed in Remark 5, and the quotient distance function d defined in (13) is identical to the distance function d^I in (14)–(18)

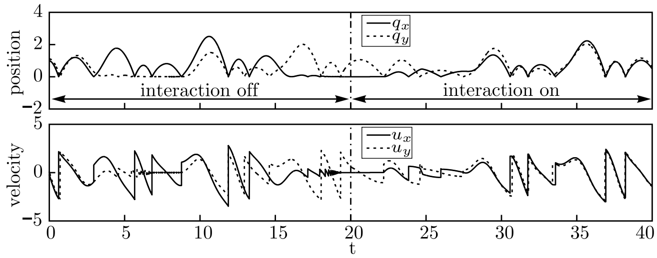


Fig. 6. Solutions $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ of the 1-DOF mechanical impact oscillators. The interaction law is switched on at $t = 20$.

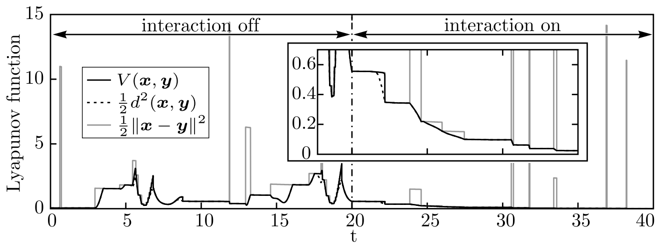


Fig. 7. The Lyapunov function $V(\boldsymbol{x}, \boldsymbol{y})$ (solid black) is non-increasing and tends to zero after the interaction law is switched on at $t = 20$ and it approximates $\frac{1}{2}d^2(\boldsymbol{x}, \boldsymbol{y})$ (dashed black). The (Euclidean distance) function $V_{00}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{y}\|^2$ (gray) shows the undesirable 'peaking behavior'.

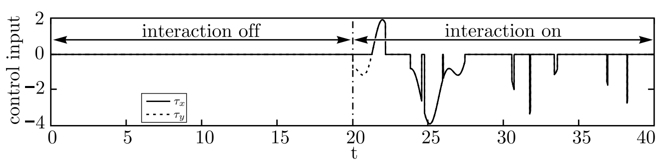


Fig. 8. Coupling forces τ_x and τ_y. The interaction law is activated at $t = 20$.

according to Theorem 1. The Lyapunov function is therefore an approximation of $\frac{1}{2}d^2(\boldsymbol{x}, \boldsymbol{y})$ (dashed black line), which is continuous in time. The (Euclidean distance) function $V_{00}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}\|\boldsymbol{x}(t) - \boldsymbol{y}(t)\|^2$ (gray line) shows the undesirable 'peaking behavior' of the Euclidean synchronization error. The Lyapunov function is initially very small and increases rapidly at $t \approx 3$, where both solutions are close to a grazing trajectory. After the interaction law is switched on at $t = 20$, the Lyapunov function is non-increasing and tends to zero as stated by Theorem 3 (see magnification in Fig. 7).

The coupling forces τ_x and τ_y may differ only if one of the contacts is persistent, which is the case right after the interaction law is switched on at $t = 20$, see Fig. 8. The coupling forces are discontinuous in time. Furthermore, the jump heights of the discontinuities are in the order of the external forcing and do not tend to zero while the Lyapunov function tends to zero. The time average of the coupling forces for a moving window with fixed width, however, tends to zero as the solutions synchronize. The influence of the coupling force does therefore tend to zero as well. Furthermore, all limit sets of the individual systems are also present in the coupled case because there are no coupling forces if the solutions are synchronized.

In consideration of the feasibility of the application of this interaction law in a digital control setting, the simulation is repeated in which the coupling forces, given by the interaction law (35)–(36), are filtered with a zero-order hold (ZOH) filter with a time interval of 0.2. Therefore, the coupling forces change no more than five times per unit time interval and they are held constant in between. The same parameters and initial conditions as in the

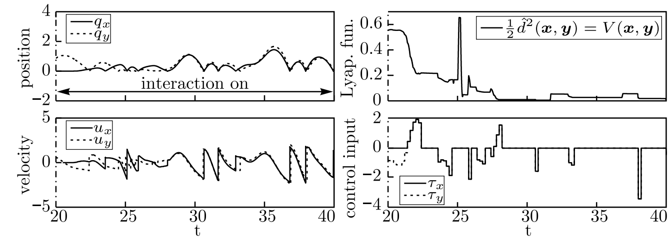


Fig. 9. Solutions $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ (left), Lyapunov function $V(\boldsymbol{x}, \boldsymbol{y})$ (right) and coupling forces τ_x and τ_y (right). The interaction law is switched on at $t = 20$ and the coupling forces are filtered by a zero-order hold filter with a time interval of 0.2.

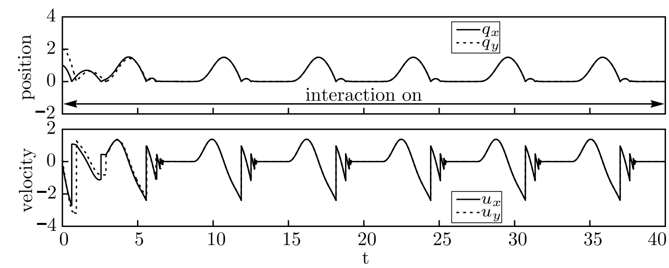


Fig. 10. The interaction law (35)–(36) does not prevent accumulation points, and finite time synchronization is achieved for the one-dimensional mechanical impact oscillator.

previous simulation are used and the results are depicted in Fig. 9. The solutions $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ approach each other and stay close to synchronization also with the ZOH filter as depicted to the left in Fig. 9. The Lyapunov function and the coupling force are shown to the right in Fig. 9. The effect of the ZOH filter is prominent in the time evolution of the coupling force. The Lyapunov function occasionally increases, especially at $t \approx 25$ where both solutions are close to the origin. A time interval of 0.2 for the ZOH filter is challenging in this example, but clearly shows the robustness of the interaction law (35)–(36).

Let us now illustrate that our results also apply in the presence of the accumulation of impact events. Fig. 10 shows the solutions for system (22)–(24) using the same damping constant and external forcing as before. The initial conditions are chosen as $\boldsymbol{x}(t_0) = (1; -0.2)$ and $\boldsymbol{y}(t_0) = (1.1; 2)$ and the coefficient of restitution is lowered to $e = 0.4$. The coupling forces are given by the interaction law (35)–(36) and the interaction law is switched on for the entire simulation. Both solutions have an accumulation point at $t \approx 7$ after which both solutions are in persistent contact for some time interval. Because the system is one-dimensional, the solutions are synchronized if they are both in persistent contact, that is, finite time synchronization is achieved. The coupling forces vanish if the solutions are synchronized, and the solutions remain synchronized in the absence of any disturbances.

6. Conclusions

In this paper, the synchronization problem for mechanical systems subjected to a single geometric unilateral constraint inducing impacts is investigated. In order to define and investigate the synchronization problem for nonsmooth systems with jumping state evolutions, it is necessary to use a distance function which is more sophisticated than the Euclidean distance function. A distance function suitable to compare discontinuous solutions is the one induced by the quotient metric, where the equivalence relation is the equivalence kernel of the impact map. The quotient distance function is continuous in time when evaluated along solutions. The resulting synchronization problem does consequently not suffer from the 'peaking phenomenon' like the Euclidean distance function. Therefore, it is suitable to define stability in the sense of Lyapunov and leads to an intuitive notion of synchrony. This quotient distance function can be simplified significantly for generally used impact laws due to their property of monotonicity and, equivalently, the non-expansivity of the corresponding impact maps.

The definition of the synchronization problem together with the distance function induced by the quotient metric is of generic value and can directly be extended to, for example, Lur'e-type systems and/or systems with multiple contacts. The presented simplification of the distance function is however not possible for multi-contact collisions for which the distance function generally comprises of the infimum over an unbounded number of intermediate points.

The synchronization problem for a 1-DOF mechanical system is investigated using Lyapunov stability analysis. The presented Lyapunov function is constructed using an approximation of the distance function. The same approach is applicable to construct a candidate Lyapunov function for mechanical systems with multiple degrees of freedom.

Local synchronization for a 1-DOF forced mechanical system is shown without any coupling forces if both solutions keep a minimal distance from the origin. This minimal distance ensures that any possible increase of the Lyapunov function due to the external forcing is compensated by the decrease due to the monotonicity of the impact law.

An interaction law is presented which achieves global synchronization also in the presence of grazing trajectories and Zeno behavior. The coupling forces compensate the external forcing whenever necessary, but they are chosen such that they do not dominate the overall dynamics. In particular, they vanish if the solutions are synchronized and all limit sets of the uncoupled systems are therefore preserved in the coupled case. In order to increase the synchronization speed, the interaction law can readily be extended by a PD-type coupling. This extension enlarges the applicability of the presented interaction law to systems with no viscous damping or vanishing stiffness such as the bouncing ball example. The coupling forces are discontinuous and do not tend to zero as the solutions approach each other. However, the time intervals with non-vanishing coupling forces tend to zero, that is, the time average for a moving window with fixed width tends to zero as well.

The feasibility in a digital control setting is shown with a numerical example where the coupling forces are filtered with a zero order hold filter for which the synchronization error still tends to and remains close to zero. This numerical example also shows the robustness of the presented interaction law.

Acknowledgments

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Appendix

Proof of Proposition 1 and Theorem 1. The infimum in the definition of the distance function d in (13) is attained if Assumption 1 is fulfilled. This result is shown in Theorem 1 and will be used in the proof of Proposition 1. The proof of Theorem 1 is therefore given before the proof of Proposition 1.

Proof of Theorem 1. Using the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_{\boldsymbol{A}}$, the distance function d in (13) is equivalent to

$$\bar{d}^A(\boldsymbol{x},\boldsymbol{y}) := \inf\left\{ \sum_{j=0}^{N} \left\|\boldsymbol{x}^j-\boldsymbol{y}^j\right\|_{\boldsymbol{A}} \,\middle|\, N\in\mathbb{N}_0, \boldsymbol{x}=\boldsymbol{x}^0, \boldsymbol{y}^j\sim\boldsymbol{x}^{j+1} \text{ for } 0\le j<N, \boldsymbol{y}^N=\boldsymbol{y}\right\}. \tag{A.1}$$

Note that, if $\boldsymbol{A}$ is the identity matrix $\boldsymbol{I}$, then d in (13) is identical to $\bar{d}^A(\boldsymbol{x},\boldsymbol{y})=\bar{d}^I(\boldsymbol{x},\boldsymbol{y})$. In the remaining proof, we will show that the distance function $\bar{d}^A$ in (A.1) is identical to the distance function d^A given by (14)–(18).

The intermediate points $(\boldsymbol{x}^{j+1};\boldsymbol{y}^j)$ in (A.1) are restricted to the synchronization set $\mathcal{S}$ and we will show that it suffices to consider intermediate points in $\mathcal{S}_{01}\cup\mathcal{S}_{10}$ only. First, we consider the two consecutive summands $\|\boldsymbol{x}^j-\boldsymbol{y}^j\|_{\boldsymbol{A}}+\|\boldsymbol{x}^{j+1}-\boldsymbol{y}^{j+1}\|_{\boldsymbol{A}}$ with an intermediate point $(\boldsymbol{x}^{j+1};\boldsymbol{y}^j)\in\mathcal{S}_{11}$. The sum is identical to $\|\boldsymbol{x}^j-\boldsymbol{y}^j\|_{\boldsymbol{A}}+\|\boldsymbol{z}-\boldsymbol{z}\|_{\boldsymbol{A}}+\|\boldsymbol{x}^{j+1}-\boldsymbol{y}^{j+1}\|_{\boldsymbol{A}}$ with two intermediate points $(\boldsymbol{z};\boldsymbol{y}^j)\in\mathcal{S}_{01}$, $(\boldsymbol{x}^{j+1};\boldsymbol{z})\in\mathcal{S}_{10}$ and $\boldsymbol{z}$ such that $\boldsymbol{z}=\bar{Z}(\boldsymbol{z})$. Therefore, every intermediate point in $\mathcal{S}_{11}$ can be replaced by two intermediate points in $\mathcal{S}_{01}\cup\mathcal{S}_{10}$. Secondly, we consider two consecutive summands $\|\boldsymbol{x}^j-\boldsymbol{y}^j\|_{\boldsymbol{A}}+\|\boldsymbol{x}^{j+1}-\boldsymbol{y}^{j+1}\|_{\boldsymbol{A}}$ with an intermediate point $(\boldsymbol{x}^{j+1};\boldsymbol{y}^j)\in\mathcal{S}_{00}$, which can be written as $\|\boldsymbol{x}^j-\boldsymbol{y}^j\|_{\boldsymbol{A}}+\|\boldsymbol{y}^j-\boldsymbol{y}^{j+1}\|_{\boldsymbol{A}}$. The triangle inequality implies that this sum is always larger than or equal to $\|\boldsymbol{x}^j-\boldsymbol{y}^{j+1}\|_{\boldsymbol{A}}$, which shows that it is not necessary to consider intermediate points in $\mathcal{S}_{00}$. Therefore, (A.1) can be simplified to

$$\bar{d}^A(\boldsymbol{x},\boldsymbol{y}) = \inf\left\{ \sum_{j=0}^{N} \left\|\boldsymbol{x}^j-\boldsymbol{y}^j\right\|_{\boldsymbol{A}} \,\middle|\, N\in\mathbb{N}_0, \boldsymbol{x}=\boldsymbol{x}^0, \begin{pmatrix}\boldsymbol{x}^{j+1}\\ \boldsymbol{y}^j\end{pmatrix}\in\mathcal{S}_{01}\cup\mathcal{S}_{10} \text{ for } 0\le j<N, \boldsymbol{y}^N=\boldsymbol{y}\right\}. \tag{A.2}$$

In the next step, we show that it is sufficient to consider at most two intermediate points in (A.2). To do so, we define $\bar{d}^A_{N<2}(\boldsymbol{x},\boldsymbol{y}) := \inf\left\{\sum_{j=0}^{N}\|\boldsymbol{x}^j-\boldsymbol{y}^j\|_{\boldsymbol{A}} \,\middle|\, N\in\{0,1\}, \boldsymbol{x}=\boldsymbol{x}^0, (\boldsymbol{x}^{j+1};\boldsymbol{y}^j)\in\mathcal{S}_{01}\cup\mathcal{S}_{10} \text{ for } 0\le j<N, \boldsymbol{y}^N=\boldsymbol{y}\right\}$ and derive a lower bound of

$$\bar{d}^A_{N\ge2}(\boldsymbol{x},\boldsymbol{y}) := \inf\left\{ \sum_{j=0}^{N} \left\|\boldsymbol{x}^j-\boldsymbol{y}^j\right\|_{\boldsymbol{A}} \,\middle|\, N\in\mathbb{N}_{\ge2}, \boldsymbol{x}=\boldsymbol{x}^0, \begin{pmatrix}\boldsymbol{x}^{j+1}\\ \boldsymbol{y}^j\end{pmatrix}\in\mathcal{S}_{01}\cup\mathcal{S}_{10} \text{ for } 0\le j<N-1, \boldsymbol{y}^N=\boldsymbol{y}\right\} \tag{A.3}$$

for the case where

$$\bar{d}^A_{N\ge2}(\boldsymbol{x},\boldsymbol{y}) < \bar{d}^A_{N<2}(\boldsymbol{x},\boldsymbol{y}). \tag{A.4}$$

Let us assume that (A.4) holds. We note that $(\boldsymbol{x}^{j+1};\boldsymbol{y}^j)\in\mathcal{S}_{01}\cup\mathcal{S}_{10}$, for $j=0,1,\ldots,N-1$, implies $\boldsymbol{x}^{j+1},\boldsymbol{y}^j\in\partial\mathcal{A}$ and the map $\bar{Z}$ restricted to $\partial\mathcal{A}$ is non-expansive in the metric $\boldsymbol{A}$ by Assumption 1. Therefore, $\|\boldsymbol{x}^j-\boldsymbol{y}^j\|_{\boldsymbol{A}}\ge\|\bar{Z}(\boldsymbol{x}^j)-\bar{Z}(\boldsymbol{y}^j)\|_{\boldsymbol{A}}$ for $1\le j<N-1$ and $\sum_{j=1}^{N-1}\|\boldsymbol{x}^j-\boldsymbol{y}^j\|_{\boldsymbol{A}}\ge\sum_{j=1}^{N-1}\|\bar{Z}(\boldsymbol{x}^j)-\bar{Z}(\boldsymbol{y}^j)\|_{\boldsymbol{A}}$ holds. The triangle inequality implies $\sum_{j=1}^{N-1}\|\bar{Z}(\boldsymbol{x}^j)-\bar{Z}(\boldsymbol{y}^j)\|_{\boldsymbol{A}}\ge\|\bar{Z}(\boldsymbol{x}^1)-\bar{Z}(\boldsymbol{y}^{N-1})\|_{\boldsymbol{A}}$. Hence, $\bar{d}^A_{N\ge2}(\boldsymbol{x},\boldsymbol{y})$ can be lower bounded by

$$\bar{d}^A_{N\ge2}(\boldsymbol{x},\boldsymbol{y}) \ge \inf\left\{ \left\|\boldsymbol{x}-\boldsymbol{y}^0\right\|_{\boldsymbol{A}} + \left\|\bar{Z}\left(\boldsymbol{x}^1\right)-\bar{Z}\left(\boldsymbol{y}^1\right)\right\|_{\boldsymbol{A}} + \left\|\boldsymbol{x}^2-\boldsymbol{y}\right\|_{\boldsymbol{A}} \,\middle|\, \begin{pmatrix}\boldsymbol{x}^1\\ \boldsymbol{y}^0\end{pmatrix}, \begin{pmatrix}\boldsymbol{x}^2\\ \boldsymbol{y}^1\end{pmatrix}\in\mathcal{S}_{01}\cup\mathcal{S}_{10}\right\}. \tag{A.5}$$

Under the assumption $(\boldsymbol{x}^1; \boldsymbol{y}^0) \in \mathcal{S}_{10}$, $\bar{Z}(\boldsymbol{x}^1) = \boldsymbol{y}^0$ together with the triangle inequality implies $\|\boldsymbol{x} - \boldsymbol{y}^0\|_{\boldsymbol{A}} + \|\bar{Z}(\boldsymbol{x}^1) - \bar{Z}(\boldsymbol{y}^1)\|_{\boldsymbol{A}} \geq \|\boldsymbol{x} - \bar{Z}(\boldsymbol{y}^1)\|_{\boldsymbol{A}}$, which violates (A.4). Therefore, if (A.4) holds, then $(\boldsymbol{x}^1; \boldsymbol{y}^0)$ in (A.3) satisfies $(\boldsymbol{x}^1; \boldsymbol{y}^0) \in \mathcal{S}_{01}$ and, thus, $\bar{Z}(\boldsymbol{x}^1) = \boldsymbol{x}^1$. Analogously, (A.4) implies $\bar{Z}(\boldsymbol{y}^1) = \boldsymbol{y}^1$. Hence, assuming (A.4), the lower bound (A.5) yields

$$\bar{d}^A_{N\geq 2}(\boldsymbol{x}, \boldsymbol{y}) \geq \inf\left\{ \left\|\boldsymbol{x} - \boldsymbol{y}^0\right\|_{\boldsymbol{A}} + \left\|\boldsymbol{x}^1 - \boldsymbol{y}^1\right\|_{\boldsymbol{A}} + \left\|\boldsymbol{x}^2 - \boldsymbol{y}\right\|_{\boldsymbol{A}} \;\middle|\; \begin{pmatrix}\boldsymbol{x}^1\\ \boldsymbol{y}^0\end{pmatrix} \in \mathcal{S}_{01},\ \begin{pmatrix}\boldsymbol{x}^2\\ \boldsymbol{y}^1\end{pmatrix} \in \mathcal{S}_{10} \right\}. \tag{A.6}$$

The lower bound (A.6) will be attained as such pairs $(\boldsymbol{x}^1; \boldsymbol{y}^0)$, $(\boldsymbol{x}^2; \boldsymbol{y}^1)$ are allowed in (A.6), that is, it is not necessary to consider more than two intermediate points if (A.4) holds. Also considering the case where (A.4) does not hold, we distinguish between the following four cases for the distance function $\bar{d}^A$:

$$\bar{d}^A(\boldsymbol{x}, \boldsymbol{y}) = \inf\Big\{ \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{A}},\ \left\|\boldsymbol{x} - \boldsymbol{y}^1\right\|_{\boldsymbol{A}} + \left\|\boldsymbol{x}^2 - \boldsymbol{y}\right\|_{\boldsymbol{A}},\ \left\|\boldsymbol{x} - \boldsymbol{y}^0\right\|_{\boldsymbol{A}} + \left\|\boldsymbol{x}^1 - \boldsymbol{y}\right\|_{\boldsymbol{A}},\ \left\|\boldsymbol{x} - \boldsymbol{y}^0\right\|_{\boldsymbol{A}} + \left\|\boldsymbol{x}^1 - \boldsymbol{y}^1\right\|_{\boldsymbol{A}} + \left\|\boldsymbol{x}^2 - \boldsymbol{y}\right\|_{\boldsymbol{A}} \;\Big|\; \begin{pmatrix}\boldsymbol{x}^1\\ \boldsymbol{y}^0\end{pmatrix} \in \mathcal{S}_{01},\ \begin{pmatrix}\boldsymbol{x}^2\\ \boldsymbol{y}^1\end{pmatrix} \in \mathcal{S}_{10} \Big\}. \tag{A.7}$$

It remains to be proven that the infimum is a minimum. To do so, we first note that $\mathcal{S}_{10} \cap \{(\boldsymbol{x}; \boldsymbol{y}) \mid \|\boldsymbol{x}\|_{\boldsymbol{A}} + \|\boldsymbol{y}\|_{\boldsymbol{A}} \geq \beta\}$ is closed for all $\beta > 0$. If

$$\inf\left\{ \left\|\boldsymbol{x} - \boldsymbol{y}^1\right\|_{\boldsymbol{A}} + \left\|\boldsymbol{x}^2 - \boldsymbol{y}\right\|_{\boldsymbol{A}} \;\middle|\; \begin{pmatrix}\boldsymbol{x}^2\\ \boldsymbol{y}^1\end{pmatrix} \in \mathcal{S}_{10} \right\} < \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{A}}, \tag{A.8}$$

then there exists a sequence $\{(\boldsymbol{x}^2_j; \boldsymbol{y}^1_j)\}_{j\in\mathbb{N}}$ with $(\boldsymbol{x}^2_j; \boldsymbol{y}^1_j) \in \mathcal{S}_{10}\ \forall j \in \mathbb{N}$ such that $\lim_{j\to\infty} \|\boldsymbol{x} - \boldsymbol{y}^1_j\|_{\boldsymbol{A}} + \|\boldsymbol{x}^2_j - \boldsymbol{y}\|_{\boldsymbol{A}} < \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{A}}$, which, using the triangle inequality $\|\boldsymbol{x}\|_{\boldsymbol{A}} + \|\boldsymbol{y}\|_{\boldsymbol{A}} \geq \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{A}}$, implies that $\|\boldsymbol{x}^2_j\|_{\boldsymbol{A}} + \|\boldsymbol{y}^1_j\|_{\boldsymbol{A}} > \beta$ for some $\beta > 0$ and any j sufficiently large. Furthermore, (A.8) yields $\|\boldsymbol{x}^2_j\|_{\boldsymbol{A}} + \|\boldsymbol{y}^1_j\|_{\boldsymbol{A}} \leq \|\boldsymbol{x} - \boldsymbol{y}^1_j\|_{\boldsymbol{A}} + \|\boldsymbol{x}\|_{\boldsymbol{A}} + \|\boldsymbol{x}^2_j - \boldsymbol{y}\|_{\boldsymbol{A}} + \|\boldsymbol{y}\|_{\boldsymbol{A}} \leq \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{A}} + \|\boldsymbol{x}\|_{\boldsymbol{A}} + \|\boldsymbol{y}\|_{\boldsymbol{A}} \leq 2(\|\boldsymbol{x}\|_{\boldsymbol{A}} + \|\boldsymbol{y}\|_{\boldsymbol{A}})$. Hence, for any sufficiently large j, it holds that $(\boldsymbol{x}^2_j; \boldsymbol{y}^1_j)$ is contained in the set $\mathcal{S}^c_{10} := \mathcal{S}_{10} \cap \{(\boldsymbol{x}^2_j; \boldsymbol{y}^1_j) \mid \beta \leq \|\boldsymbol{x}^2_j\|_{\boldsymbol{A}} + \|\boldsymbol{y}^1_j\|_{\boldsymbol{A}} \leq 2(\|\boldsymbol{x}\|_{\boldsymbol{A}} + \|\boldsymbol{y}\|_{\boldsymbol{A}})\}$. Therefore, $\lim_{j\to\infty}(\boldsymbol{x}^2_j; \boldsymbol{y}^1_j)$ lies in the compact set $\mathcal{S}^c_{10}$ and the infimum in (A.8) is a minimum. Analogously, we find that the infimum is attained if

$$\inf\left\{ \left\|\boldsymbol{x} - \boldsymbol{y}^0\right\|_{\boldsymbol{A}} + \left\|\boldsymbol{x}^1 - \boldsymbol{y}\right\|_{\boldsymbol{A}} \;\middle|\; \begin{pmatrix}\boldsymbol{x}^1\\ \boldsymbol{y}^0\end{pmatrix} \in \mathcal{S}_{01} \right\} < \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{A}}. \tag{A.9}$$

Lastly, we consider the case

$$\inf\left\{ \left\|\boldsymbol{x} - \boldsymbol{y}^0\right\|_{\boldsymbol{A}} + \left\|\boldsymbol{x}^1 - \boldsymbol{y}^1\right\|_{\boldsymbol{A}} + \left\|\boldsymbol{x}^2 - \boldsymbol{y}\right\|_{\boldsymbol{A}} \;\middle|\; \begin{pmatrix}\boldsymbol{x}^1\\ \boldsymbol{y}^0\end{pmatrix} \in \mathcal{S}_{01},\ \begin{pmatrix}\boldsymbol{x}^2\\ \boldsymbol{y}^1\end{pmatrix} \in \mathcal{S}_{10} \right\} < \inf\Big\{ \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{A}},\ \left\|\boldsymbol{x} - \boldsymbol{y}^1\right\|_{\boldsymbol{A}} + \left\|\boldsymbol{x}^2 - \boldsymbol{y}\right\|_{\boldsymbol{A}},\ \left\|\boldsymbol{x} - \boldsymbol{y}^0\right\|_{\boldsymbol{A}} + \left\|\boldsymbol{x}^1 - \boldsymbol{y}\right\|_{\boldsymbol{A}} \;\Big|\; \begin{pmatrix}\boldsymbol{x}^1\\ \boldsymbol{y}^0\end{pmatrix} \in \mathcal{S}_{01},\ \begin{pmatrix}\boldsymbol{x}^2\\ \boldsymbol{y}^1\end{pmatrix} \in \mathcal{S}_{10} \Big\} \tag{A.10}$$

and find a sequence of points $\{p_j\}_{j\in\mathbb{N}}$, with $p_j = (\boldsymbol{x}^1_j; \boldsymbol{y}^0_j; \boldsymbol{y}^1_j; \boldsymbol{x}^2_j) \in \mathcal{S}_{10} \times \mathcal{S}_{10}$, such that the limit $\lim_{j\to\infty} \|\boldsymbol{x} - \boldsymbol{y}^0_j\|_{\boldsymbol{A}} + \|\boldsymbol{x}^1_j - \boldsymbol{y}^1_j\|_{\boldsymbol{A}} + \|\boldsymbol{x}^2_j - \boldsymbol{y}\|_{\boldsymbol{A}}$ is smaller than the right-hand side of (A.10). Again using the triangular inequality, we observe that $\|\lim_{j\to\infty}(\boldsymbol{x}^2_j; \boldsymbol{y}^1_j)\| > \beta$ and $\|\lim_{j\to\infty}(\boldsymbol{x}^1_j; \boldsymbol{y}^0_j)\| > \beta$ for some $\beta > 0$. Hence, for sufficiently large j, the points p_j satisfy $p_j \in \mathcal{S}^c_{10} \times \mathcal{S}^c_{10}$, which is a compact set. Consequently, the infimum on the left-hand of the inequality (A.10) is attained.

The infimum is attained for all the possible cases $\bar{d}^A(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|$, (A.8)–(A.10). Therefore, we can replace the infimum in (A.7) by a minimum, which proves $\bar{d}^A(\boldsymbol{x}, \boldsymbol{y}) = d^A(\boldsymbol{x}, \boldsymbol{y})$ with d^A given by (14)–(18). □

Proof of Proposition 1. Conditions (ii) and (iii) are fulfilled because the quotient distance function $d(\boldsymbol{x}, \boldsymbol{y})$ is symmetric and subadditive by construction.

The implication '⇐' of condition (i) follows directly from (13) by choosing $N = 1$, $\boldsymbol{y}^0 = \boldsymbol{y}$ and $\boldsymbol{x}^1 = \boldsymbol{x}$. The implication '⇒' of condition (i) uses a result shown in the proof of Theorem 1, that is, the infimum in (13) is attained if Assumption 1 is fulfilled. Therefore, a vanishing distance function implies that there exists a set of points $\boldsymbol{y}^j \sim \boldsymbol{x}^{j+1}$ such that each summand $\|\boldsymbol{x}^j - \boldsymbol{y}^j\|$ in (13) vanishes. The transitivity of the equivalence relation concludes the proof. □

Proposition 2. *Let Assumption 1 be fulfilled. Let there exists a constant $\alpha \in (0\ 1]$ such that $\frac{1}{\alpha}\bar{Z}$ restricted to $\partial\mathcal{A}^-$ is non-contractive in the metric $\boldsymbol{A}$, i.e., $\forall (\boldsymbol{q}_1; \boldsymbol{u}^-_1), (\boldsymbol{q}_2; \boldsymbol{u}^-_2) \in \partial\mathcal{A}^-$*

$$\left\|\bar{Z}\begin{pmatrix}\boldsymbol{q}_1\\ \boldsymbol{u}^-_1\end{pmatrix} - \bar{Z}\begin{pmatrix}\boldsymbol{q}_2\\ \boldsymbol{u}^-_2\end{pmatrix}\right\|_{\boldsymbol{A}} \geq \alpha \left\|\begin{pmatrix}\boldsymbol{q}_1\\ \boldsymbol{u}^-_1\end{pmatrix} - \begin{pmatrix}\boldsymbol{q}_2\\ \boldsymbol{u}^-_2\end{pmatrix}\right\|_{\boldsymbol{A}}. \tag{A.11}$$

Then, the quotient distance function $d(\boldsymbol{x}, \boldsymbol{y})$ in (13) and the distance function $\tilde{d}(\boldsymbol{x}, \boldsymbol{y})$ in (19) are equivalent.

Proof. The quotient distance function d in (13) is equivalent to the distance function d^A in (14)–(18) as shown in Theorem 1. Furthermore, the positive definiteness of $\boldsymbol{A}$ implies that the distance function $\tilde{d}$ defined in (19) is equivalent to

$$\tilde{d}^A(\boldsymbol{x}, \boldsymbol{y}) = \min_{\boldsymbol{z}\in\mathcal{S}} \left\|\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix} - \boldsymbol{z}\right\|_{\tilde{\boldsymbol{A}}}, \quad \text{where } \tilde{\boldsymbol{A}} = \begin{pmatrix}\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A}\end{pmatrix}. \tag{A.12}$$

Hence, we will prove Proposition 2 by showing the equivalence of d^A and $\tilde{d}^A$. The synchronization set $\mathcal{S}$ in the definition (A.12) can be divided into two parts as

$$\tilde{d}^A(\boldsymbol{x}, \boldsymbol{y}) = \min\left\{\tilde{d}^A_{02}(\boldsymbol{x}, \boldsymbol{y}), \tilde{d}^A_1(\boldsymbol{x}, \boldsymbol{y})\right\},$$

where

$$\tilde{d}^A_{02}(\boldsymbol{x}, \boldsymbol{y}) = \min_{\boldsymbol{z}\in\mathcal{S}_{00}\cup\mathcal{S}_{11}} \left\|\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix} - \boldsymbol{z}\right\|_{\tilde{\boldsymbol{A}}}, \tag{A.13}$$

$$\tilde{d}^A_1(\boldsymbol{x}, \boldsymbol{y}) = \inf_{\boldsymbol{z}\in\mathcal{S}_{01}\cup\mathcal{S}_{10}} \left\|\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix} - \boldsymbol{z}\right\|_{\tilde{\boldsymbol{A}}}. \tag{A.14}$$

Assumption 1 implies that the impact map $\bar{Z}$ restricted to the domain $\mathcal{A}\backslash\partial\mathcal{A}^-$ is the identity transformation. Furthermore, the non-expansivity of $\bar{Z}$ together with the non-contractivity of $\frac{1}{\alpha}\bar{Z}$ both in the metric $\boldsymbol{A}$ and restricted to the domain $\partial\mathcal{A}^-$ implies that $\bar{Z}$ restricted to $\partial\mathcal{A}^-$ is a bijection. Therefore, the set $\mathcal{S}_{00} \cup \mathcal{S}_{11}$ is closed from which follows that the minimum (A.13) exists and it is obtained as

$$\tilde{d}^A_{02}(\boldsymbol{x}, \boldsymbol{y}) = \min_{\boldsymbol{z}^*\in\mathcal{A}} \left\|\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix} - \begin{pmatrix}\boldsymbol{z}^*\\ \boldsymbol{z}^*\end{pmatrix}\right\|_{\tilde{\boldsymbol{A}}} = \frac{1}{\sqrt{2}} d^A_{00}(\boldsymbol{x}, \boldsymbol{y}), \tag{A.15}$$

where d^A_{00} is given by (15). The infimum (A.14) can be written as $\tilde{d}^A_1(\boldsymbol{x}, \boldsymbol{y}) = \inf\left\{\left\|(\|\boldsymbol{x} - \boldsymbol{y}^0\|_{\boldsymbol{A}}; \|\boldsymbol{x}^1 - \boldsymbol{y}\|_{\boldsymbol{A}})\right\| \;\middle|\; (\boldsymbol{x}^1; \boldsymbol{y}^0) \in \mathcal{S}_{01} \cup \mathcal{S}_{10}\right\}$. Therefore, (A.14) can be bounded by (16) and (17) using the equivalence of the Euclidean norm and the 1-norm as

$$\frac{1}{\sqrt{2}} \min\{d^A_{01}, d^A_{10}\} \leq \tilde{d}^A_1 \leq \min\{d^A_{01}, d^A_{10}\}. \tag{A.16}$$

The distance function d^A_{11} can be bounded from below by a multiple of d^A_{00} by deriving a lower bound for the middle summand $\|\boldsymbol{x}^1 - \boldsymbol{y}^1\|_{\boldsymbol{A}}$ in (18) with $(\boldsymbol{x}^1; \boldsymbol{y}^0) \in \mathcal{S}_{01}$ and $(\boldsymbol{x}^2; \boldsymbol{y}^1) \in \mathcal{S}_{10}$ such that

$\boldsymbol{x}^1 = \bar{Z}(\boldsymbol{y}^0)$ and $\boldsymbol{y}^1 = \bar{Z}(\boldsymbol{x}^2)$. Exploiting the non-contractivity of $\frac{1}{\alpha}\bar{Z}$, we get $\|\boldsymbol{x}^1 - \boldsymbol{y}^1\|_{\boldsymbol{A}} = \|\bar{Z}(\boldsymbol{y}^0) - \bar{Z}(\boldsymbol{x}^2)\|_{\boldsymbol{A}} \geq \alpha\|\boldsymbol{y}^0 - \boldsymbol{x}^2\|_{\boldsymbol{A}}$. Consequently, $d_{11}^A(\boldsymbol{x},\boldsymbol{y}) \geq \inf\{\|\boldsymbol{x}-\boldsymbol{y}^0\|_{\boldsymbol{A}} + \alpha\|\boldsymbol{y}^0-\boldsymbol{x}^2\|_{\boldsymbol{A}} + \|\boldsymbol{x}^2 - \boldsymbol{y}\|_{\boldsymbol{A}} \,|\, \boldsymbol{y}^0 = \bar{Z}(\boldsymbol{y}^0) \wedge \boldsymbol{x}^2 = \bar{Z}(\boldsymbol{x}^2)\}$. Using $\alpha \in (0\ 1]$ and the triangle inequality, we obtain

$$d_{11}^A(\boldsymbol{x},\boldsymbol{y}) \geq \alpha \inf\left\{\|\boldsymbol{x}-\boldsymbol{y}^0\|_{\boldsymbol{A}} + \|\boldsymbol{y}^0-\boldsymbol{x}^2\|_{\boldsymbol{A}} + \|\boldsymbol{x}^2-\boldsymbol{y}\|_{\boldsymbol{A}} \,\middle|\, \boldsymbol{y}^0 = \bar{Z}\left(\boldsymbol{y}^0\right) \wedge \boldsymbol{x}^2 = \bar{Z}\left(\boldsymbol{x}^2\right)\right\} \geq \alpha d_{00}^A(\boldsymbol{x},\boldsymbol{y}). \tag{A.17}$$

Eq. (A.15) implies $\frac{1}{\sqrt{2}}\min\{d_{00}^A, d_{11}^A\} \leq \frac{1}{\sqrt{2}}d_{00}^A = \tilde{d}_{02}^A$. Substituting (A.17) in (A.15), we find $\tilde{d}_{02}^A = \frac{1}{\sqrt{2}}d_{00}^A \leq \frac{1}{\sqrt{2}}\frac{1}{\alpha}d_{11}^A$ and, together with $\alpha \in (0\ 1]$, we obtain $\tilde{d}_{02}^A \leq \frac{1}{\alpha}\min\{d_{00}^A, d_{11}^A\}$. The inequality $\frac{1}{\sqrt{2}}\min\{d_{00}^A, d_{11}^A\} \leq \tilde{d}_{02}^A \leq \frac{1}{\alpha}\min\{d_{00}^A, d_{11}^A\}$ together with (A.16) yields $\frac{1}{\sqrt{2}}d^A(\boldsymbol{x},\boldsymbol{y}) \leq \tilde{d}^A(\boldsymbol{x},\boldsymbol{y}) \leq \frac{1}{\alpha}d^A(\boldsymbol{x},\boldsymbol{y})$. Therefore, the quotient distance function d defined in (13) and the distance function $\tilde{d}$ defined in (19) are equivalent. □

Lemma 1. *Let the impact map $\bar{Z}$ be defined by (5) with a coefficient of restitution $e \in [0,\ 1]$. If the condition number* $\mathrm{cond}(\boldsymbol{M})$ *of the mass matrix fulfills $1 > (1-e^2)\,\mathrm{cond}(\boldsymbol{M})$, then there exists a constant $\alpha \in (0,\ 1]$ such that (A.11) holds, that is, $\frac{1}{\alpha}\bar{Z}$ restricted to $\partial\mathcal{A}^-$ is non-contractive in the metric $\boldsymbol{A} = \mathrm{diag}(\boldsymbol{K},\boldsymbol{M})$, where $\boldsymbol{K} \in \mathbb{R}^{n\times n}$ is some symmetric positive definite matrix. Furthermore, if $e = 1$, then (A.11) holds with equality and $\alpha = 1$.*

Proof. Consider the domain of the impact map $\bar{Z}$ in (5) restricted to $(\boldsymbol{q};\boldsymbol{u}^-) \in \partial\mathcal{A}^-$, i.e., $g(\boldsymbol{q}) = 0$ and $\boldsymbol{u}^- \notin \mathcal{T}_\mathcal{C}(\boldsymbol{q})$. The proximal point function in (5) simplifies to $\mathrm{prox}^{\boldsymbol{M}}_{\mathcal{T}_\mathcal{C}(\boldsymbol{q})}(\boldsymbol{u}^-) = (\boldsymbol{I} - \boldsymbol{M}^{-1}\frac{\boldsymbol{w}\boldsymbol{w}^\mathsf{T}}{G})\boldsymbol{u}^-$, where $\boldsymbol{I}$ is the identity matrix and $G = \boldsymbol{w}^\mathsf{T}\boldsymbol{M}^{-1}\boldsymbol{w}$ is the scalar Delassus-operator. Substituting into the impact map (5) yields $Z_q(\boldsymbol{u}^-) = (\boldsymbol{I} - (1+e)\boldsymbol{M}^{-1}\frac{\boldsymbol{w}\boldsymbol{w}^\mathsf{T}}{G})\boldsymbol{u}^-$. We find $\|\boldsymbol{I} - (1+e)\boldsymbol{M}^{-1}\frac{\boldsymbol{w}\boldsymbol{w}^\mathsf{T}}{G}\|^2_{\boldsymbol{M}} = \boldsymbol{M} - (1-e^2)\frac{\boldsymbol{w}\boldsymbol{w}^\mathsf{T}}{G}$, where $\frac{\boldsymbol{w}^\mathsf{T}\boldsymbol{M}^{-1}\boldsymbol{w}}{G} = 1$ has been used. Hence, for a given α and $(\boldsymbol{q}_1;\boldsymbol{u}_1^-), (\boldsymbol{q}_2;\boldsymbol{u}_2^-) \in \partial\mathcal{A}^-$,

$$\begin{aligned}
&\left\|\bar{Z}\begin{pmatrix}\boldsymbol{q}_1\\ \boldsymbol{u}_1^-\end{pmatrix} - \bar{Z}\begin{pmatrix}\boldsymbol{q}_2\\ \boldsymbol{u}_2^-\end{pmatrix}\right\|^2_{\boldsymbol{A}} - \alpha\left\|\begin{pmatrix}\boldsymbol{q}_1\\ \boldsymbol{u}_1^-\end{pmatrix} - \begin{pmatrix}\boldsymbol{q}_2\\ \boldsymbol{u}_2^-\end{pmatrix}\right\|^2_{\boldsymbol{A}}\\
&= \|Z_q(\boldsymbol{u}_1^-) - Z_q(\boldsymbol{u}_2^-)\|^2_{\boldsymbol{M}} - \alpha^2\|\boldsymbol{u}_1^- - \boldsymbol{u}_2^-\|^2_{\boldsymbol{M}}\\
&+ (1-\alpha)\|\boldsymbol{q}_1 - \boldsymbol{q}_2\|^2_{\boldsymbol{K}}\\
&= \left(\boldsymbol{u}_1^- - \boldsymbol{u}_2^-\right)^\mathsf{T}\boldsymbol{P}\left(\boldsymbol{u}_1^- - \boldsymbol{u}_2^-\right) + (1-\alpha)\|\boldsymbol{q}_1 - \boldsymbol{q}_2\|^2_{\boldsymbol{K}}
\end{aligned}$$

with $\boldsymbol{P} := ((1-\alpha^2)\boldsymbol{M} - (1-e^2)\frac{\boldsymbol{w}\boldsymbol{w}^\mathsf{T}}{G})$. Therefore, the inequality (A.11) is fulfilled if the matrix $\boldsymbol{P}$ is positive semi-definite for some $\alpha \in (0,\ 1]$. For two matrices $\boldsymbol{P},\boldsymbol{Q} \in \mathbb{R}^{n\times n}$, let $\boldsymbol{P} \succeq \boldsymbol{Q}$ denote that $\boldsymbol{P}-\boldsymbol{Q}$ is positive semi-definite. We note that $\boldsymbol{M} \succeq \lambda_{\min}(\boldsymbol{M})\boldsymbol{I}$, where $\lambda_{\min}(\boldsymbol{M})$ denotes the minimal eigenvalue of $\boldsymbol{M}$. Furthermore, since $\frac{\boldsymbol{w}\boldsymbol{w}^\mathsf{T}}{\boldsymbol{w}^\mathsf{T}\boldsymbol{w}}$ is a projection matrix with $\frac{\boldsymbol{w}\boldsymbol{w}^\mathsf{T}}{\boldsymbol{w}^\mathsf{T}\boldsymbol{w}} \preceq \boldsymbol{I}$, we obtain $\frac{\boldsymbol{w}\boldsymbol{w}^\mathsf{T}}{G} \preceq \lambda_{\max}(\boldsymbol{M})\frac{\boldsymbol{w}\boldsymbol{w}^\mathsf{T}}{\boldsymbol{w}^\mathsf{T}\boldsymbol{w}} \preceq \lambda_{\max}(\boldsymbol{M})\boldsymbol{I}$, where $\lambda_{\max}(\boldsymbol{M})$ denotes the maximal eigenvalue of $\boldsymbol{M}$.

Let $\alpha = \sqrt{1-(1-e^2)\,\mathrm{cond}(\boldsymbol{M})}$ and observe $\alpha \in (0,\ 1]$ follows from the assumption in the lemma. We obtain $\boldsymbol{P} = (1-\alpha^2)\boldsymbol{M} - (1-e^2)\frac{\boldsymbol{w}\boldsymbol{w}^\mathsf{T}}{G} \succeq ((1-\alpha^2)\lambda_{\min}(\boldsymbol{M}) - (1-e^2)\lambda_{\max}(\boldsymbol{M}))\boldsymbol{I} = \boldsymbol{0}$. Finally, $e = 1$ yields $\alpha = 1$ and (A.11) is fulfilled with equality. □

Proof of Lemma 2. The quotient distance function d is identical to d^l in (14)–(18) according to Theorem 1. Thus, we show the relation between d_{01}^l and $\hat{d}_{01}$. The definition of d_{01}^l (16) is rewritten explicitly using (9) and (24) as

$$d_{01}^l(\boldsymbol{x},\boldsymbol{y}) = \min_{z\geq 0}\left\{\left\|\begin{pmatrix}q_x\\ u_x\end{pmatrix} - \begin{pmatrix}0\\ ez\end{pmatrix}\right\| + \left\|\begin{pmatrix}0\\ -z\end{pmatrix} - \begin{pmatrix}q_y\\ u_y\end{pmatrix}\right\|\right\}.$$

Because the explicit solution to this minimization problem is cumbersome and thus not suited for differentiation necessary in the Lyapunov analysis, an approximation is used. The approximated intermediate point $z_{01} = \max\{\frac{u_x q_y - u_y q_x}{q_x + e q_y}, 0\}$ used in the approximation $\hat{d}_{01}$ of d_{01}^l is the solution to the slightly altered minimization problem $z_{01} = \mathrm{argmin}_{z\geq 0}\{\|(q_x;u_x) - (0;ez)\| + e\|(0;-z) - (q_y;u_y)\|\}$. As we are using a non-optimal minimizer, we directly obtain an upper bound $d_{01}^l \leq \hat{d}_{01}$. A lower bound for d_{01} is obtained by

$$\begin{aligned}
d_{01}^l &\geq \min_{z\geq 0}\left\{\left\|\begin{pmatrix}q_x\\ u_x\end{pmatrix} - \begin{pmatrix}0\\ ez\end{pmatrix}\right\| + e\left\|\begin{pmatrix}0\\ -z\end{pmatrix} - \begin{pmatrix}q_y\\ u_y\end{pmatrix}\right\|\right\}\\
&= \left(\left\|\begin{pmatrix}q_x\\ u_x\end{pmatrix} - \begin{pmatrix}0\\ ez\end{pmatrix}\right\| + e\left\|\begin{pmatrix}0\\ -z\end{pmatrix} - \begin{pmatrix}q_y\\ u_y\end{pmatrix}\right\|\right)\Bigg|_{z=z_{01}}\\
&\geq e\left(\left\|\begin{pmatrix}q_x\\ u_x\end{pmatrix} - \begin{pmatrix}0\\ ez\end{pmatrix}\right\| + \left\|\begin{pmatrix}0\\ -z\end{pmatrix} - \begin{pmatrix}q_y\\ u_y\end{pmatrix}\right\|\right)\Bigg|_{z=z_{01}}\\
&= e\hat{d}_{01},
\end{aligned}$$

where $e \in (0,\ 1]$ has been used. Analogously, an upper and lower bound for $d_{10}^l(\boldsymbol{x},\boldsymbol{y})$ is obtained by $\hat{d}_{10}(\boldsymbol{x},\boldsymbol{y}) \geq d_{10}^l(\boldsymbol{x},\boldsymbol{y}) \geq e\hat{d}_{10}(\boldsymbol{x},\boldsymbol{y})$.

We have already noted that $d_{00}^l = \hat{d}_{00}$. Furthermore, Eq. (A.17) in the proof of Proposition 2 (the necessary conditions for Proposition 2 are fulfilled for the impact map (24) with $e \in (0,\ 1]$ as shown in Lemma 1 given in Appendix) implies $d_{11}^l \geq \alpha d_{00}^l$ with $\alpha = e$ and, thus, we obtain $d^l = \min\{d_{00}^l, d_{01}^l, d_{10}^l, d_{11}^l\} \leq \min\{d_{00}^l, d_{01}^l, d_{10}^l\} \leq \min\{\hat{d}_{00}, \hat{d}_{01}, \hat{d}_{10}\} = \hat{d}$ as well as $\hat{d} = \min\{\hat{d}_{00}, \hat{d}_{01}, \hat{d}_{10}\} \leq \min\{d_{00}^l, \frac{1}{e}d_{01}^l, \frac{1}{e}d_{10}^l\} \leq \min\{d_{00}^l, \frac{1}{e}d_{01}^l, \frac{1}{e}d_{10}^l, \frac{1}{\alpha}d_{11}^l\} \leq \frac{1}{e}d^l$. These inequalities together with $V(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}\hat{d}^2(\boldsymbol{x},\boldsymbol{y})$ yield $\frac{1}{2}(d^l)^2 \leq V \leq \frac{1}{2}(\frac{d^l}{e})^2$, which concludes the proof. □

Proof of Lemma 3. At a given point in time, we can distinguish between four cases depending on whether the right- and left-limits of $\boldsymbol{x}$ and $\boldsymbol{y}$ agree or not. For the case $\boldsymbol{x}^- = \boldsymbol{x}^+, \boldsymbol{y}^- = \boldsymbol{y}^+$, the function V is continuous. If the state $\boldsymbol{y}$ jumps and $\boldsymbol{x}$ does not, then $u_x^- q_y - u_y^- q_x = -u_y^- q_x \geq 0$ and $u_x^+ q_y - u_y^+ q_x = -u_y^+ q_x \leq 0$ together with (31) imply $V^- = \min\{V_{00}(\boldsymbol{x},\boldsymbol{y}^-), V_{01}(\boldsymbol{x},\boldsymbol{y}^-)\}$ and $V^+ = \min\{V_{00}(\boldsymbol{x},\boldsymbol{y}^+), V_{10}(\boldsymbol{x},\boldsymbol{y}^-)\}$. Considering $u_x > 0$, we obtain $V^- = V_{01}(\boldsymbol{x},\boldsymbol{y}^-)$, which implies together with the impact law (24) that $V^+ = V_{00}(\boldsymbol{x},\boldsymbol{y}^+)$. Furthermore, the impact law yields $V_{00}(\boldsymbol{x},\boldsymbol{y}^+) = V_{01}(\boldsymbol{x},\boldsymbol{y}^-)$. Analogously, $u_x < 0$ yields $V^- = V_{00}(\boldsymbol{x},\boldsymbol{y}^-) = V_{10}(\boldsymbol{x},\boldsymbol{y}^+) = V^+$. Since the candidate Lyapunov function is symmetric, we can immediately conclude that $V^+ - V^- = 0$ also for the case where the state $\boldsymbol{x}$ jumps and $\boldsymbol{y}$ does not. The differential measure $V^+ - V^-$ is only non-zero if both solutions jump simultaneously. In this case, the density is obtained as $V^+ - V^- = V_{00}(\boldsymbol{x}^+,\boldsymbol{y}^+) - V_{00}(\boldsymbol{x}^-,\boldsymbol{y}^-) = -(1-e^2)V_{00}(\boldsymbol{x}^-,\boldsymbol{y}^-) \leq 0$, where the impact map (24) has been used. Combining all four cases concludes the proof. □

Proof of Lemma 4. For case (i) the generalized gradient of V consists of the single element $\partial V(\boldsymbol{x},\boldsymbol{y}) = \{\nabla V_{00}(\boldsymbol{x},\boldsymbol{y})\}$. Together with the equation of motion (22), we obtain (32). For case (ii) we note that the term $u_x q_y - q_x u_y$ is positive as we have seen in (31). The differential measure $\dot{V}$ evaluated using $\partial V(\boldsymbol{x},\boldsymbol{y})$

$= \{\nabla V_{01}(\boldsymbol{x},\boldsymbol{y})\}$ yields

$$\begin{aligned}\dot{V}_{01} &= \nabla V_{01}(\boldsymbol{x},\boldsymbol{y})^{\mathsf{T}}\begin{pmatrix}\dot{\boldsymbol{x}}\\ \dot{\boldsymbol{y}}\end{pmatrix}\\ &= (q_x+q_y)(u_x+u_y) - \frac{(q_x+q_y)^2}{(q_x+eq_y)^2}(u_x+eu_y)(q_x+eq_y)\\ &+ \frac{(q_x+q_y)}{(q_x+eq_y)}\frac{(u_x+eu_y)^2}{(q_x+eq_y)^2}((u_x+u_y)(q_x+eq_y)\\ &- (q_x+q_y)(u_x+eu_y)) - c\frac{(q_x+q_y)^2}{(q_x+eq_y)^2}(u_x+eu_y)^2\\ &+ \frac{(q_x+q_y)^2}{(q_x+eq_y)^2}(u_x+eu_y)\left((\lambda_x+e\lambda_y)-(1+e)f-(\tau_x+e\tau_y)\right).\end{aligned} \tag{A.18}$$

The first two terms in (A.18) can be written as $(q_x+q_y)(u_x+u_y) - \frac{(q_x+q_y)^2}{(q_x+eq_y)^2}(u_x+eu_y)(q_x+eq_y) = \frac{(q_x+q_y)}{(q_x+eq_y)}((q_x+eq_y)(u_x+u_y)-(q_x+q_y)(u_x+eu_y))$. Together with $(q_x+eq_y)(u_x+u_y)-(q_x+q_y)(u_x+eu_y) = (1-e)(u_xq_y-u_yq_x)$, we obtain $\dot{V}_{01} = \frac{(q_x+q_y)}{(q_x+eq_y)}(1-e)(u_xq_y - u_yq_x) + \frac{(q_x+q_y)}{(q_x+eq_y)}(u_x+eu_y)^2\frac{(1-e)(u_xq_y-u_yq_x)}{(q_x+eq_y)^2} - c\frac{(q_x+q_y)^2}{(q_x+eq_y)^2}(u_x+eu_y)^2 + \frac{(q_x+q_y)^2}{(q_x+eq_y)^2}(u_x+eu_y)\big(-(1+e)f-(\tau_x+e\tau_y)+(\lambda_x+e\lambda_y)\big)$, from which (33) follows. Finally, the case (iii) is symmetric to case (ii), since $V_{10}(\boldsymbol{x},\boldsymbol{y}) = V_{01}(\boldsymbol{y},\boldsymbol{x})$. □

Proof of Theorem 2. The density $(V^+ - V^-)$ is non-positive according to Lemma 3. If the minimum is attained by two minimizer, then this happens either at a (Lebesgue negligible) point in time or the solution curve runs along the boundary of two regions for which the density $\dot{V}$ is single-valued and agrees with the limits from either side of the boundary. Furthermore, the set of points in time for which the density $\dot{V}$ is undefined due to impacts is Lebesgue negligible. Therefore, the density $\dot{V}$ is defined for almost all t and is equal to either $\dot{V}_{00}$, $\dot{V}_{01}$ or $\dot{V}_{10}$ given in Lemma 4 depending on the minimizer in the definition of V.

Since the assumption $q_x+|u_x| > r_1$ and $q_y+|u_y| > r_1$ excludes persistent contact, Lemma 4(i) yields

$$\dot{V}_{00} = -c(u_x-u_y)^2. \tag{A.19}$$

In the following we will show that

$$\dot{V}_{01} \le -c\frac{(q_x+q_y)^2}{(q_x+eq_y)^2}(u_x+eu_y)^2. \tag{A.20}$$

According to Lemma 4(ii) together with $\tau_x = \tau_y = 0$ and $\lambda_x = \lambda_y = 0$, (A.20) holds if $Q := -(1-e)\frac{q_x+q_y}{q_x+eq_y}\left(1+\frac{(u_x+eu_y)^2}{(q_x+eq_y)^2}\right)(u_xq_y - q_xu_y) + \frac{(q_x+q_y)^2}{(q_x+eq_y)^2}|u_x+eu_y|(1+e)|f| \le 0$. Using $|f| < f_{\max}$ yields

$$\begin{aligned}Q \le -(1-e)\frac{(q_x+q_y)^2}{q_x+eq_y}&\Bigg(\left(1+\frac{(u_x+eu_y)^2}{(q_x+eq_y)^2}\right)\frac{u_xq_y-q_xu_y}{q_x+q_y}\\ &-\frac{|u_x+eu_y|}{q_x+eq_y}\frac{1+e}{1-e}f_{\max}\Bigg).\end{aligned} \tag{A.21}$$

In order to simplify (A.21) further, it is necessary to find a lower bound for $\frac{u_xq_y-q_xu_y}{q_x+q_y}$. Therefore, we assume that the condition

$$\hat{d}(\boldsymbol{x},\boldsymbol{y}) < \frac{1}{2}r_1 \tag{A.22}$$

is fulfilled for all $\boldsymbol{x}(t)$, $\boldsymbol{y}(t)$ and we argue later (see (A.24)) that this condition is met. The condition (A.22) together with (29) yields $q_x < \frac{1}{2}r_1$. Together with $q_x+|u_x| > r_1$, we obtain $|u_x| > r_1 - q_x > r_1 - \frac{1}{2}r_1 = \frac{1}{2}r_1$ and, analogously, we find $|u_y| > \frac{1}{2}r_1$. The condition (A.22) together with (29) also implies $|u_x+eu_y| < \frac{1}{2}r_1$, where $1 \le \frac{q_x+q_y}{q_x+eq_y} \le \frac{1}{e}$ has been used. The sign of u_x and u_y is given by $u_x > 0$ and $u_y < 0$ as any other choice leads to a contradiction. To be specific, the velocities u_x and u_y cannot have the same sign as $u_xu_y > 0$ together with $|u_x| > \frac{1}{2}r_1$, $|u_y| > \frac{1}{2}r_1$ and $|u_x+eu_y| < \frac{1}{2}r_1$ yield the contradiction $\frac{1}{2}r_1 > |u_x+eu_y| = |u_x|+e|u_y| > (1+e)\frac{1}{2}r_1$. Furthermore, $u_x < 0$ and $u_y > 0$ yield $u_xq_y - q_xu_y \le 0$, which contradicts $V = V_{01}$. The desired lower bound follows from $u_x > \frac{1}{2}r_1$, $u_y < -\frac{1}{2}r_1$ as $\frac{u_xq_y-q_xu_y}{q_x+q_y} > \frac{\frac{1}{2}r_1q_y+q_x\frac{1}{2}r_1}{q_x+q_y} = \frac{1}{2}r_1$. Substituting $\frac{u_xq_y-q_xu_y}{q_x+q_y} > \frac{1}{2}r_1$ and $f_{\max} < \frac{1-e}{1+e}r_1$ in (A.21) yields

$$\begin{aligned}Q \le -(1-e)&\frac{(q_x+q_y)^2}{q_x+eq_y}\\ &\cdot\left(\left(1+\frac{(u_x+eu_y)^2}{(q_x+eq_y)^2}\right)\frac{1}{2}r_1 - \frac{|u_x+eu_y|}{q_x+eq_y}r_1\right).\end{aligned} \tag{A.23}$$

The term Q is non-positive because the term $\left(1+\frac{(u_x+eu_y)^2}{(q_x+eq_y)^2}\right)\frac{1}{2}r_1 - \frac{|u_x+eu_y|}{q_x+eq_y}r_1 = \frac{1}{2}r_1\left(1-\frac{|u_x+eu_y|}{q_x+eq_y}\right)^2$ in (A.23) is non-negative. Therefore, (A.20) holds if the condition (A.22) is met. The symmetry of V_{01} and V_{10} immediately yields

$$\dot{V}_{10} \le -c\frac{(q_x+q_y)^2}{(eq_x+q_y)^2}(eu_x+u_y)^2. \tag{A.24}$$

Summarizing, $V^+ - V^- \le 0$ together with (A.19), (A.20) and (A.24) imply that the Lyapunov function is non-increasing, that is, $\mathrm{d}V \le 0$. Together with $d(\boldsymbol{x}^-,\boldsymbol{y}^-) \le \frac{1}{2}r_1$ assumed in the theorem and Lemma 2, we obtain that condition (A.22) is fulfilled as $\hat{d}(\boldsymbol{x},\boldsymbol{y}) \le \hat{d}(\boldsymbol{x}^-,\boldsymbol{y}^-) \le \frac{1}{e}d(\boldsymbol{x}^-,\boldsymbol{y}^-) \le \frac{1}{2}r_1$.

In order to keep the derivation concise, we will simply write $V(t)$ instead of $V(\boldsymbol{x}(t),\boldsymbol{y}(t))$ in the following. Because the candidate Lyapunov function V is non-increasing and bounded from below, it attains the limit

$$\lim_{t\to\infty} V^-(t) = V_\infty$$

with $0 \le V_\infty \le V^-(t_0)$. In the following, we show that $V_\infty = 0$. Using $(V^+ - V^-) \le 0$ and the differential measure $\mathrm{d}V$, the candidate Lyapunov function can be written as $V^-(t) = V^+(t_0) + \int_{(t_0\ t)}\mathrm{d}V \le V^-(t_0) + \int_{(t_0\ t)}\dot{V}\mathrm{d}t$. Taking the limit as $t\to\infty$ yields

$$V_\infty \le V^-(t_0) + \lim_{t\to\infty}\int_{(t_0\ t)}\dot{V}\mathrm{d}t. \tag{A.25}$$

We split the time interval $I^\infty = (t_0,\ \infty)$ into the three sets $I^\infty_{00} = \{t\in I^\infty \mid V(t) = V_{00}(t)\}$, $I^\infty_{01} = \{t\in I^\infty \mid V(t) = V_{01} \ne V_{00}(t)\}$ and $I^\infty_{10} = \{t\in I^\infty \mid V(t) = V_{10} \ne V_{00}(t)\}$ and rewrite (A.25) as

$$V_\infty \le V^-(t_0) + \int_{I^\infty_{00}}\dot{V}_{00}\mathrm{d}t + \int_{I^\infty_{01}}\dot{V}_{01}\mathrm{d}t + \int_{I^\infty_{10}}\dot{V}_{10}\mathrm{d}t. \tag{A.26}$$

The unboundedness of I^∞ implies that at least one of the cases (i) $\mathrm{d}t(I^\infty_{00}) = \infty$, (ii) $\mathrm{d}t(I^\infty_{01}) = \infty$ or (iii) $\mathrm{d}t(I^\infty_{10}) = \infty$ holds. The three integrals in (A.26) are non-positive and thus bounded from below as $V_\infty \ge 0$. Let us first consider case (i). Substituting (A.19) into $\int_{I^\infty_{00}}\dot{V}_{00}\mathrm{d}t > -\infty$ yields

$$\int_{I^\infty_{00}} c(u_x-u_y)^2\mathrm{d}t < \infty. \tag{A.27}$$

The function $(u_x(t)-u_y(t))^2$ is not absolutely continuous and therefore we cannot invoke Barbalat's lemma. However, the function V^-_{00} considered on I^∞_{00} is asymptotically absolutely continuous (i.e., it approaches an absolutely continuous function asymptotically [22]). Furthermore, $(q_x(t)-q_y(t))^2$ is absolutely continuous and $c > 0$. According to the definition $V_{00} = \frac{1}{2}(q_x - q_y)^2$

$+\frac{1}{2}(u_x-u_y)^2$, we obtain that $(u_x(t)-u_y(t))^2$ is asymptotically absolutely continuous when considered on I_{00}^∞. Therefore, we can apply the extended Barbalat's lemma[3] [22] to (A.27) and obtain $\lim_{t\to\infty, t\in I_{00}^\infty}(u_x(t)-u_y(t))^2 = 0$. This limit together with the definition of V_{00} yields $V_\infty = \lim_{t\to\infty, t\in I_{00}^\infty}\frac{1}{2}(q_x(t)-q_y(t))^2$. The non-impulsive dynamics of q_x-q_y is obtained by (22) with $\tau_x=\tau_y=0$ and $\lambda_x=\lambda_y=0$ as $(\dot{u}_x-\dot{u}_y)+c(u_x-u_y)+(q_x-q_y)=0$, from which follows that $\lim_{t\to\infty, t\in I_{00}^\infty}|\dot{u}_x(t)-\dot{u}_y(t)| = \lim_{t\to\infty, t\in I_{00}^\infty}|q_x(t)-q_y(t)| = \sqrt{2V_\infty}$. The individual velocities $u_x(t)$ and $u_y(t)$ are assumed to be functions of special locally bounded variation and, consequently, the velocity difference $u_x(t)-u_y(t)$ is a function of special locally bounded variation as well. Therefore, the limits $\lim_{t\to\infty, t\in I_{00}^\infty}|u_x(t)-u_y(t)| = 0$ and $\lim_{t\to\infty, t\in I_{00}^\infty}|\dot{u}_x(t)-\dot{u}_y(t)| = \sqrt{2V_\infty}$ imply $\lim_{t\to\infty, t\in I_{00}^\infty}|\dot{u}_x(t)-\dot{u}_y(t)| = 0$, that is, $V_\infty = 0$ in case (i).

We will use the same reasoning as in case (i) to show that $V_\infty = \lim_{t\to\infty, t\in I_{01}^\infty}\frac{1}{2}(q_x(t)+q_y(t))^2$ holds in case (ii). The inequality (A.26) implies $\int_{I_{01}^\infty}\dot{V}_{01}\mathrm{d}t > -\infty$ and, substituting (A.20), yields $\int_{I_{01}^\infty} c\frac{(q_x+q_y)^2}{(q_x+eq_y)^2}(u_x+eu_y)^2\mathrm{d}t < \infty$. The asymptotic absolute continuity of V_{01}^- and $(q_x+q_y)^2$ when considered on I_{01}^∞ together with the definition of V_{01} yields that $\frac{(q_x+q_y)^2}{(q_x+eq_y)^2}(u_x+eu_y)^2$ is asymptotic absolutely continuous as well when considered on I_{01}^∞. The extended Barbalat's lemma yields $\lim_{t\to\infty, t\in I_{01}^\infty}\frac{(q_x+q_y)^2}{(q_x+eq_y)^2}(u_x+eu_y)^2 = 0$ and, together with the definition of V_{01}, $V_\infty = \lim_{t\to\infty, t\in I_{01}^\infty}\frac{1}{2}(q_x(t)+q_y(t))^2$. In the following, we will use a proof of contradiction to show that $V_\infty = 0$ in case (ii) as well.

Let us assume that $V_\infty > r_2$ for some $r_2 > 0$. Taking the limit $\lim_{t\to\infty, t\in I_{01}^\infty}$ of (33) in Lemma 4(ii) and substituting $\lim_{t\to\infty, t\in I_{01}^\infty}|u_x+eu_y| = 0$ yields

$$\lim_{\substack{t\to\infty\\ t\in I_{01}^\infty}}\dot{V}_{01} = \lim_{\substack{t\to\infty\\ t\in I_{01}^\infty}} -(1-e)\frac{q_x+q_y}{q_x+eq_y}(u_xq_y-q_xu_y). \tag{A.28}$$

In the derivation of (A.20), we have noticed the two inequalities $\frac{u_xq_y-q_xu_y}{q_x+q_y} > \frac{1}{2}r_1$ and $q_x+q_y < \frac{1}{2}r_1$. The latter implies $\frac{1}{q_x+eq_y} \geq \frac{1}{e}\frac{1}{q_x+q_y} > \frac{1}{e}\frac{1}{\frac{1}{2}r_1}$. Substituting into (A.28) yields $\lim_{t\to\infty, t\in I_{01}^\infty}\dot{V}_{01} \leq -\frac{1-e}{e}\lim_{t\to\infty, t\in I_{01}^\infty}(q_x+q_y)^2$. Together with the assumption $V_\infty = \lim_{t\to\infty, t\in I_{01}^\infty}\frac{1}{2}(q_x(t)+q_y(t))^2 > r_2$, we obtain

$$\lim_{\substack{t\to\infty\\ t\in I_{01}^\infty}}\dot{V}_{01} \leq -2\frac{1-e}{e}r_2. \tag{A.29}$$

The limit (A.29) implies that the integral $\int_{I_{01}^\infty}\dot{V}_{01}\mathrm{d}t$ is not bounded from below, which contradicts $V_\infty \geq 0$. Therefore, $V_\infty = 0$ must hold in case (ii). The symmetry of V_{01} and V_{10} immediately yields $V_\infty = 0$ also in case (iii).

The Lyapunov function V is positive definite in the distance function d as shown in Lemma 2. Consequently, $\mathrm{d}V \leq 0$ implies $\forall\varepsilon > 0 : d(\boldsymbol{x}(t_0),\boldsymbol{y}(t_0)) < \min\{e\varepsilon, \frac{e}{2}r_1\} \leq e\varepsilon \Rightarrow d(\boldsymbol{x}(t),\boldsymbol{y}(t)) < \min\{\varepsilon, \frac{1}{2}r_1\} \leq \varepsilon$, $\forall t \geq t_0$. Furthermore, $V_\infty = 0$ implies $d(\boldsymbol{x}(t_0),\boldsymbol{y}(t_0)) < \min\{e\,\varepsilon, \frac{e}{2}r_1\} \Rightarrow \lim_{t\to\infty}d(\boldsymbol{x}(t),\boldsymbol{y}(t)) = 0$. Therefore, local synchronization is achieved according to Definition 2, which concludes the proof. □

[3] Small adjustments are necessary to apply the mentioned lemma here. Since we consider only the domain $I_{00}^\infty \subset \mathbb{R}$, the limit $\lim_{t\to\infty}$ is substituted with $\lim_{t\to\infty, t\in I_0^\infty}$ and the proof is adjusted accordingly.

Proof of Theorem 3. Let $V(t) = V(\boldsymbol{x}(t),\boldsymbol{y}(t))$ be the candidate Lyapunov function defined by (26)–(30) evaluated along the solutions of (22)–(24) with τ_x, τ_y defined by (35)–(36). As we have seen in Theorem 2, the density (V^+-V^-) is non-positive according to Lemma 3. In the following, we will show that the differential measure $\dot{V}$ is bounded from above almost everywhere by

$$\dot{V} \leq -c(u_x-u_y)^2 \quad \text{if } V = V_{00}, \tag{A.30}$$

$$\dot{V} \leq -c\frac{(q_x+q_y)^2}{(q_x+eq_y)^2}(u_x+eu_y)^2 \quad \text{if } V = V_{01} \neq V_{00}, \tag{A.31}$$

$$\dot{V} \leq -c\frac{(q_x+q_y)^2}{(eq_x+q_y)^2}(eu_x+u_y)^2 \quad \text{if } V = V_{10} \neq V_{00}. \tag{A.32}$$

First, we consider the case for which both constraints are open and the constraint forces vanish. The interaction law (35)–(36) yields $\kappa_x = \kappa_y = 0$ for $\min\{V_{01},V_{10}\} \geq V_{00}$ and $\kappa_x = \kappa_y = -f$ for $\min\{V_{01},V_{10}\} < V_{00}$. Substituting $\lambda_x = \lambda_y = 0$ and $\kappa_x = \kappa_y = 0$ into Lemma 4(i) yields that (A.30) is fulfilled. Furthermore, substituting $\lambda_x = \lambda_y = 0$ and $\kappa_x = \kappa_y = -f$ into Lemma 4(ii) and (iii) yields that (A.31) and (A.32) are fulfilled because $u_xq_y-q_xu_y > 0$ for $V_{01} < V_{00}$ and $u_xq_y-q_xu_y < 0$ for $V_{10} < V_{00}$.

Secondly, we consider the case $q_x = 0$ and $q_y > 0$. Note that we only need to consider persistent contact in the evaluation of $\dot{V}$ as $\{t \mid q_x(t) \wedge u_x(t) \neq 0\}$ and $\{t \mid q_x(t) \wedge u_x(t) = 0 \wedge \dot{u}_x(t) \neq 0\}$ are Lebesgue negligible. Persistent contact of $\boldsymbol{x}$ yields $\lambda_x - f(t) - \tau_x = 0$ according to (22). Furthermore, the definition of the candidate Lyapunov function yields $V = V_{00}$ if one state is in persistent contact and the interaction law (35)–(36) yields $\kappa_x = 0$, $\kappa_y = -f$. Substituting into Lemma 4(i) yields (A.30). Analogously, if $q_x > 0$ and $\boldsymbol{y}$ is in persistent contact, then (A.30) is fulfilled with $\kappa_x = -f$ and $\kappa_y = 0$. Finally, if both contacts are persistent, then both states are identical, the candidate Lyapunov function is zero and there are no coupling forces.

The coupling forces switch depending on the state and the solution concept of Filippov is used such that sliding modes may occur. According to the interaction law (35)–(36), we need to consider the following three switching surfaces: (a) $q_x > 0, q_y > 0$, $\min\{V_{01},V_{10}\} = V_{00}$, (b) $q_x = u_x = 0$, $q_y > 0$ and (c) $q_x > 0$, $q_y = u_y = 0$. In case (a), the convexification of (35)–(36) yields $\tau_x = \tau_y \in [-|f|, |f|]$. However, any choice of $\tau_x = \tau_y$ together with $\lambda_x = \lambda_y = 0$ fulfills (A.30) according to Lemma 4(i). In case (b), the convexification of (35)–(36) yields $\tau_x \in [-|f|, |f|]$ and $\tau_y = 0$. A persistent contact $q_x = 0$ implies $\lambda_x - f(t) - \tau_x = 0$ according to (22) and $V = V_{00}$. Together with $\lambda_y = 0$, Lemma 4(i) yields (A.30). Finally, the case (c) is symmetric to case (b). Therefore, the differential measure $\mathrm{d}V$ fulfills (A.30)–(A.32) for almost all t also on the sliding surfaces.

The remaining part of the proof is very close to the proof of Theorem 2. The candidate Lyapunov function $V(t)$ in not increasing neither during continuous flow nor at discontinuity points, that is, $\mathrm{d}V \leq 0$. Furthermore, V is bounded from below, thus attaining a limit $\lim_{t\to\infty}V^-(t) = V_\infty$. In the following, we show $V_\infty = 0$.

As in the proof of Theorem 2, the time interval $I^\infty = (t_0, \infty)$ is split into the three sets $I_{00}^\infty = \{t \in I^\infty \mid V(t) = V_{00}(t)\}$, $I_{01}^\infty = \{t \in I^\infty \mid V(t) = V_{01} \neq V_{00}(t)\}$ and $I_{10}^\infty = \{t \in I^\infty \mid V(t) = V_{10} \neq V_{00}(t)\}$. Furthermore, $\mathrm{d}t(I^\infty) = \infty$ implies that at least one of the cases (i) $\mathrm{d}t(I_{00}^\infty) = \infty$, (ii) $\mathrm{d}t(I_{01}^\infty) = \infty$ or (iii) $\mathrm{d}t(I_{10}^\infty) = \infty$ has to hold. Using the same arguments as in the proof of Theorem 2, it holds for case (i) that $(u_x(t)-u_y(t))^2$ considered on I_{00}^∞ is asymptotically absolutely continuous. The extended Barbalat's lemma [22] implies $\lim_{t\to\infty, t\in I_{00}^\infty}(u_x(t)-u_y(t))^2 = 0$ and, thus, $V_\infty = \lim_{t\to\infty, t\in I_{00}^\infty}\frac{1}{2}(q_x-q_y)^2$. Analogously for case (ii), we obtain $\lim_{\substack{t\to\infty\\ t\in I_{01}^{\infty}}} \frac{(q_x+q_y)^2}{(q_x+eq_y)^2}(u_x + eu_y)^2 = 0$ and, thus, $V_\infty = \lim_{\substack{t\to\infty\\ t\in I_{01}^{\infty}}} \frac{1}{2}(q_x+q_y)^2$. Case (iii) yields $\lim_{\substack{t\to\infty\\ t\in I_{10}^{\infty}}} \frac{(q_x+q_y)^2}{(eq_x+q_y)^2}(eu_x+u_y)^2 = 0$ and, thus, $V_\infty = \lim_{\substack{t\to\infty\\ t\in I_{10}^{\infty}}} \frac{1}{2}(q_x + q_y)^2$. As we have seen before, $V = V_{00}$ always implies $\lambda_x - f(t) - \tau_x = \lambda_y - f(t) - \tau_y$. The non-impulsive dynamics of $q_x - q_y$ is obtained by (22) as $\dot{u}_x - \dot{u}_y + cu_x - u_y + q_x - q_y = 0$, which is valid for almost all $t \in I_{00}^{\infty}$. In case (i), together with $V_\infty = \lim_{\substack{t\to\infty\\ t\in I_{00}^{\infty}}} \frac{1}{2}(q_x - q_y)^2$ and using the same arguments as in the proof of Theorem 2 yields $V_\infty = 0$. For the cases (ii) and (iii), we consider the evolution of $q_x + q_y$. Because $\lambda_x - f(t) - \tau_x = 0$ and $\lambda_y - f(t) - \tau_y = 0$ whenever $V = V_{01} \neq V_{00}$ or $V = V_{10} \neq V_{00}$, the dynamics is obtained using (22) as $\dot{u}_x + \dot{u}_y + cu_x + u_y + q_x + q_y = \tau_x + \tau_y + 2f(t) = 0$, which is valid for almost all $t \in I_{01}^{\infty} \cup I_{10}^{\infty}$. Together with $V_\infty = \lim_{\substack{t\to\infty\\ t\in I_{01}^{\infty}}} \frac{1}{2}(q_x+q_y)^2$ in case (ii) and $V_\infty = \lim_{\substack{t\to\infty\\ t\in I_{10}^{\infty}}} \frac{1}{2}(q_x + q_y)^2$ in case (iii), we obtain $V_\infty = 0$ also in these cases.

Condition (20) is fulfilled with $\delta(\varepsilon) = e\varepsilon$ using $\mathrm{d}V \leq 0$ and Lemma 2. Furthermore, $\lim_{t\to\infty} V(t) = 0$ together with Lemma 2 yield that condition (21) is fulfilled for any δ_0. Therefore, global synchronization is achieved in the sense of Definition 2. □

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