Beyond tracial states in robust self-testing and constructing derivations for quantum Markov semigroups

Vernooij, M.N.A.

DOI
10.4233/uuid:f5d98b0e-c24d-478d-976e-a48f2e0768cc

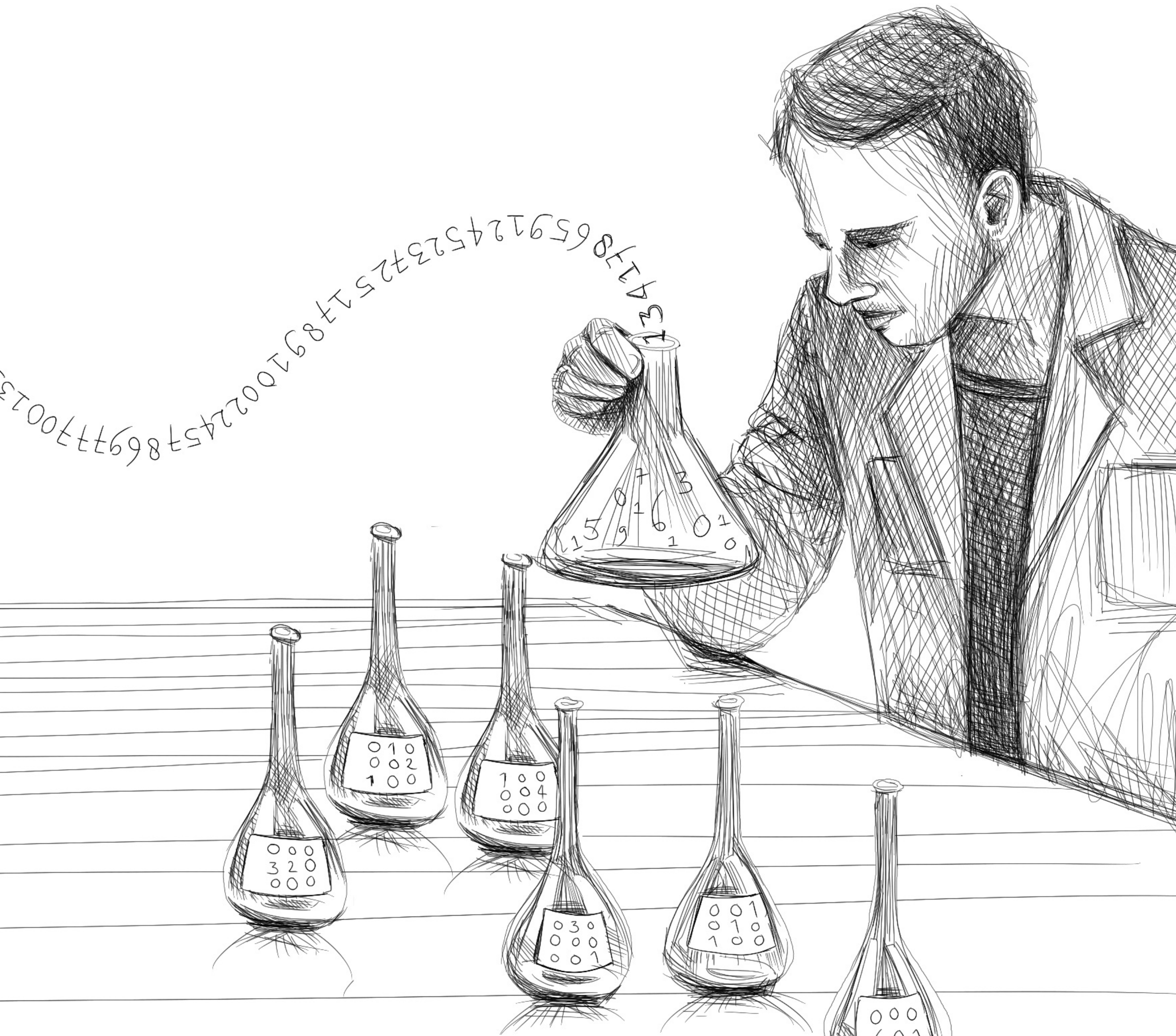
Publication date
2025

Document Version
Final published version

Citation (APA)
Vernooij, M. N. A. (2025). *Beyond tracial states in robust self-testing and constructing derivations for quantum Markov semigroups*. [Dissertation (TU Delft), Delft University of Technology]. https://doi.org/10.4233/uuid:f5d98b0e-c24d-478d-976e-a48f2e0768cc

Beyond tracial states in robust self-testing and constructing derivations for quantum Markov semigroups

Matthijs Vernooij



Beyond tracial states in robust self-testing and constructing derivations for quantum Markov semigroups

Beyond tracial states in robust self-testing and constructing derivations for quantum Markov semigroups

Proefschrift

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus prof. dr. ir. T.H.J.J. van der Hagen,
voorzitter van het College voor Promoties,
in het openbaar te verdedigen op 26 november 2025

door

Matthijs Niels Albert Vernooij

Master of Science in Mathematics,
Master of Science in Physics and Astronomy,
Radboud Universiteit, Nijmegen, Nederland
geboren te Nijmegen, Nederland.

Dit proefschrift is goedgekeurd door de promotoren.

Samenstelling promotiecommissie:

Rector Magnificus	voorzitter
Dr. M.P.T. Caspers	Technische Universiteit Delft, promotor
Prof. dr. J.M.A.M van Neerven	Technische Universiteit Delft, promotor
Onafhankelijke leden:	
Dr. P. Fima	Université Paris Cité, Frankrijk
Prof. dr. M.E. Musat	Københavns Universitet, Denemarken
Dr. C. Rouzé	Inria Saclay, Frankrijk
Prof. dr. B.M. Terhal	Technische Universiteit Delft
Prof. dr. M. Weber	Universität des Saarlandes, Duitsland
Prof. dr. M.C. Veraar	Technische Universiteit Delft, reservelid

Het onderzoek beschreven in dit proefschrift is gefinancierd door de Nederlandse Organisatie voor Wetenschappelijk Onderzoek, onder projectnummer VI.Vidi.192.018.

Keywords: Operator algebras, quantum Markov semigroups, states, Tomita-Takesaki theory, derivations, non-local games, robust self-testing, synchronous games, maximally entangled strategies, almost synchronous strategies, quantum expanders, quantum groups, representation theory of quantum groups, coideals of quantum groups

Printed by: Ipskamp Printing

Front & Back: Designed by Maaike Schouten

ISBN 978-94-6473-959-6

An electronic version of this dissertation is available at
http://repository.tudelft.nl/.

Contents

Samenvatting

Dit proefschrift speelt zich af in het vakgebied van de kwantummechanica. Dit is de natuurkundige theorie die nodig is om de wereld op de kleinste lengteschalen te beschrijven, de theorie die de spelregels bepaalt waar moleculen, atomen en nog kleinere deeltjes zich aan moeten houden. Op deze schalen gedragen deeltjes zich niet meer zoals je gewend bent. Een elektron ziet zichzelf soms meer als een golf in het water dan als een pingpongbal. In deze kwantumwereld vinden veel contra-intuïtieve dingen plaats. Het belangrijkste voorbeeld is hoe je niet meer als onafhankelijke buitenstaander het systeem kan observeren. Als je een meting doet, dan verandert daardoor het systeem. De beroemdste analogie is Schrödingers kat, die in superpositie in een doos zit en dus deels dood en deels levend is. Door te meten of de kat leeft, zorg je er met 50% kans voor dat de kat nu 100% dood is, terwijl die dat eerst niet was.

Er zijn drie verschillende onderwerpen waar dit proefschrift een steentje aan bij probeert te dragen. Als eerste bestuderen we de vraag hoe een kwantumsysteem verandert in de loop van de tijd. Als een kwantumsysteem gesloten is, wat wil zeggen dat het systeem niet in contact staat met de buitenwereld eromheen, dan wordt de tijdsevolutie beschreven door de schrödingervergelijking. De schrödingervergelijking is nu precies honderd jaar geleden geformuleerd en we begrijpen redelijk goed hoe een systeem zich gedraagt als deze vergelijking van toepassing is. Als een systeem niet gesloten maar open is, wordt de situatie een stuk ingewikkelder. Om een exacte beschrijving van een open kwantumsysteem te geven, moet je niet alleen dit systeem omschrijven, maar ook de omgeving. In essentie betekent dit dat je op zoek moet naar een groter kwantumsysteem dat wel gesloten is.

Deze aanpak is niet altijd praktisch. Een alternatief plan is om het systeem bij benadering te beschrijven. In dit geval doe je bepaalde aannames over het systeem die niet per se waar hoeven te zijn, maar wel ongeveer zouden moeten kloppen. Als we aannemen dat de omgeving van een kwantumsysteem niet echt verandert in de loop van de tijd, kunnen we de tijdsevolutie van het systeem beschrijven met behulp van een kwantummarkovhalfgroep. Ons doel is om deze kwantummarkovhalfgroepen beter te begrijpen. Een grote vooruitgang in dit streven was de ontdekking van een connectie tussen traciaal symmetrische kwantummarkovhalfgroepen en derivaties door Cipriani en Sauvageot [CS03a]. Derivaties zijn een soort algemenere variant van de afgeleide en kwantummarkovhalfgroepen kunnen traciaal symmetrisch zijn in de geïdealiseerde situatie dat de temperatuur in het systeem oneindig hoog is.

Aangezien de temperatuur in de praktijk nooit oneindig hoog is, zou het wenselijk zijn als deze verbinding standhoudt bij eindige temperaturen. Dit betekent dat we niet met traciale toestanden moeten werken, maar met algemene toestanden. Helaas toont Sectie 3.2 aan dat dit problematisch is; van de gehoopte verbinding blijft weinig over. Het blijkt dat we bij eindige temperaturen niet naar derivaties moeten kijken, maar naar een variant daarop, gedraaide derivaties genaamd. Wanneer we dat doen, blijkt in Hoofdstuk 3 dat we voor elke KMS-symmetrische kwantummarkovhalfgroep een corresponderende gedraaide derivatie kunnen maken. Dit generaliseert eerdere resultaten voor GNS-symmetrische kwantummarkovhalfgroepen. Zowel KMS-symmetrie als GNS-symmetrie komen overeen met traciale symmetrie als de temperatuur oneindig is, maar in andere gevallen voldoen veel minder systemen aan GNS-symmetrie dan aan KMS-symmetrie.

Het volgende onderwerp is het vaststellen hoe een kwantumsysteem zich gedraagt. Het is tamelijk ingewikkeld om dit direct te meten, omdat je een kwantumsysteem niet zomaar kan observeren. Als je metingen gaat doen aan je kwantumsysteem, beïnvloed je immers het systeem en gaat het zich anders gedragen dan voorheen. Aan de andere kant is het natuurlijk onmogelijk om iets over een kwantumsysteem te zeggen als je helemaal geen beschikking hebt tot informatie over het systeem. Een manier om met dit probleem om te gaan is om het kwantumsysteem te bekijken als een kwantumapparaat, een zwarte doos waar je op bepaalde momenten informatie kan invoeren en op andere momenten informatie als uitvoer krijgt. De vraag is vervolgens wat je kan zeggen over de interne werking van het apparaat op basis van de in- en uitvoer.

Als je puur op basis van de in- en uitvoer van het apparaat kan concluderen dat de interne werking bepaalde eigenschappen heeft, betekent dit dat je deze eigenschappen kunt certificeren: wanneer de uitvoer van het kwantumapparaat op de door jou bepaalde invoer is zoals verwacht, kan je de garantie geven dat het apparaat de desbetreffende eigenschappen heeft. In de praktijk kan dit lang niet altijd. De kunst is om nieuwe combinaties van invoer en uitvoer te vinden waar elk apparaat met de gezochte eigenschappen op een goede manier mee om kan gaan. Het doel is vervolgens om een protocol samen te stellen met verschillende soorten invoer en corresponderende uitvoer zodat je wel in staat bent de gewilde eigenschappen te certificeren. Idealiter is de invoer en uitvoer in een dergelijk protocol *niet* kwantummechanisch, want dan moeten de apparaten die daarmee om moeten gaan eigenlijk ook zelf weer gecertificeerd worden.

Een van de manieren om dergelijke protocollen te maken is met behulp van niet-lokale spellen. Bij een niet-lokaal spel zijn er twee spelers, elk met een kwantumapparaat, die vooraf een strategie mogen afspreken, maar tijdens het spel niet mogen communiceren. Beide spelers krijgen tegelijk elk hun eigen vraag van een scheidsrechter en moeten daar een antwoord op geven. Vervolgens is van elke combinatie van twee vragen en twee antwoorden bekend of je daarmee wint of verliest. Sommige spellen hebben de eigenschap dat er een unieke optimale strategie is en dat elke strategie die bijna optimaal is, ook erg lijkt op de optimale strategie. Zulke spellen noemen we robuuste zelftesten en ze kunnen gebruikt worden om een certificeringsprotocol voor de optimale strategie te maken.

De uitdaging is om te bewijzen dat een spel een robuuste zelftest is. Vaak wordt het een stuk makkelijker om een dergelijk bewijs te vinden als je geen rekening hoeft te houden met *alle* mogelijke strategieën die de spelers kunnen hanteren, maar alleen met strategieën die aan specifieke spelregels voldoen. Meestal heb je hier helaas niets aan, want je hebt niets aan extra spelregels als je ze niet kan handhaven. Er gebeurt echter iets speciaals bij zogeheten synchrone spellen, waar spelers verliezen als ze op dezelfde vragen verschillende antwoorden geven. Hier is het praktisch om als extra spelregels te eisen dat de spelers zogeheten maximaal verstrengelde strategieën gebruiken, spelregels die niet alleen niet te handhaven zijn, maar überhaupt onmogelijk nageleefd kunnen worden in een echte situatie. In Hoofdstuk 4 bewijzen we echter dat een bewijs met deze extra spelregels automatisch vertaalt naar een bewijs dat het spel een robuuste zelftest is *zonder* extra spelregels. Dit maakt het dus makkelijker om voor synchrone spellen te bewijzen dat het robuuste zelftesten zijn.

Tot slot gooien we het over een wat andere boeg. Er bestaan heel veel wiskundige concepten die ontworpen zijn voor een wereld die niet kwantummechanisch is. In veel gevallen is het kwantummechanische gedrag ook niet zichtbaar en dan werken deze concepten goed. Bij elk van deze concepten roept dit echter wel de vraag op of we kunnen zorgen dat het concept blijft werken als de kwantummechanische effecten relevant worden. We buigen ons hier over een specifiek concept genaamd uitdijingsgrafen. Grafen zijn wiskundige objecten waarmee je netwerkstructuren kan beschrijven en uitdijingsgrafen representeren netwerken die bijzonder goed verbonden zijn voor het aantal verbindingen dat is aangelegd in het netwerk.

Er bestaat een kwantumversie van het concept uitdijingsgrafen en de kwantumversie wordt een kwantumuitdijer genoemd. Het probleem is alleen dat er maar weinig voorbeelden bekend zijn van kwantumuitdijers, vooral niet van kwantumuitdijers met wezenlijk andere eigenschappen dan uitdijingsgrafen. Een van de manieren om uitdijingsgrafen te maken is een constructie van Margulis op basis van groepen met eigenschap (T) [Mar73] en Harrow was in staat op basis van dezelfde informatie kwantumuitdijers te maken [Har08]. Deze lijken echter nog sterk op de uitdijingsgrafen, dus het is nog de vraag of er kwantumeffecten kunnen opduiken bij kwantumuitdijers die je niet ziet bij uitdijingsgrafen.

Dit proefschrift probeert de constructies van Margulis en Harrow uit te breiden om op nieuwe manieren voorbeelden van kwantumuitdijers te kunnen vinden. De basis hiervoor zijn discrete kwantumgroepen, een generalisatie van discrete groepen, die ook eigenschap (T) kunnen hebben. In Hoofdstuk 5 geven we zowel een constructie van kwantumuitdijers op basis van discrete kwantumgroepen met eigenschap (T) geïnspireerd door die van Harrow, als een constructie geïnspireerd door die van Margulis. Helaas is het huidige begrip van de representatietheorie en coideaalstructuur van kwantumgroepen met eigenschap (T) nog niet ver genoeg gevorderd om direct interessante nieuwe voorbeelden te krijgen. De hoop is dat dit begrip er in de toekomst wel zal zijn en samen met Hoofdstuk 5 zal leiden tot kwantumuitdijers die echte kwantumeigenschappen vertonen.

SUMMARY

In this dissertation, we concern ourselves with the field of quantum mechanics. This is the theory in physics that we use to describe the world at its smallest scales; it governs the laws of nature that molecules, atoms and electrons abide by. These particles do not always behave as one would expect. At times, an electron seems more like a ripple on the water's surface than like a table tennis ball. In the quantum world, counter-intuitive things tend to happen. A prime example is how one can no longer just observe a system as an innocent bystander. When you measure something in a quantum system, you are changing that system. The most famous analogy is Schrödinger's cat, a poor creature that is locked in a box and is in a superposition of being dead and alive. If you measure how the cat is doing, whether by opening the box or by some other means, there is a 50% chance that that action causes the cat to be 100% dead, which was not the case before.

This dissertation touches upon three different topics. First, we study how quantum systems change over time. If a quantum system is closed, meaning that it does not interact with the outside world, then the time evolution is described by the Schrödinger equation. This one-hundred-year-old equation gives us a rather good way to describe how a system will behave. Things become more complicated if the system is not closed but open. Now, one has to take into account the interactions of the system with the environment, and the only way to do this exactly is to find a bigger, closed system, which we can once again describe using the Schrödinger equation.

This is not always the most practical approach. Often, it is better to forego the aim of finding an exact description and settle for a good approximation. Such approximations arise by making assumptions about the system that, while not exactly true, are almost valid. Assuming that the assumption is valid anyway gives us our approximate description of the system. We focus on the assumption that the environment does not change over time, which allows one to use quantum Markov semigroups to describe the time evolution of the quantum system. Our aim is to better understand these quantum Markov semigroups. Cipriani and Sauvageot made major progress in this direction by discovering a connection between tracially symmetric quantum Markov semigroups and derivations. A derivation is a more general version of the usual derivative, and a quantum Markov semigroup can be tracially symmetric in the idealised case where the temperature is infinite.

In real cases, the temperature is not infinite, so we would like to obtain similar results for finite temperatures. This requires us to work with general states instead of tracial states. Unfortunately, Section 3.2 shows that the connection between quantum Markov semigroups and derivations breaks down in these situations. It turns out that the problem is that we should not be looking at derivations but at the related notion of twisted derivations. Chapter 3 shows that, in this case, we can construct a corresponding twisted derivation for each KMS-symmetric quantum Markov semigroup. This generalises earlier results for GNS-symmetric quantum Markov semigroups. While both

KMS-symmetry and GNS-symmetry correspond to tracial symmetry in the infinite-temperature case, GNS-symmetry is significantly rarer in the finite-temperature case than KMS-symmetry.

Next, we are interested in determining the behaviour of a quantum system. This is somewhat complicated because one cannot just observe the quantum system. One needs to perform measurements, and this changes the behaviour of the system. On the other hand, it is of course impossible to determine anything about a system without observing something. To make sense of this, it is better to view the system as a quantum device, a black box that has certain inputs and certain outputs. From this viewpoint, the question is how much one can say about the quantum system based on the input one controls and the corresponding output one observes.

Whenever appropriately controlling the input and observing the resulting output allows you to conclude certain properties of the internal workings of the quantum device, it becomes possible to certify these properties. This combination of inputs becomes the test of the certification protocol, and a quantum device can prove that it has the desired properties by returning the correct outputs. In practise, this is not always possible. To craft a certification protocol, one often does not just need to find the right combinations of inputs and outputs but also to craft new sets of inputs and outputs that can be achieved by any quantum device with the desired properties. The goal is to ensure that having the desired properties is sufficient to pass the certification protocol but that a system is unlikely to pass if it does not have these properties. The input and output in such a protocol are preferably not quantum mechanical, because this would technically require the devices that handle the input and output to be certified as well.

One way to create such protocols is through non-local games. In these games, two players each receive a question from a referee and individually return an answer. While they can strategise beforehand, no communication between the players is allowed during the game. For each combination of two questions and two answers, it is known whether the players win or lose, and the players aim to have the highest chance of winning. Certain games have the property that there is a unique optimal strategy and that any strategy that is almost optimal is actually almost equal to the optimal strategy. Such games are called robust self-tests, and they can be used to create certification protocols.

The challenging part is to prove that a non-local game is a robust self-test. Often, finding these proofs would become easier if one is allowed to assume that the strategies players use abide by certain rules. This limits the number of strategies one has to consider and can allow more convenient mathematical descriptions. Unfortunately, there is usually no way of enforcing these rules, so these assumptions are not justified in practise. For synchronous games, where players lose if they give different answers to the same question, a natural rule is to require the players to use so-called maximally entangled strategies, which significantly simplifies the mathematics involved. However, this rule is not only nonenforcible, it is actually impossible to perfectly satisfy in real life. In Chapter 4, we get around this issue by proving that whenever a synchronous game is a robust self-test with the maximally-entangledness rule in place, it is actually a robust self-test without it as well. This makes it easier to prove that synchronous games are robust self-tests.

Lastly, we change tracks somewhat. There are many mathematical concepts that have been developed for a world that is not quantum mechanical. Even with the discov-

ery of quantum mechanics, these concepts are still useful, because in many situations quantum effects do not appear. However, this does raise the question whether we can still make sense of these concepts in situations where quantum mechanics becomes important. Here, we will focus on expander graphs. Graphs are mathematical objects that can be used to describe the structure of networks, and expander graphs represent especially well-connected networks for the limited number of connections they contain.

There exists a quantum version of the concept of expander graphs, in which case they are called quantum expanders. Unfortunately, we are currently lacking examples of quantum expanders that truly exhibit quantum features. For expander graphs, a way to construct them is using groups with property (T), which was shown by Margulis [Mar73]. These same groups can be used to obtain quantum expanders in a construction by Harrow [Har08]. However, these quantum expanders are still quite closely related to expander graphs, so it is questionable whether one can expect quantum effects in these quantum expanders that do not appear for expander graphs.

In this dissertation, we aim to extend the constructions by Margulis and Harrow in an attempt to create new ways of obtaining examples of quantum expanders. We base our constructions on discrete quantum groups, the quantum versions of discrete groups, which can also have property (T). In Chapter 5 we give a construction of quantum expanders using discrete quantum groups with property (T) based on Harrow, as well as a construction based on Margulis. The current understanding of the representation theory and coideal structure of discrete quantum groups with property (T) is unfortunately not yet sufficient to immediately obtain interesting new examples of quantum expanders. Hopefully, future research will create this understanding and, together with Chapter 5, lead to the discovery of quantum expanders with truly quantum properties.

PREFACE

As I am writing this, I have reached the end of my PhD trajectory. The aim of doing a PhD is to become an independent researcher in all its facets. This involves being able to come up with research questions, answer them, communicate about them and properly judge the implications of the conducted research. Hopefully, I have achieved these skills; I guess that by writing this, I declare that I believe it to be the case. At the end of one's PhD trajectory, a dissertation is written to prove that one has acquired some of the required skills for an independent researcher. In my case, the dissertation lies before you, and it is quite a large text.

I would find it a bit of a shame if earning my degree was the sole purpose of my dissertation. While I agree that collecting your research into one document is a good way to allow a PhD candidate to make their case that they deserve a PhD title, I think it is possible to do better than writing two hundred pages just to prove one's skills. I believe that any academic text is written for the purpose of communicating something to some audience. In the case of a dissertation, the question is: what should be communicated, and to whom? One goal is to communicate or demonstrate the appropriate skills to the PhD committee, as is also required by the TU Delft. However, I believe that my dissertation is served better by making that goal, which is already largely served by my articles, secondary to some new aims of the dissertation. These aims vary wildly per chapter, but I hope that it will allow some parts of the dissertation to actually be useful to some people.

In the introduction, I mainly discuss the physics that underlies the mathematics in this dissertation. While it may help physicists understand what kinds of mathematics appear when studying the topics we will encounter, it is mainly aimed at mathematicians. As mathematicians, it can be very easy to stay in your idealised, mathematical world; I have been guilty of this myself. However, it benefits both mathematical research and the broader scientific community if the physical motivation keeps inspiring new questions and goals. Therefore, the introduction aims to put this background sharply in the mind of any mathematician reading it.

Next, one encounters the preliminaries. Preliminaries serve to set the mathematical stage, to make sure that every reader has the foundation needed to continue. However, in most of the relevant fields for this dissertation, there are already excellent books or good expositions in articles to provide this background. What reader can one have in mind when one writes all of this down again? It turns out that there is a particularity about the work in this dissertation, especially in Chapter 3: there is a strong interplay of both the finite-dimensional and the infinite-dimensional setting. For this reason, the preliminaries are dedicated to highlighting this connection. It contains the general, infinite-dimensional theory while discussing the finite-dimensional version in parallel. I hope this may help people dealing with Tomita-Takesaki theory, or any of the other topics in Chapter 3, in finite dimensions while needing ideas or results that are described in the infinite-dimensional setting. On the other hand, the finite-dimensional setting may also provide great intuition for people learning about Tomita-Takesaki theory in full generality.

Starting with Chapter 3, the dissertation contains new research. This research is available online in published articles or preprints and can therefore already be read by anyone interested. However, this dissertation gives me the opportunity to communicate two things I have not communicated before (in writing). My first two articles, which have been combined to form Chapter 3, are really part of the same story, which I hope to tell here. More importantly, the computer experiments in the first article, [Ver24], have been adapted to discover crucial results needed in the second article, [VW23]. Previously, this approach was not available to the public because in the end the mathematical results in [VW23] stand on their own; they do not rely on computer experiments. However, I believe similar experiments still have much potential in this field, so Chapter 3 and in particular Section 3.2.4 have been written for people who might be interested in doing computer experiments in operator algebras.

Chapters 4 and 5 each stand alone, and therefore they have been directly based on their respective articles. For them, I was unable to formulate new, useful goals to serve. I just hope that the other chapters may prove useful to some people. As a physics master student, I have read one chapter of Ivan Ado's dissertation, who was my master thesis supervisor, many times. If my dissertation proves to be as useful, if it helps out just one person, then I will call it a job well done.

Thank you for reading my dissertation, and please feel free to pick and choose; you are encouraged to do so!

Matthijs Vernooij
Delft, June 2025

1

INTRODUCTION

This dissertation contains a lot of advanced or not so advanced mathematics, depending on whom you ask. The precise mathematical foundations can be found in Chapter 2, and we build upon them in the later chapters. However, it is just as important to know why you are studying something as it is to know precisely what you are studying. Therefore, the introduction is devoted to showing where the mathematical objects of study in this dissertation come from. It is intentionally mathematically imprecise at times because the reader should focus on the physical intuition behind the mathematics.

1.1. QUANTUM

We will encounter quite different research topics in the different chapters, but in the grand scheme of things they are united by the word 'quantum', or more specifically by quantum mechanics and quantum information. Quantum mechanics governs the laws of nature; everything around us plays by its rules (although even physicists do not know how this works in the case of gravity). However, one usually only sees the effects of quantum mechanics on very tiny scales, at the level of atoms and molecules. At these levels, extremely counter-intuitive things can happen. In fact, when one first learns quantum mechanics, one tends to go through the 'forget the physics and just trust the mathematics'-phase. Quantum mechanics is probabilistic, i.e. what happens depends on chance. This in itself is not so problematic, but at certain points it behaves *very differently* from things we know and understand, like a dice. Here, we will talk about what makes quantum so strange and how to talk about it mathematically, but first we need to know how to talk about things like a dice.

1.1.1. THE MATHEMATICS OF A DICE

Systems that are governed by randomness are studied in the mathematical field of probability theory. It is a very large and diverse field that allows one to deal with rather general

This section, like the other sections in this chapter, does not contain precise references. The information here can largely be found in the books [GS18] and [NC10].

and abstract situations. Here, we will avoid this abstractness and solely consider the concrete case of rolling a six-sided dice. When we think of a dice, we think of something that will generate a random number between one and six for us. Mathematically, this means that there is a set Ω of possible outcomes, and in this case

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

We also have a probability distribution p on Ω, i.e. a map from Ω to $[0,1]$ that assigns to each number between one and six the probability that we roll that number. If the dice is honest, then all of these probabilities are $1/6$, and even if the dice is not honest, the probabilities must add up to one. We have now described the numbers one through six and their probabilities, but technically their connection with the dice has not been specified. We will introduce this connection in a somewhat roundabout way, so we can clearly compare it to the quantum case that will come next. When you hold a dice in your hand, it has no value, and it is not yet 'connected' to Ω. As soon as you roll the dice, but before you look at the number of spots, the value of the dice has probability $1/6$ to equal any of the numbers one through six. This is the dice in its initial state. As soon as you look at the number of spots on the dice, you *measure* the dice. You assign to each face an *outcome* from Ω, i.e. the number of spots, and the result of the measurement is the outcome corresponding to the face that is on top. At this point in time, the dice is no longer in the state where each outcome has probability $1/6$. Instead, there is a single outcome that has probability one, and the other outcomes have probability zero.

One may think that this is the end of the story of the dice, but that is not the case. It is only the end of the story of *this particular measurement*. You can look at many different aspects when rolling a dice. For example, you can divide the table into six parts and assign an outcome to each part. The measurement then becomes looking in which part the dice ended up (see Figure 1.1).

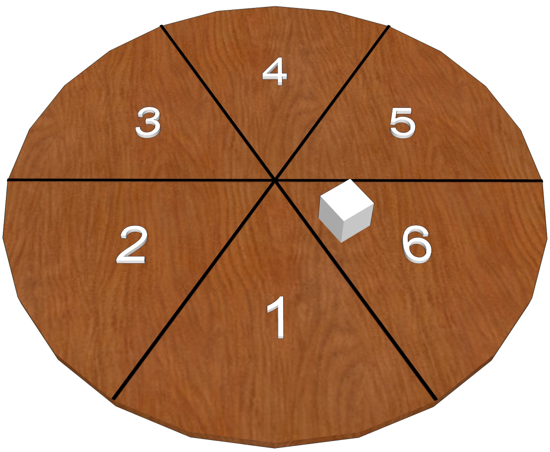


Figure 1.1: A table divided into six parts on which a dice has been rolled. In our measurement, we solely look at the position of the dice on the table, and consequently we do not see the spots on the dice. In this case, the result of the measurement is the outcome 6.

The important takeaway from this example is that these two measurements do not interfere; you can first look at the number of spots on the dice, and it will not affect its position. Conversely, first determining the position does not change the value that was rolled. This means that you can use one throw of the dice to determine both the number of spots and the value associated with its position. It may seem obvious that this should be true—you are only looking, after all—but these things will change when we turn to the quantum world.

1.1.2. THE DOUBLE-SLIT EXPERIMENT

One of the most famous experiments showing that the order of measurements matters is the double-slit experiment for single electrons. The set-up is depicted in Figure 1.2, and the procedure of the experiment is described in its caption. To understand what it tells us, we first need to know the outcomes of two related experiments. First, suppose we would have the experiment in Figure 1.2 with only one slit. One may expect that the distribution you observe is a narrow line as shown in Figure 1.3a, about the width of the slit, but this is not the case. Instead, you see a broader distribution, see Figure 1.3b, which is caused by the wave-like nature of particles.

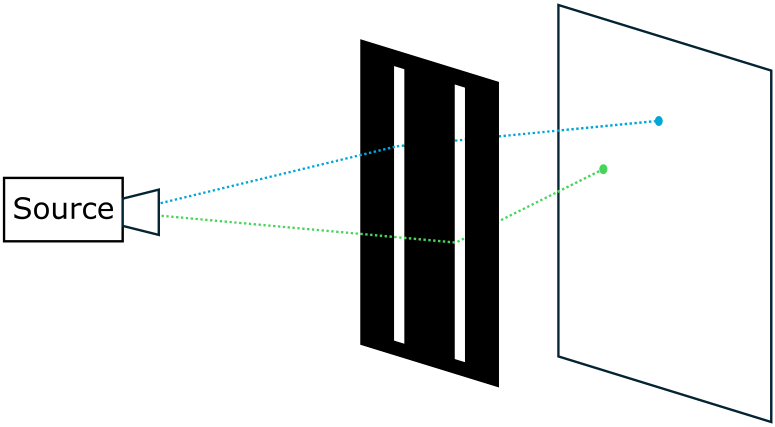


Figure 1.2: The set-up of the double-slit experiment. On the left an electron source emits individual electrons. An emitted electron travels through one of the very narrow slits (corresponding to either a path similar to the blue one or a path similar to the green one). The electron then arrives at a large screen, where the point of impact is measured. Due to a physical phenomenon called diffraction, the path the electron takes after passing a slit is somewhat random, i.e. it does not have to travel straight on through. This has been emphasised for the blue and green trajectories. After repeating the experiment many times, you obtain a distribution of where the electrons tend to hit the detector. In this experiment, one can choose to add a measurement device, determining whether the electron passes through the left or the right slit.

Next, for the double-slit experiment without an additional measurement device, you would then expect to see the two broader, overlapping distributions shown in Figure 1.3c, corresponding to the two slits. After all, it seems natural that there are two options: the electron travels through the left slit or the electron travels through the right slit. The end result should then correspond to the combination of these two possibilities. Once again, nature surprises us, and we get a striped pattern like in Figure 1.3d, where certain locations on the detector that used to be reached for a single slit are no longer reached when the second slit is added. We call this an *interference pattern*. This result is physically baffling, because it implies that an electron that may travel through one slit is influenced by the possibility that it could travel through the other slit as well. Once again, this has to do with the wave-like nature of particles, and it was very strong evidence for quantum theory.

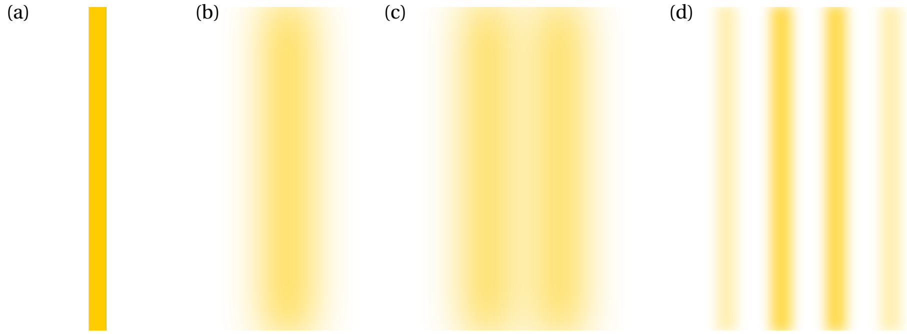


Figure 1.3: This figure is a simplified, schematic illustration of the (naively expected) results one obtains from the experiment in Figure 1.2 and the related experiment with a single slit. Instead of showing dots that depict the detection of single electrons, we show the probability that an electron is measured at a location, which corresponds to the outcome after measuring a very large number of electrons. Figure (a) shows the outcome one would naively expect for a single slit, but the real measurement outcome is shown in Figure (b). The blurring is caused by diffraction, behaviour that is typically associated with waves. When performing the double slit experiment, one would expect to observe two overlapping copies of the outcome for a single slit, which is shown in Figure (c). Instead, performing the experiments yields a striped pattern, like the one shown in Figure (d). It is important to note that there are regions of the detector that would measure electrons for the single slit, but that do not measure electrons when the second slit is added. Lastly, when one adds a measurement device determining whether the electron passes through the left or right slit, one *does* observe the outcome in Figure (c).

For us, the most important takeaway is the next step in the experiment. If we add a measurement device that keeps track of the slit the electron passes through, the interference pattern vanishes and we find the two broader, overlapping distributions in Figure 1.3c we expected initially. In other words, first measuring which slit the electron passes through affects the measurement of the position on the detector. In contrast to the case of the dice, the order of the measurements suddenly matters. Additionally, this means that it is essential to specify which measurement one will perform during a run of an experiment; in the case of the dice, this would amount to saying 'I will roll the dice and look at the number of spots' or 'I will roll the dice and look at the position'. This raises the question whether you are allowed to say that you look at both at the same time. This is a subtle question and the key difference between the so-called *classical* (non-quantum) case and the quantum case. In the classical case, one can always do this, but in the quantum case, we first need a better mathematical description to answer this question.

1.1.3. QUANTUM EXPERIMENTS IN MATHEMATICAL LANGUAGE

We start by introducing the mathematical description of quantum measurements for the dice, and afterwards generalise this to the actual quantum case. For now, we will purely focus on the number of spots on the dice. In this case, we have six possible outcomes, and if the dice is honest, all outcomes have a probability of 1/6. We denote this by

$$\rho = \begin{pmatrix} \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \end{pmatrix},$$

and we call this the *state* of the system. Next, for the measurement, we associate one outcome to each of the matrices

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, P_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, P_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, P_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

If outcome ω_j has been associated to matrix P_j, then the probability that we measure outcome ω_j for the state ρ is $\mathrm{Tr}(P_j\rho)$, where Tr stands for the trace, the sum of the diagonal entries. This description is mathematically equivalent to the one in Section 1.1.1, but it is a better starting point if we want to go to the general quantum case. In this formulation, ρ is an example of a positive (semi-definite) matrix with trace 1, and the P_j are projections that add up to the identity matrix. It turns out that these two conditions are enough to guarantee that the probabilities $\mathrm{Tr}(P_j\rho)$ are always non-negative and that they add up to one, two properties that are essential for a reasonable theory describing measurements. In quantum mechanics, we use the above properties as the foundation of our description of measurements. So, in the quantum case, the state of a system is given by a *density matrix*, a positive matrix with trace one, and a measurement is given by a *projection valued measure* (PVM), a collection of projections that adds up to the identity matrix. The use of projection valued measures corresponds to a property that we are used to in the case of a dice, namely repeatability. If you have seen that you have rolled a six, looking at the dice again will not change this. In general, we also want to be able to describe measurements that are not repeatable, in which case we need to work with a *positive operator valued measure*. All in all, a quantum state and a quantum

1

measurement with outcome set Ω are then described by

$$\text{a state } \rho \in M_n(\mathbb{C}), \text{ i.e. } \rho \geq 0 \text{ and } \mathrm{Tr}(\rho) = 1,$$
$$\text{a POVM } \{P_\omega\}_{\omega\in\Omega} \subset M_n(\mathbb{C}), \text{ i.e. } P_\omega \geq 0 \text{ and } \sum_{\omega\in\Omega} P_\omega = 1_n.$$

In this framework, two measurements can be combined if their POVMs commute. Lastly, we highlight a special class of states, called pure states. These states have only one non-zero eigenvalue, which must then automatically equal 1. In the case of the dice, an example would be a dice that always rolls a five. From a probabilistic point of view, such states may not sound very interesting, as the measurements would be predetermined. But in the quantum case, there are many measurements that are not determined at all. These pure states can be described by giving a unit vector $|\psi\rangle \in \mathbb{C}^n$. The corresponding density matrix is the linear operator $|\chi\rangle \mapsto |\psi\rangle\langle\psi|\chi\rangle$, which we denote by $|\psi\rangle\langle\psi|$. From a physics point of view, pure states correspond to the situation where we fully know the state of the system, and *mixed* (non-pure) states correspond to systems where there is some randomness describing which state we are in.

1.2. CHANGING QUANTUM SYSTEMS

Knowing how to describe a quantum system, the next step is to be able to say something about how it changes over time. If a quantum system is closed, meaning that it does not interact with the environment, the evolution of a quantum system is relatively easy to describe. However, if this is not the case, one may need to make assumptions. One of these assumptions will lead us to quantum Markov semigroups, which are the first research topic in this dissertation.

1.2.1. QUANTUM OPERATIONS

From the mathematical description of a quantum system, we can infer some properties that any physically sensible quantum operation must satisfy. First and foremost, a quantum operation needs to map states to states, as applying a quantum operation to a physical system should result in a valid description of a physical system. Next, states have a convex structure, meaning that from two states ρ_1 and ρ_2 I can obtain the random combination of the two, given by $\lambda_1\rho_1 + \lambda_2\rho_2$ for $\lambda_1 + \lambda_2 = 1$. A quantum operation Φ should also respect this structure, so applying Φ to a random combination of ρ_1 and ρ_2 should be the random combination of Φ applied to the separate states. These physical conditions (initially) translate to the following mathematical conditions on a quantum operation Φ:

(a) Φ is a linear map, i.e. $\Phi(\lambda_1\rho_1 + \lambda_2\rho_2) = \lambda_1\Phi(\rho_1) + \lambda_2\Phi(\rho_2)$.

(b) Φ is *positive*, i.e $\Phi(\rho) \geq 0$ whenever $\rho \geq 0$.

(c) Φ is trace-preserving, i.e. $\mathrm{Tr}(\Phi(\rho)) = \mathrm{Tr}(\rho)$.

The information in this section can largely be found in the books [GS18] and [NC10], together with an initial article on quantum Markov semigroups like [Lin76].

From a mathematical point of view, one may also arrive at these conditions, but it turns out that condition (b) is not quite right. When we require that states are mapped to states, we need to keep in mind that we always act on a part of a bigger system, and even on that bigger system, states should be mapped to states. The mathematical notion corresponding to this is *complete positivity* and is defined in Definition 2.2.1.

Mathematically, for an n-dimensional system (or any finite-dimensional Hilbert space), we have the *state space* of density matrices, the possible states of the system, and the *measurement algebra*, the algebra generated by the POVM elements. When we are describing how a system changes, the sensible viewpoint is to say that the state of the system changes. This is called the *Schrödinger picture*, and it is what we have used so far. However, another (dual) way to look at it is to say that the observables, the things you measure, change. This is the *Heisenberg picture*, and while it may seem unnatural to take this point of view, it can be quite useful mathematically. In this picture, a quantum operation is a *unital* completely positive map, where unital means that it maps the identity matrix to itself. These conditions correspond to the requirement that POVMs are mapped to POVMs.

1.2.2. UNITARY TIME EVOLUTION

The time evolution of a closed quantum system is given by the Schrödinger equation, one of the fundamental results in quantum mechanics. In its general form, it is given by

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle ,$$

where H is the Hamiltonian of the system, the operator measuring the energy of the system. Based on this equation, the time evolution of pure states is implemented by the unitary operator $U_t = e^{-iHt/\hbar}$. In the Heisenberg picture, the time evolution Φ_t is then given by $\Phi_t(x) = U_t^* x U_t$. Just as in the classical case with Newton's second law, the time evolution is deterministic and can be inverted. Mathematically, this means that time evolution forms a group.

In an open quantum system, the situation is not so simple. In an open quantum system, certain parts of the system interact with parts of the *environment*, by which we mean any part of the world or universe that is not included in the system. In describing such a system, the problem is that this propagates; these parts of the environment can interact with other parts of the environment, and so on. This means that one needs to identify a closed supersystem containing the original system to give an exact description of the time evolution. Mathematically, this takes the following form. If $\mathscr{H}$ is the Hilbert space for the open quantum system, then it is contained in a larger Hilbert space $\mathscr{K}$ for a closed quantum system containing the original system. For this larger system we let H be the Hamiltonian on $\mathscr{K}$, and then

$$|\psi^{\mathscr{K}}(t)\rangle = e^{-iHt/\hbar} |\psi^{\mathscr{K}}(0)\rangle .$$

In this description, we are not really describing the time evolution of the open quantum system; we have just moved to a bigger, closed quantum system and describe that. In general, this is the best you can do.

1

1.2.3. QUANTUM MARKOV SEMIGROUPS

In practice, the above approach tends to run into problems. Let $\mathcal{H}$ be the Hilbert space of the system we are interested in. Any closed quantum system containing $\mathcal{H}$ may be very large. Consequently, it may be hard to fully describe the Hamiltonian on that system, or one may not know the state on the full closed quantum system. In these situations, one needs to make physically motivated assumptions to reach a meaningful description. One common assumption is that, while the environment influences the open quantum system, the open quantum system does not (significantly) influence the environment. Typically, this is justified if the environment is large and the typical timescales in the environment are much smaller than in the open quantum system. In this case, any influence of the open quantum system on the environment, i.e. a change caused by interaction with the open quantum system, is quickly spread out over the entire environment. Because the environment is large, the effect of this change is small after being distributed over the entire environment.

Let us briefly give a (non-quantum) example of such a situation. Suppose we have a large pressurised helium tank (the environment) and we intend to inflate balloons (the open quantum system). When you inflate a balloon, helium near the valve flows into the balloon. At this instant, the environment has changed because the pressure near the valve has dropped significantly. If this were to remain the case, it would affect the next balloon we want to inflate. However, in a pressurised tank, the pressure redistributes extremely quickly, so the drop in pressure is spread out over the tank. Moreover, because the tank is large, the resulting drop in pressure is small, so we can safely assume that the tank, our environment, has not changed. Note that it is important to keep in mind the limits of your assumptions; at some point, you may have inflated so many balloons that the pressure changes noticeably.

Here, we do not worry about justifying these assumptions. Instead, we focus on describing systems where they hold. Because the environment does not change, how a system will evolve is purely dependent on its state now, and not on the states in the past. This is called the *Markov* property. Mathematically, this has the following effect. If maps $\Phi_{s\mapsto t}$ describe the time evolution of a system at time s to a system at time t, then

(a) only the difference between t and s is relevant. It does not matter at which point in time the system starts evolving, just how long the system is allowed to evolve. This means that $\Phi_{s\mapsto s+x}(\rho) = \Phi_{t\mapsto t+x}(\rho)$.

(b) predicting how a state ρ will change over time $s+t$ is the same as predicting how it will change over time s, and using that as the basis for your prediction of the evolution for the subsequent time t. This is the case because, after time s, only the state at that time is relevant for the future, not the initial state at the start of the process. This means that $\Phi_{s\mapsto t}(\Phi_{u\mapsto s}(\rho)) = \Phi_{u\mapsto t}(\rho)$.

These two properties tell us that the time evolution has to be a *semigroup*. If all of the individual maps are valid quantum operations, then we call this a quantum Markov semigroup, which is defined precisely in Definition 2.2.11. In Section 2.2.2, we introduce some of the properties quantum Markov semigroups may have, most notably KMS-symmetry. In Chapter 3, based on background knowledge in Sections 2.2.3 and 2.3, we aim to better understand KMS-symmetric quantum Markov semigroups by finding a new way to describe them.

1.3. CERTIFYING QUANTUM DEVICES

We now turn in a somewhat different direction. Having discussed a little bit how to describe the world in terms of quantum mechanics, one may wonder how we know that the world is quantum. Before quantum theory was developed, people used other theories to describe the world. Particularly at the atomic level, experiments started appearing that could not be explained. These experiments were the driving force for people to develop quantum mechanics, and using quantum mechanics people have been able to explain an incredible number of things. Consequently, we have partially written off the old, classical theories because they were falsified, i.e. they did not agree with experiments. This does not mean that we no longer use these classical theories; we consider them as very convenient approximations, but with the knowledge that they will fail in certain circumstances.

Coming back to the question of knowing whether the world is quantum, there is unfortunately no way in physics to conclude that the world must be quantum. (In fact, the description of quantum mechanics given in Section 1.1 is simplified and not sufficient to describe high energetic particles, for example.) We can only check that it corresponds with experiments, and that other theories do not. Therefore, it is better to ask ourselves whether we can prove that the world cannot be classical. Here classical roughly means that the world can be described in a way where do not fundamentally affect each other, like in Section 1.1.1. However, proving this is complicated, because in the approach described before, we are eliminating specific classical theories. In that way, one can never rule out classical theories as a whole. Luckily, one can get around this problem using the framework of non-local games. To use this framework to prove that the world cannot be classical, we have to assume that the theory of special relativity, saying that no information can travel faster than the speed of light, is true. In other words, whenever we say that we conclude that the world cannot be classical, we are actually saying that either the world is not classical, or the world does not obey special relativity.

1.3.1. NON-LOCAL GAMES

In the setting of non-local games, there are typically two players called Alice and Bob who play a cooperative game involving one referee. While playing the game, Alice and Bob are separated and cannot communicate. Alice receives a question x from the referee, and based on this question Alice returns an answer a to the referee. In the same way, Bob receives a question y from the referee and returns an answer b. The referee has a large four-dimensional table with an entry for each combination of questions x and y and answers a and b, and each entry specifies whether Alice and Bob win or lose, playing in this way. This procedure is illustrated in Figure 1.4. Alice and Bob know everything about the game, including the table D and the probability distribution ν the referee uses to choose the questions. This means that they can develop a strategy beforehand.

From the perspective of the referee, running a round of a non-local game is quite easy for the most part. In a non-local game, the questions come from a finite question

The information in this section can largely be found in the review article [ŠB20].

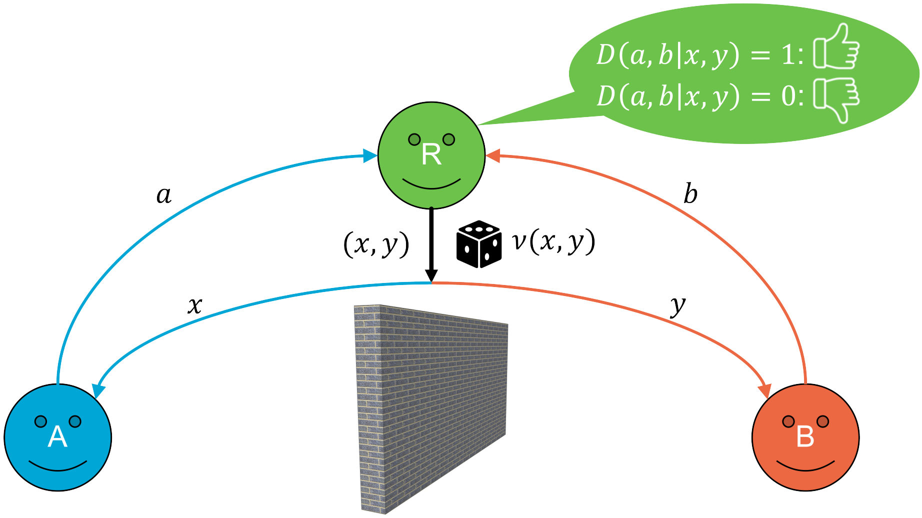


Figure 1.4: A schematic representation of a round of a non-local game. First, the questions x, y are selected and sent to Alice and Bob, then Alice and Bob choose appropriate answers a, b without communicating and finally the referee announces the result based on the decision table D.

set, so selecting random questions is straightforward. If we represent the questions and answers as natural numbers, bit sequences or anything else that is easy to work with, handing out the questions and receiving the answers do not pose any problems either, nor does checking whether they win. The only complication is how to guarantee that they cannot communicate. If we want to make claims that hold for any classical theory, things like putting Alice and Bob in different rooms do not suffice.

The solution to this problem lies in special relativity. If we can create a space-like separation between the question-and-answer exchange with Alice and the exchange with Bob, then special relativity guarantees that Alice and Bob cannot communicate during the game. Here, space-like separation means that information would need to travel faster than light to travel from Alice to Bob or vice versa during the question-and-answer windows. While it is experimentally challenging to create this separation, it is possible. This means that all assumptions of a non-local game can be valid in experiments, showing that it is a good mathematical model to study.

Classically, there is always an optimal strategy in which Alice and Bob deterministically choose their answers based on the questions they receive. On the other hand, optimal quantum strategies can be much more complicated. In the quantum case, Alice and Bob prepare a quantum state beforehand and during the game choose a measurement to perform based on the question they receive. The outcome of the measurement then determines their answer. This is described in more detail in Section 4.2.1.

An example of a non-local game that behaves differently according to quantum theory or classical theory is the so-called CHSH game, named after Clauser, Horne, Shimony and Holt. It can be described as follows:

- There are two questions, which take the form of cards: a blue card and an orange card.

- There are also two answers, once again in the form of cards: a red card and a green card.
- The questions are selected uniformly at random, so ν is the uniform distribution.
- There are two conditions Alice and Bob need to satisfy to win a round of the game:
  - If both Alice and Bob receive blue questions, they need to answer with different coloured cards. In other words, in this case the referee needs to receive one red and one green answer.
  - If at least one of them receives an orange question, they need to return the same answers, i.e. two red answers or two green answers.

Classically, the probability that Alice and Bob win a round of the game is at most 75%. One of the many ways to achieve this is if Alice and Bob always return green. However, according to quantum theory, it is possible to achieve a winning probability of approximately 85%. This means that the CHSH game exhibits *quantum advantage*, which is used in the quantum information field whenever a quantum device can do something 'better' than any classical device could. In most cases, the existence of quantum advantage is interesting because it indicates possible use cases for quantum computers or other quantum devices. In our case, the perspective is slightly different. To us, quantum advantage means that we have a way to tell apart certain quantum devices from classical devices. Indeed, whenever Alice and Bob achieve a winning probability higher than 75% in the CHSH game, they must be doing something quantum. This suggests the following experiment: set up a system that takes the roles of Alice and Bob and performs a strategy that should win with probability greater than 75%. Next, take the role of the referee and run the game. If you measure a winning probability higher than 75% while enforcing the space-like separation, you have shown that the world cannot be classical. This experiment has been performed, showing that a winning probability above 75% can indeed be achieved, and was one of the experiments for which the Nobel Prize in physics was awarded in 2022 [Nob].

1.3.2. ROBUST SELF-TESTING

This may seem like the end of the story. After all, the goal of proving that the world cannot be classical has been achieved. However, it is possible to extract even more information about the strategy Alice and Bob use from playing the CHSH game. It turns out that there is a *unique* quantum strategy $\mathscr{S}$ that achieves the optimal winning probability, at least up to an appropriate notion of being the same strategy (see Definition 4.2.1 about local dilations). Consequently, whenever we observe that Alice and Bob achieve the optimal winning probability for the CHSH game, which is $1/2+\sqrt{2}/4 \approx 85\%$, we can conclude that Alice and Bob must be using this specific strategy $\mathscr{S}$. In other words, just running this game as a referee can allow you to conclude that Alice and Bob have quantum devices implementing specific protocols.

It is good to note that this does not mean that we can in general determine or measure the employed strategy by playing the game. We cannot draw such conclusions for other strategies. Instead, we should view the game as a validation or certification procedure for the specific optimal strategy $\mathscr{S}$. If Alice and Bob claim that they can implement the strategy $\mathscr{S}$, then we want to use the game to verify their claim by seeing it as a test that they can only pass using the specific strategy $\mathscr{S}$.

At this point, real life gets in the way of the ideal, mathematical picture. There is no way to experimentally determine with any amount of statistical confidence that the winning probability of the employed strategy *exactly* equals $1/2+\sqrt{2}/4$. Moreover, there are small imperfections in any device that one creates, so we cannot expect Alice and Bob to perfectly implement the strategy $\mathscr{S}$. If we want to use non-local games to certify quantum devices in real life, we need to change the above to an approximate formulation. In the end, one arrives at the following two conditions for a non-local game G to be usable in quantum device certification:

- The game G needs to have an optimal strategy $\tilde{\mathscr{S}}$ with winning probability ω.
- Whenever a strategy $\mathscr{S}$ achieves winning probability $\omega-\varepsilon$, the distance between $\mathscr{S}$ and $\tilde{\mathscr{S}}$ must be at most $\delta(\varepsilon)$, and $\delta(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Whenever these two conditions hold, we call G a *robust self-test* (see Definition 4.2.3). Here, robust indicates that this holds for $\varepsilon>0$; the above condition purely for $\varepsilon=0$ is called self-testing.

From any non-local game that is a robust self-test, we can make a quantum device certification protocol. We first choose a threshold δ specifying the distance from the optimal strategy we want to guarantee. Next, we calculate the ε such that $\delta(\varepsilon)=\delta$, i.e. the deviation in the winning probability that still guarantees that the distance in strategies is at most δ. The following step is to use a statistics calculation to decide how many rounds of the game need to be played and which number of rounds need to be won. This depends on the desired statistical confidence on the one hand, and the preferred likelihood of passing the test when employing a strategy with winning probability between $\omega-\varepsilon$ and ω. Increasing the number of rounds improves both, while changing the number of rounds to be won results in a trade-off between the statistical confidence and the likelihood of passing the test for strategies that one would hope to pass the test.

In Chapter 4, we aim to make it easier to prove that non-local games are robust self-tests. We develop a theoretical framework for describing robust self-testing when the referee has additional information about the strategies being employed. It can be significantly easier to prove that a game is a robust self-test when the employed strategies are guaranteed to have specific properties. We then devote most of the chapter to proving that for a large class of games, the robust self-testing results remain valid when other strategies without these specific properties are allowed.

1.4. MIXING STATES

Previously, our focus has been on understanding quantum systems and recognising specific quantum systems, and our motivations really originated in quantum mechanics. Here, our motivation is somewhat different. We will encounter certain classical objects, called expander graphs, which have proven to be very useful in many different

The information here can largely be found in Sections 5.2.1 and 5.2.2, sometimes consulting the references therein.

fields of both pure and applied mathematics. In fact, they have even been used in quantum information theory. However, there also exists a quantum version of these objects, called quantum expanders. Unfortunately, their usefulness has been rather limited so far, even in quantum information theory. In Chapter 5, we aim to provide a pathway to new examples, in the hope that they will help to discover new uses. Here, we focus on characterising them from a physical point of view.

1.4.1. GRAPHS

The story starts with graphs, which are collections of points (vertices) and lines (edges). They can be used to represent anything with some sort of network structure. Any graph consists of a set of vertices V and a set of edges between these vertices E. Edges connect distinct vertices, and between each pair of vertices, there can be at most one edge. An example of a graph is given in Figure 1.5. Graphs are often used to study the connectivity of a network, and the first notion one encounters is *connectedness*. A graph is connected if one can reach any vertex from any other vertex by following the edges.

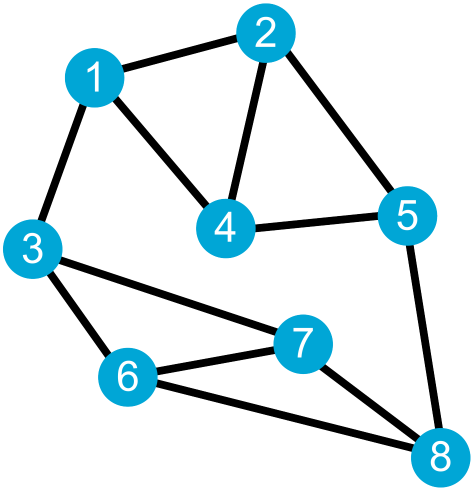


Figure 1.5: A picture of a graph. Its vertices are the blue circles numbered one through eight, and the edges are the black lines connecting them. This graph is connected and 3-regular.

We call the number of edges connected to a vertex the *degree* of that vertex, and we call a graph d-regular if any vertex has degree d. Lastly, it is important to know that there are multiple ways to represent graphs. One is in terms of a picture (see Figure 1.5), another way is to specify the vertex and edge sets V and E, but the way we need most is using the so-called adjacency matrix. In this approach, we number the vertices by 1 through n and create an $n \times n$ matrix. The i, j-th entry of the matrix is 1 if i and j are connected by an edge and 0 if they are not. The adjacency matrix A of the graph in Figure

1.5 is given by

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

1.4.2. EXPANDER GRAPHS

We are interested in particular families of graphs called expander graphs. These graphs can be seen as extremely well-connected graphs, a notion captured by the *edge expansion constant*. If you divide a graph G into two parts, then the edge expansion constant $h(G)$ of that graph gives a lower bound on the fraction of the possible connections between the parts that are present in the graph. For example, the graph in Figure 1.5 has expansion constant 1/8: if we split the graph into the parts $P_1 = \{1, 2, 4, 5\}$ and $P_2 = \{3, 6, 7, 8\}$, then 2 out of the 16 possible edges are present. Any other split would give a better fraction. In essence, the edge expansion constant is small if there are large bottlenecks in the system. If every vertex in P_1 were to attempt to communicate with every vertex in P_2, then the two edges connecting P_1 and P_2 would need to facilitate sixteen individual communications, so one edge would need to facilitate at least $8 = 1/h(G)$ communications.

Naturally, the best way to ensure that the edge expansion constant is large is to connect every vertex to every other vertex, in which case the expansion constant is always one. However, if one were to build a network in this way, then adding a new person to the network would become more and more expensive, the larger the network is, since the new person needs to get an individual connection to each other person in the network. Instead, we want to consider network layouts where the cost per person does not depend on the size of the system. In other words, we want to let each person be connected to d other persons for some fixed number d. From the graph perspective, this means that we are talking about d-regular graphs. From a network perspective, the goal of expander graphs is to find a family of layouts for larger and larger networks such that

- the cost, i.e. the number of connections, per node in the network is constant.
- the severity of the bottlenecks, i.e. the edge expansion constant of the network, is constant.

Mathematically, this means finding a family of d-regular graphs G_i with increasing size such that $h(G_i) \geq c$ for some $c > 0$. We call such a family an *expander family* and an element of such a family an *expander graph*.

Another interesting property of expander graphs is that they are very strongly mixing. If you start at any point in an expander graph and walk along a random edge at each time step, then exponentially quickly you could be anywhere in the graph with equal probability. It is related to a property of the adjacency matrix A of the graph: whenever you have a vector p of probabilities where someone could be on a graph, then $(A/d)p$ is the vector of probabilities after one step of walking along an edge. Consequently, for an expander graph, the mixing property translates to the fact that $(A/d)^n p \to (1/n, \ldots, 1/n)$ exponentially quickly. This is the property that is most readily recognisable in the quantum picture.

1.4.3. QUANTUM EXPANDERS

In the situation from before, the map $p \mapsto (A/d)p$ is a quantum operation, but then in a classical setting. This raises the question whether we can also have expanders in a quantum setting, and, indeed, we can. Where we required the graphs to be d-regular graphs in the classical case, we need the quantum operations to be unital trace-preserving completely positive maps, which we call *quantum bistochastic maps*. If $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is such a map, then there exist matrices $X_1, \ldots, X_d$ such that

$$\Phi(\rho) = \sum_{j=1}^{d} X_j^* \rho X_j,$$

and we call the least d for which this is possible the *degree* of Φ. This is the analogue of the d-regularity of the graphs.

While a quantum expander family can be defined along the same lines as a classical expander family (see Definition 5.2.16), we will stick to the intuition of a mixing property here. The quantum expansion constant of a quantum bistochastic map Φ characterises how quickly $\Phi^n(\rho)$ converges to $1_n/n$. We want this convergence to be exponential, and the rate of the exponential is linked to the quantum expansion constant. As in the classical case, a quantum expander family consists of a family of quantum bistochastic maps with degree d on larger and larger matrices such that these convergence rates are bounded away from zero. Physically, having access to a quantum expander family means that you can efficiently convert any state into a fully random state (called the maximally mixed state), and the efficiency does not depend on the size of your quantum system.

So far, the intuitive understanding of quantum expanders from a physical point of view is much more limited than the understanding of the other topics in this dissertation. While we do not directly address this issue, the hope is that Chapter 5 can contribute to this understanding indirectly. There, we develop a new approach to construct examples of quantum expanders from other mathematical objects called quantum groups. Unfortunately, this does not immediately provide us with interesting new examples, as more work needs to be done to understand the properties of quantum groups relevant for this construction (see Section 5.6).

2

Preliminaries

Mathematics is like the Mauna Kea [Gui], the tallest mountain in the world when viewed from its base, deep in the Pacific Ocean. This dissertation aims to be a few new stones on the slope of this mountain, supported by countless tons of rock. Most of the rock is hidden in the ocean, like the mathematics we assume to be known and do not even mention any more. Above the ocean, the surface of the mountain consists of the definitions and theorems we build upon. Inside, one can find all the proofs that support these results. In these preliminaries we will not dig into Mauna Kea, and we will only discuss the tiny part of the surface our stones stand upon.

On our side of the mountain, we can see the basic theory of functional analysis and C*-algebras lying just below the surface of the ocean. If one does not yet feel comfortable with this topic, I recommend 'A course in functional analysis' by Conway [Con10] and chapters 1-3 of 'C*-algebras and operator theory' by Murphy [Mur90]. Below the waves the basics of quantum information theory are also visible. The standard work in this direction is 'Quantum Information and Computing' by Nielsen and Chuang [NC10, Chapters 1, 2, 8, 9 and 11], although I am also partial to the notes on quantum information theory by Ozols and Walter [OW23], which are more focused on our specific needs.

Walking up the mountain from the sea, we first encounter a large slope dealing with von Neumann algebras, which provides the language and many important tools we use for the rest of the dissertation. Next, we take a small detour through the vast field of quantum information theory, which we need for Chapters 3 and 5. From these foundations, we start to climb the three ridges we aim to place our rocks upon. We first go over the quantum Markov semigroups and derivations (Sections 2.2.2, 2.2.3 and 2.3, Chapter 3), followed by non-local games (Chapter 4) and ending with quantum expanders and quantum groups (Chapter 5). Along the way, we put down our rocks. In principle, the preliminaries in these ridges are all relevant only for their corresponding chapters. The choice has been made to put Sections 2.2.2, 2.2.3 and 2.3 in the preliminaries instead of Chapter 3, because they strongly benefit from the 'blue box'-approach as described in the following section.

2

2.1. Von Neumann algebras

The study of C*-algebras can be called noncommutative topology because every commutative C*-algebra corresponds to the continuous functions on a topological space by Gelfand duality [Con10, Theorem VIII.2.1]. Similarly, there is a class of C*-algebras such that every commutative C*-algebra in that class corresponds to measurable functions on a reasonable measure space [Mur90, Theorem 4.4.4]. These algebras are called von Neumann algebras and describe what can be measured or 'observed' in a quantum system. There are many books that treat von Neumann algebras on an introductory level; we will follow the book 'C*-algebras and operator theory' by Murphy [Mur90] when possible. Reasonably quickly, more advanced literature will be necessary, and the golden standard in von Neumann algebras is Takesaki's three-volume work on operator algebras [Tak02; Tak03a; Tak03b].

Following these books, we will discuss several topics concerning von Neumann algebras. Treating these topics for infinite-dimensional algebras involves quite some technical details. On a first read, this can obscure the intuition behind these concepts. More importantly, large parts of quantum information theory is described in terms of finite-dimensional von Neumann algebras. To make the background knowledge required for the finite-dimensional parts of this dissertation more accessible and to provide context and intuition for the general theory, most of the theory on von Neumann algebras will be separately formulated in the finite-dimensional case. This can be found in blue boxes scattered throughout the preliminaries, such as in the following example. The aim is that one can acquire the necessary background knowledge for the finite-dimensional results in this thesis by reading the blue boxes.

Finite-dimensional version of von Neumann algebra theory

Boxes like this one contain finite-dimensional versions of general von Neumann algebra definitions or theorems. We impose the standing assumption that anything in such a box is finite dimensional without further comment.

2.1.1. Foundations

Definition 2.1.1 ([Mur90, Section 4.1], [Tak02, Section II.3])**.** Let $\mathscr{H}$ be a Hilbert space, and let $\mathscr{M}$ be a subset of the bounded operators on $\mathscr{H}$, $B(\mathscr{H})$. The *commutant* of $\mathscr{M}$, denoted by $\mathscr{M}'$, is defined as the set of operators in $B(\mathscr{H})$ that commute with all operators in $\mathscr{M}$, i.e.

$$\mathscr{M}' = \{x \in B(\mathscr{H}) \mid \forall y \in \mathscr{M} \; xy = yx\}.$$

Starting with a set $\mathscr{M}$, we can take the commutant more than once, but after two times no new objects appear; we have $\mathscr{M} \subset \mathscr{M}''$ and $\mathscr{M}' = \mathscr{M}'''$. We can now define a von Neumann algebra in terms of the double commutant.

Definition 2.1.2 ([Tak02, Definition II.3.2])**.** A *von Neumann algebra* on a Hilbert space $\mathscr{H}$ is a $*$-subalgebra $\mathscr{M}$ of $B(\mathscr{H})$ such that $\mathscr{M} = \mathscr{M}''$.

Note that this is a purely algebraic definition. However, the key point of von Neumann algebras is that the condition that $\mathscr{M} = \mathscr{M}''$ is strongly linked to the topological properties of $\mathscr{M}$.

Definition 2.1.3 ([Mur90, Chapter 4], [Tak02, Definitions II.2.1-II.2.3])**.** Let $\mathscr{H}$ be a Hilbert space. On $B(\mathscr{H})$ we define the following locally convex topologies:

- The *weak* topology is the topology generated by the seminorms

$$\{x \mapsto |\langle \xi, x\eta \rangle| \mid \xi, \eta \in \mathscr{H}\}.$$

- The σ*-weak* topology is the topology generated by the seminorms

$$\{x \mapsto |\sum_i \langle \xi_i, x\eta_i \rangle| \mid \forall i \in \mathbb{N}\ \xi_i,\ \eta_i \in \mathscr{H}, \sum_i \|\xi_i\|^2, \sum_i \|\eta_i\|^2 < \infty\}.$$

- The *strong* topology is the topology generated by the seminorms

$$\{x \mapsto \|x\xi\| \mid \xi \in \mathscr{H}\}.$$

Remark 2.1.4. A von Neumann algebra $\mathscr{M}$ has a unique predual $\mathscr{M}_*$ [Tak02, Theorem 3.5, Corollary 3.9], so it has a weak* topology [Con10, Definition V.1.1]. This topology equals the σ-weak topology [Tak02, Theorem II.2.6(iii)]. We will use these names interchangeably.

All of these topologies are coarser than the usual norm topology on $B(\mathscr{H})$, and the weak topology is coarser than the other two topologies. By the famous double commutant theorem of von Neumann, any of the topologies given above could also have been used to define von Neumann algebras.

Theorem 2.1.5 ([Mur90, Theorem 4.1.5], [Tak02, Theorem II.3.9])**.** *Let $\mathscr{H}$ be a Hilbert space and let $\mathscr{M}$ be a $*$-subalgebra of $B(\mathscr{H})$ containing the identity. The following are equivalent:*

- *$\mathscr{M}$ is a von Neumann algebra.*
- *$\mathscr{M}$ is strongly closed.*
- *$\mathscr{M}$ is σ-weakly closed.*
- *$\mathscr{M}$ is weakly closed.*

Finite-dimensional theory: von Neumann algebras

All of the topologies defined in Definition 2.1.3 are equal to the norm topology. Moreover, any $*$-subalgebra of $B(\mathscr{H})$ is closed in this topology, so any $*$-subalgebra of $B(\mathscr{H})$ is a von Neumann algebra.

We round off this section with a few definitions and the GNS construction. The GNS construction is not limited to von Neumann algebras. Indeed, it can be considered as assumed knowledge, since it is covered in ‘A course in functional analysis’ [Con10, Section IX.5]. However, it nicely sets the stage for the next section on Tomita-Takesaki theory, which is why we include it here.

Definition 2.1.6 ([Tak02, Definition III.2.15]). A norm-continuous linear map π of a von Neumann algebra into another von Neumann algebra is called *normal* if it is σ-weakly continuous.

Definition 2.1.7 ([Tak02, Definition I.9.4], [Mur90, Section 5.1]). Let $\mathcal{M} \subset B(\mathcal{H})$ be a von Neumann algebra. A *state* is a positive linear functional $\varphi : \mathcal{M} \to \mathbb{C}$ satisfying $\varphi(1) = 1$. A state φ is automatically norm-continuous. It is called *faithful* if $\forall x \in \mathcal{M} : \varphi(x^* x) = 0 \implies x = 0$ and *tracial* if $\forall x, y \in \mathcal{M} : \varphi(xy) = \varphi(yx)$. A state φ is called *pure* if for any positive linear functional f such that $f \leq \varphi$, there exists a number $c \in [0,1]$ such that $f = c\varphi$. A state that is not pure is called a *mixed* state. Any unit vector $\eta \in \mathcal{H}$ defines a *vector state* on $\mathcal{M}$ by $x \mapsto \langle \eta, x\eta \rangle$. A vector state is always normal.

Remark 2.1.8. In physics, it is common to use the bra-ket notation for vector states. In this case, $|\psi\rangle \in \mathcal{H}$ denotes a unit vector, and $\langle\psi|$ denotes the linear functional on $\mathcal{H}$ formally given by $\eta \mapsto \langle \xi, \eta \rangle$ if $\xi = |\psi\rangle$. One will never encounter the notation $\langle |\psi\rangle, \eta \rangle$. In this way, $|\psi\rangle\langle\varphi|$ becomes an element of $B(\mathcal{H})$. It is common to call $|\psi\rangle$ a state, even though mathematically the map $x \mapsto \langle\psi| x |\psi\rangle$ is the state.

Sometimes, one needs to consider what one may informally call unbounded states. This notion is made precise in the definition of a *weight*. In this dissertation, we will only work with states and we will therefore only formulate relevant results in the literature for states. However, some of these results have been proven for general weights. To do these results justice, we will remark in such cases that a more general result for (certain types of) weights holds.

Definition 2.1.9 ([Tak03a, Definition VII.1.1], [Tak02, Definition V.2.1]). Let $\mathcal{M} \subset B(\mathcal{H})$ be a von Neumann algebra. A *weight* is a map $\varphi : \mathcal{M}_+ \to [0, \infty]$ satisfying

$$\forall x, y \in \mathcal{M}_+, \lambda \in \mathbb{R}_+ : \varphi(x + y) = \varphi(x) + \varphi(y), \varphi(\lambda x) = \lambda\varphi(x).$$

A weight φ is called *semi-finite* if the set $\{x \in \mathcal{M}_+ \mid \varphi(x) < \infty\}$ generates $\mathcal{M}$, *faithful* if $\forall x \in \mathcal{M}_+ : \varphi(x) = 0 \implies x = 0$, *normal* if $\varphi(\sup x_i) = \sup \varphi(x_i)$ for every bounded increasing net $\{x_i\}$ in $\mathcal{M}_+$ and *tracial* if $\forall x \in \mathcal{M} : \varphi(x^* x) = \varphi(x x^*)$. A tracial weight is also called a *trace*.

Definition 2.1.10 ([Mur90, Section 6.2]). A von Neumann algebra $\mathcal{M} \subset B(\mathcal{H})$ is called a *factor* if $\mathcal{M} \cap \mathcal{M}' = \mathbb{C} \cdot 1$.

Two of the key properties of von Neumann algebras is that they can be decomposed into factors, and that factors can be classified into types [Tak02, Chapter V], [Tak03a, Chapter XII]. We will not need the specifics here, but record the following partial classification.

Theorem 2.1.11 ([Tak02, Chapter V]). *A factor is of type I if it is isomorphic to $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Otherwise, it is of type II if it admits a faithful normal semi-finite trace and of type III if it does not.*

We wrap up this section with the GNS construction for normal states. A more general version for normal semi-finite weights also exists. In the next subsection, we will study the Hilbert space and representation obtained in the GNS construction more closely for faithful states. A first bit of intuition can already be gained from the next blue box.

Theorem 2.1.12 ([Con10, Gelfand-Naimark-Segal Construction VIII.5.14], [Tak03a, Proposition VII.1.4]). *Let $\mathcal{M}$ be a von Neumann algebra.*

(a) *If φ is a normal state on $\mathcal{M}$, then we can construct a normal representation $\pi_\varphi : \mathcal{M} \to B(\mathcal{H}_\varphi)$ with cyclic vector e such that $\forall x \in \mathcal{M} : \varphi(x) = \langle e, \pi_\varphi(x)e\rangle$.*

(b) *Let $\pi : \mathcal{M} \to B(\mathcal{H})$ be a normal representation with cyclic vector e and let φ be the vector state corresponding to e. Then the representation constructed in (a) is unitarily equivalent to π.*

Finite-dimensional theory: factors, states and the GNS construction

- A von Neumann algebra $\mathcal{M}$ is a factor if and only if it is isomorphic to $B(\mathcal{H})$ for some Hilbert space H.
- Let $\mathcal{M} \subset B(\mathcal{H})$ be a von Neumann algebra. Any state φ corresponds uniquely to a positive element $d_\varphi \in \mathcal{M}$ with $\mathrm{Tr}(d_\varphi) = 1$ via $\varphi(x) = \mathrm{Tr}(d_\varphi x)$ for all $x \in \mathcal{M}$. Here Tr is the trace on $B(\mathcal{H})$. We call d_φ the density matrix corresponding to φ, and we sometimes identify d_φ and φ. The state is faithful if d_φ is invertible and tracial if $d_\varphi \in \mathcal{M}'$. If $\mathcal{M}$ is a factor, then a state is pure if and only if it is a vector state if and only if d_φ has rank one.
- Let φ be a state on a von Neumann algebra $\mathcal{M} \subset B(\mathcal{H})$. The Hilbert space $\mathcal{H}_\varphi$ obtained in the GNS construction is $\mathcal{M}/\ker(\varphi)$ with the inner product $\langle x, y\rangle = \varphi(x^* y)$ for $x,\ y \in \mathcal{M}$. The representation is given by left multiplication, and in this case, the cyclic vector is 1. If φ is faithful, there is a second, more natural way to view the GNS construction. From this viewpoint, the Hilbert space is $\mathcal{M}$ with the Hilbert-Schmidt inner product, and the cyclic vector is $d_\varphi^{1/2}$.

2.1.2. TOMITA-TAKESAKI THEORY

We will now turn to Tomita-Takesaki theory. This is a large and quite technical part of the theory of von Neumann algebras. In a sense, one can see it as the precise formulation and justification of concepts and operations that are easy to write down in terms of invertible density matrices. Naturally, this means that the blue boxes should help to give intuition for what these constructions are trying to achieve. In Tomita-Takesaki theory, many important unbounded operators appear. Their theory is treated nicely in [Con10, Chapter X] and [SZ19, Chapter 9] in case one needs to refresh their memory.

Throughout this subsection, we assume that $\mathcal{M}$ is a von Neumann algebra with faithful normal state φ.

HILBERT ALGEBRAS

Definition 2.1.13 ([Tak03a, Definition VI.1.1]). An algebra $\mathcal{A}$ with involution $\xi \mapsto \xi^\sharp$ (resp. $\xi \mapsto \xi^\flat$) is called a *left* (resp. *right*) *Hilbert algebra* if $\mathcal{A}$ admits an inner product satisfying the following conditions:

(a) Each fixed $\xi \in \mathscr{A}$ gives rise to a bounded operator $\pi_\ell(\xi) : \eta \in \mathscr{A} \mapsto \xi\eta \in \mathscr{A}$ (resp. $\pi_r(\xi) : \eta \in \mathscr{A} \mapsto \eta\xi \in \mathscr{A}$ by multiplying from the left (resp. right).

(b) For all $\xi, \eta, \zeta \in \mathscr{A}$ we have $\langle \xi\eta, \zeta \rangle = \langle \eta, \xi^\sharp \zeta \rangle$ (resp. $\langle \xi\eta, \zeta \rangle = \langle \xi, \zeta\eta^\flat \rangle$).

(c) The involution $\sharp$ (resp. $\flat$) is preclosed.

(d) The subalgebra $\mathscr{A}^2 = \operatorname{span}\{\xi\eta \mid \xi, \eta \in \mathscr{A}\}$ is dense in $\mathscr{A}$ with respect to the inner product.

If $\mathscr{M}$ is a von Neumann algebra with faithful normal state φ, then $\mathscr{M}$ is a left Hilbert algebra with its usual involution and inner product $\langle x, y \rangle = \varphi(x^* y)$. A similar construction is also possible for faithful semi-finite normal weights. Throughout this subsection, we will intersperse general results in terms of Hilbert algebras and descriptions for Hilbert algebras coming from von Neumann algebras with faithful normal states. We will not introduce more notation than necessary. Sometimes this causes us to formulate results purely for the faithful normal state case, when the result itself is in fact more general. The reader will be notified if this happens.

Let $\mathscr{A}$ be a fixed left Hilbert algebra. We will now construct the key objects in Tomita-Takesaki theory. This is treated in more detail in [Tak03a, Section VI.1]. Let $\mathscr{H}$ denote the completion of $\mathscr{A}$. To each $\xi \in \mathscr{A}$ corresponds a bounded operator $\pi_\ell(\xi) \in B(\mathscr{H})$, and the map $\xi \mapsto \pi_\ell(\xi)$ is a $*$-representation. The von Neumann algebra generated by $\pi_\ell(\mathscr{A})$ is called the *left von Neumann algebra*, and it is denoted by $\mathscr{R}_\ell(\mathscr{A})$. If $\mathscr{A}$ is a right Hilbert algebra instead, then to each $\eta \in \mathscr{A}$ corresponds a bounded operator $\pi_r(\eta) \in B(\mathscr{H})$, but this time, the corresponding map $\eta \mapsto \pi_r(\eta)$ is an anti $*$-representation in the sense that $\pi_r(\eta_1\eta_2) = \pi_r(\eta_2)\pi_r(\eta_1)$. In this case the von Neumann algebra generated by $\pi_r(\mathscr{A})$ is called the *right von Neumann algebra*, denoted by $\mathscr{R}_r(\mathscr{A})$.

Next, we return to the left Hilbert algebra case and consider the involution $\xi \mapsto \xi^\sharp$ and denote this by S_0 when viewed as a possibly unbounded antilinear operator on $\mathscr{H}$. This operator is preclosed by assumption. We denote the closure of S_0 and its domain by S and $D^\sharp$. From this, we can deduce several statements, which are captured in the next lemma.

Lemma 2.1.14 ([Tak03a, Lemma VI.1.15]). *We have the following results:*

(a) $S = S^{-1}$.

(b) There exists an anti-linear densely defined closed operator F with domain $D^\flat$ such that

(i) $D^\flat = \{\eta \in \mathscr{H} \mid \xi \in D^\sharp \mapsto \langle \eta, S\xi \rangle$ *is bounded*$\}$,

(ii) For all $\xi \in D^\sharp$ and $\eta \in D^\flat$ we have $\langle S\xi, \eta \rangle = \langle F\eta, \xi \rangle$.

(c) $F = F^{-1}$.

(d) $\Delta = FS$ is a linear positive non-singular self-adjoint operator such that the domain of $\Delta^{1/2}$, denoted $\operatorname{dom}(\Delta^{1/2})$*, equals $D^\sharp$.*

(e) There exists an anti-linear isometry J on $\mathscr{H}$ satisfying the following:

(i) *For all* $\xi, \eta \in \mathscr{H}$ *we have* $\langle J\xi, J\eta\rangle = \langle \eta, \xi\rangle$.

(ii) $J = J^{-1}$.

(iii) $J\Delta J = \Delta^{-1}$.

(iv) $S = J\Delta^{1/2} = \Delta^{-1/2}J$.

(v) $F = J\Delta^{-1/2} = \Delta^{1/2}J$.

(f) J *and* Δ *are uniquely determined by the property (e.iv) and* $\mathrm{dom}(\Delta^{1/2}) = D^\sharp$.

We are particularly interested in the operators J and Δ above.

Definition 2.1.15 ([Tak03a, Definition VI.1.6])**.** The operators Δ and J are called the *modular operator* and the *modular conjugation* of the left Hilbert algebra $\mathscr{A}$, respectively.

Construction 2.1.16 ([SZ19, Section 10.6])**.** Given a von Neumann algebra $\mathscr{M}$ with faithful normal state φ, we can use the GNS construction to obtain a cyclic and separating vector in a Hilbert space $\mathscr{H}$. Abusing notation a bit, we denote this vector by $\varphi^{1/2}$, which is inspired by the finite-dimensional case. Inside $\mathscr{H}$, we have the subspace $\mathscr{A} = \mathscr{M}\varphi^{1/2}$. If we endow this space with the $*$-algebra structure of $\mathscr{M}$, meaning that

$$\forall x, y \in \mathscr{M} : (x\varphi^{1/2})(y\varphi^{1/2}) = xy\varphi^{1/2} \text{ and } (x\varphi^{1/2})^\sharp = x^*\varphi^{1/2},$$

then $\mathscr{A}$ becomes a left Hilbert algebra with $\mathscr{R}_\ell(\mathscr{A}) = \mathscr{M}$. We can now take the commutant $\mathscr{M}'$ inside $B(\mathscr{H})$ and construct from this a right Hilbert algebra $\mathscr{A}' = \mathscr{M}'\varphi^{1/2}$ with

$$\forall x', y' \in \mathscr{M}' : (x'\varphi^{1/2})(y'\varphi^{1/2}) = y'x'\varphi^{1/2} \text{ and } (x'\varphi^{1/2})^\flat = x'^*\varphi^{1/2}.$$

The structure of $\mathscr{A}'$ and its relation with $\mathscr{A}$ follow from the following theorem.

Theorem 2.1.17 ([Tak03a, Lemma VI.1.14, Theorem VI.1.19], [SZ19, Section 10.6])**.** *Let* $\mathscr{A}$ *be a left Hilbert algebra with associated modular operator* Δ *and modular conjugation* J. *Then we have for all* $t \in \mathbb{R}$ *that*

$$\begin{aligned} \mathscr{R}_\ell(\mathscr{A})' &= \mathscr{R}_r(\mathscr{A}'),\\ J\mathscr{R}_\ell(\mathscr{A})J &= \mathscr{R}_\ell(\mathscr{A})',\\ J\mathscr{R}_\ell(\mathscr{A})'J &= \mathscr{R}_\ell(\mathscr{A}),\\ \Delta^{it}\mathscr{R}_\ell(\mathscr{A})\Delta^{-it} &= \mathscr{R}_\ell(\mathscr{A}),\\ \Delta^{it}\mathscr{R}_\ell(\mathscr{A})'\Delta^{-it} &= \mathscr{R}_\ell(\mathscr{A})'. \end{aligned}$$

If $\mathscr{A} = \mathscr{M}\varphi^{1/2}$, *then the one parameter unitary group* $\{\Delta^{it} \mid t \in \mathbb{R}\}$ *acts on* $\mathscr{A}$ *and* $\mathscr{A}'$ *as automorphisms and the modular conjugation* J *maps* $\mathscr{A}$ *(resp.* $\mathscr{A}'$*) to* $\mathscr{A}'$ *(resp.* $\mathscr{A}$*) anti-isomorphically in the sense that*

$$\forall x, y \in \mathscr{M} : J((x\varphi^{1/2})(y\varphi^{1/2})) = (J(x\varphi^{1/2}))(J(y\varphi^{1/2})).$$

The second part of this theorem can be formulated more generally for arbitrary left Hilbert algebras.

2

Construction 2.1.18 ([SZ19, Section 10.6], [Tak03a, Section IX.1])**.** Let $\mathscr{M}$ be a von Neumann algebra with faithful normal state φ, represented on the GNS Hilbert space. The above theorem tells us that $\mathscr{M}' = J\mathscr{M}J$. In light of this, we will denote the action of $\mathscr{M}'$ on $\varphi^{1/2}$ by $(Jx^*J)\varphi^{1/2} = \varphi^{1/2}x$ for $x \in \mathscr{M}$. Under this notation, $J(x\varphi^{1/2}y) = y^*\varphi^{1/2}x^*$ for all $x,\ y \in \mathscr{M}$. The above theorem also shows that $\sigma_t^\varphi : x \mapsto \Delta^{it}x\Delta^{-it}$ is an automorphism of $\mathscr{M}$ for each $t \in \mathbb{R}$. We call the group $\{\sigma_t^\varphi \mid t \in \mathbb{R}\}$ the *modular automorphism group* associated with φ. We will sometimes omit the superscript φ if the state is clear from the context.

TOMITA ALGEBRAS

We will now discuss Tomita algebras, which are in some sense the *nice* elements in a left Hilbert algebra. They are strongly tied to the modular operator and the modular conjugation.

Definition 2.1.19 ([Tak03a, Definition VI.2.1])**.** A left Hilbert algebra $\mathscr{A}$ is called a *Tomita algebra* if $\mathscr{A}$ admits a complex one parameter group $\{U(\alpha) \mid \alpha \in \mathbb{C}\}$ of automorphisms, not $*$-preserving, with the following properties:

(a) The function $\alpha \mapsto \langle U(\alpha)\xi, \eta\rangle$ is entire for all $\xi,\ \eta \in \mathscr{A}$.

(b) For all $\alpha \in \mathbb{C}$ and $\xi \in \mathscr{A}$ we have $(U(\alpha)\xi)^\sharp = U(\overline{\alpha})\xi^\sharp$.

(c) For all $\alpha \in \mathbb{C}$ and $\xi,\ \eta \in \mathscr{A}$ we have $\langle U(\alpha)\xi, \eta\rangle = \langle \xi, U(-\overline{\alpha})\eta\rangle$.

(d) For all $\xi,\ \eta \in \mathscr{A}$ we have $\langle \xi^\sharp, \eta^\sharp\rangle = \langle U(-i)\eta, \xi\rangle$.

The group $\{U(\alpha) \mid \alpha \in \mathbb{C}\}$ is called the modular automorphism group of $\mathscr{A}$.

Theorem 2.1.20 ([Tak03a, Theorem VI.2.2(ii)])**.** *If $\mathscr{A}$ is a Tomita algebra, then $\mathscr{A}$ becomes a right Hilbert algebra with new involution $\xi^\flat = U(-i)\xi^\sharp$ and $\mathscr{R}_\ell(\mathscr{A})' = \mathscr{R}_r(\mathscr{A})$. Moreover, the modular operator Δ is the closure of $U(-i)$.*

Theorem 2.1.21 ([Tak03a, Theorem VI.2.2(i)])**.** *Let Δ be the modular operator and J the modular conjugation for the left Hilbert algebra $\mathscr{M}\varphi^{1/2}$. If we define*

$$\mathscr{A}_0 = \{\xi \in \bigcap_{n\in\mathbb{Z}} \operatorname{dom}(\Delta^n) \mid \Delta^n\xi \in \mathscr{M}\varphi^{1/2}, n \in \mathbb{Z}\},$$

then $\mathscr{A}_0$ is a Tomita algebra with respect to $\{\Delta^{i\alpha} \mid \alpha \in \mathbb{C}\}$ such that

$$J\mathscr{A}_0 = \mathscr{A}_0,\ \mathscr{R}_\ell(\mathscr{A}_0) = \mathscr{M} \text{ and } \mathscr{R}_r(\mathscr{A}_0) = \mathscr{M}'.$$

We call $\mathscr{A}_0$ the maximal Tomita algebra *corresponding to $\mathscr{M}$ and φ.*

This theorem can be formulated for left Hilbert algebras satisfying certain certain additional conditions in place of $\mathscr{M}\varphi^{1/2}$ as well. Coming back to the von Neumann algebra picture, we find that we can define a similar subalgebra of $\mathscr{M}$ of *nice* elements.

Definition 2.1.22 ([Tak03a, Definition VIII.2.2])**.** Let $\mathscr{M}$ be a von Neumann algebra with faithful state φ and modular automorphism group σ_t^φ. An element $x \in \mathscr{M}$ is called *entire* if the function $t \mapsto \sigma_t^\varphi(x)$ can be extended to an $\mathscr{M}$-valued entire function over $\mathbb{C}$, which is also denoted by σ^φ. We denote by $\mathscr{M}_a^\varphi$ the set of all entire elements of $\mathscr{M}$.

Lemma 2.1.23 ([Tak03a, Lemma VIII.2.4])**.** *The maximal Tomita algebra $\mathscr{A}_0$ in Theorem 2.1.21 corresponds to the set of entire elements of $\mathscr{M}$ in the sense that $\mathscr{M}_a^\varphi = \pi_\ell(\mathscr{A}_0)$.*

STANDARD FORMS AND THE KMS INNER PRODUCT

We will now work our way towards the so-called standard form of a von Neumann algebra, which associates to the von Neumann algebra some of the objects derived above for a given faithful normal state, but in such a way that it does not depend on the state. This construction is also possible for faithful semi-finite normal weights, which means that a standard form can be obtained for any von Neumann algebra [Tak03a, Theorem VII.2.7]. To get there, we first need to talk about cones. We start by refreshing the definition.

Definition 2.1.24 ([Con10, Definition III.9.5])**.** Let $(X, \leq)$ be an ordered vector space. A *cone* in X is a subset C of X such that

(a) $x + y \in C$ if $x,\ y \in C$,

(b) $\lambda x \in C$ if $\lambda \geq 0$ and $x \in C$ and

(c) $x = 0$ if $x,\ -x \in C$.

Definition 2.1.25 ([Tak03a, Definition XI.1.1])**.** For a cone C in a Hilbert space $\mathscr{H}$, the *dual cone* C° is defined to be the set of all vectors $\eta \in \mathscr{H}$ such that $\langle \xi, \eta \rangle \geq 0$ for every $\xi \in C$. A cone C is called *self-dual* if $C = C^\circ$.

Theorem 2.1.26 ([Tak03a, Theorem XI.1.2])**.** *Let $\mathscr{M}$ be a von Neumann algebra with faithful normal state φ and let $\mathscr{H}$ be the corresponding GNS Hilbert space and $\mathscr{A}_0$ the Tomita algebra. We define the sets*

$$\begin{aligned} C_\varphi^\sharp &= \overline{\{\xi\xi^\sharp \mid \xi \in \mathscr{M}\varphi^{1/2}\}}, \\ C_\varphi^\flat &= \overline{\{\xi\xi^\flat \mid \xi \in \varphi^{1/2}\mathscr{M}\}} \text{ and} \\ C_\varphi &= \overline{\{\xi J\xi \mid \xi \in \mathscr{A}_0\}}. \end{aligned}$$

The sets $C_\varphi^\sharp$ and $C_\varphi^\flat$ are mutually dual cones and C_φ is a self-dual cone. Moreover, for every $x \in \mathscr{M}$ we have $xJxJC_\varphi \subset C_\varphi$.

Based on this, the definition of a standard form of a von Neumann algebra makes sense.

Definition 2.1.27 ([Tak03a, Definition XI.1.13])**.** Given a von Neumann $\mathscr{M}$, a quadruple $(\mathscr{M}, \mathscr{H}, C, J)$ of a Hilbert space $\mathscr{H}$ that $\mathscr{M}$ acts on, an anti-linear isometry J called the *modular conjugation*, and a self-dual cone C in $\mathscr{H}$ is said to be a *standard form* of $\mathscr{M}$ if the following requirements are satisfied:

(a) $J = J^{-1}$.

(b) $J\mathscr{M}J = \mathscr{M}'$

(c) For all $x \in \mathscr{M} \cap \mathscr{M}'$ we have $JxJ = x^*$.

(d) For all $\xi \in C$ we have $J\xi = \xi$.

(e) For all $x \in \mathscr{M}$ we have $xJxJC \subset C$.

One can make precise the idea that any two standard forms of a von Neumann algebra are equivalent [Tak03a, Theorem XI.1.14] and use this to construct a *canonical* standard form denoted by $(\mathcal{M}, L^2(\mathcal{M}), L^2(\mathcal{M})_+, J)$, which is also called *the* standard form of $\mathcal{M}$ [Tak03a, Definition XI.1.18]. Because the faithful normal state will be fixed in the chapters in this thesis, it is easier to work with the standard form obtained for this specific state. We will obfuscate these differences, and refer to our standard form using the notation $(\mathcal{M}, L^2(\mathcal{M}), L^2(\mathcal{M})_+, J)$. We will now discuss this standard form and its relation to the modular operator a bit further. It turns out that, informally, the modular operator acts by moving around the $\varphi^{1/2}$.

Theorem 2.1.28 ([Tak03a, Lemmas VI.1.5 and XI.1.3])**.** *The left and right Hilbert algebras for a von Neumann algebra with faithful normal state φ are related by $\varphi^{1/2}\mathcal{M} = \Delta^{1/2}\mathcal{M}\varphi^{1/2}$ and the cones by*

$$L^2(\mathcal{M})_+ = \overline{\Delta^{1/4}C_\varphi^\sharp} = \overline{\Delta^{-1/4}C_\varphi^\flat}.$$

If we define

$$\varphi^{1/4}x\varphi^{1/4} = \Delta^{1/4}x\varphi^{1/2} = \Delta^{-1/4}\varphi^{1/2}x$$

for any $x \in \mathcal{M}$, then $\varphi^{1/4}x\varphi^{1/4} \in L^2(\mathcal{M})_+$ if and only if $x \in \mathcal{M}_+$. Moreover,

$$\overline{\varphi^{1/4}\mathcal{M}_+\varphi^{1/4}} = L^2(\mathcal{M})_+.$$

Definition 2.1.29. The maps $x \mapsto x\varphi^{1/2}$, $x \mapsto \varphi^{1/2}x$ and $x \mapsto \varphi^{1/4}x\varphi^{1/4}$ are embeddings of $\mathcal{M}$ into $L^2(\mathcal{M})$. We call $x \mapsto x\varphi^{1/2}$ and $x \mapsto \varphi^{1/2}x$ the *(left) GNS embedding* and *right GNS embedding*, respectively, and $x \mapsto \varphi^{1/4}x\varphi^{1/4}$ the *KMS* or *symmetric embedding*.

While the GNS embeddings require a lot less theory than the symmetric embedding, the symmetric embedding tends to be the most natural and usually has the nicest properties when relating maps on $\mathcal{M}$ and maps on $L^2(M)$. We conclude by defining inner products on $\mathcal{M}$ based on these embeddings.

Definition 2.1.30. We call the inner products on $\mathcal{M}$ obtained from embedding $\mathcal{M}$ into $L^2(M)$ using the left GNS embedding, the right GNS embedding and the KMS embedding the *(left) GNS inner product*, *right GNS inner product* and *KMS inner product*, respectively.

Note to the reader 2.1.31. We have now come to the end of the part on Tomita-Takesaki theory in this dissertation. It does not come remotely close to stating all its main results. While I do not intend to define what should or should not be called Tomita-Takesaki theory, I think that most of [Tak03a, Sections VI.1 - XI.1 (p. 1-167)] will probably fall in this category. I believe that as far as books go, the only way to become fluent in Tomita-Takesaki theory is to thoroughly work through this part. However, the fascinating thing about Tomita-Takesaki theory is how well the intuition from easier cases (finite-dimensional von Neumann algebras, von Neumann algebras with a faithful normal state) still applies to the general case; one just needs to work harder to justify that it is true. Because of this, I think it is very valuable to become familiar with the finite-dimensional case and to understand what the described objects mean there.

2

Finite-dimensional theory: Tomita-Takesaki theory

Let $\mathscr{M}$ be a von Neumann algebra with faithful state φ and corresponding invertible density matrix d_φ and GNS Hilbert space $\mathscr{H} = \mathscr{M}$. Whereas $\varphi^{1/2}$ was just a label for the cyclic vector in the infinite-dimensional theory above, it is now actually true that $d_\varphi^{1/2}$ is equal to the cyclic vector. As such, we can now just write down the left and right Hilbert algebras as $\mathscr{M} d_\varphi^{1/2}$ and $d_\varphi^{1/2}\mathscr{M}$, which both equal $\mathscr{M}$ as vector spaces. The left multiplication becomes $(xd_\varphi^{1/2})\cdot(yd_\varphi^{1/2}) = xyd_\varphi^{1/2} \in \mathscr{M}$, or, equivalently, $x\cdot y = xd_\varphi^{-1/2}y$, for example. For the involutions we have $(xd_\varphi^{1/2})^\sharp = x^* d_\varphi^{1/2}$ and $(d_\varphi^{1/2}x)^\flat = d_\varphi^{1/2}x^*$ for the left and right Hilbert algebras, respectively.

Another way to formulate this, is as

$$x^\sharp = ((xd_\varphi^{1/2})d_\varphi^{-1/2})^\sharp = d_\varphi^{-1/2}x^* d_\varphi^{1/2} \text{ and } x^\flat = (d_\varphi^{-1/2}(d_\varphi^{1/2}x))^\flat = d_\varphi^{1/2}x^* d_\varphi^{-1/2}.$$

Now $\Delta x = (x^\sharp)^\flat = d_\varphi x d_\varphi^{-1}$ is the modular operator, and then the modular conjugation becomes $Jx = x^*$. We also see that

$$(\sigma_t^\varphi(x))(\xi) = (\Delta^{it}x\Delta^{-it})(\xi) = d_\varphi^{it}xd_\varphi^{-it}\xi d_\varphi^{it}d_\varphi^{-it} = d_\varphi^{it}xd_\varphi^{-it}\xi,$$

so the modular automorphism group becomes $\sigma_t^\varphi(x) = d_\varphi^{it}xd_\varphi^{-it} = \Delta^{it}x$. We see that the distinction between the modular automorphism group and the modular operator disappears somewhat. In the infinite-dimensional setting, this distinction exists because $\mathscr{M}$ and $\mathscr{H}$ are not the same, but here, one can identify them. Note that there are multiple ways to make this identification, we had multiple embeddings after all, but that all of these embeddings intertwine σ_t^φ and Δ^{it}.

Because d_φ is invertible, all elements are *nice*, so $\mathscr{A}_0 = \mathscr{M} d_\varphi^{1/2}$ and $\mathscr{M}_a^\varphi = \mathscr{M}$. Moreover, the commutant of $\mathscr{M}$ in $B(\mathscr{H})$ is given by right multiplication by elements in $\mathscr{M}$, and we indeed see that $JxJ\xi = \xi x^*$. For the standard form we need to be a bit careful. In the infinite-dimensional case, the Hilbert space $\mathscr{H}$ was distinct from $\mathscr{M}$, and we had three different embeddings of $\mathscr{M}$ into $\mathscr{H}$. This distinction is less clear now, since $\mathscr{H} = \mathscr{M}$ as a vector space. In the original construction, it carried the GNS inner product, but from the concept of the standard form we see that it is better to equip $\mathscr{H}$ with the KMS inner product, and that is what we will denote by $L^2(\mathscr{M})$. In this way $L^2(\mathscr{M})_+ = \mathscr{M}_+$, and the three inner products on $\mathscr{M}$ are the following:

- The left GNS inner product: $\langle x, y\rangle_{\text{GNS}} = \operatorname{Tr}(d_\varphi x^* y)$.
- The rarely used right GNS inner product: $\langle x, y\rangle_{\text{rGNS}} = \operatorname{Tr}(x^* d_\varphi y)$.
- The KMS inner product: $\langle x, y\rangle_{\text{KMS}} = \operatorname{Tr}(d_\varphi^{1/2}x^* d_\varphi^{1/2}y)$. This now clearly is the most natural, or at least the most symmetric, inner product, and therefore we also denote it by $\langle x, y\rangle_\varphi$.

2

2.2. OPERATIONS ON OPEN QUANTUM SYSTEMS

In the previous section we treated the basics of von Neumann algebras. We will now turn to the description of open quantum systems, where we use the language of von Neumann algebras to formulate the relevant mathematical properties. In this section we will stick to the dry mathematics to make the notions and results easy to find. The physical reasoning why this is the way we describe open quantum systems this way can be found in Section 1.2 of the introduction.

2.2.1. QUANTUM INFORMATION THEORY

This section recalls the basics of quantum information theory on maps describing how the state of an open quantum system can change. We will follow [OW23], which treats the mathematics behind quantum information well.

Definition 2.2.1 ([OW23, Definition 4.15], [Dav76, Section 9.2])**.** Let $\mathscr{A}$ and $\mathscr{B}$ be C*-algebras and $\Phi : \mathscr{A} \to \mathscr{B}$ be a linear map.

- Φ is called *positive* if $0 \le \Phi(x)$ for all $0 \le x \in \mathscr{A}$.
- Φ is called *completely positive* if the map $\Phi \otimes \mathrm{id} : \mathscr{A} \otimes M_n(\mathbb{C}) \to \mathscr{B} \otimes M_n(\mathbb{C})$ is positive for all $n \in \mathbb{N}$.
- Φ is called *unital* if $\Phi(1_{\mathscr{A}}) = 1_{\mathscr{B}}$.

Remark 2.2.2. The combination "unital completely positive" is often abbreviated to UCP.

Operations on open quantum systems are described by quantum channels when one views quantum mechanics in the Schrödinger picture. The corresponding maps in the Heisenberg picture become UCP maps. In the operator algebra setting, it is nicer to work in the Heisenberg picture, and we will therefore mainly focus on UCP maps. The structure of completely positive maps is well understood, as is captured in the following theorem. In finite dimensions the picture is even more clear, as one can read in the blue box that follows.

Theorem 2.2.3 (Stinespring's dilation theorem, [Dav76, Theorem 9.2.1], [Pau02, Theorem 4.1])**.** *If $\mathscr{A}$ is a C*-algebra with identity and $\Phi : \mathscr{A} \to B(\mathscr{H})$ a linear map, then Φ is completely positive if and only if it has the form*

$$\Phi(x) = V^* \pi(x) V \quad \forall x \in \mathscr{A}$$

for some representation π of $\mathscr{A}$ on a Hilbert space $\mathscr{K}$ and some bounded linear map $V : \mathscr{H} \to \mathscr{K}$. If $\mathscr{A}$ is a von Neumann algebra and Φ is normal, then π can be taken to be normal. If Φ is unital, then V is an isometry.

Remark 2.2.4. We call the triple $(\mathscr{K}, \pi, V)$ in the above theorem a *Stinespring representation* of Φ.

Finite-dimensional theory: completely positive maps

The notions in Definition 2.2.1 and Stinespring's dilation theorem do not change in the finite-dimensional case. However, we do obtain a few more characterisations.

Definition 2.2.5 ([OW23, Definition 5.1])**.** Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a linear map. We define the *Choi operator* $C^\Phi \in B(\mathcal{H} \otimes \mathcal{K})$ as

$$C^\Phi = \sum_{v,w \in B} |v\rangle\langle w| \otimes \Phi(|v\rangle\langle w|),$$

where $\{|v\rangle\}_{v \in B}$ denotes an arbitrary orthonormal basis of $\mathcal{H}$.

Remark 2.2.6. The Choi operator C^Φ depends on the choice of basis.

Lemma 2.2.7 (Choi-Jamiołkowski isomorphism [OW23, Lemma 5.2])**.** *The map*

$$B(B(\mathcal{H}), B(\mathcal{K})) \ni \Phi \mapsto C^\Phi \in B(\mathcal{H} \otimes \mathcal{K})$$

is an isomorphism with inverse given by

$$\Phi(x) = \mathrm{Tr}_{\mathcal{H}}\left((x^T \otimes I_{\mathcal{K}})C^\Phi\right) \ \forall x \in B(\mathcal{H}),$$

where the transpose is taken using the same basis as in the construction of C^Φ.

Theorem 2.2.8 ([OW23, Theorem 5.3])**.** *Let* $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ *be a linear map. The following are equivalent:*

(a) Φ *is completely positive.*

(b) $\Phi \otimes \mathrm{id}_{B(\mathcal{H})}$ *is positive.*

(c) $C^\Phi \geq 0$ *for any choice of orthonormal basis.*

(d) There exists a Kraus representation $V_1, \ldots V_r \in B(\mathcal{H}, \mathcal{K})$ *such that*

$$\forall x \in B(\mathcal{H}) : \Phi(x) = \sum_{i=1}^{r} V_i x V_i^*.$$

If the above statements hold, then there exist Kraus and Stinespring representations with $r = \mathrm{rank}(C^\Phi)$ *and* $\dim(\mathcal{K}) = \mathrm{rank}(C^\Phi)$*, respectively.*

Lastly, restricting to maps on the bounded operators of finite-dimensional Hilbert spaces allows us to introduce additional properties of such maps.

Definition 2.2.9 ([OW23, Definition 4.15])**.** Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a linear map. Φ is called *trace-preserving* if $\mathrm{Tr}(\Phi(x)) = \mathrm{Tr}(x)$ for all $x \in B(\mathcal{H})$ and a *quantum channel* (or a CPTP map) if it is completely positive and trace-preserving.

We finish this section with a notion that is slightly weaker than complete positivity.

Definition 2.2.10. A linear map $L : \mathscr{A} \to \mathscr{B}$ between C*-algebras is called *π-conditionally completely positive* for a $*$-representation $\pi : \mathscr{A} \to \mathscr{B}$ if for all $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathscr{A}, b_1, \ldots, b_n \in \mathscr{B}$ we have

$$\sum_{i,j=1}^{n} b_i^* L(a_i^* a_j) b_j \geq 0 \text{ whenever } \sum_{i=1}^{n} \pi(a_i) b_i = 0.$$

If $\mathscr{A} = \mathscr{B}$ and $\pi = \mathrm{id}$, then we call L *conditionally completely positive*. A map L is *π-conditionally completely negative* if and only if $-L$ is *π-conditionally completely positive*.

2.2.2. Quantum Markov semigroups

We are now ready to describe quantum Markov semigroups, sometimes abbreviated as QMS, which describe the time evolution of open quantum systems under the Markovian assumption, i.e. that what happens next is purely determined by the state at that point in time. Quantum Markov semigroups are key objects in this dissertation, and Chapter 3 is devoted to the study of some of their properties. While quantum Markov semigroups can be defined on C*-algebras as well, we will purely focus on von Neumann algebras.

Definition 2.2.11 ([Lin76, Section 1])**.** Let $\mathscr{M}$ be a von Neumann algebra. A *quantum Markov semigroup* is a one-parameter family of normal unital completely positive maps $(\Phi_t)_{t \in \mathbb{R}_+}$ satisfying

(a) $\Phi_s \circ \Phi_t = \Phi_{s+t}$ for all $s, t \geq 0$.

(b) $\Phi_0 = \mathrm{id}$.

(c) $\Phi_t(x) \to x$ in the σ-weak topology as $t \to 0$ for all $x \in \mathscr{M}$.

A quantum Markov semigroup is called *uniformly continuous* if $\Phi_t \to \mathrm{id}$ in operator norm as $t \to 0$.

Remark 2.2.12. The above objects are known under several names in the literature, such as dynamical semigroups [Lin76], quantum dynamical semigroups [CE79] and quantum Markov semigroups [CM17]. We stick with the last name throughout this work. We will also write $(\Phi_t)_{t \geq 0}$ or (Φ_t) instead of $(\Phi_t)_{t \in \mathbb{R}_+}$ from now on.

Definition 2.2.13 ([Lin76, Section 1], [Phi55, Section 2])**.** Let $\mathscr{M}$ be a von Neumann algebra with quantum Markov semigroup $(\Phi_t)_{t \geq 0}$. The *generator* of $(\Phi_t)_{t \geq 0}$ is the σ-weakly closed and densely defined operator L given by

$$\mathrm{dom}(L) = \left\{ x \in \mathscr{M} \;\middle|\; \lim_{t \to 0} \frac{1}{t}(x - \Phi_t(x)) \text{ exists in the } \sigma\text{-weak topology} \right\},$$
$$L(x) = \lim_{t \to 0} \frac{1}{t}(x - \Phi_t(x)) \text{ in the } \sigma\text{-weak topology.}$$

A quantum Markov semigroup is uniformly continuous if and only if the generator is bounded. In this case, we can characterise which bounded operators on $\mathscr{M}$ are generators.

Theorem 2.2.14 ([EL77b, Theorem 14.7])**.** *A bounded operator L is the generator of a uniformly continuous quantum Markov semigroup if and only if it is normal, $L(1) = 0$ and L is conditionally completely negative.*

For bounded generators of quantum Markov semigroups the structure is known.

Theorem 2.2.15 (Christensen-Evans theorem, [CE79, Theorem 3.1])**.** *Let L be the generator of a uniformly continuous quantum Markov semigroup on a von Neumann algebra $\mathcal{M}$. Then there exist a normal completely positive map $\Psi : \mathcal{M} \to \mathcal{M}$ and an element $k \in \mathcal{M}$ such that*

$$\forall x \in \mathcal{M} : L(x) = k^* x + xk - \Psi(x).$$

Finite-dimensional theory: quantum Markov semigroups

Any unital completely positive map on a von Neumann algebra $\mathcal{M}$ is normal, so a quantum Markov semigroup $(\Phi_t)_{t\geq 0}$ on $\mathcal{M}$ is just a continuous semigroup of unital completely positive maps, and it is automatically uniformly continuous. Its generator is the operator L that satisfies $\Phi_t = e^{-tL}$, and an operator is a generator if and only if $L(1) = 0$ and L is conditionally completely negative if and only if there exist a completely positive map Ψ on $\mathcal{M}$ and $k \in \mathcal{M}$ such that

$$\forall x \in \mathcal{M} : L(x) = k^* x + xk - \Psi(x).$$

QUANTUM MARKOV SEMIGROUPS ON HILBERT SPACES

Throughout this section, let $\mathcal{M}$ be a von Neumann algebra with faithful normal state φ and corresponding standard form $(\mathcal{M}, L^2(\mathcal{M}), L^2(\mathcal{M})_+, J)$.

Definition 2.2.16 ([Cip97, Definition 2.1])**.** Let T be a bounded linear map on $\mathcal{M}$. T is called *symmetric* or *KMS-symmetric* if

$$\forall x, y \in \mathcal{M}_a^\varphi : \varphi(T(x)\sigma^\varphi_{-i/2}(y)) = \varphi(\sigma^\varphi_{i/2}(x)T(y)).$$

A quantum Markov semigroup $(\Phi_t)_{t\geq 0}$ is called *symmetric* or *KMS-symmetric* if Φ_t is symmetric for every $t \geq 0$.

In the rest of the dissertation, whenever we talk about any kind of symmetry of operators or quantum Markov semigroups, the reference state on the von Neumann algebra will already have been established. If we wish to emphasise that the reference state is tracial, we call an operator or quantum Markov semigroup *tracially symmetric*.

If a bounded linear map T on $\mathcal{M}$ is symmetric, then it can be extended to $L^2(\mathcal{M})$ by the following lemma. We denote this operator on $L^2(\mathcal{M})$ by $T^{(2)}$.

Lemma 2.2.17 ([Cip97, Lemma 2.4])**.** *Let T be a symmetric linear map on $\mathcal{M}$. The induced map $T^{(2)}$ on $L^2(\mathcal{M})$ through the KMS embedding is bounded with $\|T^{(2)}\| \leq \|T\|$.*

This implies that a symmetric quantum Markov semigroup can be extended to a family of operators on $L^2(\mathcal{M})$. It turns out that we can also characterise the property of being a symmetric quantum Markov semigroup in terms of the corresponding operators on $L^2(\mathcal{M})$.

Definition 2.2.18 ([Cip97, Definitions 2.7 and 2.8])**.** A bounded operator $A \in B(L^2(\mathcal{M}))$ is called *positivity-preserving* if $A(L^2(\mathcal{M})_+) \subset L^2(\mathcal{M})_+$ and *completely positivity-preserving* if $A \otimes \mathrm{id}_n$ is positivity-preserving on $L^2(\mathcal{M}) \otimes M_n(\mathbb{C})$ for all $n \in \mathbb{N}$. It is called *Markovian* if A is completely positivity-preserving and $A\varphi^{1/2} = \varphi^{1/2}$. A semigroup $(A_t)_{t\geq 0}$ is called *Markovian* if A_t is Markovian for every $t \geq 0$.

Remark 2.2.19. In the original paper completely positivity-preservation is not required to be Markovian, only positivity-preservation. However, since a quantum Markov semigroup is required to be completely positive, it is better to reserve the name Markovian purely for completely positivity-preserving operators.

Theorem 2.2.20 ([Cip97, Theorems 2.11 and 2.12])**.** *Let $(\Phi_t)_{t\geq 0}$ be a symmetric quantum Markov semigroup on $\mathcal{M}$. Then $(\Phi_t^{(2)})_{t\geq 0}$ is a strongly continuous symmetric Markovian semigroup on $L^2(\mathcal{M})$. Conversely, if $(A_t)^{t\geq 0}$ is a strongly continuous symmetric Markovian semigroup on $L^2(\mathcal{M})$, then its restriction to $\mathcal{M}$ under the KMS embedding is a symmetric quantum Markov semigroup.*

There is also another notion of symmetry, which is called GNS symmetry, and it is related to the GNS embedding.

Definition 2.2.21. A bounded linear map T on $\mathcal{M}$ is called *GNS-symmetric* if $\varphi(xT(y)) = \varphi(T(x)y)$ for all $x,\ y \in \mathcal{M}$ and a quantum Markov semigroup $(\Phi_t)_{t\geq 0}$ is called *GNS-symmetric* if Φ_t is GNS-symmetric for every $t \geq 0$.

Lemma 2.2.22 ([Wir24, Proposition 2.2])**.** *Any GNS-symmetric unital completely positive bounded linear map T on $\mathcal{M}$ commutes with the modular automorphism group σ^φ. Consequently, any such map is KMS-symmetric.*

All maps in a GNS-symmetric quantum Markov semigroup satisfy the assumptions of the above lemma, so any GNS-symmetric quantum Markov semigroup commutes with the modular automorphism group. This makes it significantly easier to study this case. If φ is tracial, the modular automorphism group is trivial, and GNS symmetry and KMS symmetry coincide.

In the symmetric case we can also construct generators on the Hilbert space level. In this case it is possible to relate the generator and the semigroup on the Hilbert space level in a natural way.

Definition 2.2.23 ([GL95, Theorem 3.2])**.** A strongly continuous symmetric Markovian semigroup $(A_t)_{t\geq 0}$ on $L^2(\mathcal{M})$ gives rise to a *generator* L_2, which is a self-adjoint operator given by

$$\mathrm{dom}(L_2) = \left\{ \xi \in L^2(\mathcal{M}) \;\middle|\; \lim_{t\to 0} \frac{1}{t}(\xi - A_t(\xi)) \text{ exists} \right\},$$
$$L_2(\xi) = \lim_{t\to 0} \frac{1}{t}(\xi - A_t(\xi)).$$

If $(\Phi_t)_{t\geq 0}$ is a symmetric quantum Markov semigroup on $\mathcal{M}$, we call L_2 corresponding to $(\Phi_t^{(2)})_{t\geq 0}$ the *L^2-generator* of $(\Phi_t)_{t\geq 0}$.

Proposition 2.2.24 ([GL95, Theorem 3.2])**.** *Let* $(\Phi_t)_{t\geq 0}$ *be a symmetric quantum Markov semigroup on* $\mathscr{M}$ *with* L^2*-generator* L_2*. Then*

$$\Phi_t^{(2)} = e^{-tL_2}$$

using functional calculus.

Finite-dimensional theory: symmetry for quantum Markov semigroups

In the finite-dimensional case, symmetry of quantum Markov semigroups is easy to understand. If $\mathscr{M}$ is a von Neumann algebra with faithful state φ, then a quantum Markov semigroup $(\Phi_t)_{t\geq 0}$ is GNS- or KMS-symmetric if and only if all the maps are self-adjoint with respect to the corresponding inner product if and only if the generator is self-adjoint with respect to the corresponding inner product.

A GNS-symmetric quantum Markov semigroup commutes with the modular operator and is automatically KMS-symmetric. Conversely, a KMS-symmetric quantum Markov semigroup is GNS-symmetric if it commutes with the modular operator.

2.2.3. QUANTUM DIRICHLET FORMS

Each quantum Markov semigroup bijectively corresponds to a different object called a quantum Dirichlet form, which is a quadratic form satisfying certain conditions. For analytic reasons, quantum Dirichlet forms can be easier to work with than quantum Markov semigroups if the generator is unbounded.

Definition 2.2.25. A *quadratic form* on a Hilbert space $\mathscr{H}$ is a map $q : \mathscr{H} \to [0,\infty]$ such that

$$q(\lambda\eta) = |\lambda|^2 q(\eta) \text{ and } q(\xi+\eta) + q(\xi-\eta) = 2q(\xi) + 2q(\eta)$$

for all ξ, $\eta \in \mathscr{H}$ and $\lambda \in \mathbb{C}$. The form q is called *closed* if it is lower semicontinuous and *densely defined* if $\mathrm{dom}(q) = \{\eta \in \mathscr{H} \mid q(\xi) < \infty\}$ is dense.

In the second condition one can recognise the parallelogram identity [Nee22, Proposition 3.6], implying that there should be a sesquilinear form associated to q. This is indeed the case.

Proposition 2.2.26 ([Cip97, Section 4])**.** *A quadratic form* q *on* $\mathscr{H}$ *gives rise to a sesquilinear form, also denoted by* q*, on* $\mathrm{dom}(q)$ *by the polarisation identity, i.e.*

$$q(\xi,\eta) = \sum_{k=0}^{3} i^k q(i^k\xi+\eta).$$

Conversely, any sesquilinear form q *on* $\mathrm{dom}(q) \subset \mathscr{H}$ *gives rise to a quadratic form on* $\mathscr{H}$ *by*

$$q(\eta) = \begin{cases} q(\eta,\eta) & \text{if } \eta \in \mathrm{dom}(q) \\ \infty & \text{otherwise.} \end{cases}$$

We will interchangeably view such objects as quadratic forms and as sesquilinear forms, and denote both cases by the same symbol. One can define many properties quadratic forms may have, but here we will combine them to define only those quadratic forms that correspond to quantum Markov semigroups. Note that different names are used in the literature [GL95; Cip97; CS03a]; we pick names that make sense in our situation. From this point on, we once again fix a von Neumann algebra $\mathcal{M}$ with faithful normal state φ and corresponding standard form $(\mathcal{M}, L^2(\mathcal{M}), L^2(\mathcal{M})_+, J)$. We will denote quantum Dirichlet forms by $\mathcal{E}$, as this is more standard in the literature.

Notation 2.2.27. As $L^2(\mathcal{M})$ is a Hilbert space, we can project onto closed convex sets [Con10, Theorem I.2.5]. We denote by $\eta \wedge \varphi^{1/2}$ the projection of η onto the closed convex cone $\varphi^{1/2} - L^2(\mathcal{M})_+$.

Definition 2.2.28. A closed densely defined quadratic form $\mathcal{E} : L^2(\mathcal{M}) \to [0,\infty]$ is called a *conservative Dirichlet form* if

- $\mathcal{E}(J\eta) = \mathcal{E}(\eta)$ for all $\eta \in L^2(\mathcal{M})$,
- $\mathcal{E}(\eta \wedge \varphi^{1/2}) \leq \mathcal{E}(\eta)$ for all $\eta \in L^2(\mathcal{M})$,
- $\mathcal{E}(\varphi^{1/2}) = 0$.

The form $\mathcal{E}$ is called a *quantum Dirichlet form* if for every $n \in \mathbb{N}$ its amplification, the quadratic form

$$\mathcal{E}^{(n)} : L^2(M_n(\mathcal{M})) \to [0,\infty], \mathcal{E}^{(n)}((\eta_{jk})_{jk}) = \sum_{j,k=1}^{n} \mathcal{E}(\eta_{jk}),$$

is a conservative Dirichlet form.

The distinction between conservative Dirichlet forms and quantum Dirichlet forms is similar to the distinction between positive maps and completely positive maps.

The next step is to describe the connection between quantum Dirichlet forms and quantum Markov semigroups. This connection runs through the so-called generator of a quadratic form and the generator of a quantum Markov semigroup.

Construction 2.2.29. Let q be a closed densely defined quadratic form on $\mathcal{H}$. Its *generator* is the positive self-adjoint operator L_2 given by

$$\begin{aligned} \mathrm{dom}(L_2) &= \{\eta \in \mathrm{dom}(q) \mid \exists \xi \in \mathcal{H}\ \forall \zeta \in \mathrm{dom}(q) \mid q(\eta,\zeta) = \langle \xi, \zeta \rangle\}, \\ L_2\eta &= \xi \text{ as above.} \end{aligned}$$

Conversely, given a positive self-adjoint operator L_2 on $\mathcal{H}$, we can construct a closed densely defined quadratic form q on $\mathcal{H}$ by

$$q(\eta) = \begin{cases} \|L_2^{1/2}\eta\|^2 & \text{if } \eta \in \mathrm{dom}(L_2^{1/2}), \\ \infty & \text{otherwise} \end{cases}$$

with generator L_2.

Theorem 2.2.30 ([Cip97, Theorem 4.11], [GL95, Theorem 5.7])**.** *Any KMS-symmetric quantum Markov semigroup on $\mathscr{M}$ with L^2-generator L_2 gives rise to a quantum Dirichlet form on $L^2(\mathscr{M})$ through Construction 2.2.29 for L_2. Conversely, the generator of a quantum Dirichlet form is the L^2-generator of a KMS-symmetric quantum Markov semigroup.*

We end with a final, technical definition. This definition can be given for a larger class of quadratic forms [Cip97, Section 4], but that generality is not necessary here.

Definition 2.2.31. Let $\mathscr{E}$ be a quantum Dirichlet form on $L^2(\mathscr{M})$. A *form core* is a linear subspace V of $\mathrm{dom}(\mathscr{E})$ which is dense in $\mathrm{dom}(\mathscr{E})$ under the norm

$$\eta \longmapsto (\mathscr{E}(\eta) + \langle \eta, \eta \rangle)^{1/2}.$$

Finite-dimensional theory: quantum Dirichlet forms

A quadratic form on a Hilbert space $\mathscr{H}$ is a map $q : \mathscr{H} \to [0, \infty)$ such that $q(\eta) = \langle \eta, L\eta \rangle$ for some positive operator L on $\mathscr{H}$. A quadratic form $\mathscr{E}$ on $L^2(\mathscr{M})$ is called a conservative Dirichlet form if

- $\mathscr{E}(\eta^*) = \mathscr{E}(\eta)$ for all $\eta \in L^2(\mathscr{M})$.
- $\mathscr{E}(\eta \wedge \varphi^{1/2}) \leq \mathscr{E}(\eta)$ for all $\eta \in L^2(\mathscr{M})$, where $\eta \mapsto \eta \wedge \varphi^{1/2}$ is the projection onto the closed convex set $\varphi^{1/2} - \mathscr{M}_+$.
- $\mathscr{E}(\varphi^{1/2}) = 0$.

It is a quantum Dirichlet form if all matrix amplifications of $\mathscr{E}$ are conservative Dirichlet forms. However, in finite dimensions, or for any uniformly continuous quantum Markov semigroup for that matter, there is almost never a reason to worry about quantum Dirichlet forms. Consequently, they will not appear in any finite-dimensional part.

2.3. DERIVATIONS

The final topic in these preliminaries concerns derivations, which are maps that satisfy some sort of product or Leibniz rule. This comes in three parts: we first need to describe the spaces derivations map into, then we can define derivations and we finish by describing a way to construct new derivations based on a family of derivations.

2.3.1. BIMODULE STRUCTURES

Let $\mathscr{A}$ and $\mathscr{B}$ be algebras. Intuitively, an $\mathscr{A}$-$\mathscr{B}$ bimodule is a vector space where one can multiply from the left by elements in $\mathscr{A}$ and from the right by elements in $\mathscr{B}$. This will be made precise in the following definitions.

Definition 2.3.1 ([Tak03a, Definition IX.3.1(i)])**.** Let $\mathscr{A}$ be an algebra. The *opposite* algebra $\mathscr{A}^\circ$ is the algebra obtained by reversing the product in $\mathscr{A}$, i.e. $\mathscr{A}^\circ = \mathscr{A}$ as a linear space, and we denote the element in $\mathscr{A}^\circ$ corresponding to $x \in \mathscr{A}$ by x°. Then the product of $x^\circ, y^\circ \in \mathscr{A}^\circ$ is defined as

$$x^\circ y^\circ = (yx)^\circ.$$

If $\mathscr{A}$ is a $*$-algebra or a von Neumann algebra, then the $*$-operation and in the second case also the norm on $\mathscr{A}^\circ$ are carried over from $\mathscr{A}$, just like the linear structure.

We will only consider the case where our algebras are $*$-algebras, and we will require that their involution interacts nicely with the left or right multiplication. Usually, one requires that the bimodule, the vector space the algebras act on, is equipped with a Hilbert space structure in these cases. However, for our purposes it is useful to also have a notion of bimodules that only come with a positive sesquilinear form (see Remark 2.3.4 and the blue box). Unfortunately, this forces us to use non-standard nomenclature, in part due to the existing but different notion of Hilbert bimodules [RW98, Chapter 2]. As in [Tak03a], we will only consider faithful bimodules.

Definition 2.3.2 (Based on [Tak03a, Definition IX.3.1(iii) and Notes on Chapter IX])**.** Let $\mathscr{A}, \mathscr{B}$ be $*$-algebras.

(a) An *$\mathscr{A}$-$\mathscr{B}$-bimodule* is a vector space V equipped with a positive sesquilinear form $\langle\cdot,\cdot\rangle$ and algebra homomorphisms π and π' from $\mathscr{A}$ and $\mathscr{B}^\circ$ to the linear maps on V, respectively, such that

  (i) $\pi(x)$ and $\pi'(y^\circ)$ commute for all $x \in \mathscr{A}$, $y \in \mathscr{B}$,

  (ii) $\langle \pi(x^*)\eta, \xi\rangle = \langle \eta, \pi(x)\xi\rangle$ for all $x \in \mathscr{A}$, $\eta, \xi \in V$,

  (iii) $\langle \pi'((y^\circ)^*)\eta, \xi\rangle = \langle \eta, \pi'(y^\circ)\xi\rangle$ for all $y \in \mathscr{B}$, $\eta, \xi \in V$.

  (iv) $\pi(x)$ and $\pi'(y^\circ)$ are bounded and non-zero for all $x \in \mathscr{A}\backslash\{0\}, y \in \mathscr{B}\backslash\{0\}$ with respect to the seminorm on V generated by $\langle\cdot,\cdot\rangle$.

(b) A *complete $\mathscr{A}$-$\mathscr{B}$-bimodule* is an $\mathscr{A}$-$\mathscr{B}$-bimodule V such that the positive sesquilinear form turns V into a Hilbert space.

(c) If $\mathscr{A}$ and $\mathscr{B}$ are von Neumann algebras, we call a complete $\mathscr{A}$-$\mathscr{B}$-bimodule an *$\mathscr{A}$-$\mathscr{B}$-correspondence* if π and π' are normal.

Remark 2.3.3. We will simplify notation for in bimodules and write $x\eta y$ for $\pi(x)\pi'(y^\circ)\eta$. If confusion might arise, we may write $x \cdot \eta \cdot y$ instead of $x\eta y$.

Remark 2.3.4. It can be convenient to work with a degenerate positive sesquilinear form when one has a natural and conveniently describable vector space in mind for the bimodule. One could of course quotient out the kernel of the sesquilinear form to obtain a complete bimodule, but this would change the vector space. The price one pays is that one does not have a norm on the bimodule but only a seminorm.

Definition 2.3.5. Let $\mathscr{M}$ be a von Neumann algebra. A pair $(\mathscr{H}, \mathscr{J})$ consisting of an $\mathscr{M}$-$\mathscr{M}$-correspondence $\mathscr{H}$ and an anti-unitary involution on $\mathscr{H}$ is called a *self-dual $\mathscr{M}$-$\mathscr{M}$-correspondence* if $\mathscr{J}(x\eta y) = y^* \mathscr{J}(\eta) x^*$ for all $x, y \in \mathscr{M}$ and $\eta \in \mathscr{H}$.

Finite-dimensional theory: Bimodule structures

Definitions 2.3.1, 2.3.2(a), 2.3.2(b) and 2.3.5 do not change in the finite-dimensional case, but now every complete $\mathscr{A}$-$\mathscr{B}$-bimodule is an $\mathscr{A}$-$\mathscr{B}$-correspondence. Note that the boundedness condition in 2.3.2(a) still plays a role in the finite-dimensional case, as it prevents a vector $\eta \in V$ with $\langle \eta, \eta \rangle = 0$ from being mapped to a non-zero vector.

Choosing to work with degenerate positive sesquilinear forms, as discussed in Remark 2.3.4, can be even more advantageous in de finite-dimensional case. While going to a complete bimodule would change the vector space, allowing the positive sesquilinear form to be degenerate is often not problematic, because such forms can be described in terms of a positive semidefinite matrix after identifying the vector space with $\mathbb{C}^n$ for some $n \in \mathbb{N}$. In this case degeneracy of the positive sesquilinear form corresponds to non-invertibility of the matrix. This will be explored concretely in Section 3.2.

2.3.2. DERIVATIONS AND TWISTED DERIVATIONS

Using the structure of bimodules, we can now define what a derivation is, which intuitively is a map satisfying the product rule. We subsequently introduce a notion of *twisted derivations* for von Neumann algebras in standard form. In the finite-dimensional theory section, we describe the notion more generally, which will clarify the name. Using twisted derivations, we define the concept of a first order differential calculus for KMS-symmetric quantum Markov semigroups.

Definition 2.3.6. Let $\mathscr{A}$ be a $*$-algebra. A *derivation* δ from $\mathscr{A}$ into an $\mathscr{A}$-$\mathscr{A}$-bimodule is a linear map satisfying

$$\delta(xy) = x\delta(y) + \delta(x)y$$

for all $x, y \in \mathscr{A}$.

Derivations appear in many places in mathematics, and there are even long-standing open questions connected to them (for example [Pau02, p. 125]). However, we will purely consider them as tools to better understand the generators of KMS-symmetric quantum Markov semigroups. If the faithful normal state that is used is tracial, then derivations are the appropriate tools (see Theorem 2.3.10). However, in Chapter 3 we will see that this is no longer true if the state is not tracial. Simultaneously, Wirth discovered the appropriate modification of the definition [Wir22], which we will call twisted derivations. Note that one could come up with a more algebraic notion of a twisted derivation, but a key observation by Wirth was that this is not the right viewpoint for applications in KMS-symmetric quantum Markov semigroups. In the finite-dimensional case, one can use an algebraic definition (see Definition 2.3.12). In this algebraic version, the twists are represented by algebra homomorphisms affecting the left and right multiplication in the product rule.

Definition 2.3.7. Let $\mathscr{M}$ be a von Neumann algebra with faithful normal state φ. A *twisted derivation* is a closed operator $\delta : L^2(\mathscr{M}) \supset \mathrm{dom}(\delta) \to \mathscr{H}$ to a complete $\mathscr{M}$-$\mathscr{M}$-

bimodule such that for all $\eta \in \mathscr{M}\varphi^{1/2}$ and $\xi \in \varphi^{1/2}\mathscr{M}$ we have

$$\delta(\eta\xi) = \pi_\ell(\eta) \cdot \delta(\xi) + \delta(\eta) \cdot J\pi_r(\xi)^* J.$$

2

Let us briefly unpack this definition. The starting point of the definition is that δ is a possibly unbounded operator between Hilbert spaces. However, one cannot multiply elements in a general Hilbert space, so we have to find an appropriate way to describe the product rule. Note that to formulate something like a product rule in the first place, we need to associate elements in $\mathscr{M}$ to (certain) elements in the Hilbert space. For this we can use the $*$-representation π_ℓ and the $*$-antirepresentation π_r described above Lemma 2.1.14. Plugging an element from $\mathscr{M}\varphi^{1/2}$ into π_ℓ immediately gives an element in $\mathscr{M}$. However, π_r has two problems. It does not map into $\mathscr{M}$, but into $\mathscr{M}'$, and it is an $*$-*anti*representation. However, we can resolve both issues using the relations $J\mathscr{M}'J = \mathscr{M}$ and $\mathscr{R}_r(\varphi^{1/2}\mathscr{M}) = \mathscr{M}'$ from Theorem 2.1.17. These tell us that $\xi \mapsto J\pi_r(\xi)^* J$ is a $*$-representation into $\mathscr{M}$. With this, the product rule in Definition 2.3.7 is well-defined.

Remark 2.3.8. In this dissertation, we sharply distinguish derivations and twisted derivations. We do this because part of the point of Chapter 3 is that certain things are possible for twisted derivations, while they are not possible for derivations. This may suggest that one should think about derivations and twisted derivations as separate things, but that is misleading. One should instead view twisted derivations as the proper way of taking the concept of derivations from the tracial state case (where both definitions agree) and generalising it to the non-tracial state case. In other words, 'twisted derivations' should be seen as nothing else than the right way to think about derivations in the setting of KMS-symmetric quantum Markov semigroups. Therefore, while we distinguish between derivations and twisted derivations in precise statements in this dissertation, in general the word 'twisted' may be implied by context.

Using the notion of a twisted derivation, we can now define a first-order differential calculus. It was introduced in this setting in [VW23], which forms the basis of most of Chapter 3, but as it provides context for Theorem 2.3.10 and can be used to formulate its finite-dimensional version, Theorem 2.3.13, it is useful to record it here. We will also repeat the definition in its original location in Chapter 3 in order to preserve its natural point of introduction (see Definition 3.5.6).

Definition 2.3.9. Let $\mathscr{M}$ be a von Neumann algebra with faithful normal state φ. We call a triple $(\mathscr{H}, \mathscr{J}, \delta)$ consisting of a self-dual $\mathscr{M}$-$\mathscr{M}$-correspondence and a twisted derivation δ into $\mathscr{H}$ a *first-order differential calculus* if

(a) $J\,\mathrm{dom}(\delta) = \mathrm{dom}(\delta)$ and $\delta(J\eta) = \mathscr{J}(\delta(\eta))$ for all $\eta \in \mathrm{dom}(\delta)$,

(b) $\mathrm{span}\{\delta(\eta)x \mid \eta \in \mathrm{dom}(\delta),\ x \in \mathscr{M}\}$ is dense in $\mathscr{H}$.

We call a first-order differential calculus *bounded* if δ is bounded.

We will now state a theorem by Cipriani and Sauvageot that is the main inspiration for Chapter 3. We reformulate the result slightly to be in line with our nomenclature. Note that we do not use the terminology of a first-order differential calculus because the left and right actions for the complete $\mathscr{M}$-$\mathscr{M}$-bimodule that appears may not be normal.

Theorem 2.3.10 ([CS03a, Theorems 8.2 and 8.3])**.** *Let $\mathscr{M}$ be a von Neumann algebra with faithful normal* tracial *state τ. Let $\mathscr{E}$ be a quantum Dirichlet form on $L^2(\mathscr{M})$. Then there exists a complete $\mathscr{M}$-$\mathscr{M}$-bimodule $\mathscr{H}$ with anti-unitary involution $\mathscr{J}$ and a twisted derivation δ into $\mathscr{H}$ such that*

- $\mathscr{J}(x\eta y) = y^* \mathscr{J}(\eta) x^*$ *for all* $x, y \in \mathscr{M}$ *and* $\eta \in \mathscr{H}$,
- $J \operatorname{dom}(\delta) = \operatorname{dom}(\delta)$ *and* $\delta(J\eta) = \mathscr{J}(\delta(\eta))$ *for all* $\eta \in \operatorname{dom}(\delta)$,
- $\operatorname{span}\{\delta(\eta)x \mid \eta \in \operatorname{dom}(\delta),\ x \in \mathscr{M}\}$ *is dense in* $\mathscr{H}$,
- $\mathscr{E}(\eta) = \|\delta(\eta)\|^2$ *for all* $\eta \in \operatorname{dom}(\mathscr{E})$,
- $L_2 = \delta^* \circ \delta$, *where* L_2 *is the generator of* $\mathscr{E}$.

Remark 2.3.11. Their result is somewhat more general in several ways. For now, we will point out that τ could also have been a faithful normal tracial weight instead of a state.

In the above theorem, $D = \operatorname{dom}(\mathscr{E}) \cap \tau^{1/4}\mathscr{M}\tau^{1/4}$ is a form core for $\mathscr{E}$. D is a $*$-algebra, and when δ is restricted to D, it is a derivation in the sense of Definition 2.3.6. Generalising this theorem to the case where one considers a non-tracial state is the reason twisted derivations were created. Section 3.2 focusses on the question whether a theorem like Theorem 2.3.10 is possible for GNS-symmetric quantum Markov semigroups for a non-tracial state φ if one demands to obtain a true derivation structure. The remainder of Chapter 3 aims to generalise Theorem 2.3.10 to KMS-symmetric quantum Markov semigroups for non-tracial states φ.

Finite-dimensional theory: Derivations and twisted derivations

Definition 2.3.6 does not change in the finite-dimensional case. However, we can now define twisted derivations in an algebraic way.

Definition 2.3.12. Let $\mathscr{A}$ be a $*$-algebra. A *twisted derivation* δ from $\mathscr{A}$ into an $\mathscr{A}$-$\mathscr{A}$-bimodule is a linear map such that there exist algebra homomorphisms α, β on $\mathscr{A}$ satisfying

$$\delta(xy) = \alpha(x)\delta(y) + \delta(x)\beta(y).$$

For a von Neumann algebra $\mathscr{M}$ with faithful state φ, a twisted derivation in the sense of Definition 2.3.7 is a twisted derivation on $\mathscr{M}$ with $\alpha = \sigma_{-i/4}$ and $\beta = \sigma_{i/4}$. Unless specified otherwise, in this work a twisted derivation will mean a twisted derivation in the sense of Definition 2.3.7, although we may view it through the lens of Definition 2.3.12.

A first-order differential calculus $(\mathscr{H}, \mathscr{J}, \delta)$ is a twisted derivation δ into a self-dual correspondence $(\mathscr{H}, \mathscr{J})$ that intertwines J and $\mathscr{J}$ and whose image generates $\mathscr{H}$ as a bimodule. We can now formulate Theorem 2.3.10 in terms of first-order differential calculi.

Theorem 2.3.13. *Let $\mathscr{M}$ be a von Neumann algebra with faithful tracial state τ and let $(\Phi_t)_{t\geq 0}$ be a symmetric quantum Markov semigroup with generator L. Then there exists a first-order differential calculus $(\mathscr{H}, \mathscr{J}, \delta)$ such that $\delta^* \circ \delta = L$.*

2

2.3.3. ULTRAPRODUCTS OF HILBERT SPACES

We conclude the preliminaries by describing a way to take some sort of limit of Hilbert spaces, called an ultraproduct. This is usually introduced through ultrafilters, which belong to the mathematical field of logic. However, there is a functional analytic connection that we can exploit here to give the definitions we need while avoiding ultrafilters.

Consider $\ell^\infty(\mathbb{N})$, which is a commutative von Neumann algebra, and as such, a commutative C*-algebra. Consequently, $\ell^\infty(\mathbb{N}) \cong C(X)$ for some compact Hausdorff space X [Con10, Theorem VII.2.1], which is called the Stone-Čech compactification and denoted by $\beta\mathbb{N}$ [Con10, Stone-Čech Compactification, p. 137]. It consists of all non-zero homomorphisms from $\ell^\infty(\mathbb{N})$ to $\mathbb{C}$ [Con10, Proposition VII.8.2] and as such naturally contains $\mathbb{N}$. In fact, $\mathbb{N}$ is dense in $\beta\mathbb{N}$, as any element of $C(\beta\mathbb{N}) \cong \ell^\infty(\mathbb{N})$ is fully determined by its values on $\mathbb{N}$. Consequently, whenever we have a bounded sequence of complex numbers $(a_n)_{n\in\mathbb{N}}$, we can in principle find its 'limit' $\lim_{n\to\omega} a_n$ at any point $\omega \in \beta\mathbb{N}\backslash\mathbb{N}$. We will use this to define ultraproducts of Hilbert spaces.

Construction 2.3.14. Let $(\mathcal{H}_n)_{n\in\mathbb{N}}$ be a sequence of Hilbert spaces. Let $K = \{(\eta_n) \in \prod_{n\in\mathbb{N}} \mathcal{H}_n \mid \sup_{n\in\mathbb{N}} \|\eta_n\| < \infty\}$, which is a Banach space under the supremum norm. Now fix an $\omega \in \beta\mathbb{N}\backslash\mathbb{N}$. We can define a positive sesquilinear form on K by $\langle(\eta_n)_{n\in\mathbb{N}}, (\xi_n)_{n\in\mathbb{N}}\rangle = \lim_{n\to\omega}\langle\eta_n, \xi_n\rangle$, which has kernel $I = \{(\eta_n) \in \prod_{n\in\mathbb{N}} \mathcal{H}_n \mid \lim_{n\to\omega}\|\eta_n\| = 0\}$. As I is closed, we know that $H = K/I$ is a Hilbert space, and we call this the *ultraproduct* of $(\mathcal{H}_n)_{n\in\mathbb{N}}$, denoted by $\prod_{n\to\omega} \mathcal{H}_n$.

Notation 2.3.15. We denote the equivalence class of $(\eta_n)_{n\in\mathbb{N}}$ in $\prod_{n\to\omega}\mathcal{H}_n$ by $[\eta_n]$.

2.4. NOTATION OVERVIEW

Here one can find a brief overview of some common notation used in this dissertation. In some exceptional cases we may deviate from this notation.

- $\mathcal{H}, \mathcal{K}$ - Hilbert spaces
- η, ξ - elements of a Hilbert space
- $B(\mathcal{H})$ - bounded operators on a Hilbert space
- $\langle\cdot,\cdot\rangle$ - inner product
- $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ - inner product on Hilbert space H
- $M_n(\mathbb{C})$ - the set of $n \times n$ matrices
- A, B, C, D, E, X, Y - matrices
- $M^{\text{sym}}_{n\times n}(\mathbb{R})$ - symmetric $n \times n$ real-valued matrices
- X^T for $X \in M_n(\mathbb{C})$ - transpose
- Tr - trace
- tr - normalised trace

- $\mathscr{A}, \mathscr{B}$ - algebras, $*$-algebras, C*-algebras or answer sets
- $\mathscr{M}, \mathscr{N}$ - von Neumann algebras
- $L^2(\mathscr{M})$ - standard Hilbert space associated to $\mathscr{M}$ with fixed faithful normal state
- $\mathscr{X}, \mathscr{Y}$ - question sets
- $x,\ y$ - elements of a von Neumann algebra or questions from a question set
- $a,\ b$ - elements of a von Neumann algebra or answers from an answer set; sometimes elements of $L^2(\mathscr{M})$
- c - element of a von Neumann algebra or a constant
- d - element of a von Neumann algebra or a dimension
- $\mathscr{A} \odot \mathscr{B}$ - algebraic tensor product
- $\mathscr{M} \otimes \mathscr{N}$ - spacial tensor product of von Neumann algebras
- $\mathscr{H} \otimes \mathscr{K}$ - tensor product of Hilbert spaces
- $\mathscr{M} \otimes_{\text{bin}} \mathscr{N}$ - binormal tensor product
- $x \otimes y$ - tensor product of elements x and y in any kind of tensor product
- $\mathrm{Tr}_{\mathscr{H}_2}$ on $B(\mathscr{H}_1) \otimes B(\mathscr{H}_2)$ - partial trace ([NC10, Section 8.3.1])
- 1 - number, identity in an algebra
- $1_n,\ 1_{\mathscr{H}},\ 1_{\mathscr{A}}$ - identity in the corresponding algebra $(M_n(\mathbb{C}),\ B(\mathscr{H}),\ \mathscr{A})$
- id - identity map (usually on an algebra)
- id_n, $\mathrm{id}_{\mathscr{A}}$ - identity map on the corresponding algebra $(M_n(\mathbb{C}),\ \mathscr{A})$
- $\|\cdot\|$ - norm (operator norm on a C*-algebra)
- $\|\cdot\|_p$ - L^p-norm based on the trace
- $\|\cdot\|_\tau$ - 2-norm based on tracial state τ.
- $\varphi,\ \rho$ - state, positive linear functional or weight; sometimes density matrix
- $\varphi^{1/2},\ \rho^{1/2}$ - cyclic vector for the state or square root of density matrix
- τ - tracial state, positive linear functional or weight
- d_φ - density matrix
- $\langle\cdot,\cdot\rangle_\rho$ - KMS inner product with respect to state ρ
- Φ - (completely positive) map on C*-algebra

- $(\Phi_t)_{t\geq 0}$ - quantum Markov semigroup
- (Φ_t) - quantum Markov semigroup
- L - generator on von Neumann algebra
- L_2 - generator on Hilbert space
- x^* - adjoint
- $\Phi^\dagger$ - adjoint of map on C*-algebra
- $\eta^\sharp$, $\eta^\flat$ - involutions in left or right Hilbert algebras
- Δ - modular operator or comultiplication
- σ_t^φ - modular automorphism group
- χ_Λ - characteristic function of the subset Λ
- $\mathscr{C}$ - class of strategies
- $\mathbb{F}_q$ - finite field of order q for some prime power q
- G - game or (quantum) graph
- Γ - discrete group
- $\mathscr{C}(\Gamma, S)$ - Cayley graph of Γ with generating set S
- $\mathbf{G}$ - compact quantum group
- $\mathbf{\Gamma}$ - discrete quantum group
- (S, ε) - Kazhdan pair for a group
- (E, ε) - Kazhdan pair for a quantum group
- $|S|$ - number of elements of a set S

General conventions

We also record a few general conventions adhered to throughout this dissertation.

- When talking about positivity of matrices, we always mean positive as an element of the C*-algebra of $M_n(\mathbb{C})$.
- Inner products are linear in the second component.
- $\mathbb{R}_+$ includes zero.

3

Derivations for KMS-symmetric quantum Markov semigroups

This chapter is based on the following articles:

- Matthijs Vernooij. “On the existence of derivations as square roots of generators of state-symmetric quantum Markov semigroups”. In: *Infinite Dimensional Analysis, Quantum Probability and Related Topics* 27.01 (2024), p. 2350003.
- Matthijs Vernooij and Melchior Wirth. “Derivations and KMS-Symmetric Quantum Markov Semigroups”. In: *Communications in Mathematical Physics* 403.1 (2023), pp. 381–416.

Cipriani and Sauvageot have shown that the generator of the L^2 implementation of a tracially symmetric quantum Markov semigroup can be expressed as the square of a derivation with values in a complete bimodule. Here, we study the existence of (twisted) derivations corresponding to quantum Markov semigroups which satisfy symmetry properties with respect to a non-tracial state.

First, we show that this construction of a derivation can in general not be generalised to quantum Markov semigroups that are symmetric with respect to a non-tracial state. In particular we show that all derivations to complete bimodules can be assumed to have a concrete form, and then we use this form to show that in the finite-dimensional case the existence of such a derivation is equivalent to the existence of a positive matrix solution of a system of linear equations. We solve this system of linear equations for concrete examples using Mathematica to complete the proof.

Second, we prove that the generator of the L^2 implementation of a KMS-symmetric quantum Markov semigroup can be expressed as the square of a twisted derivation with values in a complete bimodule, extending a subsequently released result by Wirth for GNS-symmetric semigroups. This result hinges on the introduction of a new completely positive *map on the algebra of bounded operators on the GNS Hilbert space. This transformation maps symmetric Markovian operators to symmetric Markovian operators and is essential to obtain the required inner product on the complete bimodule. This transformation was discovered by adapting the approach in the first part to the twisted derivations introduced by Wirth.*

3.1. INTRODUCTION

Quantum Markov semigroups are a versatile tool that has found applications not only in quantum statistical mechanics, where they were originally introduced in the description of certain open quantum systems [Ali76; GKS76; Kos72; Lin76], but also in various purely mathematical fields such as noncommutative harmonic analysis [JX07; JMP14], noncommutative probability [Bia03; CFK14], noncommutative geometry [Arh23; CS03a; Sau96] and the structure theory of von Neumann algebras [CS15; CS17; Pet09a].

One central question from the beginning was to describe the generators of quantum Markov semigroups. For quantum Markov semigroups acting on matrix algebras, a characterisation of their generators was given by Lindblad [Lin76] and Gorini-Kossakowski-Sudarshan [GKS76] and later extended to generators of uniformly continuous quantum Markov semigroups on arbitrary von Neumann algebras by Christensen-Evans [CE79]. While partial results are known in particular for type I factors [AZ15; Dav79; Hol95], a similarly explicit description of unbounded generators of quantum Markov semigroups on arbitrary von Neumann algebras seems out of reach.

Both for the modelling of open quantum systems and purely mathematical questions in noncommutative probability, operator algebra theory, etc., one is often not interested in arbitrary quantum Markov semigroups, but quantum Markov semigroups that are symmetric with respect to a reference state or weight. In quantum statistical mechanics, these describe open systems coupled to a heat bath in thermal equilibrium. From the mathematical standpoint, symmetry with respect to a reference state allows to extend the semigroups to symmetric semigroups on the GNS Hilbert space, which makes the powerful tools for self-adjoint Hilbert space operators available.

If the reference state or weight is a trace, there is an unambiguous notion of symmetry, called tracial symmetry. The study of tracially symmetric quantum Markov semigroups through their associated quadratic forms, so-called Dirichlet forms, was initiated by Albeverio and Høegh-Krohn [AH77] and further developed by Lindsay and Davies [DL92; DL93]. The (Hilbert space) generators of tracially symmetric quantum Markov semigroups have been characterised by Cipriani and Sauvageot [CS03a] to be of the form $\delta^*\delta$, where δ is a derivation with values in a complete bimodule. This result has lead to a wide range of applications from analysis on fractals [HT13] and metric graphs [BK19] over noncommutative geometry [CS03b], noncommutative probability [Dab10; JZ15], quantum optimal transport [Wir21; WZ21] to the structure theory of von Neumann algebras and in particular Popa's deformation and rigidity theory [Pet09b; DI16; Cas21; CIW21].

Despite this success, the notion of tracial symmetry is somewhat limiting. For one, von Neumann algebras with a type III summand do not even admit a (faithful normal) trace. But even on semi-finite von Neumann algebras, plenty of quantum Markov semigroups of interest are not tracially symmetric. For example, in many models of open quantum systems one considers quantum Markov semigroups symmetric with respect to a Gibbs state, which is only a trace in the infinite temperature limit. For these reasons, we now turn our attention to the setting where our von Neumann algebras are equipped with a general, not necessarily tracial, state.

In this setting, we will address two different questions in this chapter. In Section 3.1.1 of the introduction, followed by 3.2, we investigate whether a Cipriani and Sauvageot type result can be achieved in its most obvious formulation. We show, using certain computer experiments, that this is impossible. Section 3.1.2 of the introduction, together with the remainder of the chapter, treats how to generalise Cipriani and Sauvageot to the non-tracial setting in the correct formulation. This result, achieved together with Wirth, extends his results in [Wir22], which was published after the release of [Ver24], upon which Section 3.2 is based. To properly reflect how the results in this dissertation intertwine with Wirth's results, a modular structure has been chosen, where Sections 3.1.1 and 3.2 can be read largely independently from the other sections. Moreover, a discussion on the connection between the two parts has been added in the form of Section 3.2.4.

3.1.1. (COUNTER)EXAMPLES FOR THE EXISTENCE OF DERIVATIONS

Here, we pose the following problem, which has already been mentioned by Caspers [Cas21, p. 279] and by Skalski and Viselter [SV19, p. 62]:

Problem 1 (Abstract version)**.** Is it possible to generalise the construction by Cipriani and Sauvageot of a derivation that is the square root of a generator of a QMS to non-tracial states?

There are two things that need to be specified to make Problem 1 concrete. Let φ be a faithful normal state on a von Neumann algebra $\mathscr{M}$. There are many natural inner products on $\mathscr{M}$ based on φ depending on the embedding of $\mathscr{M}$ in its L^2 space, as was observed by Kosaki [Kos84] (see also Section 2.1.2), and we need to choose which one we want to use. One class of inner products can be obtained by the embeddings $x \mapsto \Delta^{s/2} x \varphi^{1/2}$ of $\mathscr{M}$ into $L^2(\mathscr{M})$ for every $s \in [0,1]$. In the matrix algebra setting, these inner products are also described by Carlen and Maas [CM17] as the inner products given by

$$\langle a, b\rangle_s = \varphi(\sigma^{\varphi}_{si}(a^*)b),$$

for $s \in [0,1]$. $\langle\cdot,\cdot\rangle_0$ and $\langle\cdot,\cdot\rangle_{\frac{1}{2}}$ correspond to the GNS or KMS inner product, respectively, and we will mainly focus on these inner products.

We also need to decide which requirements we put on the codomain $\mathscr{H}$ of the derivation. Usually, one requires $\mathscr{H}$ to be a (complete) $\mathscr{M}$-$\mathscr{M}$-bimodule (see Definition 2.3.6). However, Carlen and Maas have shown that when $\mathscr{M}$ is finite-dimensional and $(\Phi_t)_{t\geq 0}$ is a QMS with detailed balance, i.e. it is symmetric with respect to the GNS inner product, then there exists a derivation δ from $\mathscr{M}$ with the KMS inner product to something akin to an $\mathscr{M}$-$\mathscr{M}$-bimodule $\mathscr{H}$ with an inner product such that the generator of $(\Phi_t)_{t\geq 0}$ is given by $\delta^* \circ \delta$ [CM20]. Unfortunately, the left and right actions of $\mathscr{M}$ on $\mathscr{H}$ are not $*$-homomorphisms, so $\mathscr{H}$ is not truly an $\mathscr{M}$-$\mathscr{M}$-bimodule. This would cause analytical issues if one were to go to infinite dimensions, and it is an indication that one is looking at the problem in the wrong way (see also the results by Wirth described in 3.1.2). Here, we insist that the codomain of the derivation is a complete $\mathscr{M}$-$\mathscr{M}$-bimodule, as was the case in the construction of Cipriani and Sauvageot, and investigate whether a derivation with similar properties still exists. This leads us to the following concrete version of Problem 1:

Problem 2 (Concrete version)**.** Let $s \in [0,1]$. For a von Neumann algebra $\mathscr{M}$ with faithful normal state φ and a GNS-symmetric quantum Markov semigroup $(\Phi_t)_{t\geq 0}$ on $\mathscr{M}$ with L^2-generator L_2, do there always exist a complete $\mathscr{M}$-$\mathscr{M}$-bimodule $\mathscr{H}$ and a densely defined derivation $\delta : \mathscr{M} \to \mathscr{H}$ such that $L_2 = \delta^* \circ \overline{\delta}$ when we consider δ as an operator on $L^2(\mathscr{M}, \langle\cdot,\cdot\rangle_s)$?

We consider the case of GNS-symmetric quantum Markov semigroups (see Section 2.2.2), because this is the strongest symmetry condition we can impose on the quantum Markov semigroups. Consequently, if one were to provide a negative answer to this problem, one immediately provides a negative answer to the related problem for KMS-symmetric quantum Markov semigroups. This is precisely what we do in Section 3.2. We provide a negative answer to Problem 2 for $s = 0$ and $s = 1/2$. Additionally, the method we use for $s = 0$ and $s = 1/2$ can be applied directly to the other cases, and we hypothesise that the answer is negative for all $s \in [0,1]$. Lastly, we show that the same method can be used to analyse twisted derivations (see Definition 2.3.7) after minor modifications.

We conclude the introduction by giving a short overview of the proof. We will first show that we only have to consider Hilbert spaces of a concrete form in Section 3.2.1. Next, we will prove Theorem 3.2.6, which states that for finite-dimensional von Neumann algebra the question whether such a derivation exists can be transformed to the solvability of a system of linear equations, with the additional requirement that the resulting solution matrix is positive. Together this gives a method to check for any finite-dimensional von Neumann algebra $\mathscr{M}$ with state φ, inner product $\langle\cdot,\cdot\rangle_s$ and generator of a QMS L whether there exists a derivation δ to a complete $\mathscr{M}$-$\mathscr{M}$-bimodule such that $L_2 = \delta^* \circ \delta$. We conclude by applying these methods to GNS-self-adjoint generators of QMSs on $M_2(\mathbb{C})$ and $M_3(\mathbb{C})$ with the GNS or KMS inner product in Section 3.2.3. We show that these derivations sometimes, but not always, exist by using Mathematica to solve the systems of linear equations that appear. This shows that Problem 2 has a negative answer in the $M_3(\mathbb{C})$ case. A more detailed analysis of the $M_3(\mathbb{C})$ case with the GNS inner product shows that states and corresponding generators for which such a derivation exist are rare. This is made precise in Example 3.2.11.

After completion of [Ver24], upon which Section 3.2 is based, a preprint by Wirth became available, which shows that the construction of Cipriani and Sauvageot can be generalised for GNS-symmetric QMSs if one generalises the derivation to a twisted derivation and looks for a first-order differential calculus (see Definition 2.3.9) [Wir22]. This result is described in detail in Section 3.1.2. Our results can be seen as proof that this generalisation to twisted derivations cannot be avoided and suggest that the result by Wirth is the best one can hope to achieve. For this more general class of strategies, one can use the same approach to investigate whether a first-order differential calculus exists in the KMS-symmetric case. This led directly to the discovery of the $\mathscr{V}$-transform in finite dimensions (see Definition 3.3.2). The background for these computer experiments has been added as Section 3.2.4 for completeness.

3.1.2. TWISTED DERIVATIONS FOR KMS-SYMMETRIC SEMIGROUPS

We now continue where we left off before Section 3.1.1, where we were interested in quantum Markov semigroups that satisfy some symmetry property with respect to a non-tracial state. In this case, there are several non-equivalent notions of symmetry, for example GNS symmetry and KMS symmetry (see Section 2.2.2). If the state is of the form $\mathrm{Tr}(\cdot\,\rho)$ on a type I factor, then GNS symmetry is symmetry with respect to the inner product $(x, y) \mapsto \mathrm{Tr}(x^* y\rho)$, while KMS symmetry is symmetry with respect to the inner product $(x, y) \mapsto \mathrm{Tr}(x^* \rho^{1/2} y \rho^{1/2})$.

GNS symmetry is the strongest one of the symmetry conditions usually considered, and it also implies commutation with the modular group, which makes the structure of GNS-symmetric quantum Markov semigroups particularly nice and rich. For GNS-symmetric quantum Markov semigroups on matrix algebras, Alicki's theorem [Ali76] gives a characterisation of their generators in the spirit of the results of Lindblad and Gorini-Kossakowski-Sudarshan. But it can also be recast as a representation of the generator as square of a derivation, thus presenting an analogue of the result of Cipriani and Sauvageot mentioned above (see [CM17]). In this process one loses the property that the left and right action are $*$-homomorphisms, but this is unavoidable, as is discussed in Section 3.2.

This result of Alicki has played a central role in recent research motivated by quantum information theory, in particular for the development of a dynamical quantum optimal transport distance [CM17], relating hypercontractivity and logarithmic Sobolev inequalities for quantum systems [Bar17] and the proof of the complete modified logarithmic Sobolev inequality for finite-dimensional GNS-symmetric quantum Markov semigroups [GR22].

Moving beyond matrix algebras, Wirth established a version of the Christensen-Evans theorem for generators of uniformly continuous GNS-symmetric quantum Markov semigroups [Wir24] and a generalisation of the result of Cipriani and Sauvageot for generators of arbitrary GNS-symmetric quantum Markov semigroups [Wir22].

Let us describe the latter result in some more detail. If a quantum Markov semigroup (Φ_t) is GNS-symmetric with respect to the state (or more generally weight) φ on a von Neumann algebra $\mathcal{M}$, then it has a GNS implementation as a strongly continuous semigroup $(\Phi_t^{(2)})$ on the GNS Hilbert space $L^2(\mathcal{M})$. Let $\mathcal{E}$ denote the associated quadratic form and $\mathcal{A}_0$ the maximal Tomita algebra induced by φ. It is shown in [Wir22] that

$$\mathcal{A}_{\mathcal{E}} = \{a \in \mathcal{A}_0 \mid \Delta^z(a) \in \mathrm{dom}(\mathcal{E}) \text{ for all } z \in \mathbb{C}\}$$

is a Tomita algebra and a form core for $\mathcal{E}$. Moreover, there exists a Hilbert space $\mathcal{H}$ with commuting left and right actions of $\mathcal{A}_{\mathcal{E}}$ and a closable operator $\delta\colon \mathcal{A}_{\mathcal{E}} \to \mathcal{H}$ satisfying the Leibniz rule $\delta(ab) = a\delta(b) + \delta(a)b$ such that

$$\mathcal{E}(a, b) = \langle \delta(a), \delta(b) \rangle_{\mathcal{H}}$$

for $a, b \in \mathcal{A}_{\mathcal{E}}$. Furthermore, $\mathcal{H}$ carries an anti-unitary involution and a strongly continuous unitary group with certain compatibility conditions that reflect the commutation of $(\Phi_t^{(2)})$ with the modular operator and modular conjugation. In contrast to the tracially symmetric case, the left and right action of $\mathfrak{A}_{\mathcal{E}}$ are $*$-homomorphisms with respect to two different involutions on $\mathcal{A}_{\mathcal{E}}$, owing to the fact that $\mathcal{A}_{\mathcal{E}}$ carries both the structure of a left and a right Hilbert algebra, which are different if the reference state is not a trace.

It is then natural to ask whether this relation between L^2 generators of quantum Markov semigroups and twisted derivations can be extended to KMS-symmetric semigroups as KMS symmetry can be seen as the more natural assumption in some contexts. For one, every completely positive map can be decomposed as a linear combination of KMS-symmetric ones using the Accardi-Cecchini adjoint [AC82], while for GNS symmetry the commutation with the modular group poses an algebraic constraint. This makes KMS-symmetric quantum Markov semigroups more suitable for various applications, such as the characterisation of the Haagerup property in terms of KMS-symmetric quantum Markov semigroups [CS15], while the same property for GNS-symmetric semigroups is more restrictive. But also in quantum statistical mechanics, irreversible open quantum systems are often modelled by quantum Markov semigroups that are only KMS-symmetric rather than GNS-symmetric, such as the heat-bath dynamics introduced in [KB16] for example.

The lack of commutation with the modular group poses a serious challenge. For example, many questions regarding noncommutative L^p spaces can be reduced to L^p spaces with respect to a trace by Haagerup's reduction method [HJX10], but commutation with the modular group is necessary for maps to be compatible with this reduction procedure.

Even for KMS-symmetric quantum Markov semigroups on type I factors, when explicit representations of the generator are known [FU10; AC21], it is not obvious if the generator can be expressed in the form $\delta^*\delta$ for a twisted derivation.

For these reasons it has not been clear if one should even expect that a Cipriani-Sauvageot-type result holds for KMS-symmetric quantum Markov semigroups. In the latter parts of this chapter we show that this is indeed the case, not only on matrix algebras, but arbitrary von Neumann algebras. For generators of uniformly continuous quantum Markov semigroups, our main result is the following (Theorems 3.5.2, 3.5.9 in the main part). These results are most naturally formulated in the language of Hilbert algebras, and we use this formulation in the main part, in which case the triple $(\mathcal{H}, \mathcal{J}, \delta)$ below is a bounded first-order differential calculus (see Definition 2.3.9). Here, we will avoid Hilbert algebras for the sake of accessibility and express the theorems in terms of embeddings of $\mathcal{M}$ into $L^2(\mathcal{M})$. We write Ω_φ for the cyclic and separating vector implementing φ and denote the KMS implementation of L on $L^2(\mathcal{M})$ by L_2.

Theorem 3.A. *Let (Φ_t) be a uniformly continuous quantum Markov semigroup on $\mathcal{M}$ that is KMS-symmetric with respect to φ, and let L denote its generator. There exists a Hilbert space $\mathcal{H}$ with commuting normal left and right actions of $\mathcal{M}$, an anti-linear involution $\mathcal{J}\colon \mathcal{H} \to \mathcal{H}$ and a bounded operator $\delta\colon L^2(\mathcal{M}) \to \mathcal{H}$ satisfying*

(a) $\mathcal{J}(x\xi y) = y^*(\mathcal{J}\xi)x^*$ *for all* $x, y \in \mathcal{M}$ *and* $\xi \in \mathcal{H}$,

(b) $\delta(\Omega_\varphi x^*) = \mathcal{J}\delta(x\Omega_\varphi)$ *for all* $x \in \mathcal{M}$,

(c) $\delta(x\Omega_\varphi y) = x\delta(\Omega_\varphi y) + \delta(x\Omega_\varphi)y$ *for all* $x, y \in \mathcal{M}$,

(d) $\overline{\mathrm{lin}}\{\delta(a)x \mid a \in L^2(\mathcal{M}), x \in \mathcal{M}\} = \mathcal{H}$

such that

$$L_2 = \delta^*\delta.$$

Moreover, there exists $\xi \in \mathcal{H}$ *such that*

$$\delta(a) = x\xi - \xi y$$

whenever $a = x\Omega_\varphi = \Omega_\varphi y$ *for* $x,\ y \in \mathcal{M}$.

Furthermore, a triple $(\mathcal{H}, \mathcal{J}, \delta)$ *satisfying (a)–(d) is uniquely determined by* $\delta^*\delta$ *up to isomorphism.*

For KMS-symmetric quantum Markov semigroups that are not uniformly continuous we do not have a uniqueness result, but we can still prove existence in the following form (Theorems 3.6.1, 3.6.3 in the main part).

Theorem 3.B. *Let* (Φ_t) *be a KMS-symmetric quantum Markov semigroup on* $\mathcal{M}$ *with generator* L. *If* $a\Omega_\varphi \in \mathrm{dom}(L_2^{1/2}) \cap \mathcal{M}\Omega_\varphi$ *and* $\Omega_\varphi b \in \mathrm{dom}(L_2^{1/2}) \cap \Omega_\varphi\mathcal{M}$, *then* $a\Omega_\varphi b \in \mathrm{dom}(L_2^{1/2})$.

Moreover, there exists a Hilbert space $\mathcal{H}$ *with commuting left and right actions of* $\mathcal{M}$, *an anti-unitary involution* $\mathcal{J}\colon \mathcal{H} \to \mathcal{H}$ *and a closed operator* $\delta\colon \mathrm{dom}(L_2^{1/2}) \to \mathcal{H}$ *satisfying*

(a) $\mathcal{J}(x\xi y) = y^*(\mathcal{J}\xi)x^*$ *for* $x,\ y \in \mathcal{M}$ *and* $\xi \in \mathcal{H}$,

(b) $\delta(\Omega_\varphi x^*) = \mathcal{J}\delta(x\Omega_\varphi)$ *for* $x \in \mathcal{M}$ *with* $x\Omega_\varphi \in \mathrm{dom}(L_2^{1/2})$,

(c) $\delta(x\Omega_\varphi y) = x\delta(\Omega_\varphi y) + \delta(x\Omega_\varphi)y$ *for* $x,\ y \in \mathcal{M}$ *with* $x\Omega_\varphi, \Omega_\varphi y \in \mathrm{dom}(L_2^{1/2})$,

(d) $\overline{\mathrm{lin}}\{\delta(a)x \mid a \in \mathrm{dom}(L_2^{1/2}),\ x \in \mathcal{M}\} = \mathcal{H}$

such that

$$L_2 = \delta^*\delta.$$

Establishing these results requires a fundamentally new tool in the form of a quantum channel on $B(L^2(\mathcal{M}))$, which we call the $\mathcal{V}$*-transform*. Formally, the $\mathcal{V}$-transform of $T \in B(L^2(\mathcal{M}))$ is the solution S of the equation

$$\frac{1}{2}(\Delta^{1/4}S\Delta^{-1/4} + \Delta^{-1/4}S\Delta^{1/4}) = T,$$

where Δ is the modular operator.

A remarkable fact about this map is that it maps positivity-preserving maps (with respect to the self-dual cone induced by φ) to positivity-preserving maps. This property is key for the existence of the Hilbert space $\mathcal{H}$ in our main results.

Let us briefly summarise the outline of the remainder of this chapter after Section 3.2, which deals with the computer experiments. To make the proof strategy transparent without the technical difficulties occurring for general von Neumann algebras, but also to make the main results more accessible for researchers in the quantum information theory community, we first develop the $\mathcal{V}$-transform and prove our main result for matrix algebras in Section 3.3. In Section 3.4 we define the $\mathcal{V}$-transform on general von Neumann algebras and establish some of its properties in particular regarding positivity preservation. In Section 3.5 we prove our main results on existence and uniqueness of twisted derivations associated with uniformly continuous KMS-symmetric quantum Markov semigroups. Finally, in Section 3.6 we show the existence of twisted derivations associated with not necessarily uniformly continuous semigroups.

3

3.2. (COUNTER)EXAMPLES FOR THE EXISTENCE OF DERIVATIONS FOR QUANTUM MARKOV SEMIGROUPS

3.2.1. THE GENERAL FORM OF A DERIVATION

In this section we aim to show that any derivation from a $*$-algebra $\mathscr{A}$ to a Hilbert space can be viewed as a derivation from $\mathscr{A}$ to $\mathscr{A}\odot\mathscr{A}$ with an appropriate left-multiplication. This bimodule structure on $\mathscr{A}\odot\mathscr{A}$ and derivation were also used in the work of Cipriani and Sauvageot [CS03a]. More precisely, we will prove Theorem 3.2.2, but first we will recall the notion of a nondegenerate right action.

Definition 3.2.1. Let $\mathscr{A}$ be an algebra that acts on a Hilbert space $\mathscr{H}$ from the right. We call the action *nondegenerate* if for every non-zero $\eta\in\mathscr{H}$ there exists an $a\in\mathscr{A}$ such that $\eta a\neq 0$.

Theorem 3.2.2. *Let $\mathscr{A}$ be a $*$-algebra and $\mathscr{H}$ a complete $\mathscr{A}$-$\mathscr{A}$-bimodule with nondegenerate right action. Suppose that $\delta:\mathscr{A}\to\mathscr{H}$ is a derivation. Then there exist an $\mathscr{A}$-$\mathscr{A}$-bimodule structure on $\mathscr{A}\odot\mathscr{A}$ with positive sesquilinear form $\langle\cdot,\cdot\rangle$, an isometric bimodule homomorphism $\varphi:\mathscr{A}\odot\mathscr{A}\to\mathscr{H}$, which extends to $\overline{\mathscr{A}\odot\mathscr{A}/\langle\cdot,\cdot\rangle}$, and a derivation $\partial:\mathscr{A}\to\overline{\mathscr{A}\odot\mathscr{A}/\langle\cdot,\cdot\rangle}$ such that $\delta=\varphi\circ\partial$ and $\mathscr{A}\delta(\mathscr{A})\mathscr{A}\subset\varphi(\mathscr{A}\odot\mathscr{A})$. The left and right multiplication of $\mathscr{A}$ on $\mathscr{A}\odot\mathscr{A}$ is given by*

$$a(b\otimes c)d=ab\otimes cd-a\otimes bcd$$

for all $a,b,c,d\in\mathscr{A}$. If $\mathscr{A}$ is unital, then ∂ maps via $\mathscr{A}\odot\mathscr{A}$ and is given by

$$\partial(a)=[a\otimes 1]\in\mathscr{A}\odot\mathscr{A}/\langle\cdot,\cdot\rangle,$$

where $[\cdot]$ denotes the equivalence class in the quotient.

Remark 3.2.3. Note that for unital $*$-algebras $\mathscr{A}$ the nondegeneracy condition is equivalent to the right action being unital, as can be seen by a short calculation.

Proof. Let $\mathscr{H}_0\subset\mathscr{H}$ be the linear subspace generated by $\delta(\mathscr{A})\mathscr{A}$. In other words, $\mathscr{H}_0$ is given by

$$\mathscr{H}_0=\{\sum_{i=1}^{n}\delta(a_i)b_i\mid n\in\mathbb{N},a_i,b_i\in\mathscr{A},i\leq n\}.$$

Due to the Leibniz rule, we have that

$$\delta(ab)c-\delta(a)bc=a\delta(b)c=a\delta(bc)-ab\delta(c),$$

showing that $\mathscr{H}_0$ contains $\mathscr{A}\delta(\mathscr{A})\mathscr{A}$. Consequently, $\mathscr{H}_0$ is an $\mathscr{A}$-$\mathscr{A}$-bimodule.

First, we construct an $\mathscr{A}$-$\mathscr{A}$-bimodule $\mathscr{K}$. As the vector space, we take $\mathscr{A}\odot\mathscr{A}$, and we endow it with the left and right multiplication given by

$$a(b\otimes c)=ab\otimes c-a\otimes bc \text{ and } (a\otimes b)c=a\otimes bc$$

for all $a,\ b,\ c\in\mathscr{A}$, respectively. The right multiplication is clearly associative, and for the left multiplication the computation

$$\begin{aligned}a(b(c\otimes d))=a(bc\otimes d-b\otimes cd)&=abc\otimes d-a\otimes bcd-ab\otimes cd+a\otimes bcd\\&=abc\otimes d-ab\otimes cd=(ab)(c\otimes d)\end{aligned}$$

shows that it is indeed associative. This allows us to define the linear map $\varphi : \mathscr{K} \to \mathscr{H}_0$ by $\varphi(a \otimes b) = \delta(a)b$. This is clearly a right $\mathscr{A}$-module homomorphism, and the fact that it is a left $\mathscr{A}$-module homomorphism follows from

$$\varphi(a(b \otimes c)) = \varphi(ab \otimes c - a \otimes bc) = \delta(ab)c - \delta(a)bc = a\delta(b)c = a\varphi(b \otimes c).$$

φ is surjective by definition of $\mathscr{H}_0$. Since $\mathscr{A}\delta(\mathscr{A})\mathscr{A} \subset \mathscr{H}_0$, we know that $\mathscr{A}\delta(\mathscr{A})\mathscr{A} \subset \varphi(\mathscr{A} \odot \mathscr{A})$.

The next step is to define a positive sesquilinear form $\langle \cdot, \cdot \rangle_{\mathscr{K}}$ on $\mathscr{K}$ to turn $\mathscr{K}$ into an $\mathscr{A}$-$\mathscr{A}$-bimodule , which we do by setting

$$\langle a \otimes b, c \otimes d \rangle_{\mathscr{K}} = \langle \delta(a)b, \delta(c)d \rangle_{\mathscr{H}} = \langle \varphi(a \otimes b), \varphi(c \otimes d) \rangle_{\mathscr{H}}.$$

Because φ is a bimodule homomorphism, it also follows that the left and right multiplications on $\mathscr{K}$ are $*$-homomorphisms (see Definition 2.3.2(a,ii-iii)). It is immediately clear that φ is isometric by the definition of the positive sesquilinear form on $\mathscr{K}$. Because φ is isometric, we find that $\ker(\varphi) = \{x \in \mathscr{K} \mid \langle x, x \rangle_{\mathscr{K}} = 0\}$. Since φ is a bimodule homomorphism, we know that this kernel is a subbimodule. Therefore, $\mathscr{K}' = \mathscr{K} / \langle \cdot, \cdot \rangle$ is a bimodule and φ is an isometric bimodule isomorphism from $\mathscr{K}'$ to $\mathscr{H}_0$. Consequently, we can extend φ from the completion of $\mathscr{K}'$, $\overline{\mathscr{K}'}$, to $\overline{\mathscr{H}_0}$. Because $\pi(a)$ and $\pi'(a)$, denoting left and right multiplication by a, are bounded for all $a \in \mathscr{A}$, we know that $\overline{\mathscr{K}'}$ and $\overline{\mathscr{H}_0}$ are complete bimodules. By the same reasoning we see that $\overline{\mathscr{K}'}$ is an $\mathscr{A}$-$\mathscr{A}$-bimodule.

Lastly, we need to define the derivation $\partial : \mathscr{A} \to \overline{\mathscr{A} \odot \mathscr{A} / \langle \cdot, \cdot \rangle}$. First suppose that $\mathscr{A}$ is unital. In this case we will define $\partial : \mathscr{A} \to \mathscr{A} \odot \mathscr{A} / \langle \cdot, \cdot \rangle$ by $\partial(a) = [a \otimes 1]$, which then automatically also maps into $\overline{\mathscr{A} \odot \mathscr{A} / \langle \cdot, \cdot \rangle}$. This is a derivation, because

$$a(b \otimes 1) + (a \otimes 1)b = ab \otimes 1 - a \otimes b + a \otimes b = ab \otimes 1$$

and taking the equivalence class on both sides gives the Leibniz rule. Furthermore, we see that $\delta = \varphi \circ \partial$ indeed holds. This concludes the proof for the unital case.

Now we turn to the non-unital case. To define our derivation, we look at the densely defined linear functional l_a on $\overline{\mathscr{K}'}$ given by

$$l_a([b \otimes c]) = \langle \delta(a), \delta(b)c \rangle_{\mathscr{H}}$$

for all $b,\ c \in \mathscr{A}$. We know that $\|l_a\| \leq \langle \delta(a), \delta(a) \rangle_{\mathscr{H}}^{\frac{1}{2}}$ by the Cauchy-Schwarz inequality, so l_a can be extended to a bounded linear functional. Consequently, there exists a unique $x \in \overline{\mathscr{K}'}$ such that

$$l_a([b \otimes c]) = \langle x, [b \otimes c] \rangle_{\mathscr{K}'}$$

holds for all $b,\ c \in \mathscr{A}$, since $\overline{\mathscr{K}'}$ is a Hilbert space. This allows us to define $\partial(a)$ for each $a \in \mathscr{A}$ as the unique element in $\overline{\mathscr{K}'}$ such that

$$\langle \partial(a), [b \otimes c] \rangle_{\mathscr{K}'} = \langle \delta(a), \delta(b)c \rangle_{\mathscr{H}}$$

holds for all $b,\ c \in \mathscr{A}$. We can now do the required computation to show that ∂ is in fact a derivation. Here we will use that both $\mathscr{H}$ and $\overline{\mathscr{K}'}$ are $\mathscr{A}$-$\mathscr{A}$-bimodules and that φ is a bimodule homomorphism. Let $a,\ b,\ c,\ d \in \mathscr{A}$ be arbitrary. Then we have

$$\begin{aligned}\langle a\partial(b)+\partial(a)b,[c\otimes d]\rangle_{\mathscr{K}'} &= \langle a\partial(b),[c\otimes d]\rangle_{\mathscr{K}'}+\langle \partial(a)b,[c\otimes d]\rangle_{\mathscr{K}'}\\ &= \langle \partial(b),a^*([c\otimes d])\rangle_{\mathscr{K}'}+\langle \partial(a),[c\otimes db^*]\rangle_{\mathscr{K}'}\\ &= \langle \delta(b),\varphi(a^*([c\otimes d]))\rangle_{\mathscr{H}}+\langle \delta(a),\varphi([c\otimes db^*])\rangle_{\mathscr{H}}\\ &= \langle \delta(b),a^*\varphi([c\otimes d])\rangle_{\mathscr{H}}+\langle \delta(a),\varphi([c\otimes d])b^*\rangle_{\mathscr{H}}\\ &= \langle a\delta(b),\varphi([c\otimes d])\rangle_{\mathscr{H}}+\langle \delta(a)b,\varphi([c\otimes d])\rangle_{\mathscr{H}}\\ &= \langle \delta(ab),\varphi([c\otimes d])\rangle_{\mathscr{H}}\\ &= \langle \partial(ab),[c\otimes d]\rangle_{\mathscr{K}'},\end{aligned}$$

which shows that $\partial(ab)=\partial(a)b+a\partial(b)$. So we have shown that ∂ is a derivation. All that remains is to prove that $\delta=\varphi\circ\partial$. For all $a,\ b,\ c\in\mathscr{A}$ we have

$$\langle \varphi(\partial(a)),\delta(b)c\rangle_{\mathscr{H}}=\langle \varphi(\partial(a)),\varphi([b\otimes c])\rangle_{\mathscr{H}}=\langle \partial(a),[b\otimes c]\rangle_{\mathscr{K}'}=\langle \delta(a),\delta(b)c\rangle_{\mathscr{H}}.$$

Consequently, for all $a,\ b,\ c\in\mathscr{A}$ we see that

$$\langle \varphi(\partial(a))-\delta(a),\delta(b)c\rangle_{\mathscr{H}}=0.$$

But then we also have for all $a,\ b,\ c,\ d\in\mathscr{A}$ that

$$\langle (\varphi(\partial(a))-\delta(a))d^*,\delta(b)c\rangle_{\mathscr{H}}=\langle \varphi(\partial(a))-\delta(a),\delta(b)cd\rangle_{\mathscr{H}}=0,$$

which shows that for any $\eta\in\overline{\mathscr{H}_0}$ and $a,\ d\in\mathscr{A}$ we have

$$\langle (\varphi(\partial(a))-\delta(a))d,\eta\rangle_{\mathscr{H}}=0.$$

Since $\delta(a)d$ is contained in $\overline{\mathscr{H}_0}$ for all $a,\ d\in\mathscr{A}$, we have

$$\langle (\varphi(\partial(a))-\delta(a))d,(\varphi(\partial(a))-\delta(a))d\rangle_{\mathscr{H}}=0.$$

Therefore we can conclude that $(\varphi(\partial(a))-\delta(a))d=0$ for all $a,\ d\in\mathscr{A}$. But we know that the right action is nondegenerate, so this proves that $\varphi(\partial(a))=\delta(a)$, which is what we wanted to show. □

Remark 3.2.4. From now on we will refer to $\mathscr{A}\odot\mathscr{A}$ with the left and right multiplication of $\mathscr{A}$ on $\mathscr{A}\odot\mathscr{A}$ given by

$$a(b\otimes c)d=ab\otimes cd-a\otimes bcd$$

for all $a,\ b,\ c,\ d\in\mathscr{A}$ as the *canonical algebraic bimodule* $\mathscr{A}\odot\mathscr{A}$. We use this nomenclature as it requires a positive sesquilinear form to become an $\mathscr{A}$-$\mathscr{A}$-bimodule according to Definition 2.3.2.

Corollary 3.2.5. *Let $\mathscr{A}$ be a unital $*$-algebra and $f:\mathscr{A}\times\mathscr{A}\to\mathbb{C}$ a sesquilinear map. There exists a derivation $\delta:\mathscr{A}\to\mathscr{H}$ from $\mathscr{A}$ to a complete $\mathscr{A}$-$\mathscr{A}$-bimodule $\mathscr{H}$ with unital right action such that $\langle\delta(a),\delta(b)\rangle=f(a,b)$ for all $a,\ b\in\mathscr{A}$ if and only if there exist a positive sesquilinear form $\langle\cdot,\cdot\rangle_{\mathscr{A}\odot\mathscr{A}}$ on the canonical algebraic bimodule $\mathscr{A}\odot\mathscr{A}$ such that $\mathscr{A}\odot\mathscr{A}$ is an $\mathscr{A}$-$\mathscr{A}$-bimodule and $\langle a\otimes 1,b\otimes 1\rangle_{\mathscr{A}\odot\mathscr{A}}=f(a,b)$ holds for all $a,\ b\in\mathscr{A}$.*

Proof. The only if direction follows from Theorem 3.2.2. The converse direction follows by completing $\mathscr{A}\odot\mathscr{A}/\langle\cdot,\cdot\rangle$. This gives a bimodule because $\{x\in\mathscr{A}\odot\mathscr{A}\mid\langle x,x\rangle=0\}=\{x\in\mathscr{A}\odot\mathscr{A}\mid\langle x,y\rangle=0\ \forall y\in\mathscr{A}\odot\mathscr{A}\}$ is a subbimodule and the left and right actions extend to the completion because they are bounded. □

3.2.2. EXISTENCE OF DERIVATIONS AS SOLUTION OF SYSTEM OF LINEAR EQUATIONS

Our interest in the existence of derivations comes from the question whether a derivation exists that is the square root of a generator L of a quantum Markov semigroup. This means that we are looking for a derivation δ from a $*$-algebra $\mathscr{A}$ to a complete bimodule $\mathscr{H}$ such that $\langle\delta(a),\delta(b)\rangle_{\mathscr{H}} = \langle L^{\frac{1}{2}}(a), L^{\frac{1}{2}}(b)\rangle_{\mathscr{A}}$ for all $a,\ b \in \mathscr{A}$. Now Theorem 3.2.2 allows us to consider only bimodules of a specific form. This will be essential to formulate a procedure to either find such a derivation or show that it does not exist.

Theorem 3.2.6. *Let $\mathscr{A}$ be a unital finite-dimensional $*$-algebra with dimension m and let $f : \mathscr{A} \times \mathscr{A} \to \mathbb{C}$ be a sesquilinear map. Fix an isomorphism $\psi : \mathscr{A} \otimes \mathscr{A} \to \mathbb{C}^{m^2}$ and a basis $\mathbf{Q} = \{Q_1, \ldots, Q_m\}$ of $\mathscr{A}$. Then the following are equivalent:*

(a) There exists a positive $m^2 \times m^2$ matrix X that is a solution to the system of $2m^5 + m^2$ equations given by

$$\bigcup_{i_1,\ldots,i_5=1}^{m} \Big\{\psi(Q_{i_2}^* \otimes Q_{i_3}^*)^* X \psi(Q_{i_1}(Q_{i_4} \otimes Q_{i_5})) - \psi(Q_{i_1}^*(Q_{i_2}^* \otimes Q_{i_3}^*))^* X \psi(Q_{i_4} \otimes Q_{i_5}) = 0,$$
$$\psi(Q_{i_2}^* \otimes Q_{i_3}^*)^* X \psi((Q_{i_4} \otimes Q_{i_5})Q_{i_1}) - \psi((Q_{i_2}^* \otimes Q_{i_3}^*)Q_{i_1}^*)^* X \psi(Q_{i_4} \otimes Q_{i_5}) = 0,$$
$$\psi(Q_{i_1}^* \otimes 1)^* X \psi(Q_{i_2} \otimes 1) - f(Q_{i_1}^*, Q_{i_2}) = 0\Big\}, \tag{3.1}$$

which is a linear system in X. Here $\mathscr{A} \otimes \mathscr{A}$ is endowed with the canonical algebraic bimodule structure.

(b) There exists a derivation δ, given by $\delta(a) = a \otimes 1$, to the canonical algebraic bimodule $\mathscr{A} \otimes \mathscr{A}$ and a positive sesquilinear form $\langle\cdot,\cdot\rangle_{\mathscr{A}\otimes\mathscr{A}}$ on $\mathscr{A} \otimes \mathscr{A}$ such that $\mathscr{A} \otimes \mathscr{A}$ with this positive sesquilinear form is an $\mathscr{A}$-$\mathscr{A}$-bimodule and for all $a,\ b \in \mathscr{A}$ we have $\langle\delta(a),\delta(b)\rangle_{\mathscr{A}\otimes\mathscr{A}} = f(a,b)$.

Proof. By Theorem 3.2.2 we know that we can assume without loss of generality that our derivation δ will have the form $\delta(a) = a \otimes 1$ for all $a \in \mathscr{A}$. This means that we already have a derivation from $\mathscr{A}$ to $\mathscr{A} \otimes \mathscr{A}$, so the positive sesquilinear form $\langle\cdot,\cdot\rangle_{\mathscr{A}\otimes\mathscr{A}}$ is all that is left to choose. The idea is to write down linear equations that capture the properties that we require of the positive sesquilinear form. These properties are:

(a) The left multiplication is a $*$-homomorphism, i.e. $\langle a(b \otimes c), d \otimes e\rangle_{\mathscr{A}\otimes\mathscr{A}} = \langle b \otimes c, a^*(d \otimes e)\rangle_{\mathscr{A}\otimes\mathscr{A}}$ for all $a,\ b,\ c,\ d,\ e \in \mathscr{A}$.

(b) The right multiplication is a $*$-homomorphism, i.e. $\langle (b \otimes c)a, d \otimes e\rangle_{\mathscr{A}\otimes\mathscr{A}} = \langle b \otimes c, (d \otimes e)a^*\rangle_{\mathscr{A}\otimes\mathscr{A}}$ for all $a,\ b,\ c,\ d,\ e \in \mathscr{A}$.

(c) The positive sesquilinear form takes the required values on the range of δ, i.e. $\langle\delta(a),\delta(b)\rangle_{\mathscr{A}\otimes\mathscr{A}} = f(a,b)$ for all $a,\ b \in \mathscr{A}$.

Any positive sesquilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^{m^2}$ can be represented by a positive $m^2 \times m^2$ matrix X by the relation

$$\langle v, w\rangle = v^* X w,$$

if we view $v,\ w \in \mathbb{C}^{m^2}$ as column vectors. We can therefore try to describe the positive sesquilinear form by finding linear equations for the matrix X such that for all $a,\ b,\ c,\ d \in \mathscr{A}$ we have

$$\langle a \otimes b, c \otimes d \rangle_{\mathscr{A} \otimes \mathscr{A}} = \psi(a \otimes b)^* X \psi(c \otimes d).$$

We will start with the left multiplication. Consider the map $\varphi_{\mathrm{l}} : \mathscr{A}^5 \to \mathbb{C}$ given by

$$\varphi_{\mathrm{l}}(a,b,c,d,e) = \langle a^*(b^* \otimes c^*), d \otimes e \rangle_{\mathscr{A} \otimes \mathscr{A}} - \langle b^* \otimes c^*, a(d \otimes e) \rangle_{\mathscr{A} \otimes \mathscr{A}}.$$

We know that the left multiplication is a $*$-homomorphism if and only if φ_{l} is the zero function. In fact, since φ_{l} is linear in each of the five coordinates, we know that φ_{l} is the zero function if and only if $\varphi_{\mathrm{l}}(a,b,c,d,e) = 0$ for all combinations $a,\ b,\ c,\ d,\ e \in \mathbf{Q}$. Because $\mathscr{A}$ is finite-dimensional, this condition only imposes a finite number of conditions on the positive sesquilinear form. Converting $\varphi_{\mathrm{l}}(a,b,c,d,e) = 0$ to an equation containing X, we see that we obtain the set of equations

$$R_{\mathrm{l}} = \bigcup_{i_1,\ldots,i_5=1}^{m} \{\psi(Q_{i_2}^* \otimes Q_{i_3}^*)^* X \psi(Q_{i_1}(Q_{i_4} \otimes Q_{i_5})) - \psi(Q_{i_1}^*(Q_{i_2}^* \otimes Q_{i_3}^*))^* X \psi(Q_{i_4} \otimes Q_{i_5}) = 0\}.$$

So we know that the positive sesquilinear form given by X turns the left multiplication into a $*$-homomorphism if and only if all equations in R_{l} hold. Analogously, we find that the positive sesquilinear form by X turns the right multiplication into a $*$-homomorphism if and only if all equations in

$$R_{\mathrm{r}} = \bigcup_{i_1,\ldots,i_5=1}^{m} \{\psi(Q_{i_2}^* \otimes Q_{i_3}^*)^* X \psi((Q_{i_4} \otimes Q_{i_5})Q_{i_1}) - \psi((Q_{i_2}^* \otimes Q_{i_3}^*)Q_{i_1}^*)^* X \psi(Q_{i_4} \otimes Q_{i_5}) = 0\}.$$

hold.

This leaves property (c), for which we look at the map $\varphi_{\mathrm{f}} : \mathscr{A}^2 \to \mathbb{C}$ given by

$$\varphi_{\mathrm{f}}(a,b) = \langle a^* \otimes 1, b \otimes 1 \rangle - f(a^*, b).$$

This is once again a linear map in both coordinates, and property (c) holds if and only if φ_{f} is the zero function. By the same reasoning as before, we find that property (c) holds if and only if $\varphi_{\mathrm{f}}(a,b) = 0$ for all pairs $a,\ b \in \mathbf{Q}$. Translated to X this means that property (c) holds if and only if all equations in

$$R_{\mathrm{f}} = \bigcup_{i_1,i_2=1}^{m} \{\psi(Q_{i_1}^* \otimes 1)^* X \psi(Q_{i_2} \otimes 1) - f(Q_{i_1}^*, Q_{i_2}) = 0\}$$

hold.

Combining all these equations, we find that there exists a positive sesquilinear form on $\mathscr{A} \otimes \mathscr{A}$ satisfying (a), (b) and (c) if and only if there exists a positive matrix X that is a simultaneous solution of all equations in R_{l}, R_{r} and R_{f}. □

Corollary 3.2.7. *Let $\mathscr{A}$ be a unital finite-dimensional $*$-algebra with dimension m and let $f : \mathscr{A} \times \mathscr{A} \to \mathbb{C}$ be a sesquilinear map. Fix an isomorphism $\psi : \mathscr{A} \otimes \mathscr{A} \to \mathbb{C}^{m^2}$ and a basis $\mathbf{Q} = \{Q_1, \ldots, Q_m\}$ of $\mathscr{A}$. If condition (a) of Theorem 3.2.6 is not satisfied, then no complete $\mathscr{A}$-$\mathscr{A}$-bimodule $\mathscr{H}$ with unital right action and derivation $\delta : \mathscr{A} \to \mathscr{H}$ exist such that $\langle \delta(a), \delta(b) \rangle_{\mathscr{H}} = f(a,b)$.*

Proof. This is immediate from Corollary 3.2.5 and Theorem 3.2.6. □

3.2.3. EXISTENCE OF A SQUARE ROOT OF A GENERATOR OF A QMS

The developed theory will allow us to tackle Problem 2. In the finite-dimensional case, the inner products $\langle\cdot,\cdot\rangle_s$ on $\mathscr{A}$ mentioned in the introduction are given by

$$\langle a, b\rangle_s = \mathrm{Tr}(d_\varphi^{1-s} a^* d_\varphi^s b),$$

where Tr is the normalised trace on $\mathscr{A}$ (see the finite-dimensional theory box in Section 2.1.2). For a fixed $s \in [0,1]$ and using the notation above, we then want to know whether, given a finite-dimensional C*-algebra $\mathscr{A}$, a faithful state φ on $\mathscr{A}$ and a generator L of a QMS such that for all $a,\ b \in \mathscr{A}$: $\langle L(a), b\rangle_0 = \langle a, L(b)\rangle_0$, we can find an $\mathscr{A}$-$\mathscr{A}$-bimodule $\mathscr{H}$ with positive sesquilinear form $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ and a derivation $\delta : \mathscr{A} \to \mathscr{H}$ such that $\mathrm{Tr}(d_\varphi^{1-s} a^* d_\varphi^s L(b)) = \langle\delta(a), \delta(b)\rangle_{\mathscr{H}}$ for all $a,\ b \in \mathscr{A}$.

Using concrete examples for the case that φ is not tracial, we will show that this is sometimes, but not always, possible for both $s = 0$ and $s = \frac{1}{2}$. To obtain these concrete generators, we need part of a theorem stated by Carlen and Maas [CM17, Theorem 3.1] and originally proven by Alicki [Ali76]. This theorem was based on the description of generators of QMSs by Lindblad [Lin76] and more directly on the work of Gorini, Kossakowski and Sudarshan [GKS76].

Theorem 3.2.8. *Let φ be a faithful state on $M_n(\mathbb{C})$. Let $\mathscr{J}$ be a finite index set and let $\{V_j\}_{j\in\mathscr{J}} \subset M_n(\mathbb{C})$ and $\{\omega_j\}_{j\in\mathscr{J}} \subset \mathbb{R}$ be such that*

$$\{V_j\}_{j\in\mathscr{J}} = \{V_j^*\}_{j\in\mathscr{J}} \text{ and } \sigma_{-i}^{\varphi}(V_j) = e^{-\omega_j} V_j. \tag{3.2}$$

Then the operator $L : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, given by

$$L(a) = -\sum_{j\in\mathscr{J}} e^{-\frac{\omega_j}{2}} \left(V_j^*[a, V_j] + [V_j^*, a]V_j\right),$$

is a generator of a QMS that is GNS-symmetric. Conversely, any generator of a QMS on $M_n(\mathbb{C})$ that is GNS-symmetric is of the above form.

We can now give the examples that prove the claim that the desired derivation sometimes, but not always, exists. Because of Theorem 3.2.8, we can construct a concrete generator L of a QMS that is GNS-symmetric by choosing an $n \in \mathbb{N}$, a faithful state φ on $M_n(\mathbb{C})$, an index set $\mathscr{J}$ and two sets $\{V_j\}_{j\in\mathscr{J}} \subset M_n(\mathbb{C})$ and $\{\omega_j\}_{j\in\mathscr{J}} \subset \mathbb{R}$ that satisfy Equation (3.2). To use Corollary 3.2.7 or Theorem 3.2.6 we also need to choose an isomorphism $\psi : M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \to \mathbb{C}^{n^4}$ and a basis $\mathbf{Q}$ for $M_n(\mathbb{C})$. For $f : \mathscr{A} \times \mathscr{A} \to \mathbb{C}$ we can then pick $f(a,b) = \varphi(a^* L(b))$ if we consider the GNS inner product on $M_n(\mathbb{C})$ or $f(a,b) = \varphi(\sigma_{\frac{i}{2}}^{\varphi}(a^*)b)$ if we look at the KMS inner product. We then find the system of linear equations (3.1), which we solve using Mathematica. All computations using Mathematica are symbolic. We will numerically evaluate some eigenvalues, but solely to provide some feeling for the results. The code that was used can be found at `www.doi.org/10.4121/19323878`. Note that the code is written in a way that represents inner products that are linear in the first component, but this does not affect the results.

Example 3.2.9 (Generator with derivation as square root). For this example we pick

$$n=2,\ \mathscr{J}=\{1,2\},\ e^{-\omega_1}=\frac{\pi+1}{\pi-1}, e^{-\omega_2}=\frac{\pi-1}{\pi+1},\ \psi(E_{ij}\otimes E_{kl})=e_{8i+4k+2j+l-14},$$
$$\mathbf{Q}=\{E_{ij}\mid 1\le i,j\le 2\},\ \varphi(a)=\frac{1}{2}\mathrm{Tr}\left(\begin{pmatrix}1+\frac{1}{\pi} & 0\\ 0 & 1-\frac{1}{\pi}\end{pmatrix}a\right),\ V_1=\begin{pmatrix}0&1\\0&0\end{pmatrix} \text{ and } V_2=V_1^*$$

and we consider the GNS inner product on $M_2(\mathbb{C})$. If we pick $f:\mathscr{A}\times\mathscr{A}\to\mathbb{C}: f(a,b)=\varphi(a^*L(b))$, then we can concretely check statement (1) of Theorem 3.2.6 using Mathematica. We obtain a self-adjoint matrix X that is a solution of the system of equations and whose eigenvalues can be expressed algebraically in terms of π and are all non-negative. They are approximately given by

$$\{1.96,1.96,0.96,0.96,0.64,0.64,0.31,0.31,0.07,0.07,0.07,0.07,0,0,0,0\},$$

showing that X is indeed positive. By Theorem 3.2.6 this shows that there exists a derivation δ from $M_2(\mathbb{C})$ to an $M_2(\mathbb{C})-M_2(\mathbb{C})$-bimodule such that $L=\delta^*\circ\delta$ with respect to the GNS inner product on $M_2(\mathbb{C})$.

Alternatively, we can consider the KMS inner product on $M_2(\mathbb{C})$ and consequently replace f by $f(a,b)=\varphi(\sigma^{\varphi}_{\frac{i}{2}}(a^*)L(b))$. In this case we also find a self-adjoint matrix X that solves Equations (3.1) and whose eigenvalues can be expressed algebraically in terms of π and are all non-negative. Approximately, the eigenvalues are given by

$$\{1.44,1.44,1.44,1.44,0.47,0.47,0.47,0.47,0.03,0.03,0.03,0.03,0,0,0,0\}.$$

So Theorem 3.2.6 shows that we can also find a derivation δ from $M_2(\mathbb{C})$ to an $M_2(\mathbb{C})-M_2(\mathbb{C})$-bimodule such that $L=\delta^*\circ\delta$ with respect to the KMS inner product on $M_2(\mathbb{C})$.

Example 3.2.10 (Generator without derivation as square root). For this example we pick

$$n=3,\ \mathscr{J}=\{1,2\},\ e^{-\omega_1}=\frac{\pi^2}{e^2}, e^{-\omega_2}=\frac{e^2}{\pi^2},$$
$$\psi(E_{ij}\otimes E_{kl})=e_{27i+9k+3j+l-39},\ \mathbf{Q}=\{E_{ij}\mid 1\le i,j\le 3\},$$
$$\varphi(a)=\frac{1}{1+\pi^2+e^2}\mathrm{Tr}\left(\begin{pmatrix}1&0&0\\0&\pi^2&0\\0&0&e^2\end{pmatrix}a\right),\ V_1=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix} \text{ and } V_2=V_1^*.$$

If we pick $f: M_3(\mathbb{C})\times M_3(\mathbb{C})\to\mathbb{C}: f(a,b)=\varphi(a^*L(b))$, then we can concretely check statement (1) of Theorem 3.2.6 using Mathematica. We find that there does not exist any matrix X that satisfies the system of equations. Using Corollary 3.2.7 we can now conclude that there do not exist an $M_3(\mathbb{C})-M_3(\mathbb{C})$-bimodule $\mathscr{H}$ and a derivation $\delta: M_3(\mathbb{C})\to\mathscr{H}$ such that $L=\delta^*\circ\delta$ with respect to the GNS inner product on $M_3(\mathbb{C})$.

Alternatively, if we consider the KMS inner product on $M_3(\mathbb{C})$ and consequently use $f(a,b)=\varphi(\sigma^{\varphi}_{\frac{i}{2}}(a^*)L(b))$, then we do find a subspace $\mathscr{X}$ of matrices that satisfy Equations (3.1). However, for any matrix $X\in\mathscr{X}$ we have that

$$\psi(E_{12}\otimes E_{22}+E_{13}\otimes E_{32})^*X\psi(E_{12}\otimes E_{22}+E_{13}\otimes E_{32})=-\frac{e^2+2e(\pi-1)+\pi(\pi-2)}{1+\pi^2+e^2}<0.$$

Consequently, we see that X is not positive definite, and therefore that the corresponding sesquilinear form on $M_3(\mathbb{C}) \otimes M_3(\mathbb{C})$ is not positive. Since this holds for any $X \in \mathscr{X}$, we conclude that that there do not exist an $M_3(\mathbb{C}) - M_3(\mathbb{C})$-bimodule $\mathscr{H}$ and a derivation $\delta : M_3(\mathbb{C}) \to \mathscr{H}$ such that $L = \delta^* \circ \delta$ with respect to the KMS inner product on $M_3(\mathbb{C})$.

Example 3.2.10 gives us a concrete counterexample and therefore a negative answer to problem 2. In the next example, we analyse this situation in somewhat more detail, and show that this counterexample is not *special*, in the sense that a random example one comes up with is very likely to be a counterexample.

Example 3.2.11. Let us consider the $M_3(\mathbb{C})$ case and a state φ given by

$$\varphi(a) = \frac{1}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \operatorname{Tr}\left(\begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix} a\right)$$

for some $\lambda_i > 0$. Without loss of generality we can assume that $\lambda_1 = 1$. First we note that choosing $V = \operatorname{diag}(x, y, z)$ with $x,\ y,\ z \in \mathbb{R}$ (and J a singleton set) in Theorem 3.2.8 gives the same generator as choosing

$$V_1 = \sqrt{\frac{1}{2}((x-y)^2 + (x-z)^2 - (y-z)^2)}\operatorname{diag}(1,0,0),$$
$$V_2 = \sqrt{\frac{1}{2}((x-y)^2 + (y-z)^2 - (x-z)^2)}\operatorname{diag}(0,1,0) \text{ and}$$
$$V_3 = \sqrt{\frac{1}{2}((x-z)^2 + (y-z)^2 - (x-y)^2)}\operatorname{diag}(0,0,1).$$

Consequently, if all ratios of the λ_i are different and unequal to 1, then by Theorem 3.2.8 any GNS-symmetric generator of a QMS can be obtained by using the collection of V_j's given by $\{\sqrt{Y_{ij}}E_{ij} \mid 1 \le i, j \le 3\}$ for some symmetric matrix $Y \in M_{3\times 3}(\mathbb{R}_{\ge 0})$.

Let ω_{kl} be such that

$$e^{-\omega_{kl}} E_{kl} = \sigma^{\varphi}_{-i}(E_{kl}) = \frac{\lambda_k^2}{\lambda_l^2} E_{kl}.$$

We will now consider a slightly more general situation than the generators of GNS-symmetric QMSs. For any symmetric Y in $M_3(\mathbb{R})$ define the operator $L_Y : M_3(\mathbb{C}) \to M_3(\mathbb{C})$ by

$$L_Y(a) = -\sum_{i,j=1}^{3} Y_{ij} e^{-\frac{\omega_{ij}}{2}} \left(E_{ij}^*[a, E_{ij}] + [E_{ij}^*, a]E_{ij}\right).$$

Note that any generator of a QMS that is GNS-symmetric can be obtained by choosing the appropriate Y. In order to use Theorem 3.2.6 and Corollary 3.2.7 we pick the basis $\mathbf{Q} = \{E_{ij} \mid 1 \le i, j \le 3\}$ of $M_3(\mathbb{C})$ and the isomorphism $\psi : M_3(\mathbb{C}) \otimes M_3(\mathbb{C}) \to \mathbb{C}^{3^4} : \psi(E_{ij} \otimes E_{kl}) = e_{27i+9k+3j+l-39}$, where the e_i form the standard basis of $\mathbb{C}^{81}$, and we fix the explicit state φ by choosing $\lambda_2 = \pi$ and $\lambda_3 = e^{\pi}$. Lastly, we define the function $f_Y : M_3(\mathbb{C}) \times M_3(\mathbb{C}) \to \mathbb{C} : f_Y(a,b) = \varphi(a^* L_Y(b))$. Now Equations (3.1) are linear in the pair (X, Y). Consequently, we can solve the system of linear equations to find which conditions on Y must hold to make sure that an X exists that satisfies the equations. Solving this system of linear equations with Mathematica we find that such an X exists if and only if

3

$$\frac{(1-\lambda_3^2-\lambda_2^2)(\lambda_3^2-\lambda_2^2)}{\lambda_3\lambda_2}Y_{23}+\frac{(\lambda_3^2-1-\lambda_2^2)(\lambda_2^2-1)}{\lambda_2}Y_{12}+\frac{(\lambda_2^2-1-\lambda_3^2)(1-\lambda_3^2)}{\lambda_3}Y_{13} \\ +(\lambda_3^2-\lambda_2^2)Y_{11}+(1-\lambda_3^2)Y_{22}+(\lambda_2^2-1)Y_{33}=0 \quad (3.3)$$

with $\lambda_2 = \pi$ and $\lambda_3 = e^\pi$. These calculations have been executed for a specific state φ. However, due to our choice of φ, using the fact that π and e^π are algebraically independent [Nes96], we can obtain some more general conclusions.

Proposition 3.2.12. *There exists a set $P \subset \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ such that*

$$|\{x \in \mathbb{R}_{>0} \mid |\{y \in \mathbb{R}_{>0} \mid (x,y) \in P \text{ or } (y,x) \in P\}| = \infty\}| < \infty \quad (3.4)$$

and for all $(\lambda_2, \lambda_3) \in (\mathbb{R}_{>0} \times \mathbb{R}_{>0}) \setminus P$ and symmetric $Y \in M_3\mathbb{R}$ the following are equivalent:

(a) There exists a solution X of Equations (3.1) with the function $f = f_Y$.

(b) Y satisfies Equation (3.3).

Let us briefly discuss the above proposition and the role of P. The proposition states that (a) and (b) are equivalent in most cases. The exceptions to the equivalence are captured in the set P. Equation (3.4) states that the set P is small in the following sense: for each $x > 0$ and $y > 0$ one can wonder whether (x, y) or (y, x) is a pair for which the equivalence does not hold. If this is the case, we require $(x, y) \in P$. For some $x > 0$, there are infinitely many $y > 0$ such that $(x, y) \in P$. The point of Equation (3.4) is to guarantee that there are only finitely many of such x. In other words, for all but finitely many $x > 0$ we have that for all but finitely many $y > 0$, (a) and (b) are equivalent. To prove this proposition, we need a lemma.

Lemma 3.2.13. *Let $m,\ n \in \mathbb{N}$, $A \in M_{n,m}(\mathbb{C})$ and $b : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^n$ be an entrywise rational function. Then there exists a finite family of rational functions $r_1, \ldots, r_l$ in two variables such that for all $\lambda_1,\ \lambda_2 \in \mathbb{R}$ the system of linear equations $Ax = b(\lambda_1, \lambda_2)$ with $x \in \mathbb{C}^m$ has a solution if and only if $r_i(\lambda_1, \lambda_2) = 0$ for all $1 \le i \le l$.*

Proof. Let k be the rank of A. By the Rouché-Capelli theorem, we know that our system of linear equations has a solution if and only if $(A|b(\lambda_1, \lambda_2))$ also has rank k. Since the rank of $(A|b(\lambda_1, \lambda_2))$ is at least k, this is equivalent to the statement that all $(k+1) \times (k+1)$ minors are zero. Since all of the entries of $b(\lambda_1, \lambda_2)$ are rational functions in λ_1 and λ_2 and A does not depend on λ_1 and λ_2, this means that all of these minors are rational functions in λ_1 and λ_2. Choosing these minors as our finite family of rational functions gives the desired result. □

Proof of Proposition 3.2.12. Let $M_{3\times 3}^{\text{sym}}(\mathbb{R}) = \{A \in M_3(\mathbb{R}) \mid A = A^*\}$ and $\mathcal{M}_{\lambda_2,\lambda_3} \subset M_{3\times 3}^{\text{sym}}(\mathbb{R})$ be the subset of matrices satisfying Equation (3.3). We can find a set $\mathcal{T}$ of functions from $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$ to $M_{3\times 3}^{\text{sym}}(\mathbb{R})$ such that for all $\lambda_2, \lambda_3 \in \mathbb{R}_{>0}$ we have that $\{T(\lambda_2, \lambda_3) \mid T \in \mathcal{T}\}$ is a basis for $\mathcal{M}_{\lambda_2,\lambda_3}$, and T_{ij} is a rational function of λ_2 and λ_3 for all $1 \le i,\ j \le 3$ and $T \in \mathcal{T}$. For each $T \in \mathcal{T}$ and $Q_i,\ Q_j \in \mathbf{Q}$ we have that $(\lambda_2, \lambda_3) \mapsto f_{T(\lambda_2,\lambda_3)}(Q_i, Q_j)$ is a rational function in λ_2 and λ_3. By Lemma 3.2.13 we now know that for all $T \in \mathcal{T}$ there exist a family of rational functions $\mathscr{R}_T$ such that for all $\lambda_2, \lambda_3 \in \mathbb{R}_{>0}$: there exists a solution for Equations (3.1) with the function $f = f_{T(\lambda_2,\lambda_3)}$ if and only if $r(\lambda_2, \lambda_3) = 0$ for all $r \in \mathscr{R}_T$. Consequently, by linearity, we find for all $\lambda_2, \lambda_3 \in \mathbb{R}_{>0}$ that $r(\lambda_2, \lambda_3) = 0$ for all $r \in \mathscr{R} = \bigcup_{T\in\mathscr{T}} \mathscr{R}_T$ if and only if for all $Y \in \mathscr{M}_{\lambda_2,\lambda_3}$ the system of linear Equations (3.1) with $f = f_Y$ has a solution. We know that this holds for $\lambda_2 = \pi$ and $\lambda_3 = e^\pi$. Since these numbers are algebraically independent, we must have that the numerators of all of these rational functions are equal to the zero function. Therefore we conclude that (b) implies (a) for all $\lambda_2, \lambda_3 \in \mathbb{R}_{>0}$.

For the other direction, we will show that (b) is a necessary requirement for (a). We can analogously find a finite set of rational functions $\mathscr{R}'$ such that $r(\lambda_2, \lambda_3) = 0$ for all $r \in \mathscr{R}'$ if and only if for all $Y \in M^{\text{sym}}_{3\times3}(\mathbb{R})$ a solution exists to Equations (3.1) with $f = f_Y$. We know by the beginning of this example that the latter part is not true for $\lambda_2 = \pi$ and $\lambda_3 = e^\pi$, so there exists an $r \in \mathscr{R}'$ such that $r(\pi, e^\pi) \neq 0$. Let P be defined by $P = \{(\kappa_1, \kappa_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \mid r(\kappa_1, \kappa_2) = 0\}$. Then P satisfies

$$|\{x \in \mathbb{R}_{>0} \mid |\{y \in \mathbb{R}_{>0} \mid (x, y) \in P \text{ or } (y, x) \in P\}| = \infty\}| < \infty.$$

Now let $(\lambda_2, \lambda_3) \in (\mathbb{R}_{>0} \times \mathbb{R}_{>0}) \setminus P$ be arbitrary. By definition of P we know that there exists a $Y \in M^{\text{sym}}_{3\times3}(\mathbb{R})$ such that Equations (3.1) for $f = f_Y$ do not have a solution. Since these equations are linear in Y, this means that the subspace of $M^{\text{sym}}_{3\times3}(\mathbb{R})$ for which these equations have a solution has codimension greater or equal to 1 in $M^{\text{sym}}_{3\times3}(\mathbb{R})$. But we already know that there exists a solution for all $Y \in \mathscr{M}_{\lambda_2,\lambda_3}$, and $\mathscr{M}_{\lambda_2,\lambda_3}$ has codimension 1 in $M^{\text{sym}}_{3\times3}(\mathbb{R})$. All in all this shows that for all $(\lambda_2, \lambda_3) \in (\mathbb{R}_{>0} \times \mathbb{R}_{>0}) \setminus P$ and $Y \in M^{\text{sym}}_{3\times3}(\mathbb{R})$ a solution to Equations (3.1) with $f = f_Y$ exists if and only if $Y \in \mathscr{M}_{\lambda_2,\lambda_3}$. This proves the proposition. □

We conclude by returning to the setting of generators of QMSs. Using Proposition 3.2.12 and Corollary 3.2.7 we see that outside of the set P, which has measure zero, the corresponding states do not allow a generator L_Y, for a symmetric Y with positive entries not satisfying Equation (3.3), to be written as $\delta^* \circ \delta$ for some derivation δ to a $*$-bimodule. Note that we cannot use Theorem 3.2.6 to definitively conclude that such a derivation always exists if Equation (3.3) is satisfied, because we do not know if the condition that the solution matrix of the system of equations is positive, is satisfied.

Remark 3.2.14. A superficial investigation into the 4×4 matrix case indicates that one would find two linear equations, each similar to Equation (3.3), that describe when a matrix X exists that satisfies the conditions of Theorem 3.2.6. Conversely, in the 2×2 case, no equations appear; the matrix X always exists, even though it is not always positive. This suggests that the number of equations increases with the size of the matrices and it gives some intuition as to why we were able to find examples in the 2×2 case where the sought derivation does exist.

Remark 3.2.15. The line of reasoning in Example 3.2.11 does not work exclusively for the GNS inner product. Any generator given by Theorem 3.2.8 is self-adjoint with respect to $\langle\cdot,\cdot\rangle_s$ for all $s \in [0, 1]$ [CM17, Theorem 2.9]. It is therefore possible to work out the above examples for the inner product $\langle\cdot,\cdot\rangle_s$ for any $s \in [0, 1]$, which gives similar results as above for all values of s that have been tried, apart from one exception. When one considers the $M_3(\mathbb{C})$ case for $s = \frac{1}{2}$, then one finds that there always exists a solution X

to Equations (3.1), but that this X is not always positive. Unfortunately, the system of equations does not depend linearly on s, so there does not seem to be a clear way to prove that $s = \frac{1}{2}$ is the only special value. However, for any fixed s one can use the above method to investigate when a derivation exists.

3.2.4. Existence of twisted derivations

In this final section, we discuss how to alter the constructions in Sections 3.2.1 and 3.2.2 to the case of twisted derivations. We will purely work in the finite-dimensional case, and we can therefore use Definition 2.3.12. Let us recall the definition here.

3

Definition 3.2.16. Let $\mathscr{A}$ be a finite-dimensional $*$-algebra. A *twisted derivation* δ from $\mathscr{A}$ into an $\mathscr{A}$-$\mathscr{A}$-bimodule is a linear map such that there exist algebra homomorphisms α, β on $\mathscr{A}$ satisfying

$$\delta(xy) = \alpha(x)\delta(y) + \delta(x)\beta(y).$$

This definition involves the existence of the algebra homomorphisms α and β. Unfortunately, we cannot account for this freedom, as being an algebra homomorphism is a non-linear condition. Moreover, we multiply $\alpha(x)$ and $\delta(y)$, which is another non-linear interaction. However, we know from [Wir22] that we ought to be looking for a specific twist (see Definition 2.3.7). We want $\alpha = \sigma_{-i/4}$ and $\beta = \sigma_{i/4}$. By fixing these algebra homomorphisms, it becomes possible to obtain a result along the lines of Corollary 3.2.7. We will discuss the changes to the constructions, but we will not go through the proofs again, as these are analogous to the non-twisted case. Throughout this section, we fix a finite-dimensional von Neumann algebra $\mathscr{M}$ with faithful state φ.

In Theorem 3.2.2, the idea is that $a \otimes b$ represents $\delta(a)b$ for $a, b \in \mathscr{M}$. One can derive that the left multiplication in the non-twisted case has to be

$$a(b \otimes c) \cong a\delta(b)c = (\delta(ab) - \delta(a)b)c \cong ab \otimes c - a \otimes bc,$$

where we use $\cong$ to indicate that we are identifying $x \otimes y$ and $\delta(x)y$. In the twisted case, this is no longer the right computation. Instead, we compute

$$a(b \otimes c) \cong a\delta(b)c = (\delta(\sigma_{i/4}(a)b) - \delta(\sigma_{i/4}(a))\sigma_{i/4}(b))c \cong \sigma_{i/4}(a)b \otimes c - \sigma_{i/4}(a) \otimes \sigma_{i/4}(b)c.$$

Consequently, by following the proof of Theorem 3.2.2, we find the following theorem.

Theorem 3.2.17. *Let $\mathscr{H}$ be an $\mathscr{M}$-$\mathscr{M}$-correspondence with non-degenerate right action. Suppose that $\delta : \mathscr{M} \to \mathscr{H}$ is a twisted derivation with twists $\alpha = \sigma_{-i/4}$ and $\beta = \sigma_{i/4}$. Then there exists an $\mathscr{M}$-$\mathscr{M}$-bimodule structure on $\mathscr{M} \otimes \mathscr{M}$ and an isometric bimodule homomorphism $T : \mathscr{M} \otimes \mathscr{M} \to \mathscr{H}$ such that*

$$\delta(a) = T(a \otimes 1)$$

for all $a \in \mathscr{M}$. The left and right multiplication of $\mathscr{M}$ on $\mathscr{M} \otimes \mathscr{M}$ is given by

$$a(b \otimes c)d = \sigma_{i/4}(a)b \otimes cd - \sigma_{i/4}(a) \otimes \sigma_{i/4}(b)cd$$

for all $a, b, c, d \in \mathscr{M}$.

Theorem 3.2.6 changes even less. One just needs to change the canonical algebraic bimodule structure to the algebraic bimodule structure described here. In that case, one does not have to change the proof at all, apart from referring to Theorem 3.2.17 instead of 3.2.2. These theorems, together with the corollaries analogous to Corollaries 3.2.5 and 3.2.7, allow one to perform the same procedure as in Section 3.2.3 to look for twisted derivations.

One thing becomes apparent when one tries to perform this procedure in the twisted case for KMS-symmetric quantum Markov semigroups, however: it is not so easy to find counterexamples. In fact, one does not find any. This raises the question: does there always exist a twisted derivation for a (finite-dimensional) KMS-symmetric quantum Markov semigroup? I will wrap up this chapter by briefly explaining how one can generate reasonable computer evidence for this question, and how one can use this computer evidence to discover the structure of these derivations, leading to the $\mathcal{V}$-transform (see Definition 3.3.2) and the remainder of this chapter.

To generate reasonable computer evidence for the existence of twisted derivations in the KMS-symmetric case, one needs two important things. First, one needs to be able to determine that a twisted derivation really exists in an example. This may not sound problematic, since we can make our computer check if the system of linear equations has a solution. However, when one finds a solution, one typically finds a subspace of solution matrices, and for these twisted derivations to exist, one needs a solution matrix that is positive. Without any mathematical theory, this can quickly become an intractable problem. Fortunately, the field of semidefinite optimisation is designed precisely to answer questions like these. This gives us a way to definitively determine (up to numerical precision) whether a twisted derivation exists in an example.

The second requirement is to have a good way to randomly pick an example from all possible examples. In our case, this is provided by [AC21], which characterises all pure KMS-symmetric quantum Markov semigroups on $n \times n$ matrices in terms of a single $n \times n$ matrix W satisfying certain properties. As the existence of twisted derivations for quantum Markov semigroups is preserved under convex combinations, one can take a random W satisfying these properties, and from there construct the KMS-symmetric quantum Markov semigroup. All in all, this means that we have a good way to generate computer evidence, with one caveat: our approach scales very poorly with n. The number of equations scales with n^{10} in Theorem 3.2.6. While some improvement is possible here, one can still only work with very small n.

As evidence that twisted derivations for finite-dimensional KMS-symmetric quantum Markov semigroups, this approach is therefore of limited value. However, it has much more potential when one attempts to use it to discover the structure behind the inner products one obtains from the algorithm. To do this, one first needs to find an algebraic expression for the remaining degrees of freedom in the inner product in terms of the entries of W. This involves performing some reasonable guesses based on the algebraic expressions in the solution space of the linear equations. Next, one needs to find the structure in the algebraic expression, and move from an individual description of each entry to a description in terms of general matrix objects. There is no clear procedure to do these things, and one might be inclined to call it more art than science. However, I would argue that this is very common in experimental physics or any other experimental scientific field. One observes things, looks for a formula describing these observations, and then for an theoretical explanation of this formula. As such, I would view this approach as experimental mathematics, and its fruits can be found in the next chapter, which I believe would not have existed for some time without this approach.

3.3. QUANTUM MARKOV SEMIGROUPS AND DERIVATIONS ON MATRIX ALGEBRAS

3

In this section we demonstrate the connection between KMS-symmetric quantum Markov semigroups and twisted derivations in the case of matrix algebras. We first prove a finite-dimensional version of the main result of this chapter, which allows to express generators of KMS-symmetric quantum Markov semigroups as squares of twisted derivations in a suitable sense (Theorem 3.3.4). We then use the simple structure of bimodules over matrix algebras to give a more explicit expression for the quadratic form associated with the generator of a KMS-symmetric quantum Markov semigroups (Theorem 3.3.5). As in the general case treated in the next sections, the crucial technical tool is the $\mathcal{V}$-transform, which will be introduced in Subsection 3.3.1.

Let us start with some notation. We write $M_n(\mathbb{C})$ for the algebra of $n\times n$ matrices over the complex numbers, 1_n for the identity matrix in $M_n(\mathbb{C})$, and id_n for the identity map from $M_n(\mathbb{C})$ to itself. The norm $\|\cdot\|$ always denotes the operator norm, either for elements of $M_n(\mathbb{C})$ or for linear maps from $M_n(\mathbb{C})$ to itself.

The finite-dimensional theory of the modular automorphism group and of KMS-symmetric quantum Markov semigroups can be found in the finite-dimensional boxes in Sections 2.1.2 and 2.2. Here we introduce the main objects we will work with in this section. Fix a density matrix $\rho\in M_n(\mathbb{C})$, that is, a positive matrix with trace 1, and assume that ρ is invertible. The KMS inner product induced by ρ is defined as

$$\langle\cdot,\cdot\rangle_\rho\colon M_n(\mathbb{C})\times M_n(\mathbb{C})\to\mathbb{C},\ (A,B)\mapsto \mathrm{Tr}(A^*\rho^{1/2}B\rho^{1/2}).$$

If $\Phi\colon M_n(\mathbb{C})\to M_n(\mathbb{C})$ is a linear map, we write $\Phi^\dagger$ for its adjoint with respect to $\langle\cdot,\cdot\rangle_\rho$ and we say that Φ is *KMS-symmetric* if $\Phi^\dagger=\Phi$.

A quantum Markov semigroup (Φ_t) is called *KMS-symmetric* if Φ_t is KMS-symmetric for all $t\geq 0$. Equivalently, (Φ_t) is KMS-symmetric if and only if its generator is KMS-symmetric.

The *modular group* (or rather its analytic continuation) of ρ is the family $(\sigma_z)_{z\in\mathbb{C}}$ of algebra homomorphisms on $M_n(\mathbb{C})$ defined by

$$\sigma_z(A)=\rho^{iz}A\rho^{-iz}$$

for $A\in M_n(\mathbb{C})$ and $z\in\mathbb{C}$. Note that $\sigma_z^\dagger=\sigma_{-\bar z}$.

If a KMS-symmetric operator on $M_n(\mathbb{C})$ commutes with the modular group, then it is called GNS-symmetric (this is equivalent to the usual definition of GNS symmetry by [CM17, Lemma 2.5]).

According to [AC21, Theorem 4.4], the generator L of a KMS-symmetric quantum Markov semigroup is of the form

$$L(A)=(\mathrm{id}_n+\sigma_{-i/2})^{-1}(\Psi(1_n))A+A(\mathrm{id}_n+\sigma_{i/2})^{-1}(\Psi(1_n))-\Psi(A)$$

for some KMS-symmetric completely positive map $\Psi\colon M_n(\mathbb{C})\to M_n(\mathbb{C})$.

The main goal of this section is to show that the sesquilinear form associated with L can be written as

$$\langle L(A), B\rangle_\rho = \sum_{j=1}^{N} \langle [V_j, A], [V_j, B]\rangle_\rho$$

with matrices $V_1, \dots, V_N \in M_n(\mathbb{C})$.

3.3.1. THE $\mathscr{V}$-TRANSFORM

We write $B(M_n(\mathbb{C}))$ for the space of all linear maps from $M_n(\mathbb{C})$ to itself. This space is generated by left and right multiplication operators in the following sense. For $A \in M_n(\mathbb{C})$ let

$$\mathbb{L}_A, \mathbb{R}_A \colon M_n(\mathbb{C}) \to M_n(\mathbb{C}), \mathbb{L}_A(X) = AX, \mathbb{R}_A(X) = XA.$$

By [CM17, Lemma A.1], the linear span of $\{\mathbb{L}_A \mathbb{R}_B \mid A, B \in M_n(\mathbb{C})\}$ is $B(M_n(\mathbb{C}))$.

The $\mathscr{V}$-transform is a linear map on $B(M_n(\mathbb{C}))$, which is most conveniently defined through its inverse. Let

$$\mathscr{W} \colon B(M_n(\mathbb{C})) \to B(M_n(\mathbb{C})), \Phi \mapsto \frac{1}{2}(\sigma_{i/4}\Phi\sigma_{-i/4} + \sigma_{-i/4}\Phi\sigma_{i/4}).$$

In particular, if $\Phi = \mathbb{L}_A \mathbb{R}_B$, then

$$\mathscr{W}(\Phi) = \frac{1}{2}(\mathbb{L}_{\sigma_{i/4}(A)}\mathbb{R}_{\sigma_{i/4}(B)} + \mathbb{L}_{\sigma_{-i/4}(A)}\mathbb{R}_{\sigma_{-i/4}(B)}).$$

Proposition 3.3.1. *The map $\mathscr{W}$ is invertible with inverse given by*

$$\mathscr{W}^{-1}(\Phi) = 2\int_0^\infty \sigma_{-i/4} e^{-r\sigma_{-i/2}} \Phi \sigma_{-i/4} e^{-r\sigma_{-i/2}}\, dr$$

for $\Phi \in B(M_n(\mathbb{C}))$.

In particular, if A, $B \in M_n(\mathbb{C})$, then

$$\mathscr{W}^{-1}(\mathbb{L}_A \mathbb{R}_B) = 2\int_0^\infty \mathbb{L}_{\sigma_{i/4}(e^{-r\sigma_{i/2}}(A))} \mathbb{R}_{\sigma_{-i/4}(e^{-r\sigma_{-i/2}}(B))}\, dr.$$

Proof. Since $\sigma_{-i/2}$ is an invertible operator on $M_n(\mathbb{C})$ and $\mathrm{Tr}(\sigma_{-i/2}(A)^* A) \geq 0$ for all $A \in M_n(\mathbb{C})$, the spectrum of $\sigma_{-i/2}$ consists of strictly positive numbers. Let λ denote the smallest eigenvalue of $\sigma_{-i/2}$. By the spectral theorem,

$$\|e^{-r\sigma_{-i/2}}\| \leq e^{-\lambda r}$$

for all $r \geq 0$. It follows that for $\Phi \in B(M_n(\mathbb{C}))$ we have

$$\|\sigma_{-i/4} e^{-r\sigma_{-i/2}} \Phi \sigma_{-i/4} e^{-r\sigma_{-i/2}}\| \leq e^{-2\lambda r} \|\sigma_{-i/4}\|^2 \|\Phi\|.$$

Therefore the integral

$$\int_0^\infty \sigma_{-i/4} e^{-r\sigma_{-i/2}} \Phi \sigma_{-i/4} e^{-r\sigma_{-i/2}}\, dr$$

converges absolutely.

Moreover,

$$\begin{aligned}\mathscr{W}\left(2\int_0^\infty \sigma_{-i/4}e^{-r\sigma_{-i/2}}\Phi\sigma_{-i/4}e^{-r\sigma_{-i/2}}\,dr\right)\\ &= \int_0^\infty e^{-r\sigma_{-i/2}}(\sigma_{-i/2}\Phi+\Phi\sigma_{-i/2})e^{-r\sigma_{-i/2}}\,dr\\ &= -\int_0^\infty \frac{d}{dr}(e^{-r\sigma_{-i/2}}\Phi e^{-r\sigma_{-i/2}})\,dr\\ &= \Phi.\end{aligned}$$

3

Thus $\mathscr{W}$ is invertible and the claimed integral expression for the inverse holds. Similar arguments yield the integral formula for $\mathscr{W}^{-1}(\mathbb{L}_A\mathbb{R}_B)$. □

Definition 3.3.2. We call the inverse of $\mathscr{W}$ the $\mathscr{V}$-transform and denote it by $\mathscr{V}$. For $\Phi\in B(M_n(\mathbb{C}))$ we also write $\check{\Phi}$ for $\mathscr{V}(\Phi)$.

Lemma 3.3.3. *The $\mathscr{V}$-transform is a bijective linear map on $B(M_n(\mathbb{C}))$ with the following properties.*

(a) If $\Phi\in B(M_n(\mathbb{C}))$ is KMS-symmetric, then $\mathscr{V}(\Phi)$ is KMS-symmetric.

(b) If $\Phi\in B(M_n(\mathbb{C}))$ is completely positive, then $\mathscr{V}(\Phi)$ is completely positive.

(c) $\mathscr{V}(\mathrm{id}_n)=\mathrm{id}_n$.

Proof. (a) Let $\Phi\in B(M_n(\mathbb{C}))$. By Proposition 3.3.1 we have

$$\mathscr{V}(\Phi)=2\int_0^\infty \sigma_{-i/4}e^{-r\sigma_{-i/2}}\Phi\sigma_{-i/4}e^{-r\sigma_{-i/2}}\,dr.$$

Since $\sigma_{-i/4}$ is KMS-symmetric, so is $e^{-r\sigma_{-i/2}}$. Thus the KMS adjoint of $\mathscr{V}(\Phi)$ satisfies

$$\mathscr{V}(\Phi)^\dagger=2\int_0^\infty \sigma_{-i/4}e^{-r\sigma_{-i/2}}\Phi^\dagger\sigma_{-i/4}e^{-r\sigma_{-i/2}}\,dr.$$

In particular, if Φ is KMS-symmetric, so is $\mathscr{V}(\Phi)$.

(b) If Φ is completely positive, by Kraus' theorem there exist $V_1,\dots,V_N\in M_n(\mathbb{C})$ such that

$$\Phi=\sum_{j=1}^N \mathbb{L}_{V_j^*}\mathbb{R}_{V_j}.$$

From Proposition 3.3.1 and the identity $\sigma_z(a)^*=\sigma_{\overline{z}}(a^*)$ we deduce

$$\mathscr{V}(\Phi)=\sum_{j=1}^N\int_0^\infty \mathbb{L}_{\sigma_{-i/4}(e^{-r\sigma_{-i/2}}(V_j))^*}\mathbb{R}_{\sigma_{-i/4}(e^{-r\sigma_{-i/2}}(V_j))}\,dr.$$

Since maps of the form $\mathbb{L}_{A^*}\mathbb{R}_A$ are completely positive and positive linear combinations and limits of completely positive maps are again completely positive, it follows that $\mathscr{V}(\Phi)$ is completely positive.

(c) The identity $\mathscr{W}(\mathrm{id}_n)=\mathrm{id}_n$ is immediate from the definition, from which $\mathscr{V}(\mathrm{id}_n)=\mathrm{id}_n$ follows directly. □

3.3.2. DERIVATIONS FOR A KMS-SYMMETRIC QMS ON A MATRIX ALGEBRA

We are now in the position to prove the existence of a twisted derivation that implements the Dirichlet form associated with a KMS-symmetric quantum Markov semigroup on $M_n(\mathbb{C})$. We first present an abstract version that will later be generalised to quantum Markov semigroups on arbitrary von Neumann algebras. A more explicit version tailored for matrix algebras will be discussed below.

Theorem 3.3.4. *Let (Φ_t) be a KMS-symmetric quantum Markov semigroup on $M_n(\mathbb{C})$ and let L denote its generator. There exists a Hilbert space $\mathcal{H}$, a unital $*$-homomorphism $\pi_l\colon M_n(\mathbb{C}) \to B(\mathcal{H})$, a unital $*$-antihomomorphism $\pi_r\colon M_n(\mathbb{C}) \to B(\mathcal{H})$, and anti-linear isometric involution $\mathcal{J}\colon \mathcal{H} \to \mathcal{H}$ and a linear map $\delta\colon M_n(\mathbb{C}) \to \mathcal{H}$ satisfying*

(a) $\pi_l(A)\pi_r(B) = \pi_r(B)\pi_l(A)$ *for all* $A, B \in M_n(\mathbb{C})$,

(b) $\mathcal{J}(\pi_l(A)\pi_r(B)\xi) = \pi_l(B)^*\pi_r(A)^*\mathcal{J}\xi$ *for all* $A, B \in M_n(\mathbb{C})$ *and* $\xi \in \mathcal{H}$,

(c) $\delta(A^*) = \mathcal{J}\delta(A)$ *for all* $A \in M_n(\mathbb{C})$,

(d) $\delta(AB) = \pi_l(\sigma_{-i/4}(A))\delta(B) + \pi_r(\sigma_{i/4}(B))\delta(A)$ *and*

(e) $\mathcal{H} = \mathrm{lin}\{\pi_l(A)\delta(B) \mid A, B \in M_n(\mathbb{C})\}$

such that

$$\langle A, L(B)\rangle_\rho = \langle \delta(A), \delta(B)\rangle_{\mathcal{H}} \tag{3.5}$$

for all $A, B \in M_n(\mathbb{C})$.

Proof. First, we claim that $\check{L}$ is a KMS-symmetric conditionally completely negative map (see Definition 2.2.10). The KMS-symmetry follows from Lemma 3.3.3(a). By Lemma 3.3.3(b) $\check{\Phi}_t$ is completely positive for all $t \geq 0$. Since

$$\check{L} = \lim_{t\to 0} \frac{1}{t}(\mathrm{id}_n - \check{\Phi}_t)$$

by Lemma 3.3.3(c), it follows from the definitions that $\check{L}$ is conditionally completely negative, proving the claim. We also observe that

$$\check{L}(1_n) = 2\int_0^\infty \sigma_{-i/4} e^{-r\sigma_{-i/2}}(L(\sigma_{-i/4} e^{-r\sigma_{-i/2}}(1_n)))\, dr = 0$$

by Proposition 3.3.1.

Next, we define a sesquilinear form $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on the algebraic tensor product $M_n(\mathbb{C}) \odot M_n(\mathbb{C})$ by

$$\langle A_1 \otimes B_1, A_2 \otimes B_2\rangle_{\mathcal{H}} = -\frac{1}{2}\mathrm{Tr}(B_1^*\rho^{1/2}\check{L}(A_1^* A_2)\rho^{1/2}B_2).$$

Now consider the subspace

$$N = \left\{\sum_j A_j \otimes B_j \,\middle|\, \sum_j A_j \sigma_{-i/2}(B_j) = 0\right\}.$$

3

Because $\check{L}$ is completely conditionally negative, we see that for any $\sum_j A_j \otimes B_j \in N$ we have

$$\begin{aligned}\sum_{jk} \langle A_j \otimes B_j, A_k \otimes B_k \rangle_{\mathscr{H}} &= -\frac{1}{2} \sum_{jk} \mathrm{Tr}(B_j^* \rho^{1/2} \check{L}(A_j^* A_k) \rho^{1/2} B_k) \\ &= -\frac{1}{2} \sum_{jk} \mathrm{Tr}(\rho^{1/2} \sigma_{-i/2}(B_j)^* \check{L}(A_j^* A_k) \sigma_{-i/2}(B_k) \rho^{1/2}) \\ &\geq 0.\end{aligned}$$

Therefore, this sesquilinear form is positive semidefinite on N. Now let

$$\mathscr{H} = N / \{u \in N \mid \langle u, u \rangle_{\mathscr{H}} = 0\}.$$

Then $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ induces an inner product on $\mathscr{H}$, which turns $\mathscr{H}$ into a Hilbert space (as $\mathscr{H}$ is finite-dimensional). We write $\sum_j A_j \otimes_{\mathscr{H}} B_j$ for the image of $\sum_j A_j \otimes B_j$ in $\mathscr{H}$ under the quotient map.

Define π_l and π_r by

$$\pi_l(X) \sum_j A_j \otimes_{\mathscr{H}} B_j = \sum_j X A_j \otimes_{\mathscr{H}} B_j, \pi_r(X) \sum_j A_j \otimes_{\mathscr{H}} B_j = \sum_j A_j \otimes_{\mathscr{H}} B_j X$$

for $X \in M_n(\mathbb{C})$ and $\sum_j A_j \otimes B_j \in N$. These maps are well defined. Indeed, they preserve N, and for all $u \in N$ with $\langle u, u \rangle_{\mathscr{H}} = 0$ we have

$$\langle \pi_l(X) \pi_r(Y) u, \pi_l(X) \pi_r(Y) u \rangle_{\mathscr{H}} = \langle \pi_l(X^* X) \pi_r(Y Y^*) u, u \rangle_{\mathscr{H}} \leq 0$$

by Cauchy-Schwarz. From the definitions of π_l, π_r and $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ we now conclude that π_l is a unital $*$-homomorphism and π_r is a unital $*$-antihomomorphism.

Subsequently, we will define the anti-linear isometric involution $\mathscr{J}$ on $\mathscr{H}$. Consider the map

$$A \otimes B \mapsto -B^* \otimes A^*$$

from $M_n(\mathbb{C}) \odot M_n(\mathbb{C})$ to itself. This is an isometry since $\check{L}$ is KMS-symmetric. Moreover, it preserves N. Consequently, it acts in a well-defined manner on the equivalence classes of $\mathscr{H}$, and we call this map $\mathscr{J}$. It is clear that $\mathscr{J}$ is an anti-linear involution.

Lastly, we define the map $\delta : M_n(\mathbb{C}) \to \mathscr{H}$ by

$$\delta(A) = \sigma_{-i/4}(A) \otimes_{\mathscr{H}} 1_n - 1_n \otimes \sigma_{i/4}(A).$$

With this definition properties (a)-(d) are immediate from the definitions. For property (e) note that

$$\sum_j A_j \otimes_{\mathscr{H}} B_j = -\sum_j \pi_l(A_j) \delta(\sigma_{-i/4}(B_j)) - A_j \sigma_{-i/2}(B_j) \otimes_{\mathscr{H}} 1_n = -\sum_j \pi_l(A_j) \delta(\sigma_{-i/4}(B_j))$$

for any $\sum_j A_j \otimes_{\mathscr{H}} B_j \in \mathscr{H}$.

To complete the proof of the theorem, we need to show that Equation (3.5) holds. Let $A, B \in M_n(\mathbb{C})$. Then we have

$$\begin{aligned}\langle \delta(A), \delta(B) \rangle_{\mathscr{H}} &= \langle \sigma_{-i/4}(A) \otimes_{\mathscr{H}} 1_n - 1_n \otimes_{\mathscr{H}} \sigma_{i/4}(A), \sigma_{-i/4}(B) \otimes_{\mathscr{H}} 1_n - 1_n \otimes_{\mathscr{H}} \sigma_{i/4}(B) \rangle_{\mathscr{H}} \\ &= \frac{1}{2} \Big(\mathrm{Tr}(\sigma_{-i/4}(A^*) \rho^{1/2} \check{L}(\sigma_{-i/4}(B)) \rho^{1/2}) + \mathrm{Tr}(\rho^{1/2} \check{L}(\sigma_{i/4}(A^*)) \rho^{1/2} \sigma_{i/4}(B)) \\ &\quad - \mathrm{Tr}(\rho^{1/2} \check{L}(\sigma_{i/4}(A^*) \sigma_{-i/4}(B)) \rho^{1/2}) - \mathrm{Tr}(\sigma_{-i/4}(A^*) \rho^{1/2} \check{L}(1_n) \rho^{1/2} \sigma_{i/4}(B)) \Big).\end{aligned}$$

The last two terms are zero because $\check{L}$ is KMS-symmetric and $\check{L}(1_n) = 0$. For the first two terms we use the KMS-symmetry of $\check{L}$ and the fact that $\mathrm{Tr}(\sigma_{it}(C)D) = \mathrm{Tr}(C\sigma_{-it}(D))$ for all $t \in \mathbb{R}$ and $C, D \in M_n(\mathbb{C})$ to conclude that

$$\begin{aligned}\langle \delta(A), \delta(B)\rangle_{\mathscr{H}} &= \frac{1}{2}\mathrm{Tr}(A^*\rho^{1/2}\sigma_{i/4}(\check{L}(\sigma_{-i/4}(B)))\rho^{1/2}) + \frac{1}{2}\mathrm{Tr}(\rho^{1/2}A^*\rho^{1/2}\sigma_{-i/4}(\check{L}(\sigma_{i/4}(B))))\\ &= \mathrm{Tr}(A^*\rho^{1/2}\mathscr{W}(\check{L})(B)\rho^{1/2})\\ &= \langle A, L(B)\rangle_\rho,\end{aligned}$$

as desired. □

3

As a consequence of the previous result, we get a more explicit expression for the quadratic form associated with the generator of a KMS-symmetric quantum Markov semigroup on $M_n(\mathbb{C})$. An analogous expression can be found in [CM17, Eq. (5.3)], [CM20, Prop. 2.5] for the special case of GNS-symmetric quantum Markov semigroups.

Theorem 3.3.5. *If (Φ_t) is a KMS-symmetric quantum Markov semigroup on $M_n(\mathbb{C})$ with generator L, then there exist matrices $V_1, \dots, V_N \in M_n(\mathbb{C})$ such that $\{V_j\}_{j=1}^N = \{V_j^*\}_{j=1}^N$ and*

$$\langle A, L(B)\rangle_\rho = \sum_{j=1}^N \langle [V_j, A], [V_j, B]\rangle_\rho$$

for all $A, B \in M_n(\mathbb{C})$.

Proof. Let $\mathscr{H}$, π_l, π_r, $\mathscr{J}$ and δ be as in the last Theorem. The map $\pi_l \otimes \pi_r$ is a unital representation of $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})^\circ$. It follows from the representation theory of matrix algebras [Bla06, Proposition IV.1.2.2.] that there exists an auxiliary Hilbert space $\mathscr{K}$ such that $\mathscr{H} \cong M_n(\mathbb{C}) \otimes \mathscr{K}$, where $M_n(\mathbb{C})$ is endowed with the Hilbert-Schmidt inner product, and

$$\begin{aligned}\pi_l(A)(B\otimes\xi) &= AB\otimes\xi,\\ \pi_r(A)(B\otimes\xi) &= BA\otimes\xi\end{aligned}$$

for $A, B \in M_n(\mathbb{C})$ and $\xi \in \mathscr{K}$ under this identification.

Moreover, since $\mathscr{H}$ is finite-dimensional by property (e), the space $\mathscr{K}$ is finite-dimensional, say $\mathscr{K} = \mathbb{C}^N$.

Thus $\delta(A) = \sum_{j=1}^N \delta_j(A) \otimes e_j$ with the canonical orthonormal basis (e_j) on $\mathbb{C}^N$. It follows from property (d) that

$$\rho^{-1/4}\delta_j(AB)\rho^{-1/4} = A\rho^{-1/4}\delta_j(B)\rho^{-1/4} + \rho^{-1/4}\delta_j(A)\rho^{-1/4}B$$

for $A, B \in M_n(\mathbb{C})$. In other words, $A \mapsto \rho^{-1/4}\delta_j(A)\rho^{-1/4}$ is a derivation. By the derivation theorem [Kap53, Theorem 9] there exists $V_j \in M_n(\mathbb{C})$ such that $\delta_j(A) = \rho^{1/4}[V_j, A]\rho^{1/4}$.

We conclude that

$$\langle A, L(B)\rangle_\rho = \langle \delta(A), \delta(B)\rangle_{\mathscr{H}} = \sum_{j=1}^N \mathrm{Tr}(\delta_j(A)^*\delta_j(B)) = \sum_{j=1}^N \langle [V_j, A], [V_j, B]\rangle_\rho.$$

Since $\langle [V_j, A], [V_j, B]\rangle_\rho = \langle [V_j^*, B^*], [V_j^*, A^*]\rangle_\rho$, we have

$$\langle \delta(A), \delta(B)\rangle_{\mathcal{H}} = \langle \mathcal{J}\delta(B), \mathcal{J}\delta(A)\rangle_{\mathcal{H}} = \sum_{j=1}^N \langle [V_j, B^*], [V_j, A^*]\rangle_\rho = \sum_{j=1}^N \langle [V_j^*, A], [V_j^*, B]\rangle_\rho.$$

This shows that $V_1, \ldots, V_N$ can be chosen such that $\{V_j\}_{j=1}^N = \{V_j^*\}_{j=1}^N$. □

3

Remark 3.3.6. This theorem can also be proven without the use of Theorem 3.3.4. In the appendix we include a proof of Theorem 3.3.5 using the $\mathcal{V}$-transform and the structure of generators of KMS-symmetric quantum Markov semigroups described in [AC21, Theorem 4.4].

3.4. THE $\mathcal{V}$-TRANSFORM

As we have seen in Section 3.3 in the case of matrix algebras, the key ingredient to construct the twisted derivation associated with a KMS-symmetric quantum Markov semigroup is the $\mathcal{V}$-transform. This remains the case for semigroups on general von Neumann algebras, but the definition and properties of the $\mathcal{V}$-transform become much more delicate as it involves in general unbounded operators like the analytic generator of the modular group. In particular the fact that the $\mathcal{V}$-transform preserves completely positive maps, which is crucial for defining an inner product, requires new arguments in this general setting.

For technical convenience, we first define the $\mathcal{V}$-transform for bounded operators on the Hilbert space $L^2(\mathcal{M})$, before we transfer it to KMS-symmetric unital completely positive maps on $\mathcal{M}$.

Throughout this section we fix a σ-finite von Neumann algebra $\mathcal{M}$ and a faithful normal state φ on $\mathcal{M}$. We write Δ and J for the modular operator and modular conjugation associated with φ, and $L^2_+(\mathcal{M})$ for the standard self-dual positive cone in $L^2(\mathcal{M})$. We also write φ for the element in $L^1(\mathcal{M})$ representing φ so that $\varphi^{1/2}$ is the unique positive vector in $L^2(\mathcal{M})$ representing φ, $\Delta^{it}\cdot\Delta^{-it}$ is the modular group etc.

Let us very briefly recap some basic definitions from modular theory. A more detailed account can be found in Section 2.1.2. The standard Hilbert space $L^2(\mathcal{M})$ can be identified with the GNS Hilbert space associated with φ in such a way that $\varphi^{1/2}$ corresponds to the cyclic and separating vector induced by φ and the left action of $\mathcal{M}$ on $L^2(\mathcal{M})$ corresponds to the GNS representation induced by φ.

The operator

$$\mathcal{M}\varphi^{1/2} \to L^2(\mathcal{M}),\ x\varphi^{1/2} \mapsto x^*\varphi^{1/2}$$

is a densely defined closable anti-linear operator, whose closure is denoted by S.

The operator S has a polar decomposition $S = J\Delta^{1/2}$ with an anti-unitary involution J, called the *modular conjugation*, and a non-singular positive self-adjoint operator Δ, called the *modular operator*.

The *modular group* is the group σ^φ of φ-preserving $*$-automorphisms of $\mathcal{M}$ given by $\sigma_t^\varphi(x) = \Delta^{it}x\Delta^{-it}$ for $x \in \mathcal{M}$ and $t \in \mathbb{R}$. An element x of $\mathcal{M}$ is called *entire analytic* for σ^φ if the map $t \mapsto \sigma_t^\varphi(x)$ has an analytic continuation to the complex plane. This analytic continuation is then unique, and its value at $z \in \mathbb{C}$ is denoted by $\sigma_z^\varphi(x)$.

3.4.1. $\mathcal{V}$-TRANSFORM OF BOUNDED OPERATORS ON $L^2(\mathcal{M})$

In this subsection we define the $\mathcal{V}$-transform on $B(L^2(\mathcal{M}))$ and discuss some of its properties. The $\mathcal{V}$-transform in this setting is the map

$$\mathcal{V}\colon B(L^2(\mathcal{M})) \to B(L^2(\mathcal{M})),\; T \mapsto \check{T} = 2\int_0^\infty \Delta^{1/4} e^{-r\Delta^{1/2}} T \Delta^{1/4} e^{-r\Delta^{1/2}}\,dr.$$

We first prove that it is well-defined. Note that $\Delta^{1/4}e^{-r\Delta^{1/2}}$ is bounded (and self-adjoint) for every $r > 0$, so that the integrand is a bounded operator and the only difficulty is integrability.

3

Lemma 3.4.1. *If $T \in B(L^2(\mathcal{M}))$ and $\xi, \eta \in L^2(\mathcal{M})$, then*

$$r \mapsto \langle \xi, \Delta^{1/4} e^{-r\Delta^{1/2}} T \Delta^{1/4} e^{-r\Delta^{1/2}} \eta\rangle$$

is integrable on $(0,\infty)$, and

$$2\left|\int_0^\infty \langle \xi, \Delta^{1/4} e^{-r\Delta^{1/2}} T \Delta^{1/4} e^{-r\Delta^{1/2}} \eta\rangle\,dr\right| \leq \|T\|\|\xi\|\|\eta\|.$$

Proof. Let E denote the spectral measure of $\Delta^{1/2}$. By the spectral theorem we have

$$\begin{aligned}
&2\int_0^\infty |\langle \xi, \Delta^{1/4} e^{-r\Delta^{1/2}} T \Delta^{1/4} e^{-r\Delta^{1/2}} \eta\rangle|\,dr\\
&\quad\leq \|T\|\int_0^\infty (\|\Delta^{1/4}e^{-r\Delta^{1/2}}\xi\|^2 + \|\Delta^{1/4}e^{-r\Delta^{1/2}}\eta\|^2)\,dr\\
&\quad= \|T\|\int_0^\infty\int_0^\infty \lambda e^{-2\lambda r}\,d(\langle \xi, E(\lambda)\xi\rangle + \langle \eta, E(\lambda)\eta\rangle)\,dr\\
&\quad= \|T\|\int_0^\infty \lambda \int_0^\infty e^{-2\lambda r}\,dr\,d(\langle \xi, E(\lambda)\xi\rangle + \langle \eta, E(\lambda)\eta\rangle)\\
&\quad= \frac{1}{2}\|T\|(\|\xi\|^2 + \|\eta\|^2).
\end{aligned}$$

Thus $r \mapsto \langle \xi, \Delta^{1/4} e^{-r\Delta^{1/2}} T \Delta^{1/4} e^{-r\Delta^{1/2}} \eta\rangle$ is integrable on $(0,\infty)$, and the desired inequality follows from the usual rescaling trick $\xi \mapsto \alpha\xi$, $\eta \mapsto \eta/\alpha$. □

Definition 3.4.2. For $T \in B(L^2(\mathcal{M}))$ let $\check{T}$ denote the unique bounded linear operator on $L^2(\mathcal{M})$ such that

$$\langle \xi, \check{T}\eta\rangle = 2\int_0^\infty \langle \xi, \Delta^{1/4} e^{-r\Delta^{1/2}} T \Delta^{1/4} e^{-r\Delta^{1/2}} \eta\rangle\,dr$$

for all $\xi, \eta \in L^2(\mathcal{M})$.

We call the map

$$\mathcal{V}\colon B(L^2(\mathcal{M})) \to B(L^2(\mathcal{M})),\; T \mapsto \check{T}$$

the $\mathcal{V}$-transform.

Note that if T commutes with the modular operator, then $\check{T} = T$.

Proposition 3.4.3. *The $\mathcal{V}$-transform on $B(L^2(\mathcal{M}))$ is a normal unital completely positive trace-preserving map.*

Proof. Let E denote the spectral measure of $\Delta^{1/2}$. By the spectral theorem,

$$\begin{aligned} 2\int_0^\infty \langle \Delta^{1/4}e^{-r\Delta^{1/2}}\xi, \Delta^{1/4}e^{-r\Delta^{1/2}}\eta\rangle\, dr &= 2\int_0^\infty\int_0^\infty \lambda e^{-2\lambda r}\, d\langle\xi, E(\lambda)\eta\rangle\, dr \\ &= 2\int_0^\infty \lambda \int_0^\infty e^{-2\lambda r}\, dr\, d\langle \xi, E(\lambda)\eta\rangle \\ &= \langle \xi, \eta\rangle \end{aligned}$$

for all ξ, $\eta \in L^2(\mathcal{M})$. Thus $\check{1}_{B(L^2(\mathcal{M}))} = 1_{B(L^2(\mathcal{M}))}$.

Since $\Delta^{1/4}e^{-r\Delta^{1/2}}$ is self-adjoint, the map $T \mapsto \Delta^{1/4}e^{-r\Delta^{1/2}}T\Delta^{1/4}e^{-r\Delta^{1/2}}$ is completely positive. Hence $\mathcal{V}$ is completely positive.

To prove that $\mathcal{V}$ is trace-preserving, let $T \in B(L^2(\mathcal{M}))$ be positive. By the previous part, $\mathcal{V}(T)$ is positive and we have

$$\begin{aligned} \mathrm{Tr}(\mathcal{V}(T)) &= 2\int_0^\infty \mathrm{Tr}(\Delta^{1/4}e^{-r\Delta^{1/2}}T\Delta^{1/4}e^{-r\Delta^{1/2}})\, dr \\ &= 2\int_0^\infty \mathrm{Tr}(T^{1/2}\Delta^{1/2}e^{-2r\Delta^{1/2}}T^{1/2})\, dr \\ &= \mathrm{Tr}(T^{1/2}\mathcal{V}(1)T^{1/2}) \\ &= \mathrm{Tr}(T). \end{aligned}$$

Interchanging the integral and the sum defining the trace is justified by Fubini's theorem since the integrand is positive in each case.

To prove that the $\mathcal{V}$-transform is normal, note first that it restricts to a bounded linear map $\mathcal{V}_*$ on the space of trace-class operators on $L^2(\mathcal{M})$ by the previous part. Moreover, if S, $T \in B(L^2(\mathcal{M}))$ and S is trace-class, then

$$\mathrm{Tr}(\mathcal{V}_*(S)T) = 2\int_0^\infty \mathrm{Tr}(\Delta^{1/4}e^{-r\Delta^{1/2}}S\Delta^{1/4}e^{-r\Delta^{1/2}}T)\, dr = \mathrm{Tr}(S\mathcal{V}(T)).$$

Thus $\mathcal{V} = (\mathcal{V}_*)^*$, which implies that $\mathcal{V}$ is normal. □

Lemma 3.4.4 (Key property). *(a) If $T \in B(L^2(\mathcal{M}))$, then*

$$\frac{1}{2}\langle \Delta^{1/4}\xi, \check{T}\Delta^{-1/4}\eta\rangle + \frac{1}{2}\langle \Delta^{-1/4}\xi, \check{T}\Delta^{1/4}\eta\rangle = \langle \xi, T\eta\rangle$$

for all $\xi, \eta \in \mathrm{dom}(\Delta^{1/4}) \cap \mathrm{dom}(\Delta^{-1/4})$.

(b) If $R \in B(L^2(\mathcal{M}))$ such that

$$\frac{1}{2}\langle \Delta^{1/4}\xi, R\Delta^{-1/4}\eta\rangle + \frac{1}{2}\langle \Delta^{-1/4}\xi, R\Delta^{1/4}\eta\rangle = \langle \xi, T\eta\rangle$$

for all $\xi, \eta \in \bigcap_{n\in\mathbb{Z}} \mathrm{dom}(\Delta^n)$, then $R = \check{T}$.

Proof. (a) If $\xi,\eta\in\operatorname{dom}(\Delta^{1/4})\cap\operatorname{dom}(\Delta^{-1/4})$, then

$$\begin{aligned}&\langle\Delta^{1/4}\xi,\check{T}\Delta^{-1/4}\eta\rangle+\langle\Delta^{-1/4}\xi,\check{T}\Delta^{1/4}\eta\rangle\\&\quad=2\int_0^\infty(\langle\Delta^{1/2}e^{-r\Delta^{1/2}}\xi,Te^{-r\Delta^{1/2}}\eta\rangle+\langle e^{-r\Delta^{1/2}}\xi,T\Delta^{1/2}e^{-r\Delta^{1/2}}\eta\rangle)\,dr\\&\quad=-2\int_0^\infty\frac{d}{dr}\langle e^{-r\Delta^{1/2}}\xi,Te^{-r\Delta^{1/2}}\eta\rangle\,dr\\&\quad=2\langle\xi,T\eta\rangle.\end{aligned}$$

Here we used that since $\Delta^{1/2}$ is non-singular, $e^{-r\Delta^{1/2}}\zeta\to0$ as $r\to\infty$ for every $\zeta\in L^2(\mathcal{M})$.

(b) If $\xi,\eta\in\bigcap_{n\in\mathbb{Z}}\operatorname{dom}(\Delta^n)$, then $\Delta^{1/4}e^{-r\Delta^{1/2}}\xi,\Delta^{1/4}e^{-r\Delta^{1/2}}\eta\in\operatorname{dom}(\Delta^{1/4})\cap\operatorname{dom}(\Delta^{-1/4})$ and hence

$$\begin{aligned}\langle\xi,\check{T}\eta\rangle&=2\int_0^\infty\langle\Delta^{1/4}e^{-r\Delta^{1/2}}\xi,T\Delta^{1/4}e^{-r\Delta^{1/2}}\eta\rangle\,dr\\&=\int_0^\infty(\langle\Delta^{1/2}e^{-r\Delta^{1/2}}\xi,Re^{-r\Delta^{1/2}}\eta\rangle+\langle e^{-r\Delta^{1/2}}\xi,R\Delta^{1/2}e^{-r\Delta^{1/2}}\eta\rangle)\,dr.\end{aligned}$$

From here we conclude $\langle\xi,\check{T}\eta\rangle=\langle\xi,R\eta\rangle$ as above. Since $\bigcap_{n\in\mathbb{Z}}\operatorname{dom}(\Delta^n)$ is dense in $L^2(\mathcal{M})$, the equality $R=\check{T}$ follows. □

Informally, the identity from the previous lemma reads

$$\frac{1}{2}\Delta^{1/4}\check{T}\Delta^{-1/4}+\frac{1}{2}\Delta^{-1/4}\check{T}\Delta^{1/4}=T.$$

Proposition 3.4.5. *Let $T\in B(L^2(\mathcal{M}))$.*

(a) $\check{T}=\int_0^\infty\Delta^{-1/4}e^{-r\Delta^{-1/2}}T\Delta^{-1/4}e^{-r\Delta^{-1/2}}\,dr$ in the weak operator topology.

(b) If $JT=TJ$, then $J\check{T}=\check{T}J$.

(c) If $T\varphi^{1/2}=\varphi^{1/2}$, then $\check{T}\varphi^{1/2}=\varphi^{1/2}$.

Proof. (a) Let $R=\int_0^\infty\Delta^{-1/4}e^{-r\Delta^{-1/2}}T\Delta^{-1/4}e^{-r\Delta^{1/2}}\,dr$. The existence is justified by the same arguments as for $\check{T}$. Replacing Δ by Δ^{-1} in Lemma 3.4.4(a) we obtain

$$\frac{1}{2}\langle\Delta^{1/4}\xi,R\Delta^{-1/4}\eta\rangle+\frac{1}{2}\langle\Delta^{-1/4}\xi,R\Delta^{1/4}\eta\rangle=\langle\xi,T\eta\rangle$$

for $\xi,\eta\in\operatorname{dom}(\Delta^{1/4})\cap\operatorname{dom}(\Delta^{-1/4})$. Now $R=\check{T}$ follows from Lemma 3.4.4(b).

(b) One has $J\Delta^{1/4}=\Delta^{-1/4}J$, and by the spectral theorem also $Je^{-r\Delta^{1/2}}=e^{-r\Delta^{-1/2}}J$. Thus

$$\begin{aligned}J\check{T}&=\int_0^\infty\Delta^{-1/4}e^{-r\Delta^{-1/2}}JT\Delta^{1/4}e^{-r\Delta^{1/2}}\,dr\\&=\int_0^\infty\Delta^{-1/4}e^{-r\Delta^{-1/2}}TJ\Delta^{1/4}e^{-r\Delta^{1/2}}\,dr\\&=\int_0^\infty\Delta^{-1/4}e^{-r\Delta^{-1/2}}T\Delta^{-1/4}e^{-r\Delta^{-1/2}}\,dr\,J.\end{aligned}$$

Now $J\check{T}=\check{T}J$ follows from (a).

(c) This is immediate from the definition of $\check{T}$. □

We are particularly interested in the action of the $\mathscr{V}$-transform on Markovian operators. Let us first recall the definition (see also Definition 2.2.18).

Definition 3.4.6. An operator $T \in B(L^2(\mathscr{M}))$ is *Markovian* if it is completely positivity-preserving and $T\varphi^{1/2} = \varphi^{1/2}$.

To prove that the $\mathscr{V}$-transform also preserves Markovian operators, we need a series of lemmas. Recall that the cones $C_\varphi^\sharp$ and $C_\varphi^\flat$ are defined in Definition 2.1.26 as

$$C_\varphi^\sharp = \overline{\{x\varphi^{1/2} \mid x \in \mathscr{M}_+\}},\ C_\varphi^\flat = \overline{\{\varphi^{1/2}x \mid x \in \mathscr{M}_+\}},$$

and further recall that $C_\varphi^\sharp$ and $C_\varphi^\flat$ are dual cones (see Definition 2.1.25), that is, $\xi \in C_\varphi^\sharp$ if and only if $\langle \xi, \eta\rangle \geq 0$ for all $\eta \in C_\varphi^\flat$, and vice versa.

Lemma 3.4.7. *Let $T \in B(L^2(\mathscr{M}))$ be positivity-preserving. If $\xi \in C_\varphi^\sharp$ (resp. $\xi \in C_\varphi^\flat$) and $\eta \in C_\varphi^\flat$ (resp. $\eta \in C_\varphi^\sharp$), then*

$$\mathrm{Re}\langle \xi, \check{T}\eta\rangle \geq 0.$$

Proof. Let $x, y \in \mathscr{M}_+$. By Lemma 3.4.4(a) we have

$$\frac{1}{2}\langle x\varphi^{1/2}, \check{T}(\varphi^{1/2}y)\rangle + \frac{1}{2}\langle \varphi^{1/2}x, \check{T}(y\varphi^{1/2})\rangle = \langle \varphi^{1/4}x\varphi^{1/4}, T(\varphi^{1/4}y\varphi^{1/4})\rangle \geq 0.$$

Since T is positivity-preserving, it commutes with J. Hence so does $\check{T}$ by Proposition 3.4.5(b). If we apply this to the first summand from the previous equation, we obtain

$$\langle x\varphi^{1/2}, \check{T}(\varphi^{1/2}y)\rangle = \langle J\check{T}(\varphi^{1/2}y), J(x\varphi^{1/2})\rangle = \langle \check{T}(y\varphi^{1/2}), \varphi^{1/2}x\rangle.$$

Therefore

$$\mathrm{Re}\langle x\varphi^{1/2}, \check{T}(\varphi^{1/2}y)\rangle \geq 0.$$

This settles the claim for $\xi = x\varphi^{1/2}$ and $\eta = \varphi^{1/2}y$. For arbitrary $\xi \in C_\varphi^\sharp$ and $\eta \in C_\varphi^\flat$ the inequality follows by approximation. The proof for $\xi \in C_\varphi^\flat$ and $\eta \in C_\varphi^\sharp$ is analogous. □

Lemma 3.4.8. *If $R \in B(L^2(\mathscr{M}))$ such that*

$$\mathrm{Re}\langle \xi, R\eta\rangle \geq 0, \qquad \xi \in C_\varphi^\sharp, \eta \in C_\varphi^\flat,$$
$$\mathrm{Re}\langle \xi, R\eta\rangle \geq 0, \qquad \xi \in C_\varphi^\flat, \eta \in C_\varphi^\sharp,$$

then $\mathrm{Re}\langle \xi, R\eta\rangle \geq 0$ *for all* $\xi, \eta \in L^2_+(\mathscr{M})$.

Proof. Let $S = \{z \in \mathbb{C} \mid 0 < \mathrm{Re}\, z < 1/2\}$ and

$$f : \overline{S} \to \mathbb{C},\ f(z) = e^{-\langle J\Delta^z(x\varphi^{1/2}), R\Delta^{\bar{z}}(y\varphi^{1/2})\rangle}$$

for $x,\ y \in \mathscr{M}_+$. The function f is continuous on $\overline{S}$, holomorphic on S and satisfies

$$\begin{aligned}|f(z)| &= e^{-\operatorname{Re}\langle J\Delta^z(x\varphi^{1/2}), R\Delta^z(y\varphi^{1/2})\rangle}\\ &\leq e^{\|J\Delta^z(x\varphi^{1/2})\|\|R\Delta^z(y\varphi^{1/2})\|}\\ &\leq e^{\|R\|\|\Delta^{\operatorname{Re} z}(x\varphi^{1/2})\|\|\Delta^{\operatorname{Re} z}(y\varphi^{1/2})\|}.\end{aligned}$$

By the spectral theorem,

$$\|\Delta^{\operatorname{Re} z}(x\varphi^{1/2})\|^2 \leq \|x\varphi^{1/2}\|^2 + \|\Delta^{1/2}(x\varphi^{1/2})\|^2$$

and likewise for $y\varphi^{1/2}$. Thus f is bounded on $\overline{S}$.

If $\operatorname{Re} z = 0$, then $\Delta^z(y\varphi^{1/2}) = \sigma^\varphi_{\operatorname{Im} z}(y)\varphi^{1/2} \in C^\sharp_\varphi$ and $J\Delta^z(x\varphi^{1/2}) = \varphi^{1/2}\sigma^\varphi_{\operatorname{Im} z}(x) \in C^\flat_\varphi$. Thus $|f(z)| \leq 1$ by assumption. Similarly, $|f(z)| \leq 1$ if $\operatorname{Re} z = 1/2$.

It follows from the Phragmén-Lindelöf principle that $|f(1/4)| \leq 1$, which means

$$\operatorname{Re}\langle\Delta^{1/4}(x\varphi^{1/2}), R\Delta^{1/4}(y\varphi^{1/2}) = \operatorname{Re}\langle J\Delta^{1/4}(x\varphi^{1/2}), R\Delta^{1/4}(y\varphi^{1/2})\rangle\rangle \geq 0.$$

This settles the claim for $\xi = \Delta^{1/4}(x\varphi^{1/2})$, $\eta = \Delta^{1/4}(y\varphi^{1/2})$ with $x,\ y \in \mathscr{M}_+$. For arbitrary $\xi,\ \eta \in L^2_+(\mathscr{M})$ the claim follows by approximation by Theorem 2.1.28. □

Lemma 3.4.9. *If $R \in B(L^2(\mathscr{M}))$ such that $\operatorname{Re}\langle\xi, R\eta\rangle \geq 0$ for all $\xi,\ \eta \in L^2_+(\mathscr{M})$ and $JR = RJ$, then R is positivity-preserving.*

Proof. For $\xi,\ \eta \in L^2_+(\mathscr{M})$ we have $J\xi = \xi$, $J\eta = \eta$, and hence

$$\langle R\eta, \xi\rangle = \langle J\xi, JR\eta\rangle = \langle\xi, R\eta\rangle.$$

Therefore $\langle\xi, R\eta\rangle$ is real and thus positive by assumption. As $L^2_+(\mathscr{M})$ is self-dual, the claim follows. □

Proposition 3.4.10. *The $\mathscr{V}$-transform maps symmetric Markovian operators to symmetric Markovian operators.*

Proof. If $T \in B(L^2(\mathscr{M}))$ is positivity-preserving, it follows from the previous three lemmas that $\check{T}$ is positivity-preserving as well. If T is completely positivity-preserving, the same argument applied to the amplifications $T \otimes \mathrm{id}_n$ shows that $\check{T}$ is completely positivity-preserving. That $T\varphi^{1/2} = \varphi^{1/2}$ implies $\check{T}\varphi^{1/2} = \varphi^{1/2}$ was established in Proposition 3.4.5(c). □

3.4.2. $\mathscr{V}$-TRANSFORM OF KMS-SYMMETRIC OPERATORS ON $\mathscr{M}$

Formally, the $\mathscr{V}$-transform of a bounded linear operator Φ on $\mathscr{M}$ should be given by

$$2\int_0^\infty \sigma_{-i/4}e^{-r\sigma_{-i/2}}\Phi\sigma_{-i/4}e^{-r\sigma_{-i/2}}\,dr,$$

where we simply replaced the modular operator in the definition of $\mathscr{V}$ by the analytic generator of the modular group on $\mathscr{M}$. However, it seems hard to make this formula rigorous. It is not even clear if $\sigma_{-i/2}$ generates a semigroup on $\mathscr{M}$ in a suitable sense.

Instead, we take a different approach that relies on the correspondence between certain operators in $B(\mathscr{M})$ and operators in $B(L^2(\mathscr{M}))$. This way we can only define the $\mathscr{V}$-transform for KMS-symmetric unital completely positive maps and generators of KMS-symmetric quantum Markov semigroups, but this suffices for our purposes.

Let us first recall the definition of a KMS-symmetric map in this setting (see Definition 2.2.16. A linear map $\Phi\colon \mathscr{M}\to\mathscr{M}$ is called *KMS-symmetric* (with respect to φ) if

$$\varphi(T(x)\sigma^{\varphi}_{-i/2}(y)) = \varphi(\sigma^{\varphi}_{i/2}(x)T(y)).$$

for all $x,\ y\in\mathscr{M}^{\varphi}_a$. Using the trace-like functional Tr on the Haagerup L^1 space, this can compactly be rewritten as

$$\mathrm{Tr}(\Phi(x)^*\varphi^{1/2}y\varphi^{1/2}) = \mathrm{Tr}(x^*\varphi^{1/2}\Phi(y)\varphi^{1/2})$$

in analogy with the definition for matrix algebras.

There is the following correspondence between symmetric completely positivity-preserving maps on $L^2(\mathscr{M})$ and KMS-symmetric completely positive maps on $\mathscr{M}$ (see Section 2.2.2): if Φ is KMS-symmetric unital completely positive map on $\mathscr{M}$, then there exists a unique bounded linear operator T on $L^2(\mathscr{M})$ such that

$$T(\varphi^{1/4}x\varphi^{1/4}) = \varphi^{1/4}\Phi(x)\varphi^{1/4}$$

for all $x\in\mathscr{M}$. This operator is a symmetric Markovian operator, which we denote by $\Phi^{(2)}$.

Conversely, if T is a symmetric Markovian operator on $L^2(\mathscr{M})$, then there exists a KMS-symmetric unital completely positive map Φ on $\mathscr{M}$ such that $\Phi^{(2)} = T$.

If Φ is a KMS-symmetric unital completely positive map on $\mathscr{M}$, then $\mathscr{V}(\Phi^{(2)})$ is a symmetric Markovian operator by Proposition 3.4.10. This justifies the following definition.

Definition 3.4.11. Let Φ be a KMS-symmetric unital completely positive map on $\mathscr{M}$. Its $\mathscr{V}$-transform $\check{\Phi}$ is the unique KMS-symmetric unital completely positive map Ψ on $\mathscr{M}$ such that

$$\mathscr{V}(\Phi^{(2)}) = \Psi^{(2)}.$$

We also write $\mathscr{V}(\Phi)$ for $\check{\Phi}$.

The main object of interest of this chapter are semigroups of unital completely positive maps. While the $\mathscr{V}$-transform preserves unitality and complete positivity, it does not, in general, preserve the semigroup property. We will discuss in the following how one can still $\mathscr{V}$-transform generators of a class of such semigroups in a way that preserves complete positivity of the generated semigroup. The relevant background on quantum Markov semigroups can be found in Section 2.2.2. Here we highlight that the generators of uniformly continuous quantum Markov semigroups can be characterised as follows (see Theorem 2.2.14): a bounded operator L on $\mathscr{M}$ is the generator of a uniformly continuous quantum Markov semigroup if and only it is normal, $L(1) = 0$ and L is conditionally completely negative.

Let (Φ_t) be a uniformly continuous KMS-symmetric Markov semigroup on $\mathscr{M}$ and (T_t) the associated symmetric Markov semigroup on $L^2(\mathscr{M})$. Let L and L_2 denote the generator of (Φ_t) on $\mathcal{M}$ and (T_t) on $L^2(\mathcal{M})$, respectively. By the uniform continuity assumption, both are bounded linear operators. Thus we can form the $\mathcal{V}$-transform of L_2, and the continuity of the $\mathcal{V}$-transform implies

$$\check{L}_2 = \lim_{t\to 0} \frac{1}{t}(1 - \check{T}_t)$$

in operator norm. However, since $(\check{T}_t)$ is not a semigroup in general, this does not imply directly that $\check{L}_2$ generates a symmetric Markov semigroup. The lack of semigroup property can be taken care of by a suitable rescaling of the time parameter. More precisely, we have the following result.

3

Proposition 3.4.12. *If (T_t) is a symmetric Markov semigroup on $L^2(\mathcal{M})$, then the semigroup generated by $\check{L}_2$ satisfies*

$$e^{-t\check{L}_2}\xi = \lim_{n\to\infty} \check{T}^n_{t/n}\xi$$

for all $t \geq 0$ and $\xi \in L^2(\mathcal{M})$. In particular, $(e^{-t\check{L}_2})$ is a symmetric Markov semigroup.

Proof. Since the $\mathcal{V}$-transform is a contraction, we have $\|\check{T}_t\| \leq \|T_t\| \leq 1$ for all $t \geq 0$. It follows from the Chernoff product formula [Engel, Nagel, Theorem III.5.2] that $e^{-t\check{L}_2}\xi = \lim_{n\to\infty} \check{T}^n_{t/n}\xi$ for all $t \geq 0$ and $\xi \in L^2(\mathcal{M})$. Finally, as $\check{T}_s$ is a symmetric Markovian operator for all $s \geq 0$, so is $e^{-t\check{L}_2}$ for all $t \geq 0$. □

As a consequence of the previous theorem, there exists a unique KMS-symmetric quantum Markov semigroup (Ψ_t) on $\mathcal{M}$ such that

$$\varphi^{1/4}\Psi_t(x)\varphi^{1/4} = e^{-t\check{L}_2}(\varphi^{1/4}x\varphi^{1/4})$$

for all $x \in \mathcal{M}$ and $t \geq 0$. Moreover, (Ψ_t) is uniformly continuous. This justifies the following definition.

Definition 3.4.13. If L is the generator of a uniformly continuous KMS-symmetric Markov semigroup on $\mathcal{M}$, its $\mathcal{V}$-transform $\check{L}$ is the unique normal linear operator on $\mathcal{M}$ such that

$$\varphi^{1/4}\check{L}(x)\varphi^{1/4} = \check{L}_2(\varphi^{1/4}x\varphi^{1/4})$$

for all $x \in \mathcal{M}$.

We have now defined the $\mathcal{V}$-transform for two different classes of operators on $\mathcal{M}$, generators of KMS-symmetric quantum Markov semigroups and KMS-symmetric unital completely positive maps, in different ways. Because any generator L of a KMS-symmetric quantum Markov semigroup satisfies $L(1) = 0$, we know that these two classes are disjoint. Therefore there is no conflict between our definitions of the $\mathcal{V}$-transform of completely positive maps and Markov generators.

3.5. DERIVATIONS FOR UNIFORMLY CONTINUOUS QUANTUM MARKOV SEMIGROUPS

In this section we study the correspondence between uniformly KMS-symmetric quantum Markov semigroups and certain twisted derivations with values in bimodules. We show that — like in the case of matrix algebras — every uniformly continuous KMS-symmetric quantum Markov semigroup gives rise to a twisted derivation (Theorem 3.5.2) and that this twisted derivation is unique (Theorem 3.5.9).

Throughout this section let $\mathcal{M}$ be a von Neumann algebra and φ a faithful normal state on $\mathcal{M}$. KMS symmetry is always understood with respect to this state φ and multiplication of elements in $L^2(\mathcal{M})$ is understood as the multiplication induced by the full left Hilbert algebra associated with φ. For a left-bounded vector $a \in L^2(\mathcal{M})$ we write $\pi_l(a)$ for the bounded operator of left multiplication by a. Likewise, if $a \in L^2(\mathcal{M})$ is right-bounded, we write $\pi_r(a)$ for the right multiplication operator.

3.5.1. EXISTENCE AND INNERNESS

After we established the existence and properties of the $\mathscr{V}$-transform in the previous section, the proof for the existence of a twisted derivation associated with a uniformly continuous KMS-symmetric quantum Markov semigroup follows the same strategy as in the finite-dimensional case.

To establish that the twisted derivation is inner, we need the following result, which is an easy consequence of the Christensen-Evans theorem [CE79]. Recall that a bounded linear map between C^*-algebras is called *decomposable* if it is a linear combination of completely positive maps [Haa85].

Proposition 3.5.1. *Every generator of a uniformly continuous quantum Markov semigroup on $\mathcal{M}$ is the difference of two normal completely positive maps. In particular, it is decomposable.*

Proof. Let L be a normal conditionally negative definite map on $\mathcal{M}$. By [CE79, Theorem 3.1] there exists $k \in \mathcal{M}$ and a completely positive map $\Phi\colon \mathcal{M} \to \mathcal{M}$ such that

$$L(x) = k^* x + xk - \Phi(x)$$

for all $x \in \mathcal{M}$. Since L is normal, so is Φ. It follows from [Pis20, Proposition 6.10] that L is the difference of two normal completely positive maps. □

Every correspondence (see Definition 2.3.2(c)) gives rise to a representation of the *binormal tensor product* $\mathcal{M} \otimes_{\text{bin}} \mathcal{M}^\circ$, which is defined as follows (see [EL77a]). A linear functional ω from the algebraic tensor product $\mathcal{M} \odot \mathcal{M}^\circ$ to $\mathbb{C}$ is called a *binormal state* if $\omega(u^* u) \geq 0$ for all $u \in \mathcal{M} \odot \mathcal{M}^\circ$, $\omega(1) = 1$ and the maps $x \mapsto \omega(x \otimes y_0^\circ)$, $y^\circ \mapsto \omega(x_0 \otimes y^\circ)$ are weak* continuous for every $x_0, y_0 \in \mathcal{M}$. For u in the algebraic tensor product $\mathcal{M} \odot \mathcal{M}^\circ$ let

$$\|u\|_{\text{bin}} = \sup\{\omega(u^* u)^{1/2} \mid \omega \text{ binormal state on } \mathcal{M} \odot \mathcal{M}^\circ\}.$$

The binormal tensor product $\mathcal{M} \otimes_{\text{bin}} \mathcal{M}^\circ$ is the completion of $\mathcal{M} \odot \mathcal{M}^\circ$ with respect to the norm $\|\cdot\|_{\text{bin}}$.

Theorem 3.5.2. *Let (Φ_t) be a uniformly continuous KMS-symmetric quantum Markov semigroup on $\mathcal{M}$ and let L denote its generator. There exists a correspondence $\mathcal{H}$, an anti-unitary involution $\mathcal{J}\colon \mathcal{H}\to\mathcal{H}$ and a bounded operator $\delta\colon L^2(\mathcal{M})\to\mathcal{H}$ satisfying*

(a) $\mathcal{J}(x\xi y)=y^*(\mathcal{J}\xi)x^*$ *for all* $x,\ y\in\mathcal{M}$ *and* $\xi\in\mathcal{H}$,

(b) $\delta(Ja)=\mathcal{J}\delta(a)$ *for all* $a\in L^2(\mathcal{M})$,

(c) $\delta(ab)=\pi_l(a)\cdot\delta(b)+\delta(a)\cdot J\pi_r(b)^*J$ *for all* $a\in\mathcal{M}\varphi^{1/2}$, $b\in\varphi^{1/2}\mathcal{M}$,

(d) $\overline{\operatorname{lin}}\{\delta(a)x\mid a\in L^2(\mathcal{M}),\, x\in\mathcal{M}\}=\mathcal{H}$

such that

$$L_2=\delta^*\delta.$$

Moreover, there exists $\xi_0\in\mathcal{H}$ *such that*

$$\delta(a)=\pi_l(a)\cdot\xi_0-\xi_0\cdot J\pi_r(a)^*J$$

for $a\in\mathcal{M}\varphi^{1/2}\cap\varphi^{1/2}\mathcal{M}$.

Proof. Let

$$\omega\colon \mathcal{M}\odot\mathcal{M}^\circ\to\mathbb{C},\ x\otimes y^\circ\mapsto -\frac{1}{2}\langle\varphi^{1/2},\check{L}(x)\varphi^{1/2}y\rangle.$$

We first show that ω extends continuously to $\mathcal{M}\otimes_{\mathrm{bin}}\mathcal{M}^\circ$.

Since $\check{L}$ is the generator of a uniformly continuous quantum Markov semigroup, by Proposition 3.5.1 there exist normal completely positive maps $\Psi_1,\ \Psi_2$ on $\mathcal{M}$ such that $\check{L}=\Psi_2-\Psi_1$. For $j\in\{1,2\}$, the maps

$$\omega_j\colon \mathcal{M}\odot\mathcal{M}^\circ\to\mathbb{C},\ x\otimes y^\circ\mapsto \frac{1}{2}\langle\varphi^{1/2},\Psi_j(x)\varphi^{1/2}y\rangle$$

are positive and separately weak* continuous. Hence they extend to positive functionals on $\mathcal{M}\otimes_{\mathrm{bin}}\mathcal{M}^\circ$ by definition of $\|\cdot\|_{\mathrm{bin}}$. Thus ω extends to a bounded linear map on $\mathcal{M}\otimes_{\mathrm{bin}}\mathcal{M}^\circ$. We continue to denote the extension of ω to $\mathcal{M}\otimes_{\mathrm{bin}}\mathcal{M}^\circ$ by ω.

Let

$$q\colon \mathcal{M}\odot\mathcal{M}^\circ\to L^2(\mathcal{M}),\ x\otimes y^\circ\mapsto x\varphi^{1/2}y.$$

Clearly, the kernel of q is a left ideal of $\mathcal{M}\odot\mathcal{M}^\circ$. Let I denote its closure, which is a closed left ideal of $\mathcal{M}\otimes_{\mathrm{bin}}\mathcal{M}^\circ$.

If $u=\sum_{j=1}^n x_j\otimes y_j^\circ\in\ker q$, then

$$\begin{aligned}\omega(u^*u)&=-\frac{1}{2}\sum_{j,k=1}^n\langle\varphi^{1/2}y_j,\check{L}(x_j^*x_k)\varphi^{1/2}y_k\rangle\\&=\lim_{t\to0}\frac{1}{2t}\left(\sum_{j,k=1}^n\langle\varphi^{1/2}y_j,\check{\Phi}_t(x_j^*x_k)\varphi^{1/2}y_k\rangle-\|q(u)\|^2\right)\\&=\lim_{t\to0}\frac{1}{2t}\sum_{j,k=1}^n\langle\varphi^{1/2}y_j,\check{\Phi}_t(x_j^*x_k)\varphi^{1/2}y_k\rangle.\end{aligned}$$

The last expression is positive since $\check{\Phi}_t$ is completely positive for all $t \geq 0$. By continuity, it follows that $\omega(u^* u) \geq 0$ for all $u \in I$.

Let $\mathcal{H}$ be the GNS Hilbert space associated with $\omega|_I$ and π_ω the GNS representation of $\mathcal{M} \otimes_{\text{bin}} \mathcal{M}^\circ$ on $\mathcal{H}$, that is, $\mathcal{H}$ is the completion of I with respect to the inner product $(x, y) \mapsto \omega(x^* y)$ and $\pi_\omega(x)[y]_{\mathcal{H}} = [xy]_{\mathcal{H}}$, were $[\cdot]_{\mathcal{H}}$ denotes the canonical map $I \to \mathcal{H}$. As ω is separately weak* continuous, it follows easily that the actions $x \mapsto \pi_\omega(x \otimes 1)$ and $y^\circ \mapsto \pi_\omega(1 \otimes y^\circ)$ are normal. These actions make $\mathcal{H}$ into an $\mathcal{M}$-$\mathcal{M}$-correspondence.

Moreover, let (e_λ) be a right approximate identity for I consisting of positive contractions. Since $\|[e_\lambda]_{\mathcal{H}}\| = \omega(e_\lambda^2)^{1/2} \leq \|\omega\|^{1/2}$, we may assume additionally that $[e_\lambda]_{\mathcal{H}}$ converges weakly to some vector $\xi_\omega \in \mathcal{H}$. If $u \in I$, then

$$[u]_{\mathcal{H}} = [\lim_\lambda u e_\lambda] = \lim_\lambda \pi_\omega(u)[e_\lambda]_{\mathcal{H}} = \pi_\omega(u)\xi_\omega.$$

In particular, ξ_ω is a cyclic vector for π_ω.

To define $\mathcal{J}$, first define an anti-linear map $\mathcal{J}_0$ on $\mathcal{M} \odot \mathcal{M}^\circ$ by $\mathcal{J}_0(x \otimes y^\circ) = y^* \otimes (x^*)^\circ$. A direct computation shows $q\mathcal{J}_0 = Jq$. In particular, $\mathcal{J}_0$ leaves $\ker q$ invariant.

Furthermore, if $x_1, x_2, y_1, y_2 \in \mathcal{M}$, then

$$\begin{aligned}
\omega(\mathcal{J}_0(x_1 \otimes y_1^\circ)^* \mathcal{J}_0(x_2 \otimes y_2^\circ)) &= -\frac{1}{2}\langle \varphi^{1/2}, \check{L}(y_1 y_2^*)\varphi^{1/2} x_2^* x_1\rangle \\
&= -\frac{1}{2}\langle \varphi^{1/2} x_1^* x_2, \check{L}(y_1 y_2^*)\varphi^{1/2}\rangle \\
&= -\frac{1}{2}\langle \varphi^{1/2} \check{L}(x_1^* x_2), y_1 y_2^* \varphi^{1/2}\rangle \\
&= -\frac{1}{2}\langle \varphi^{1/2}, y_1 y_2^* \varphi^{1/2} \check{L}(x_2^* x_1)\rangle \\
&= -\frac{1}{2}\langle J(y_1 y_2^* \varphi^{1/2} \check{L}(x_2^* x_1)), J\varphi^{1/2}\rangle \\
&= -\frac{1}{2}\langle \check{L}(x_1^* x_2)\varphi^{1/2} y_2 y_1^*, \varphi^{1/2}\rangle \\
&= -\frac{1}{2}\langle \varphi^{1/2}, \check{L}(x_2^* x_1)\varphi^{1/2} y_1 y_2^*\rangle \\
&= \omega((x_2 \otimes y_2^\circ)^*(x_1 \otimes y_1^\circ)).
\end{aligned}$$

Therefore the map

$$[I]_{\mathcal{H}} \to [I]_{\mathcal{H}}, [x \otimes y^\circ]_{\mathcal{H}} \mapsto [y^* \otimes (x^*)^\circ]_{\mathcal{H}}$$

extends to an isometric anti-linear operator $\mathcal{J}$ on $\mathcal{H}$. Obviously, $\mathcal{J}$ is an involution and property (a) follows directly from the definition.

Finally let us construct δ. For $a \in \mathcal{M}\varphi^{1/2} \cap \varphi^{1/2}\mathcal{M}$ let

$$\delta(a) = \pi_\omega(\pi_l(a) \otimes 1 - 1 \otimes (J\pi_r(a)^* J)^\circ)\xi_\omega.$$

Note that

$$q(\pi_l(a) \otimes 1 - 1 \otimes (J\pi_r(a)^* J)^\circ) = \pi_l(a)\varphi^{1/2} - \pi_r(a)\varphi^{1/2} = 0$$

so that $\pi_l(a) \otimes 1 - 1 \otimes (J\pi_r(a)^* J)^\circ \in I$ and $\delta(a) = [\pi_l(a) \otimes 1 - 1 \otimes (J\pi_r(a)^* J)^\circ]_{\mathcal{H}}$.

We have

$$\begin{aligned}
\|\delta(a)\|_{\mathscr{H}}^2 &= \|[\pi_l(a)\otimes 1 - 1\otimes (J\pi_r(a)^* J)^\circ]_{\mathscr{H}}\|^2 \\
&= \omega(|\pi_l(a)\otimes 1 - 1\otimes (J\pi_r(a)^* J)^\circ|^2) \\
&= \omega(\pi_l(a^\sharp a)\otimes 1) + \omega(1\otimes (J\pi_r(aa^\flat)J)^\circ) \\
&\quad - \omega(\pi_l(a^\sharp)\otimes (J\pi_r(a)^* J)^\circ) - \omega(\pi_l(a)\otimes (J\pi_r(a^\flat)^* J)^\circ) \\
&= -\frac{1}{2}\langle \varphi^{1/2}, \check{L}_2(\Delta^{1/4}(a^\sharp a))\rangle - \frac{1}{2}\langle \Delta^{1/4}J(aa^\flat), \check{L}_2\varphi^{1/2}\rangle \\
&\quad + \frac{1}{2}\langle \Delta^{1/4}Ja, \check{L}_2\Delta^{1/4}(a^\sharp)\rangle + \frac{1}{2}\langle \Delta^{1/4}Ja^\flat, \check{L}_2\Delta^{1/4}a\rangle \\
&\overset{(1)}{=} \frac{1}{2}\langle J\Delta^{-1/4}a, \check{L}_2 J\Delta^{1/4}a\rangle + \frac{1}{2}\langle \Delta^{-1/4}a, \check{L}_2\Delta^{1/4}a\rangle \\
&\overset{(2)}{=} \frac{1}{2}\langle \Delta^{1/4}a, \check{L}_2\Delta^{-1/4}a\rangle + \frac{1}{2}\langle \Delta^{-1/4}a, \check{L}_2\Delta^{1/4}a\rangle \\
&\overset{(3)}{=} \langle a, L_2(a)\rangle.
\end{aligned}$$

Here we used the symmetry of $\check{L}_2$ and $\check{L}_2\varphi^{1/2} = 0$ for (1), the symmetry of $\check{L}_2$ and $\check{L}_2 J = J\check{L}_2$ for (2) and the key property from Lemma 3.4.4 for (3).

Therefore the map δ extends to a bounded linear operator from $L^2(\mathscr{M})$ to $\mathscr{H}$, and this extension, still denoted by δ, satisfies $\delta^*\delta = L_2$. Clearly,

$$\delta(ab) = \pi_l(a)\cdot\delta(b) + \delta(a)\cdot J\pi_r(b)^* J$$

for $a, b \in \mathscr{M}\varphi^{1/2} \cap \varphi^{1/2}\mathscr{M}$. If we only have $a \in \mathscr{M}\varphi^{1/2}$ and $b \in \varphi^{1/2}\mathscr{M}$, a standard approximation argument [Haa75, Lemma 1.3] shows that this identity continues to hold, which settles property (c).

Property (b) is clear from the definition if $a \in \mathscr{M}\varphi^{1/2} \cap \varphi^{1/2}\mathscr{M}$, and can be extended to $a \in L^2(\mathscr{M})$ again by approximation.

Finally, if $u = \sum_{j=1}^n x_j \otimes y_j^\circ \in \ker q$ and x_j is entire analytic for σ^φ for all $j \in \{1, \dots, n\}$, then

$$\begin{aligned}
u &= \sum_{j=1}^n x_j \otimes y_j^\circ \\
&= \sum_{j=1}^n (x_j \otimes y_j^\circ - 1\otimes (\sigma^\varphi_{i/2}(x_j)y_j)^\circ),
\end{aligned}$$

where we used $0 = q(u) = \varphi^{1/2}\sum_{j=1}^n \sigma^\varphi_{i/2}(x_j)y_j$. Thus

$$[u]_{\mathscr{H}} = \sum_{j=1}^n \delta(x_j\varphi^{1/2})y_j.$$

Since π_ω is normal and $[\ker q]_{\mathscr{H}}$ is dense in $\mathscr{H}$ by definition, property (d) follows. □

Remark 3.5.3. In terms of the GNS embedding of $\mathscr{M}$ into $L^2(\mathscr{M})$, the product rule (c) can be expressed as

$$\delta(xy\varphi^{1/2}) = x\cdot\delta(y\varphi^{1/2}) + \delta(x\varphi^{1/2})\cdot\sigma^\varphi_{i/2}(y)$$

for $x, y \in \mathcal{M}$ with y entire analytic for σ^φ. Similarly, for the symmetric embedding one obtains

$$\delta(\varphi^{1/4}xy\varphi^{1/4}) = \sigma^\varphi_{-i/4}(x)\cdot\delta(\varphi^{1/4}y\varphi^{1/4}) + \delta(\varphi^{1/4}x\varphi^{1/4})\cdot\sigma^\varphi_{i/4}(y)$$

for $x, y \in \mathcal{M}$ entire analytic for σ^φ.

Remark 3.5.4. Inner derivations of the form occurring in the theorem have been considered before in the construction of completely Dirichlet forms, in particular when $\mathcal{H} = L^2(\mathcal{M})$. For example, in [Cip97, Section 5], maps of the form

3

$$\delta\colon L^2(\mathcal{M}) \to L^2(\mathcal{M}),\ a \mapsto i(\mu x a - \lambda a x)$$

for $x \in \mathcal{M}$ and $\lambda, \mu > 0$ are studied. If $x\varphi^{1/2}$ is an eigenvector of Δ to the eigenvalue λ^2/μ^2 (the case considered in [Cip97, Proposition 5.3 (iv)]), then

$$\delta(a) = \pi_l(a)\cdot\xi_0 - \xi_0\cdot J\pi_r(a)^* J$$

for $\xi_0 = -i\mu x\varphi^{1/2} = -i\lambda\varphi^{1/2}x$.

Note however that in this case, $\delta^*\delta$ commutes with the modular group so that the semigroup generated by $\delta^*\delta$ is GNS-symmetric.

Remark 3.5.5. If one only wants to show the existence of the twisted derivation δ, one could work directly with the GNS representation of ω on $\ker q$ without passing to the binormal tensor product. Hence Proposition 3.5.1 and thus the Christensen-Evans theorem is only needed to show that the twisted derivation is inner.

3.5.2. UNIQUENESS

We show next that the triple $(\mathcal{H}, \mathcal{J}, \delta)$ constructed in Theorem 3.5.2 is uniquely determined by the semigroup (Φ_t) up to isomorphism. Let us first introduce some terminology for triples of this kind (see Definitions 2.3.5 and 2.3.9).

Definition 3.5.6. We call a triple $(\mathcal{H}, \mathcal{J}, \delta)$ consisting of a self-dual $\mathcal{M}$-$\mathcal{M}$-correspondence $(\mathcal{H}, \mathcal{J})$ and a closed operator $\delta\colon \operatorname{dom}(\delta) \subset L^2(\mathcal{M}) \to \mathcal{H}$ a *first-order differential calculus* if

(a) $J\operatorname{dom}(\delta) = \operatorname{dom}(\delta)$ and $\delta(Ja) = \mathcal{J}\delta(a)$ for all $a \in L^2(\mathcal{M})$,

(b) Whenever $a \in \operatorname{dom}(\delta)\cap\mathcal{M}\varphi^{1/2}$, $b \in \operatorname{dom}(\delta)\cap\varphi^{1/2}\mathcal{M}$, then $ab \in \operatorname{dom}(\delta)$ and $\delta(ab) = \pi_l(a)\delta(b) + \delta(a)J\pi_r(b)^*J$,

(c) $\overline{\operatorname{lin}}\{\delta(a)x \mid a \in \operatorname{dom}(\delta),\ x \in \mathcal{M}\} = \mathcal{H}$.

With this definition, Theorem 3.5.2 says that for every bounded Markov generator L_2 on $L^2(\mathcal{M})$ there exists a first-order differential calculus $(\mathcal{H}, \mathcal{J}, \delta)$ such that $L_2 = \delta^*\delta$. In this subsection we will show that $(\mathcal{H}, \mathcal{J}, \delta)$ is uniquely determined by L_2.

To lighten the notation, we write $a\xi$ for $\pi_l(a)\xi$ if $a \in \mathcal{M}\varphi^{1/2}$ and ξb for $\xi\cdot J\pi_r(b)^*J$ if $b \in \mathcal{M}\varphi^{1/2}$ for the remainder of this subsection.

The first step towards uniqueness is a purely algebraic consequence of the properties (a) and (b) of a first-order differential calculus.

Lemma 3.5.7. *If $(\mathcal{H},\mathcal{J},\delta)$ is a first-order differential calculus and δ is bounded, then*

$$\begin{aligned}&\langle\delta(\Delta^{1/4}a)\Delta^{1/4}b,\delta(\Delta^{-1/4}c)\rangle_{\mathcal{H}}+\langle\delta(\Delta^{-1/4}a)\Delta^{-1/4}b,\delta(\Delta^{1/4}c)\rangle_{\mathcal{H}}\\&\quad=\langle\delta(\Delta^{-1/4}(ab)),\delta(\Delta^{1/4}c)\rangle_{\mathcal{H}}+\langle\delta(\Delta^{1/4}a),\delta(\Delta^{-1/4}cJ(\Delta^{-1/4}b))\rangle_{\mathcal{H}}\\&\quad\quad-\langle\delta(J(\Delta^{1/4}c)\Delta^{1/4}a),\delta(J(\Delta^{-1/4}b))\rangle_{\mathcal{H}}\end{aligned}$$

for all $a,b,c\in\mathcal{M}_a^\varphi\varphi^{1/2}$, where $\mathcal{M}_a^\varphi$ denotes the set of all entire analytic elements for σ^φ.

Proof. As $a,b,c\in\mathcal{M}_a^\varphi\varphi^{1/2}$, these elements lie in $\bigcap_{n\in\mathbb{Z}}\operatorname{dom}(\Delta^n)$, and arbitrary powers of the modular operator map them to left- and right-bounded vectors. In particular, all expressions in the claimed equation are well-defined.

Using properties (a) and (b) of a first-order differential calculus, we can do the following computations. Let $x,y,z\in\mathcal{M}_a^\varphi\varphi^{1/2}$. Then we have

$$\begin{aligned}\langle\delta(x)y,\delta(z)\rangle_{\mathcal{H}}&=\langle\delta(x)y,\delta(z)\rangle_{\mathcal{H}}+\langle x\delta(y),\delta(z)\rangle_{\mathcal{H}}-\langle\mathcal{J}(\delta(z)),\mathcal{J}(x\delta(y))\rangle_{\mathcal{H}}\\&=\langle\delta(xy),\delta(z)\rangle_{\mathcal{H}}-\langle\delta(Jz),\delta(Jy)Jx\rangle_{\mathcal{H}}\\&=\langle\delta(xy),\delta(z)\rangle_{\mathcal{H}}-\langle\delta(Jz)\Delta^{1/2}x,\delta(Jy)\rangle_{\mathcal{H}}\end{aligned}$$

and

$$\begin{aligned}\langle\delta(x)y,\delta(z)\rangle_{\mathcal{H}}&=\langle\delta(x),\delta(z)J\Delta^{-1/2}y\rangle_{\mathcal{H}}+\langle\delta(x),z\delta(J\Delta^{-1/2}y)\rangle_{\mathcal{H}}-\langle\delta(x),z\delta(J\Delta^{-1/2}y)\rangle_{\mathcal{H}}\\&=\langle\delta(x),\delta(zJ\Delta^{-1/2}y)\rangle_{\mathcal{H}}-\langle\delta(x),z\delta(J\Delta^{-1/2}y)\rangle_{\mathcal{H}}\\&=\langle\delta(x),\delta(zJ\Delta^{-1/2}y)\rangle_{\mathcal{H}}-\langle J(\Delta^{1/2}z)\delta(x),\delta(J\Delta^{-1/2}y)\rangle_{\mathcal{H}}.\end{aligned}$$

If we take $x=\Delta^{-1/4}a$, $y=\Delta^{-1/4}b$, $z=\Delta^{1/4}c$ in the first identity and $x=\Delta^{1/4}a$, $y=\Delta^{1/4}b$, $z=\Delta^{-1/4}c$ in the second identity and add them up, we obtain

$$\begin{aligned}&\langle\delta(\Delta^{-1/4}a)\Delta^{-1/4}b,\delta(\Delta^{1/4}c)\rangle_{\mathcal{H}}+\langle\delta(\Delta^{1/4}a)\Delta^{1/4}b,\delta(\Delta^{-1/4}c)\rangle_{\mathcal{H}}\\&=\langle\delta(\Delta^{-1/4}(ab)),\delta(\Delta^{1/4}c)\rangle_{\mathcal{H}}-\langle\delta(J\Delta^{1/4}c)\Delta^{1/4}a,\delta(J\Delta^{-1/4}b)\rangle_{\mathcal{H}}\\&\quad+\langle\delta(\Delta^{1/4}a),\delta((\Delta^{-1/4}c)J\Delta^{-1/4}b)\rangle_{\mathcal{H}}-\langle(J\Delta^{1/4}c)\delta(\Delta^{1/4}a),\delta(J\Delta^{-1/4}b)\rangle_{\mathcal{H}}\\&=\langle\delta(\Delta^{-1/4}(ab)),\delta(\Delta^{1/4}c)\rangle_{\mathcal{H}}+\langle\delta(\Delta^{1/4}a),\delta((\Delta^{-1/4}c)J\Delta^{-1/4}b)\rangle_{\mathcal{H}}\\&\quad-\langle\delta((J\Delta^{1/4}c)\Delta^{1/4}a),\delta(J\Delta^{-1/4}b)\rangle_{\mathcal{H}},\end{aligned}$$

where we used again property (b) of a first-order differential calculus in the last step. □

The significance of this result is that the right side depends only on the inner product of elements from the range of δ and not the bimodule generated by the range. If the modular operator Δ is trivial, that is, φ is a trace, one can conclude uniqueness of the twisted derivation directly from this lemma. In the general case, substantially more work is needed. In particular, there are analytical difficulties that are absent in the case of tracially symmetric (or more generally GNS-symmetric) quantum Markov semigroups.

One tool we use are spectral subspaces of the analytic generator of the modular group. We recall the definition and some of their properties here. See [CZ76] for more details.

For $0 < \lambda_1 < \lambda_2$ let

$$\mathcal{M}[\lambda_1, \lambda_2] = \left\{ x \in \bigcap_{t\in\mathbb{R}} \mathrm{dom}(\sigma^\varphi_{it}) \,\middle|\, \varlimsup_{t\to\infty} \|\sigma^\varphi_{it}(x)\|^{1/t} \leq \frac{1}{\lambda_1}, \varlimsup_{t\to\infty} \|\sigma^\varphi_{-it}(x)\|^{1/t} \leq \lambda_2 \right\}.$$

This is a norm closed subspace of $\mathcal{M}$, invariant under σ^φ, and the spectrum of the restriction of σ^φ_{-i} to $\mathcal{M}[\lambda_1,\lambda_2]$ is contained in $[\lambda_1,\lambda_2]$ (see [CZ76, (iii)–(v), p. 351]). Moreover, the union $\bigcup_{0<\lambda_1\leq\lambda_2<\infty} \mathcal{M}[\lambda_1,\lambda_2]$ is weak* dense in $\mathcal{M}$ [CZ76, (vi), p. 356]. Additionally, σ^φ_t is given by e^{itH} for some $H \in B(\mathcal{M}[\lambda_1,\lambda_2])$ with $\mathrm{sp}(H) \subset [-\ln(\lambda_1), \ln(\lambda_2)]$ [CZ76, Theorem 5.2, p. 349]. For convenience, we introduce similar notation in the $L^2(\mathcal{M})$-case and write $L^2(\mathcal{M})[\lambda_1,\lambda_2]$ for $\chi_{[\lambda_1,\lambda_2]}(\Delta)L^2(\mathcal{M})$.

For completeness, we also recall the definition of the projective cross norm of Banach spaces: if X, Y are Banach spaces, then the projective cross norm of $z \in X \odot Y$ is given by

$$\|z\| = \inf\left\{ \sum_{i=1}^n \|x_i\|\|y_i\| \,\middle|\, z = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

Lemma 3.5.8. *Let $0 < \lambda_1 < \lambda_2$. Define X to be the completion of $L^2(\mathcal{M})[\lambda_1,\lambda_2] \odot \mathcal{M}[\lambda_1,\lambda_2]$ with the projective cross norm. Then the bounded operator $T : X \to X$ defined on pure tensors by*

$$T(\eta \otimes a) = \Delta^{1/4}(\eta) \otimes \sigma^\varphi_{-i/4}(a)$$

is well defined and $\mathrm{sp}(T) \subset (0,\infty)$.

Proof. The spectrum of $\Delta^{1/4}$ restricted to $\chi_{[\lambda_1,\lambda_2]}(\Delta)L^2(\mathcal{M})$ is contained in $[\lambda_1^{1/4}, \lambda_2^{1/4}]$ by definition. Since the restriction of $\sigma^\varphi_{-i/4}$ to $\mathcal{M}[\lambda_1,\lambda_2]$ is $e^{1/4H}$ for some $H \in B(\mathcal{M}[\lambda_1,\lambda_2])$ with $\mathrm{sp}(H) \subset [-\ln(\lambda_1), \ln(\lambda_2)]$, we know that the spectrum of the restriction of $\sigma^\varphi_{-i/4}$ to $\mathcal{M}[\lambda_1,\lambda_2]$ is also contained in $[\lambda_1^{1/4}, \lambda_2^{1/4}]$. Then $\Delta^{1/4} \otimes \mathrm{id}$ and $\mathrm{id} \otimes \sigma^\varphi_{-i/4}$ are well defined and have spectra contained in $[\lambda_1^{1/4}, \lambda_2^{1/4}]$, so $\Delta^{1/4} \otimes \sigma^\varphi_{-i/4}$ has spectrum contained in $(0,\infty)$ [Sch69, p. 96]. □

Theorem 3.5.9. *Let $(\mathcal{H}_1, \mathcal{J}_1, \delta_1)$ and $(\mathcal{H}_2, \mathcal{J}_2, \delta_2)$ be first order differential calculi for $\mathcal{M}$ such that δ_1 and δ_2 are bounded and $\delta_1^*\delta_1 = \delta_2^*\delta_2$. Then there exists a unitary bimodule map $\Theta : \mathcal{H}_1 \to \mathcal{H}_2$ intertwining $\mathcal{J}_1$ and $\mathcal{J}_2$ such that $\Theta(\delta_1(a)) = \delta_2(a)$ for all $a \in L^2(\mathcal{M})$.*

Proof. The unitary bimodule map Θ will be given by

$$\Theta(\delta_1(a)b) = \delta_2(a)b$$

on elements of the form $\delta_1(a)b$ with $a,\ b \in L^2(\mathcal{M})$. The difficult part of the proof is to show that this map is isometric; the other properties will follow naturally.

Let $0 < \lambda_1 < \lambda_2$ be arbitrary. Let X and T be as in Lemma 3.5.8. Note that T is invertible since $0 \notin \mathrm{sp}(T)$. On $L^2(\mathcal{M})[\lambda_1,\lambda_2] \odot \mathcal{M}[\lambda_1,\lambda_2] \subset X$ we can define the maps q_1 and q_2 to $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, by

$$q_1(\eta \otimes a) = \delta_1(\eta)a \text{ and } q_2(\eta \otimes a) = \delta_2(\eta)a.$$

Because δ_1 and δ_2 are bounded and right multiplication is bounded in the operator norm, we can boundedly extend q_1 and q_2 to X.

Using Lemma 3.5.7 we can now show that for all $x,\ y \in X$ we have

$$\begin{aligned}\langle q_1(T(x)), q_1(T^{-1}(y))\rangle_{\mathcal{H}_1} + \langle q_1(T^{-1}(x)), q_1(T(y))\rangle_{\mathcal{H}_1} \\ = \langle q_2(T(x)), q_2(T^{-1}(y))\rangle_{\mathcal{H}_2} + \langle q_2(T^{-1}(x)), q_2(T(y))\rangle_{\mathcal{H}_2}.\end{aligned} \tag{3.6}$$

Indeed, for all $j \in \{1, 2\}$, $\eta,\ \xi \in L^2(\mathcal{M})[\lambda_1, \lambda_2]$ and $a, b \in \mathcal{M}[\lambda_1, \lambda_2]$ we have

$$\begin{aligned}&\langle q_j(T(\eta\otimes a)), q_j(T^{-1}(\xi\otimes b))\rangle_{\mathcal{H}_j} + \langle q_j(T^{-1}(\eta\otimes a)), q_j(T(\xi\otimes b))\rangle_{\mathcal{H}_j} \\ &= \langle \delta_j(\Delta^{1/4}\eta)\sigma^{\varphi}_{-i/4}(a), \delta_j(\Delta^{-1/4}\xi)\sigma^{\varphi}_{i/4}(b)\rangle_{\mathcal{H}_j} + \langle \delta_j(\Delta^{-1/4}\eta)\sigma^{\varphi}_{i/4}(a), \delta_j(\Delta^{1/4}\xi)\sigma^{\varphi}_{-i/4}(b)\rangle_{\mathcal{H}_j} \\ &= \langle \delta_j(\Delta^{1/4}\eta)\sigma^{\varphi}_{-i/4}(ab^*), \delta_j(\Delta^{-1/4}\xi)\rangle_{\mathcal{H}_j} + \langle \delta_j(\Delta^{-1/4}\eta)\sigma^{\varphi}_{i/4}(ab^*), \delta_j(\Delta^{1/4}\xi))\rangle_{\mathcal{H}_j} \\ &= \langle \delta_j(\Delta^{-1/4}(\eta\varphi^{1/2}ab^*)), \delta_j(\Delta^{1/4}\xi)\rangle_{\mathcal{H}_j} + \langle \delta_j(\Delta^{1/4}\eta), \delta_j(\Delta^{-1/4}\xi J(\Delta^{-1/4}(\varphi^{1/2}ab^*)))\rangle_{\mathcal{H}_j} \\ &\quad - \langle \delta_j(J(\Delta^{1/4}\xi)\Delta^{1/4}\eta), \delta_j(J(\Delta^{-1/4}\varphi^{1/2}ab^*))\rangle_{\mathcal{H}_j},\end{aligned}$$

where the last step follows from Lemma 3.5.7 and the fact that $va = v \cdot J\pi_r(\varphi^{1/2}a)^* J$ for $v \in \mathcal{H}_j$ and $a \in \mathcal{M}$. Since we have for all $\eta,\ \xi \in L^2(\mathcal{M})$ that

$$\langle \delta_1(\eta), \delta_1(\xi)\rangle_{\mathcal{H}_1} = \langle \eta, \delta_1^*\delta_1(\xi)\rangle = \langle \eta, \delta_2^*\delta_2(\xi)\rangle = \langle \delta_2(\eta), \delta_2(\xi)\rangle_{\mathcal{H}_2},$$

we can now conclude that

$$\begin{aligned}&\langle q_1(T(\eta\otimes a)), q_1(T^{-1}(\xi\otimes b))\rangle_{\mathcal{H}_1} + \langle q_1(T^{-1}(\eta\otimes a)), q_1(T(\xi\otimes b))\rangle_{\mathcal{H}_1} \\ &\qquad = \langle q_2(T(\eta\otimes a)), q_2(T^{-1}(\xi\otimes b))\rangle_{\mathcal{H}_2} + \langle q_2(T^{-1}(\eta\otimes a)), q_2(T(\xi\otimes b))\rangle_{\mathcal{H}_2}.\end{aligned}$$

By linearity and density of $L^2(\mathcal{M})[\lambda_1, \lambda_2] \odot \mathcal{M}[\lambda_1, \lambda_2]$ in X we find that Equation (3.6) holds.

The next part of the proof is to show that Equation (3.6) implies that

$$\langle q_1(x), q_1(y)\rangle_{\mathcal{H}_1} = \langle q_2(x), q_2(y)\rangle_{\mathcal{H}_2}$$

for $x,\ y \in X$. For this, we consider the operator Te^{-sT^2} for $s > 0$, defined by holomorphic functional calculus. We start with the observation that

$$\begin{aligned}\langle q_j(T(Te^{-sT^2}(x))), q_j(T^{-1}(Te^{-sT^2}(y)))\rangle_{\mathcal{H}_j} + \langle q_j(T^{-1}(Te^{-sT^2}(x))), q_j(T(Te^{-sT^2}(y)))\rangle_{\mathcal{H}_j} \\ = -\frac{d}{ds}\langle q_j(e^{-sT^2}(x)), q_j(e^{-sT^2}(y))\rangle_{\mathcal{H}_j}\end{aligned} \tag{3.7}$$

for $j \in \{1, 2\}$ and $x,\ y \in X$. Since T is a bounded operator on X and $\mathrm{sp}(T) \subset (0, \infty)$, we know that $\lim_{s\to\infty} \|e^{-sT^2}\| = 0$ [Nee22, Theorem 6.24]. Consequently, we have for $j \in \{1, 2\}$ and $x,\ y \in X$ that

$$-\lim_{r\to\infty}\int_0^r \frac{d}{ds}\langle q_j(e^{-sT^2}(x)), q_j(e^{-sT^2}(y))\rangle_{\mathcal{H}_j}\, ds = \langle q_j(x), q_j(y)\rangle_{\mathcal{H}_j}.$$

Since the integrand of the above integral is equal for $j = 1$ and $j = 2$ by Equation (3.6) and Equation (3.7), we deduce that

$$\langle q_1(x), q_1(y)\rangle_{\mathcal{H}_1} = \langle q_2(x), q_2(y)\rangle_{\mathcal{H}_2}$$

for all $x,\ y \in X$. Therefore, we have for all $\eta,\ \xi \in L^2(\mathcal{M})[\lambda_1,\lambda_2]$ and $a,\ b \in \mathcal{M}[\lambda_1,\lambda_2]$ that

$$\langle \delta_1(\eta)a, \delta_1(\xi)b\rangle_{\mathcal{H}_1} = \langle \delta_2(\eta)a, \delta_2(\xi)b\rangle_{\mathcal{H}_2}.$$

So far we have shown that Θ preserves the inner product on certain subsets of $\mathcal{H}_1$, and the goal is to extend this to all of $\mathcal{H}_1$. This takes a few steps. First, note that Θ preserves the inner product on all of

3

$$\operatorname{lin}\{\delta_1(\eta)a \mid \lambda_1,\lambda_2 > 0, \eta \in L^2(\mathcal{M})[\lambda_1,\lambda_2], a \in \mathcal{M}[\lambda_1,\lambda_2]\},$$

since for each $\eta_1,\eta_2 \in \bigcup_{\lambda_1,\lambda_2>0} L^2(\mathcal{M})[\lambda_1,\lambda_2]$ and $a_1,a_2 \in \bigcup_{\lambda_1,\lambda_2>0}\mathcal{M}[\lambda_1,\lambda_2]$ we can find $\lambda_1',\ \lambda_2' > 0$ such that $\eta_1,\eta_2 \in (\chi_{[\lambda_1',\lambda_2']}(\Delta)L^2(\mathcal{M}))$ and $a_1,a_2 \in \mathcal{M}[\lambda_1',\lambda_2']$. Next, since $\bigcup_{\lambda_1,\lambda_2>0} L^2(\mathcal{M})[\lambda_1,\lambda_2]$ is dense in $L^2(\mathcal{M})$ and δ_1 is bounded, we can extend this to the subspace $\operatorname{lin}\{\delta_1(\eta)a \mid \eta \in L^2(\mathcal{M}), a \in \bigcup_{\lambda_1,\lambda_2>0}\mathcal{M}[\lambda_1,\lambda_2]\}$ and subsequently to the subspace $\operatorname{lin}\{\delta_1(\eta)a \mid \eta \in L^2(\mathcal{M}), a \in \mathcal{M}\}$ because $\bigcup_{\lambda_1,\lambda_2>0}\mathcal{M}[\lambda_1,\lambda_2]$ is weak* dense in $\mathcal{M}$. Lastly, by property (c) of a first-order differential calculus we conclude that Θ is isometric on all of $\mathcal{H}_1$.

We will finish the proof by discussing the other desired properties of Θ. By property (c) of a first-order differential calculus, $\operatorname{lin}\{\delta_2(\eta)a \mid \eta \in L^2(\mathcal{M}), a \in \mathcal{M}\}$ is dense in $\mathcal{H}_2$. Because the image of an isometric map is closed, we know that Θ is surjective and therefore that it is a linear isometric isomorphism. By property (b) of a first-order differential calculus it is a unitary bimodule map, and it is clear that it intertwines $\mathcal{J}_1$ and $\mathcal{J}_2$. □

3.6. DERIVATIONS FOR QUANTUM MARKOV SEMIGROUPS WITH UNBOUNDED GENERATORS

In this section we study twisted derivations for quantum Markov semigroups that are not necessarily uniformly continuous. In this case the generator can be unbounded, and it is convenient to work with the associated quadratic forms on $L^2(\mathcal{M})$, which we call quantum Dirichlet forms (see Section 2.2.3). We show that the bounded vectors in the form domain form an algebra (Theorem 3.6.1), which gives a suitable domain for a twisted derivation. We then show that there exists a (possibly unbounded) first-order differential calculus associated with our given quantum Dirichlet form (Theorem 3.6.3).

We keep the notation from the previous section. In particular, $\mathcal{M}$ is a von Neumann algebra and φ is a normal faithful state on $\mathcal{M}$.

Theorem 3.6.1. *Let $\mathcal{M}$ be a von Neumann algebra, φ a normal faithful state on $\mathcal{M}$ and $\mathcal{E}$ a quantum Dirichlet form on $L^2(\mathcal{M})$. If $a \in \operatorname{dom}(\mathcal{E}) \cap \mathcal{M}\varphi^{1/2}$, $b \in \operatorname{dom}(\mathcal{E}) \cap \varphi^{1/2}\mathcal{M}$, then $ab \in \operatorname{dom}(\mathcal{E})$ and*

$$\mathcal{E}(ab)^{1/2} \le \|\pi_l(a)\|\mathcal{E}(b)^{1/2} + \mathcal{E}(a)^{1/2}\|\pi_r(b)\|.$$

In particular, $\operatorname{dom}(\mathcal{E}) \cap \mathcal{M}\varphi^{1/2} \cap \varphi^{1/2}\mathcal{M}$ with the involution J is a $*$-algebra.

Proof. Let (T_t) be the strongly continuous semigroup associated with $\mathcal{E}$ and let $\mathcal{E}_t(a) = \frac{1}{t}\langle a, a - T_t a\rangle$ for $a \in L^2(\mathcal{M})$. By the spectral theorem, $\mathcal{E}_t(a) \nearrow \mathcal{E}(a)$ as $t \searrow 0$.

Let (Φ_t) be the KMS-symmetric quantum Markov semigroup on $\mathcal{M}$ associated with (T_t). Since Φ_t is completely positive, $\frac{1}{t}(I - \Phi_t)$ is conditionally completely negative.

Thus $\frac{1}{t}(I-\Phi_t)$ generates a quantum Markov semigroup on $\mathcal{M}$, which is clearly KMS-symmetric, and the associated symmetric Markov semigroup on $L^2(\mathcal{M})$ has generator $\frac{1}{t}(I-T_t)$.

By Theorem 3.5.2 there exists a bounded first-order differential calculus $(\mathcal{H}_t,\mathcal{J}_t,\delta_t)$ such that $\mathcal{E}_t(a)=\|\delta_t(a)\|^2_{\mathcal{H}_t}$ for $a\in L^2(\mathcal{M})$. Thus, if $a\in\operatorname{dom}(\mathcal{E})\cap\mathcal{M}\varphi^{1/2}$ and $b\in\operatorname{dom}(\mathcal{E})\cap\varphi^{1/2}\mathcal{M}$, then

$$\begin{aligned}\mathcal{E}_t(ab)^{1/2}&=\|\delta_t(ab)\|_{\mathcal{H}_t}\\&=\|\pi_l(a)\delta_t(b)+\delta_t(a)J\pi_r(b)^*J\|_{\mathcal{H}_t}\\&\leq\|\pi_l(a)\|\|\delta_t(b)\|_{\mathcal{H}_t}+\|\delta_t(a)\|_{\mathcal{H}_t}\|\pi_r(b)\|\\&=\|\pi_l(a)\|\mathcal{E}_t(b)^{1/2}+\mathcal{E}_t(a)^{1/2}\|\pi_r(b)\|.\end{aligned}$$

The claim follows by taking the limit $t\searrow 0$ of both sides. □

Remark 3.6.2. In contrast to the case of GNS-symmetric quantum Markov semigroups [Wir22, Theorem 6.3] we could not show that $\operatorname{dom}(\mathcal{E})\cap\mathcal{M}\varphi^{1/2}\cap\varphi^{1/2}\mathcal{M}$ is a form core for $\mathcal{E}$. The space $\operatorname{dom}(\mathcal{E})\cap\varphi^{1/4}\mathcal{M}\varphi^{1/4}$ is always a form core, but we do not expect it to be an algebra in general.

Theorem 3.6.3. *Let $\mathcal{M}$ be a von Neumann algebra and φ a faithful normal state on $\mathcal{M}$. If $\mathcal{E}$ is a quantum Dirichlet form on $L^2(\mathcal{M})$, then there exists a complete $\mathcal{M}$-$\mathcal{M}$-bimodule $\mathcal{H}$, an anti-unitary involution $\mathcal{J}\colon\mathcal{H}\to\mathcal{H}$ such that*

$$\mathcal{J}(x\xi y)=y^*(\mathcal{J}\xi)x^*$$

for $x,\ y\in\mathcal{M}$ and $\xi\in\mathcal{H}$, a closed operator $\delta\colon\operatorname{dom}(\mathcal{E})\to\mathcal{H}$ such that $\mathcal{J}\delta=\delta J$ and

$$\delta(ab)=\pi_l(a)\cdot\delta(b)+\delta(a)\cdot J\pi_r(a)^*J$$

for $a\in\operatorname{dom}(\mathcal{E})\cap\mathcal{M}\varphi^{1/2}$, $b\in\operatorname{dom}(\mathcal{E})\cap\varphi^{1/2}\mathcal{M}$, and

$$\mathcal{E}(a,b)=\langle\delta(a),\delta(b)\rangle_{\mathcal{H}}$$

for $a,\ b\in\operatorname{dom}(\mathcal{E})$.

Remark 3.6.4. We do not claim that the actions of $\mathcal{M}$ on $\mathcal{H}$ are normal so that $\mathcal{H}$ is in general not a correspondence. This is not only an artefact of our proof or a feature of KMS symmetry specifically, but happens even for symmetric Markov semigroups on commutative von Neumann algebras. For GNS-symmetric quantum Markov semigroups, the normality of the action is linked to the existence of the carré du champ (see [Wir22, Section 7]).

Proof. Let (T_t) be the strongly continuous semigroup associated with $\mathcal{E}$. As in the proof of Theorem 3.6.1, we consider the quantum Dirichlet form $\mathcal{E}_t$ given by $\mathcal{E}_t(a)=\frac{1}{t}\langle a,a-T_ta\rangle$ and the associated first-order differential calculus $(\mathcal{H}_t,\mathcal{J}_t,\delta_t)$. We will use an ultraproduct construction to define $(\mathcal{H},\mathcal{J},\delta)$ for $\mathcal{E}$.

Choose $\omega\in\beta\mathbb{N}\setminus\mathbb{N}$ and a null sequence (t_n) in $(0,\infty)$ and let $\mathcal{H}$ be the ultraproduct $\prod_{n\to\omega}\mathcal{H}_{t_n}$ (see Construction 2.3.14). We write $[\xi_n]$ for the equivalence class of (ξ_n) in $\mathcal{H}$.

We can define commuting left and right actions of $\mathcal{M}$ on $\mathcal{H}$ by

$$x[\xi_n]y = [x\xi_n y].$$

As noted in the previous remark, these actions are not necessarily normal.

Moreover, let

$$\mathcal{J}\colon \mathcal{H} \to \mathcal{H},\, \mathcal{J}[\xi_n] = [\mathcal{J}_{t_n}\xi_n].$$

It is easy to verify that $\mathcal{J}$ is an anti-unitary involution such that $\mathcal{J}(x\xi y) = y^*(\mathcal{J}\xi)x^*$ for $x,\ y \in \mathcal{M}$ and $\xi \in \mathcal{H}$.

Finally, if $a \in \mathrm{dom}(\mathcal{E})$, let $\delta(a) = [\delta_{t_n}(a)]$. Since

$$\|\delta_{t_n}(a)\|^2_{\mathcal{H}_{t_n}} = \mathcal{E}_{t_n}(a) \leq \mathcal{E}(a),$$

the map $\delta\colon \mathrm{dom}(\mathcal{E}) \to \mathcal{H}$ is well-defined. Furthermore,

$$\langle \delta(a), \delta(b)\rangle_{\mathcal{H}} = \lim_{n\to\omega} \langle \delta_{t_n}(a), \delta_{t_n}(b)\rangle_{\mathcal{H}_{t_n}} = \lim_{n\to\omega} \mathcal{E}_{t_n}(a,b) = \mathcal{E}(a,b)$$

The operator δ is closed because $\mathcal{E}$ is closed. The other properties of δ can be checked componentwise. □

Remark 3.6.5. At this point we do not know if the triple $(\mathcal{H}, \mathcal{J}, \delta)$ is uniquely determined. Given the known results for tracially symmetric or more generally GNS-symmetric quantum Markov semigroups, it seems reasonable to suspect that the left and right action are not uniquely determined on $\mathcal{M}$, but only on some $*$-subalgebra.

APPENDIX

3.A. ALTERNATIVE PROOF OF THEOREM 3.3.5

The proof of Theorem 3.3.5 shows that this result can be easily obtained from Theorem 3.3.4, and therefore that one can prove it in a basis-independent fashion. On the other hand, it gives very little intuition for the V_j mentioned in the theorem. To provide a bit more feeling for the V_j, we include an alternative proof of the result.

Recall that by [AC21, Theorem 4.4] the generator L of a KMS-symmetric quantum Markov semigroup on $M_n(\mathbb{C})$ can be written as

$$L(A) = (\mathrm{id}_n + \sigma_{-i/2})^{-1}(\Psi(1_n))A + A(\mathrm{id}_n + \sigma_{i/2})^{-1}(\Psi(1_n)) - \Psi(A) \tag{3.8}$$

with a KMS-symmetric completely positive map $\Psi\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$.

Alternative proof of Theorem 3.3.5. Let Ψ be a KMS-symmetric completely positive map such that Equation (3.8) holds. By Lemma 3.3.3(b), $\check{\Psi}$ is also completely positive and KMS-symmetric. Next, we define the completely positive map Ξ by

$$\Xi(A) = \rho^{1/4}\check{\Psi}(\rho^{-1/4}A\rho^{-1/4})\rho^{1/4}$$

for all $A \in M_n(\mathbb{C})$. Now for all $A,\ B \in M_n(\mathbb{C})$ we have

$$\mathrm{Tr}(A\Xi(B)) = \mathrm{Tr}(\rho^{-1/4}A\rho^{-1/4}\rho^{1/2}\check{\Psi}(\rho^{-1/4}B\rho^{-1/4})\rho^{1/2}) = \mathrm{Tr}(\Xi(A)B)$$

by the KMS-symmetry of $\check{\Psi}$.

Let $V_1,\ \dots,\ V_N$ be the Kraus representation of Ξ, meaning that

$$\Xi(A) = \sum_{j=1}^{N} V_j^* A V_j$$

for all $A \in M_n(\mathbb{C})$. Since $\mathrm{Tr}(A\Xi(B)) = \mathrm{Tr}(\Xi(A)B)$, we can assume without loss of generality that for each j there exists an index j^* such that $V_j^* = V_{j^*}$ and $(j^*)^* = j$. Note that $\sigma_{-i/4}(V_1),\dots,\sigma_{-i/4}(V_N)$ is a Kraus representation of $\check{\Psi}$. Calculating $\sum_j \langle [V_j,A],[V_j,B]\rangle_\rho$ then gives

$$\begin{aligned}\sum_j \langle [V_j,A],[V_j,B]\rangle_\rho =& \sum_j \mathrm{Tr}(\rho^{1/2}(A^*V_j^* - V_j^*A^*)\rho^{1/2}(V_jB - BV_j))\\ =& \sum_j \mathrm{Tr}(\rho^{1/2}A^*\rho^{1/2}(\sigma_{i/2}(V_j^*)V_jB + BV_j\sigma_{-i/2}(V_j)\\ &- \sigma_{i/2}(V_j^*)BV_j - V_jB\sigma_{-i/2}(V_j^*)))\\ =& \sum_j \mathrm{Tr}(\rho^{1/2}A^*\rho^{1/2}(\sigma_{i/2}(V_j^*)V_jB + BV_j^*\sigma_{-i/2}(V_j)\\ &- \sigma_{i/2}(V_j^*)BV_j - V_j^*B\sigma_{-i/2}(V_j))).\end{aligned}$$

We observe that

$$\begin{aligned}\sum_j \sigma_{i/2}(V_j^*)BV_j + V_j^*B\sigma_{-i/2}(V_j) =& \sum_j \sigma_{i/4}(\sigma_{-i/4}(V_j)^*)B\sigma_{i/4}(\sigma_{-i/4}(V_j))\\ &- \sigma_{-i/4}(\sigma_{-i/4}(V_j)^*)B\sigma_{-i/4}(\sigma_{-i/4}(V_j))\\ =& \mathscr{W}(\check{\Psi})(B)\end{aligned}$$

and consequently that

$$(\mathrm{id}_n + \sigma_{i/2})(\sum_j V_j^*\sigma_{-i/2}(V_j)) = \sum_j \sigma_{i/2}(V_j^*)V_j + V_j^*\sigma_{-i/2}(V_j) = \mathscr{W}(\check{\Psi})(1_n).$$

Therefore we have

$$\begin{aligned}\sum_j V_j^*\sigma_{-i/2}(V_j) &= (\mathrm{id}_n + \sigma_{i/2})^{-1}(\Psi(1_n)),\\ \sum_j \sigma_{i/2}(V_j^*)V_j &= (\mathrm{id}_n + \sigma_{-i/2})^{-1}(\Psi(1_n)).\end{aligned}$$

Coming back to our original expression we find

$$\begin{aligned}\textstyle\sum_j \langle [V_j,A],[V_j,B]\rangle_\rho =& \mathrm{Tr}(\rho^{1/2}A^*\rho^{1/2}((1+\sigma_{-i/2})^{-1}(\Psi(1_n))B\\ &+ B(\mathrm{id}_n + \sigma_{i/2})^{-1}(\Psi(1_n)) - \Psi(B)))\\ =& \langle L(A),B\rangle_\rho.\end{aligned}$$

□

4

Lifting the maximally-entangledness assumption in robust self-testing for synchronous games

This chapter is based on the following article:

- Matthijs Vernooij and Yuming Zhao. *Lifting the maximally-entangledness assumption in robust self-testing for synchronous games.* 2025. arXiv: 2505.05994 [quant-ph].

Robust self-testing in non-local games allows a classical referee to certify that two untrustworthy players are able to perform a specific quantum strategy up to high precision. Proving robust self-testing results becomes significantly easier when one restricts the allowed strategies to symmetric projective maximally entangled (PME) strategies, which allow natural descriptions in terms of tracial von Neumann algebras. This has been exploited in the celebrated MIP=RE paper and related articles to prove robust self-testing results for synchronous games when restricting to PME strategies. However, the PME assumptions are not physical, so these results need to be upgraded to make them physically relevant. In this chapter, we do just that: we prove that any perfect synchronous game which is a robust self-test when restricted to PME strategies, is in fact a robust self-test for all strategies. We then apply our result to the Quantum Low Degree Test to find an efficient n-qubit test.*

4.1. Introduction

In a non-local game G, two cooperating but distant players respond to questions drawn from a known distribution to satisfy a known winning condition determined by a referee. A *quantum strategy* allows the players to share an entangled state and perform local measurements, often leading to a higher winning probability than classically achievable. Remarkably, certain non-local games exhibit an even stronger guarantee: they admit a unique optimal quantum strategy, making it possible to certify the underlying quantum state and measurements solely from the observed statistics. This is the essence of *self-testing*, a concept whose roots trace back to foundational work by Summers and Werner [SW87; SW88], Popescu and Rohrlich [PR92], and Tsirelson [Tsi93], and was later introduced by Mayers and Yao [MY04] from a cryptographic perspective.

Self-testing is arguably the strongest form of device-independent quantum certification, where one aims to classically verify the behaviour of a quantum device without making any assumptions about its internal workings. This idea has found impactful applications in device-independent cryptography [BŠCA18a; BŠCA18b], verifiable quantum delegation [RUV13; CGJV19; BMZ24], and quantum complexity theory, including the recent breakthrough $\mathrm{MIP}^* = \mathrm{RE}$ [Ji+22].

4

Due to the inevitable noise and imperfections in real-world implementations, the strategy executed in an experiment may only win the given non-local game nearly optimally. As a result, all the aforementioned applications require self-tests to be *robust*: any nearly optimal strategy must be close—under a suitable notion of distance—to the unique optimal strategy. We say that a game G κ*-robustly self-tests* an ideal optimal strategy $\tilde{\mathscr{S}}$ for a class of employed strategies $\mathscr{C}$ if every ε-optimal strategy in $\mathscr{C}$ is $\kappa(\varepsilon)$-close (up to local isometries) to $\tilde{\mathscr{S}}$. Here, $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$ and is called the *robustness* of this self-test.

To mathematically prove that a game is a robust self-test, one typically imposes additional assumptions on the class $\mathscr{C}$ of employed strategies. These assumptions simplify the analysis because the resulting strategies often admit nice algebraic forms. For instance, the existing self-testing results typically assume that the employed strategies $\mathscr{S} = (|\psi\rangle, \{A^x_a\}, \{B^y_b\})$ are *projective*, meaning that the players' measurement operators $\{A^x_a\}$, $\{B^y_b\}$ are projection-valued measures (PVMs). In this setting, strategies for a game with n questions and m answers correspond to representations of the group algebra $\mathbb{C}[\mathbb{Z}_m^{*n} \times \mathbb{Z}_m^{*n}]$. Another common assumption is that the employed strategies $\mathscr{S} = (|\psi\rangle, \{A^x_a\}, \{B^y_b\})$ are *full-rank*, in the sense that the shared entangled state $|\psi\rangle$ has full Schmidt rank. Such strategies are centrally-supported [PSZZ24], so for every measurement operator A^x_a of Alice, there exists some operator $\hat{A}^x_a$ acting on Bob's registers such that $A^x_a \otimes 1 |\psi\rangle = 1 \otimes \hat{A}^x_a |\psi\rangle$. However, from a device-independent perspective, such assumptions limit the scope of security and soundness guarantees provided by self-testing. For example, in quantum key distribution, an adversarial device might deviate from the assumed behaviour, potentially compromising the protocol; in quantum interactive proof systems, these assumptions may fail to capture the behaviour of general malicious provers, thereby weakening soundness. This gives rise to a trade-off: stronger assumptions make robust self-testing results easier to prove but less generally applicable. A central question is, therefore, how to lift such assumptions while preserving the robustness of self-tests.

Recent progress has been made on this front. In particular, [PSZZ24] establishes that for binary output games and synchronous games, self-testing for projective strategies implies self-testing for general POVM strategies. This was later extended in [Bap+23], where the authors show that the projectivity assumption and the full-rankness assumption can both be lifted in robust self-testing, provided that the non-local game has an optimal strategy that is simultaneously projective and full-rank.

In this chapter, we focus on *synchronous games* [Pau+16] and robust self-testing for their *perfect* quantum strategies. Such games exhibit a rich algebraic structure. To each synchronous game G one can associate a $*$-algebra $\mathscr{A}(G)$—known as the synchronous algebra—whose tracial states correspond to perfect strategies within different mathematical models for entanglement [KPS18; HMPS19]. This algebraic framework plays a central role in the recent developments connecting quantum interactive proofs to operator algebras [Ji+22; MNY22; NZ23].

One of the essential building blocks of synchronous games is the class of *PME* strategies, where the players share a *maximally entangled* state and perform *projective, symmetric* measurements. It is well-known that any perfect strategy for a synchronous game is a convex mixture of PME strategies [Pau+16], and recent results [Vid22; Pad25] demonstrate that any near-perfect strategy is close to a convex mixture of PME strategies. This has been further extended to infinite-dimensional models [MS23; Lin24]. PME strategies are also particularly tractable in the context of self-testing: if we assume employed strategies are PME, then the robustness is closely related to the stability of the synchronous algebra in the normalised Hilbert-Schmidt norm. Therefore, it is often more straightforward to prove that a synchronous game is a robust self-test for PME strategies, using techniques from approximate representation theory (see e.g., [CVY23]). However, perfectly maximally entangled states are not physically realisable in practice. Meanwhile, noise on maximally entangled states can significantly reduce the power of quantum interactive proof systems [QY21; QY23; Don+24]. This motivates our central question about lifting the PME assumption in synchronous self-testing.

Question 1. If a synchronous game robustly self-tests a perfect strategy $\tilde{\mathscr{S}}$ for PME strategies, does it follow that it robustly self-tests $\tilde{\mathscr{S}}$ for general POVM strategies?

4.1.1. MAIN RESULTS

We answer the above question affirmatively and quantitatively.

Theorem 4.1.1. *Let G be a synchronous game. If G κ-robustly self-tests a perfect quantum strategy $\tilde{\mathscr{S}}$ for PME strategies, then G κ'-robustly self-tests $\tilde{\mathscr{S}}$ for general POVM strategies, where κ' is polynomially related to κ.*

The precise relationship[1] between κ and κ' is established in Corollary 4.5.5. Crucially, this relationship is independent of the size (i.e., the number of questions and answers) of the game; it depends only on the *synchronicity* of G, which quantifies how frequently the referee checks the players' consistency by sending them the same question.

This notion of synchronicity is also closely tied to another fundamental aspect of robust self-testing: the probability distribution over question pairs. A non-local game is defined with respect to such a distribution ν, determining how often each type of question is asked. Separately, when evaluating the closeness between an employed strategy

[1] We remark that the results in [Zha24] and [Kar25] show that for any perfect synchronous game G, both robust self-testing for PME strategies and robust self-testing for general strategies are equivalent to the uniqueness of amenable tracial state on the synchronous algebra $\mathscr{A}(G)$. However, their results do not establish a quantitative relationship between the robustness of the two cases.

and the ideal optimal strategy, a second distribution $\hat{\nu}$ can be used to weight the "distance" (see Definition 4.2.1). Historically, most robust self-testing results assume that the game distribution ν and the distance distribution $\hat{\nu}$ coincide. However, in many natural scenarios—especially when certain questions only exist to enforce the desired structure of optimal strategies on the other questions—it is useful to consider the more general case where $\nu \neq \hat{\nu}$.

To capture such situations, we define a game to be a $(\kappa, \hat{\nu})$-*robust self-test* if every ε-optimal strategy (with respect to the game distribution ν) is $\kappa(\varepsilon)$-close to the ideal strategy with respect to $\hat{\nu}$ (see Definition 4.2.3). Most of the results in this chapter are stated and proved with respect to this generalised notion of robustness. This framework is particularly useful in the analysis of the Quantum Low Degree Test [CVY23], which is one of the key ingredients in the MIP*=RE proof. Within this generalised notion of robust self-testing, we show that this test can be used to verify that the players approximately have access to maximally entangled qubits and Pauli operators acting on those qubits.

4

Theorem 4.1.2 (Precise statement in Corollary 4.6.3)**.** *Performing the Quantum Low Degree Test and a synchronicity test with equal probability κ'-robustly self-tests the maximally entangled state on n qubits together with a generating set of Pauli operators on those qubits. Here κ' depends polynomially on κ, the robustness of the Quantum Low Degree Test restricted to maximally entangled states.*

4.1.2. PROOF APPROACH AND TECHNICAL CONTRIBUTIONS

Our main result—that robust self-testing for PME strategies implies robust self-testing for general strategies—is proved in three steps, corresponding to Sections 4.3, 4.4, and 4.5, respectively.

We begin by formulating PME strategies—more generally, any strategy employing a maximally entangled state—using tracial von Neumann algebras. We adapt the notion in [CVY23] of distance between families of unitaries to define a '*von Neumann distance*' between PME strategies (see Definition 4.3.1). Our first main technical contribution is to show that this von-Neumann distance is equivalent, up to a constant-factor trade-off, to the standard distance defined in the Hilbert-space formalism (Lemma 4.3.2). Therefore, we can reformulate robust self-testing for PME strategies in the von Neumann algebraic language. This algebraic formulation allows us to exploit the symmetry of PME strategies by working with the algebra generated by a single player's measurements and reduced state, and it enables us to connect our results to the main result from [CVY23].

As an intermediate step toward proving Theorem 4.1.1, our second main technical contribution is to show that κ', the robustness for general strategies, is controlled by the robustness κ for PME strategies and the *spectral gap* of the ideal perfect strategy $\tilde{\mathscr{S}}$ (Theorem 4.4.6). Here, $\tilde{\mathscr{S}}$ having spectral gap Δ means that any strategy $\mathscr{S}$ that uses the same measurements as $\tilde{\mathscr{S}}$ but employs a quantum state orthogonal to the maximally entangled state in $\tilde{\mathscr{S}}$ can achieve a winning probability of up to $1-\Delta$. Intuitively, such a strategy $\mathscr{S}$ is "far from" $\tilde{\mathscr{S}}$, so a small spectral gap indicates poor robustness in self-testing.

Our last step is to show that, perhaps surprisingly, if a synchronous game G κ-robustly self-tests an ideal perfect strategy $\tilde{\mathscr{S}}$ for PME strategies, then the spectral gap of $\tilde{\mathscr{S}}$ admits a lower bound that is polynomially related to κ (Theorem 4.5.4). As a result, in

Theorem 4.1.1, the robustness κ' is controlled by—indeed, polynomially related to—the original robustness κ. In the case of Quantum Low Degree Test based on a linear code of relative distance d, we explicitly compute its spectral gap to be $d/2$ (Theorem 4.6.1), which further yields our second main result Theorem 4.1.2.

4.2. PRELIMINARIES

4.2.1. NON-LOCAL GAMES AND STRATEGIES

In this chapter, given a Hilbert space $\mathscr{H}$, we denote by $B(\mathscr{H})$ the algebra of bounded operators on $\mathscr{H}$. We write $1_{\mathscr{H}}$ for the identity operator on $\mathscr{H}$, and simply 1 if the underlying space is clear from the context. We also use 1 for the identity element of a von Neumann algebra. A collection of operators $\{P_i\}_{i=1}^k \subset B(\mathscr{H})$ is a positive operator-valued measure (POVM) if every $P_i \geq 0$ and $\sum_{i=1}^k P_i = 1$.

4

A two-player (commonly called Alice and Bob) non-local game G is specified by a tuple $G = (\mathscr{X}, \mathscr{Y}, \nu, \mathscr{A}, \mathscr{B}, D)$, where $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{A}$, and $\mathscr{B}$ are finite sets, ν is a probability distribution on $\mathscr{X} \times \mathscr{Y}$, and $D : \mathscr{X} \times \mathscr{Y} \times \mathscr{A} \times \mathscr{B} \to \{0,1\}$ is a predicate. Alice and Bob know all the data in G and they can strategise together before the game begins, but they are not allowed to communicate once the game starts. During the game, Alice and Bob receive questions $x \in \mathscr{X}$ and $y \in \mathscr{Y}$ respectively from a referee with probability $\nu(x, y)$, and they return answers $a \in \mathscr{A}$ and $b \in \mathscr{B}$ respectively. Based on the predicate D, the referee determines whether they win ($D(a, b|x, y) = 1$) or lose ($D(a, b|x, y) = 0$). In some cases, the sets of feasible answers are determined by questions. When this happens, we think of $\mathscr{A}$ and $\mathscr{B}$ as collections $\mathscr{A} = \{\mathscr{A}(x) \mid x \in \mathscr{X}\}$ and $\mathscr{B} = \{\mathscr{B}(y) \mid y \in \mathscr{Y}\}$), where for each question $x \in \mathscr{X}$ and $y \in \mathscr{Y}$, Alice and Bob can only return some $a \in \mathscr{A}(x)$ and $b \in \mathscr{B}(y)$ respectively.

In quantum mechanics, the strategy Alice and Bob employ for a non-local game $G = (\mathscr{X}, \mathscr{Y}, \nu, \mathscr{A}, \mathscr{B}, D)$ is described by a tuple

$$\mathscr{S} = \left(|\psi\rangle \in \mathscr{H}_A \otimes \mathscr{H}_B, A = \{A_a^x\}, B = \{B_b^y\}\right)$$

where

(a) Alice and Bob share a quantum state (unit vector) $|\psi\rangle \in \mathscr{H}_A \otimes \mathscr{H}_B$, and

(b) for each $x \in \mathscr{X}$ (resp. $y \in Y$) Alice measures her register using a POVM $\{A_a^x \mid a \in \mathscr{A}\}$ on $\mathscr{H}_A$ (resp. Bob measures his register using a POVM $\{B_b^y \mid b \in \mathscr{B}\}$ on $\mathscr{H}_B$), and we shorten $\{\{A_a^x\}_{a \in \mathscr{A}} \mid x \in \mathscr{X}\}$ to $\{A_a^x\}$ (resp. $\{\{B_b^y\}_{b \in \mathscr{B}} \mid y \in \mathscr{Y}\}$ to $\{B_b^y\}$).

By Born's rule, if the players employ a strategy $\mathscr{S} = (|\psi\rangle, A, B)$, then the probability that they response $a \in \mathscr{A}$ and $b \in \mathscr{B}$ upon receiving $x \in \mathscr{X}$ and $y \in \mathscr{Y}$ is given by

$$C_{x,y,a,b} = \langle\psi| A_a^x \otimes B_b^y |\psi\rangle. \tag{4.1}$$

The collection $C = \{C_{x,y,a,b}\} \in \mathbb{R}^{\mathscr{X} \times \mathscr{Y} \times \mathscr{A} \times \mathscr{B}}$ is called the correlation induced by $\mathscr{S}$. The winning probability of C for $G = (\mathscr{X}, \mathscr{Y}, \nu, \mathscr{A}, \mathscr{B}, D)$ is given by

$$\omega(G; C) = \mathop{\mathbb{E}}_{(x,y)\sim\nu} \sum_{a,b} D(a, b|x, y) C_{x,y,a,b},$$

where $\mathbb{E}_{(x,y)\sim\nu} \cdot = \sum_{x,y} \nu(x,y) \cdot$ is the expectation with respect to ν. The winning probability $\omega(G;\mathscr{S})$ of a strategy $\mathscr{S}$ for a game G refers to the winning probability of the correlation induced by $\mathscr{S}$. When the game G is clear from the context, we just write $\omega(\mathscr{S})$ for $\omega(G;\mathscr{S})$. In this chapter, we assume all strategies employ *finite-dimensional* systems. We denote by $C_q(\mathscr{X},\mathscr{Y},\mathscr{A},\mathscr{B})$ the set of correlations induced by strategies through Equation (4.1). The quantum value $\omega_q(G)$ of a non-local game G is the supremum of $w(G;C)$ over all $C \in C_q$. A strategy $\mathscr{S}$ or a correlation C is said to be optimal for G if its winning probability achieves the quantum value $w_q(G)$. When $w_q(G) = 1$, we replace "optimal" with "perfect". We call a game perfect if it admits a perfect (finite-dimensional) strategy.

Given a strategy $\mathscr{S} = (|\psi\rangle, A, B)$, we refer to the element

$$T_{G,\mathscr{S}} := \mathbb{E}_{(x,y)\sim\nu} \sum_{a,b} D(a,b|x,y) A^x_a \otimes B^y_b$$

in $B(\mathscr{H}_A \otimes \mathscr{H}_B)$ as the *game polynomial* of $\mathscr{S}$ for G. The winning probability of any strategy $\mathscr{S}' = (|\psi'\rangle, A, B)$ for G is equal to $\langle\psi'| T_{G,\mathscr{S}} |\psi'\rangle$. We will be particularly interested in the *spectral gap* of $T_{G,\mathscr{S}}$, which is the difference between the largest and second largest eigenvalues of $T_{G,\mathscr{S}}$, accounting for multiplicities. This means that any self-adjoint operator has spectral gap equal to zero if the largest eigenvalue has multiplicity larger than one.

If a POVM $\{P_i \mid 1 \le i \le k\}$ on $\mathscr{H}$ consists of mutually orthogonal projections in the sense that $P_i^2 = P_i$ and $P_iP_j = 0$ for $i \neq j$, then it is called a PVM. Any k-outcome PVM $\{P_i \mid 1 \le i \le k\}$ corresponds a unitary U of order k via the Fourier transform

$$U = \sum_{j=1}^{k} \exp\left(\frac{2\pi\sqrt{-1}}{k} j\right) P_j$$

and vice versa via spectral decomposition. A strategy $\mathscr{S} = (|\psi\rangle, A, B)$ is said to be *projective* if $\{A^x_a \mid a \in \mathscr{A}\}$ and $\{B^y_b \mid b \in \mathscr{B}\}$ are PVMs for all $x \in \mathscr{X}$, $y \in \mathscr{Y}$. In this case, we often specify the measurement operators $\{A^x_a \mid a \in \mathscr{A}\}$ and $\{B^y_b \mid b \in \mathscr{B}\}$ using their corresponding unitaries $U(A^x)$ and $U(B^y)$.

Given two finite-dimensional Hilbert spaces $\mathscr{H}_A$ and $\mathscr{H}_B$, every vector $|\psi\rangle \in \mathscr{H}_A \otimes \mathscr{H}_B$ has a Schmidt decomposition

$$|\psi\rangle = \sum_{i=1}^{k} \lambda_i |\alpha_i\rangle \otimes |\beta_i\rangle$$

where the Schmidt coefficients λ_i's are positive real numbers, and $\{|\alpha_i\rangle \mid 1 \le i \le k\}$ and $\{|\beta_i\rangle \mid 1 \le i \le k\}$ are orthonormal subsets of $\mathscr{H}_A$ and $\mathscr{H}_B$ respectively. A unit vector $|\psi\rangle \in \mathscr{H}_A \otimes \mathscr{H}_B$ is *maximally entangled* if $\dim(\mathscr{H}_A) = \dim(\mathscr{H}_B) =: d$ and $|\psi\rangle$ has a Schmidt decomposition

$$|\psi\rangle = \sum_{i=1}^{d} \frac{1}{\sqrt{d}} |\alpha_i\rangle \otimes |\beta_i\rangle.$$

A strategy $\mathscr{S} = (|\psi\rangle, A, B)$ is said to be *maximally entangled* if $|\psi\rangle \in \mathscr{H}_A \otimes \mathscr{H}_B$ is a maximally entangled state.

If $\mathscr{X} = \mathscr{Y}$ and $\mathscr{A} = \mathscr{B}$ in a non-local game $G = (\mathscr{X},\mathscr{Y},\nu,\mathscr{A},\mathscr{B},D)$, then we write $G = (\mathscr{X},\nu,\mathscr{A},D)$. It is often convenient to work with symmetric games and symmetric strategies. A non-local game $G = (\mathscr{X},\nu,\mathscr{A},D)$ is *symmetric* if

(a) $\nu(x,y) = \nu(y,x)$, and

(b) $D(a,b|x,y) = D(b,a|y,x)$ for all $a,\ b \in \mathscr{A}$ and $x,\ y \in \mathscr{X}$.

A strategy $\mathscr{S} = (|\psi\rangle, A, B)$ for a non-local game $G = (\mathscr{X}, \nu, \mathscr{A}, D)$ is *symmetric* if

(a) $\mathscr{H}_A = \mathscr{H}_B := \mathscr{H}$,

(b) $|\psi\rangle = \sum_i \sqrt{\lambda_i}\,|u_i\rangle \otimes |u_i\rangle$ where $\lambda_i \geq 0$ for and $\{|u_i\rangle\}_i$ is an orthonormal basis for $\mathscr{H}$, and

(c) $A_a^x = (B_a^x)^T$ for all $a \in \mathscr{A}$ and $x \in \mathscr{X}$, where the transpose is taken with respect to the basis $\{|u_i\rangle\}$.

Note that in this case, the reduced density matrix ρ of $|\psi\rangle$ on Alice and Bob's sides are both $\sum_i \lambda_i\,|u_i\rangle\langle u_i|$, and we have

$$\langle\psi|\,S \otimes T\,|\psi\rangle = \mathrm{Tr}(S\rho^{1/2}T^T\rho^{1/2})$$

for $S,\ T \in B(\mathscr{H})$, which is called *Ando's formula*. Since Bob's measurements are completely determined by Alice's measurements, we write any symmetric strategy as $\mathscr{S} = (|\psi\rangle, A)$. Given any strategy $\mathscr{S} = (|\psi\rangle, A, B)$ with Schmidt decomposition given by $|\psi\rangle = \sum_{i=1}^d \lambda_i\,|\alpha_i\rangle \otimes |\beta_i\rangle$, we define its *associated symmetric strategies* $\mathscr{S}_A = (|\psi_A\rangle, A)$ and $\mathscr{S} = (|\psi_B\rangle, B)$ where $|\psi_A\rangle = \sum_{i=1}^d \lambda_i\,|\alpha_i\rangle \otimes |\alpha_i\rangle$ and $|\psi_B\rangle = \sum_{i=1}^d \lambda_i\,|\beta_i\rangle \otimes |\beta_i\rangle$.

Let $\mathscr{S} = (|\psi\rangle, A)$ be a symmetric strategy. If $|\psi\rangle$ is maximally entangled, we call $\mathscr{S}$ an *ME* strategy. If an ME strategy is projective, we call it a *PME* strategy. Note that, given a symmetric strategy $\mathscr{S} = (|\psi\rangle, A)$, Ando's formula tells us that choosing a different symmetric state $|\psi'\rangle$ with the same reduced density matrix gives rise to an equivalent strategy $\mathscr{S}' = (|\psi'\rangle, A)$, in the sense that they are related through a unitary on Bob's side. This may seem strange at first, but this happens because the transpose depends on the state, so the operators for Bob will also change if one changes the state. In particular, this means that for any two maximally entangled state $|\psi\rangle$ and $|\psi'\rangle$, the strategies $(|\psi\rangle, A)$ and $(|\psi'\rangle, A)$ are equivalent.

In this chapter, we focus on *synchronous games* and *synchronous correlations*. A symmetric game $G = (\mathscr{X}, \nu, \mathscr{A}, D)$ is synchronous if $\nu(x,x) > 0$ and $D(a,a'|x,x) = 0$ for all x and $a \neq a'$. Given $\beta \in (0,1)$, we say a game $G = (\mathscr{X}, \nu, \mathscr{A}, D)$ is β-synchronous if it is synchronous and

$$\nu(x,x) \geq \beta \sum_{y \in \mathscr{X}} \nu(x,y)$$

for all $x \in \mathscr{X}$. A correlation $C \in C_q(\mathscr{X}, \mathscr{X}, \mathscr{A}, \mathscr{A})$ is said to be synchronous if $C_{x,x,a,a'} = 0$ for all x and $a \neq a'$. We use $C_q^s(\mathscr{X}, \mathscr{A})$ to denote the set of synchronous correlations in $C_q(\mathscr{X}, \mathscr{X}, \mathscr{A}, \mathscr{A})$. As shown in [Pau+16], a correlation $C \in C_q(\mathscr{X}, \mathscr{X}, \mathscr{A}, \mathscr{A})$ is synchronous if and only if there is a finite-dimensional unital tracial $*$-algebra $(\mathscr{M}, \tau)$ and PVMs $\{E_a^x \mid a \in \mathscr{A}\}, x \in \mathscr{X}$ in $\mathscr{M}$ such that

$$C_{x,y,a,b} = \tau(E_a^x E_b^y)$$

for all $x,\ y \in \mathscr{X}$ and $a,\ b \in \mathscr{A}$.

Any ME strategy $\mathscr{S} = (|\psi\rangle, A)$ induces a tracial state τ on the algebra generated by A_a^x's via $\tau(\alpha) = \langle\psi|\,\alpha\otimes 1\,|\psi\rangle$. Furthermore, $\langle\psi|\,A_a^x\otimes B_b^y\,|\psi\rangle = \tau(A_a^x A_b^y)$. This gives an alternative way to describe ME and PME strategies in terms of tracial von Neumann algebras.

A tracial von Neumann algebra $(\mathscr{M},\tau)$ is a von Neumann algebra $\mathscr{M}$ together with a normal faithful tracial state τ on $\mathscr{M}$. The corresponding trace-norm (or so called 2-norm) $\|\cdot\|_\tau$ on $\mathscr{M}$ is given by $\|\alpha\|_\tau := \sqrt{\tau(\alpha^*\alpha)}$. For example, $(M_n(\mathbb{C}),\mathrm{tr})$ is the von Neumann algebra of $n\times n$ matrices with the normalised trace $\mathrm{tr}(\alpha) = \frac{1}{n}\mathrm{Tr}(\alpha)$. We also work with Schatten p-norm in $M_n(\mathbb{C})$ for $p\in[1,\infty]$. In particular, $\|X\|_1 = \mathrm{Tr}(|X|)$, $\|X\|_2 = \sqrt{\mathrm{Tr}(X^*X)}$, and $\|X\|_\infty$ is the largest singular value of X.

Any ME (resp. PME) strategy $\mathscr{S}$ for a game $G = (\mathscr{X},\nu,\mathscr{A},D)$ can be specified by a tuple $(\mathscr{M},\tau,A)$ where $\mathscr{M} = B(\mathscr{H})$ for some finite-dimensional space $\mathscr{H}$, τ is the normalised trace on $\mathscr{M}$, and $A = \{\{A_a^x\}_{a\in\mathscr{A}}\subset\mathscr{M} \mid x\in\mathscr{X}\}$ are POVMs (resp. PVMs) in $\mathscr{M}$. The correlation induced by $\mathscr{S} = (\mathscr{M},\tau,A)$ is given by

$$C_{x,y,a,b} = \tau(A_a^x A_b^y).$$

In fact, any such triple $\mathscr{S} = (\mathscr{M},\tau,A)$ gives rise to a strategy in this way, even if $\mathscr{M}$ is a finite-dimensional von Neumann algebra that is not of the form $B(\mathscr{H})$ for some finite-dimensional Hilbert space $\mathscr{H}$. One can recover the usual formulation of a strategy $\mathscr{S} = (|\psi\rangle, A', B')$ on $\mathscr{H}_A\otimes\mathscr{H}_B$ using the GNS construction. Let $\mathscr{M}\subset B(\mathscr{H})$. Then $\mathscr{H}_A = \mathscr{H}_B = \mathscr{H}$, $A' = A$, $B' = A^T$ and $|\psi\rangle$ is the cyclic vector corresponding to the GNS construction by identifying $B(\mathscr{H})$ (equipped with the Hilbert-Schmidt inner product) with $\mathscr{H}\otimes\mathscr{H}$. Since $\mathscr{M}\subset B(\mathscr{H})$, the cyclic vector becomes a vector in $\mathscr{H}\otimes\mathscr{H}$. Note that while $\mathscr{H}\otimes\overline{\mathscr{H}}$ and $B(\mathscr{H})$ are naturally isomorphic, $\mathscr{H}$ and $\overline{\mathscr{H}}$ are not. One needs to specify a basis to identify $B(\mathscr{H})$ and $\mathscr{H}\otimes\mathscr{H}$ and to take the transpose in $B' = A^T$. Consequently, a triple $\mathscr{S} = (\mathscr{M},\tau,A)$ defines $\mathscr{S} = (|\psi\rangle, A', B')$ up to unitary equivalence on Bob's side. We call the state $|\psi\rangle$ obtained in this way a GNS state for $(\mathscr{M},\tau)$, and it satisfies

$$\tau(X) = \langle\psi|\,(X\otimes 1)\,|\psi\rangle = \langle\psi|\,(1\otimes X^T)\,|\psi\rangle$$

for all $X\in\mathscr{M}$.

Note that not all strategies $\mathscr{S} = (|\psi\rangle, A, B)$ can be written in von Neumann algebra terms. A strategy is of the form $(\mathscr{M},\tau,A)$ if and only if it is a classical convex combination of ME strategies, intuitively meaning that each round an ME strategy is selected based on classical shared randomness.

Definition 4.2.1 (Local dilation)**.** For two strategies $\mathscr{S} = (|\psi\rangle, A, B)$ and $\tilde{\mathscr{S}} = (|\tilde\psi\rangle, \tilde A, \tilde B)$, we say that $\tilde{\mathscr{S}}$ is a local (ε,ν)-dilation of $\mathscr{S}$ for some $\varepsilon\geq 0$ and distribution ν on $\mathscr{X}\times\mathscr{Y}$ if there are isometries $V_A : \mathscr{H}_A\to\tilde{\mathscr{H}}_A\otimes\mathscr{K}_A$ and $V_B : \mathscr{H}_B\to\tilde{\mathscr{H}}_B\otimes\mathscr{K}_B$ and a unit vector $|aux\rangle\in\mathscr{K}_A\otimes\mathscr{K}_B$ such that

$$\|(V_A\otimes V_B)\,|\psi\rangle - |\tilde\psi\rangle\otimes|aux\rangle\| \leq \varepsilon,$$

$$\left(\mathop{\mathbb{E}}_{x\sim\nu_A}\sum_a \|(V_A\otimes V_B)(A_a^x\otimes 1)\,|\psi\rangle - \left((\tilde A_a^x\otimes 1)\,|\tilde\psi\rangle\right)\otimes|aux\rangle\|^2\right)^{1/2} \leq \varepsilon,$$

$$\left(\mathop{\mathbb{E}}_{y\sim\nu_B}\sum_b \|(V_A\otimes V_B)(1\otimes B_b^y)\,|\psi\rangle - \left((1\otimes\tilde B_b^y)\,|\tilde\psi\rangle\right)\otimes|aux\rangle\|^2\right)^{1/2} \leq \varepsilon,$$

where ν_A and ν_B are marginal distributions of ν on $\mathscr{X}$ and $\mathscr{Y}$ respectively.

Remark 4.2.2. The above definition is different from [Vid22, Definition 2.4], where the last two inequalities are replaced by the single inequality

$$\left(\mathop{\mathbb{E}}_{(x,y)\sim\nu}\sum_{a,b}\|(V_A\otimes V_B)(A_a^x\otimes B_b^y)|\psi\rangle-\big((\tilde{A}_a^x\otimes\tilde{B}_b^y)|\tilde{\psi}\rangle\big)\otimes|aux\rangle\|^2\right)^{1/2}\leq\varepsilon.$$

The reason we choose this different definition is that we need to relate local dilations in this framework to local dilations in the von Neumann algebra framework, as will be defined in Definition 4.3.1. This is only possible using our definition of a local dilation. We will now show that a local (ε,ν)-dilation in the sense of Definition 4.2.1 is a local $(3\varepsilon,\nu)$-dilation in the sense of [Vid22, Definition 2.4], but that the reverse implication does not hold for any constant trade-off for POVM strategies. We do not know if such a separation exists for projective strategies as well.

Let $\tilde{\mathcal{S}}=(|\tilde{\psi}\rangle,\tilde{A},\tilde{B})$ be a local (ε,ν)-dilation of $\mathcal{S}=(|\psi\rangle,A,B)$ in the sense of Definition 4.2.1. Using the fact that

$$\sum_b(V_BB_b^yV_B^*)^2\leq 1,\sum_a(\tilde{A}_a^x)^2\leq 1\text{ and }\sum_{a,b}(\tilde{A}_a^x\otimes V_BB_b^y)^*(\tilde{A}_a^x\otimes V_BB_b^y)\leq 1,$$

it follows from the inequalities of Definition 4.2.1 that

$$\left(\mathop{\mathbb{E}}_{(x,y)\sim\nu}\sum_{a,b}\|(V_A\otimes V_B)(A_a^x\otimes B_b^y)|\psi\rangle-(\tilde{A}_a^x\otimes 1_{\mathcal{K}_A}\otimes V_BB_b^yV_B^*)|\tilde{\psi}\rangle\otimes|aux\rangle\|^2\right)^{1/2}\leq\varepsilon,$$

$$\left(\mathop{\mathbb{E}}_{(x,y)\sim\nu}\sum_{a,b}\|(\tilde{A}_a^xV_A\otimes V_BB_b^y)|\psi\rangle-\big((\tilde{A}_a^x\otimes\tilde{B}_b^y)|\tilde{\psi}\rangle\big)\otimes|aux\rangle\|^2\right)^{1/2}\leq\varepsilon$$

and

$$\left(\mathop{\mathbb{E}}_{(x,y)\sim\nu}\sum_{a,b}\|(\tilde{A}_a^x\otimes V_BB_b^y)\big((V_A\otimes 1_{\mathcal{H}})|\psi\rangle-(1_{\tilde{\mathcal{H}}\otimes\mathcal{K}_A}\otimes V_B^*)|\tilde{\psi}\rangle\otimes|aux\rangle\big)\|^2\right)^{1/2}\leq\varepsilon.$$

Using the triangle inequality now yields

$$\left(\mathop{\mathbb{E}}_{(x,y)\sim\nu}\sum_{a,b}\|(V_A\otimes V_B)(A_a^x\otimes B_b^y)|\psi\rangle-\big((\tilde{A}_a^x\otimes\tilde{B}_b^y)|\tilde{\psi}\rangle\big)\otimes|aux\rangle\|^2\right)^{1/2}\leq 3\varepsilon.$$

To disprove an implication with constant trade-off in the converse direction, let $\tilde{\mathcal{S}}=(|\tilde{\psi}\rangle,\tilde{A},\tilde{B})$ be a local (ε,ν)-dilation of $\mathcal{S}=(|\psi\rangle,A,B)$ in the sense of [Vid22, Definition 2.4]. Consider now the families of strategies $\{\tilde{\mathcal{S}}_n\}_{n\geq 1}$ and $\{\mathcal{S}_n\}_{n\geq 1}$, given by

$$\mathcal{S}_n=(|\psi\rangle,A^{(n)},B^{(n)}),\ (A^{(n)})_{a,i}^x=\frac{1}{n}A_a^x,\ (B^{(n)})_{b,i}^y=\frac{1}{n}B_b^y\text{ and}$$
$$\tilde{\mathcal{S}}_n=(|\psi\rangle,\tilde{A}^{(n)},\tilde{B}^{(n)}),\ (\tilde{A}^{(n)})_{a,i}^x=\frac{1}{n}\tilde{A}_a^x,\ (\tilde{B}^{(n)})_{b,i}^y=\frac{1}{n}\tilde{B}_b^y$$

for answer sets $\mathcal{A}_n=\mathcal{A}\times\{1,\ldots,n\}$ and $\mathcal{B}_n=\mathcal{B}\times\{1,\ldots,n\}$. We see that $\tilde{\mathcal{S}}_n$ is a local $(\varepsilon/n,\nu)$-dilation of $\mathcal{S}_n$ in the sense of [Vid22, Definition 2.4]. On the other hand, if $\tilde{\mathcal{S}}$ is a local (δ,ν)-dilation of $\mathcal{S}$ in the sense of Definition 4.2.1 for some optimal $\delta>0$, then $\tilde{\mathcal{S}}_n$ is only a local $(\delta/\sqrt{n},\nu)$-dilation of $\mathcal{S}_n$ in the sense of Definition 4.2.1, proving that it is impossible to universally bound δ from above by $C\varepsilon$ for some constant C.

Definition 4.2.3 (Self-testing)**.** Given a non-local game $G = (\mathscr{X},\mathscr{Y},\nu,\mathscr{A},\mathscr{B},D)$, a class of strategies $\mathscr{C}$, an $\tilde{\mathscr{S}} \in \mathscr{C}$ that is optimal for G, a probability distribution $\hat{\nu}$ on $\mathscr{X}\times\mathscr{Y}$ and a function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$, we say that G $(\kappa,\hat{\nu})$-robustly self-tests $\tilde{\mathscr{S}}$ for the class $\mathscr{C}$ if for any $\varepsilon \geq 0$ and $\mathscr{S} \in \mathscr{C}$ with $\omega(G;\mathscr{S}) \geq w_q(G)-\varepsilon$, $\tilde{\mathscr{S}}$ is a local $(\kappa(\varepsilon),\hat{\nu})$-dilation of $\mathscr{S}$. We say that G κ-robustly self-tests $\tilde{\mathscr{S}}$ for class $\mathscr{C}$ if $\hat{\nu}$ is equal to ν, the distribution used in the non-local game.

The optimal strategy $\tilde{\mathscr{S}}$ in the above definition is usually referred to as the ideal optimal strategy for G. In this chapter, we are primarily interested in the class of all PME strategies $\mathscr{C}_{PME}$ and the class of all strategies $\mathscr{C}_{all}$. We simply say that G $(\kappa,\hat{\nu})$-robustly self-tests $\tilde{\mathscr{S}}$ (or G is a $(\kappa,\hat{\nu})$-robust self-test) if G $(\kappa,\hat{\nu})$-robustly self-tests $\tilde{\mathscr{S}}$ for $\mathscr{C}_{all}$. We say that G $(\kappa,\hat{\nu})$-PME-robustly self-tests $\tilde{\mathscr{S}}$ (or G is a $(\kappa,\hat{\nu})$-PME-robust self-test) if G $(\kappa,\hat{\nu})$-robustly self-tests $\tilde{\mathscr{S}}$ for $\mathscr{C}_{PME}$.

4

In Section 4.2.1 the perhaps somewhat uncommon notion of a β-synchronous non-local game was introduced. If one views robust self-testing from the viewpoint of certification protocols, the β-synchronous condition is not significantly stronger than being synchronous. This is captured in the following definition and lemma.

Definition 4.2.4. Let $G = (\mathscr{X},\nu,\mathscr{A},D)$ be a synchronous game, let ν_A be the marginal distribution of ν on $\mathscr{X}$ and let $\beta \in (0,1)$. Let ν' be the probability distribution on $\mathscr{X}\times\mathscr{X}$ defined by $\nu'(x,y) = \beta\nu_A(x)\delta_{x,y} + (1-\beta)\nu(x,y)$. Then we call $(\mathscr{X},\nu',\mathscr{A},D)$ the β-synchronised version of G.

Lemma 4.2.5. *Let $G = (\mathscr{X},\nu,\mathscr{A},D)$ be a synchronous game and let $\beta \in (0,1)$. Let $\hat{\nu}$ be a probability distribution on $\mathscr{X}\times\mathscr{X}$ and let G' be the β-synchronised version of G. If G $(\kappa,\hat{\nu})$-robustly self-tests an optimal synchronous strategy $\mathscr{S}$ for class $\mathscr{C}$, then G' $(\kappa',\hat{\nu})$-robustly self-tests $\mathscr{S}$ for class $\mathscr{C}$ with $\kappa'(\varepsilon) = \kappa\left(\frac{\varepsilon}{1-\beta}\right)$.*

Proof. This is immediate after realising that for any strategy $\hat{\mathscr{S}}$ we have the implication

$$|\omega(G';\mathscr{S}) - \omega(G';\hat{\mathscr{S}})| \leq \varepsilon \implies |\omega(G;\mathscr{S}) - \omega(G;\hat{\mathscr{S}})| \leq \frac{\varepsilon}{1-\beta}. \qquad \square$$

Historically, robust self-testing has been studied in the case where the probability distribution of the game equals the probability distribution used in the local dilation. However, one can conceive situations where one wants to verify that some parts of a strategy match the ideal one, and that the other questions are merely present to ensure this behaviour. We will encounter an example of this behaviour when we come to the Quantum Low Degree Test in Section 4.2.3.

Definition 4.2.1 is about measuring the “distance" between two strategies. As we will discuss in Section 4.3, for ME strategies, their distance can be described in the framework of von Neumann algebras. So we provide more background on von Neumann algebras for subsequent use. We use $\ell^2(\mathbb{N})$ to denote the Hilbert space of sequences in $\mathbb{C}^{\mathbb{N}}$ that are convergent in the Euclidean norm and we use $\{|e_i\rangle \mid i \in \mathbb{N}\}$ to denote the standard basis. The trace Tr on the von Neumann algebra $B(\ell^2(\mathbb{N}))$ of bounded operators on $\ell^2(\mathbb{N})$ is given by $\mathrm{Tr}(x) = \sum_{i\in\mathbb{N}} \langle e_i|x|e_i\rangle$. In general, the tensor product of two tracial von Neumann algebras is viewed as a tracial von Neumann algebra by taking the spacial tensor product and equipping it with the tensor product of the traces. For any tracial von

Neumann algebra $(\mathscr{M},\tau^{\mathscr{M}})$, we denote by $\mathscr{M}^\infty$ the von Neumann algebra $\mathscr{M}\overline{\otimes}B(\ell^2(\mathbb{N}))$, i.e. the σ-weak closure of $\mathscr{M}\otimes B(\ell^2(\mathbb{N}))$ equipped with the trace $\tau^\infty=\tau^{\mathscr{M}}\otimes\mathrm{Tr}$. Let $I_{\mathscr{M}}$ be the projection onto the 1st coordinate in $\mathbb{C}^{\mathbb{N}}$. We usually identify $\mathscr{M}$ with $\mathscr{M}\otimes I_{\mathscr{M}}$ in $\mathscr{M}^\infty$ and write $I_{\mathscr{M}}$ for $1\otimes I_{\mathscr{M}}\in\mathscr{M}^\infty$. For any projection $P\in\mathscr{M}^\infty$, an operator $V\in\mathscr{M}^\infty P$ is an isometry if $V^*V=P$.

4.2.2. ALMOST SYNCHRONOUS CORRELATIONS

To demonstrate that a game is a robust self-test, we need to study strategies that are nearly optimal. In particular, we consider correlations and strategies that are *almost synchronous*. Given a distribution ν on $\mathscr{X}$, the asynchronicity of a correlation $C\in C_q(\mathscr{X},\mathscr{X},\mathscr{A},\mathscr{A})$ with respect to ν is

$$\delta_{\text{sync}}(C;\nu):=\mathop{\mathbb{E}}_{x\sim\nu}\sum_{a\neq b}C_{x,x,a,b}=1-\mathop{\mathbb{E}}_{x\sim\nu}\sum_{a}C_{x,x,a,a}.$$

The asynchronicity $\delta_{\text{sync}}(\mathscr{S};\nu)$ of a strategy $\mathscr{S}=(|\psi\rangle,A,B)$ refers to the asynchronicity of the correlation induced by $\mathscr{S}$. If $\mathscr{S}=(\mathscr{M},\tau,A)$ is an ME strategy, then

$$\delta_{\text{sync}}(\mathscr{S};\nu)=\mathop{\mathbb{E}}_{x\sim\nu}\sum_{a\neq b}\tau(A^x_aA^x_b)=1-\mathop{\mathbb{E}}_{x\sim\nu}\sum_{a}\tau\left((A^x_a)^2\right). \tag{4.2}$$

From Equation (4.2), it is easy to see that the asynchronicity of any PME strategy is 0.

Lemma 4.2.6. *Let $G=(\mathscr{X},\nu,\mathscr{A},D)$ be a β-synchronous game and let ν_A be the marginal distribution of ν on $\mathscr{X}$. Then*

$$\omega(G;C)\le 1-\beta\delta_{\text{sync}}(C;\nu_A)$$

for any correlation C.

Proof. Since G is β-synchronous, we have

$$\nu(x,x)\ge\beta\sum_{a}\nu(x,y)=\beta\nu_A(x)$$

for all $x\in\mathscr{X}$. So

$$\delta:=\delta_{\text{sync}}(C;\nu_A)=1-\sum_a\nu_A(x)\sum_a C_{x,x,a,a}\ge 1-\frac{1}{\beta}\sum_a\nu(x,x)\sum_a C_{x,x,a,a}.$$

It follows that

$$\begin{aligned}1-\omega(G;C)&=\sum_{x,y}\nu(x,y)\sum_{D(a,b|x,y)=0}C_{x,y,a,b}\\&\ge\sum_{x}\nu(x,x)\sum_{D(a,b|x,x)=0}C_{x,x,a,b}\\&=\sum_{x}\nu(x,x)\sum_{a\neq b}C_{x,x,a,b}\\&\ge\sum_{x}\beta\nu_A(x)\sum_{a\neq b}C_{x,x,a,b}\\&=\beta\delta.\end{aligned}$$

This proves the inequality. $\square$

Lemma 4.2.7. *Let $\mathscr{S} = (|\psi\rangle, A, B)$ be a strategy and let $\mathscr{S}_A$ and $\mathscr{S}_B$ be the associated symmetric strategies. Then for any distribution ν on $\mathscr{X}$,*

$$1-\delta_{\text{sync}}(\mathscr{S};\nu) \le \sqrt{1-\delta_{\text{sync}}(\mathscr{S}_A;\nu)}\sqrt{1-\delta_{\text{sync}}(\mathscr{S}_B,\nu)}, \tag{4.3}$$

$$\delta_{\text{sync}}(\mathscr{S}_A;\nu) \le 2\delta_{\text{sync}}(\mathscr{S};\nu), \text{ and} \tag{4.4}$$

$$\delta_{\text{sync}}(\mathscr{S}_B;\nu) \le 2\delta_{\text{sync}}(\mathscr{S};\nu). \tag{4.5}$$

Proof. Equation (4.3) was proved in [Vid22, Corollary 3.2]. Since $\sqrt{1-\delta_{\text{sync}}(\mathscr{S}_B,\nu)} \le 1$,

$$\begin{aligned} 1-\delta_{\text{sync}}(\mathscr{S};\nu) &\le \sqrt{1-\delta_{\text{sync}}(\mathscr{S}_A;\nu)}\sqrt{1-\delta_{\text{sync}}(\mathscr{S}_B,\nu)} \\ &\le \sqrt{1-\delta_{\text{sync}}(\mathscr{S}_A;\nu)} \le 1-\frac{1}{2}\delta_{\text{sync}}(\mathscr{S}_A;\nu). \end{aligned}$$

This proves Equation (4.4). Equation (4.5) holds similarly. □

The following “replacement" lemma is similar to [Vid22, Lemma 2.10]. For the subsequent use, we provide specific constants instead of big-$\mathscr{O}$ in the asymptotic analysis.

Lemma 4.2.8. *Let $\mathscr{S} = (|\psi\rangle, A, B)$ be a strategy for a symmetric game $G = (\mathscr{X}, \nu, \mathscr{A}, D)$. Let ν_A be the marginal distribution of ν on $\mathscr{X}$, and let ρ_A and ρ_B be the densities of $|\psi\rangle$ on $\mathscr{H}_A$ and $\mathscr{H}_B$ respectively. If $\hat{A} = \{\hat{A}_a^x\}$ is a family of POVMs on $\mathscr{H}_A$ and $\hat{B} = \{\hat{B}_b^y\}$ is a family of POVMs on $\mathscr{H}_B$ with*

$$\gamma_A := \mathop{\mathbb{E}}_{x\sim\nu_A}\sum_a \operatorname{Tr}\left((A_a^x - \hat{A}_a^x)^2\rho_A\right),$$

$$\gamma_B := \mathop{\mathbb{E}}_{y\sim\nu_A}\sum_b \operatorname{Tr}\left((B_b^y - \hat{B}_b^y)^2\rho_B\right),$$

then

$$\mathop{\mathbb{E}}_{x,y\sim\nu}\sum_{a,b}|C_{x,y,a,b} - \hat{C}_{x,y,a,b}| \le 12\delta_{\text{sync}}(\mathscr{S};\nu_A) + 4\sqrt{\gamma_A} + 4\sqrt{\gamma_B},$$

where C and $\hat{C}$ are the correlations induced by $\mathscr{S}$ and $\hat{\mathscr{S}} = (|\psi\rangle, \hat{A}, \hat{B})$, respectively.

Proof. Let $\mathscr{S}_A$ and $\mathscr{S}_B$ be the associated symmetric strategies of $\mathscr{S}$, and let C' be the correlations induced by the strategy $(|\psi\rangle, \hat{A}, B)$. Then by [Vid22, Lemma 2.10], we have

$$\mathop{\mathbb{E}}_{x,y\sim\nu}\sum_{a,b}|C_{x,y,a,b} - C'_{x,y,a,b}| \le 3\delta_{\text{sync}}(\mathscr{S}_A;\nu_A) + 4\sqrt{\gamma_A},$$

$$\mathop{\mathbb{E}}_{x,y\sim\nu}\sum_{a,b}|C'_{x,y,a,b} - \hat{C}_{x,y,a,b}| \le 3\delta_{\text{sync}}(\mathscr{S}_B;\nu_A) + 4\sqrt{\gamma_B}.$$

The rest follows from the triangle inequality and Lemma 4.2.7. □

Let $\mathscr{S}$ be an almost synchronous strategy. Lemma 4.2.8 implies that small perturbations on the measurement operators will result in small perturbations on the correlation. This is particularly useful when we want to orthogonalise the measurements (i.e., find a projective strategy nearby).

Lemma 4.2.9. *Let $\mathscr{S} = (|\psi\rangle, A, B)$ be a strategy for a symmetric game $G = (x, \nu, \mathscr{A}, D)$ with reduced densities ρ_A and ρ_B on $\mathscr{H}_A$ and $\mathscr{H}_B$, respectively. Let ν_A be the marginal of ν on $\mathscr{X}$ and let $\delta = \delta_{\text{sync}}(\mathscr{S}, \nu_A)$. Then there exists a projective strategy $\hat{\mathscr{S}} = (|\psi\rangle, \hat{A}, \hat{B})$ such that $\hat{\delta} := \delta_{\text{sync}}(\hat{\mathscr{S}}, \nu_A) = \mathscr{O}(\delta^{\frac{1}{8}})$,*

$$\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \mathrm{Tr}\left((A_a^x - \hat{A}_a^x)^2 \rho_A\right) = \mathscr{O}(\delta^{\frac{1}{4}}), \tag{4.6}$$

$$\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \mathrm{Tr}\left((B_a^x - \hat{B}_a^x)^2 \rho_B\right) = \mathscr{O}(\delta^{\frac{1}{4}}), \tag{4.7}$$

and

$$\mathop{\mathbb{E}}_{x,y\sim\nu} \sum_{a,b} |C_{x,y,a,b} - \hat{C}_{x,y,a,b}| \le \mathscr{O}(\delta^{\frac{1}{8}}), \tag{4.8}$$

where C and $\hat{C}$ are the correlations induced by $\mathscr{S}$ and $\hat{\mathscr{S}}$, respectively.

4

Proof. Let $\mathscr{S}_A$ and $\mathscr{S}_B$ be the associated symmetric strategies of $\mathscr{S}$. Lemma 4.2.7 implies that $\delta_A := \delta_{\text{sync}}(\mathscr{S}_A, \nu_A) = \mathscr{O}(\delta)$ and $\delta_B := \delta_{\text{sync}}(\mathscr{S}_B, \nu_A) = \mathscr{O}(\delta)$. Then by [Vid22, Lemma 3.4], there are PVMs $\hat{A} = \{\hat{A}_a^x\}$ on $\mathscr{H}_A$ such that

$$\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \mathrm{Tr}\left((A_a^x - \hat{A}_a^x)^2 \rho_A\right) = \mathscr{O}(\delta_A^{\frac{1}{4}}) = \mathscr{O}(\delta^{\frac{1}{4}}).$$

This proves Equation (4.6). The existence of PVMs $\hat{B}$ and Equation (4.7) follows similarly. Since every B_a^x and $\hat{A}_a^x$ are measurement operators,

$$\begin{aligned}
|\hat{\delta} - \delta| &= \left| \mathop{\mathbb{E}}_{x\in\nu_A} \sum_a \left(\langle\psi| A_a^x \otimes B_a^x |\psi\rangle - \langle\psi| \hat{A}_a^x \otimes \hat{B}_a^x |\psi\rangle \right) \right| \\
&= \left| \mathop{\mathbb{E}}_{x\in\nu_A} \sum_a \left(\langle\psi| (A_a^x - \hat{A}_a^x) \otimes B_a^x |\psi\rangle + \langle\psi| \hat{A}_a^x \otimes (B_a^x - \hat{B}_a^x) |\psi\rangle \right) \right| \\
&\le \left(\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \mathrm{Tr}\left((A_a^x - \hat{A}_a^x)^2 \rho_A\right) \right)^{1/2} + \left(\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \mathrm{Tr}\left((B_a^x - \hat{B}_a^x)^2 \rho_B\right) \right)^{1/2} \\
&= \mathscr{O}(\delta^{1/8}).
\end{aligned}$$

Hence $\hat{\delta} = \mathscr{O}(\delta^{1/8})$. Then Equation (4.8) follows by Lemma 4.2.8. □

Equation (4.8) in the above lemma says that the correlations induced by $\mathscr{S}$ and $\hat{\mathscr{S}}$ are close. The following lemma further demonstrates that for any non-local game G, the winning probabilities of $\mathscr{S}$ and $\hat{\mathscr{S}}$ are close.

Lemma 4.2.10. *Suppose C and C' are correlations that are induced by two strategies $\mathscr{S}$ and $\mathscr{S}'$, respectively, for a non-local game $G = (\mathscr{X}, \mathscr{Y}, \nu, \mathscr{A}, \mathscr{B}, D)$ such that*

$$\mathop{\mathbb{E}}_{x,y\sim\nu} \sum_{a,b} |C_{x,y,a,b} - C'_{a,b,x,y}| \le \varepsilon.$$

Then $|\omega(G;\mathscr{S}) - \omega(G;\mathscr{S}')| \le \varepsilon$.

Proof. Since every $D(a,b|x,y)$ is either 1 or 0, this lemma is an immediate consequence of the Hölder inequality. □

We call the collection of positive operators $\{A_i \mid 1 \le i \le m\}$ an *incomplete POVM* if $\sum_{i=1}^m A_i \le 1$. It can be completed to a POVM $\{A_i \mid 1 \le i \le m+1\}$ by adding an outcome $m+1$ where $A_{m+1} = 1 - \left(\sum_{i=1}^m A_i\right)$. The following lemma gives an upper bound of the "distance" between any two incomplete POVMs.

Lemma 4.2.11. *Let $\{A_i \mid 1 \le i \le m\}$ and $\{\tilde{A}_i \mid 1 \le i \le m\}$ be two collections of positive operators on some Hilbert space $\mathscr{H}$ such that $\sum_{i=1}^m A_i \le 1$ and $\sum_{i=1}^m \tilde{A}_i \le 1$. Then*

$$\left\| \sum_{i=1}^m (A_i - \tilde{A}_i)^2 \right\|_\infty \le 4.$$

Proof. We assume without loss of generality that $\{A_i\}$ and $\{\tilde{A}_i\}$ are POVMs, as completing them can only increase the norm. We start by considering just the sum and expanding the square to find that

$$0 \le \sum_{i=1}^m (A_i - \tilde{A}_i)^2 = \sum_{i=1}^m A_i^2 + \tilde{A}_i^2 - A_i\tilde{A}_i - \tilde{A}_i A_i \le 2 - \left(\sum_{i=1}^m A_i\tilde{A}_i + \tilde{A}_i A_i\right),$$

having used that $0 \le A_i,\ \tilde{A}_i \le 1$, so $A_i^2 \le A_i$ and $\tilde{A}_i^2 \le \tilde{A}_i$. Consequently,

$$\left\| \sum_{i=1}^m (A_i - \tilde{A}_i)^2 \right\|_\infty \le 2 + 2\left\| \sum_{i=1}^m A_i\tilde{A}_i \right\|_\infty. \tag{4.9}$$

By the Cauchy-Schwarz inequality for inner product C*-modules [RW98, Lemma 2.5], we know that

$$\left(\sum_{i=1}^m A_i\tilde{A}_i\right)^* \left(\sum_{i=1}^m A_i\tilde{A}_i\right) \le \left\| \sum_{i=1}^m A_i^2 \right\|_\infty \sum_{i=1}^m \tilde{A}_i^2 \le 1,$$

so by the C*-identity we have

$$\left\| \sum_{i=1}^m A_i\tilde{A}_i \right\|_\infty \le 1.$$

Plugging this into Equation (4.9) proves the lemma. □

We now describe the results in [Vid22, Section 3]. These are essential for our proof of Theorem 4.4.2. However, we need to use the explicit constructions and results in the proofs of Vidick's main theorems, instead of the final results. For this reason, we combine the results we need in the appropriate form in the following theorem. In this theorem, and in the rest of this chapter, we say that $x \le \mathrm{poly}(\varepsilon)$ if there exist universal constants $C,\ c > 0$ such that $x \le C\varepsilon^c$.

Theorem 4.2.12 (Vidick)**.** *Let $\mathscr{X}$ and $\mathscr{A}$ be finite question and answer sets and ν be a measure on $\mathscr{X} \times \mathscr{X}$ with marginal ν_A on the first entry. Let $\mathscr{S} = (|\psi\rangle, A, B)$ be a projective strategy on $\mathscr{H}_A \otimes \mathscr{H}_B$ and $\delta = \delta_{\mathrm{sync}}(\mathscr{S}, \nu_A)$. Let ρ_A be the reduced density of $|\psi\rangle$ on $\mathscr{H}_A$, let $P_\lambda = \chi_{\ge\lambda}(\rho_A)$ be its spectral projections for $\lambda \ge 0$ and let $\mathscr{H}_\lambda = P_\lambda \mathscr{H}_A$. Then the following are true:*

(a) *The strategies $\mathscr{S}_\lambda = (|\psi_\lambda\rangle, P_\lambda A P_\lambda)$ are ME strategies, where $|\psi_\lambda\rangle$ is the maximally entangled state on $\mathscr{H}_\lambda \otimes \mathscr{H}_\lambda$.*

(b) *Let ρ_λ be the maximally mixed state on $\mathscr{H}_\lambda$. Then*

$$\rho_A = \int_0^\infty \rho_\lambda d\mu(\lambda),$$

where μ is the probability measure defined by $d\mu(\lambda) = \mathrm{Tr}(P_\lambda)d\lambda$.

(c) *The strategies $\mathscr{S}_\lambda$ provide an approximate decomposition of $\mathscr{S}$ as a convex sum of maximally entangled strategies in the following sense:*

$$\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \int_0^\infty \mathrm{Tr}\left(\left(A_a^x - P_\lambda A_a^x P_\lambda\right)^2 \rho_\lambda\right) d\mu(\lambda) \leq \sqrt{2\delta}.$$

(d) *Let C be the correlation of $\mathscr{S}$. The correlations C^λ of $\mathscr{S}_\lambda$ satisfy*

$$\mathop{\mathbb{E}}_{(x,y)\sim\nu} \sum_{a,b} \left| C_{x,y,a,b} - \int_0^\infty C^\lambda_{x,y,a,b} d\mu(\lambda)\right| \leq \mathrm{poly}(\delta).$$

(e) *There exist PME strategies $\mathscr{S}'_\lambda = (|\psi_\lambda\rangle, A^\lambda)$ such that (c) and (d) hold with $\mathscr{S}_\lambda$, $P_\lambda A_a^x P_\lambda$ and $\sqrt{2\delta}$ replaced by $\mathscr{S}'_\lambda$, $A_a^{\lambda,x}$ and $\mathrm{poly}(\delta)$, respectively.*

Remark 4.2.13. The main omission from the results in [Vid22, Section 3] is that similar results with worse dependence on δ hold for non-projective strategies $\mathscr{S}$. Furthermore, one should note that this theorem can also be applied to the B-side of the strategy $\mathscr{S}$.

Remark 4.2.14. Note that

$$\frac{1}{\mathrm{Tr}(P_\lambda)} \| A_a^x P_\lambda - P_\lambda A_a^x \|_2^2 = 2\,\mathrm{Tr}\left(\left(A_a^x - P_\lambda A_a^x P_\lambda\right)^2 \rho_\lambda\right),$$

so the above theorem also shows that

$$\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \int_0^\infty \frac{1}{\mathrm{Tr}(P_\lambda)} \| A_a^x P_\lambda - P_\lambda A_a^x \|_2^2 d\mu(\lambda) \leq 2\sqrt{2\delta},$$

which is the form we will use in our proofs.

4.2.3. QUANTUM LOW DEGREE TEST

A kind of strategy that we would like to self-test in particular is strategies that use the spectral projections of elements of the Pauli group and the maximally entangled state on the corresponding number of qubits. After making this statement more precise, we present the recently discovered Quantum Low Degree Test, which is a PME-robust self-test.

Let $k \in \mathbb{N}$ and let $X, Z \in M_2(\mathbb{C})$ be the corresponding Pauli matrices. For a prime power q we let $\mathbb{F}_q$ denote the finite field with q elements. For an $\mathbf{a} \in \mathbb{F}_2^k$ we denote the operator $\bigotimes_{j=1}^k X^{a_j}$ by $\sigma^X(\mathbf{a})$ and $\bigotimes_{j=1}^k Z^{a_j}$ by $\sigma^Z(\mathbf{a})$. The Pauli group on k qubits, P_k, is then given by

$$P_k = \{(-1)^a \sigma^X(\mathbf{b})\sigma^Z(\mathbf{c}) \mid a \in \mathbb{F}_2, \mathbf{b}, \mathbf{c} \in \mathbb{F}_2^k\}.$$

Informally, having access to P_k means that you have access to k qubits, since it is a basis for $B((\mathbb{C}^2)^{\otimes k})$ [CRSV17], so a qubit test should in some sense verify that you have access to the Pauli group. In the non-local game setting, a qubit test verifies that both Alice and Bob have access to k qubits, and that those qubits are maximally entangled. The definition was introduced in [CVY23], where they only consider PME strategies. Here, we will include the class of strategies in the definition.

Definition 4.2.15. Let $k \in \mathbb{N}, \kappa : [0,1] \to \mathbb{R}_+$ and $\mathscr{C}$ be a class of strategies. A $(k,\kappa)-\mathscr{C}-$qubit test is a synchronous game $G = (\mathscr{X}, \nu, \mathscr{A}, D)$ such that there exist

- two sets $S_X,\ S_Z \subset \mathbb{F}_2^k$ that each span $\mathbb{F}_2^k$,
- an injection $\varphi : (\{X\} \times S_X) \cup (\{Z\} \times S_Z) \to \mathscr{X}$ such that $\mathscr{A}(\varphi(X, \mathbf{a})) = \mathscr{A}(\varphi(Z, \mathbf{b})) = \mathbb{F}_2$ for all $\mathbf{a} \in S_X,\ \mathbf{b} \in S_Z$,

and the following hold:

- (Completeness:) There exists a strategy $\tilde{\mathscr{S}} = (|\tilde{\psi}\rangle, \tilde{A}, \tilde{B})$ for G on $(\mathbb{C}^{2^k} \otimes \mathscr{H}_A) \otimes (\mathbb{C}^{2^k} \otimes \mathscr{H}_B)$ for some Hilbert spaces $\mathscr{H}_A$ and $\mathscr{H}_B$ such that $\omega(\tilde{\mathscr{S}}) = 1$, $U(\tilde{A}^{\varphi(W,\mathbf{a})}) = \sigma^W(\mathbf{a})$ for every $W \in \{X, Z\}$ and $\mathbf{a} \in S_W$ and $\mathrm{Tr}_{\mathscr{H}_A, \mathscr{H}_B}(|\tilde{\psi}\rangle\langle\tilde{\psi}|)$ is maximally entangled on $\mathbb{C}^{2^k} \otimes \mathbb{C}^{2^k}$.
- (Soundness:) Let ν' be the renormalised restriction of ν on the image of φ. For any strategy $\mathscr{S} = (|\psi\rangle, A, B)$ in the class $\mathscr{C}$ with $\omega(\mathscr{S}) = 1 - \varepsilon$ for some $\varepsilon \geq 0$, we have that $\tilde{\mathscr{S}}$ is a local (ε, ν')-dilation of $\mathscr{S}$. In other words, G (κ, ν')-robustly self-tests $\tilde{\mathscr{S}}$ for the class $\mathscr{C}$.

We now turn towards the Quantum Low Degree Test, which has been introduced in [CVY23]. This test is based on linear codes, so we will first briefly introduce them. More details can be found in [CVY23].

Let $n,\ k,\ d \in \mathbb{N}$ and q be a prime power. An $[n,k,d]_q$-linear code $C \subset \mathbb{F}_q^n$ is a k-dimensional subspace such that for all $x \neq 0$ the number of non-zero elements of x (called the Hamming weight) is at least d, the distance of the code. We call d/n the relative distance. A parity check matrix for a code C is a matrix $h \in \mathbb{F}_q^{m\times n}$ such that $\ker(h) = C$. It is an r-local tester if the Hamming weight of each row is at most r. A generating matrix for a code C is a matrix $E \in \mathbb{F}_q^{n\times k}$ such that the rows of E form a basis for C. One example of a code is the binary version of the Reed-Muller code C_{RM2}, which is an $[2^{t(m+1)}, t(d+1)^m, D]_2$-linear code with $D \geq \frac{1}{2}(1 - \frac{md}{2^t})2^{t(m+1)}$ and an $(d+2)$-local tester with $2^{t(m+2)}(1+m)$ rows [CVY23].

From the code C_{RM2} and a generating matrix E for this code one can construct the Quantum Low Degree Test $G_{\mathrm{qldt}} = (\mathscr{X}_{\mathrm{qldt}}, \nu_{\mathrm{qldt}}, \mathscr{A}_{\mathrm{qldt}}, D_{\mathrm{qldt}})$, for which we refer to [CVY23]. This non-local game is a (k,κ)-$\mathscr{C}_{PME}$-qubit test with $\kappa(\varepsilon) = \mathrm{poly}(m,d,t) \cdot \mathrm{poly}(\varepsilon, 2^{-t})$ and $k = t(d+1)^m$. One can choose $m,\ d$ and t such that the non-local game has $2^{\mathrm{poly}(\log(k))}$ questions and $\kappa(\varepsilon) = \mathrm{poly}(\log(k)) \cdot \mathrm{poly}(\varepsilon)$. As G_{qldt} is a PME-qubit test, there exists a renormalised restriction of ν_{qldt} on the image of φ as in Definition 4.2.15, which we denote by ν'_{qldt}. The main properties we need is that this game is symmetric, that $S_X = S_Z = \{(E_{ij})_{i=1}^k \mid 1 \leq j \leq n\}$ where $n = 2^{t(m+1)}$, the length of the code C_{RM2} and that the ideal strategy does not use auxiliary Hilbert spaces.

4.3. SELF-TESTING IN THE VON NEUMANN ALGEBRA PICTURE

For a large part of our analysis it is necessary to work with the measurement operators of a single party and with the reduced density matrix of the state for that party. In this case, Definition 4.2.1 is not a convenient way to view local dilations. Instead, we would like a formulation in terms of the measurement operators on Alice's side and the reduced state for Alice. This is not possible for all strategies, but it is possible for strategies that are classical convex combinations of ME strategies, i.e. the strategies that can be described as a triple $(\mathscr{M}, \tau, A)$ for a tracial von Neumann algebra $(\mathscr{M}, \tau)$ and a family of POVMs A in $\mathscr{M}$.

Definition 4.3.1. Given two strategies $\tilde{\mathscr{S}} = (\mathscr{M}, \tau^{\mathscr{M}}, \tilde{A})$ and $\mathscr{S} = (\mathscr{N}, \tau^{\mathscr{N}}, A)$, a distribution ν on $\mathscr{X}$, and an $\varepsilon \geq 0$, we say that $\tilde{\mathscr{S}}$ is a local (ε, ν)-vNA-dilation of $\mathscr{S}$ if the following statements hold. There exist a finite-dimensional von Neumann algebra $\mathscr{M}_0$ with tracial state $\tau^{\mathscr{M}_0}$, a projection $P \in (\mathscr{M} \otimes \mathscr{M}_0)^\infty$ of finite trace such that $\mathscr{N} \cong P(\mathscr{M} \otimes \mathscr{M}_0)^\infty P$ and $\tau^{\mathscr{N}} = \tau^\infty / \tau^\infty(P)$, and a partial isometry $W \in P(\mathscr{M} \otimes \mathscr{M}_0)^\infty I_{\mathscr{M} \otimes \mathscr{M}_0}$ such that

$$\mathop{\mathbb{E}}_{x\sim\nu} \sum_a \| \tilde{A}^x_a \otimes I_{\mathscr{M}_0} - W^* A^x_a W \|^2_{\tau^{\mathscr{M}\otimes\mathscr{M}_0}} \leq \varepsilon \tag{4.10}$$

and

$$\tau^{\mathscr{N}}(P - WW^*) \leq \varepsilon, \tau^{\mathscr{M}\otimes\mathscr{M}_0}(I_{\mathscr{M}\otimes\mathscr{M}_0} - W^* W) \leq \varepsilon. \tag{4.11}$$

The above definition is modelled on the notions of closeness of strategies and soundness of a qubit test in [CVY23, Definition 5.3 and 5.6] in the sense that soundness in the qubit test implies that the corresponding strategies are approximate local vNA-dilations. The following lemma shows that for ME strategies, there is a square dependence between the notions of local dilation and local vNA-dilation, with constants that are independent of the question and answer sizes.

Lemma 4.3.2. *Let $G = (\mathscr{X}, \nu, \mathscr{A}, D)$ be a symmetric game, $\hat{\nu}$ a symmetric probability distribution on $\mathscr{X} \times \mathscr{X}$ with marginal distribution $\hat{\nu}_A$ on $\mathscr{X}$, and let $\tilde{\mathscr{S}} = (\mathscr{M}, \tau^{\mathscr{M}}, \tilde{A})$ and $\mathscr{S} = (\mathscr{N}, \tau^{\mathscr{N}}, A)$ be two ME strategies for G, where $\mathscr{M} = B(\tilde{\mathscr{H}})$ and $\mathscr{N} = B(\mathscr{H})$ for some finite-dimensional spaces $\mathscr{H}$ and $\tilde{\mathscr{H}}$.*

(a) If $\tilde{\mathscr{S}}$ is a local $(\varepsilon, \hat{\nu})$-dilation of $\mathscr{S}$, then $\tilde{\mathscr{S}}$ is a local $(1700\varepsilon^2, \hat{\nu}_A)$-vNA-dilation of $\mathscr{S}$.

(b) If $\tilde{\mathscr{S}}$ is an local $(\varepsilon, \hat{\nu}_A)$-vNA-dilation of $\mathscr{S}$, then $\tilde{\mathscr{S}}$ is a local $((4+\sqrt{2})\sqrt{\varepsilon}, \hat{\nu})$-dilation of $\mathscr{S}$.

Remark 4.3.3. We prove this lemma in the appendix because it takes quite some effort. This is mainly due to the fact that an auxiliary state in a local dilation does not have to be maximally entangled, even if both $\mathscr{S}$ and $\tilde{\mathscr{S}}$ are ME strategies. As we want to obtain a tracial state on $\mathscr{M} \otimes \mathscr{M}_0$, this is a problem. We get around this by proving that the auxiliary state can always be taken to be maximally entangled (see Lemmas 4.A.2 and 4.A.6), but this comes at a price in terms of the constant for the approximate local dilation, which is the reason why the constant 1700 appears in the theorem. It is good to note that the auxiliary state is not required to be fully maximally entangled if the auxiliary von Neumann algebra $\mathscr{M}_0$ is not a factor, i.e. $B(\mathscr{H}_0)$ for some Hilbert space $\mathscr{H}_0$. Unfortunately, it is unclear how this freedom can be exploited.

We now formulate the first lifting result for robust self-testing, where we show that we can use robust self-testing for the class of PME strategies to say something about almost synchronous ME strategies as well. Note that the most natural case is when $\hat{\nu} = \nu$, but to apply this result to the Quantum Low Degree Test, we need this more general version.

Lemma 4.3.4. *Let $G = (\mathscr{X}, \nu, \mathscr{A}, \lambda)$ be a symmetric game with the marginal distribution ν_A on $\mathscr{X}$ and let $\hat{\nu}$ be a symmetric probability distribution on $\mathscr{X} \times \mathscr{X}$ and $c \geq 1$ such that $\hat{\nu}_A \leq c\nu_A$, where $\hat{\nu}_A$ is the marginal distribution of $\hat{\nu}$ on $\mathscr{X}$. Suppose $\tilde{\mathscr{S}} = (\mathscr{M}, \tau^{\mathscr{M}}, \tilde{A})$ is a PME strategy that is optimal for G and G $(\kappa, \hat{\nu})$-PME-robustly self-tests $\tilde{\mathscr{S}}$. Then for any ME strategy $\mathscr{S} = (\mathscr{N}, \tau^{\mathscr{N}}, A)$ with $\delta = \delta_{\text{sync}}(\mathscr{S}, \nu_A)$ and $\omega(G;\mathscr{S}) \geq \omega(G, \tilde{S}) - \varepsilon$, $\tilde{\mathscr{S}}$ is a $(\kappa'(\varepsilon, \delta), \hat{\nu})$-local dilation of $\mathscr{S}$, where $\kappa'(\varepsilon, \delta) = \kappa\left(24\sqrt{\delta} + \varepsilon\right) + 9c\delta$.*

Proof. Let $\delta_x := 1 - \sum_a \tau^{\mathscr{N}}\left((A_a^x)^2\right)$ for all $x \in \mathscr{X}$. Then the asynchronicity of $\mathscr{S}$ is $\delta := \delta_{\text{sync}}(\mathscr{S}; \nu_A) = \mathbb{E}_{x\sim\nu_A} \delta_x$. By [dlS22a, Theorem 1.2], there is a family of PVMs $P = \{P_a^x \mid a \in \mathscr{A}\}_{x\in\mathscr{X}}$ in $\mathscr{N}$ such that

$$\sum_a \| A_a^x - P_a^x \|_{\tau^{\mathscr{N}}}^2 \leq 9\delta_x$$

for all $x \in \mathscr{X}$. So

$$\gamma := \mathbb{E}_{x\sim\nu_A} \sum_a \| A_a^x - P_a^x \|_{\tau^{\mathscr{N}}}^2 \leq 9\delta. \tag{4.12}$$

Consider the PME strategy $\mathscr{S}' := (\mathscr{N}, \tau^{\mathscr{N}}, P)$. Since $\delta_{\text{sync}}(\mathscr{S}', \nu_A) = 0$ and both $\mathscr{S}$ and $\mathscr{S}'$ are symmetric, by Lemmas 4.2.8 and 4.2.10,

$$|\omega(G;\mathscr{S}) - \omega(G;\mathscr{S}')| \leq 8\sqrt{\gamma} = 24\sqrt{\delta}.$$

Hence, $\omega(G;\mathscr{S}') \geq \omega(G;\tilde{\mathscr{S}}) - \varepsilon - 24\sqrt{\delta}$. Let $\lambda = \varepsilon + 24\sqrt{\delta}$. Let $|\psi\rangle \in \mathscr{H} \otimes \mathscr{H}$ and $|\tilde{\psi}\rangle \in \tilde{\mathscr{H}} \otimes \tilde{\mathscr{H}}$ be GNS states for $(\mathscr{N}, \tau^{\mathscr{N}})$ and $(\mathscr{M}, \tau^{\mathscr{M}})$, respectively. Since G $(\kappa, \hat{\nu})$-PME robustly self-tests $\tilde{\mathscr{S}}$, there are isometries $V_A : \mathscr{H} \to \tilde{\mathscr{H}} \otimes \mathscr{K}_A$ and $V_B : \mathscr{H} \to \tilde{\mathscr{H}} \otimes \mathscr{K}_B$ and a unit vector $|aux\rangle \in \mathscr{K}_A \otimes \mathscr{K}_B$ such that

$$\|(V_A \otimes V_B)|\psi\rangle - |\tilde{\psi}\rangle \otimes |aux\rangle\| \leq \kappa(\lambda),$$

$$\left(\mathbb{E}_{x\sim\hat{\nu}_A} \sum_a \|(V_A \otimes V_B)(P_a^x \otimes 1)|\psi\rangle - \left((\tilde{A}_a^x \otimes 1)|\tilde{\psi}\rangle\right) \otimes |aux\rangle\|^2\right)^{1/2} \leq \kappa(\lambda),$$

$$\left(\mathbb{E}_{y\sim\hat{\nu}_A} \sum_b \|(V_A \otimes V_B)(1 \otimes (P_b^y)^T)|\psi\rangle - \left((1 \otimes (\tilde{A}_b^y)^T)|\tilde{\psi}\rangle\right) \otimes |aux\rangle\|^2\right)^{1/2} \leq \kappa(\lambda).$$

Since $V_A \otimes V_B$ is an isometry, together with Equation (4.12) and the fact that $\hat{\nu}_A \leq c\nu_A$, we have

$$\left(\mathbb{E}_{x\sim\hat{\nu}_A} \sum_a \|(V_A \otimes V_B)(A_a^x \otimes 1)|\psi\rangle - \left((\tilde{A}_a^x \otimes 1)|\tilde{\psi}\rangle\right) \otimes |aux\rangle\|^2\right)^{1/2} \leq \kappa(\lambda) + 9c\delta,$$

and by symmetry

$$\left(\mathbb{E}_{y\sim\hat{\nu}_A} \sum_b \|(V_A \otimes V_B)(1 \otimes (A_b^y)^T)|\psi\rangle - \left((1 \otimes (\tilde{A}_b^y)^T)|\tilde{\psi}\rangle\right) \otimes |aux\rangle\|^2\right)^{1/2} \leq \kappa(\lambda) + 9c\delta.$$

As $\lambda = 24\sqrt{\delta} + \varepsilon$, we conclude that $\tilde{\mathscr{S}}$ is a $\left(\kappa\left(24\sqrt{\delta} + \varepsilon\right) + 9c\delta, \hat{\nu}\right)$-local dilation of $\mathscr{S}$. □

The next lemma shows that local (ε,ν)-vNA-dilations can equivalently be defined in terms of the distance between the measurement operators in $\mathcal{N}$ instead of in $\mathcal{M}\otimes\mathcal{M}_0$.

Lemma 4.3.5. *Let $\tilde{\mathcal{S}} = (\mathcal{M},\tau^{\mathcal{M}},\tilde{A})$ and $\mathcal{S} = (\mathcal{N},\tau^{\mathcal{N}},A)$ be strategies and $\varepsilon \geq 0$. Suppose that there exist a finite-dimensional von Neumann algebra $\mathcal{M}_0$ with tracial state $\tau^{\mathcal{M}_0}$, a projection $P \in (\mathcal{M}\otimes\mathcal{M}_0)^\infty$ of finite trace such that $\mathcal{N} \cong P(\mathcal{M}\otimes\mathcal{M}_0)^\infty P$ and $\tau^{\mathcal{N}} = \tau^\infty/\tau^\infty(P)$, and a partial isometry $W \in P(\mathcal{M}\otimes\mathcal{M}_0)^\infty I_{\mathcal{M}\otimes\mathcal{M}_0}$ such that*

$$\tau^{\mathcal{N}}(P-WW^*) \leq \varepsilon, \tau^{\mathcal{M}\otimes\mathcal{M}_0}(I_{\mathcal{M}\otimes\mathcal{M}_0} - W^*W) \leq \varepsilon.$$

Then for all $x \in \mathcal{X}$ we have

$$\sum_a \|\tilde{A}^x_a \otimes I_{\mathcal{M}_0} - W^* A^x_a W\|^2_{\tau^{\mathcal{M}\otimes\mathcal{M}_0}} \leq 2\varepsilon + \frac{1}{1-\varepsilon}\sum_a \|W(\tilde{A}^x_a\otimes I_{\mathcal{M}_0})W^* - A^x_a\|^2_{\tau^{\mathcal{N}}} \text{ and}$$
$$\sum_a \|W(\tilde{A}^x_a\otimes I_{\mathcal{M}_0})W^* - A^x_a\|^2_{\tau^{\mathcal{N}}} \leq \frac{2\varepsilon}{1-\varepsilon} + \frac{1}{1-\varepsilon}\sum_a \|\tilde{A}^x_a \otimes I_{\mathcal{M}_0} - W^* A^x_a W\|^2_{\tau^{\mathcal{M}\otimes\mathcal{M}_0}}.$$

Proof. Let $x \in \mathcal{X}$ be arbitrary. Let us identify $\tilde{A}^x_a \otimes 1$ and $\tilde{A}^x_a$. Note that

$$\begin{aligned}\left|\sum_a \tau^\infty\left(W^*W\tilde{A}^x_a W^*W\tilde{A}^x_a - (\tilde{A}^x_a)^2\right)\right| \leq& \left|\sum_a \tau^\infty\left(W^*W\tilde{A}^x_a W^*W\tilde{A}^x_a - \tilde{A}^x_a W^*W\tilde{A}^x_a\right)\right| \\ &+ \left|\sum_a \tau^\infty\left(\tilde{A}^x_a W^*W\tilde{A}^x_a - (\tilde{A}^x_a)^2\right)\right| \\ \leq& \tau^\infty(I_{\mathcal{M}\otimes\mathcal{M}_0} - W^*W)\|\sum_a (\tilde{A}^x_a)^2\|_\infty \\ &+ \tau^\infty(I_{\mathcal{M}\otimes\mathcal{M}_0} - W^*W)\|\sum_a \tilde{A}^x_a W^*W\tilde{A}^x_a\|_\infty \\ \leq& 2\varepsilon.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\sum_a \tau^{\mathcal{M}\otimes\mathcal{M}_0}\left((\tilde{A}^x_a - W^*A^x_aW)^2\right) =& \sum_a \tau^\infty\Big((\tilde{A}^x_a)^2 + WW^*A^x_aWW^*A^x_a \\ &- \tilde{A}^x_a W^*A^x_aW - W^*A^x_aW\tilde{A}^x_a\Big) \\ \leq& 2\varepsilon + \sum_a \tau^\infty\Big(W^*W\tilde{A}^x_aW^*W\tilde{A}^x_a + (A^x_a)^2 - 2\tilde{A}^x_aW^*A^x_aW\Big) \\ =& 2\varepsilon + \sum_a \tau^\infty(P)\tau^{\mathcal{N}}\left((W\tilde{A}^x_aW^* - A^x_a)^2\right)\end{aligned}$$

and analogously

$$\tau^{\mathcal{N}}\left((W\tilde{A}^x_aW^* - A^x_a)^2\right) \leq \frac{2\varepsilon}{\tau^\infty(P)} + \frac{1}{\tau^\infty(P)}\tau^{\mathcal{M}\otimes\mathcal{M}_0}\left((\tilde{A}^x_a - W^*A^x_aW)^2\right).$$

Since $1-\varepsilon \leq \tau^\infty(P) \leq (1-\varepsilon)^{-1}$ by Lemma 4.A.7, we have shown the desired result. □

4.4. ROBUST SELF-TESTING BASED ON THE SPECTRAL GAP OF THE GAME POLYNOMIAL

In this section we show that a PME-robust self-test whose game polynomial has spectral gap is automatically a robust self-test. The proof consists of three steps. First, we show that any almost perfect strategy $\mathscr{S}$ for a PME-robust self-test can be approximately decomposed into 'orthogonal' symmetric maximally entangled strategies $\mathscr{S}_i$ with high winning probability. Next, we use the PME-robust self-testing properties of the game to find partial isometries from $\mathscr{S}_i$ to the ideal strategy and combine and extend them to obtain isometries from $\mathscr{S}$ to the ideal strategy. The spectral gap of the game polynomial then allows us to conclude that these isometries form a local dilation.

In the first step of the proof, we will use [Vid22, Theorem 3.1], presented as Theorem 4.2.12 in this chapter, and intermediate results of its proof to construct the decomposition into the strategies $\mathscr{S}_i$. One of these intermediate results provides a bound on the commutator of the measurement operators of $\mathscr{S}$ and the spectral projections of its reduced density matrix. To make proper use of this, we need the following lemma, which records its consequences.

Lemma 4.4.1. *Let $\{A_a^x\}$ be a collection of PVMs on $\mathscr{H}_A$ and ν a symmetric probability measure on $\mathscr{X}\times\mathscr{X}$ with marginal ν_A on $\mathscr{X}$. Let $\gamma \geq 0$ and let R be a projection on $\mathscr{H}_A$ such that*

$$\mathop{\mathbb{E}}_{x\sim\nu_A}\sum_a \frac{1}{\mathrm{Tr}(R)}\|A_a^xR - RA_a^x\|_2^2 \leq \gamma.$$

If $|\psi\rangle \in (RH_A)\otimes(RH_A)$ is a maximally entangled state, then

$$\delta_{\mathrm{sync}}(\mathscr{S},\nu_A) = \mathop{\mathbb{E}}_{x\sim\nu_A}\sum_a \frac{1}{\mathrm{Tr}(R)}\left|\mathrm{Tr}(RA_a^x) - \mathrm{Tr}(RA_a^xRA_a^x)\right| \leq \frac{1}{2}\gamma$$

for the ME strategy $\mathscr{S} = (|\psi\rangle, \{RA_a^xR\})$.

Proof. Observe that

$$\mathrm{Tr}(A_a^xR - A_a^xRA_a^xR) = \frac{1}{2}\|A_a^xR - RA_a^x\|_2^2. \tag{4.13}$$

Since

$$\delta_{\mathrm{sync}}(\mathscr{S},\nu_A) = 1 - \mathop{\mathbb{E}}_{x\sim\nu_A}\sum_a \langle\psi|\, A_a^x\otimes(A_a^x)^T\,|\psi\rangle = \mathop{\mathbb{E}}_{x\sim\nu_A}\sum_a \frac{1}{\mathrm{Tr}(R)}\left(\mathrm{Tr}(RA_a^x) - \mathrm{Tr}(RA_a^xRA_a^x)\right)$$

by Ando's formula, we immediately see from Equation (4.13) that $\delta_{\mathrm{sync}}(\mathscr{S},\nu_A) \leq \frac{1}{2}\gamma$. □

We will now prove the main theorem of this section. Because the proof requires many computations, we will include several claims in the proof to guide the reader. They serve as announcements of the next step in the proof and indicate the general flow of the argument. For the remainder of this section and the next, we will be working with two probability distributions on $\mathscr{X}\times\mathscr{X}$; one determining the winning probability and one for the local dilations. The most natural situation is when both distributions are identical, but the Quantum Low Degree Test requires us to treat the case with distinct distributions.

Theorem 4.4.2. *Let $\mathscr{S} = (|\psi\rangle, A, B)$ be a strategy for a perfect symmetric non-local game $G = (\mathscr{X}, \nu, \mathscr{A}, D)$ with winning probability $1-\varepsilon$. Let ρ_A and ρ_B be the reduced density matrices on $\mathscr{H}_A$ and $\mathscr{H}_B$, respectively, ν_A the marginal of ν on $\mathscr{X}$ and $\delta = \delta_{\text{sync}}(\mathscr{S}, \nu_A)$. Let $\hat{\nu}$ be a symmetric probability distribution on $\mathscr{X}\times\mathscr{X}$ with marginal $\hat{\nu}_A$ on $\mathscr{X}$ and $c \geq 1$ such that $\hat{\nu} \leq c\nu$. Suppose that G $(\kappa,\hat{\nu})$-PME-robustly self-tests the optimal strategy $\tilde{\mathscr{S}} = (|\tilde{\psi}\rangle, \tilde{A}, \tilde{B})$ on $\tilde{\mathscr{H}}_A \otimes \tilde{\mathscr{H}}_B$. Then there exist Hilbert spaces $\mathscr{K}_A$ and $\mathscr{K}_B$ and isometries $V_A : \mathscr{H}_A \to \tilde{\mathscr{H}}_A \otimes \mathscr{K}_A$ and $V_B : \mathscr{H}_B \to \tilde{\mathscr{H}}_B \otimes \mathscr{K}_B$ such that for the game $G_{\hat{\nu}} = (\mathscr{X}, \hat{\nu}, \mathscr{A}, D)$ the strategy $\hat{\mathscr{S}} = (|\psi\rangle, V_A^*(\tilde{A}\otimes 1_{\mathscr{K}_A})V_A, V_B^*(\tilde{B}\otimes 1_{\mathscr{K}_B})V_B)$ has winning probability $\omega(G_{\hat{\nu}};\hat{\mathscr{S}}) \geq 1-\mathscr{O}\big((c\cdot\text{id}+\kappa)(2\varepsilon+\text{poly}(\delta))\big)$, and*

$$\mathop{\mathbb{E}}_{x\sim\hat{\nu}_A}\sum_a \text{Tr}\left(\left(A_a^x - V_A^*(\tilde{A}_a^x\otimes 1_{\mathscr{K}_A})V_A\right)^2\rho_A\right) = \mathscr{O}\left((c\cdot\text{id}+\kappa)^2(\text{poly}(\delta)+2\varepsilon)\right),$$
$$\mathop{\mathbb{E}}_{y\sim\hat{\nu}_A}\sum_b \text{Tr}\left(\left(B_b^y - V_B^*(\tilde{B}_a^x\otimes 1_{\mathscr{K}_B})V_B\right)^2\rho_B\right) = \mathscr{O}\left((c\cdot\text{id}+\kappa)^2(\text{poly}(\delta)+2\varepsilon)\right).$$

Proof. Let $\mathscr{S}' = (|\psi\rangle, A', B')$ be the projective strategy given by Lemma 4.2.9 and let $\delta' = \delta_{\text{sync}}(\mathscr{S}', \nu_A)$. By Lemma 4.2.9, we know that $\delta' = \mathscr{O}(\delta^{\frac{1}{8}})$. Let the probability measure μ on $\mathbb{R}_+$, the projections P_λ of $\mathscr{H}_A$ onto $\mathscr{H}_\lambda$ and the family of strategies $\mathscr{S}_\lambda = (|\psi_\lambda\rangle, P_\lambda A P_\lambda)$ on $\mathscr{H}_\lambda$ with reduced density matrix ρ_λ be as in Theorem 4.2.12. Let C, C' and C^λ be the correlations of $\mathscr{S}$, $\mathscr{S}'$ and $\mathscr{S}_\lambda$, respectively. Let

$$\mathop{\mathbb{E}}_{x,y\sim\nu}\sum_{a,b}|C_{x,y,a,b} - \int_0^\infty C^\lambda_{x,y,a,b}d\mu(\lambda)| = \alpha$$

and

$$\mathop{\mathbb{E}}_{x\sim\nu_A}\sum_a\int_0^\infty \frac{1}{\text{Tr}(P_\lambda)}\|A_a'^x P_\lambda - P_\lambda A_a'^x\|_2^2 d\mu(\lambda) = \beta.$$

Claim 4.4.3. *The set $\Lambda \subset \mathbb{R}_+$, defined by*

$$\Lambda = \{\lambda\geq 0 | \omega(\mathscr{S}_\lambda) \geq 1-\sqrt{\alpha}-\varepsilon \textit{ and } \mathop{\mathbb{E}}_{x\sim\nu_A}\sum_a \frac{1}{\text{Tr}(P_\lambda)}\|A_a'^x P_\lambda - P_\lambda A_a'^x\|_2^2 \leq \sqrt{\beta}\},$$

satisfies $\mu(\Lambda) \geq 1-\sqrt{\alpha}-\sqrt{\beta}$.

By Theorem 4.2.12(d), Lemma 4.2.9, Lemma 4.2.10 and the triangle inequality, we know that $\alpha \leq \text{poly}(\delta') + \mathscr{O}(\delta^{\frac{1}{8}}) \leq \text{poly}(\delta)$. Lemma 4.2.10 tells us that for any $t > 0$ we have that

$$\mu(\{\lambda \geq 0 \mid \omega(\mathscr{S}_\lambda) \geq 1-\varepsilon-t\}) \geq 1-\frac{\alpha}{t}.$$

Here we have used the observation that

$$t\mu(\{\lambda \mid f(\lambda) \geq s\}) \leq \int_0^\infty f(\lambda)d\mu(\lambda). \tag{4.14}$$

Next, Theorem 4.2.12, Remark 4.2.14 and Lemma 4.2.9 state that $\beta \leq 2\sqrt{2\delta'} \leq \text{poly}(\delta)$, and we also have for every $s > 0$ that

$$\mu(\{\lambda\geq 0|\mathop{\mathbb{E}}_{x\sim\nu_A}\sum_a\frac{1}{\text{Tr}(P_\lambda)}\|A_a'^x P_\lambda - P_\lambda A_a'^x\|_2^2 \leq s\}) \geq 1-\frac{\beta}{s}$$

by Equation (4.14). Choosing s and t is a trade-off between the strength of the bound and the measure of the set for which the bound holds. We choose $s = \sqrt{\beta}$ and $t = \sqrt{\alpha}$, but many choices are possible here. If we define

$$\Lambda = \{\lambda \geq 0 | \omega(\mathscr{S}_\lambda) \geq 1 - \sqrt{\alpha} - \varepsilon \text{ and } \mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \frac{1}{\operatorname{Tr}(P_\lambda)} \| A'^x_a P_\lambda - P_\lambda A'^x_a \|_2^2 \leq \sqrt{\beta}\},$$

we find that $\mu(\Lambda) \geq 1 - \sqrt{\alpha} - \sqrt{\beta}$. From now on we will purely work with $\mathscr{S}_\lambda$ for $\lambda \in \Lambda$. For $\lambda \in \mathbb{R}_+ \setminus \Lambda$ the crudest estimates will suffice.

This proves Claim 4.4.3.

The strategies $\mathscr{S}_\lambda$ for $\lambda \in \Lambda$ give an approximate decomposition of the strategy $\mathscr{S}$. However, this is not the right decomposition for us when we want to construct isometries. In essence, the current decomposition is a decomposition into smaller and smaller strategies. What we need is an approximate decomposition into 'orthogonal' strategies, so our next aim is to construct this. Note that the projections P_λ are ordered; $\lambda \leq \lambda'$ implies that $P_\lambda \geq P_{\lambda'}$, so the dimensions of the Hilbert spaces $\mathscr{H}_\lambda$ are also ordered. We recursively define a partition $\{\Lambda_i\}_{i=1}^k$ of Λ by setting $\lambda_1 = \min(\Lambda)$,

$$\Lambda_i = \{\lambda \in \Lambda \mid \dim \mathscr{H}_{\lambda_i} \geq \dim \mathscr{H}_\lambda > \frac{1}{2} \dim \mathscr{H}_{\lambda_i}\} \text{ and } \lambda_{i+1} = \min(\Lambda \setminus \bigcup_{j=1}^{i} \Lambda_j),$$

where $k \in \mathbb{N}$ is such that $\Lambda = \bigcup_{j=1}^k \Lambda_j$. We now define projections $Q_i = P_{\lambda_i} - P_{\lambda_{i+1}}$, where $1 \leq i \leq k-1$ and $Q_k = P_{\lambda_k}$ and set $\mathscr{H}_i = Q_i \mathscr{H}_A$. Let $|\psi_i\rangle \in \mathscr{H}_i \otimes \mathscr{H}_i$ be a maximally entangled state. We first aim to prove that the strategies $\mathscr{S}_i = (|\psi_i\rangle, Q_i A' Q_i)$ have high winning probability. An intermediate step is to show that Q_i almost commutes with A'^x_a in an averaged sense.

Claim 4.4.4. *For $1 \leq i \leq k$ the Q_i and $\mathscr{S}_i$ constructed above satisfy*

$$\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \frac{1}{\operatorname{Tr}(Q_i)} \| A'^x_a Q_i - Q_i A'^x_a \|_2^2 \leq (\sqrt{2}+1)^2 \sqrt{\beta}$$

and

$$\delta_{\text{sync}}(\mathscr{S}_i, \nu_A) \leq \frac{1}{2}(\sqrt{2}+1)^2 \sqrt{\beta} \text{ and } \omega(\mathscr{S}_i) \geq 1 - 2\varepsilon - 2\sqrt{\alpha} - 6\sqrt{\beta} - 8\sqrt{\frac{5}{2}+\sqrt{2}}\sqrt[4]{\beta}.$$

Let $1 \leq i \leq k-1$. Since $\lambda_i, \lambda_{i+1} \in \Lambda$, we know that

$$\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \frac{1}{\operatorname{Tr}(P_\lambda)} \| A'^x_a P_\lambda - P_\lambda A'^x_a \|_2^2 \leq \sqrt{\beta}$$

for $\lambda \in \{\lambda_i, \lambda_{i+1}\}$. From this we get

$$\begin{aligned} \left(\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \| A'^x_a Q_i - Q_i A'^x_a \|_2^2 \right)^{\frac{1}{2}} &\leq \sqrt{\operatorname{Tr}(P_{\lambda_i})}\sqrt[4]{\beta} + \sqrt{\operatorname{Tr}(P_{\lambda_{i+1}})}\sqrt[4]{\beta} \\ &\leq (\sqrt{2}+1)\sqrt{\operatorname{Tr}(Q_i)}\sqrt[4]{\beta} \end{aligned}$$

by using the triangle inequality and the estimate $\dim \mathcal{H}_{\lambda_i} \geq 2 \dim \mathcal{H}_{\lambda_{i+1}}$ in the first and second inequality, respectively. Taking $\sqrt{\mathrm{Tr}(Q_i)}$ to the other side gives us

$$\left(\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \frac{1}{\mathrm{Tr}(Q_i)} \| A'^x_a Q_i - Q_i A'^x_a \|_2^2 \right)^{\frac{1}{2}} \leq (\sqrt{2}+1) \sqrt[4]{\beta}. \tag{4.15}$$

By Lemma 4.4.1, we find that

$$\delta_{\mathrm{sync}}(\mathcal{S}_i, \nu_A) \leq \frac{(\sqrt{2}+1)^2}{2} \sqrt{\beta}.$$

Using Equation (4.15) and the fact that $\omega(\mathcal{S}_{\lambda_i}), \omega(\mathcal{S}_{\lambda_{i+1}}) \geq 1 - \sqrt{\alpha} - \varepsilon$, we are now ready to show that $\omega(\mathcal{S}_i) \geq 1 - \sqrt{\alpha} - \varepsilon - (1 + \frac{3}{2}\sqrt{2}) \sqrt[4]{\beta}$. We will do this by showing that the winning probability of $\mathcal{S}_{\lambda_i}$ is close to the winning probability of the strategy $(|\psi_{\lambda_i}\rangle, Q_i A Q_i + P_{\lambda_{i+1}} A P_{\lambda_{i+1}})$. For this, we compute

$$\begin{aligned}
&\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \frac{1}{\mathrm{Tr}(P_{\lambda_i})} \mathrm{Tr}\left((P_{\lambda_i} A'^x_a P_{\lambda_i} - Q_i A'^x_a Q_i - P_{\lambda_{i+1}} A'^x_a P_{\lambda_{i+1}})^2 P_{\lambda_i} \right) \\
&\qquad = \mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \frac{1}{\mathrm{Tr}(P_{\lambda_i})} \mathrm{Tr}\left(P_{\lambda_i} A'^x_a P_{\lambda_i} A'^x_a - Q_i A'^x_a Q_i A'^x_a - P_{\lambda_{i+1}} A'^x_a P_{\lambda_{i+1}} A'^x_a \right),
\end{aligned}$$

since Q_i and $P_{\lambda_{i+1}}$ are orthogonal. Using Lemma 4.4.1 for each of the three terms, we find that

$$\begin{aligned}
&\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \frac{1}{\mathrm{Tr}(P_{\lambda_i})} \mathrm{Tr}\left((P_{\lambda_i} A'^x_a P_{\lambda_i} - Q_i A'^x_a Q_i - P_{\lambda_{i+1}} A'^x_a P_{\lambda_{i+1}})^2 P_{\lambda_i} \right) \\
&\qquad \leq (\frac{5}{2} + \sqrt{2}) \sqrt{\beta} + \mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \left| \frac{1}{\mathrm{Tr}(P_{\lambda_i})} \mathrm{Tr}\left(P_{\lambda_i} A'^x_a - Q_i A'^x_a - P_{\lambda_{i+1}} A'^x_a \right) \right| \\
&\qquad = (\frac{5}{2} + \sqrt{2}) \sqrt{\beta}.
\end{aligned}$$

By Lemma 4.2.8 and Lemma 4.2.10, we conclude that

$$|\omega(\mathcal{S}_{\lambda_i}) - \omega((|\psi_{\lambda_i}\rangle, Q_i A Q_i + P_{\lambda_{i+1}} A P_{\lambda_{i+1}}))| \leq 6\sqrt{\beta} + 8\sqrt{\frac{5}{2} + \sqrt{2}} \sqrt[4]{\beta}.$$

As

$$\omega((|\psi_{\lambda_i}\rangle, Q_i A Q_i + P_{\lambda_{i+1}} A P_{\lambda_{i+1}})) = \frac{\mathrm{Tr}(Q_i)}{\mathrm{Tr}(P_{\lambda_i})} \omega(\mathcal{S}_i) + \frac{\mathrm{Tr}(P_{\lambda_{i+1}})}{\mathrm{Tr}(P_{\lambda_i})} \omega(\mathcal{S}_{\lambda_{i+1}}),$$

since it is a classical mixture of strategies, we find that

$$\begin{aligned}
\omega(\mathcal{S}_i) &\geq \frac{\mathrm{Tr}(P_{\lambda_i})}{\mathrm{Tr}(Q_i)} (1 - \varepsilon - \sqrt{\alpha}) - \frac{\mathrm{Tr}(P_{\lambda_{i+1}})}{\mathrm{Tr}(Q_i)} - 6\sqrt{\beta} - 8\sqrt{\frac{5}{2} + \sqrt{2}} \sqrt[4]{\beta} \\
&\geq 1 - 2\varepsilon - 2\sqrt{\alpha} - 6\sqrt{\beta} - 8\sqrt{\frac{5}{2} + \sqrt{2}} \sqrt[4]{\beta}.
\end{aligned}$$

Note that this proof works for $1 \leq i \leq k-1$. However, $Q_k = P_{\lambda_k}$ and $\mathcal{S}_k = \mathcal{S}_{\lambda_k}$, so in that case the result holds immediately since $\lambda_i \in \Lambda$.

This proves Claim 4.4.4.

For convenience, we define the constant

$$\gamma = 2\varepsilon + 2\sqrt{\alpha} + 6\sqrt{\beta} + 8\sqrt{\frac{5}{2} + \sqrt{2}}\sqrt[4]{\beta}$$

going forward.

Having obtained our approximate decomposition into orthogonal strategies that have high winning probability, we are in a position to use the PME-robust self-testing properties of G. Our goal will be to construct a single partial isometry W that works well for each set of measurement operators $P_{\lambda_i} A P_{\lambda_i}$. Let $1 \le i \le k$. Define $\mathscr{H}_i = Q_i \mathscr{H}_A$, consider $(B(\mathscr{H}_i), \mathrm{tr}_i)$, where tr_i is the normalised trace on $B(\mathscr{H}_i)$, and let $\delta_i = \delta_{\mathrm{sync}}(\mathscr{S}_i, \nu_A)$. Since $\tilde{\mathscr{S}}$ and $\mathscr{S}_i$ are ME strategies, we can view them in the von Neumann algebra picture as strategies $(\mathscr{M}, \tau^{\mathscr{M}}, \tilde{A})$ and $(\mathscr{N}_i, \tau^{\mathscr{N}_i}, A_i)$, respectively. Since G is a $(\kappa, \hat{\nu})$-PME-robust self-test for $\tilde{\mathscr{S}}$, we know by Lemmas 4.3.2 and 4.3.4 that $\tilde{\mathscr{S}}$ is a local $(\theta_i, \hat{\nu}_A)$-vNA-dilation of $\mathscr{S}_i$, with

$$\theta_i = 1700\left(\kappa\left(24\sqrt{\delta_i} + (1 - \omega(\mathscr{S}_i))\right) + 9c\delta_i\right)^2.$$

This means that there exists a finite-dimensional von Neumann algebra $\mathscr{M}_i$, a corresponding von Neumann algebra $(\mathscr{M} \otimes \mathscr{M}_i)^\infty$ with trace denoted by τ_i^∞, a projection $R_i \in (\mathscr{M} \otimes \mathscr{M}_i)^\infty$ of finite trace such that $\mathscr{N}_i \cong R_i (\mathscr{M} \otimes \mathscr{M}_i)^\infty R_i$ and $\mathrm{tr}^{\mathscr{N}_i} = \tau_i^\infty / \tau_i^\infty(R_i)$, and a partial isometry $V_i \in R_i (\mathscr{M} \otimes \mathscr{M}_i)^\infty I_{\mathscr{M} \otimes \mathscr{M}_i}$ such that

$$\frac{1}{\mathrm{Tr}(Q_i)} \mathrm{Tr}(Q_i - V_i V_i^*) = \tau^{\mathscr{N}_i}(R_i - V_i V_i^*) \le \theta_i, \tau_i^\infty(I_{\mathscr{M} \otimes \mathscr{M}_i} - V_i^* V_i) \le \theta_i$$

and

$$\begin{aligned} &\mathop{\mathbb{E}}_{x \sim \hat{\nu}_A} \sum_a \frac{1}{\mathrm{Tr}(Q_i)} \mathrm{Tr}\left((Q_i A'^x_a Q_i - V_i \tilde{A}^x_a V_i^*)^2 Q_i\right) \qquad (4.16)\\ &\qquad = \mathop{\mathbb{E}}_{x \sim \hat{\nu}_A} \sum_a \tau^{\mathscr{N}_i}\left(\left((A_i)^x_a - V_i \tilde{A}^x_a V_i^*\right)^2\right) \le 2\theta_i + \frac{\theta_i}{1 - \theta_i}, \end{aligned}$$

by Lemma 4.3.5. Note that we have identified $B(\mathscr{H}_i) = \mathscr{N}_i$ and $R_i (\mathscr{M} \otimes \mathscr{M}_i)^\infty R_i$, leading us to identify Q_i and R_i. We have also identified $\tilde{A}^x_a \in \mathscr{M}$ and $\tilde{A}^x_a \otimes 1_{\mathscr{H}_i} \otimes 1_{B(\ell^2(\mathbb{N}))} \in (\mathscr{M} \otimes \mathscr{M}_i)^\infty$, and we will continue to do so in the rest of this proof. Lastly, remark that the expectations in Equation (4.16) are with respect to $\hat{\nu}_A$ instead of ν_A. From now on, we will mainly need expectations with respect to $\hat{\nu}_A$. We will freely use that we can obtain estimates on expectations with respect to $\hat{\nu}_A$ by multiplying expectations with respect to ν_A by c, as $\hat{\nu}_A \le c\nu_A$.

Define

$$\hat{\mathscr{M}} = \bigoplus_{i=1}^k \mathscr{M}_i, \ \check{\mathscr{M}} = (\mathscr{M} \otimes \hat{\mathscr{M}})^\infty \text{ and } \hat{R} = \sum_{i=1}^k R_i.$$

We observe that $R\check{\mathscr{M}}R \cong B(\mathscr{H}_{\lambda_1})$, and we are now ready to define our desired partial isometry V by

$$V = \sum_{i=1}^k V_i \in R\check{\mathscr{M}} I_{\mathscr{M} \otimes \hat{\mathscr{M}}}.$$

The definition of the λ_i allows us to compute that

$$\frac{1}{\mathrm{Tr}(P_{\lambda_1})}\mathrm{Tr}(P_{\lambda_1}-VV^*)=\frac{1}{\mathrm{Tr}(P_{\lambda_1})}\sum_{i=1}^{k}\mathrm{Tr}(Q_i-V_iV_i^*)\leq\sum_{i=1}^{k}\frac{\mathrm{Tr}(Q_i)}{\mathrm{Tr}(P_{\lambda_1})}\theta_i.$$

Let us define

$$\theta=1700\left(\kappa\left(24(1+\frac{1}{2}\sqrt{2})\sqrt[4]{\beta}+\gamma\right)+\frac{9}{2}(\sqrt{2}+1)^2c\sqrt{\beta}\right)^2,$$

so $\theta_i\leq\theta$ for all i. We also have that $\sum_{i=j}^{k}\mathrm{Tr}(Q_i)=\mathrm{Tr}(P_{\lambda_j})$ for all $1\leq j\leq k$, so we find that

$$\frac{1}{\mathrm{Tr}(P_{\lambda_j})}\mathrm{Tr}(P_{\lambda_j}-VV^*)\leq\theta.$$

If $\lambda\in\Lambda_j$, then it follows from this equation that

$$\frac{1}{\mathrm{Tr}(P_\lambda)}\mathrm{Tr}(P_\lambda-VV^*P_\lambda VV^*)=\frac{1}{\mathrm{Tr}(P_\lambda)}\mathrm{Tr}((P_{\lambda_j}-VV^*)P_\lambda)\leq 2\theta, \tag{4.17}$$

where we use the cyclicity of the trace in the first step and Hölder together with $\mathrm{Tr}(P_{\lambda_j})\leq 2\,\mathrm{Tr}(P_\lambda)$ in the second.

Next, we set out to show a relation between A^λ and $V\tilde{A}V^*$. Our goal is to prove the following claim, which roughly states that V is a good partial isometry on each subspace $\mathcal{H}_\lambda$ with $\lambda\in\Lambda$.

Claim 4.4.5. *The partial isometry V satisfies*

$$\mathop{\mathbb{E}}_{x\sim\hat{\nu}_A}\sum_a\frac{1}{\mathrm{Tr}(P_\lambda)}\mathrm{Tr}\big((A'^x_a-V\tilde{A}^x_aV^*)^2P_\lambda\big)\leq(\sqrt{2}+1)^2c\sqrt{\beta}+4\theta+\frac{2\theta}{1-\theta}.$$

By the way V is constructed, we have

$$V\tilde{A}^x_aV^*Q_i=V_i\tilde{A}^x_aV_i^*Q_i.$$

Let $\lambda\in\Lambda$ and let $1\leq j\leq k$ be such that $\lambda\in\Lambda_j$. By decomposing P_λ into $Q_{j+1},\ldots,Q_k$ and $P_\lambda-P_{\lambda_{j+1}}$, we find that

$$\begin{aligned}\mathop{\mathbb{E}}_{x\sim\hat{\nu}_A}\sum_a\mathrm{Tr}\big((A'^x_a-V\tilde{A}^x_aV^*)^2P_\lambda\big)&=\mathop{\mathbb{E}}_{x\sim\hat{\nu}_A}\sum_a\mathrm{Tr}\Big((A'^x_a-V_j\tilde{A}^x_aV_j^*)^2(P_\lambda-P_{\lambda_{j+1}})\Big)\\&\quad+\mathop{\mathbb{E}}_{x\sim\hat{\nu}_A}\sum_a\sum_{i=j+1}^{k}\mathrm{Tr}\big((A'^x_a-V_i\tilde{A}^x_aV_i^*)^2Q_i\big).\end{aligned}$$

Since $(A'^x_a-V_j\tilde{A}^x_aV_j^*)^2$ is positive and $P_\lambda\leq P_{\lambda_j}$, we have the estimate

$$\mathop{\mathbb{E}}_{x\sim\hat{\nu}_A}\sum_a\mathrm{Tr}\Big((A'^x_a-V_j\tilde{A}^x_aV_j^*)^2(P_\lambda-P_{\lambda_{j+1}})\Big)\leq\mathop{\mathbb{E}}_{x\sim\hat{\nu}_A}\sum_a\mathrm{Tr}\Big((A'^x_a-V_j\tilde{A}^x_aV_j^*)^2Q_j\Big),$$

so

$$\mathop{\mathbb{E}}_{x\sim\hat{\nu}_A}\sum_a\mathrm{Tr}\big((A'^x_a-V\tilde{A}^x_aV^*)^2P_\lambda\big)\leq\mathop{\mathbb{E}}_{x\sim\hat{\nu}_A}\sum_a\sum_{i=j}^{k}\mathrm{Tr}\big((A'^x_a-V_i\tilde{A}^x_aV_i^*)^2Q_i\big).$$

Note that $V_i^* Q_i = V_i^*$. Using this knowledge, expanding the inner term of the right hand side gives

$$\mathrm{Tr}\left((A'^x_a - V_i \tilde{A}^x_a V_i^*)^2 Q_i\right) = \mathrm{Tr}\left(A'^x_a Q_i - A'^x_a V_i \tilde{A}^x_a V_i^* - V_i \tilde{A}^x_a V_i^* A'^x_a + V_i \tilde{A}^x_a V_i^* V_i \tilde{A}^x_a V_i^*\right).$$

Note that

$$\delta_{\mathrm{sync}}(\mathscr{S}_i, \hat{\nu}_A) = \mathop{\mathbb{E}}_{x\sim\hat{\nu}_A} \sum_a \frac{1}{\mathrm{Tr}(Q_i)} \mathrm{Tr}(A'^x_a Q_i - A'^x_a Q_i A'^x_a Q_i).$$

By using Claim 4.4.4 and Lemma 4.4.1, we find

$$\mathop{\mathbb{E}}_{x\sim\hat{\nu}_A} \sum_a \mathrm{Tr}\left((A'^x_a - V_i \tilde{A}^x_a V_i^*)^2 Q_i\right) \le \mathrm{Tr}(Q_i) \frac{(\sqrt{2}+1)^2}{2} c\sqrt{\beta} + \mathop{\mathbb{E}}_{x\sim\hat{\nu}_A} \sum_a \| Q_i A'^x_a Q_i - V_i \tilde{A}^x_a V_i^* \|_2^2.$$

Summing the first term on the right hand side gives

$$\sum_{i=j}^k \mathrm{Tr}(Q_i) \frac{(\sqrt{2}+1)^2}{2} c\sqrt{\beta} = \mathrm{Tr}(P_{\lambda_j}) \frac{(\sqrt{2}+1)^2}{2} c\sqrt{\beta} \le \mathrm{Tr}(P_\lambda)(\sqrt{2}+1)^2 c\sqrt{\beta}.$$

Summing the second term we use Equation (4.16) and an analogous summation estimate to see that

$$\mathop{\mathbb{E}}_{x\sim\hat{\nu}_A} \sum_a \sum_{i=j}^k \| Q_i A'^x_a Q_i - V_i \tilde{A}^x_a V_i^* \|_2^2 \le 2\,\mathrm{Tr}(P_\lambda)\left(2\theta + \frac{\theta}{1-\theta}\right).$$

All in all we conclude that

$$\mathop{\mathbb{E}}_{x\sim\hat{\nu}_A} \sum_a \frac{1}{\mathrm{Tr}(P_\lambda)} \mathrm{Tr}\left((A'^x_a - V \tilde{A}^x_a V^*)^2 P_\lambda\right) \le (\sqrt{2}+1)^2 c\sqrt{\beta} + 4\theta + \frac{2\theta}{1-\theta}$$

by pulling $\mathrm{Tr}(P_\lambda)$ to the other side.

This proves Claim 4.4.5.

We are now ready to finish the proof of the theorem. What is left to do is to pad V by an arbitrary partial isometry which has full support on the kernel of V. This only introduces a small error because $\mu(\mathbb{R}_+ \setminus \Lambda) \le \sqrt{\alpha} + \sqrt{\beta}$ by Claim 4.4.3 and because $\mathrm{Tr}(P_{\lambda_1} - VV^*) \le \theta\,\mathrm{Tr}(P_{\lambda_1})$. Let $P^\perp = 1_{B(\mathscr{H}_A)} - VV^*$, $R^\perp \in \check{\mathscr{M}}$ a projection such that $R^\perp \check{\mathscr{M}} R^\perp \cong P^\perp B(\mathscr{H}_A) P^\perp$ and $\hat{R} R^\perp = 0$. We identify $P^\perp B(\mathscr{H}_A) P^\perp$ and $R^\perp \check{\mathscr{M}} R^\perp$ and let $\mathscr{M}^\perp \in \hat{\mathscr{M}} \otimes B(\ell^2(\mathbb{N}))$ be a von Neumann algebra and $V^\perp \in R^\perp \check{\mathscr{M}} I_{\mathscr{M}\otimes\mathscr{M}^\perp}$ a partial isometry such that $V^\perp (V^\perp)^* = R^\perp$ and $I_{\hat{\mathscr{M}}} \mathscr{M}^\perp I_{\hat{\mathscr{M}}} = \{0\}$. Let $W = V + V^\perp \in \check{\mathscr{M}} I_{\mathscr{M}\otimes(\hat{\mathscr{M}}\oplus\mathscr{M}^\perp)}$. First, observe that

$$\begin{aligned} \|\rho_A - VV^* \rho_A VV^*\|_1 &\le \|\int_{\lambda\in\mathbb{R}_+\setminus\Lambda} \rho_\lambda d\mu(\lambda)\|_1 + \int_{\lambda\in\Lambda} \|\rho_\lambda - VV^*\rho_\lambda VV^*\|_1 d\mu(\lambda) \\ &\le \sqrt{\alpha} + \sqrt{\beta} + 2\theta \end{aligned}$$

by construction of Λ and Equation (4.17), and

$$\|\sum_a (A'^x_a - \tilde{W} \tilde{A}^x_a \tilde{W}^*)^2\|_\infty \le 4$$

by Lemma 4.2.11. This allows us to compute that

$$\begin{aligned}\mathop{\mathbb{E}}_{x\sim\hat\nu_A}\sum_a \operatorname{Tr}\big((A'^x_a-\tilde W\tilde A^x_a\tilde W^*)^2\rho_A\big)\\ &\le 4(\sqrt{\alpha}+\sqrt{\beta}+2\theta)+\mathop{\mathbb{E}}_{x\sim\hat\nu_A}\sum_a\int_{\lambda\in\Lambda}\operatorname{Tr}\big((A'^x_a-\tilde W\tilde A^x_a\tilde W^*)^2VV^*\rho_\lambda VV^*\big)\,d\mu(\lambda)\\ &\le 4(\sqrt{\alpha}+\sqrt{\beta}+2\theta)+(\sqrt2+1)^2c\sqrt{\beta}+4\theta+\frac{2\theta}{1-\theta}.\end{aligned}$$

By the construction of $\mathscr{S}'$ using Lemma 4.2.9, the triangle inequality and the fact that the estimate becomes trivial if $\theta\ge 0.28$, we obtain

$$\mathop{\mathbb{E}}_{x\sim\hat\nu_A}\sum_a \operatorname{Tr}\big((A^x_a-\tilde W\tilde A^x_a\tilde W^*)^2\rho_A\big)=\mathscr{O}(\sqrt{\alpha}+c\sqrt{\beta})+15\theta,$$

showing that $\tilde W$ is a good isometry. Now define $V_A=\tilde W$, and by repeating the proof for the B-side, we obtain an isometry V_B with the corresponding properties. The spaces $\mathscr{K}_A$ and $\mathscr{K}_B$ are found by taking the Hilbert space upon which $\hat{\mathscr{M}}\oplus\mathscr{M}^\perp$ acts. We now define the strategy $\hat{\mathscr{S}}=(|\psi\rangle, V_A\tilde A V_A^*, V_B\tilde B V_B^*)$. For an estimate on the success probability of $\hat{\mathscr{S}}$, we can just use Lemma 4.2.8 to show that $\omega(G_{\hat\nu};\hat{\mathscr{S}})=1-c\varepsilon-\mathscr{O}(\delta+\sqrt{\sqrt{\alpha}+c\sqrt{\beta}+\theta})$, so $\omega(G_{\hat\nu};\hat{\mathscr{S}})=1-\mathscr{O}\big((c\cdot\mathrm{id}+\kappa)(\mathrm{poly}(\delta)+2\varepsilon)\big)$. □

Theorem 4.4.6. *Let $G=(\mathscr{X},\nu,\mathscr{A},D)$ be a perfect symmetric non-local game and let $\mathscr{S}$ be a strategy for G with winning probability $1-\varepsilon$. Let ν_A be the marginal of ν on $\mathscr{X}$ and let $\delta=\delta_{\mathrm{sync}}(\mathscr{S},\nu_A)$. Let $\hat\nu$ be a symmetric probability distribution on $\mathscr{X}\times\mathscr{X}$ with marginal $\hat\nu_A$ on $\mathscr{X}$ and $c\ge1$ such that $\hat\nu\le c\nu$. Define $G_{\hat\nu}=(\mathscr{X},\hat\nu,\mathscr{A},D)$. Suppose that G $(\kappa,\hat\nu)$-PME-robustly self-tests a strategy $\tilde{\mathscr{S}}$ and that the game polynomial $T_{G_{\hat\nu},\tilde{\mathscr{S}}}$ has spectral gap α. Then $\tilde{\mathscr{S}}$ is a local $\Big(\mathscr{O}\big(\frac{1}{\sqrt\alpha}\sqrt{c\cdot\mathrm{id}+\kappa}(2\varepsilon+\mathrm{poly}(\delta))\big),\hat\nu\Big)$-dilation of $\mathscr{S}$.*

Proof. Let the optimal strategy $\tilde{\mathscr{S}}$ be given by $\tilde{\mathscr{S}}=(|\tilde\psi\rangle,\tilde A,\tilde B)$. Because $\mathscr{S}$ and G satisfy the requirements of Theorem 4.4.2, we know that there exist Hilbert spaces $\mathscr{K}_A$ and $\mathscr{K}_B$ and isometries $V_A:\mathscr{H}_A\to\tilde{\mathscr{H}}_A\otimes\mathscr{K}_A$ and $V_B:\mathscr{H}_B\to\tilde{\mathscr{H}}_B\otimes\mathscr{K}_B$ such that $\hat{\mathscr{S}}=(|\psi\rangle, V_A^*(\tilde A\otimes 1_{\mathscr{K}_A})V_A, V_B^*(\tilde B\otimes 1_{\mathscr{K}_B})V_B)$ has winning probability $\omega(\hat{\mathscr{S}})=1-\mathscr{O}\big((c\cdot\mathrm{id}+\kappa)(2\varepsilon+\mathrm{poly}(\delta))\big)$ and

$$\mathop{\mathbb{E}}_{x\sim\hat\nu_A}\sum_a \operatorname{Tr}\big((A^x_a-V_A^*\tilde A^x_aV_A)^2\rho_A\big)=\mathscr{O}\big((c\cdot\mathrm{id}+\kappa)^2(\mathrm{poly}(\delta)+2\varepsilon)\big),\tag{4.18}$$

$$\mathop{\mathbb{E}}_{y\sim\hat\nu_A}\sum_b \operatorname{Tr}\big((B^y_b-V_B^*\tilde B^x_aV_B)^2\rho_B\big)=\mathscr{O}\big((c\cdot\mathrm{id}+\kappa)^2(\mathrm{poly}(\delta)+2\varepsilon)\big).\tag{4.19}$$

Here we have identified $\tilde A^x_a\in B(\tilde{\mathscr{H}}_A)$ and $\tilde A^x_a\otimes 1_{\mathscr{K}_A}\in B(\tilde{\mathscr{H}}_A\otimes\mathscr{K}_A)$ and $\tilde B^y_b\in B(\tilde{\mathscr{H}}_B)$ and $\tilde B^y_b\otimes 1_{\mathscr{K}_B}\in B(\tilde{\mathscr{H}}_B\otimes\mathscr{K}_B)$, and we will continue to do so in this proof. From the winning probability estimate it follows that

$$\langle\psi|\left(\sum_{x,y}\hat\nu(x,y)\sum_{a,b}D(a,b|x,y)(V_A\otimes V_B)^*(\tilde A^x_a\otimes\tilde B^y_b)(V_A\otimes V_B)\right)|\psi\rangle \\ \ge 1-\mathscr{O}\big((c\cdot\mathrm{id}+\kappa)(2\varepsilon+\mathrm{poly}(\delta))\big),$$

which implies that

$$\langle\psi|(V_A\otimes V_B)^*(T_{G_{\hat{v}},\tilde{\mathscr{S}}}\otimes 1_{\mathcal{K}_A\otimes\mathcal{K}_B})(V_A\otimes V_B)|\psi\rangle \geq 1-\mathcal{O}\big((c\cdot\mathrm{id}+\kappa)(2\varepsilon+\mathrm{poly}(\delta))\big).$$

Let $P\in B(\tilde{\mathcal{H}}_A\otimes\tilde{\mathcal{H}}_B\otimes\mathcal{K}_A\otimes\mathcal{K}_B)$ be the projection onto the 1-eigenspace of $T_{G,\tilde{\mathscr{S}}}\otimes 1_{\mathcal{K}_A\otimes\mathcal{K}_B}$, which is the largest eigenvalue since G is perfect. Then we have

$$\langle\psi|(V_A\otimes V_B)^*(1-P)(T_{G_{\hat{v}},\tilde{\mathscr{S}}}\otimes 1_{\mathcal{K}_A\otimes\mathcal{K}_B})(1-P)(V_A\otimes V_B)|\psi\rangle \leq (1-\alpha)\|(1-P)(V_A\otimes V_B)|\psi\rangle\|^2$$

and

$$\langle\psi|(V_A\otimes V_B)^*P(T_{G_{\hat{v}},\tilde{\mathscr{S}}}\otimes 1_{\mathcal{K}_A\otimes\mathcal{K}_B})P(V_A\otimes V_B)|\psi\rangle = \|P(V_A\otimes V_B)|\psi\rangle\|^2,$$

so

$$\langle\psi|(V_A\otimes V_B)^*(T_{G_{\hat{v}},\tilde{\mathscr{S}}}\otimes 1_{\mathcal{K}_A\otimes\mathcal{K}_B})(V_A\otimes V_B)|\psi\rangle \leq 1-\alpha\|(1-P)(V_A\otimes V_B)|\psi\rangle\|^2.$$

This directly implies that

$$\|(1-P)(V_A\otimes V_B)|\psi\rangle\| \leq \mathcal{O}\Big(\frac{1}{\sqrt{\alpha}}\sqrt{c\cdot\mathrm{id}+\kappa}(2\varepsilon+\mathrm{poly}(\delta))\Big).$$

Since the 1-eigenspace of $T_{G,\tilde{\mathscr{S}}}$ is one-dimensional, we know that there exists a state $|aux\rangle\in\mathcal{K}_A\otimes\mathcal{K}_B$ such that

$$\frac{1}{\|P(V_A\otimes V_B)|\psi\rangle\|}P(V_A\otimes V_B)|\psi\rangle = |\tilde{\psi}\rangle\otimes|aux\rangle.$$

By the triangle inequality, we find that

$$\|(V_A\otimes V_B)|\psi\rangle-|\tilde{\psi}\rangle\otimes|aux\rangle\| \leq \mathcal{O}\Big(\frac{1}{\sqrt{\alpha}}\sqrt{c\cdot\mathrm{id}+\kappa}(2\varepsilon+\mathrm{poly}(\delta))\Big).$$

Next, note that

$$\mathrm{Tr}\big((A_a^x-V_A^*\tilde{A}_a^xV_A)^2\rho_A\big) = \|(V_A(A_a^x-V_A^*\tilde{A}_a^xV_A)\otimes V_B)|\psi\rangle\|^2 \tag{4.20}$$

since V_A is an isometry. Moreover, observe that

$$\sum_a\|(V_AV_A^*\tilde{A}_a^x\otimes 1_{\tilde{\mathcal{H}}_B\otimes\mathcal{K}_B})(V_A\otimes V_B|\psi\rangle-|\tilde{\psi}\rangle\otimes|aux\rangle)\|^2 \tag{4.21}$$
$$\leq \|(V_A\otimes V_B)|\psi\rangle-|\tilde{\psi}\rangle\otimes|aux\rangle\|^2,$$

using the fact that $V_AV_A^*$ is a contraction and

$$\sum_a(\tilde{A}_a^x)^2 \leq \sum_a\tilde{A}_a^x = 1_{\tilde{\mathcal{H}}_A\otimes\mathcal{K}_A}. \tag{4.22}$$

Consequently, our goal will be to provide an estimate for

$$\sum_a\|((V_AV_A^*\tilde{A}_a^x-\tilde{A}_a^x)\otimes 1_{\tilde{\mathcal{H}}_B\otimes\mathcal{K}_B})|\tilde{\psi}\rangle\otimes|aux\rangle\|^2.$$

First, we get rid of the sum over a. Let $m = \dim(\tilde{\mathcal{H}}_A)$. Since $|\tilde{\psi}\rangle$ is maximally entangled, we have

$$\sum_a \|((V_A V_A^* \tilde{A}_a^x - \tilde{A}_a^x) \otimes 1_{\tilde{\mathcal{H}}_B \otimes \mathcal{K}_B}) |\tilde{\psi}\rangle \otimes |aux\rangle\|^2 \tag{4.23}$$

$$= \sum_a \operatorname{Tr}\Big(((V_A V_A^* \tilde{A}_a^x - \tilde{A}_a^x)^* (V_A V_A^* \tilde{A}_a^x - \tilde{A}_a^x) \otimes 1_{\tilde{\mathcal{H}}_B \otimes \mathcal{K}_B}) |\tilde{\psi}\rangle \otimes |aux\rangle \langle \tilde{\psi}| \otimes \langle aux|\Big)$$

$$= \frac{1}{m} \sum_a \operatorname{Tr}\Big(((V_A V_A^* \tilde{A}_a^x - \tilde{A}_a^x)^* (V_A V_A^* \tilde{A}_a^x - \tilde{A}_a^x) \otimes 1_{\tilde{\mathcal{H}}_B \otimes \mathcal{K}_B})(1_{\tilde{\mathcal{H}}_A} \otimes |aux\rangle \langle aux|)\Big) \tag{4.24}$$

$$\leq \frac{1}{m} \operatorname{Tr}\Big(((V_A V_A^* - 1_{\tilde{\mathcal{H}}_A \otimes \mathcal{K}_A})^* (V_A V_A^* - 1_{\tilde{\mathcal{H}}_A \otimes \mathcal{K}_A}) \otimes 1_{\tilde{\mathcal{H}}_B \otimes \mathcal{K}_B})(1_{\tilde{\mathcal{H}}_A} \otimes |aux\rangle \langle aux|)\Big) \tag{4.25}$$

$$= \|((V_A V_A^* - 1_{\tilde{\mathcal{H}}_A \otimes \mathcal{K}_A}) \otimes 1_{\tilde{\mathcal{H}}_B \otimes \mathcal{K}_B}) |\tilde{\psi}\rangle \otimes |aux\rangle\|^2,$$

where we use the cyclicity of the trace and the standard estimate (4.22) for the inequality. By adding 0, we find

$$\begin{aligned}\|((V_A V_A^* - 1_{\tilde{\mathcal{H}}_A \otimes \mathcal{K}_A}) \otimes 1_{\tilde{\mathcal{H}}_B \otimes \mathcal{K}_B}) |\tilde{\psi}\rangle \otimes |aux\rangle\|^2 \\ = \|((V_A V_A^* - 1_{\tilde{\mathcal{H}}_A \otimes \mathcal{K}_A}) \otimes 1_{\tilde{\mathcal{H}}_B \otimes \mathcal{K}_B})(|\tilde{\psi}\rangle \otimes |aux\rangle - (V_A \otimes V_B) |\psi\rangle) \\ + ((V_A V_A^* - 1_{\tilde{\mathcal{H}}_A \otimes \mathcal{K}_A}) \otimes 1_{\tilde{\mathcal{H}}_B \otimes \mathcal{K}_B})(V_A \otimes V_B) |\psi\rangle\|^2.\end{aligned}$$

Since V_A is an isometry, $V_A^* V_A = 1_{\tilde{\mathcal{H}}_A \otimes \mathcal{K}_A}$, so the second term is zero. Furthermore,

$$0 \leq V_A V_A^* \leq 1_{\tilde{\mathcal{H}}_A \otimes \mathcal{K}_A},$$

so

$$\|1_{\tilde{\mathcal{H}}_A \otimes \mathcal{K}_A} - V_A V_A^*\|_\infty \leq 1.$$

Together with Equation (4.25), this gives

$$\|((V_A V_A^* - 1_{\tilde{\mathcal{H}}_A \otimes \mathcal{K}_A}) \otimes 1_{\tilde{\mathcal{H}}_B \otimes \mathcal{K}_B}) |\tilde{\psi}\rangle \otimes |aux\rangle\| \leq \|(V_A \otimes V_B) |\psi\rangle - |\tilde{\psi}\rangle \otimes |aux\rangle\|.$$

This equation, together with Equations (4.20) and (4.21), the starting estimate (4.18) and the triangle inequality yields

$$\begin{aligned}\mathop{\mathbb{E}}_{x\sim\hat{\nu}_A} \Big(\sum_a \|(V_A \otimes V_B)(A_a^x \otimes 1_{\mathcal{H}_B}) |\psi\rangle - (\tilde{A}_a^x \otimes 1_{\tilde{\mathcal{H}}_B \otimes \mathcal{K}_B}) |\tilde{\psi}\rangle \otimes |aux\rangle\|^2\Big)^{\frac{1}{2}} \\ \leq \mathcal{O}\Big(\frac{1}{\sqrt{\alpha}} \sqrt{c \cdot \mathrm{id} + \kappa}(2\varepsilon + \mathrm{poly}(\delta))\Big).\end{aligned}$$

By doing the analogous proof for the B-side, which crucially gives us the same auxiliary state $|aux\rangle$, we have obtained the three inequalities that show that $\tilde{\mathcal{S}}$ is a local $(\mathcal{O}\big(\alpha^{-1/2}\sqrt{c \cdot \mathrm{id} + \kappa}(2\varepsilon + \mathrm{poly}(\delta))\big), \hat{\nu})$-dilation of $\mathcal{S}$. □

Corollary 4.4.7. *Let $G = (\mathcal{X}, \nu, \mathcal{A}, D)$ be a perfect β-synchronous non-local game $(\kappa, \hat{\nu})$-PME-robust self-testing $\tilde{\mathcal{S}}$ and let $G_{\hat{\nu}} = (\mathcal{X}, \hat{\nu}, \mathcal{A}, D)$. Suppose that the game polynomial $T_{G_{\hat{\nu}}, \tilde{\mathcal{S}}}$ has spectral gap α and $\hat{\nu} \leq c\nu$ for $c \geq 1$. Then G is a $(\kappa', \hat{\nu})$-robust self-test for $\kappa'(\varepsilon) = \mathcal{O}(\alpha^{-1/2}\sqrt{c \cdot \mathrm{id} + \kappa}(\mathrm{poly}(\beta^{-1}\varepsilon))$.*

Proof. This is immediate from the previous theorem after realising that the asynchronicity is bounded above by $\beta^{-1}\varepsilon$ for a β-synchronous game as shown in Lemma 4.2.6. □

Remark 4.4.8. The constants hidden in $\mathcal{O}$ and poly are universal and do not depend on any property of the game.

4.5. SPECTRAL GAP FOR ROBUST SYNCHRONOUS SELF-TESTS

In the previous section we proved that every β-synchronous PME-robust self-test whose game polynomial has spectral gap is a robust self-test. This raises the question whether the spectral gap condition is necessary. Are there examples of such games for which the spectral gap can be arbitrarily small, or is there some lower bound on the spectral gap? It turns out that a lower bound exists if the non-local game is a κ-PME-robust self-test, i.e. if the probability distribution for robust self-test is the same as the probability distribution of the game. This is a consequence of several elements of the proof of Theorem 4.4.2 under the additional assumption that the Hilbert space $\mathscr{H}_A \otimes \mathscr{H}_B$ for a strategy $\mathscr{S}$ equals the Hilbert space $\tilde{\mathscr{H}}_A \otimes \tilde{\mathscr{H}}_A$ for the strategy $\tilde{\mathscr{S}}$ self-tested by the game. We rely on a dimension estimate, given in Lemma 4.5.1, to prove Lemma 4.5.2, stating that a PME strategy exists with high winning probability such that its reduced density matrix is close to the one of $\mathscr{S}$. From there we are able to prove the desired theorem.

Lemma 4.5.1. *Let $|\psi\rangle \in \mathscr{H} \otimes \mathscr{H}$ and $|\tilde{\psi}\rangle \in \tilde{\mathscr{H}} \otimes \tilde{\mathscr{H}}$ be maximally entangled states on Hilbert spaces such that* $\dim(\mathscr{H}) \leq \dim(\tilde{\mathscr{H}})$ *and* $\varepsilon \geq 0$. *Suppose that there exist isometries $V_A : \mathscr{H} \to \tilde{\mathscr{H}} \otimes \mathscr{K}_A$ and $V_B : \mathscr{H} \to \tilde{\mathscr{H}} \otimes \mathscr{K}_B$ and a state $|aux\rangle \in \mathscr{K}_A \otimes \mathscr{K}_B$ such that*

$$\|(V_A \otimes V_B)|\psi\rangle - |\tilde{\psi}\rangle \otimes |aux\rangle\| \leq \varepsilon.$$

Then $\dim(\mathscr{H}) \geq (1-\varepsilon^2)\dim(\tilde{\mathscr{H}})$.

Proof. Let $n = \dim(\mathscr{H})$ and $m = \dim(\tilde{\mathscr{H}})$, so we know that $n \leq m$. Let $d = \dim(\tilde{\mathscr{H}} \otimes \mathscr{K}_A)$ and

$$(V_A \otimes V_B)|\psi\rangle = \sum_{i=1}^{d} \lambda_i |u_i\rangle \otimes |v_i\rangle \text{ and } |\tilde{\psi}\rangle \otimes |aux\rangle = \sum_{i=1}^{d} \mu_i |\tilde{u}_i\rangle \otimes |\tilde{v}_i\rangle$$

be Schmidt decompositions, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_d$. Since $|\psi\rangle$ is maximally entangled on $\mathscr{H} \otimes \mathscr{H}$, we know that $\lambda_i = 1/\sqrt{n}$ if $i \leq n$ and $\lambda_i = 0$ otherwise. Since $|\tilde{\psi}\rangle$ is maximally entangled on $\tilde{\mathscr{H}} \otimes \tilde{\mathscr{H}}$, we know that $\mu_1 \leq 1/\sqrt{m}$. Consequently,

$$\|(\lambda_i)_i - (\mu_i)_i\|_2^2 \geq n\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}}\right)^2 + \frac{m-n}{m} \geq \frac{m-n}{m}.$$

Combining this with Lemma 4.A.1, we conclude that

$$m \cdot (1-\varepsilon^2) \leq n \leq m.$$

□

Lemma 4.5.2. *Let $\mathscr{S} = (|\psi\rangle, A, B)$ be a strategy for a perfect symmetric non-local game $G = (\mathscr{X}, \nu, \mathscr{A}, D)$ with winning probability $1-\varepsilon$. Let ρ_A be the reduced density matrix on $\mathscr{H}_A$, ν_A the marginal of ν on $\mathscr{X}$, and $\delta = \delta_{\text{sync}}(\mathscr{S}, \nu_A)$. Let $\hat{\nu}$ be a probability distribution on $\mathscr{X} \times \mathscr{X}$. Suppose that G $(\kappa, \hat{\nu})$-PME-robustly self-tests the optimal PME strategy $\tilde{\mathscr{S}} =$*

$(|\tilde{\psi}\rangle, \tilde{A})$ *on* $\tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_A$ *and that* $\dim(\tilde{\mathcal{H}}_A) = \dim(\mathcal{H}_A)$. *Then there exists a PME strategy* $\hat{\mathcal{S}}_A = (|\hat{\psi}\rangle, \hat{A})$ *on* $\hat{\mathcal{H}}_A \subset \mathcal{H}_A$ *with reduced density matrix* $\hat{\rho}_A$*, such that*

$$\|\rho_A - \hat{\rho}_A\|_1 \leq \mathcal{O}\big(\kappa(\varepsilon + \mathrm{poly}(\delta))^2 + \mathrm{poly}(\delta)\big) \text{ and} \tag{4.26}$$

$$\omega(G; \hat{\mathcal{S}}_A) \geq 1 - \varepsilon - \mathrm{poly}(\delta). \tag{4.27}$$

Remark 4.5.3. In both the above and the subsequent lemma, $\hat{\nu}$ does not affect the conclusions. This happens because the condition that G is a $(\kappa, \hat{\nu})$-PME-robust self-test is slightly stronger than necessary, since we do not need the equations in the definition of a local dilation concerning the measurement operators. As we are not aware of any use of the slightly more general statement, we refrain from introducing the nomenclature required to formally state this more general statement.

Proof. Let $\mathcal{S}' = (|\psi\rangle, A', B')$ be the projective strategy given by Lemma 4.2.9 and let $\delta' = \delta_{\mathrm{sync}}(\mathcal{S}', \nu_A)$. By Lemma 4.2.9, we know that $\delta' = \mathcal{O}(\delta^{\frac{1}{8}})$. Let the measure μ on $\mathbb{R}_+$, the projections P_λ of $\mathcal{H}_A$ onto $\mathcal{H}_\lambda$ and the family of strategies $\mathcal{S}'_\lambda = (|\psi_\lambda\rangle, A^\lambda)$ on $\mathcal{H}_\lambda$ with reduced density matrix ρ_λ be as in Theorem 4.2.12. Let C, C' and C^λ be the correlations of $\mathcal{S}$, $\mathcal{S}'$ and $\mathcal{S}'_\lambda$, respectively. Let

$$\mathop{\mathbb{E}}_{x,y\sim\nu} \sum_{a,b} |C_{x,y,a,b} - \int_0^\infty C^\lambda_{x,y,a,b} d\mu(\lambda)| = \alpha.$$

By Theorem 4.2.12(e), Lemma 4.2.9 and the triangle inequality, we know that $\alpha \leq \mathrm{poly}(\delta') + \mathcal{O}(\delta^{\frac{1}{8}}) \leq \mathrm{poly}(\delta)$. Lemma 4.2.10 tells us that for any $t > 0$ we have that

$$\mu(\{\lambda \geq 0 \mid \omega(\mathcal{S}'_\lambda) \geq 1 - \varepsilon - t\}) \geq 1 - \frac{\alpha}{t}.$$

As in the proof of Theorem 4.4.2, choosing t is a trade-off between the strength of the bound and the measure of the set for which the bound holds. We choose $t = \sqrt{\alpha}$, but many choices are possible here. If we define

$$\Lambda = \{\lambda \geq 0 \mid \omega(\mathcal{S}'_\lambda) \geq 1 - \sqrt{\alpha} - \varepsilon\},$$

we find that $\mu(\Lambda) \geq 1 - \sqrt{\alpha}$.

Let $\lambda \in \Lambda$ and let m be the dimension of $\tilde{\mathcal{H}}_A$. Since $\omega(\mathcal{S}'_\lambda) \geq 1 - \sqrt{\alpha} - \varepsilon$ and G is a $(\kappa, \hat{\nu})$-PME-robust self-test, we know that $\tilde{\mathcal{S}}$ is a local $(\kappa(\sqrt{\alpha} + \varepsilon), \hat{\nu})$-dilation of $\mathcal{S}'_\lambda$. Consequently, there exist isometries $V_A : \mathcal{H}_\lambda \to \tilde{\mathcal{H}}_A \otimes \mathcal{K}_A$ and $V_B : \mathcal{H}_\lambda \to \tilde{\mathcal{H}}_A \otimes \mathcal{K}_B$ and a unit vector $|aux\rangle \in \mathcal{K}_A \otimes \mathcal{K}_B$ such that

$$\|(V_A \otimes V_B)|\psi_\lambda\rangle - |\tilde{\psi}\rangle \otimes |aux\rangle\| \leq \kappa(\sqrt{\alpha} + \varepsilon).$$

By Lemma 4.5.1, we know that $\dim(\mathcal{H}_\lambda) \geq (1 - \kappa(\sqrt{\alpha} + \varepsilon)^2)m$.

Let P_λ be the projection of $\mathcal{H}_A$ onto $\mathcal{H}_\lambda$, so $\rho_\lambda = \mathrm{Tr}(P_\lambda)^{-1} P_\lambda$. The P_λ are constructed in Theorem 4.2.12 as the spectral projections of ρ_A, so $P_\lambda = \chi_{\geq\lambda}(\rho_A)$. This implies that $P_{\lambda_1} \leq P_{\lambda_2}$ if and only if $\lambda_1 \geq \lambda_2$. Let $\lambda_0 = \min(\Lambda)$, which exists because Λ is closed. Using the estimate of $\dim(\mathscr{H}_\lambda)$ and the order on the projections P_λ, we now calculate that

$$\begin{aligned}\|\rho_\lambda-\rho_{\lambda_0}\|_1 &= \|\frac{1}{\mathrm{Tr}(P_\lambda)}P_\lambda-\frac{1}{\mathrm{Tr}(P_{\lambda_0})}P_{\lambda_0}\|_1\\ &\le \left(\frac{1}{\mathrm{Tr}(P_\lambda)}-\frac{1}{\mathrm{Tr}(P_{\lambda_0})}\right)\|P_\lambda\|_1+\frac{1}{\mathrm{Tr}(P_{\lambda_0})}\|P_\lambda-P_{\lambda_0}\|_1\\ &\le \frac{1}{m}\left(\frac{1}{1-\kappa(\varepsilon+\sqrt{\alpha})^2}-1\right)m+\frac{m\left(1-\left(1-\kappa(\varepsilon+\sqrt{\alpha})^2\right)\right)}{m(1-\kappa(\varepsilon+\sqrt{\alpha})^2)}\\ &= \frac{2\kappa(\varepsilon+\sqrt{\alpha})^2}{1-\kappa(\varepsilon+\sqrt{\alpha})^2}\le 4\kappa(\varepsilon+\sqrt{\alpha})^2,\end{aligned}$$

if $\kappa(\varepsilon+\sqrt{\alpha})^2\le 1/2$. By directly estimating that $\|\rho_\lambda-\rho_{\lambda_0}\|_1\le 4\kappa(\varepsilon+\sqrt{\alpha})^2$ if $\kappa(\varepsilon+\sqrt{\alpha})^2\ge 1/2$, we know that

$$\|\rho_\lambda-\rho_{\lambda_0}\|_1\le 4\kappa(\varepsilon+\sqrt{\alpha})^2$$

always holds. Since $\mu(\Lambda)\ge 1-\sqrt{\alpha}$ and

$$\rho_A=\int_0^\infty \rho_\lambda\, d\mu(\lambda),$$

it follows that $\|\rho_A-\rho_{\lambda_0}\|_1\le 4\kappa(\varepsilon+\sqrt{\alpha})^2+\sqrt{\alpha}$. Since $\alpha,\beta\le \mathrm{poly}(\delta)$, this shows Equation (4.26). Note that we automatically satisfy Equation (4.27) since $\lambda_0\in\Lambda$, so the proof is complete. □

Theorem 4.5.4. *Let $G=(\mathscr{X},\nu,\mathscr{A},D)$ be a β-synchronous non-local game and let $\hat{\nu}$ be a probability distribution on $\mathscr{X}\times\mathscr{X}$. Suppose that G $(\kappa,\hat{\nu})$-PME-robustly self-tests a perfect strategy $\mathscr{S}$. There exists universal constants C_1, C_2, $\zeta>0$ such that the spectral gap of the game polynomial $T_{G,\mathscr{S}}$ is at least $C_1\beta((\mathrm{id}+\kappa^2)^{-1}(C_2))^\zeta$.*

Proof. Let $\mathscr{S}=(|\psi_0\rangle, A)$ on $\mathscr{H}_A\otimes\mathscr{H}_A$ be the perfect strategy for G, n be the dimension of $\mathscr{H}_A$, $T=T_{G,\mathscr{S}}$ and Δ its spectral gap. Let $|\psi_1\rangle$ be a state in $\mathscr{H}_A\otimes\mathscr{H}_A$ orthogonal to $|\psi_0\rangle$ such that $\langle\psi_1|T|\psi_1\rangle=1-\Delta$. Our goal is to use $|\psi_0\rangle$ and $|\psi_1\rangle$ to construct a state $|\psi\rangle$ with high winning probability, but whose reduced density matrix on $\mathscr{H}_A$ is not close to the normalised identity. Lemma 4.5.2 allows us to relate this to κ, from which we can derive the theorem. Note that we have the freedom to multiply $|\psi_1\rangle$ by a phase, which we will use later.

Let $|u_j\rangle_{j=1}^n$ be an orthonormal basis of $\mathscr{H}_A$ such that

$$|\psi_0\rangle=\frac{1}{\sqrt{n}}\sum_j |u_j\rangle|u_j\rangle$$

and let $(S_{jk})_{jk}$ be a matrix such that

$$|\psi_1\rangle=\sum_{jk}S_{jk}|u_j\rangle|u_k\rangle.$$

Define

$$|\psi\rangle=\frac{1}{\sqrt{2}}|\psi_0\rangle+\frac{1}{\sqrt{2}}|\psi_1\rangle,$$

so

$$|\psi\rangle = \frac{1}{\sqrt{2}}\sum_{jk}(S_{jk} + \frac{1}{\sqrt{n}}\delta_{jk})\,|u_j\rangle\,|u_k\rangle\,.$$

First note that the winning probability of $(|\psi\rangle\,, A)$ is given by

$$\omega((|\psi\rangle\,, A)) = \langle\psi|\,T\,|\psi\rangle = \frac{1}{2}(\langle\psi_0|\,T\,|\psi_0\rangle + \langle\psi_1|\,T\,|\psi_1\rangle) = 1 - \frac{1}{2}\Delta,$$

and that this probability does not depend on the phase of $|\psi_1\rangle$. Next, we calculate its reduced density matrix ρ. For a state $|\varphi\rangle$, given by

$$|\varphi\rangle = \sum_{jk} F_{jk}\,|u_j\rangle\,|u_k\rangle\,,$$

the reduced density matrix on $\mathscr{H}_A$ is given by

$$\mathrm{Tr}_B(|\varphi\rangle\,\langle\varphi|) = \sum_{jkj'k'} F_{jk}\overline{F_{j'k'}}\delta_{kk'}\,|u_j\rangle\,\langle u_{j'}| = \sum_{jj'}(FF^*)_{jj'}\,|u_j\rangle\,\langle u_{j'}|\,.$$

Consequently, we find that

$$\rho = \frac{1}{2}(S + \frac{1}{\sqrt{n}})(S + \frac{1}{\sqrt{n}})^*$$

with respect to the basis $|u_j\rangle$. Therefore,

$$\|\frac{1}{n} - \rho\|_1 = \|\frac{1}{n} - \frac{1}{2}(S + \frac{1}{\sqrt{n}})(S + \frac{1}{\sqrt{n}})^*\|_1 = \|\frac{1}{2n} - \frac{1}{2}SS^* - \frac{1}{2\sqrt{n}}(S + S^*)\|_1.$$

If we consider the state $|\psi'\rangle$, given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}\,|\psi_0\rangle - \frac{1}{\sqrt{2}}\,|\psi_1\rangle$$

and its reduced density matrix ρ', then we have that

$$\|\frac{1}{n} - \rho'\|_1 = \|\frac{1}{n} - \frac{1}{2}(-S + \frac{1}{\sqrt{n}})(-S + \frac{1}{\sqrt{n}})^*\|_1 = \|\frac{1}{2n} - \frac{1}{2}SS^* + \frac{1}{2\sqrt{n}}(S + S^*)\|_1.$$

Using the triangle inequality, it follows that

$$\frac{1}{\sqrt{n}}\|S + S^*\|_1 \le \|\frac{1}{n} - \rho\|_1 + \|\frac{1}{n} - \rho'\|_1.$$

We will now use our freedom in the phase of $|\psi_1\rangle$ to assume some properties of S without loss of generality. It holds in general that $\|A + iB\|_2^2 = \|A\|_2^2 + \|B\|_2^2$ for self-adjoint matrices A and B, so we can assume that $\|S + S^*\|_2^2 \ge 2$, since $\|S\|_2^2 = \mathrm{Tr}(SS^*) = 1$ and $S + S^*$ is twice the self-adjoint part of S. Next, we can possibly multiply S by -1 to assume that

$$\frac{1}{2\sqrt{n}}\|S + S^*\|_1 \le \|\frac{1}{n} - \rho\|_1.$$

We now claim that at least one of

$$\frac{1}{16+12\sqrt{2}} \leq \frac{1}{2\sqrt{n}}\|S+S^*\|_1 \text{ and } \frac{1}{8+6\sqrt{2}} \leq \frac{1}{2}\|\frac{1}{n}-SS^*\|_1$$

holds. Let $(\lambda_j)_{j=1}^n$ be the eigenvalues of $S+S^*$ and define the sets

$$\Lambda_> = \{1 \leq j \leq n \mid |\lambda_j| \geq \frac{8+6\sqrt{2}}{\sqrt{n}}\} \text{ and } \Lambda_< = \{1 \leq j \leq n \mid |\lambda_j| < \frac{8+6\sqrt{2}}{\sqrt{n}}\}.$$

Now we have that

$$\sum_{j\in\Lambda_>} \lambda_j^2 \geq 1 \text{ or } \sum_{j\in\Lambda_<} \lambda_j^2 \geq 1.$$

In the first case, we use that the sequence of eigenvalues of $4SS^*$ majorises the squares of the eigenvalues of both the positive and negative parts of $S+S^*$ [Bha97, Theorem III.5.1], so

$$\|\frac{1}{n}-SS^*\|_1 \geq \sum_{j\in\Lambda_>}\left(\frac{\lambda_j^2}{8}-\frac{1}{n}\right) \geq \sum_{j\in\Lambda_>}\left(\frac{\lambda_j^2}{8}-\frac{\lambda_j^2}{136+96\sqrt{2}}\right) \geq \frac{1}{8}-\frac{1}{136+96\sqrt{2}} = \frac{2}{8+6\sqrt{2}}.$$

In the other case we find that

$$1 \leq \sum_{j\in\Lambda_<} \lambda_j^2 \leq \sum_{j\in\Lambda_<} \frac{8+6\sqrt{2}}{\sqrt{n}}|\lambda_j|,$$

so $(8+6\sqrt{2})\|S+S^*\|_1 \geq \sqrt{n}$. This shows the claim. Since

$$\|\frac{1}{n}-\rho\|_1 \geq \frac{1}{2}\|\frac{1}{n}-SS^*\|_1 - \frac{1}{2\sqrt{n}}\|S+S^*\|_1,$$

we can now conclude that

$$\|\frac{1}{n}-\rho\|_1 \geq \frac{1}{16+12\sqrt{2}} > \frac{1}{33} \tag{4.28}$$

in either case of the claim. We have now constructed a state $|\psi\rangle$ with high winning probability such that the reduced density matrix is not close to the normalised identity.

We are now in a position to use Lemma 4.5.2. Note that the winning probability controls the asynchronicity by Lemma 4.2.6 since G is β-synchronous, so $\delta_{\text{sync}}((|\psi\rangle, A)) \leq \Delta/(2\beta)$. Using Lemma 4.5.2 we obtain a maximally entangled strategy $\hat{\mathscr{S}} = (|\hat{\psi}\rangle, \hat{A})$ on $\hat{\mathscr{H}} \otimes \hat{\mathscr{H}}$ with reduced density matrix $\hat{\rho}$ such that $\omega(\hat{\mathscr{S}}) \geq 1-\text{poly}(\Delta/\beta)$ and

$$\|\rho-\hat{\rho}\|_1 = \mathcal{O}(\kappa(\text{poly}(\frac{\Delta}{\beta}))^2 + \text{poly}(\frac{\Delta}{\beta})).$$

By the $(\kappa,\hat{\nu})$-PME-robust self-testing of G and Lemma 4.5.1, we know that $\dim(\hat{\mathscr{H}}) \geq (1-\kappa(\text{poly}(\Delta/\beta))^2)n$. From this it follows that

$$\|\hat{\rho}-\frac{1}{n}\|_1 = \dim(\hat{\mathscr{H}})\left(\frac{1}{\dim(\hat{\mathscr{H}})}-\frac{1}{n}\right) + (n-\dim(\hat{\mathscr{H}}))\frac{1}{n} \leq 2\kappa\left(\text{poly}\left(\frac{\Delta}{\beta}\right)\right)^2,$$

so

$$\|\rho - \frac{1}{n}\|_1 = \mathcal{O}(\kappa(\mathrm{poly}(\frac{\Delta}{\beta}))^2 + \mathrm{poly}(\frac{\Delta}{\beta})).$$

By Equation (4.28), we have now shown that

$$\frac{1}{33} = \mathcal{O}(\kappa(\mathrm{poly}(\frac{\Delta}{\beta}))^2 + \mathrm{poly}(\frac{\Delta}{\beta})).$$

Inverting this inequality shows that there exist constants C_1, C_2, $\zeta > 0$ such that

$$\Delta \geq C_1 \beta((\mathrm{id} + \kappa^2)^{-1}(C_2))^{\zeta},$$

proving the theorem. □

Corollary 4.5.5. *There exist universal constants C_1, C_2, C_3, ζ_1, $\zeta_2 > 0$ such that every β-synchronous κ-PME-robust self-test is a κ'-robust self-test with*

$$\kappa'(\varepsilon) \leq C_1 \frac{\sqrt{(\mathrm{id}+\kappa)}(C_2(\frac{\varepsilon}{\beta})^{\zeta_1})}{\beta((\mathrm{id}+\kappa^2)^{-1}(C_3))^{\zeta_2}}.$$

4.6. SPECTRAL GAP OF THE QUANTUM LOW DEGREE TEST

In Corollary 4.5.5, we have found that every κ-PME-robust self-test is also a robust self-test with a related robustness. However, it is not known that the Quantum Low Degree Test is a $(\kappa, \nu_{\mathrm{qldt}})$-PME-robust self-test. Using the notation in Definition 4.2.15, we only know that the Quantum Low Degree Test is a (κ, ν')-PME-robust self-test. We are therefore forced to explicitly compute a lower bound for the spectral gap for this distribution. In fact, we are able to calculate the exact spectral gap.

Theorem 4.6.1. *Let $G_{\mathrm{qldt}} = (\mathcal{X}_{\mathrm{qldt}}, \nu_{\mathrm{qldt}}, \mathcal{A}_{\mathrm{qldt}}, D_{\mathrm{qldt}})$ be the Quantum Low Degree Test for k qubits with optimal strategy $\mathcal{S}$ and let $G' = (\mathcal{X}_{\mathrm{qldt}}, \nu'_{\mathrm{qldt}}, \mathcal{A}_{\mathrm{qldt}}, D_{\mathrm{qldt}})$. Let d be the relative distance of the code used in the Quantum Low Degree Test. Then the spectral gap of the game polynomial $T_{G',\mathcal{S}}$ is $\frac{d}{2}$.*

Proof. For an element $a \in \mathbb{F}_2$ let $\overline{a} = 1 - a$. Let $B = \{|\psi_{ab}\rangle \mid a, b \in \mathbb{F}_2\}$ be the Bell basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$ given by

$$|\psi_{ab}\rangle = \frac{1}{2}\left(|0a\rangle + (-1)^b |1\overline{a}\rangle\right)$$

for all a, $b \in \mathbb{F}_2$. For $\mathbf{a}, \mathbf{b} \in \mathbb{F}_2^k$ define

$$|\psi_{\mathbf{ab}}\rangle = \bigotimes_{i=1}^{k} |\psi_{a_i b_i}\rangle.$$

Note that

$$\langle \psi_{ab}| X \otimes X |\psi_{ab}\rangle = (-1)^b \text{ and } \langle \psi_{ab}| Z \otimes Z |\psi_{ab}\rangle = (-1)^a$$

for all a, $b \in \mathbb{F}_2$, so

$$\langle \psi_{\mathbf{ab}}| \sigma^X(\mathbf{c}) \otimes \sigma^X(\mathbf{c}) |\psi_{\mathbf{ab}}\rangle = (-1)^{\mathbf{b}\cdot\mathbf{c}} \text{ and } \langle \psi_{\mathbf{ab}}| \sigma^Z(c) \otimes \sigma^Z(c) |\psi_{\mathbf{ab}}\rangle = (-1)^{\mathbf{a}\cdot\mathbf{c}} \tag{4.29}$$

for all $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{F}_2^k$.

The game polynomial $T_{G',\mathscr{S}}$ is given by

$$T_{G',\mathscr{S}} = \mathop{\mathbb{E}}_{(x,y)\sim\nu'_{\text{qldt}}} \sum_{a,b} D_{\text{qldt}}(a,b|x,y) A_a^x \otimes (A_b^y)^T,$$

where $\{A^x\}_{x\in\mathscr{X}_{\text{qldt}}}$ are the measurement operators of the ideal strategy. Let E be the generating matrix for the code C_{RM2} used in the construction of the Quantum Low Degree Test, so $S_X = S_Z = \{(E_{ij})_{i=1}^k \mid 1 \le j \le n\}$, and let n be the length of the code. For each $W \in \{X,Z\}$ and $\mathbf{a} \in S_W$ there are two questions $x^1_{W,\mathbf{a}}$ and $x^2_{W,\mathbf{a}}$ in $\mathscr{X}_{\text{qldt}}$ such that

$$U(A^y) = \sigma^W(\mathbf{a}) \text{ for } y \in \{x^1_{W,\mathbf{a}}, x^2_{W,\mathbf{a}}\}.$$

Moreover,

$$\nu'_{\text{qldt}}(x^1_{W,\mathbf{a}}, x^2_{W,\mathbf{a}}) = \nu'_{\text{qldt}}(x^2_{W,\mathbf{a}}, x^1_{W,\mathbf{a}}) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{n},$$

where this probability is obtained by multiplying the probability that this W is chosen, the probability that question 1 goes to Alice and the probability that $\mathbf{a}$ is chosen from S_W. For these questions, Alice and Bob win if they give the same answer. Since

$$A_0^{x^1_{W,\mathbf{a}}} \otimes (A_0^{x^2_{W,\mathbf{a}}})^T + A_1^{x^1_{W,\mathbf{a}}} \otimes (A_1^{x^2_{W,\mathbf{a}}})^T = \frac{1}{2}\left(1 + \sigma^W(\mathbf{a}) \otimes \sigma^W(\mathbf{a})\right),$$

this implies that

$$T_{G',\mathscr{S}} = \frac{1}{2} + \frac{1}{4n} \sum_{W\in\{X,Z\}} \sum_{\mathbf{a}\in S_X} \sigma^W(\mathbf{a}) \otimes \sigma^W(\mathbf{a}),$$

where the factor arises from the fact that we are considering both $W = X$ and $W = Z$ and the observation that both orders of questions give the same expression.

By Equation (4.29), the above inequality tells us that

$$\langle \psi_{\mathbf{ab}} | T_{G',\mathscr{S}} | \psi_{\mathbf{ab}} \rangle = \frac{1}{2} + \frac{1}{4n} \sum_{\mathbf{c}\in S_X} \left((-1)^{\mathbf{a}\cdot\mathbf{c}} + (-1)^{\mathbf{b}\cdot\mathbf{c}} \right).$$

Therefore, the question we need to answer is for given $\mathbf{a} \in \mathbb{F}_2^k$, how many $\mathbf{c} \in S_X$ are there such that $\mathbf{a}\cdot\mathbf{c} = 1$? For this, we need to recall how to calculate the physical representation $\mathbf{a}_{\text{phys}} \in \mathbb{F}_2^n$ of a logical code word $\mathbf{a}_{\text{logic}} \in \mathbb{F}_2^k$. This connection is defined by the generating matrix and is given by

$$\mathbf{a}_{\text{phys}} = \sum_{i=1}^{k} a_{\text{logic},i} \cdot (E_{ij})_{j=1}^n.$$

Another way to represent this is by describing each $a_{\text{phys},i}$, which gives

$$a_{\text{phys},i} = \mathbf{a}_{\text{logic}} \cdot (E_{ij})_{i=1}^k.$$

This means that the number of $\mathbf{c} \in S_X$, i.e. the number of columns of E, for which $\mathbf{a}\cdot\mathbf{c} = 1$ precisely equals the number of ones in the physical representation of the logical code word $\mathbf{a}$. By the nature of the code, this is at least dn if $\mathbf{a} \neq 0$. Consequently,

$$\langle \psi_{\mathbf{ab}} | T_{G',\mathscr{S}} | \psi_{\mathbf{ab}} \rangle \le 1 - \frac{d}{2}$$

unless $\mathbf{a} = 0 = \mathbf{b}$. Since the Quantum Low Degree Test is perfect and the distance in a code is attained, this shows that the spectral gap is $\frac{d}{2}$. □

Remark 4.6.2. The above proof is closely related to [dlS22b, Example 1.2]. Since $\mathbf{a} \mapsto \sigma^W(\mathbf{a}) \otimes \sigma^W(\mathbf{a})$ is a unitary representation of $\mathbb{F}_2^k$, [dlS22b, Example 1.2] tells us that the operator

$$\frac{1}{n} \sum_{\mathbf{a} \in S_X} \sigma^W(\mathbf{a}) \otimes \sigma^W(\mathbf{a})$$

has spectral gap $2d$, if you do not account for multiplicities. Doing this for both $W = X$ and $W = Z$ and arguing that $T_{G',\mathscr{S}}$ only has a one-dimensional eigenspace for eigenvalue 1, using much of the proof above, also yields 4.6.1.

Corollary 4.6.3. *The* $1/2$*-synchronised version of the Quantum Low Degree Test is a* $\big(k, \mathrm{poly}(\log(k)) \cdot \mathrm{poly}(\varepsilon)\big)$*-qubit test.*

Proof. This is the consequence of combining Corollary 4.4.7, Theorem 4.6.1 and Lemma 4.2.5 after observing that $\nu'_{\mathrm{qldt}} \leq 4\nu_{\mathrm{qldt}}$. □

APPENDIX

4.A. PROOF OF LEMMA 4.3.2

In this appendix, we prove Lemma 4.3.2. The proof relies on several technical lemmas that we outline below.

Lemma 4.A.1. *Let* $|\psi\rangle$ *and* $|\tilde{\psi}\rangle$ *be two unit vectors in* $\mathbb{C}^d \otimes \mathbb{C}^d$ *with Schmidt decompositions* $|\psi\rangle = \sum_{i=1}^d \lambda_i |u_i\rangle \otimes |v_i\rangle$ *and* $|\tilde{\psi}\rangle = \sum_{i=1}^d \mu_i |\tilde{u}_i\rangle \otimes |\tilde{v}_i\rangle$*, where* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$ *and* $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_d \geq 0$*. Then the* ℓ_2*-distance of the sequences* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)$ *and* $\boldsymbol{\mu} = (\mu_1, \ldots . \mu_d)$ *is bounded above by the norm-distance of* $|\psi\rangle$ *and* $|\tilde{\psi}\rangle$*. That is,*

$$\|(\lambda_i)_i - (\mu_i)_i\|_2 \leq \||\psi\rangle - |\tilde{\psi}\rangle\|. \tag{4.30}$$

Proof. Let $a_{k\ell} := \mathrm{Re}\big(\langle u_k|\tilde{u}_\ell\rangle \langle v_k|\tilde{v}_\ell\rangle\big)$ for all $1 \leq k,\ \ell \leq d$. We first show that there exists a bistochastic matrix $X = (x_{k\ell})_{k,\ell}$ such that $a_{k\ell} \leq x_{k\ell}$ for all $k,\ \ell$. For any ℓ,

$$\begin{aligned} c_\ell := \sum_{k=1}^d a_{kl} &\leq \sum_{k=1}^d |\langle u_k|\tilde{u}_\ell\rangle| \cdot |\langle v_k|\tilde{v}_\ell\rangle| \\ &\leq \left(\sum_{k=1}^d |\langle u_k|\tilde{u}_\ell\rangle|^2\right)^{1/2} \cdot \left(\sum_{k=1}^d |\langle v_k|\tilde{v}_\ell\rangle|^2\right)^{1/2} \\ &= \||\tilde{u}_\ell\rangle\| \cdot \||\tilde{v}_\ell\rangle\| = 1. \end{aligned}$$

The second line follows from the Cauchy-Schwartz inequality. The third line holds because $\{|u_k\rangle \mid 1 \leq k \leq d\}$ and $\{|v_k\rangle \mid 1 \leq k \leq d\}$ are orthonormal bases for $\mathbb{C}^d$. Similarly,

$$r_k := \sum_{\ell=1}^d a_{k\ell} \leq 1$$

for all k. So

$$T := \sum_{k,\ell} a_{k\ell} = \sum_{\ell} c_{\ell} = \sum_{k} r_k \le d.$$

If $T = d$, then $c_\ell = r_k = 1$ for all k, ℓ and hence $(a_{kl})_{k,l}$ is a bistochastic matrix. Now assume $T < d$. Then

$$b_{k\ell} := \frac{(1-c_\ell)(1-r_k)}{d-T} \ge 0$$

for all k, ℓ, and

$$\sum_{k=1}^{d} (a_{k\ell} + b_{k\ell}) = c_\ell + \frac{\sum_{k=1}^{d}(1-r_k)}{d-T}(1-c_\ell) = c_\ell + \frac{d-T}{d-T}(1-c_\ell) = 1$$

for all ℓ. Similarly, $\sum_{\ell=1}^{d}(a_{k\ell} + b_{k\ell}) = 1$ for all k. Hence $(a_{k\ell} + b_{k\ell})_{k,\ell}$ is a bistochastic matrix. We conclude that $(a_{k\ell})_{k,\ell}$ is entrywise smaller than some bistochastic matrix X.

4

By Birkhoff–von Neumann theorem, $X = \theta_1 P_1 + \cdots + \theta_t P_t$ is a convex combination of some permutation matrices $P_1, \ldots, P_t$. It follows that

$$\begin{aligned}
\operatorname{Re}\big(\langle\psi|\tilde{\psi}\rangle\big) &= \sum_{k,\ell} \lambda_k \mu_\ell a_{k\ell} = \begin{pmatrix}\lambda_1 & \cdots & \lambda_d\end{pmatrix} (a_{k\ell})_{k,\ell} \begin{pmatrix}\mu_1 \\ \vdots \\ \mu_d\end{pmatrix} \\
&\le \begin{pmatrix}\lambda_1 & \cdots & \lambda_d\end{pmatrix} X \begin{pmatrix}\mu_1 \\ \vdots \\ \mu_d\end{pmatrix} = \sum_{i=1}^{t} \theta_i \begin{pmatrix}\lambda_1 & \cdots & \lambda_d\end{pmatrix} P_i \begin{pmatrix}\mu_1 \\ \vdots \\ \mu_d\end{pmatrix} \\
&\le \sum_{i=1}^{t} \theta_i \begin{pmatrix}\lambda_1 & \cdots & \lambda_d\end{pmatrix} \begin{pmatrix}\mu_1 \\ \vdots \\ \mu_d\end{pmatrix} = \sum_{k=1}^{d} \lambda_k \mu_k.
\end{aligned}$$

The last line follows from the rearrangement inequality. This implies

$$\| |\psi\rangle - |\tilde{\psi}\rangle \|^2 = 2 - 2\operatorname{Re}\big(\langle\psi|\tilde{\psi}\rangle\big) \ge 2 - 2\sum_{k=1}^{d} \lambda_k \mu_k = \|\boldsymbol{\lambda} - \boldsymbol{\mu}\|_2^2.$$

So Equation (4.30) follows. □

Lemma 4.A.2. *Let $\mathcal{S} = (|\psi\rangle, A, B)$ and $\tilde{\mathcal{S}} = (|\tilde{\psi}\rangle, \tilde{A}, \tilde{B})$ be two maximally entangled strategies on $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_B$ such that $\tilde{\mathcal{S}}$ is a local (ε, ν)-dilation of $\mathcal{S}$ for some $\varepsilon \ge 0$ and distribution ν on $\mathcal{X} \times \mathcal{Y}$ with isometries V_A and V_B and unit vector $|aux\rangle \in \mathcal{K}_A \otimes \mathcal{K}_B$. Let ν_A and ν_B be the marginal distributions of ν on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Then there exist subspaces $\mathcal{K}'_A \subset \mathcal{K}_A$ and $\mathcal{K}'_B \subset \mathcal{K}_B$ and a maximally entangled state $|aux'\rangle \in \mathcal{K}'_A \otimes \mathcal{K}'_B$ such that*

$$\begin{aligned}
\|(V_A \otimes V_B)|\psi\rangle - |\tilde{\psi}\rangle \otimes |aux'\rangle\| &\le 3\varepsilon, \\
\left(\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_{a} \|(V_A \otimes V_B)(A^x_a \otimes 1_{\mathcal{H}_B})|\psi\rangle - \big((\tilde{A}^x_a \otimes 1_{\tilde{\mathcal{H}}_B})|\tilde{\psi}\rangle\big) \otimes |aux'\rangle\|^2\right)^{1/2} &\le 3\varepsilon, \\
\left(\mathop{\mathbb{E}}_{y\sim\nu_B} \sum_{b} \|(V_A \otimes V_B)(1_{\mathcal{H}_A} \otimes B^y_b)|\psi\rangle - \big((1_{\tilde{\mathcal{H}}_A} \otimes \tilde{B}^y_b)|\tilde{\psi}\rangle\big) \otimes |aux'\rangle\|^2\right)^{1/2} &\le 3\varepsilon.
\end{aligned}$$

Proof. Let

$$(V_A \otimes V_B)|\psi\rangle = \sum_i \lambda_i |u_i\rangle|v_i\rangle \text{ and } |\tilde{\psi}\rangle \otimes |aux\rangle = \sum_i \kappa_i |\tilde{u}_i\rangle|\tilde{v}_i\rangle$$

be the Schmidt decompositions of $(V_A \otimes V_B)|\psi\rangle$ and $|\tilde{\psi}\rangle \otimes |aux\rangle$, where $\boldsymbol{\lambda} = (\lambda_i)_i$ and $\boldsymbol{\kappa} = (\kappa_i)_i$ are decreasing sequences. Because $\tilde{\mathscr{S}}$ is a local (ε, ν)-dilation of $\mathscr{S}$, we know that

$$\|\boldsymbol{\lambda} - \boldsymbol{\kappa}\|_2 \leq \|(V_A \otimes V_B)|\psi\rangle - |\tilde{\psi}\rangle \otimes |aux\rangle\| \leq \varepsilon$$

by Lemma 4.A.1.

Since $|\psi\rangle$ is maximally entangled, we know that there exists an $m \in \mathbb{N}$ such that $\lambda_i = \frac{1}{\sqrt{m}}$ for $1 \leq i \leq m$ and $\lambda_i = 0$ for $i > m$. On the other side we know that the multiplicity of each value in $\boldsymbol{\kappa}$ is divisible by $n = \dim(\tilde{\mathscr{H}}_A)$. We want to show that there exists a sequence $\boldsymbol{\kappa}'$ of Schmidt coefficients which takes exactly one non-zero value, has multiplicities divisible by n and is close to $\boldsymbol{\kappa}$. First suppose that $m \geq n$. In this case, $\boldsymbol{\lambda}$ would be a good candidate, but the multiplicity of λ_i does not have to be divisible by n. To remedy this, we define

$$\kappa_i'' = \begin{cases} \frac{1}{\sqrt{m}} & i \leq \lfloor \frac{m}{n} \rfloor n \text{ or } \left(\lfloor \frac{m}{n} \rfloor n < i \leq \lfloor (\frac{m}{n}+1) \rfloor n \text{ and } \kappa_i \geq \frac{1}{2\sqrt{m}} \right) \\ 0 & i > \lfloor (\frac{m}{n}+1) \rfloor n \text{ or } \left(\lfloor \frac{m}{n} \rfloor n < i \leq \lfloor (\frac{m}{n}+1) \rfloor n \text{ and } \kappa_i < \frac{1}{2\sqrt{m}} \right) \end{cases}$$

and observe that $\|\boldsymbol{\kappa}'' - \boldsymbol{\kappa}\|_2 \leq \|\boldsymbol{\lambda} - \boldsymbol{\kappa}\|_2$. Unfortunately, $\|\boldsymbol{\kappa}''\|_2 \neq 1$ in general. However, by the triangle inequality, $1 - \varepsilon \leq \|\boldsymbol{\kappa}''\|_2 \leq 1 + \varepsilon$, so defining $\boldsymbol{\kappa}'$ as the normalisation of $\boldsymbol{\kappa}''$ gives us the desired sequence with $\|\boldsymbol{\kappa}' - \boldsymbol{\kappa}\|_2 \leq 2\varepsilon$. For the case where $m < n$, we immediately define

$$\kappa_i' = \begin{cases} \frac{1}{\sqrt{n}} & i \leq n \\ 0 & i > n \end{cases}$$

which is normalised and satisfies $\|\boldsymbol{\kappa}' - \boldsymbol{\kappa}\|_2 \leq \|\boldsymbol{\lambda} - \boldsymbol{\kappa}\|_2 \leq \varepsilon$.

Let

$$|aux\rangle = \sum_i \mu_i |w_i\rangle|w_i'\rangle$$

be the Schmidt decomposition of $|aux\rangle$. Having obtained the sequence $\boldsymbol{\kappa}'$ with the desired properties, we can now define

$$|aux'\rangle = \sum_i \sqrt{n}\kappa'_{ni}|w_i\rangle|w_i'\rangle .$$

The state $|\tilde{\psi}\rangle \otimes |aux'\rangle$ is then given by

$$|\tilde{\psi}\rangle \otimes |aux'\rangle = \sum_i \kappa_i' |\tilde{u}_i\rangle|\tilde{v}_i\rangle ,$$

and therefore we have $\||\tilde{\psi}\rangle \otimes |aux'\rangle - |\tilde{\psi}\rangle \otimes |aux\rangle\|_2 \leq 2\varepsilon$. By the triangle inequality, we find that

$$\|(V_A \otimes V_B)|\psi\rangle - |\tilde{\psi}\rangle \otimes |aux'\rangle\| \leq 3\varepsilon.$$

Next, observe that

$$\sum_a \|((\tilde{A}^x_a \otimes 1_{\tilde{\mathcal{H}}_B})|\tilde{\psi}\rangle) \otimes (|aux'\rangle - |aux\rangle)\|^2 = \sum_a \langle\tilde{\psi}|(\tilde{A}^x_a \otimes 1_{\tilde{\mathcal{H}}_B})^2|\tilde{\psi}\rangle \, \||aux'\rangle - |aux\rangle\|^2.$$

Since $\tilde{A}$ is a POVM, we know that

$$\sum_a \langle\tilde{\psi}|(\tilde{A}^x_a \otimes 1_{\tilde{\mathcal{H}}_B})^2|\tilde{\psi}\rangle \le \sum_a \langle\tilde{\psi}|(\tilde{A}^x_a \otimes 1_{\tilde{\mathcal{H}}_B})|\tilde{\psi}\rangle = 1,$$

so we find that

$$\sum_a \|((\tilde{A}^x_a \otimes 1_{\tilde{\mathcal{H}}_B})|\tilde{\psi}\rangle) \otimes (|aux'\rangle - |aux\rangle)\|^2 \le 4\varepsilon^2.$$

We have

$$\left(\mathbb{E}_{x\sim\nu_A} \sum_a \|(V_A \otimes V_B)(A^x_a \otimes 1_{\mathcal{H}_B})|\psi\rangle - \big((\tilde{A}^x_a \otimes 1_{\tilde{\mathcal{H}}_B})|\tilde{\psi}\rangle\big) \otimes |aux\rangle\|^2\right)^{1/2} \le \varepsilon$$

since $\tilde{\mathcal{S}}$ is a local (ε,ν)-dilation of $\mathcal{S}$. Combining these equations using the triangle inequality for the vector space

$$\bigoplus_{x,a} \tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_B \otimes \mathcal{K}_A \otimes \mathcal{K}_B$$

with norm given by

$$\|v\| = \left(\mathbb{E}_{x\sim\nu_A} \sum_a \|v(x,a)\|^2\right)^{\frac{1}{2}},$$

we obtain

$$\left(\mathbb{E}_{x\sim\nu_A} \sum_a \|(V_A \otimes V_B)(A^x_a \otimes 1_{\mathcal{H}_B})|\psi\rangle - \big((\tilde{A}^x_a \otimes 1_{\tilde{\mathcal{H}}_B})|\tilde{\psi}\rangle\big) \otimes |aux'\rangle\|^2\right)^{1/2} \le 3\varepsilon.$$

The other inequality can be proved analogously. □

Theorem 4.A.3 (Polar decomposition, [Con10, Theorem VIII.3.11])**.** *Let $A \in B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. There is a partial isometry W with* $\ker(W) = \ker(A)$ *and* $\overline{\mathrm{ran}(W)} = \overline{\mathrm{ran}(A)}$ *such that $A = W|A|$. $(W,|A|)$ is called the* polar decomposition *of A.*

Remark 4.A.4. If $A : \mathcal{H} \to \mathcal{K}$ is a bounded linear map between Hilbert spaces, one can also take the polar decomposition by viewing A as element of $B(\mathcal{H} \oplus \mathcal{K})$.

Lemma 4.A.5. *Let $V : \mathcal{H} \to \mathcal{H}'$ be an isometry, let $P \in B(\mathcal{H}')$ be a projection and let $X = W|X|$ be the polar decomposition for $X = PV$. Then*

$$(V - W)^*(V - W) \le 2(V - PV)^*(V - PV).$$

Proof. We start by expanding the left hand side, which gives us

$$\begin{aligned}(V - W)^*(V - W) &= V^*V - V^*W - W^*V + W^*W \\ &= V^*V - V^*PW - W^*PV + W^*W \\ &= V^*V - |X|W^*W - W^*W|X| + W^*W \\ &= V^*V - |X| - |X| + W^*W\end{aligned}$$

Because $\||X|\|_\infty \le 1$ and W is a partial isometry, we have the inequalities

$$-|X^*| \le -|X^*|^2 \text{ and } W^*W \le 1_{\mathcal{H}} = V^*V.$$

Consequently, we have

$$\begin{aligned}(V-W)^*(V-W) &\le V^*V - |X|^2 - |X|^2 + V^*V \\ &= V^*V - |X|W^*W|X| - |X|W^*W|X| + V^*V \\ &= 2(V^*V - V^*PV) \\ &= 2(V-PV)^*(V-PV).\end{aligned}$$

□

4

Lemma 4.A.6. *Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, |\tilde{\psi}\rangle \in \tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_B$ be states such that $|\psi\rangle$ is maximally entangled, let ν_A be a probability measure on $\mathcal{X}$ and let A, $\tilde{A}$ be POVMs on $\mathcal{H}_A$ and $\tilde{\mathcal{H}}_A$, respectively. Let $V_A : \mathcal{H}_A \to \tilde{\mathcal{H}}_A \otimes \mathcal{K}_A$ and $V_B : \mathcal{H}_B \to \tilde{\mathcal{H}}_B \otimes \mathcal{K}_B$ be isometries and $|aux\rangle \in \mathcal{K}'_A \otimes \mathcal{K}'_B \subset \mathcal{K}_A \otimes \mathcal{K}_B$ such that*

$$\|(V_A \otimes V_B)|\psi\rangle - |\tilde{\psi}\rangle \otimes |aux\rangle\| \le \varepsilon,$$

$$\left(\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \|(V_A \otimes V_B)(A^x_a \otimes 1_{\mathcal{H}_B})|\psi\rangle - \big((\tilde{A}^x_a \otimes 1_{\tilde{\mathcal{H}}_B})|\tilde{\psi}\rangle\big) \otimes |aux\rangle\|^2\right)^{1/2} \le \varepsilon.$$

Then there exist partial isometries $V'_A : \mathcal{H}_A \to \tilde{\mathcal{H}}_A \otimes \mathcal{K}'_A$ and $V'_B : \mathcal{H}_B \to \tilde{\mathcal{H}}_B \otimes \mathcal{K}'_B$ such that

$$\|(V'_A \otimes V'_B)|\psi\rangle - |\tilde{\psi}\rangle \otimes |aux\rangle\| \le (1+2\sqrt{2})\varepsilon, \tag{4.31}$$

$$\left(\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \|(V'_A A^x_a \otimes V'_B)|\psi\rangle - \big((\tilde{A}^x_a \otimes 1_{\tilde{\mathcal{H}}_B})|\tilde{\psi}\rangle\big) \otimes |aux\rangle\|^2\right)^{\frac{1}{2}} \le (1+4\sqrt{2})\varepsilon, \tag{4.32}$$

$$1 - \|(V'_A \otimes 1_{\mathcal{H}_B})|\psi\rangle\|^2 \le 4\varepsilon^2, \tag{4.33}$$

$$1 - \|\big((V'_A)^* \otimes 1_{\tilde{\mathcal{H}}_B \otimes \mathcal{K}_B}\big)|\tilde{\psi}\rangle \otimes |aux\rangle\|^2 \le \varepsilon^2. \tag{4.34}$$

Moreover, if $|\tilde{\psi}\rangle \otimes |aux\rangle$ is maximally entangled on $(\tilde{\mathcal{H}}_A \otimes \mathcal{K}'_A) \otimes (\tilde{\mathcal{H}}_B \otimes \mathcal{K}'_B)$, then

$$\|(1_{\mathcal{H}_A} \otimes V'_B)|\psi\rangle - \big((V'_A)^* \otimes 1_{\tilde{\mathcal{H}}_B \otimes \mathcal{K}_B}\big)|\tilde{\psi}\rangle \otimes |aux\rangle\| \le 7\varepsilon. \tag{4.35}$$

Proof. Let $n = \dim(\mathcal{H}_A)$. Let P_A and P_B be the projections onto $\tilde{\mathcal{H}}_A \otimes \mathcal{K}'_A$ and $\tilde{\mathcal{H}}_B \otimes \mathcal{K}'_B$, respectively. Because P_A and P_B are contractions, we have

$$\|(V_A \otimes V_B)|\psi\rangle - (P_A V_A \otimes V_B)|\psi\rangle\| \le 2\varepsilon, \tag{4.36}$$

$$\|(V_A \otimes V_B)|\psi\rangle - (V_A \otimes P_B V_B)|\psi\rangle\| \le 2\varepsilon \text{ and} \tag{4.37}$$

$$\|(V_A \otimes V_B)|\psi\rangle - (P_A V_A \otimes P_B V_B)|\psi\rangle\| \le 2\varepsilon \tag{4.38}$$

by the triangle inequality, using the estimate $\|(V_A \otimes V_B)|\psi\rangle - |\tilde{\psi}\rangle \otimes |aux\rangle\| \le \varepsilon$. Now let $X_A = P_A V_A$, $X_B = P_B V_B$ and let $X_A = V'_A|X_A|$ and $X_B = V'_B|X_B|$ be the polar decompositions of their adjoints. By Lemma 4.A.5, we have that

$$\|(V_A \otimes V_B)|\psi\rangle - (V'_A \otimes V'_B)|\psi\rangle\|^2 \le 2\|(V_A \otimes V_B)|\psi\rangle - (P_A V_A \otimes P_B V_B)|\psi\rangle\|^2.$$

Using the triangle inequality, this implies that

$$\|(V_A'\otimes V_B')|\psi\rangle - |\tilde\psi\rangle\otimes|aux'\rangle\| \le (1+2\sqrt{2})\varepsilon,$$

proving Equation (4.31). Next, let Q be the projection onto the orthogonal complement of the kernel of X_A. We see that

$$\|(V_A'\otimes 1_{\mathcal{H}_B})|\psi\rangle\|^2 = \langle\psi|(Q\otimes 1_{\mathcal{H}_B})|\psi\rangle \ge \langle\psi|(X_A^*X_A\otimes 1_{\mathcal{H}_B})|\psi\rangle = \|(P_AV_A\otimes V_B)|\psi\rangle\|^2.$$

Since V_A and V_B are isometries, the Pythagorean theorem combined with Equation (4.36) tells us that

$$\|(V_A'\otimes 1_{\mathcal{H}_B})|\psi\rangle\|^2 \ge \|(P_AV_A\otimes V_B)|\psi\rangle\|^2 \ge 1-4\varepsilon^2,$$

proving Equation (4.33). Similarly, we compute

$$\begin{aligned}\|\big((V_A')^*\otimes 1_{\tilde{\mathcal{H}}_B\otimes\mathcal{K}_B}\big)|\tilde\psi\rangle\otimes|aux\rangle\| &\ge \|\big(|X_A|(V_A')^*\otimes 1_{\tilde{\mathcal{H}}_B\otimes\mathcal{K}_B}\big)|\tilde\psi\rangle\otimes|aux\rangle\|\\ &= \|\big(V_A^*P_A\otimes 1_{\tilde{\mathcal{H}}_B\otimes\mathcal{K}_B}\big)|\tilde\psi\rangle\otimes|aux\rangle\|\\ &= \|\big(V_AV_A^*\otimes 1_{\tilde{\mathcal{H}}_B\otimes\mathcal{K}_B}\big)|\tilde\psi\rangle\otimes|aux\rangle\|.\end{aligned}$$

Note that $1_{\tilde{\mathcal{H}}_A\otimes\mathcal{K}_A} - V_AV_A^*$ is a contraction and $(1_{\tilde{\mathcal{H}}_A\otimes\mathcal{K}_A} - V_AV_A^*)V_A = 0$, so we have

$$\|\Big((1_{\tilde{\mathcal{H}}_A\otimes\mathcal{K}_A} - V_AV_A^*)\otimes 1_{\tilde{\mathcal{H}}_B\otimes\mathcal{K}_B}\Big)|\tilde\psi\rangle\otimes|aux\rangle\| \le \|(V_A\otimes V_B)|\psi\rangle - |\tilde\psi\rangle\otimes|aux\rangle\| \le \varepsilon. \quad (4.39)$$

These things combine to yield Equation (4.34), i.e.

$$1-\|\big((V_A')^*\otimes 1_{\tilde{\mathcal{H}}_B\otimes\mathcal{K}_B}\big)|\tilde\psi\rangle\otimes|aux\rangle\|^2 \le \varepsilon^2.$$

For the other estimates of this lemma, we need to use that $|\psi\rangle$ is maximally entangled. First, note that for any partial isometry $w_B : \mathcal{H}_B \to \tilde{\mathcal{H}}_B\otimes\mathcal{K}_B'$ we have

$$\|\big((V_A-V_A')\otimes w_B\big)(A_a^x\otimes 1_{\mathcal{H}_B})|\psi\rangle\| \le \|((V_A-V_A')\otimes 1_{\mathcal{H}_B})(A_a^x\otimes 1_{\mathcal{H}_B})|\psi\rangle\|,$$

since $1_{\tilde{\mathcal{H}}_A\otimes\mathcal{K}_A}\otimes w_B$ is a contraction. Consequently,

$$\begin{aligned}&\sum_a\|((V_A-V_A')\otimes V_B')(A_a^x\otimes 1_{\mathcal{H}_B})|\psi\rangle\|^2\\ &\le \sum_a\langle\psi|A_a^x(V_A-V_A')^*(V_A-V_A')A_a^x\otimes 1_{\mathcal{H}_B}|\psi\rangle\\ &= \sum_a \mathrm{Tr}_{(\tilde{\mathcal{H}}_A\otimes\mathcal{K}_A)\otimes\mathcal{H}_B}\big((V_A-V_A')A_a^x\otimes 1_{\mathcal{H}_B}|\psi\rangle\langle\psi|A_a^x(V_A-V_A')^*\otimes 1_{\mathcal{H}_B}\big)\\ &= \frac{1}{n}\sum_a \mathrm{Tr}_{\tilde{\mathcal{H}}_A\otimes\mathcal{K}_A}\big((V_A-V_A')A_a^x 1_{\mathcal{H}_A}A_a^x(V_A-V_A')^*\big)\\ &\le \frac{1}{n}\mathrm{Tr}_{\tilde{\mathcal{H}}_A\otimes\mathcal{K}_A}\big((V_A-V_A')(V_A-V_A')^*\big)\\ &= \frac{1}{n}\mathrm{Tr}_{\mathcal{H}_A}\big((V_A-V_A')^*(V_A-V_A')\big)\\ &= \|\big((V_A-V_A')\otimes 1_{\mathcal{H}_B}\big)|\psi\rangle\|^2,\end{aligned}$$

where we used in the third step that $|\psi\rangle$ is a maximally entangled state. An analogous calculation yields

$$\|(V_A \otimes V_B)|\psi\rangle - (P_A V_A \otimes V_B)|\psi\rangle\|^2 = \frac{1}{n}\mathrm{Tr}_{\mathcal{H}_A}\left((V_A - P_A V_A)^*(V_A - P_A V_A)\right),$$

so by Lemma 4.A.5, we have

$$\|\left((V_A - V'_A) \otimes 1_{\mathcal{H}_B}\right)|\psi\rangle\| \le 2\sqrt{2}\varepsilon,$$

and

$$\sum_a \|((V_A - V'_A) \otimes V'_B)(A^x_a \otimes 1_{\mathcal{H}_B})|\psi\rangle\|^2 \le 8\varepsilon^2.$$

Analogously, we get

$$\|\left(1_{\mathcal{H}_A} \otimes (V_B - V'_B)\right)|\psi\rangle\| \le 2\sqrt{2}\varepsilon \text{ and} \tag{4.40}$$
$$\sum_a \|(V_A \otimes (V_B - V'_B))(A^x_a \otimes 1_{\mathcal{H}_B})|\psi\rangle\|^2 \le 8\varepsilon^2.$$

This in turn implies that

$$\left(\mathop{\mathbb{E}}_{x\sim\nu_A}\sum_a \|(V_A \otimes V_B - V'_A \otimes V'_B)(A^x_a \otimes 1_{\mathcal{H}_B})|\psi\rangle\|^2\right)^{\frac{1}{2}} \le 4\sqrt{2}\varepsilon$$

by the triangle inequality, and another application of it yields Equation (4.32).

Now assume that $|\tilde{\psi}\rangle \otimes |aux\rangle$ is maximally entangled. Let $m = \dim(\tilde{\mathcal{H}}_A \otimes \mathcal{K}'_A)$. Then for any $X : \mathcal{H}_A \to \tilde{\mathcal{H}}_A \otimes \mathcal{K}_A$ we have

$$\begin{aligned}
\|(X \otimes 1_{\mathcal{H}_B})|\psi\rangle\|^2 &= \mathrm{Tr}_{\tilde{\mathcal{H}}_A\otimes\mathcal{K}_A\otimes\mathcal{H}_B}\left((X \otimes 1_{\mathcal{H}_B})|\psi\rangle\langle\psi|(X \otimes 1_{\mathcal{H}_B})^*\right)\\
&= \frac{1}{n}\mathrm{Tr}_{\tilde{\mathcal{H}}_A\otimes\mathcal{K}_A}\left(XX^*\right)\\
&\ge \frac{m}{n}\mathrm{Tr}_{\tilde{\mathcal{H}}_A\otimes\mathcal{K}_A\otimes\tilde{\mathcal{H}}_B\otimes\mathcal{K}_B}\left((XX^* \otimes 1_{\tilde{\mathcal{H}}_B\otimes\mathcal{K}_B})(|\tilde{\psi}\rangle \otimes |aux\rangle)(\langle\tilde{\psi}| \otimes \langle aux|)\right)\\
&= \frac{m}{n}\|(X^* \otimes 1_{\tilde{\mathcal{H}}_B\otimes\mathcal{K}_B})|\tilde{\psi}\rangle \otimes |aux\rangle\|^2.
\end{aligned}$$

To use the above result, we will need an estimate relating m and n. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be the sequences of Schmidt coefficients of $|\psi\rangle$ and $|\tilde{\psi}\rangle \otimes |aux\rangle$, respectively. By Lemma 4.A.1, observing that local isometries preserve the Schmidt coefficients, and the assumptions in Lemma 4.A.6, we know that

$$\varepsilon^2 \ge \|\boldsymbol{\lambda} - \boldsymbol{\mu}\|_2^2.$$

Since both $|\psi\rangle$ and $|\tilde{\psi}\rangle \otimes |aux\rangle$ are maximally entangled on Hilbert spaces of dimension n and m, respectively, we find that (see also the proof of Lemma 4.5.1)

$$\|\boldsymbol{\lambda} - \boldsymbol{\mu}\|_2^2 \ge \max\left(\frac{m-n}{m}, \frac{n-m}{n}\right),$$

so $(1-\varepsilon^2)m \le n \le (1-\varepsilon^2)^{-1}m$.

Consequently,

$$\|\big((V_A^* - (V_A')^*) \otimes 1_{\tilde{\mathcal{H}}_B \otimes \mathcal{K}_B}\big) |\tilde{\psi}\rangle \otimes |aux\rangle\| \leq \frac{1}{1-\varepsilon^2} \|\big((V_A - V_A') \otimes 1_{\mathcal{H}_B}\big) |\psi\rangle\| \leq \frac{2\sqrt{2}\varepsilon}{1-\varepsilon^2}. \quad (4.41)$$

So for the “Moreover" part, we have

$$\begin{aligned}
&\|(1_{\mathcal{H}_A} \otimes V_B') |\psi\rangle - \big((V_A')^* \otimes 1_{\tilde{\mathcal{H}}_B \otimes \mathcal{K}_B}\big) |\tilde{\psi}\rangle \otimes |aux\rangle\| \\
&\qquad \leq \|\big((V_A^* - (V_A')^*) \otimes 1_{\tilde{\mathcal{H}}_B \otimes \mathcal{K}_B}\big) |\tilde{\psi}\rangle \otimes |aux\rangle\| \\
&\qquad + \|(V_A^* \otimes 1_{\tilde{\mathcal{H}}_B \otimes \mathcal{K}_B})\big((V_A \otimes V_B) |\psi\rangle - |\tilde{\psi}\rangle \otimes |aux\rangle\big)\| \\
&\qquad + \|\big(1_{\mathcal{H}_A} \otimes (V_B - V_B')\big) |\psi\rangle\| \\
&\qquad \leq (1 + 2\sqrt{2} + \frac{2\sqrt{2}}{1-\varepsilon^2})\varepsilon,
\end{aligned}$$

using that V_A^* is a contraction, together with Equations (4.40) and (4.41). Since the above estimate becomes trivial if $\varepsilon > 0.29$, we conclude that

$$\|(1_{\mathcal{H}_A} \otimes V_B') |\psi\rangle - \big((V_A')^* \otimes 1_{\tilde{\mathcal{H}}_B \otimes \mathcal{K}_B}\big) |\tilde{\psi}\rangle \otimes |aux\rangle\| \leq 7\varepsilon,$$

proving Equation (4.35). □

Now we are ready to prove part (a) of Lemma 4.3.2.

Proof of Lemma 4.3.2, part (a). Let $|\tilde{\psi}\rangle \in \tilde{\mathcal{H}} \otimes \tilde{\mathcal{H}}$ and $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ be maximally entangled states such that $\tau^{\mathcal{M}}(x) = \langle\tilde{\psi}| x \otimes 1_{\mathcal{H}} |\tilde{\psi}\rangle$ for all $x \in \mathcal{M} \subset B(\tilde{\mathcal{H}})$ and $\tau^{\mathcal{N}}(x) = \langle\psi| x \otimes 1_{\tilde{\mathcal{H}}} |\psi\rangle$ for all $x \in \mathcal{N} \subset B(\mathcal{H})$. Since $\tilde{\mathcal{S}}$ is an (ε, ν)-local dilation of $\tilde{\mathcal{S}}$, by Lemma 4.A.2, there are isometries $V_A : \mathcal{H} \to \tilde{\mathcal{H}} \otimes \mathcal{K}_A$ and $V_B : \mathcal{H} \to \tilde{\mathcal{H}} \otimes \mathcal{K}_B$ and a maximally entangled state $|aux\rangle \in \mathcal{K}_A' \otimes \mathcal{K}_B' \subset \mathcal{K}_A \otimes \mathcal{K}_B$ such that

$$\begin{aligned}
\|(V_A \otimes V_B) |\psi\rangle - |\tilde{\psi}\rangle \otimes |aux\rangle\| &\leq 3\varepsilon, \\
\Big(\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \|(V_A \otimes V_B)(A_a^x \otimes 1) |\psi\rangle - \big((\tilde{A}_a^x \otimes 1) |\tilde{\psi}\rangle\big) \otimes |aux\rangle\|^2\Big)^{1/2} &\leq 3\varepsilon.
\end{aligned}$$

Then by Lemma 4.A.6, there exist partial isometries $V_A' : \mathcal{H} \to \tilde{\mathcal{H}} \otimes \mathcal{K}_A'$ and $V_B' : \mathcal{H} \to \tilde{\mathcal{H}} \otimes \mathcal{K}_B'$ such that

$$\begin{aligned}
\|(V_A' \otimes V_B') |\psi\rangle - |\tilde{\psi}\rangle \otimes |aux\rangle\| &\leq (3 + 6\sqrt{2})\varepsilon, \\
\Big(\mathop{\mathbb{E}}_{x\sim\nu_A} \sum_a \|(V_A' \otimes V_B')(A_a^x \otimes 1_{\mathcal{H}}) |\psi\rangle - \big((\tilde{A}_a^x \otimes 1_{\tilde{\mathcal{H}}}) |\tilde{\psi}\rangle\big) \otimes |aux\rangle\|^2\Big)^{1/2} &\leq (3 + 12\sqrt{2})\varepsilon, \\
\|(1_{\mathcal{H}} \otimes V_B') |\psi\rangle - \big((V_A')^* \otimes 1_{\tilde{\mathcal{H}} \otimes \mathcal{K}_B}\big) |\tilde{\psi}\rangle \otimes |aux\rangle\| &\leq 21\varepsilon, \\
1 - \|(V_A' \otimes 1_{\mathcal{H}}) |\psi\rangle\|^2 &\leq 36\varepsilon^2, \\
1 - \|\big((V_A')^* \otimes 1_{\tilde{\mathcal{H}} \otimes \mathcal{K}_B'}\big) |\tilde{\psi}\rangle \otimes |aux\rangle\|^2 &\leq 9\varepsilon^2.
\end{aligned}$$

Let $\mathcal{M}_0 := B(\mathcal{K}_A') \subset B(\ell^2(\mathbb{N}))$ and $P \in (\mathcal{M} \otimes \mathcal{M}_0)^\infty$ be a projection such that $\mathcal{N} \cong P(\mathcal{M} \otimes \mathcal{M}_0)^\infty P$. After identifying $\mathcal{N}$ and $P(\mathcal{M} \otimes \mathcal{M}_0)^\infty P$, we can define $W = (V_A')^* \in P(\mathcal{M} \otimes \mathcal{M}_0)^\infty I_{\mathcal{M} \otimes \mathcal{M}_0}$.

Since $|\tilde{\psi}\rangle \otimes |aux\rangle$ is an maximally entangled state in $\tilde{\mathscr{H}} \otimes \mathscr{K}'_A$, and $|\psi\rangle$ is a maximally entangled state in $\mathscr{H} \otimes \mathscr{H}$, we have

$$\tau^{\mathscr{N}}(P - WW^*) = 1 - \|(V'_A \otimes 1_{\mathscr{H}})|\psi\rangle\|^2 \leq 36\varepsilon^2,$$
$$\tau^{\mathscr{M}\otimes\mathscr{M}_0}(I_{\mathscr{M}\otimes\mathscr{M}_0} - W^*W) = 1 - \|\big((V'_A)^* \otimes 1_{\tilde{\mathscr{H}}\otimes\mathscr{K}'_B}\big)|\tilde{\psi}\rangle \otimes |aux\rangle\|^2 \leq 9\varepsilon^2,$$

and

$$\begin{aligned}
&\Big(\mathop{\mathbb{E}}_{x\sim\nu_A}\sum_a \|\tilde{A}^x_a \otimes I_{\mathscr{M}_0} - W^* A^x_a W\|^2_{\tau^{\mathscr{M}\otimes\mathscr{M}_0}}\Big)^{1/2}\\
=&\Big(\mathop{\mathbb{E}}_{x\sim\nu_A}\sum_a \|\big(V'_A A^x_a (V'_A)^* \otimes 1_{\tilde{\mathscr{H}}\otimes\mathscr{K}'_A}\big)|\tilde{\psi}\rangle \otimes |aux\rangle - \big((\tilde{A}^x_a \otimes 1_{\tilde{\mathscr{H}}})|\tilde{\psi}\rangle\big) \otimes |aux\rangle\|^2\Big)^{1/2}\\
\leq&\Big(\mathop{\mathbb{E}}_{x\sim\nu_A}\sum_a \|\big(V'_A A^x_a \otimes V'_B\big)|\psi\rangle - \big((\tilde{A}^x_a \otimes 1_{\tilde{\mathscr{H}}})|\tilde{\psi}\rangle\big) \otimes |aux\rangle\|^2\Big)^{1/2} + 21\varepsilon\\
\leq&(24 + 12\sqrt{2})\varepsilon,
\end{aligned}$$

where the first inequality uses that $\sum_a A^x_a (V'_A)^* V'_A A^x_a \leq 1$. We conclude that $\tilde{\mathscr{S}}$ is a local $\big((24+12\sqrt{2})^2\varepsilon^2, \nu_A\big)$-vNA-dilation of $\mathscr{S}$. Rounding $(24+12\sqrt{2})^2$ to 1700 gives the desired results. □

For the proof of the second part of Lemma 4.3.2 we first need the following lemma.

Lemma 4.A.7. *Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras with tracial states $\tau^{\mathscr{M}}$ and $\tau^{\mathscr{N}}$. Suppose that there is a projection $P \in \mathscr{M}^\infty$ with finite trace such that $\mathscr{N} \cong P\mathscr{M}^\infty P$ and $\tau^{\mathscr{N}} = (\tau^\infty(P))^{-1}\tau^\infty$ after identification of $\mathscr{N}$ and $P\mathscr{M}^\infty P$. If there exist a $\delta < 1$ and a partial isometry $w \in P\mathscr{M}^\infty I_{\mathscr{M}}$ such that*

$$\tau^{\mathscr{M}}(I_{\mathscr{M}} - w^*w) \leq \delta \text{ and } \tau^{\mathscr{N}}(P - ww^*) \leq \delta,$$

then $1 - \delta \leq \tau^\infty(P) \leq (1-\delta)^{-1}$.

Proof. We show that $(1-\delta) \leq \tau^\infty(P)$. The other inequality is analogous. We compute

$$\begin{aligned}
\delta &\geq \tau^{\mathscr{M}}(I_{\mathscr{M}} - w^*w)\\
&= 1 - \tau^\infty(w^*w)\\
&= 1 - \tau^\infty(P)\tau^{\mathscr{N}}(ww^*)\\
&= 1 - \tau^\infty(P) + \tau^\infty(P)\tau^{\mathscr{N}}(P - ww^*)\\
&\geq 1 - \tau^\infty(P),
\end{aligned}$$

so $(1-\delta) \leq \tau^\infty(P)$. □

Proof of Lemma 4.3.2, part (b). Let $|\tilde{\psi}\rangle$ and $|\psi\rangle$ be GNS states for $(\mathscr{M}, \tau^{\mathscr{M}})$ and $(\mathscr{N}, \tau^{\mathscr{N}})$, respectively. Since $\tilde{\mathscr{S}}$ is a local (ε, ν_A)-vNA-dilation of $\mathscr{S}$, there exist a finite-dimensional von Neumann algebra $\mathscr{M}_0$ with tracial state $\tau^{\mathscr{M}_0}$, a projection $P \in (\mathscr{M} \otimes \mathscr{M}_0)^\infty$ of finite trace such that $\mathcal{N} \cong P(\mathcal{M}\otimes\mathcal{M}_0)^\infty P$ and $\tau^{\mathcal{N}} = \tau^\infty/\tau^\infty(P)$, and a partial isometry $W \in P(\mathcal{M}\otimes\mathcal{M}_0)^\infty I_{\mathcal{M}\otimes\mathcal{M}_0}$ such that

$$\mathop{\mathbb{E}}_{x\sim\nu_A}\sum_a \|\tilde{A}^x_a\otimes I_{\mathcal{M}_0} - W^* A^x_a W\|^2_{\tau^{\mathcal{M}\otimes\mathcal{M}_0}} \le \varepsilon \tag{4.42}$$

and

$$\tau^{\mathcal{N}}(P - WW^*) \le \varepsilon, \tau^{\mathcal{M}\otimes\mathcal{M}_0}(I_{\mathcal{M}\otimes\mathcal{M}_0} - W^*W) \le \varepsilon. \tag{4.43}$$

Our first step is to turn W^* into an isometry. Let $P_1 = P - WW^*$ and let $\check{\mathcal{M}}_1 \subset (1 - I_{\mathcal{M}\otimes\mathcal{M}_0})(\mathcal{M}\otimes\mathcal{M}_0)^\infty(1 - I_{\mathcal{M}\otimes\mathcal{M}_0})$ be a finite-dimensional von Neumann algebra of the form $\mathcal{M}\otimes B(\mathcal{H}_1)$ such that $\dim P_1\mathcal{N}P_1 \le \dim\check{\mathcal{M}}_1$. Let $\mathcal{M}_1 = B(\mathcal{H}_1)$. We can now choose a partial isometry $W_1 \in P_1(\mathcal{M}\otimes\mathcal{M}_0)^\infty I_{\check{\mathcal{M}}_1}$ such that $W_1W_1^* = P_1$. Define $V = W^* + W_1^*$, which now satisfies $P = V^*V$. Observe that

$$\tau^{\mathcal{M}\otimes\mathcal{M}_0}(x) = \tau^\infty((I_{\mathcal{M}\otimes\mathcal{M}_0} + I_{\check{\mathcal{M}}_1})I_{\mathcal{M}\otimes\mathcal{M}_0} x I_{\mathcal{M}\otimes\mathcal{M}_0}(I_{\mathcal{M}\otimes\mathcal{M}_0} + I_{\check{\mathcal{M}}_1})) \text{ for } x \in \mathcal{M}\otimes\mathcal{M}_0 \text{ and}$$
$$\tau^{\mathcal{N}}(x) = \frac{1}{\tau^\infty(P)}\tau^\infty((I_{\mathcal{M}\otimes\mathcal{M}_0} + I_{\check{\mathcal{M}}_1})VxV^*(I_{\mathcal{M}\otimes\mathcal{M}_0} + I_{\check{\mathcal{M}}_1})) \text{ for } x \in P(\mathcal{M}\otimes\mathcal{M}_0)^\infty P.$$

If we now perform the GNS construction using the state

$$x \mapsto \tau^\infty(I_{\check{\mathcal{M}}_1} x I_{\check{\mathcal{M}}_1})$$

on $\check{\mathcal{M}}_1$, we obtain a cyclic vector $|\varphi\rangle \in (\tilde{\mathcal{H}}\otimes\mathcal{H}_1)\otimes(\tilde{\mathcal{H}}\otimes\mathcal{H}_1)$ such that

$$\tau^\infty(I_{\check{\mathcal{M}}_1} x I_{\check{\mathcal{M}}_1}) = \langle\varphi|(x\otimes 1_{\tilde{\mathcal{H}}\otimes\mathcal{H}_1})|\varphi\rangle = \langle\varphi|(1_{\tilde{\mathcal{H}}\otimes\mathcal{H}_1}\otimes x^T)|\varphi\rangle.$$

Note that $|\varphi\rangle$ is **not** a unit vector, but we will still use bra-ket notation for convenience. Let $|\psi_0\rangle \in \mathcal{H}_0\otimes\mathcal{H}_0$ be a GNS state for $(\mathcal{M}_0, \tau^{\mathcal{M}_0})$. All in all, this means that

$$\tau^\infty(x) = (\langle\tilde{\psi}|\otimes\langle\psi_0| + \langle\varphi|)(x\otimes 1_{\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1)})(|\tilde{\psi}\rangle\otimes|\psi_0\rangle + |\varphi\rangle)$$

for all $x \in \mathcal{M}\otimes(\mathcal{M}_0\oplus\mathcal{M}_1)$. Since

$$x \mapsto \tau^\infty(P)\langle\psi|(V^*\otimes\overline{V}^*)\left(x\otimes 1_{\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\otimes\mathcal{H}_1)}\right)(V\otimes\overline{V})|\psi\rangle$$

and

$$x \mapsto (\langle\tilde{\psi}|\otimes\langle\psi_0| + \langle\varphi|)\left(VV^*xVV^*\otimes 1_{\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1)}\right)(|\tilde{\psi}\rangle\otimes|\psi_0\rangle + |\varphi\rangle)$$

implement the same positive linear functional on $\mathcal{M}\otimes(\mathcal{M}_0\oplus\mathcal{M}_1)$ and both

$$(V\otimes\overline{V})|\psi\rangle, \left(VV^*\otimes 1_{\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1)}\right)(|\tilde{\psi}\rangle\otimes|\psi_0\rangle + |\varphi\rangle) \in (\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1))\otimes(\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1)),$$

we know by the uniqueness of the GNS construction that there exists a unitary $U \in B(\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1))$ such that

$$\left(1_{\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1)}\otimes U\right)(V\otimes\overline{V})|\psi\rangle = \left(VV^*\otimes 1_{\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1)}\right)(|\tilde{\psi}\rangle\otimes|\psi_0\rangle + |\varphi\rangle).$$

We now aim to show that $\left(VV^*\otimes 1_{\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1)}\right)(|\tilde{\psi}\rangle\otimes|\psi_0\rangle + |\varphi\rangle)$ is close to $|\tilde{\psi}\rangle\otimes|\psi_0\rangle$. Note that

$$VV^* = (W^* + W_1^*)(W + W_1) = W^*W + W_1^*W_1.$$

Furthermore, we have

$$\left(W\otimes 1_{\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1)}\right)|\varphi\rangle = 0 = \left(W_1\otimes 1_{\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1)}\right)|\tilde{\psi}\rangle\otimes|\psi_0\rangle\,,$$

so

$$\begin{aligned}\left(VV^*\otimes 1_{\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1)}\right)(|\tilde{\psi}\rangle\otimes|\psi_0\rangle+|\varphi\rangle) = &\left(W^*W\otimes 1_{\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1)}\right)|\tilde{\psi}\rangle\otimes|\psi_0\rangle\\ &+\left(W_1^*W_1\otimes 1_{\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1)}\right)|\varphi\rangle\,.\end{aligned}$$

We then compute

$$\begin{aligned}\langle\varphi|\left(W_1^*W_1\otimes 1_{\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1)}\right)^2|\varphi\rangle &= \tau^\infty(W_1^*W_1)\\ &=\tau^\infty(P)\tau^{\mathcal{N}}(W_1W_1^*)\\ &=\tau^\infty(P)\tau^{\mathcal{N}}(P-WW^*)\le\tau^\infty(P)\varepsilon\end{aligned}$$

and

$$\begin{aligned}\langle\tilde{\psi}|\otimes\langle\psi_0|\left((1_{\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1)}-W^*W)\otimes 1_{\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1)}\right)^2|\tilde{\psi}\rangle\otimes|\psi_0\rangle &= \tau^\infty((I_{\mathcal{M}\otimes\mathcal{M}_0}-W^*W)^2)\\ &\le\varepsilon.\end{aligned}$$

Consequently,

$$\|\left(1_{\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1)}\otimes U\right)(V\otimes\overline{V})|\psi\rangle-|\tilde{\psi}\rangle\otimes|\psi_0\rangle\|\le\sqrt{\varepsilon}+\sqrt{\tau^\infty(P)\varepsilon}\le 3\sqrt{\varepsilon},$$

where we use Lemma 4.A.7 for the last inequality. Define $V_A=V$ and $V_B=U\overline{V}$, so

$$\|(V_A\otimes V_B)|\psi\rangle-|\tilde{\psi}\rangle\otimes|\psi_0\rangle\|\le 3\sqrt{\varepsilon}.$$

By the introduction of $|\tilde{\psi}\rangle$ and $|\psi_0\rangle$, Equation (4.42) becomes

$$\left(\mathop{\mathbb{E}}_{x\sim\nu_A}\sum_a\|(\tilde{A}_a^x\otimes 1_{\mathcal{H}_0}\otimes 1_{\tilde{\mathcal{H}}\otimes\mathcal{H}_0})|\tilde{\psi}\rangle\otimes|\psi_0\rangle-(W^*A_a^xW\otimes 1_{\tilde{\mathcal{H}}\otimes\mathcal{H}_0})|\tilde{\psi}\rangle\otimes|\psi_0\rangle\|^2\right)^{1/2}\le\sqrt{\varepsilon}.$$

Since

$$\sum_a(W^*A_a^xW)^2\le 1,$$

we know that

$$\left(\sum_a\|\left(W^*A_a^xW\otimes 1_{\tilde{\mathcal{H}}\otimes(\mathcal{H}_0\oplus\mathcal{H}_1)}\right)(|\tilde{\psi}\rangle\otimes|\psi_0\rangle-(V_A\otimes V_B)|\psi\rangle)\|^2\right)^{1/2}\le 3\sqrt{\varepsilon}.$$

The last step is to find an estimate for

$$\begin{aligned}&\sum_a\|\left((W^*A_a^xWV_A-V_AA_a^x)\otimes V_B\right)|\psi\rangle\|^2\\ &\qquad=\sum_a\tau^{\mathcal{N}}\left((W^*A_a^xWV_A-V_AA_a^x)^*(W^*A_a^xWV_A-V_AA_a^x)\right).\end{aligned}$$

We compute

$$
\begin{aligned}
\|((W^*A_a^xWV_A - V_AA_a^x)\otimes V_B)\,|\psi\rangle\|^2 =&\tau^{\mathcal{N}}\Big(V_A^*W^*A_a^xWW^*A_a^xWV_A + A_a^xV_A^*V_AA_a^x \\
&- A_a^xV_A^*W^*A_a^xWV_A - V_A^*W^*A_a^xWV_AA_a^x\Big) \\
=&\tau^{\mathcal{N}}\Big(WW^*A_a^xWW^*A_a^x + (A_a^x)^2 \\
&- A_a^xWW^*A_a^xWW^* - WW^*A_a^xWW^*A_a^x\Big) \\
=&\tau^{\mathcal{N}}\left((A_a^x)^2 - WW^*A_a^xWW^*A_a^x\right) \\
\leq&\tau^{\mathcal{N}}\left((P-WW^*)(A_a^x)^2\right) \\
&+\tau^{\mathcal{N}}\left((P-WW^*)A_a^xWW^*A_a^x\right) \\
\leq&2\tau^{\mathcal{N}}\left((P-WW^*)(A_a^x)^2\right).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\sum_a \|(W^*A_a^xWV_A - V_AA_a^x)\otimes V_B)\,|\psi\rangle\|^2 &\leq 2\sum_a \tau^{\mathcal{N}}\left((P-WW^*)(A_a^x)^2\right) \\
&\leq 2\tau^{\mathcal{N}}(P-WW^*) \leq 2\varepsilon.
\end{aligned}
$$

By the triangle inequality, we have shown that

$$
\left(\mathop{\mathbb{E}}_{x\sim\nu_A}\sum_a \left\|\left(\tilde{A}_a^x\otimes 1_{\mathcal{H}_0}\otimes 1_{\tilde{\mathcal{H}}\otimes\mathcal{H}_0}\right)|\tilde{\psi}\rangle\otimes|\psi_0\rangle - (V_AA_a^x\otimes V_B)\,|\psi\rangle\right\|^2\right)^{1/2} \leq (4+\sqrt{2})\sqrt{\varepsilon}.
$$

Analogously, we obtain

$$
\left(\mathop{\mathbb{E}}_{x\sim\nu_A}\sum_a \left\|1_{\tilde{\mathcal{H}}\otimes\mathcal{H}_0}\otimes((\tilde{A}_a^x)^T\otimes 1_{\mathcal{H}_0})\,|\tilde{\psi}\rangle\otimes|\psi_0\rangle - (V_A\otimes V_B(A_a^x)^T)\,|\psi\rangle\right\|^2\right)^{1/2} \leq (4+\sqrt{2})\sqrt{\varepsilon},
$$

proving the lemma. □

5

Quantum expanders and property (T) discrete quantum groups

This chapter is based on the following article:

- Michael Brannan, Eric Culf and Matthijs Vernooij. *Quantum expanders and property (T) discrete quantum groups*. 2025. arXiv: 2502.01974 [math.OA].

Families of expander graphs were first constructed by Margulis from discrete groups with property (T). Within the framework of quantum information theory, several authors have generalised the notion of an expander graph to the setting of quantum channels. In this chapter, we use discrete quantum groups with property (T) to construct quantum expanders in two ways. The first approach obtains a quantum expander family by constructing the requisite quantum channels directly from finite-dimensional irreducible unitary representations, extending earlier work of Harrow using groups. The second approach directly generalises Margulis' original construction and is based on a quantum analogue of a Schreier graph using the theory of coideals. To obtain examples of quantum expanders, we apply our machinery to discrete quantum groups with property (T) coming from compact bicrossed products.

5.1. Introduction

Expander graphs (cf. Section 5.2.1) are particularly well-connected sparse graphs. They appear in many applications, such as robustly connected networks, error correcting codes, the complexity of local Hamiltonians, pseudorandomness, the Baum-Connes conjecture and Monte-Carlo simulations [HLW06; AE15]. In the field of quantum information theory, the notion of quantum expanders (cf. Section 5.2.2) arises naturally as an extension of classical expander graphs [Has07a; Har08]. Just as in the classical case, randomness, in this case in the form of random matrix techniques, provides an important way to construct quantum expanders [Has07b; LY24].

To give explicit examples of expander families, we need deterministic constructions. These constructions tend to be based on the representation theory of groups, for example Margulis's original idea to construct classical expanders using property (T) groups (cf. Section 5.2.1) and Harrow's construction of quantum expanders using finite groups (cf. Section 5.2.2) [Mar73; Har08]. Both of these approaches use the fact that the geometry of the representation theory of a group Γ (e.g., finiteness, amenability, property (T), and so on) provides information about the spectral gap of certain convolution operators on Γ (or its homogeneous spaces), and this in turn provides (via Cheeger inequalities) bounds on the expansion constants of related graphs or quantum channels.

This chapter aims to develop the representation-theoretic tools for constructing general quantum expanders. Discrete quantum groups and their representations can be used to construct more general examples of bistochastic quantum channels (cf. Section 5.2.4), which, in contrast to quantum channels coming from groups, are often not mixed unitary. Since property (T) makes sense for discrete quantum groups (cf. Section 5.2.3), it is natural to ask if Margulis' construction is possible in the quantum setting. We show that this is indeed the case.

5

More precisely, we are able to construct quantum expanders from discrete quantum groups with property (T) in two ways. First, we extend the construction of quantum expanders by Harrow to obtain the following result (see Theorem 5.3.7 for the precise formulation):

Theorem 5.A. *Let $\mathbf{\Gamma}$ be an infinite property (T) discrete quantum group with fixed symmetric Kazhdan pair $(E,\varepsilon) \in \mathrm{Irred}(\mathbf{G}) \times \mathbb{R}_{>0}$, where $\mathrm{Irred}(\mathbf{G})$ denotes the set of equivalence classes of irreducible unitary representations of the compact quantum group $\mathbf{G}$ dual to $\mathbf{\Gamma}$. Let $(U_i)_{i\in\mathbb{N}}$ be a sequence of finite-dimensional irreducible unitary representations of $\mathbf{\Gamma}$ in $M_{n_i}(\mathbb{C})$ with $\lim_{i\to\infty} n_i = \infty$. Then one can construct a family of bistochastic quantum channels $\mathscr{F} = (\Psi_i : M_{n_i}(\mathbb{C}) \to M_{n_i}(\mathbb{C}))_{i\in\mathbb{N}}$ which is a bounded degree quantum expander family with quantum edge expansion*

$$h_Q(\mathscr{F}) := \inf_{i\in\mathbb{N}} h_Q(\Psi_i) \geq K\varepsilon^2,$$

where $K = (4\sum_{x\in E} \dim x)^{-1}$.

When $\mathbf{\Gamma}$ is a classical discrete group, we have $\dim x = 1$ for all $x \in \mathrm{Irred}(\mathbf{G})$, giving $K = \frac{1}{4|E|}$, which is the familiar constant appearing in previous constructions of expander graphs using property (T) groups. When $\mathbf{\Gamma}$ is a genuine discrete quantum group (i.e., not actually a group), the resulting family of expanders $\mathscr{F}$ appearing above has the interesting feature that it may no longer be of mixed unitary type. Thus Theorem A can be seen as means to construct explicit quantum expanders of a rather different flavour than those currently existing in the literature. We use Theorem A to construct and examine quantum expanders arising from compact bicrossed product quantum groups in Section 5.4.

In recent years, a theory of (finite) *quantum graphs* has emerged, based on the notion of a *quantum adjacency matrix* [MRV18; Was23; Daw24]. The fundamental idea is to consider a non-commutative vertex set modelled by a finite dimensional von Neumann algebra $\mathcal{M}$, equipped with a completely positive linear map $A: \mathcal{M} \to \mathcal{M}$ satisfying a certain Schur-idempotency condition. The pair $(\mathcal{M}, A)$ is then called a quantum graph. We review these ideas in Section 5.2.2. From the quantum adjacency matrix view on quantum graphs, notions like regularity, connectedness, and spectral gap all have natural generalisations. The study of quantum expanders from the perspective of quantum adjacency matrices has yet to be studied in any detail in the literature, and we provide some first contributions in this direction. First, we show that the commonly used notions of degree (= the Kraus rank) for bistochastic quantum channels on $M_n(\mathbb{C})$ and for quantum adjacency matrices on $M_n(\mathbb{C})$ agree (Proposition 5.2.22). Next, using Wasilewski's recent formulation of quantum Cayley graphs using quantum adjacency matrices [Was23] and the notion of a coideal in a discrete quantum group, we formulate a natural notion of a *quantum Schreier graph*. We then use these ideas to prove Theorem 5.5.7, which can be regarded as a more direct generalisation of Margulis' original construction of expander graphs as Schreier coset graphs:

Theorem 5.B. *Let $\mathbf{\Gamma}$ be an infinite property (T) discrete quantum group with symmetric Kazhdan pair (E, ε). Denote by $A: \ell^\infty(\mathbf{\Gamma}) \to \ell^\infty(\mathbf{\Gamma})$ the convolution operator $Ax = p_E \star x$, ($x \in \ell^\infty(\mathbf{\Gamma})$), describing the quantum Cayley graph $\mathscr{C}(\mathbf{\Gamma}, E)$, where p_E is the central projection associated to E. For each $i \in \mathbb{N}$, let $\mathcal{M}_i \subset \ell^\infty(\mathbf{\Gamma})$ be a finite-dimensional coideal, and assume moreover $\lim_{i\to\infty} \dim(\mathcal{M}_i) = \infty$. Then the family $(\mathcal{M}_i, A|_{\mathcal{M}_i})$ of quantum Schreier graphs forms a quantum expander family of bounded degree.*

As we explain in Section 5.5, when $\mathbf{\Gamma}$ is classical, our quantum Schreier graphs are in one-to-one correspondence with the finite Schreier coset graphs of $\mathbf{G}$, recovering Margulis' result. In the quantum case, this will generally yield quantum expander graphs over multimatrix algebras.

The remainder of the chapter is organised as follows: Section 5.2 contains a detailed discussion of classical expander graphs, Cheeger inequalities, and their quantisations. We also provide a detailed recap of the relevant facts about quantum groups, their associated quantum channels, and property (T) in Section 5.2.3. Section 5.3 is dedicated to proving Theorem 5.A, while Section 5.4 applies these results to examples coming from bicrossed products. Section 5.5 is then dedicated to introducing the notion of a quantum Schreier graph, and proving Theorem 5.B. Finally, we conclude the chapter with a brief discussion and outlook for future work in Section 5.6.

5.2. PRELIMINARIES

5.2.1. CLASSICAL EXPANDERS

In this section, we recall the definition of a (classical) expander graph, and outline the construction of an expander family from a property (T) group due to Margulis [Mar73]. Our main reference for this section is the book of Kowalski [Kow19].

EXPANDER GRAPHS

Definition 5.2.1. A (simple undirected loop-free) *graph* is a pair of sets $G = (V, E)$, where V is the set of vertices and E is the set of edges, corresponding to pairs of distinct elements from V. We assume V is finite unless otherwise specified. For a graph G, we use the following notation:

- The *neighbourhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid \{u, v\} \in E\}$.
- We say G is d*-regular* if $|N(v)| = d$ for all $v \in V$.
- Given subsets $X,\ Y \subset V$, write $E(X, Y) = \big\{\{x,\ y\} \in E \,\big|\, x \in X, y \in Y\big\}$ for the set of edges between X and Y.
- The *adjacency matrix*, written $A(G)$, is the $|V| \times |V|$ matrix with entries labelled by pairs of vertices such that $A(G)_{x,y} = 1$ if and only if $\{x,\ y\} \in E$ and 0 otherwise.

Definition 5.2.2. The *expansion constant* (or Cheeger constant) of a graph G is

$$h(G) = \min\left\{ \frac{|E(U, V \setminus U)|}{\min\{|U|, |V \setminus U|\}} \;\middle|\; \varnothing \subsetneq U \subsetneq V \right\}.$$

Note that we have to restrict to connected graphs for this to be non-zero: the expansion constant of any disconnected graph is 0.

The expansion constant is closely related to the spectral gap of the adjacency matrix of the graph. This relationship gives an important tool for studying expansion properties. First, note that if G is d-regular, then the largest eigenvalue of $A(G)$ is d, and if G is connected, the multiplicity of this eigenvalue is 1. Denote the second-largest eigenvalue of $A(G)$ by $\lambda_2(G)$. Then, the spectral gap is $d - \lambda_2(G)$, which relates to $h(G)$ as follows.

Theorem 5.2.3 (Discrete Cheeger inequalities [Dod84; AM85])**.** *Let G be a connected d-regular graph. Then,*

$$\frac{1}{2}(d - \lambda_2(G)) \leq h(G) \leq \sqrt{2d(d - \lambda_2(G))}.$$

There is a slightly stronger form of the upper bound, given by $\sqrt{d^2 - \lambda_2(G)^2}$ due to [Moh89]. See [Kow19] for extensions of these inequalities to non-regular graphs.

Definition 5.2.4. Let $\eta > 0$ and $d \in \mathbb{N}$. We say a graph G is a (d, η)*-expander* if it is d-regular and $h(G) \geq \eta$. A (d, η)*-expander family* is a sequence (G_n) of d-regular graphs such that $|G_n| \to \infty$ and $h(G_n) \geq \eta$.

An important use of expanders is in *expander random walk sampling*. Informally, a random walk on an expander graph reaches any vertex in the graph very quickly. One application of this is to reduce the amount of randomness needed to take an average of a function over the vertices, by taking the average over a short random walk [AKS87]. See [Gil98] for the formal statement and a full proof. Importantly for our context, in the random walk picture, a d-regular graph G gives rise to a bistochastic matrix $d^{-1}A(G)$, which is the description which quantises most naturally.

PROPERTY (T) GROUPS AND EXPANDERS

A natural way to construct examples of regular graphs is from groups.

Definition 5.2.5. Let Γ be a finitely-generated group with a generating set S that is symmetric, *i.e.* $S^{-1} = S$. The *Cayley graph* of Γ with respect to S is the graph $\mathscr{C}(\Gamma, S) = (\Gamma, E)$ with $E = \big\{\{g, gs\} \,\big|\, g \in \Gamma,\ s \in S\big\}$.

Note that $\mathscr{C}(\Gamma, S)$ is finite if Γ is finite and is infinite for infinite finitely-generated groups. Also, the Cayley graph can be defined with respect to an arbitrary set S, but if it is not a generating set, $\mathscr{C}(\Gamma, S)$ will not be connected. As will be noted below for quotient spaces, the notion of a Cayley graph can be extended to sets on which Γ acts.

The Margulis construction of expanders [Mar73; Lub10] relies on a group with Kazhdan's property (T). We give only the definition for discrete groups here, as this is relevant to our situation.

Definition 5.2.6. Let Γ be a discrete group and $\pi : \Gamma \to B(\mathscr{H})$ be a unitary representation. For a finite set $S \subset \Gamma$ and $\varepsilon > 0$, we say that $v \in \mathscr{H} \setminus \{0\}$ is an (S, ε)*-invariant vector* for π if $\|\pi(g)v - v\| < \varepsilon \|v\|$ for all $g \in S$.

We say Γ has *property (T)* if for every unitary representation π on a Hilbert space $\mathscr{H}$ such that there exists an (S, ε)-invariant vector for all finite sets $S \subset \Gamma$ and $\varepsilon > 0$, there exists an invariant vector $v \in \mathscr{H} \setminus \{0\}$, that is $\pi(g)v = v$ for all $g \in \Gamma$.

Property (T) is equivalent to the existence of a *Kazhdan pair*: a fixed pair (S, ε) with S a finite symmetric generating set, and $\varepsilon > 0$ such that if a unitary representation has an (S, ε)-invariant vector, then it has an invariant vector.

We now state the main result of Margulis.

Theorem 5.2.7 (Margulis [Mar73])**.** *Let Γ be a discrete group with property (T), and let (S, ε) be a Kazhdan pair for Γ such that S is a generating set. For any subgroup of finite index $H \leq \Gamma$,*

$$h(\mathscr{C}(\Gamma/H, S)) \geq \frac{\varepsilon^2}{4}.$$

In the above theorem, $\mathscr{C}(\Gamma/H, S)$ denotes the induced *Schreier coset graph* on the quotient space Γ/H. The vertices are given by the cosets $\{gH$
$midg \in \Gamma\} = \Gamma/H$, and the edges are given by

$$E = \big\{\{gH, sgH\} \bigm| s \in S\big\}.$$

Hence, this construction gives rise to an $(|S|, 4^{-1}\varepsilon^2)$-expander, and if Γ has a sequence of subgroups with arbitrarily large finite index, then the construction gives an $(|S|, 4^{-1}\varepsilon^2)$-expander family.

Example 5.2.8. The group $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ has property (T), and admits a Kazhdan pair (S, ε), where $S = \big\{I \pm e_{ij} \bigm| i \neq j\big\}$, for e_{ij} the canonical matrix units, and $\varepsilon = (42\sqrt{3} + 860)^{-1} > 0.001$ [Sha99; Kas05]. Also, by taking the quotients $\mathrm{SL}_3(\mathbb{Z}) \to \mathrm{SL}_3(\mathbb{Z}_p)$ for p prime we get an infinite sequence of quotient groups of strictly increasing order. Thus, by the theorem above, we get a $(12, 0.25 \times 10^{-6})$-expander family.

In Section 5.5 we extend Margulis' theorem to the setting of discrete quantum groups (cf. Theorem 5.5.7).

5.2.2. QUANTUM EXPANDERS

In this section, we define what we mean by a quantum expander and recall the construction of Harrow [Har08] that allows us to construct a quantum expander from a classical expander constructed from a finite group, as in the previous section. In Section 5.3 we generalise this class of quantum expanders to those arising from discrete quantum groups.

QUANTUM BISTOCHASTIC MAPS

The notion of a quantum expander was first introduced in [Has07a]. There are two different ways that quantum expanders are represented in the literature: they can be represented as tuples of operators, as in [Has07a; Has07b; Li+25], or they can be represented as quantum channels, as in [Har08]. We use the latter definition, as all of the spectral and expansion properties are most readily phrased in this setting, and it naturally generalises the presentation of an expander graph as a bistochastic matrix.

Definition 5.2.9. Let $\mathcal{K}$ be a finite-dimensional Hilbert space. We call a linear map $\Phi : B(\mathcal{K}) \to B(\mathcal{K})$ a *quantum bistochastic map* if it is unital, completely positive and trace-preserving. We say Φ is *undirected* if it is hermitian with respect to the Hilbert-Schmidt inner product. We say Φ is *connected* if its 1-eigenspace is one-dimensional.

Example 5.2.10 ([Li+25]). Given a d-regular graph $G = (V, E)$, there is a natural way to construct a quantum bistochastic map from it. First, we construct a family of disjoint vertex cycle covers for G inductively. Pick an arbitrary vertex cycle cover G_1 of G, that is a 2-regular subgraph that contains all the vertices of G. It must be composed of a disjoint union of cycles. Now, remove the edges of G_1 from G, giving a $d-2$-regular graph. We repeat this process $\lfloor d/2 \rfloor$ times to get disjoint cycle covers $G_1, \ldots, G_{\lfloor d/2 \rfloor}$. If d is even, there are no remaining edges. If d is odd, the remaining graph is 1-regular, consisting of a disjoint union of pairs of vertices with one edge between them. Let this graph be G'. For each i, we can write the adjacency matrix $A(G_i) = P_{2i-1} + P_{2i}$, where P_{2i-1} is a permutation matrix corresponding to traversing the cycles in one direction, and $P_{2i} = P_{2i-1}^*$ is the permutation matrix corresponding to travelling in the other direction. In the case that d is odd, let $P_d = A(G')$: this is a hermitian permutation matrix. Now, we can define the map $\Phi_G : B(\mathbb{C}^V) \to B(\mathbb{C}^V)$ by

$$\Phi_G(\rho) = \frac{1}{d} \sum_{i=1}^{d} P_i \rho P_i^*$$

Note that this map is not uniquely determined by G but depends on the choice of permutations P_i, which are not uniquely determined by the above procedure. By construction, Φ_G is a completely positive linear map, and as the permutation matrices are unitary, this is also a mixed-unitary channel, and hence both trace-preserving and unital. Thus, Φ_G is a quantum bistochastic map.

Also, we see that the action of Φ_G on the diagonal subspace $D = \operatorname{span}\{|x\rangle\langle x| \mid x \in V\} \cong \mathbb{C}^V$ is exactly the action of the normalised adjacency matrix $d^{-1}A(G)$ on $\mathbb{C}^V$ (where $\mathbb{C}^V$ is identified with the diagonal subalgebra of $B(\mathbb{C}^V)$). In fact,

$$\Phi_G(|x\rangle\langle x|) = \frac{1}{d} \sum_{i=1}^{d} P_i |x\rangle\langle x| P_i^* = \frac{1}{d} \sum_{y \in N(x)} |y\rangle\langle y|.$$

Next, we see that Φ_G is undirected. For $i = 1, \ldots, \lfloor d/2 \rfloor$, $P_{2i-1}^* = P_{2i}$ and if d is odd, $P_d^* = P_d$, so Φ_G is hermitian.

However, note that Φ_G may not be connected even if G is. For example, consider the graph G with two vertices and one edge connecting them. Then, the adjacency matrix equals $A(G) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. As such, $\Phi_G(\rho) = A(G)\rho A(G)$, whose 1-eigenspace is $\operatorname{span}\{I, A(G)\}$

and hence has dimension two. Nevertheless, as will be seen in the next subsection, if we see the P_i as corresponding to the generators of a group in the left-regular representation, we can restrict to a block corresponding to an irreducible representation and hence recover connectedness (and expansion).

We now come to a natural notion of expansion constant for quantum bistochastic maps introduced in [Has07b]. See also [Li+25].

Definition 5.2.11. The *quantum (edge) expansion* of a quantum bistochastic map $\Phi : B(\mathcal{K}) \to B(\mathcal{K})$ is

$$h_Q(\Phi) = \min\left\{ \frac{\operatorname{Tr}((I-\Pi)\Phi(\Pi))}{\min\{\operatorname{Tr}(\Pi), \operatorname{Tr}(I-\Pi)\}} \,\middle|\, \Pi \in B(\mathcal{K}) \text{ projection}, \Pi \neq 0, I \right\}.$$

Remark 5.2.12. One can make sense of the above definition of expansion in the setting of a finite von Neumann algebra $\mathcal{M}$ equipped with a faithful normal tracial state $\tau : \mathcal{M} \to \mathbb{C}$, and a τ-preserving completely positive map $\Phi : \mathcal{M} \to \mathcal{M}$. In this case, τ replaces the usual trace Tr in Definition 5.2.11. The advantage of this more general setup is that it also captures the classical notion of expansion for d-regular graphs $G = (V, E)$, where one takes $\mathcal{M} = C(V)$ (the algebra of functions on V) and τ the uniform probability on V. See the discussion in [Has07b] for more details.

Note that, as for a classical graph, there is nontrivial expansion only if the map is connected, according to our definition of connectedness.

Lemma 5.2.13. *If a quantum bistochastic map Φ is not connected, $h_Q(\Phi) = 0$.*

Proof. First, note that I is an eigenvector of Φ with eigenvalue 1. If the dimension of the 1-eigenspace is not one, there exists a hermitian matrix X such that $\Phi(X) = X$, as Φ is hermitian-preserving. Thus, taking a linear combination of I and X, there is a positive matrix P that is not full rank such that $\Phi(P) = P$. Let Π be the projection onto the support of P. Then there exist λ, $\mu > 0$ such that $\lambda\Pi \leq P \leq \mu\Pi$. As such, $\Phi(\Pi) \leq \frac{1}{\lambda}\Phi(P) = \frac{1}{\lambda}P \leq \frac{\mu}{\lambda}\Pi$, giving that $\operatorname{Tr}((I-\Pi)\Phi(\Pi)) \leq \frac{\mu}{\lambda}\operatorname{Tr}((I-\Pi)\Pi) = 0$ and hence $h_Q(\Phi) = 0$. □

The notion of expansion above gives rise to a natural analogue of the Cheeger inequalities. As seen classically in the previous section, there is a relationship between the second-largest eigenvalue $\lambda_2(\Phi)$ and the quantum expansion.

Theorem 5.2.14 ([Has07b])**.** *Let $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ be a connected undirected quantum bistochastic map. Then,*

$$\frac{1}{2}(1-\lambda_2(\Phi)) \leq h_Q(\Phi) \leq \sqrt{2(1-\lambda_2(\Phi))}.$$

Note that the statement of the above theorem differs by a factor of d from Theorem 5.2.3, the case of classical graphs. This is because $h_Q(\Phi)$ is defined for the quantum analogue of the bistochastic matrix $d^{-1}A(G)$, and not for the adjacency matrix $A(G)$.

In order to properly speak about expanders in the quantum setting, we must first have a good notion of degree for a quantum bistochastic map. Let $\mathcal{K}$ be a finite dimensional Hilbert space and $\Phi : B(\mathcal{K}) \to B(\mathcal{K})$ be a quantum channel. Let $\{K_i\}_{i=1}^d \subset B(\mathcal{K})$

be a family of Kraus operators for Φ. That is, Φ can be represented as

$$\Phi(\rho) = \sum_{i=1}^{d} K_i \rho K_i^*.$$

The minimal size $1 \leq d \leq (\dim \mathcal{K})^2$ of a family of Kraus operators representing Φ is called the *Kraus rank* of Φ. Note that the Kraus rank of Φ is the rank of the Choi matrix $C^\Phi = \sum_{ij} e_{ij} \otimes \Phi(e_{ij}) \in B(\mathcal{K}) \otimes B(\mathcal{K})$. Following [Har08; LY24; Li+25], we make the following definition.

5

Definition 5.2.15. The *degree* of quantum bistochastic map $\Phi : B(\mathcal{K}) \to B(\mathcal{K})$, $\deg \Phi$, is defined to be the Kraus rank of Φ.

Note that for classical d-regular graphs G, the somewhat complicated extension of $d^{-1}A(G)$ to a quantum bistochastic map Φ_G in Example 5.2.10 has the nice property that it is degree-preserving. On the other hand, the "canonical" extension of $d^{-1}A(G)$ to a different quantum bistochastic map $\tilde{\Phi}_G$ by using the Kraus operators $\{d^{-1/2} e_{ij} \mid \{i,j\} \in E\}$ is not degree-preserving.

With the notion of degree in hand, we can finally define quantum expanders (also called quantum edge expanders).

Definition 5.2.16. Let $\eta > 0$ and $d \in \mathbb{N}$. We say a quantum bistochastic map Φ is a (d,η)-*expander* if $\deg \Phi = d$ and $h_Q(\Phi) \geq \eta$. A (constant degree d) (d,η)-*expander family* is a sequence $\Phi_n : B(\mathcal{K}_n) \to B(\mathcal{K}_n)$ of (d,η)-expanders such that $\dim \mathcal{K}_n \to \infty$.

Remark 5.2.17. This definition extends the definition of a classical expander family when one uses Example 5.2.10 to construct a family of quantum bistochastic maps on the corresponding function algebras. As discussed in the remark below Definition 5.2.11, the notion of quantum edge expansion on function algebras equals the classical notion of edge expansion.

QUANTUM ADJACENCY MATRICES

The above definition of degree for quantum bistochastic maps may seem somewhat unnatural, given that it assigns a global degree d to *every* quantum bistochastic map Φ (even those coming from non-regular graphs). Instead, one might try to restrict to a special class of quantum bistochastic maps Φ which have the property that there exists $d > 0$ such that $d\Phi$ is some sort of quantum analogue of a "$\{0,1\}$-matrix". This idea has been formalised in [MRV18], and leads to the notion of a *quantum graph* and a *quantum adjacency matrix*. See also [Daw24; Was23; CW22] for a treatment that is closer to ours below. In the following definition, we restrict to what are called *tracial* quantum graphs, as they are what will appear in all our examples.

Definition 5.2.18. [MRV18] Let $\mathcal{M} = \bigoplus_a M_{n_a}(\mathbb{C})$ be a finite-dimensional von Neumann algebra, let $\psi : \mathcal{M} \to \mathbb{C}$ be the faithful trace given by $\psi = \sum_a n_a \mathrm{Tr}_{M_{n_a}(\mathbb{C})}(\cdot)$, and let $m : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ denote the multiplication map. A completely positive map $A : \mathcal{M} \to \mathcal{M}$ is called a *quantum adjacency matrix* if it satisfies

$$m(A \otimes A)m^* = A,$$

where $m^* : \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}$ denotes the Hilbert space adjoint of m induced by the Hilbert space structures coming from the traces ψ and $\psi \otimes \psi$. The pair $G = (\mathcal{M}, A)$ is called a *quantum graph*. We say that a quantum graph $G = (\mathcal{M}, A)$ is *undirected* if A is hermitian with respect to the canonical Hilbert space structure on $\mathcal{M}$ induced by ψ.

When $\mathcal{M}$ is abelian, the above notions reduce to the usual ones for graphs and adjacency matrices. See the discussion in [MRV18; Daw24] for example

For quantum graphs $G = (\mathcal{M}, A)$, the notion of regularity is quite natural to define.

Definition 5.2.19. An undirected quantum graph $G = (\mathcal{M}, A)$ is d*-regular* if $A1 = A^*1 = d1$ for some $d > 0$.

Remark 5.2.20. In [Was23] the definition of quantum graph is extended to the setting of infinite-dimensional $\mathcal{M}$ being given by infinite direct products of matrix algebras. Here the technical issue is that m^* is unbounded, so care must be taken in interpreting the definition of a quantum adjacency matrix. The theory of infinite quantum graphs is important and quite natural as it naturally models the notion of a Cayley graph for an infinite discrete quantum group [Was23]. This notion of quantum Cayley graph will be introduced and used in Section 5.5.

5

Example 5.2.21. Consider the map Φ_G constructed in Theorem 5.2.10 from a d-regular graph. Set $A = d\Phi_G$. Then the pair $(B(\mathbb{C}^V), A)$ is a d-regular quantum graph. Indeed, using $m^*(|u\rangle\langle v|) = |V|^{-1}\sum_{w\in V} |u\rangle\langle w| \otimes |w\rangle\langle v|$, we have

$$\begin{aligned} m \circ (A \otimes A) \circ m^*(|u\rangle\langle v|) &= |V|^{-1} \sum_{w,i,j} P_i |u\rangle\langle w| P_i^* P_j |w\rangle\langle v| P_j^* \\ &= |V|^{-1} \sum_{i,j} \mathrm{Tr}(P_i^* P_j) P_i |u\rangle\langle v| P_j^* \\ &= d\Phi_G(|u\rangle\langle v|) \\ &= A(|u\rangle\langle v|), \end{aligned}$$

as $\mathrm{Tr}(P_i^* P_j) = |V|\delta_{i,j}$ since the $\{P_i\}_i$ are disjoint permutations.

Note that if $G = (\mathcal{M}, A)$ is a d-regular quantum graph, then $\Phi = d^{-1}A$ will define a quantum bistochastic map on $\mathcal{M}$. When $\mathcal{M} = B(\mathcal{K})$ is a full matrix algebra, one can ask whether the two notions of degree we have introduced agree. Fortunately they do, and this is the next result.

Proposition 5.2.22. *A completely positive map $A : B(\mathcal{K}) \to B(\mathcal{K})$ is a quantum adjacency matrix if and only if the normalised Choi matrix*

$$P_A := \frac{1}{\dim \mathcal{K}} \sum_{ij} e_{ij} \otimes A(e_{ij})$$

is a projection. If $(B(\mathcal{K}), A)$ is moreover a d-regular quantum graph, then

$$d = \mathrm{rank}(P_A) = \text{the Kraus rank of } A.$$

Proof. The first claim is a well-known fact about quantum graphs on full matrix algebras. See for example [CW22, Lemma 1.6 and Proposition 1.7]. The second claim is just a computation. Indeed, if $A1 = d1$, then

$$\begin{aligned}\text{Kraus rank of } A &= \text{rank}(P_A)\\ &= (\text{Tr}\otimes\text{Tr})(P_A)\\ &= \frac{1}{\dim\mathcal{K}}\sum_i \text{Tr}(A(e_{ii}))\\ &= d\end{aligned}$$
□

In particular, for regular quantum graphs $(B(\mathcal{K}), A)$ over full matrix algebras, the degree d is an integer between 1 and $(\dim\mathcal{K})^2$, and it is easy to see that every value of d in this range is attained, for example by taking Φ to be the mixed-unitary channel corresponding to an equal mixture of d orthogonal unitaries.

It is important to note that for general quantum graphs, $G = (\mathcal{M}, A)$ there does not seem to exist a definition of the expansion constant $h_Q(G)$ (although a natural one is implicit in our discussion above - see the Remark following Definition 5.2.11.) More importantly, a version of the Cheeger inequality Theorem 5.2.14 beyond the case of matricial quantum graphs (i.e., those with $\mathcal{M} = B(\mathcal{K})$ - which are already covered by Theorem 5.2.14) does not exist. A general version of the Cheeger inequality for (even non-tracial) quantum graphs has recently been announced in forthcoming work of Junk [Jun24]. In any case, even without expansion constants and Cheeger inequalities, we can still talk about spectral gap for the bistochastic map $\Phi = d^{-1}A$ associated to a d-regular quantum $G = (\mathcal{M}, A)$, and use this spectral gap to define (d,η)-expanders in this case. This is what we shall do in Section 5.5, where we consider quantum analogues of Schreier graphs associated to quantum Cayley graphs of discrete quantum groups.

QUANTUM EXPANDERS FROM CLASSICAL EXPANDERS

In this final section, we recall the construction of a quantum expander based on a finite group due to Harrow [Har08].

Proposition 5.2.23 ([Har08]). *Let Γ be a finite group with symmetric generating set S, and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a nontrivial irreducible unitary representation. Then, the quantum channel*

$$\Phi(\sigma) = \frac{1}{|S|}\sum_{g\in S}\pi(g)\sigma\pi(g)^*$$

is a connected undirected quantum bistochastic map such that $\lambda_2(\Phi) \le |S|^{-1}\lambda_2(\mathcal{C}(\Gamma, S))$.

We outline the idea of the proof below, as it contains many of the key conceptual ideas that will be used throughout the rest of the chapter. Note that the adjacency matrix of the Cayley graph $\mathcal{C}(\Gamma, S)$, when viewed as an operator on the Hilbert space $\ell^2(\Gamma)$, is given by $\sum_{g\in S}\lambda(g)$, where λ is the left-regular representation. The left-regular representation decomposes as a direct sum of all irreducible representations with multiplicity given by the dimension. Hence the trivial representation $g \mapsto 1$ corresponds to the one-dimensional 1-eigenspace of the normalised adjacency matrix, and for any nontrivial unitary irreducible representation ρ, the largest eigenvalue of $\sum_{g\in S}\rho(g)$ is upperbounded by $\lambda_2(\mathcal{C}(\Gamma, S))$. Then, the matrix representation of Φ (viewed as a Hilbert space operator on $B(\mathscr{H}) \cong \mathscr{H} \otimes \bar{\mathscr{H}})$, is given by $|S|^{-1}\sum_{g\in S}\pi(g)\otimes\overline{\pi(g)}$. Representation theory then tells us that this operator can be decomposed as a direct sum of blocks of the form $\sum_{g\in S}\rho(g)$ for irreducible ρ, and there is only a single one-dimensional block in this decomposition corresponding to the trivial representation. Hence, we get the bound on the second-largest eigenvalue.

5.2.3. QUANTUM GROUPS

This section recalls some well known facts about quantum groups based on notes by Maes and Van Daele [MvD98], adjusting the notation to be more in line with recent papers [Fim10; Was23]. See also the book [NT13] for many of the results stated below without proof. We will focus on compact quantum groups, discrete quantum groups, the duality between them and their representations, and end the section with the definition of property (T) for discrete quantum groups due to Fima [Fim10].

COMPACT QUANTUM GROUPS

Definition 5.2.24. A *compact quantum group* $\mathbf{G}$ is a pair $(\mathscr{A}, \Delta)$ of a unital C*-algebra $\mathscr{A}$ and a $*$-homomorphism $\Delta : \mathscr{A} \to \mathscr{A} \otimes_{\min} \mathscr{A}$, called the *comultiplication*, satisfying

(a) $(\mathrm{id}\otimes\Delta)\circ\Delta = (\Delta\otimes\mathrm{id})\circ\Delta$,

(b) The sets $(\mathscr{A}\otimes 1)\Delta(\mathscr{A})$ and $(1\otimes\mathscr{A})\Delta(\mathscr{A})$ are linearly dense in $\mathscr{A}\otimes_{\min}\mathscr{A}$.

In the following, the algebra $\mathscr{A}$ will be denoted by $C(\mathbf{G})$ as it generalises the continuous functions on a compact group G. In this classical case, $\Delta : C(G) \to C(G)\otimes_{\min} C(G) = C(G\times G)$ is given by $\Delta f(s,t) = f(st)$ for $f \in C(G)$ and $s,\ t\in G$. Then properties (a) and (b) for Δ are just reformulations of the associativity and left/right cancellation properties (respectively), which completely characterise continuous group laws on compact Hausdorff spaces.

Any compact quantum group admits an analogue of the unique Haar probability measure. See for example [NT13, Theorem 1.2.1].

Theorem 5.2.25. *For any compact quantum group* $\mathbf{G}$*, there exists a unique state* $h : C(\mathbf{G}) \to \mathbb{C}$*, called the Haar state, satisfying*

$$(\mathrm{id}\otimes h)(\Delta(a)) = h(a)1 = (h\otimes\mathrm{id})(\Delta(a))$$

for all $a \in C(\mathbf{G})$.

If the Haar state is tracial, then the compact quantum group is said to be of *Kac type.*

Two notions of representation will play a role in this chapter. The first is the usual notion of a (continuous) unitary representation derived from the theory of locally compact groups. The second one corresponds to ordinary $*$-representations of the C*-algebra $C(\mathbf{G})$. These two notions of representation are dual to each other in the sense of Pontryagin duality, as formulated in Proposition 5.2.31. Below we will make use of the leg numbering notation, and denote by $B_0(\mathscr{H})$ the C*-algebra of compact operators on a Hilbert space $\mathscr{H}$.

Definition 5.2.26. A *unitary representation* of $\mathbf{G}$ on a Hilbert space $\mathscr{H}$ is a unitary element $u \in M(B_0(\mathscr{H}) \otimes C(\mathbf{G}))$ such that

$$(\mathrm{id} \otimes \Delta)(u) = u_{12}u_{13},$$

where $M(\mathscr{A})$ denotes the C*-algebra of multipliers of a given C*-algebra $\mathscr{A}$. A closed subspace $\mathscr{K} \subset \mathscr{H}$ is called invariant if $(p \otimes 1)u(p \otimes 1) = u(p \otimes 1)$, where p is the orthogonal projection onto $\mathscr{K}$. A representation u is called *irreducible* if the only invariant closed subspaces are $\{0\}$ and $\mathscr{H}$.

The next definition is not standard, but it is a useful concept nonetheless. Note that transpose maps with respect to different bases are related by unitary conjugation, so the definition is well-defined.

Definition 5.2.27. Let $T : B(\mathscr{H}) \to B(\mathscr{H})$ be the transpose map with respect to an orthonormal basis. A unitary representation u of a compact quantum group $\mathbf{G}$ on a finite-dimensional Hilbert space $\mathscr{H}$ is called a *bi-unitary representation* if $(T \otimes \mathrm{id})(u)$ is also unitary.

5

As with groups, there is a concept of intertwiners for compact quantum groups and a version of Schur's lemma (Lemma 5.2.30).

Definition 5.2.28. Let v, w be unitary representations of a compact quantum group $\mathbf{G}$ on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. An *intertwiner* between v and w is an element $S \in B(\mathscr{H}_1, \mathscr{H}_2)$ such that $(S \otimes 1)v = w(S \otimes 1)$. Two unitary representations are called *unitarily equivalent* if there exists a intertwiner between them that is unitary. The space of intertwiners from u to v is denoted $\mathrm{Mor}(u, v)$.

Notation 5.2.29. $\mathrm{Irred}(\mathbf{G})$ denotes the equivalence classes of irreducible (unitary) representations of $\mathbf{G}$. For any $x \in \mathrm{Irred}(\mathbf{G})$ we fix a representative u^x on Hilbert space $\mathscr{H}_x$.

Lemma 5.2.30. *A unitary representation u is irreducible if and only if all intertwiners between u and itself are scalar multiples of the identity.*

Denote by $\mathscr{O}(\mathbf{G})$ the vector space spanned by the matrix coefficients of the irreducible representations of $\mathbf{G}$, *i.e.*

$$\mathscr{O}(\mathbf{G}) = \mathrm{span}\{u^x_{\xi,\eta} = (\omega_{\xi,\eta} \otimes \mathrm{id})u^x \mid x \in \mathrm{Irred}(\mathbf{G}),\ \xi, \eta \in \mathscr{H}_x\} \subset C(\mathbf{G}),$$

where $\omega_{\xi,\eta}(a) = \langle \xi | a | \eta \rangle$ for $a \in B(\mathscr{H}_x)$. $\mathscr{O}(\mathbf{G})$ is a dense unital $*$-algebra of $C(\mathbf{G})$, and the comultiplication Δ restricts to a unital $*$-homomorphism $\Delta : \mathscr{O}(\mathbf{G}) \to \mathscr{O}(\mathbf{G}) \otimes \mathscr{O}(\mathbf{G})$, turning $\mathscr{O}(\mathbf{G})$ into a *Hopf $*$-algebra*. The fact that $\mathscr{O}(\mathbf{G})$ is a $*$-algebra follows from the existence of the tensor product, contragredient, and complete reducibility of finite dimensional representations. It turns out that $\mathscr{O}(\mathbf{G})$ is the unique dense Hopf $*$-subalgebra of $C(\mathbf{G})$.

Since $\mathscr{O}(\mathbf{G})$ is spanned by coefficients of unitary operators, it admits a universal C*-completion, denoted by $C(\mathbf{G}_{\max})$. The comultiplication Δ extends continuously to a comultiplication $\Delta_{\max}$ on $C(\mathbf{G}_{\max})$, making the pair $(C(\mathbf{G}_{\max}), \Delta_{\max})$ a compact quantum group, called the *maximal version of* $\mathbf{G}$. The Haar measure h restricts to a faithful state on $\mathcal{O}(G)$, and the resulting C*-completion of $\mathcal{O}(G)$ obtained by performing the GNS construction with respect to h is denoted by $C(\mathbf{G}_{\min})$. The comultiplication Δ on $\mathcal{O}(\mathbf{G})$ extends continuously to a comultiplication $\Delta_{\min}$ on $C(\mathbf{G}_{\min})$, and the resulting compact quantum group is called the *minimal (or reduced) version of* $\mathbf{G}$.

For any initial C*-realisation $(C(\mathbf{G}), \Delta)$ of a compact quantum group $\mathbf{G}$, one has quotient maps

$$C(\mathbf{G}_{\max}) \to C(\mathbf{G}) \to C(\mathbf{G}_{\min})$$

extending the identity map on $\mathcal{O}(\mathbf{G})$ and intertwining the comultiplications. Since $\mathcal{O}(\mathbf{G})$ is uniquely determined by *any* of these C*-completions and conversely any of these C*-completions can be recovered from $\mathcal{O}(\mathbf{G})$, they are all just different C*-algebraic manifestations of a single compact quantum group structure $\mathbf{G}$. This is analogous to the fact that a discrete group Γ can be encoded C*-algebraically in more than one way (*e.g.* via its full C*-algebra $C^*(\Gamma)$, or its reduced C*-algebra $C^*_r(\Gamma)$).

DISCRETE QUANTUM GROUPS, DUALITY, AND PROPERTY (T)

Discrete quantum groups can be defined axiomatically using the language of multiplier Hopf $*$-algebras [VDa96], or equivalently as structures dual to compact quantum groups. We follow the latter approach, as it best suits our needs.

Let $\mathbf{G}$ be a compact quantum group. Associated to $\mathbf{G}$, we define the C*-algebra

$$C_0(\hat{\mathbf{G}}) = \bigoplus_{x\in\mathrm{Irred}(\mathbf{G})}^{c_0} B(\mathcal{H}_x)$$

and the von Neumann algebra

$$\ell^\infty(\hat{\mathbf{G}}) = \prod_{x\in\mathrm{Irred}(\mathbf{G})}^{\ell^\infty} B(\mathcal{H}_x)$$

as the norm and weak* closure of the algebraic direct sum

$$\bigoplus_{x\in\mathrm{Irred}(\mathbf{G})} = \left\{ \sum_{i=1}^n a_i \;\middle|\; x_i \in \mathrm{Irred}(\mathbf{G}), a_i \in B(\mathcal{H}_{x_i}) \right\},$$

respectively. For $x \in \mathrm{Irred}(\mathbf{G})$, the minimal central projection associated to $B(\mathcal{H}_x) \subset C_0(\hat{\mathbf{G}}) \subset \ell^\infty(\hat{\mathbf{G}})$ is denoted by p_x. More generally, for $E \subset \mathrm{Irred}(\mathbf{G})$, we write $p_E = \sum_{x\in E} p_x$, with the sum converging σ-weakly in $\ell^\infty(\hat{\mathbf{G}})$. We also denote by Tr_x the canonical un-normalised trace on $B(\mathcal{H}_x)$, $\dim x = \mathrm{Tr}_x(1) = \dim \mathcal{H}_x$, and $\dim E = \sum_{x\in E} \dim x$ for any subset $E \subset \mathrm{Irred}(\mathbf{G})$. At times, we will also work with $C_{00}(\hat{\mathbf{G}}) \subset C_0(\hat{\mathbf{G}})$, the dense $*$-subalgebra of finitely supported elements.

There exists a normal injective co-associative $*$-homomorphism $\hat{\Delta}$ from $\ell^\infty(\hat{\mathbf{G}})$ to $\ell^\infty(\hat{\mathbf{G}}) \otimes \ell^\infty(\hat{\mathbf{G}})$ given by $\hat{\Delta}(ap_x)S = Sap_x$ for all $a \in \ell^\infty(\hat{\mathbf{G}})$, $S \in \mathrm{Mor}(u^x, u^y \otimes u^z)$, $x, y, z \in \mathrm{Irred}(\mathbf{G})$. Equivalently, $\hat{\Delta}$ can be defined in terms of the unitary

$$\mathcal{V} = \bigoplus_{x\in\mathrm{Irred}(\mathbf{G})} u^x \in M(C_0(\hat{\mathbf{G}}) \otimes C(\mathbf{G}_{\max})).$$

Then $\hat{\Delta}$ is uniquely determined by the identity

$$(\hat{\Delta} \otimes \mathrm{id})\mathcal{V} = \mathcal{V}_{13}\mathcal{V}_{23}.$$

Although we will have little need for them here, we mention that $\ell^\infty(\hat{\mathbf{G}})$ comes equipped with left and right invariant weights, $\hat{h}_L$ and $\hat{h}_R$, satisfying the formal identities

$$(\hat{h}_R \otimes \mathrm{id})\hat{\Delta}(a) = \hat{h}_R(a)1 \quad \& \quad (\mathrm{id} \otimes \hat{h}_L)\hat{\Delta}(a) = \hat{h}_L(a)1 \qquad (a \in \ell^\infty(\hat{\mathbf{G}})).$$

The quadruple $\mathbf{\Gamma} = \hat{\mathbf{G}} = (\ell^\infty(\hat{\mathbf{G}}), \hat{\Delta}, \hat{h}_L, \hat{h}_R)$ is the *discrete quantum group dual to* $\mathbf{G}$. We also denote by $\ell^1(\hat{\mathbf{G}}) = (\ell^\infty(\hat{\mathbf{G}}))_*$, the predual of $\ell^\infty(\hat{\mathbf{G}})$. $\ell^1(\hat{\mathbf{G}})$ is a completely contractive Banach algebra with convolution product given by

$$\psi \star \varphi = (\psi \otimes \varphi) \circ \hat{\Delta} \qquad (\psi, \varphi \in \ell^1(\hat{\mathbf{G}})).$$

A *unitary representation* of $\hat{\mathbf{G}}$ on a Hilbert space $\mathscr{H}$ is a unitary $U \in M(C_0(\hat{\mathbf{G}}) \otimes B_0(\mathscr{H}))$ such that

$$(\hat{\Delta} \otimes \mathrm{id})(U) = U_{13}U_{23}.$$

A closed subspace $\mathscr{K} \subset \mathscr{H}$ is called invariant if $(1 \otimes p)U(1 \otimes 1) = U(1 \otimes p)$, where p is the orthogonal projection onto $\mathscr{K}$. A representation u is called *irreducible* if the only invariant closed subspaces are $\{0\}$ and $\mathscr{H}$.

5

By faithfully representing $C(\mathbf{G}_{\max})$ on a Hilbert space $\mathscr{H}$, one can regard the unitary $\mathscr{V}$ defined above is a special example of a *unitary representation* of $\hat{\mathbf{G}}$. $\mathscr{V}$ is a multiplicative unitary in the sense of [BS93].

For a discrete quantum group $\hat{\mathbf{G}}$, one can also define co-unit and (generally unbounded) antipode maps. The *co-unit* is the normal state $\hat{\varepsilon} : \ell^\infty(\hat{\mathbf{G}}) \to \mathbb{C}$ given by $\hat{\varepsilon}(x) = xp_0$, where p_0 is the rank one central projection corresponding to the trivial representation of $\mathbf{G}$. For our purposes, it suffices to densely define the antipode $\hat{S} : \ell^\infty(\hat{\mathbf{G}}) \to \ell^\infty(\hat{\mathbf{G}})$ by $(\hat{S} \otimes \mathrm{id})U = U^*$ for any unitary representation of $\hat{\mathbf{G}}$.

As one would expect, unitary representations of a discrete quantum group $\hat{\mathbf{G}}$ are in one-to-one correspondence with $*$-representations of the C*-algebra of the dual compact quantum group $\mathbf{G}$, and vice versa. This duality is encoded precisely in terms of the multiplicative unitary $\mathscr{V}$.

Proposition 5.2.31. *For any unitary representation U of $\hat{\mathbf{G}}$ on $\mathscr{H}$ there exists a unique $*$-homomorphism $\pi : C(\mathbf{G}_{\max}) \to \mathscr{H}$ such that* $(\mathrm{id} \otimes \pi)(\mathscr{V}) = U$. *Conversely, for any unitary representation u of $\mathbf{G}$ on $\mathscr{H}$ there exists a unique $*$-homomorphism $\rho : \ell^\infty(\hat{\mathbf{G}}) \to B(\mathscr{H})$ such that* $(\rho \otimes \mathrm{id})(\mathscr{V}) = u$.

Just as for compact quantum groups, we can introduce the notion of an intertwiners for unitary representations of discrete quantum groups. Let U_1, U_2 be unitary representations on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. An intertwiner between U_1 and U_2 is an element $S \in B(\mathscr{H}_1, \mathscr{H}_2)$ such that $(1 \otimes S)U_1 = U_2(1 \otimes S)$. Schur's lemma also holds in the discrete quantum group case: A unitary representation U is irreducible if and only if all intertwiners between U and itself are scalar multiples of the identity (see for example [Des03]).

We conclude this recap of quantum groups by giving the definitions of property (T) and Kazhdan pairs for discrete quantum groups, which were introduced by Fima [Fim10]. For a unitary representation U of a discrete quantum group $\hat{\mathbf{G}}$ on $\mathscr{H}$, we write U^x for Up_x as element of $B(\mathscr{H}_x) \otimes B(\mathscr{H})$. Below, in the quantum case, we deviate slightly from our previous notation (S, ε) for Kazhdan pairs, and instead write (E, ε). This is to avoid possible confusion with the antipodes S and $\hat{S}$.

Definition 5.2.32. Let $\hat{\mathbf{G}}$ be a discrete quantum group and U a unitary representation of $\hat{\mathbf{G}}$ on $\mathscr{H}$.

- We say that U has an *invariant* vector if there exists a unit vector $\xi \in \mathscr{H}$ such that for all $x \in \mathrm{Irred}(\mathbf{G})$ and $\eta \in \mathscr{H}_x$, we have that
$$U^x(\eta \otimes \xi) = \eta \otimes \xi.$$
- Let $E \subset \mathrm{Irred}(\mathbf{G})$ be a finite subset and $\varepsilon > 0$. We say that U has an (E, ε)*-invariant vector* if there exists a unit vector $\xi \in \mathscr{H}$ such that for all $x \in E$ and $\eta \in \mathscr{H}_x$ we have that
$$\|U^x \eta \otimes \xi - \eta \otimes \xi\| < \varepsilon \|\eta\|.$$
- We say that U has *almost invariant vectors* if, for all finite $E \subset \mathrm{Irred}(\mathbf{G})$ and all $\varepsilon > 0$, U has an (E, ε)-invariant vector.
- We say that $\hat{\mathbf{G}}$ has *property (T)* if every unitary representation of $\hat{\mathbf{G}}$ having almost invariant vectors has a non-zero invariant vector.
- A pair (E, ε), where $E \subset \mathrm{Irred}(\mathbf{G})$ is finite and $\varepsilon > 0$ is called a *Kazhdan pair* if every unitary representation of $\hat{\mathbf{G}}$ having an (E, ε)-invariant vector also has a non-zero invariant vector.

As shown in [Fim10], discrete quantum groups with property (T) are *unimodular* and *finitely generated*. Here unimodular means that the Haar measure h on $\mathbf{G}$ is a tracial state. Equivalently, unimodularity is characterised at the level of $\hat{\mathbf{G}}$ by the equality of the left and right Haar weights. In this case, the Haar weight $\hat{h} = \hat{h}_L = \hat{h}_R$ is the semifinite trace given by

$$\hat{h}(ap_x) = \dim x \, \mathrm{Tr}_x(ap_x) \qquad (x \in \mathrm{Irred}(\mathbf{G}), a \in \ell^\infty(\hat{\mathbf{G}})).$$

A discrete quantum group $\hat{\mathbf{G}}$ is unimodular if and only if $\mathscr{O}(\mathbf{G})$ is finitely generated as a $*$-algebra. Equivalently, there exists a finite subset $E \subset \mathrm{Irred}(\mathbf{G})$ containing the trivial representation such that any finite dimensional unitary representation of $\mathbf{G}$ is generated by E by taking direct sums, tensor products and subrepresentations. Such a set E is called a generating set for $\hat{\mathbf{G}}$.

As is in the case of ordinary discrete groups, discrete quantum groups with property (T) always admit a Kazhdan pair.

Proposition 5.2.33 ([Fim10])**.** *A discrete quantum group* $\hat{\mathbf{G}}$ *has property (T) if and only if there exists a finite generating set* $E \subset \mathrm{Irred}(\mathbf{G})$ *and an* $\varepsilon > 0$ *such that* (E, ε) *is a Kazhdan pair.*

A consequence of unimodularity for property (T) discrete quantum groups is the bi-unitarity of all finite-dimensional unitary representations of the compact dual quantum group.

Proposition 5.2.34. *Let* $\mathbf{G}$ *be a compact quantum group such that the Haar measure is a tracial state. Then all unitary representations of* $\mathbf{G}$ *are bi-unitary.*

Proof. Let $u = \sum_{ij} e_{ij} \otimes u_{ij} \in M_n(\mathbb{C}) \otimes \mathcal{O}(\mathbf{G})$ be a finite dimensional unitary representation of $\mathbf{G}$, and let T be the transpose map on $M_n(\mathbb{C})$. Put $v = (T \otimes \mathrm{id})(u^*) = \sum_{ij} e_{ij} \otimes u^*_{ij}$. Then v is a representation of $\mathbf{G}$ (the *contragredient of* u) and v is clearly unitary if and only if $(T \otimes \mathrm{id})(u)$ is unitary.

On the other hand, by the proof of [MvD98, Proposition 6.4], $R^{1/2} v R^{-1/2}$ is unitary, where $R = (\mathrm{id} \otimes h)(v^* v)$. But using the fact that the Haar measure h is tracial, we obtain

$$R = (\mathrm{id} \otimes h)(v^* v) = \sum_{ijk} e_{ij} \otimes h(u_{ki} u^*_{kj}) = \sum_{ijk} e_{ij} \otimes h(u^*_{kj} u_{ki}) = (T \otimes h)(u^* u) = 1. \qquad \square$$

Corollary 5.2.35. *Let* $\mathbf{G}$ *be a compact quantum group such that* $\hat{\mathbf{G}}$ *has property (T). Then all unitary representations of* $\mathbf{G}$ *are bi-unitary.*

5.2.4. QUANTUM CHANNELS FROM QUANTUM GROUPS

As in the case of groups discussed in Section 5.2, quantum groups and their representations naturally give rise to interesting quantum channels. There are many ways in which one can construct quantum channels of various flavours from these more general algebraic structures – see for example [Ver22; BCLY20; CN13; LY22]. Here we will just introduce the natural analogue of the mixed unitary channels associated to representations of finite or discrete groups. We emphasise, however, that in this more general setup, the resulting quantum channels are not necessarily mixed unitary. Below, we present two constructions of quantum channels, from compact and discrete quantum groups, and show that due to the duality these constructions are equivalent. Ultimately, we see that the duality allows two perspectives on a class of channels, which can become useful when studying examples.

We first begin with the construction of quantum channels from compact quantum groups.

Proposition 5.2.36. *Let* $\mathbf{G}$ *be a compact quantum group,* u *a finite-dimensional unitary representation of* $\mathbf{G}$ *on a Hilbert space* $\mathcal{H}$*,* π *a* $*$*-representation of* $C(\mathbf{G}_{\max})$ *on a finite-dimensional Hilbert space* $\mathcal{K}$*, and* φ *a state on* $B(\mathcal{H})$*. Then* $\Phi = \Phi_{u,\varphi,\pi} : B(\mathcal{K}) \to B(\mathcal{K})$*, given by*

$$\Phi(x) = (\varphi \otimes \mathrm{id})\big((\mathrm{id} \otimes \pi)(u)(1 \otimes x)(\mathrm{id} \otimes \pi)(u^*)\big),$$

is a normal unital completely positive map. If u *is a bi-unitary (e.g. if* $\mathbf{G}$ *is of Kac type* $\iff$ $\hat{\mathbf{G}}$ *is unimodular), then* Φ *is trace-preserving.*

Proof. The map Φ is a composition of completely positive maps, so it is completely positive. Next,

$$\Phi(1) = (\varphi \otimes \mathrm{id})\big((\mathrm{id} \otimes \pi)(u)(1 \otimes 1)(\mathrm{id} \otimes \pi)(u^*)\big) = (\varphi \otimes \mathrm{id})\big((\mathrm{id} \otimes \pi)(uu^*)\big) = 1,$$

so Φ is unital. If u is bi-unitary, write u as $u = \sum_{ij} e_{ij} \otimes u_{ij}$. Then

$$\mathrm{Tr}(\Phi(x)) = \sum_{ijkl} \varphi(e_{ij} e_{lk}) \,\mathrm{Tr}(\pi(u_{ij}) x \pi(u^*_{kl})) = \sum_{ijk} \varphi(e_{ik}) \,\mathrm{Tr}(x \pi(u^*_{kj} u_{ij})) = \sum_{i} \varphi(e_{ii}) \,\mathrm{Tr}(x),$$

showing that Φ is trace-preserving, since φ is a state. $\square$

One can equivalently describe the above class of UCP maps in terms of the dual discrete quantum group. Let $\mathbf{\Gamma} = \hat{\mathbf{G}}$ be a discrete quantum group, $U \in M(C_0(\mathbf{\Gamma}) \otimes B_0(\mathcal{K}))$ a unitary representation of $\mathbf{\Gamma}$ on a finite dimensional Hilbert space $\mathcal{K}$, and $\psi \in \ell^1(\mathbf{\Gamma})$ a normal state. Then it is easy to see that $\Psi = \Psi_{\psi,U} : B(\mathcal{K}) \to B(\mathcal{K})$, given by

$$\Psi(x) = (\psi \otimes \mathrm{id})(U(1 \otimes x)U^*),$$

is a normal unital completely positive map.

In the next proposition, we will need the notion of a *finitely supported* element $\psi \in \ell^1(\hat{\mathbf{G}})$: There exists a finite rank central projection $p \in C_{00}(\hat{\mathbf{G}})$ such that $\psi(xp) = \psi(x)$ for all $x \in \ell^\infty(\hat{\mathbf{G}})$

Proposition 5.2.37. *Let $\mathbf{G}$ be a compact quantum group. For every map $\Phi = \Phi_{u,\varphi,\pi}$ arising in Proposition 5.2.36, there exists a finitely supported state $\psi \in \ell^1(\hat{\mathbf{G}})$ and a unitary representation U of $\hat{\mathbf{G}}$ such that $\Phi = \Psi_{\psi,U}$. Conversely, given a finitely supported state $\psi \in \ell^1(\hat{\mathbf{G}})$ and a finite-dimensional unitary representation U of $\hat{\mathbf{G}}$ on $\mathcal{K}$, then the UCP map $\Psi_{\psi,U} : B(\mathcal{K}) \to B(\mathcal{K})$ is given by $\Psi_{\psi,U} = \Phi_{u,\varphi,\pi}$ for some u, φ, π as in Proposition 5.2.36.*

Proof. Let u, π, φ, and Φ be as in Proposition 5.2.36. By Proposition 5.2.31, $U = (\mathrm{id} \otimes \pi)(\mathcal{V})$ is a unitary representation of $\hat{\mathbf{G}}$ and there exists a (normal) *-homomorphism $\rho : \ell^\infty(\hat{\mathbf{G}}) \to B(\mathcal{H})$ such that $(\rho \otimes \mathrm{id})(\mathcal{V}) = u$. Now $\psi = \varphi \circ \rho$ is a normal state in $\ell^1(\hat{\mathbf{G}})$ and we find

$$\Phi(x) = (\varphi \otimes \mathrm{id})\big((\rho \otimes \pi)(\mathcal{V})(1 \otimes x)(\rho \otimes \pi)(\mathcal{V}^*)\big) = (\psi \otimes \mathrm{id})(U(1 \otimes x)U^*) = \Psi(x).$$

Note that ψ finitely supported because it is supported on the central summands of $\ell^\infty(\hat{\mathbf{G}})$ associated to the irreducible subrepresentations of u.

Conversely, if we start with a pair $\{\psi, U\}$ as in the statement of the proposition, we obtain a unique morphism $\pi : C(\mathbf{G}_{\max}) \to B(\mathcal{K})$ from U via Pontryagin duality, and we obtain a pair $\{\varphi, u\}$ from the state ψ via the GNS representation. Note that the finite support condition on ψ endures that u is a finite dimensional representation of $\mathbf{G}$. □

We end this section with a description of the fixed points of the UCP maps $\Psi_{\psi,U}$ for the cases that will concern us. To do this, we recall a well-known result about fixed points of quantum channels.

Theorem 5.2.38. *[Wat18, Theorem 4.25] Let $\Phi : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ be a unital CPTP map with Kraus decomposition $\Phi(\rho) = \sum_i K_i \rho K_i^*$. The set of fixed points of Φ is the commutant of the Kraus operators $\{K_i\}'$.*

Note that this result is independent of the choice of Kraus representation.

Now let $\hat{\mathbf{G}}$ be a discrete quantum group with *finite* generating set $E \subset \mathrm{Irred}(G)$, and let $U \in M(C_0(\hat{\mathbf{G}}) \otimes B_0(\mathcal{K}))$ a finite-dimensional representation. Let $p_E = \sum_{x \in E} p_x \in C_{00}(\hat{\mathbf{G}})$ be the largest central projection supported on E, and let $\psi \in \ell^1(\hat{\mathbf{G}})$ be a state supported on E. That is, $\psi(p_E x) = \psi(x)$ for all $x \in \ell^\infty(\hat{\mathbf{G}})$.

In the following two results, we consider the associated normal UCP map $\Psi_{\psi,U} : B(\mathcal{K}) \to B(\mathcal{K})$. This map should be interpreted as the quantum group generalisation of the mixed unitary channel

$$\rho \mapsto \sum_{s \in E} \psi(s)\pi(s)\rho\pi(s)^*, \tag{5.1}$$

where ψ is some probability density supported on a generating set E of a group Γ.

Theorem 5.2.39. *Assume that the restriction of ψ to $p_E\ell^\infty(\hat{\mathbf{G}})p_E = \bigoplus_{s\in E} B(\mathscr{H}_s)$ is faithful and that $\Psi_{\psi,U}$ is trace-preserving (which holds automatically if $\hat{\mathbf{G}}$ is unimodular). Then $\rho \in B(\mathscr{K})$ is a fixed point of $\Psi_{\psi,U}$ if and only if it is an intertwiner of U.*

Proof. Let $U^s \in B(\mathscr{H}_s) \otimes B(\mathscr{K})$ denote the s-th component of U for $s \in \mathrm{Irred}(\mathbf{G})$. Also let ψ_s be the restriction of ψ to $B(\mathscr{H}_s)$. Choosing appropriate bases of matrix units e^s_{ij} of $B(\mathscr{H}_s)$, we may assume that the density of ψ_s is diagonal and so $\psi_s(e_{ij}) = \lambda_{i,s}\delta_{i,j} > 0$. (These coefficients are all non-zero by the faithfulness assumption.) Then we compute

$$\begin{aligned}\Psi_{\psi,U}(\rho) &= \sum_{s\in E}\sum_{a,b,c} \psi_s(e^s_{ab})U^s_{ac}\rho(U^s_{bc})^* \\ &= \sum_{s\in E}\sum_{a,c} \lambda_{a,s}U^s_{ac}\rho(U^s_{ac})^*.\end{aligned}$$

By Theorem 5.2.38, ρ is a fixed point of $\Psi_{\psi,U}$ if and only if $\sqrt{\lambda_{a,s}}U^s_{ac}\rho = \rho\sqrt{\lambda_{a,s}}U^s_{ac}$ for all $s \in E$ and $1 \le a,\ c \le \dim \mathscr{H}_s$. As E is generating, we conclude that $(1\otimes\rho)U = U(1\otimes\rho)$. The reverse implication is immediate. □

5

An immediate consequence of interest in the next section is:

Corollary 5.2.40. *If U is an irreducible finite dimensional representation of a unimodular discrete quantum group $\hat{\mathbf{G}}$, the eigenvalue 1 of the unital channel $\Psi_{\psi,U}$ has multiplicity 1.*

5.3. QUANTUM EXPANDERS FROM PROPERTY (T) QUANTUM GROUPS

Throughout this section we fix a discrete quantum group $\hat{\mathbf{G}}$ with property (T), along with a fixed Kazhdan pair (E,ε) for $\hat{\mathbf{G}}$. We furthermore can and will always take E to be *symmetric*. That is, $E = \bar{E}$, where $\bar{E} = \{\bar{x} \mid x \in E\}$ and $\bar{x}$ denotes the conjugate representation associated to $x \in \mathrm{Irred}(\mathbf{G})$. In the following proposition, we let $\mathscr{H}_E = \bigoplus_{x\in E}\mathscr{H}_x$ and consider the natural embedding $p_E\ell^\infty(\hat{\mathbf{G}})p_E \subset B(\mathscr{H}_E)$. Let ψ_E be the normal tracial state on $\ell^\infty(\hat{\mathbf{G}})$ defined by $\psi_E(a) = \frac{1}{\dim \mathscr{H}_E}\mathrm{Tr}_{B(\mathscr{H}_E)}(ap_E)$. Note that ψ_E is nothing other than the normalised trace on $\mathscr{H}_E$, lifted to $\ell^\infty(\hat{\mathbf{G}})$ in the obvious way. We are interested in studying the expansion properties of the channel

$$\Psi_{\psi_E,U} : B(\mathscr{K}) \to B(\mathscr{K}), \qquad \Psi_{\psi_E,U}(\rho) = (\psi_E \otimes \mathrm{id})(U(1\otimes\rho)U^*) \qquad (\rho \in B(\mathscr{K})), \tag{5.2}$$

where $U \in M(C_0(\hat{\mathbf{G}}) \otimes B_0(\mathscr{K}))$ is a finite-dimensional irreducible unitary representation of $\hat{\mathbf{G}}$ on $\mathscr{K}$.

We begin with a definition and a couple of observations.

Definition 5.3.1. A normal linear functional $\psi \in \ell^1(\hat{\mathbf{G}})$ is called *symmetric* if

$$\psi \circ \hat{S} = \psi.$$

Lemma 5.3.2. *Let $\psi \in \ell^1(\hat{\mathbf{G}})$ be a symmetric state. For any unitary representation $V \in M(C_0(\hat{\mathbf{G}}) \otimes B_0(\mathscr{K}))$, the map $X = (\psi\otimes\mathrm{id})V \in B(\mathscr{K})$ is self-adjoint.*

Proof. Note that $\hat{S}$ is (densely) defined by the equations $(\hat{S}\otimes \mathrm{id})V = V^*$ for all unitary representations of $\hat{\mathbf{G}}$. Because ψ is positive, we have

$$X^* = ((\psi\otimes \mathrm{id})V)^* = (\psi\otimes \mathrm{id})(V^*) = (\psi\circ\hat{S}\otimes \mathrm{id})V = (\psi\otimes \mathrm{id})V = X. \qquad \square$$

Note that the above lemma applies to the case where $\psi = \psi_E$, since $\psi_E\circ\hat{S} = \psi_{\bar{E}} = \psi_E$. Here we use the symmetry condition $E = \bar{E}$

Lemma 5.3.3. *Let $\mathscr{H}$ be a finite dimensional Hilbert space and $0 < \varepsilon < 1$. Suppose $A \in B(\mathscr{H})$ satisfies $\|A\| \le 1$ and there exists a unit vector $\eta \in \mathscr{H}$ such that $\mathrm{Re}\langle\eta|A|\eta\rangle \le 1-\varepsilon$. Then,*

$$\mathrm{Re}\,\mathrm{tr}(A) \le 1 - \frac{\varepsilon}{\dim\mathscr{H}}.$$

Proof. Write $d = \dim\mathscr{H}$. Extend η to an orthonormal basis $\eta_1 = \eta$, η_2, ..., η_d for $\mathscr{H}$. Then,

$$\mathrm{Re}\,\mathrm{tr}(A) = \frac{1}{d}\mathrm{Re}(\langle\eta_1|A|\eta_1\rangle + \ldots + \langle\eta_d|A|\eta_d\rangle) \le \frac{1}{d}(1-\varepsilon+d-1) = 1-\frac{\varepsilon}{d}. \qquad \square$$

We now come to an estimate which will lead to the main result of the section.

Proposition 5.3.4. *Let $V \in M(C_0(\hat{\mathbf{G}})\otimes B_0(\mathscr{K}))$ be a unitary representation of $\hat{\mathbf{G}}$ on a Hilbert space $\mathscr{K}$. Let $\mathscr{K}_0 \subset \mathscr{K}$ be the closed invariant subspace of invariant vectors for V. Then, for any unit vector $\xi \in \mathscr{K}_0^\perp$,*

$$\mathrm{Re}\,\langle\xi|(\psi_E\otimes \mathrm{id})(V)|\xi\rangle \le 1 - \frac{\varepsilon^2}{2\dim\mathscr{H}_E}.$$

Proof. Consider the restriction of V to $\mathscr{K}_0^\perp$, on which it remains a representation of $\hat{\mathbf{G}}$, but now without invariant vectors. Fix a unit vector $\xi \in \mathscr{K}_0^\perp$. Since (E,ε) is a Kazhdan pair for $\hat{\mathbf{G}}$, there exists some unit vector $\eta \in \mathscr{H}_E$ such that $\|V\eta\otimes\xi - \eta\otimes\xi\| \ge \varepsilon$. This gives

$$\varepsilon^2 \le \|V\eta\otimes\xi - \eta\otimes\xi\|^2 = 2 - 2\,\mathrm{Re}\,\langle\eta\otimes\xi|\,V\,|\eta\otimes\xi\rangle \iff \mathrm{Re}\,\langle\eta\otimes\xi|\,V\,|\eta\otimes\xi\rangle \le 1 - \frac{\varepsilon^2}{2}$$

Applying Lemma 5.3.3 to the contraction $A = (\mathrm{id}\otimes\omega_{\xi,\xi})V \in B(\mathscr{H}_E)$ and noting that ψ_E is the normalised trace on $\mathscr{H}_E$, we obtain the result. $\square$

Applying a Frobenius reciprocity argument, we can use Proposition 5.3.4 to immediately deduce the spectral gap of the channels of interest $\Psi_{\psi_E,U}$.

Theorem 5.3.5. *Let $U \in M(C_0(\hat{\mathbf{G}})\otimes B_0(\mathscr{K}))$ be a finite-dimensional irreducible representation. Then*

$$\lambda_2(\Psi_{\psi_E,U}) \le 1 - \frac{\varepsilon^2}{2\dim\mathscr{H}_E}.$$

Proof. Consider the unitary representation $V = U\otimes\bar{U}$ on the Hilbert space $\mathscr{K}\otimes\bar{\mathscr{K}}$. Here $\bar{U}$ is the conjugate representation of U acting on $\bar{\mathscr{K}}$ associated to the discrete quantum group (see for example [DSV17, Definition 2.2] for the construction of $\bar{U}$). Under the canonical unitary isomorphism of Hilbert spaces $B(\mathscr{K}) \cong \mathscr{K}\otimes\bar{\mathscr{K}}$; $|\xi\rangle\langle\eta| \mapsto \xi\otimes\bar{\eta}$ (where

$B(\mathscr{K})$ is equipped with the usual trace-inner product), we have that $\Psi_{\psi_E,U}$ is identified with the self-adjoint map

$$Y=(\psi_E\otimes \mathrm{id})(V)\in B(\mathscr{K}\otimes\bar{\mathscr{K}}).$$

Now, by self-adjointness of $\Psi_{\psi_E,U}$ (or, equivalently, Y), we have

$$\lambda_2(\Psi_{\psi_E,U})=\sup\left\{\operatorname{Re}\operatorname{Tr}\left(X^*\Psi_{\psi_E,U}(X)\right)\right\},$$

where the supremum is taken over $X\in B(\mathscr{K})$ such that $\|X\|_2=1$ and X is orthogonal to the 1 eigenspace of $\Psi_{\psi_E,U}$. By Corollary 5.2.40, this eigenspace is exactly $\mathbb{C}1$, and corresponds to the one-dimensional subspace $(\mathscr{K}\otimes\bar{\mathscr{K}})_0$ of invariant vectors for Y. Thus, we may apply Proposition 5.3.4 to deduce

$$\operatorname{Re}\operatorname{Tr}\left(X^*\Psi_{\psi,U}(X)\right)\le 1-\frac{\varepsilon^2}{2\dim\mathscr{H}_E},$$

for all such X. □

5

Applying Theorem 5.2.14, we immediately obtain the following

Corollary 5.3.6. *The channels $\Psi_{\psi_E,U}$ considered in Theorem 5.3.5 have quantum edge expansion*

$$h_Q(\Psi_{\psi_E,U})\ge\frac{\varepsilon^2}{4\dim\mathscr{H}_E}$$

in the sense of [Li+25].

Note that the lower bound on the expansion constant is independent of the size of the Hilbert space $\mathscr{K}$ on which the representation U acts. Moreover, the channels $\Psi_{\psi_E,U}$ have degree uniformly bounded by $|E|:=\sum_{x\in E}(\dim\mathscr{H}_x)^2$. Indeed, this follows from the fact that, after choosing appropriate bases, $\Psi_{\psi_E,U}$ has the explicit Kraus form

$$\Psi_{\psi_E,U}(\rho)=\frac{1}{\dim\mathscr{H}_E}\sum_{s\in E}\sum_{1\le i,j\le\dim x}U^x_{ij}\rho(U^x_{ij})^*\qquad(\rho\in B(\mathscr{K})),$$

where $U^x=(U^x_{ij})\in B(\mathscr{H}_x)\otimes B(\mathscr{K})$ is the block of U associated to a given $x\in\operatorname{Irred}(\mathbf{G})$, as in the proof of Theorem 5.2.39.

In particular, the problem of construction of explicit bounded degree expander families reduces to that of finding an increasing family of finite-dimensional irreducible representations of $\hat{\mathbf{G}}$.

Theorem 5.3.7. *Let $\hat{\mathbf{G}}$ be an infinite property (T) discrete quantum group with symmetric Kazhdan pair (E,ε). Let $(U_i)_{i\in\mathbb{N}}$ be a sequence of finite-dimensional irreducible unitary representations on Hilbert spaces $\mathscr{K}_i$ with $\lim_i\dim\mathscr{K}_i=\infty$. Then the family of bistochastic quantum channels $(\Psi_i)_{i\in\mathbb{N}}$, with $\Psi_i=\Psi_{\psi_E,U_i}$ forms a bounded degree quantum expander family with quantum edge expansion*

$$\inf_{i\in\mathbb{N}}h_Q(\Psi_i)\ge\frac{\varepsilon^2}{4\dim\mathscr{H}_E}.$$

The conditions of Theorem 5.3.7 are of course immediately satisfied if $\hat{\mathbf{G}}$ is *residually finite* in the sense of [BBCW20], that is, when $\mathcal{O}(\mathbf{G})$ is residually finite dimensional as a $*$-algebra, meaning that there exists a separating family of finite dimensional representations.

Corollary 5.3.8. *Let $\hat{\mathbf{G}}$ be an infinite discrete quantum group with property (T). If $\hat{\mathbf{G}}$ is residually finite, then $\hat{\mathbf{G}}$ gives rise to a bounded degree family of quantum expanders.*

Remark 5.3.9. Note that in the presence of property (T), the results of [BBCW20] show that residual finiteness of $\hat{\mathbf{G}}$ is in fact equivalent to $\hat{\mathbf{G}}$ having the a priori weaker *Kirchberg Factorisation Property*. In terms of the Haar state on $\mathbf{G}$, this latter condition is equivalent to the Haar state $h : C(\mathbf{G}_{\max}) \to \mathbb{C}$ being a so-called *amenable trace*. At this time, it is not clear to us whether checking for the Factorisation Property is any easier than directly verifying residual finiteness. In the next section, we will explore some natural examples quantum groups with property (T), arising from compact bicrossed products of residually finite discrete groups, and their associated expanders.

5

5.3.1. Spectral gap for channels associated to non-tracial states

In the previous section, we only considered expansion properties of the channels $\Psi_{\psi_E,U} : B(\mathcal{K}) \to B(\mathcal{K})$, where U is an irreducible finite-dimensional unitary representation of $\hat{\mathbf{G}}$ on $\mathcal{K}$, and ψ_E is the tracial state on $\ell^\infty(\hat{\mathbf{G}})$ associated to the normalised trace on $B(\mathcal{H}_E)$. In this section, we outline how much of the same analysis may be extended if one replaces ψ_E with any faithful symmetric state ψ supported on $p_E \ell^\infty(\hat{\mathbf{G}}) p_E \subset B(\mathcal{H}_E)$. We shall see from our estimates below that any deviation of ψ from the canonical trace ψ_E gives rise to weaker lower bound on the spectral gap. This might suggest that the tracial case is optimal when considering expansion properties. On the other hand, the weaker bound obtained below in the non-tracial case may just be a consequence of our crude estimates and could possibly be improved.

The main technical ingredient we need is a non-tracial extension of Lemma 5.3.3.

Lemma 5.3.10. *Let $\mathcal{H}$ be a finite dimensional Hilbert space, $0 < \varepsilon < 1$, and ρ a density matrix in $B(\mathcal{H})$ with smallest eigenvalue λ. Suppose $A \in B(\mathcal{H})$ satisfies $\|A\| \le 1$ and there exists a unit vector $\eta \in \mathcal{H}$ such that $\mathrm{Re}\,\langle \eta | A | \eta \rangle \le 1 - \varepsilon$. Then,*

$$\mathrm{Re}\,\mathrm{Tr}(\rho A) \le 1 - \lambda\varepsilon.$$

Proof. Let $d = \dim \mathcal{H}$. Extend η to an orthonormal basis $\eta_1 = \eta, \ldots, \eta_d$ for $\mathcal{H}$. Write $\rho = \lambda 1 + (1 - d\lambda)\rho'$, where $\rho' = \frac{\rho - \lambda 1}{1 - d\lambda}$ is also a density matrix. Then,

$$\mathrm{Re}\,\mathrm{Tr}(\rho A) = \lambda\,\mathrm{Re}\,\mathrm{Tr}(A) + (1 - d\lambda)\,\mathrm{Re}\,\mathrm{Tr}(\rho' A) \le 1 - d\lambda + \lambda\,\mathrm{Re}\sum_{i=1}^{d} \langle \eta_i | A | \eta_i \rangle \le 1 - \varepsilon\lambda. \qquad \square$$

Let $\psi : B(\mathcal{H}_E) \to \mathbb{C}$ be a fixed faithful state, which we also regard as a normal state on $\ell^\infty(\hat{\mathbf{G}})$ with central support p_E. Assume that ψ is symmetric. Write $\psi = \mathrm{Tr}(\sigma \cdot)$ for a unique density $\sigma \in B(\mathcal{H}_E)$, and let $0 < \lambda \le \frac{1}{\dim \mathcal{H}_E}$ be the minimal eigenvalue of σ.

Theorem 5.3.11. *Let $U \in M(C_0(\hat{\mathbf{G}}) \otimes B_0(\mathcal{K}))$ be a finite-dimensional irreducible representation. Then for the channel*

$$\Psi_{\psi,U} : B(\mathcal{K}) \to B(\mathcal{K}), \qquad \Psi_{\psi,U}(\rho) = (\psi \otimes \mathrm{id})(U(1 \otimes \rho)U^*) \qquad (\rho \in B(\mathcal{K})),$$

$$\lambda_2(\Psi_{\psi,U}) \le 1 - \frac{\lambda\varepsilon^2}{2}.$$

Proof. As in the case of $\psi = \psi_E$ studied in the previous subsection, it suffices to show that

$$\mathrm{Re}\,\langle\xi|(\psi \otimes \mathrm{id})(V)|\xi\rangle \le 1 - \frac{\lambda\varepsilon^2}{2}, \tag{5.3}$$

for any representation $V \in M(C_0(\hat{\mathbf{G}}) \otimes B_0(\mathcal{K}))$ and any unit vector $\xi \in (\mathcal{K}_0)^\perp$, the complement of the fixed vectors. This latter estimate follows immediately from Lemma 5.3.10 and property (T). □

5

Remark 5.3.12. One may also observe that the channels $\Psi_{\psi,U}$ still have degree bounded by $|E| = \sum_{x \in E}(\dim \mathcal{H}_x)^2$.

5.4. EXPANDERS ASSOCIATED TO BICROSSED PRODUCT QUANTUM GROUPS

In this section, we apply the general theory of the previous section to analyse concrete examples of quantum expanders. As we have seen in Theorem 5.3.7, the construction of an explicit expander family simply requires as input a residually finite discrete quantum group $\hat{\mathbf{G}}$ with property (T). While there are nowadays many new and interesting examples of genuine infinite discrete quantum groups with property (T) [VV19; FMP17; RV24], surprisingly little seems to be known about their representation theory and their residual finiteness. One class of examples where these properties are nevertheless well-understood are quantum groups arising from bicrossed products. We now very briefly introduce the notion of a (compact) bicrossed product quantum group. For more details we refer to [FMP17, Section 3].

Let G and Γ be a compact and a discrete group, respectively, forming a *matched pair* in the sense that they are realised as trivially-intersecting closed subgroups of a locally compact group H, with the property that the product ΓG has full Haar measure in H. This amounts to giving a left action α of Γ on G and a right action β of G on Γ satisfying certain compatibility conditions (*i.e.* [FMP17, Proposition 3.3]).

From a quadruple $(\Gamma, G, \alpha, \beta)$ as above, one can construct a compact quantum group $\mathbf{G} = \mathbf{G}(\Gamma, G, \alpha, \beta)$, as explained in [VV03] or [FMP17, Section 3.2]; we will denote it by $\mathbf{G}(\Gamma, G, \alpha, \beta)$ when we wish to be explicit about the matched pair structure, and reserve the present notation of Γ, G, α, β, and $\mathbf{G}$ throughout the current section. The dual discrete quantum groups $\hat{\mathbf{G}}$ will provide, under certain conditions, non-trivial examples of discrete quantum groups with property (T).

Let us now discuss the construction of $\mathbf{G}$. Let $\mathcal{A}_m = \Gamma \ltimes_{\alpha,f} C(G)$ be the full crossed product C*-algebra and $\mathcal{A} = \Gamma \ltimes_\alpha C(G)$ the reduced crossed product C*-algebra. Let ω denote the canonical injective maps from $C(G)$ to $\mathcal{A}_m$ and from $C(G)$ to $\mathcal{A}$. For $\gamma \in$

Γ we denote by u_γ the canonical unitaries in either $\mathscr{A}_m$ or $\mathscr{A}$. The bicrossed product construction endows $\mathscr{A}_m$ with a comultiplication making it a compact quantum group. This is done as follows. First, for each coset $\gamma \cdot G \in \Gamma/G$, we define

$$v^{\gamma\cdot G} = (v_{rs})_{r,s\in\gamma\cdot G} \in M_{|\gamma\cdot G|}(\mathbb{C}) \otimes C(G),$$

where $v_{rs} = \chi_{A_{r,s}}$ is the characteristic function of the set $A_{r,s} = \{g \in G \mid \beta_g(r) = s\}$. Note that $v^{\gamma\cdot G}$ is a *magic unitary* over $C(G)$, that is, a matrix whose rows and columns are projection-valued measures (PVMs). Finally, the comultiplication is given by the unique unital $*$-homomorphism $\Delta_m : \mathscr{A}_m \to \mathscr{A}_m \otimes \mathscr{A}_m$ such that

$$\Delta_m \circ \omega = (\omega \otimes \omega) \circ \Delta_G \text{ and } \Delta_m(u_\gamma) = \sum_{v\in\gamma\cdot G} u_\gamma \omega(v_{\gamma,r}) \otimes u_r \qquad (\gamma \in \Gamma). \tag{5.4}$$

Then $\mathbf{G} = (\mathscr{A}_m, \Delta_m)$ is a compact quantum group, called the *(compact) bicrossed product* associated to the matched pair (Γ, G).

The final piece of information we need concerns the description of the irreducible unitary representations of $\mathbf{G}$. For each $\gamma \in \Gamma$, we consider the orbit $\gamma \cdot G \in \Gamma/G$ and define

$$V^{\gamma\cdot G} = \sum_{r,s\in\gamma\cdot G} e_{r,s} \otimes u_r \omega(v_{r,s}) \in M_{|\gamma\cdot G|}(\mathbb{C}) \otimes \mathscr{A}_m.$$

Then $V^{\gamma\cdot G}$ determines an irreducible unitary representation of $\mathbf{G}$, and the representations in the family $\{V^{\gamma\cdot G}\}_{\gamma\cdot G\in\Gamma/G}$ of representations are pairwise inequivalent and satisfy $\overline{V^{\gamma\cdot G}} \simeq V^{\gamma^{-1}\cdot G}$. Moreover, any irreducible unitary representation of $\mathbf{G}$ is equivalent to a subrepresentation of $V^{\gamma\cdot G} \otimes v^x$ for some $\gamma \cdot G \in \Gamma/G$ and $x \in \mathrm{Irred}(G)$, where $v^x = (\mathrm{id} \otimes \omega)(u^x)$.

5.4.1. PROPERTY (T) AND THE BICROSSED PRODUCT CONSTRUCTION

To the best of our knowledge, a complete classification of property (T) for the duals $\hat{\mathbf{G}}$ of a compact bicrossed product $\mathbf{G}(\Gamma, G, \alpha, \beta)$ in terms of the input data is not known. However, several important results have been obtained [FMP17; VV19]. In particular, [FMP17, Theorem 4.3] shows that $\hat{\mathbf{G}}$ has property (T) whenever Γ has property (T) and G is finite. Moreover, in this case, the proof of [FMP17, Theorem 4.3] shows that if (E, ε) is a Kazhdan pair for Γ containing the identity element, and v is a fundamental representation of G, then (the irreducible components of) $\{V^{\gamma\cdot G} \otimes v\}_{\gamma\in E}$ is a Kazhdan set for $\hat{\mathbf{G}}$.

As a concrete example, one can take any natural number $n \geq 3$ and any prime number $p \geq 3$. Let $\Gamma = SL_n(\mathbb{Z})$, $G = SL_n(\mathbb{F}_p)$, and consider the action α given by $\alpha_\gamma(g) = [\gamma]g[\gamma]^{-1}$ for $\gamma \in \Gamma$ and $g \in G$, and β being the trivial action. The resulting discrete quantum group $\hat{\mathbf{G}}$ then has property (T). Moreover, $\hat{\mathbf{G}}$ is residually finite by [BBCW20, Theorem 4.2].

In the work [VV19], a very large class of compact bicrossed products $\mathbf{G}$ whose duals have property (T) was discovered using triangle presentations and related constructions. These examples are somewhat more interesting than the above one, because they arise from matched pairs where G is also infinite (and therefore $\hat{\mathbf{G}}$ is not commensurable with a classical property (T) group). As an example (cf. [VV19, Remark 6.3]), one can let $\mathbb{K}$ be a commutative local field with ring of integers $\mathcal{O}$. Put $H = PGL(3,\mathbb{K})$ and $G = PGL(3,\mathcal{O})$. Let $\Gamma < H$ be a subgroup such that $H = \Gamma G$ and $\Gamma \cap G = \{e\}$. Then the dual of the compact bicrossed product $\mathbf{G}$ associated to the matched pair (Γ, G) has property (T).

5.4.2. CHANNELS CORRESPONDING TO THE BICROSSED PRODUCT CONSTRUCTION

Let us briefly describe the general flavour of the channels that one can expect from bicrossed product quantum groups. We will use the notation of the previous section. Let U be a finite dimensional unitary representation of $\hat{\mathbf{G}}$ on a finite-dimensional Hilbert space $\mathscr{K}$, and let $\pi : C(\mathbf{G}) \to B(\mathscr{K})$ be the associated $*$-homomorphism.

Let $\gamma \in \Gamma$ be arbitrary, and let us analyse what kind of channels we can obtain if we use the construction in Proposition 5.2.36 with $E = \{V^{\gamma\cdot G}\}$. Let φ be a state on $M_{|\gamma\cdot G|}(\mathbb{C})$. Using Proposition 5.2.36, we find the quantum channel $\Phi = \Phi_{V^{\gamma\cdot G},\varphi,\pi}$ is given by

$$\begin{aligned}
\Phi(\rho) &= (\varphi\otimes \mathrm{id})((\mathrm{id}\otimes\pi)(V^{\gamma\cdot G})(I\otimes\rho)(\mathrm{id}\otimes\pi)(V^{\gamma\cdot G})^*) \\
&= (\varphi\otimes\mathrm{id})\left(\sum_{r_1,r_2,s_1,s_2\in\gamma\cdot G} (e_{r_1,s_1}\otimes\pi(u_{r_1}\omega(v_{r_1,s_1})))(I\otimes\rho)(e_{r_2,s_2}\otimes\pi(u_{r_2}\omega(v_{r_2,s_2})))^*\right) \\
&= (\varphi\otimes\mathrm{id})\left(\sum_{r_1,r_2,s\in\gamma\cdot G} e_{r_1,r_2}\otimes(\pi(u_{r_1}\omega(v_{r_1,s}))\rho\pi(u_{r_2}\omega(v_{r_2,s}))^*)\right) \\
&= (\varphi\otimes\mathrm{id})\left(\sum_{r_1,r_2,s\in\gamma\cdot G} e_{r_1,r_2}\otimes(\pi(u_{r_1})\pi(\omega(v_{r_1,s}))\rho\pi(\omega(v_{r_2,s}))^*\pi(u_{r_2}))^*)\right).
\end{aligned}$$

Proposition 5.4.1. *If φ is a trace, then Φ is a mixed unitary channel.*

To prove this proposition, we first need an easy lemma.

Lemma 5.4.2. *Let $p_1, \dots, p_n$ be orthogonal projections in $M_N(\mathbb{C})$ that sum to the identity. Then the channel*

$$\rho \mapsto \sum_{i=1}^n p_i\rho p_i$$

is a mixed unitary channel.

Proof. Define the operator

$$U = \sum_{k=1}^n e^{2\pi i k/n} p_k.$$

This is clearly a unitary. Moreover,

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^n U^k\rho(U^k)^* &= \frac{1}{n}\sum_{k,l,m=1}^n e^{2\pi i k l/n} p_l \rho e^{-2\pi i k m/n} p_m \\
&= \frac{1}{n}\sum_{k,l,m=1}^n e^{2\pi i k(l-m)/n} p_l\rho p_m \\
&= \sum_{l=1}^n p_l\rho p_l.
\end{aligned}$$

This shows that the channel is indeed a mixed unitary channel. □

Proof of Proposition 5.4.1. We will now assume that φ is the trace. In this case, the equation for Φ above Proposition 5.4.1 becomes

$$\Phi(\rho) = \frac{1}{|\gamma \cdot G|} \sum_{r,s \in \gamma \cdot G} \pi(u_r)\pi(\omega(v_{r,s}))\rho\pi(\omega(v_{r,s}))^*\pi(u_r)^*. \tag{5.5}$$

By Lemma 5.4.2 the channels

$$\rho \mapsto \sum_{s \in \gamma \cdot G} \pi(\omega(v_{r,s}))\rho\pi(\omega(v_{r,s}))^* \qquad (s \in \gamma \cdot G)$$

are mixed unitaries, which in turn shows that Φ is a convex combination of compositions of unitary channels, and hence itself mixed unitary. □

Remark 5.4.3. The conclusion of Proposition 5.4.1 holds more generally if we consider channels built from any direct sums of building block representations of the form $V^{\gamma \cdot G} \otimes v^x$, $(\gamma \in \Gamma, x \in \mathrm{Irred}(G))$ and traces on the underlying multi-matrix algebras. Indeed, the channels associated to single tensor products $V^{\gamma \cdot G} \otimes v^x$ and traces are easily seen to be compositions of mixed unitary channels, and passing to direct sums amounts to convex combinations of mixed unitary channels.

5

Remark 5.4.4. Proposition 5.4.1 does not rule out the possibility of obtaining non-mixed unitary expanders from bicrossed product quantum groups when we work with non-tracial states φ on the spaces $M_N(\mathbb{C})$. However, we know from the results of Section 5.3.1 that our bounds for spectral gap are governed by the smallest eigenvalue of a φ (i.e., the tracial part of φ in the proof of Lemma 5.3.10), so even if the resulting channel Φ was non-random unitary with good expansion, we would really only still be looking at a convex combination of a mixed unitary quantum expander and a non-mixed unitary channel whose expansion properties are unknown.

5.5. COIDEALS AND SPECTRAL GAP FOR QUANTUM SCHREIER GRAPHS

In Section 5.3, we saw how irreducible finite-dimensional representations of property (T) discrete quantum groups can be used to construct quantum channels with uniform lower bounds on their spectral gap. In this section, we outline another approach to constructing quantum expanders which is closer in spirit to the classical construction of expander graphs of [Lub10; Mar73] obtained by considering finite Schreier coset graphs associated to discrete groups with property (T). The key idea is to consider the appropriate quantum analogue of the (finite) quotient spaces Γ/H, where Γ is a property (T) discrete group, and $H \leq \Gamma$ is a (finite-index) subgroup. For discrete quantum groups, this is captured by the notion of a *coideal*.

We use the notations of the previous sections for quantum groups. In particular, $\mathbf{G}$ always denotes a compact quantum group and $\hat{\mathbf{G}}$ denotes its dual discrete quantum group.

Definition 5.5.1. Let $\hat{\mathbf{G}}$ be a discrete quantum group. A *(left) coideal* of $\hat{\mathbf{G}}$ is a von Neumann subalgebra $\mathcal{M} \subset \ell^\infty(\hat{\mathbf{G}})$ such that

$$\hat{\Delta}(\mathcal{M}) \subset \ell^\infty(\hat{\mathbf{G}})\overline{\otimes}\mathcal{M}.$$

It is well-known [DKS12] that when $\hat{\mathbb{G}} = \Gamma$ is a classical discrete group, coideals in Γ are in one-to-one correspondence with the homogeneous spaces Γ/H, where $H \le \Gamma$ is a subgroup. The identification is given by $\mathcal{M} = \ell^\infty(\Gamma/H) \subset \ell^\infty(\Gamma)$, the von Neumann subalgebra of functions constant on the left cosets of H. For general discrete quantum groups $\hat{\mathbb{G}}$, this analogy breaks down, and the theory of coideals turns out to be much richer than the study of quotients of $\hat{\mathbb{G}}$ by quantum subgroups. In particular, while the notion of *quantum subgroup* $\hat{H} \subset \hat{\mathbb{G}}$ is well understood [DKS12], and gives rise to a "quotient-type" coideal $\mathcal{M} = \ell^\infty(\hat{\mathbb{G}}/\hat{H})$, not all coideals arise from quantum subgroups. See for example [Kas17; AK24], and references therein.

Nonetheless, we still regard general coideals $\mathcal{M} \subset \ell^\infty(\hat{\mathbb{G}})$ as a kind of quantum homogeneous space. In particular, every coideal $\mathcal{M} \subset \ell^\infty(\hat{\mathbb{G}})$ admits a natural ergodic "translation action" of $\hat{\mathbb{G}}$ on $\mathcal{M}$, which we now describe.

Definition 5.5.2. A *(left) action* of a discrete quantum group $\hat{\mathbb{G}}$ on a von Neumann algebra $\mathcal{M}$ is a normal unital $*$-homomorphism

$$\alpha : \mathcal{M} \to \ell^\infty(\hat{\mathbb{G}}) \overline{\otimes} \mathcal{M}$$

satisfying the coassociativity condition

$$(\hat{\Delta} \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \alpha)\alpha.$$

We moreover call the action α *ergodic* if the fixed point algebra $\mathcal{M}^\alpha = \{x \in \mathcal{M} \mid \alpha(x) = 1 \otimes x\}$ is trivial, i.e.,

$$\mathcal{M}^\alpha = \mathbb{C}1.$$

The above definition generalises the usual notion of an action of a discrete group Γ on $\mathcal{M}$ by $*$-automorphisms. Indeed, if Γ is a classical discrete group and $\Gamma \curvearrowright^\alpha \mathcal{M}$ is an action given by

$$\Gamma \times \mathcal{M} \ni (g, x) \mapsto \alpha_g(x) \in \mathcal{M}, \tag{5.6}$$

then one can define a map (still denoted by α) $\alpha : \mathcal{M} \to \ell^\infty(\Gamma) \overline{\otimes} \mathcal{M} \cong \prod_{g \in \Gamma}^{\ell^\infty} \mathcal{M}$ via

$$\alpha(x) = (\alpha_g(x))_{g \in \Gamma} \qquad (x \in \mathcal{M}). \tag{5.7}$$

Then one can easily verify that action (5.7) satisfies Definition 5.5.2 if and only if Equation 5.6 defines an action of Γ on $\mathcal{M}$ in the usual sense.

Returning to coideals $\mathcal{M} \subset \ell^\infty(\hat{\mathbb{G}})$ of a discrete quantum group $\hat{\mathbb{G}}$, the restriction of the comultiplication defines *the canonical (left) action* $\alpha := \hat{\Delta}|_{\mathcal{M}}$ of $\hat{\mathbb{G}}$ on $\mathcal{M}$. Again in the classical case of a discrete group Γ, with $\mathcal{M} = \ell^\infty(\Gamma/H)$ the coideal associated to a subgroup $H \le \Gamma$, the action α is just the canonical left translation action of Γ on $\ell^\infty(\Gamma/H)$.

Actions of quantum groups on von Neumann algebras are strongly linked to the theory of unitary representations. Indeed, if $U \in M(C_0(\hat{\mathbb{G}}) \otimes B_0(\mathcal{K}))$ is a unitary representation of a discrete quantum group on a Hilbert space $\mathcal{K}$, then

$$\alpha : B(\mathcal{K}) \to \ell^\infty(\hat{\mathbb{G}}) \overline{\otimes} B(\mathcal{K}); \quad x \mapsto U^*(1 \otimes x)U \qquad (x \in B(\mathcal{K}))$$

defines an action of $\hat{\mathbb{G}}$ on the von Neumann algebra $B(\mathcal{K})$. Conversely, a deep result of Vaes [Vae01] shows that any action α of $\hat{\mathbb{G}}$ on a von Neumann algebra $\mathcal{M}$ is *unitarily* *implemented* in the above sense. That is, one can find a faithful representation $\mathcal{M} \subset B(\mathcal{K})$ and a unitary representation $U \in M(C_0(\hat{\mathbf{G}}) \otimes B_0(\mathcal{K}))$ such that

$$\alpha(x) = U^*(1 \otimes x)U \qquad (x \in \mathcal{M}). \tag{5.8}$$

Remark 5.5.3. In the above discussion, one could have equivalently defined unitarily implemented actions (5.8) via the formula $x \mapsto U(1 \otimes x)U^*$. The convention (5.8) just ensures that U is a representation of $\hat{\mathbf{G}}$, as opposed to a representation of the opposite quantum group $\hat{\mathbf{G}}^{\mathrm{op}}$.

When $\mathcal{M}$ has admits a normal faithful α-*invariant* state φ in the sense that $(\mathrm{id} \otimes \varphi) \circ \alpha = \varphi(\cdot)1$, the unitary representation implementing α can be described somewhat explicitly [DSV17, Section 3]: Here one may take $\mathcal{K} = L^2(\mathcal{M}, \varphi)$, the GNS Hilbert space of φ. Then U is given via its adjoint by

$$U^*(\xi \otimes \Lambda(x)) = \alpha(x)(\xi \otimes \Lambda(1)) \qquad (x \in \mathcal{M},\ \xi \in \ell^2(\hat{\mathbf{G}})), \tag{5.9}$$

where $\Lambda : \mathcal{M} \to L^2(\mathcal{M}, \varphi)$ is the GNS map and $\ell^2(\hat{\mathbf{G}})$ is the GNS Hilbert space of the (right) Haar weight of $\hat{\mathbf{G}}$.

Let $\mathcal{M}$, α, φ, U be as in the previous paragraph, and let $\psi \in \ell^1(\hat{\mathbf{G}})$ be a normal state. Then, just as in Section 5.3.1, we obtain a normal UCP φ-preserving map

$$\Psi : \mathcal{M} \to \mathcal{M}; \qquad \Psi(x) = (\psi \otimes \mathrm{id})\alpha(x) \qquad (x \in \mathcal{M}). \tag{5.10}$$

Thanks to Equation 5.9, the canonical extension of Ψ to a bounded map $\tilde{\Psi} \in B(L^2(\mathcal{M}, \varphi))$ (densely defined by the formula $\tilde{\Psi}(\Lambda(x)) = \Lambda(\Psi(x))$) is given in terms of the implementing unitary representation U by the formula

$$\tilde{\Psi} = (\psi \otimes \mathrm{id})(U^*). \tag{5.11}$$

In particular, we have $(\tilde{\Psi})^* = (\psi \otimes \mathrm{id})U$, so $\tilde{\Psi}$ is self-adjoint when $\psi = \psi \circ \hat{S}$.

A special case of interest to us in the sequel in which actions are always guaranteed to admit faithful invariant states is when the underlying von Neumann algebra is finite-dimensional. This result is surely folklore, but we provide a proof for self-containment.

Lemma 5.5.4. *Let α be an action of a unimodular discrete quantum group $\hat{\mathbf{G}}$ on a finite-dimensional von Neumann algebra $\mathcal{M}$. Then $\mathcal{M}$ admits a faithful α-invariant state.*

Proof. Let $\mathcal{M} \subset B(\mathcal{K})$ and let $U \in M(C_0(\hat{\mathbf{G}}) \otimes B_0(\mathcal{K}))$ be an implementing unitary for the action α. We make use of the *quantum Bohr compactification* $b\hat{\mathbf{G}}$ of $\hat{\mathbf{G}}$ [Soł05]. By definition, $b\hat{\mathbf{G}}$ is the compact quantum group with $C(b\hat{\mathbf{G}})$ given by the C*-subalgebra of $\ell^\infty(\hat{\mathbf{G}})$ generated by the matrix coefficients of the so-called finite dimensional *admissible* unitary representations of $\hat{\mathbf{G}}$, with comultiplication given by the restriction of $\hat{\Delta}$. Here, admissible means that the transpose of U, when viewed as a matrix, is invertible. Since $\hat{\mathbf{G}}$ is unimodular, every finite-dimensional representation U is admissible [Vis17, Theorem 2.1], and since the action α is implemented by U, which is a representation of $b\hat{\mathbf{G}}$, it can also be regarded as an action of $b\hat{\mathbf{G}}$ on $\mathcal{M}$. The claim now follows from the fact that any action of a compact quantum group on a finite-dimensional von Neumann algebra $\mathcal{M}$ admits a faithful invariant state ψ. Indeed, one can just take $\psi = (h \otimes \psi_0) \circ \alpha$, where h is the Haar state of $b\hat{\mathbf{G}}$ and ψ_0 is any faithful state on $\mathcal{M}$. □

5.5.1. QUANTUM CAYLEY GRAPHS AND SCHREIER GRAPHS

We now introduce quantum Schreier graphs. A quantum Schreier graph is a natural extension of the notion of a quantum Cayley graph, as described in [Was23], and we start off by discussing these. We note that a version of quantum Cayley graphs was also developed earlier by Vergnioux [Ver05], using a rather different-looking, but ultimately equivalent language. We will use the notation and terminology of [Was23] below.

Let $\hat{\mathbf{G}}$ be a unimodular discrete quantum group. For $x \in C_{00}(\hat{\mathbf{G}})$, we consider the normal functional $\omega_x := \hat{h}(\cdot x) \in \ell^1(\hat{\mathbf{G}})$. Given $x, y \in C_{00}(\hat{\mathbf{G}})$, there exists a unique $z \in C_{00}(\hat{\mathbf{G}})$ such that $\omega_z = \omega_x \star \omega_y$. This defines an associative product on $C_{00}(\hat{\mathbf{G}})$, also called convolution, and denoted $z = x \star y$ [Ver05; Ver07]. In fact, when $\hat{G}$ is unimodular, we have [BR17, Lemma 4.3]

$$x \star y = (\omega_x \circ \hat{S} \otimes \mathrm{id})\hat{\Delta}(y). \tag{5.12}$$

In particular, the left convolution map $y \mapsto x \star y$ is nothing other than the normal completely bounded map $\Psi_{\omega_x} : \ell^\infty(\hat{\mathbf{G}}) \to \ell^\infty(\hat{\mathbf{G}})$ associated to the action $\alpha = \hat{\Delta}$ of $\hat{\mathbf{G}}$ on $\ell^\infty(\hat{\mathbf{G}})$. This map is completely positive when $\omega_x \circ \hat{S}$ is a positive functional.

To define a quantum Cayley graph over $\hat{\mathbf{G}}$, we fix a finite rank projection $p \in C_{00}(\hat{\mathbf{G}})$ such that $\hat{S}(p) = p$, $\hat{\varepsilon}(p) = 0$ and $\bigvee_{n\in\mathbb{N}} p^{\star n} = 1_{\ell^\infty(\hat{\mathbf{G}})}$. Define the normal completely positive map $A : \ell^\infty(\hat{\mathbf{G}}) \to \ell^\infty(\hat{\mathbf{G}})$ by

$$Ax = p \star x \qquad (x \in \ell^\infty(G)).$$

Definition 5.5.5 ([Was23])**.** The pair $(\ell^\infty(\hat{\mathbf{G}}), A)$ is called a *quantum Cayley graph over* $\hat{\mathbf{G}}$.

As explained in [Was23], when $\Gamma = \hat{\mathbf{G}}$ is a classical discrete group, the above construction yields $p = \chi_E$, the characteristic function of a symmetric generating set $e \notin E = E^{-1}$ of Γ, and $Af(g) = (\chi_E \star f)(g) = \sum_{s\in E} f(sg)$ is the usual adjacency operator of the Cayley graph $\mathscr{C}(\Gamma, E)$. We remark that quantum Cayley graphs are always *d-regular quantum graphs* with $d = \hat{h}(p)$, since

$$A1 = \hat{h}(p)1.$$

We also note that

$$\hat{h}(Ax) = \hat{h}(p \star x) = \hat{h}(p)\hat{h}(x),$$

for all $x \in C_{00}(\hat{\mathbf{G}})$. As a consequence, the normalised adjacency operator is $d^{-1}A$ is a bistochastic quantum channel relative to the tracial Haar weight $\hat{h}$.

Let $\mathscr{M} \subset \ell^\infty(\hat{\mathbf{G}})$ be a coideal. It is clear from the definitions that the quantum adjacency matrix A associated to the quantum Cayley graph in Definition 5.5.5 leaves $\mathscr{M}$ invariant. In particular, in the case of classical discrete groups, restriction $A|_{\mathscr{M}}$ is the natural analogue of the adjacency matrix of the associated Schreier coset graph. This leads us to the following definition.

Definition 5.5.6. Let $(\ell^\infty(\hat{\mathbf{G}}), A)$ be a (d-regular) quantum Cayley graph over $\hat{\mathbf{G}}$. For any coideal $\mathscr{M} \subset \ell^\infty(\hat{\mathbf{G}})$, the pair $(\mathscr{M}, A|_{\mathscr{M}})$ is called the *(d-regular) quantum Schreier graph* associated to the generating projection p.

5.5.2. SPECTRAL GAP FOR QUANTUM SCHREIER GRAPHS

We now come to the main result of the section, which asserts that quantum Schreier graphs associated to property (T) quantum groups give rise to bounded degree expanders.

Let $\hat{\mathbf{G}}$ be a discrete quantum group with property (T). Let (E,ε) be a fixed Kazhdan pair for $\hat{\mathbf{G}}$ with $E=\bar{E}$. Let $p_E \in C_{00}(\hat{\mathbf{G}})$ be the central support of E, let $\mathcal{M} \subset \ell^\infty(\hat{\mathbf{G}})$ be a coideal, and let $(\mathcal{M}, A|_{\mathcal{M}})$ be the quantum Schreier graph associated to the generating projection p_E. Finally, assume that $\mathcal{M}$ admits a normal invariant state φ. In this case (thanks to the assumption $\hat{S}(p)=p$), we have that $A|_{\mathcal{M}}$ extends to a self-adjoint map $\widetilde{A|_{\mathcal{M}}} : L^2(\mathcal{M},\varphi) \to L^2(\mathcal{M},\varphi)$. By Lemma 5.5.4, this is always true if $\dim \mathcal{M} < \infty$. We denote by $\lambda_2(\mathcal{M},E) \le 1$ the second largest eigenvalue of $d^{-1}\widetilde{A|_{\mathcal{M}}}$, where $d=\hat{h}(p_E)=\sum_{s\in E}(\dim s)^2$ is the degree.

Theorem 5.5.7. *Let $\hat{\mathbf{G}}$ and (E,ε) be as above. For any finite dimensional quantum Schreier graph $(\mathcal{M}, A|_{\mathcal{M}})$ be associated to p_E, we have*

$$\lambda_2(\mathcal{M},E) \le 1 - \frac{\lambda\varepsilon^2}{2}$$

where $\lambda = \min_{s\in E} \frac{\dim \mathcal{H}_s}{d}$.

Proof. The proof is similar in spirit to the proofs of Theorems 5.3.5 and 5.3.11. Let φ be a faithful invariant state on $\mathcal{M}$ with respect to the action $\alpha = \hat{\Delta}|_{\mathcal{M}}$. Let $U \in M(C_0(\hat{\mathbf{G}}) \otimes B_0(L^2(\mathcal{M},\varphi)))$ be the unitary implementation of α. Then by Equation 5.11, we need to compute the second largest eigenvalue $\lambda_2(\tilde{\Psi})$ of the self-adjoint map

$$\tilde{\Psi} = (\psi\otimes \mathrm{id})(U^*) = (\psi \otimes \mathrm{id})(U),$$

where we are working with the $\hat{S}$-invariant state $\psi = d^{-1}\hat{h}(\cdot p_E) \in \ell^1(\hat{\mathbf{G}})$. Note that since the action $\alpha = \hat{\Delta}|_{\mathcal{M}}$ is ergodic, it follows from Equation 5.9 (describing U) that the subspace $L^2(\mathcal{M},\varphi)_0$ of U-invariant vectors is exactly $\Lambda(\mathcal{M}^\alpha) = \mathbb{C}\Lambda(1)$.

We thus need to establish the inequality

$$\inf_{\xi\in\Lambda(1)^\perp,\ \|\xi\|=1} \mathrm{Re}\,\langle \xi | (\psi\otimes\mathrm{id})(U)|\xi\rangle \le 1-\frac{\lambda\varepsilon^2}{2} \tag{5.13}$$

Recalling now the notation $\mathcal{H}_E = \bigoplus_{s\in E}\mathcal{H}_s$, and the canonical embedding $p_E\ell^\infty(\hat{\mathbf{G}})p_E \subset B(\mathcal{H}_E)$ from Section 5.3, we may canonically extend ψ to a (non-tracial state) on $B(\mathcal{H}_E)$ with eigenvalues given (without multiplicity) by $\{\frac{\dim\mathcal{H}_s}{d}\}_{s\in E}$. (The extended ψ is nothing more than $\psi\circ\mathbb{E}$, where $\mathbb{E} : B(\mathcal{H}_E) \to p_E\ell^\infty(\hat{\mathbf{G}})p_E$ is the trace-preserving conditional expectation.) With all this setup in place, we now see that the desired inequality (5.13) is just a special case of the inequality (5.3) obtained in the proof of Theorem 5.3.11. □

5.6. DISCUSSION

In this chapter, we provide two general approaches to construct quantum expanders from discrete quantum groups with property (T). The first approach (Section 5.3) takes as input a property (T) discrete quantum group together with a family of irreducible finite-dimensional representations with growing dimension, while the second approach (Section 5.5) takes as input a property (T) discrete quantum group together with a family of finite-dimensional coideals with unbounded dimension. Ultimately, what is needed for either method to produce examples is a good understanding of the finite-dimensional representation theory of a given property (T) discrete quantum group. However, at this time, it appears that there is a dearth of non-classical examples where the finite dimensional representation theory is well understood. In this section we will briefly discuss how the example we do have (quantum expanders coming from bicrossed products) should be viewed and what steps need to be taken to find more examples.

5.6.1. QUANTUM EXPANDERS COMING FROM BICROSSED PRODUCTS

The most natural quantum expanders coming from bicrossed products are convex combinations of quantum channels of the form given in Equation (5.5),

$$\Phi(\rho) = \frac{1}{|\gamma \cdot G|} \sum_{r,s \in \gamma \cdot G} \pi(u_r)\pi(\omega(v_{r,s}))\rho\pi(\omega(v_{r,s}))^* \pi(u_r)^*.$$

5

However, taking the same convex combination of quantum channels of the form

$$\Phi'(\rho) = \frac{1}{|\gamma \cdot G|} \sum_{r \in \gamma \cdot G} \pi(u_r)\rho\pi(u_r)^*,$$

which are quantum channels corresponding to the classical property (T) discrete group Γ, already gives a quantum expander. If one composes two unital trace-preserving completely positive maps Φ and Ψ, then a spectral gap of either Φ or Ψ ensures a spectral gap of at least the same size in $\Phi \circ \Psi$. How should we then view the quantum expanders coming from bicrossed products? The starting point is the quantum expander coming from the property (T) discrete group. This channel is a mixed unitary channel, and in the quantum expander coming from the bicrossed product each unitary conjugation is precomposed by a *different* quantum channel. In general, doing this would no longer give you a quantum expander, but our results show that precomposing by these particular quantum channels preserves the spectral gap.

5.6.2. REPRESENTATION THEORY OF PROPERTY (T) QUANTUM GROUPS

As mentioned previously, there are several recent geometric constructions which give rise to examples of property (T) discrete quantum groups that are different from the bicrossed product construction. These include the discrete quantum groups associated to triangle presentations [VV19], and the quantisations of discrete groups [RV24]. As pointed out in [RV24], there is also the quantum automorphism group $\mathrm{Aut}^+(HS)$ of the Higman-Sims graph HS, whose dual is an infinite discrete quantum group with property (T) . All of these examples provide new potential candidates for the application of our machinery, as soon as their representation theory is understood. In particular, we think it is a very interesting problem to better understand the structure of the quantum automorphism group of the Higman-Sims graph HS, $\mathrm{Aut}^+(HS)$, and its discrete dual. For example, does the Hopf*-algebra $\mathcal{O}(\mathrm{Aut}^+(HS))$ have many non-abelian finite-dimensional representations? Is it residually finite dimensional?

5.6.3. FINDING EXAMPLES OF QUANTUM SCHREIER GRAPHS

The approach to quantum expanders via quantum Schreier graphs is very promising because, on the one hand it gives a direct generalisation of the classical construction of quantum expanders of Margulis [Mar73], while on the other hand it yields potentially new types of quantum expanders acting on general finite-dimensional von Neumann algebras (and not just full matrix algebras, as is standard in the literature). The problem here again is to find non-trivial examples to apply these tools. We leave open the problem of finding non-classical discrete quantum groups with property (T) with large finite-dimensional coideals. Even for the property (T) quantum groups coming from bicrossed products, we do not know much about the structure of their coideals.

ACKNOWLEDGEMENTS

Doing a PhD is quite a journey. Sometimes you are stringing together nice results, and at other times the mathematics just does *not* want to budge. I have learnt that navigating this alone is not a good place for me to be. Fortunately, I did not have to; I have met so many people along the way. The thread of my life has become interwoven with countless others, and this did not stop the people whose lives were already intertwined with mine from happily weaving on. What better end to creating this dissertation than to tackle the daunting task of temporarily unravelling this great weave?

Naturally, we start with the person who made this PhD possible in the first place. Martijn, you have given me a lot of freedom and encouragement, even when this led me further and further from your usual research, allowing me to discover what topics I really care about. In the meantime, I could always knock on your door and come in (although I wonder if you will ever get used to the locked-door situation). I would usually have a short question, but somehow I could never manage to leave within half an hour; we would always end up talking about three different questions or a few completely irrelevant sidetracks. I think you may feel like it is just standard stuff for a PhD supervisor, but I am pretty sure it is quite unique. Of course, I cannot leave out encouraging me to go to Canada, which completely changed the course of my PhD. All I can say is: Erik was right, you are an excellent choice of supervisor, and I have been quite lucky.

I also want to take the opportunity to thank Melchior, the Tomita-Takesaki wizard. Thank you for your very useful questions, great ideas and overall nice conversations during conferences. I have learnt a lot from you, and I have always enjoyed thinking together about some kind of problem. I still had a lot to learn when we started working together, and you were great at considering and encouraging my sometimes wild ideas and were always ready to explain a bit of necessary theory.

For most of my PhD I sat side by side with the *other* Matthijs, with whom I shared an uncanny number of things: a supervisor, a PhD mentor group and my hallway and kitchen in the city centre. I do not quite know how you managed to make that happen, Matthijs, but I will definitely not complain. Across from me was the real-analysis-teaching, Terraforming-Mars-playing duo of Gerrit and Carel. I have had a lot of fun coming up with wacky counterexamples, playing board games and being distracted when you decided that Project Euler was the way to practise programming, Gerrit. It was a joy to share my office with you. Enli filled the first gap in the office, and I want to say ‘thank you’ for trying to teach me Chinese, even though I got stuck at the second word.

Of course, my PhD environment extended beyond the office. Thank you, Bálint, for starting our football team and not finding a goalie, it has been great; Milan, Marco, Sebastian, for being there for the team when we were always short on players, I still think back to the good old days from time to time; Joris, for accidentally coming up with a project when I was fed up with working alone, it made a big difference; Esmée, for the nice initiatives, for taking over the FIKA when I was away, and for the nice conversations.

Thank you to Amudhan, Antonio, Bas, Chenxi, Emiel, Fabian, Floris, Foivos, Francesca, Jan, Jesse, Jordy, Joshua, Katharina, Lukas, Mario, Mark, Max, Noé, Sophie, Teun and Wolter for the many nice conversations and for making the FIKA a great success, it has been a lot of fun. A special shout-out to Kristin for starting the FIKA and bringing the group closer together and to Cindy for making the university seem a lot more automated than it actually is.

When I came to Delft for my PhD, I knew nobody..., apart from my PhD mentor group. Luckily, Amey, Avelon and Frederik, you guys are great. We were completed by Matthijs and Ivet, and I want to thank you for the many fun things we did together. Tied to that is the PhD forum, which was co-founded by Amey. Thank you to Elisa, Frederik, Lasse, Naqi, and Vicente for the many fun things we organised together and thank you to the PhDs who are now filling our places.

At this point, I can no longer postpone addressing the second major influence on my academic life, Mike. Allowing me to come to Waterloo completely changed the path of my PhD, and I will forever be grateful for it. I was able to discover new areas of mathematics that suited me much better, experience a true collaborative research environment and rekindle my love for doing research. You are a great, welcoming guy and it was a lot of fun to be working together. I still cherish our amazing farewell picture, and I happily forgive you for the promised *deep snow* being a year late. A shout-out goes to William for making it possible to get an office in IQC, which was the final piece of the puzzle to make the stars align.

As I have said many times, Waterloo is not a very impressive city. To enjoy your time there, you need to be surrounded by nice people. In this respect, Waterloo is absolutely stacked. It is hard to comprehend that you can get to know so many great people in about five months. I want to thank all of you for giving me an absolutely amazing time. Here is to everyone at the board game club and especially Liam for the many great games and for introducing me to spirit island; to Alex, Arsalan, Athena, Ernest, James, Paula, Shlok and Srijita for all the fun Sunday afternoon IQC board game sessions; to Calum, Gabe, Hugh, Jeremy, Michelle and all the others from the UWaterloo climbing club for some great climbing sessions and for going to see the total solar eclipse with me, I will never forget that; to everyone at WatSFiC for letting me rediscover D&D; and from Merellien to Ben, Ernest, Ewan, Rahul and Srijita (and Biscuit) for one of my favourite bits of D&D I have ever played. A special thanks goes to Ewan, my office mate (until he fled to the light). We are not a great pair if you want any conversation to be brief, but we exchanged many fun ideas and I am happy we got stuck in the basement together.

The people of Waterloo do not end there. I want to thank Devashish, Kieran, Shlok, and all the others at IQC for many nice conversations about all kinds of quantum things; Eric for being enthusiastic about quantum expanders together with me and working together to create our paper; Alex and Yuming for painstakingly telling me all about non-local games and (robust) self-testing, I have learnt a ton from you; Adam for thinking with me about using computers to find cool examples of non-local games, even though we were unable to get it to work in the end; and Yuming in particular for working together on our paper and allowing me to visit Copenhagen after you moved. Moving not cities but just buildings, I want to thank Aareyan, ~~Adam Alesia~~ Aleksa, Annie, Erik, Jashan, Jennifer, Larissa and Thomas, i.e. the MC squad for all the fun things we did. From the lunches, the crosswords and connections, the fitness and power yoga to the board games, the escape room and the football, it was amazing to share all of it with you. You have been so welcoming and I really felt at home with you guys. Thank you to everyone in Canada for eating all the stuff I baked, and I hope to run into you again!

One of the keys to navigating PhD life is to not navigate *PhD life*. You want to be navigating life in all its glory, and sometimes you just happen to be doing some work for your PhD. It does not tend to be easy avoiding the pitfall of living a PhD life, but for me it was not very hard; I have been surrounded by many friends, and spending time with them has always felt, and is, more important than working on some proof. You made doing a PhD easy in many ways, while it tends to be quite hard. I have now thanked my friends in Waterloo, so it is time to transition from one side of the pond to the other. Here, I am surrounded by so many nice people that we need some structure, so I will proceed roughly in anti-chronological order.

This allows me to start with Slopend, the student outdoor sports association. I am still surprised how an association can be filled with so many nice, encouraging, friendly and enthusiastic people. I want to thank everyone for truly making Delft a new home, but there are a few highlights. Benjamin, Diana, Fionnah, Otto and Tinko, thank you for being a KliCo (climbing committee) with me. We did many fun things, with the climbing weekend as the highlight so far. You taught me that the most important point on the agenda is to see how everyone is doing, and I am very happy that you are there to tell me off if I for some reason think it is better if I do everything. Of course, I am not forgetting David, Jeffrey, Julian, Meike and Niels, who joined me in my first KliCo and all came together to make the best committee T-shirt ever. Thanks to you, I became a true part of Slopend, and it gave me a number one boulder buddy in the form of Jeffrey, which has been great. Lastly, I want to thank the man who made all of this happen in the first place. Doeke, thanks for being a *nieuwsgierig aagje* and a super nice guy. Without you, I probably would not have discovered Slopend, and now I wish I would never have to leave.

Next to Slopend, another constant factor in Delft was the Monday evening football competition. Thank you to Bálint for gathering a ragtag team from all over the place and to all of the team for giving me many nice games and allowing me to get back into the goal, I really loved it. In particular, I want to thank Abi, Dylan, Kiki, Maaike, Mirte, Sjoerd and Youri for being present almost all of the time and creating a lot of fun moments. Lastly, I want to thank my C1-buddy turned climbing buddy, a real-life Yetiaan, Paris. While we do not quite see eye to eye as to what constitutes a salad (I think you would find a way to claim that my pastries are salads), it has been awesome to go climbing all the time, and you have become a great friend.

It is hard to grasp that in just four years, my life has become intertwined with all of these people. I am extremely grateful for the pattern it has created, and I am curious how it will develop in the future. However, this pattern is intermingled with the threads of people I had met before, and who definitely did not stop weaving their threads around mine because I decided to depart to the other side of the country. I am very happy that I was able to keep, sustain and develop so many of these friendships, irrespective of the physical distance.

First of all, I want to thank everyone at the NijSAC. The NijSAC has allowed me to discover a very fun sport. I have learnt a lot about it from all kinds of people these last six years, and I very much enjoy slowly being able to give back some of that knowledge.

Thank you in particular to Annabel, Daniel, Irene, Jasper, Marloes, Robin, Twan, Willemijn and everyone at one of the summer campsites for the amazing climbing holidays in many beautiful places; they were great to get the mathematics out of my head for a while. Thank you to everyone at the NijSAC who has been (very) important in my life at some point in time; there are so many fun sides to the NijSAC and I have developed a lot since joining. A special thanks goes to Amber and Willemijn, who have been great friends since the start, and to Helen, who always smiles in the face of a challenge and proves that some friendships are worth holding on to.

Around the same time I got to know a bunch of mathematics students who have become very close friends. Thank you to Amber, Anne, Christiaan, Dirk, Erik, Maaike, Simone, Timo and Thijs for the fun birthdays, holidays, random activities and for celebrating Sinterklaas with me. It has been, and hopefully will continue to be, an absolute blast. Special thanks go to Amber for immediately becoming my friend and for in general being insanely enthusiastic about *almost* everything. Of course I cannot forget Josefien, I really enjoyed visiting you in Stockholm, and thank you to both you and Dirk for all the Woordgraptogrammen we solved together, keeping me sane during Covid times. Lastly, Maaike, thank you for turning this dissertation into a work of art!

Almost seven years ago I travelled to Leeds for my Erasmus exchange, which changed my life. A large part of this was that I got to meet an amazing friend, Maartje. I am so glad that we happened to meet, and I have learnt a ridiculous number of things from you that have nothing to do with mathematics. I really enjoy all of the stuff we undertake together and love that it encourages me to try new things. You are one of the friends I cannot imagine life without, and I count myself lucky being able to call you my friend.

And now it is time to turn to my high school friends, which somehow naturally includes Aldo. If there is any group of friends amongst whom life is automatic, where, for all our occasional frustrations, friendship is a given and you get to just be, it is you guys. We have played *so* many board games, have been on countless vacations, undertaken several wacky activities, and I am always happy if we get to be together again. It tends to take away all other worries and concerns. I remember how, just after I started my PhD in 2021, we were able to get together again for a short holiday after (or in between) Covid. Getting together after a year and a half made us realise just how much we missed it. I hope, and trust, that this is a friendship that will always be there for the rest of my life. I also want to highlight Jelle, Bart en Brian (which I somehow cannot write down in any other order), a somewhat separate group of friends with whom the board game evenings have been going strong for about 16 years, albeit somewhat irregularly at times. I hope there will be many more years to come.

I end with the two friends who transcend everything, who will forever be my friends whether I talk to them or not: Joep en Jeroen. Our contact has been irregular during my PhD, but just the thought that whenever we happen to meet, we will still be friends warms my heart and puts a smile upon my face. It may seem weird, but just knowing these things means a lot and makes me feel secure when I am moving to a new place. You could knock on my door in ten years time, and you would be welcome. I would not be surprised if the same would happen if I would knock on one of your parents' door. We have largely walked the path to this point together, and I most likely would not be here without you. Thank you for being my friends, and you will always be able to call me one.

Lastly, I want to thank my family, all of the Vernooijtjes and Van Oersjes, all of the uncles, aunts, cousins and additions to the family. You were always interested in how I was doing, academically but also very much non-academically, and it is very nice to have so many interested people around you. Chief amongst them were of course my dad, sister and especially my mum, who would be interested in every story about my PhD you can think of. It is nice to know that when you feel like your research is not interesting to any scientist, you at least have someone who thinks it is cool, even though they do not understand most of it. Another nice thing about family is that you can very easily spend a birthday thinking about anything but the research which might be very annoying at that point in time, and, I am happy to say, we celebrate a *lot* of birthdays. With this, I have mentioned the prime people who have decided to interweave the threads of their life with mine. I am very grateful for the way the weave around me turned out so far, and I hope it will be just as nice in the future.

Hoewel ik nu iedereen heb gehad, kan ik het toch niet hierbij laten. Ik wil nog twee mensen specifiek in het zonnetje zetten. Paula, het is niet altijd handig dat ik helemaal in Delft zit. Ik ben je super dankbaar dat jij altijd klaarstaat als er weer een reden is dat het makkelijker was geweest als ik in Nijmegen zou promoveren. Ik weet niet wat ik zonder jou zou moeten. Pappa, ik ben je natuurlijk voor ontelbaar veel dingen dankbaar, maar ik wil je in het bijzonder bedanken voor het feit dat ik effectief in twee steden tegelijk kon wonen, zonder vragen, eisen of gedoe. Een behoorlijk deel van wat hierboven beschreven staat is mede mogelijk gemaakt door jouw flexibiliteit. Ik zeg het misschien niet zo vaak, maar ik ben je er altijd dankbaar voor.

Bibliography

[AC21] Érik Amorim and Eric A. Carlen. "Complete positivity and self-adjointness". In: *Linear Algebra Appl.* 611 (2021), pp. 389–439.

[AC82] Luigi Accardi and Carlo Cecchini. "Conditional expectations in von Neumann algebras and a theorem of Takesaki". In: *J. Functional Analysis* 45.2 (1982), pp. 245–273.

[AE15] Dorit Aharonov and Lior Eldar. "The commuting local Hamiltonian problem on locally expanding graphs is approximable in NP". In: *Quantum Information Processing* 14.1 (2015), pp. 83–101.

[AH77] Sergio Albeverio and Raphael Høegh-Krohn. "Dirichlet forms and Markov semigroups on C^*-algebras". In: *Comm. Math. Phys.* 56.2 (1977), pp. 173–187.

[AK24] Benjamin Anderson-Sackaney and Fatemeh Khosravi. "Topological boundaries of representations and coideals". In: *Adv. Math.* 452 (2024), Paper No. 109830, 39.

[AKS87] Miklós Ajtai, János Komlós and Endre Szemerédi. "Deterministic simulation in LOGSPACE". In: *Proceedings of the nineteenth annual ACM symposium on Theory of computing*. 1987, pp. 132–140.

[Ali76] Robert Alicki. "On the detailed balance condition for non-Hamiltonian systems". In: *Rep. Math. Phys.* 10.2 (1976), pp. 249–258.

[AM85] Noga Alon and Vitali D. Milman. "$\lambda 1$, isoperimetric inequalities for graphs, and superconcentrators". In: *Journal of Combinatorial Theory, Series B* 38.1 (1985), pp. 73–88.

[Arh23] Cédric Arhancet. *Spectral triples, Coulhon-Varopoulos dimension and heat kernel estimates*. 2023. arXiv: 2209.12263 [math.OA].

[AZ15] George Androulakis and Matthew Ziemke. "Generators of quantum Markov semigroups". In: *Journal of Mathematical Physics* 56.8 (2015), p. 083512.

[Bap+23] Pedro Baptista, Ranyiliu Chen, Jedrzej Kaniewski, David Rasmussen Lolck, Laura Mančinska, Thor Gabelgaard Nielsen and Simon Schmidt. *A mathematical foundation for self-testing: Lifting common assumptions*. 2023. arXiv: 2310.12662 [quant-ph].

[Bar17] Ivan Bardet. *Estimating the decoherence time using non-commutative Functional Inequalities*. 2017. arXiv: 1710.01039 [quant-ph].

[BBCW20] Angshuman Bhattacharya, Michael Brannan, Alexandru Chirvasitu and Shuzhou Wang. "Property (T), property (F) and residual finiteness for discrete quantum groups". In: *J. Noncommut. Geom.* 14.2 (2020), pp. 567–589.

[BCLY20] Michael Brannan, Benoît Collins, Hun Hee Lee and Sang-Gyun Youn. "Temperley-Lieb quantum channels". In: *Comm. Math. Phys.* 376.2 (2020), pp. 795–839.

[BCV25] Michael Brannan, Eric Culf and Matthijs Vernooij. *Quantum expanders and property (T) discrete quantum groups.* 2025. arXiv: 2502.01974 [math.OA].

[Bha97] Rajendra Bhatia. *Matrix analysis.* English. Graduate texts in mathematics; 169. New York: Springer, 1997.

[Bia03] Philippe Biane. "Logarithmic Sobolev inequalities, matrix models and free entropy". In: vol. 19. 3. International Workshop on Operator Algebra and Operator Theory (Linfen, 2001). 2003, pp. 497–506.

[BK19] Fabrice Baudoin and Daniel J. Kelleher. "Differential one-forms on Dirichlet spaces and Bakry-Émery estimates on metric graphs". In: *Trans. Amer. Math. Soc.* 371.5 (2019), pp. 3145–3178.

[Bla06] Bruce Blackadar. *Operator algebras.* Vol. 122. Encyclopaedia of Mathematical Sciences. Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III. Springer-Verlag, Berlin, 2006, pp. xx+517.

[BMZ24] Anne Broadbent, Arthur Mehta and Yuming Zhao. "Quantum Delegation with an Off-The-Shelf Device". In: *19th Conference on the Theory of Quantum Computation, Communication and Cryptography (TQC 2024).* Ed. by Frédéric Magniez and Alex Bredariol Grilo. Vol. 310. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2024, 12:1–12:23.

[BR17] Michael Brannan and Zhong-Jin Ruan. "L_p-representations of discrete quantum groups". In: *J. Reine Angew. Math.* 732 (2017), pp. 165–210.

[BS93] Saad Baaj and Georges Skandalis. "Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres". In: *Ann. Sci. École Norm. Sup. (4)* 26.4 (1993), pp. 425–488.

[BŠCA18a] Joseph Bowles, Ivan Šupić, Daniel Cavalcanti and Antonio Acín. "Device-independent entanglement certification of all entangled states". In: *Physical review letters* 121.18 (2018), p. 180503.

[BŠCA18b] Joseph Bowles, Ivan Šupić, Daniel Cavalcanti and Antonio Acín. "Self-testing of Pauli observables for device-independent entanglement certification". In: *Physical Review A* 98.4 (2018), p. 042336.

[Cas21] Martijn Caspers. "Gradient forms and strong solidity of free quantum groups". In: *Math. Ann.* 379.1-2 (2021), pp. 271–324.

[CE79] Erik Christensen and David E. Evans. "Cohomology of Operator Algebras and Quantum Dynamical Semigroups". In: *Journal of the London Mathematical Society* s2-20.2 (1979), pp. 358–368.

[CFK14] Fabio Cipriani, Uwe Franz and Anna Kula. "Symmetries of Lévy processes on compact quantum groups, their Markov semigroups and potential theory". In: *J. Funct. Anal.* 266.5 (2014), pp. 2789–2844.

[CGJV19] Andrea Coladangelo, Alex B. Grilo, Stacey Jeffery and Thomas Vidick. "Verifier-on-a-Leash: New Schemes for Verifiable Delegated Quantum Computation, with Quasilinear Resources". In: *EUROCRYPT 2019*. Vol. 3. 2019, pp. 247–277.

[Cip97] Fabio Cipriani. "Dirichlet Forms and Markovian Semigroups on Standard Forms of von Neumann Algebras". In: *Journal of Functional Analysis* 147.2 (1997), pp. 259–300.

[CIW21] Martijn Caspers, Yusuke Isono and Mateusz Wasilewski. "L_2-cohomology, derivations, and quantum Markov semi-groups on q-Gaussian algebras". In: *Int. Math. Res. Not. IMRN* 9 (2021), pp. 6405–6441.

[CM17] Eric A. Carlen and Jan Maas. "Gradient flow and entropy inequalities for quantum Markov semigroups with detailed balance". In: *Journal of Functional Analysis* 273.5 (2017), pp. 1810–1869.

[CM20] Eric A. Carlen and Jan Maas. "Non-commutative Calculus, Optimal Transport and Functional Inequalities in Dissipative Quantum Systems". In: *J. Stat. Phys.* 178.2 (2020), pp. 319–378.

[CN13] Jason Crann and Matthias Neufang. "Quantum channels arising from abstract harmonic analysis". In: *J. Phys. A* 46.4 (2013), pp. 045308, 22.

[Con10] John B. Conway. *A course in functional analysis*. English. 2nd ed. Graduate texts in mathematics; 96. New York: Springer Science+Business Media, 2010.

[CRSV17] Rui Chao, Ben W. Reichardt, Chris Sutherland and Thomas Vidick. "Overlapping Qubits". In: *Information Technology Convergence and Services*. 2017.

[CS03a] Fabio Cipriani and Jean-Luc Sauvageot. "Derivations as square roots of Dirichlet forms". In: *J. Funct. Anal.* 201.1 (2003), pp. 78–120.

[CS03b] Fabio Cipriani and Jean-Luc Sauvageot. "Noncommutative potential theory and the sign of the curvature operator in Riemannian geometry". In: *Geom. Funct. Anal.* 13.3 (2003), pp. 521–545.

[CS15] Martijn Caspers and Adam Skalski. "The Haagerup approximation property for von Neumann algebras via quantum Markov semigroups and Dirichlet forms". In: *Comm. Math. Phys.* 336.3 (2015), pp. 1637–1664.

[CS17] Fabio Cipriani and Jean-Luc Sauvageot. "Amenability and subexponential spectral growth rate of Dirichlet forms on von Neumann algebras". In: *Adv. Math.* 322 (2017), pp. 308–340.

[CVY23] Michael Chapman, Thomas Vidick and Henry Yuen. *Efficiently stable presentations from error-correcting codes*. 2023. arXiv: 2311.04681 [quant-ph].

[CW22] Alexandru Chirvasitu and Mateusz Wasilewski. "Random quantum graphs". In: *Trans. Amer. Math. Soc.* 375.5 (2022), pp. 3061–3087.

[CZ76] Ioana Ciorănescu and László Zsidó. "Analytic generators for one-parameter groups". In: *Tohoku Math. J. (2)* 28.3 (1976), pp. 327–362.

[Dab10] Yoann Dabrowski. *A non-commutative Path Space approach to stationary free Stochastic Differential Equations*. 2010. arXiv: 1006.4351 [math.OA].

[Dav76] E. Brian Davies. *Quantum theory of open systems*. English. London: Academic Press, 1976.

[Dav79] E. Brian Davies. “Generators of dynamical semigroups”. In: *Journal of Functional Analysis* 34.3 (1979), pp. 421–432.

[Daw24] Matthew Daws. “Quantum graphs: different perspectives, homomorphisms and quantum automorphisms”. In: *Commun. Am. Math. Soc.* 4 (2024), pp. 117–181.

[Des03] Pieter Desmedt. “Aspects of the theory of locally compact quantum groups : amenability - plancherel measure”. PhD thesis. KU Leuven, 2003.

[DI16] Yoann Dabrowski and Adrian Ioana. “Unbounded derivations, free dilations, and indecomposability results for II_1 factors”. In: *Trans. Amer. Math. Soc.* 368.7 (2016), pp. 4525–4560.

[DKS12] Matthew Daws, Adam Kasprzak Pawełand Skalski and Piotr M. Sołtan. “Closed quantum subgroups of locally compact quantum groups”. In: *Adv. Math.* 231.6 (2012), pp. 3473–3501.

[DL92] E. Brian Davies and J. Martin Lindsay. “Noncommutative symmetric Markov semigroups”. In: *Math. Z.* 210.3 (1992), pp. 379–411.

[DL93] E. Brian Davies and J. Martin Lindsay. “Superderivations and symmetric Markov semigroups”. In: *Comm. Math. Phys.* 157.2 (1993), pp. 359–370.

[dlS22a] Mikael de la Salle. “Orthogonalization of Positive Operator Valued Measures”. en. In: *Comptes Rendus. Mathématique* 360 (2022), pp. 549–560.

[dlS22b] Mikael de la Salle. *Spectral gap and stability for groups and non-local games*. 2022. arXiv: 2204.07084 [math.OA].

[Dod84] Jozef Dodziuk. “Difference equations, isoperimetric inequality and transience of certain random walks”. In: *Transactions of the American Mathematical Society* 284.2 (1984), pp. 787–794.

[Don+24] Yangjing Dong, Honghao Fu, Anand Natarajan, Minglong Qin, Haochen Xu and Penghui Yao. “The Computational Advantage of MIP^* Vanishes in the Presence of Noise”. In: *39th Computational Complexity Conference (CCC 2024)*. Ed. by Rahul Santhanam. Vol. 300. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2024, 30:1–30:71.

[DSV17] Matthew Daws, Adam Skalski and Ami Viselter. “Around property (T) for quantum groups”. In: *Comm. Math. Phys.* 353.1 (2017), pp. 69–118.

[EL77a] Edward G. Effros and E. Christopher Lance. “Tensor products of operator algebras”. In: *Adv. Math.* 25.1 (1977), pp. 1–34.

[EL77b] David E. Evans and John T. Lewis. “Dilations of irreversible evolutions in algebraic quantum theory”. In: *Communications Dublin Inst. Advanced Studies. Ser. A* 24 (1977), pp. v+104.

[Fim10] Pierre Fima. “Kazhdan’s property T for discrete quantum groups”. In: *International Journal of Mathematics* 21.01 (2010), pp. 47–65.

[FMP17] Pierre Fima, Kunal Mukherjee and Issan Patri. “On compact bicrossed products”. In: *Journal of Noncommutative Geometry* 11.4 (2017), pp. 1521–1591.

[FU10] Franco Fagnola and Veronica Umanità. “Generators of KMS symmetric Markov semigroups on $B(h)$ symmetry and quantum detailed balance”. In: *Comm. Math. Phys.* 298.2 (2010), pp. 523–547.

[Gil98] David Gillman. “A Chernoff bound for random walks on expander graphs”. In: *SIAM Journal on Computing* 27.4 (1998), pp. 1203–1220.

[GKS76] Vittorio Gorini, Andrzej Kossakowski and E. C. George Sudarshan. “Completely positive dynamical semigroups of N-level systems”. In: *J. Mathematical Phys.* 17.5 (1976), pp. 821–825.

[GL95] Stanislaw Goldstein and J. Martin Lindsay. “KMS-symmetric Markov semigroups”. In: *Mathematische Zeitschrift* 219.4 (1995), pp. 591–608.

[GR22] Li Gao and Cambyse Rouzé. “Complete entropic inequalities for quantum Markov chains”. In: *Arch. Ration. Mech. Anal.* 245.1 (2022), pp. 183–238.

[GS18] David J. Griffiths and Darrell F. Schroeter. *Introduction to quantum mechanics*. English. Third edition. Cambridge, United Kingdom: Cambridge University Press, 2018.

[Gui] Guiness World Records. *Tallest mountain*. URL: https://www.guinnessworldrecords.com/world-records/tallest-mountain. (accessed: 17-12-2024).

[Haa75] Uffe Haagerup. “The standard form of von Neumann algebras”. In: *Math. Scand.* 37.2 (1975), pp. 271–283.

[Haa85] Uffe Haagerup. “Injectivity and decomposition of completely bounded maps”. In: *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*. Vol. 1132. Lecture Notes in Math. Springer, Berlin, 1985, pp. 170–222.

[Har08] Aram W. Harrow. “Quantum expanders from any classical Cayley graph expander”. In: *Quantum Info. Comput.* 8.8 (2008), pp. 715–721.

[Has07a] Matthew B. Hastings. “Entropy and entanglement in quantum ground states”. In: *Physical Review B—Condensed Matter and Materials Physics* 76.3 (2007), p. 035114.

[Has07b] Matthew B. Hastings. “Random unitaries give quantum expanders”. In: *Physical Review A* 76.3 (2007), p. 032315.

[HJX10] Uffe Haagerup, Marius Junge and Quanhua Xu. “A reduction method for noncommutative L_p-spaces and applications”. In: *Trans. Amer. Math. Soc.* 362.4 (2010), pp. 2125–2165.

[HLW06] Shlomo Hoory, Nathan Linial and Avi Wigderson. “Expander graphs and their applications”. In: *Bull. Amer. Math. Soc. (N.S.)* 43.4 (2006), pp. 439–561.

[HMPS19] J. William Helton, Kyle P. Meyer, Vern I. Paulsen and Matthew Satriano. “Algebras, synchronous games, and chromatic numbers of graphs”. In: *New York Journal of Mathematics* 25 (2019), pp. 328–361.

[Hol95] Alexander S. Holevo. “On the structure of covariant dynamical semigroups”. In: *J. Funct. Anal.* 131.2 (1995), pp. 255–278.

[HT13] Michael Hinz and Alexander Teplyaev. “Vector analysis on fractals and applications”. In: *Fractal geometry and dynamical systems in pure and applied mathematics. II. Fractals in applied mathematics.* Vol. 601. Contemp. Math. Amer. Math. Soc., Providence, RI, 2013, pp. 147–163.

[Ji+22] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright and Henry Yuen. *MIP*=RE.* 2022. arXiv: 2001.04383 [quant-ph].

[JMP14] Marius Junge, Tao Mei and Javier Parcet. “An invitation to harmonic analysis associated with semigroups of operators”. In: *Harmonic analysis and partial differential equations.* Vol. 612. Contemp. Math. Amer. Math. Soc., Providence, RI, 2014, pp. 107–122.

[Jun24] Luca Junk. *Cheeger inequalities for quantum graphs.* Private Communication. 2024.

[JX07] Marius Junge and Quanhua Xu. “Noncommutative maximal ergodic theorems”. In: *J. Amer. Math. Soc.* 20.2 (2007), pp. 385–439.

[JZ15] Marius Junge and Qiang Zeng. “Noncommutative martingale deviation and Poincaré type inequalities with applications”. In: *Probab. Theory Related Fields* 161.3-4 (2015), pp. 449–507.

[Kap53] Irving Kaplansky. “Modules over operator algebras”. In: *Amer. J. Math.* 75 (1953), pp. 839–858.

[Kar25] Prem Nigam Kar. *Robust Self-testing for Synchronous Correlations and Games.* 2025. arXiv: 2503.23500 [quant-ph].

[Kas05] Martin Kassabov. “Kazhdan constants for $\mathrm{SL}_n(\mathbb{Z})$”. In: *International Journal of Algebra and Computation* 15.05n06 (2005), pp. 971–995.

[Kas17] Fatemeh Kasprzak Pawełand Khosravi. “Coideals, quantum subgroups and idempotent states”. In: *Q. J. Math.* 68.2 (2017), pp. 583–615.

[KB16] Michael J. Kastoryano and Fernando G. S. L. Brandão. “Quantum Gibbs samplers: the commuting case”. In: *Comm. Math. Phys.* 344.3 (2016), pp. 915–957.

[Kos72] Andrzej Kossakowski. “On quantum statistical mechanics of non-Hamiltonian systems”. In: *Reports on Mathematical Physics* 3.4 (1972), pp. 247–274.

[Kos84] Hideki Kosaki. “Applications of the complex interpolation method to a von Neumann algebra: Non-commutative L^p-spaces”. In: *Journal of Functional Analysis* 56.1 (1984), pp. 29–78.

[Kow19] Emmanuel Kowalski. *An introduction to expander graphs.* Société mathématique de France Paris, 2019.

[KPS18] Se-Jin Kim, Vern Paulsen and Christopher Schafhauser. “A synchronous game for binary constraint systems”. In: *Journal of Mathematical Physics* 59.3 (2018), p. 032201.

[Li+25] Yinan Li, Youming Qiao, Avi Wigderson, Yuval Wigderson and Chuanqi Zhang. “On Linear-Algebraic Notions of Expansion”. In: *Theory of Computing* 21.1 (2025), pp. 1–28.

[Lin24] Junqiao Lin. *Tracial embeddable strategies: Lifting MIP* tricks to MIPco*. 2024. arXiv: 2304.01940 [quant-ph].

[Lin76] Gorän Lindblad. “On the generators of quantum dynamical semigroups”. In: *Communications in Mathematical Physics* 48.2 (1976), pp. 119–130.

[Lub10] Alexander Lubotzky. *Discrete groups, expanding graphs and invariant measures*. Modern Birkhäuser Classics. With an appendix by Jonathan D. Rogawski, Reprint of the 1994 edition. Birkhäuser Verlag, Basel, 2010, pp. iii+192.

[LY22] Hun Hee Lee and Sang-Gyun Youn. “Quantum channels with quantum group symmetry”. In: *Comm. Math. Phys.* 389.3 (2022), pp. 1303–1329.

[LY24] Cécilia Lancien and Pierre Youssef. *A note on quantum expanders*. 2024. arXiv: 2302.07772 [quant-ph].

[Mar73] Grigorii Aleksandrovich Margulis. “Explicit construction of concentrators”. In: *Problem Inf. Trans.* 9 (1973), pp. 325–332.

[MNY22] Hamoon Mousavi, Seyed Sajjad Nezhadi and Henry Yuen. “Nonlocal Games, Compression Theorems, and the Arithmetical Hierarchy”. In: *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing (STOC)*. 2022, pp. 1–11.

[Moh89] Bojan Mohar. “Isoperimetric numbers of graphs”. In: *Journal of combinatorial theory, Series B* 47.3 (1989), pp. 274–291.

[MRV18] Benjamin Musto, David Reutter and Dominic Verdon. “A compositional approach to quantum functions”. In: *J. Math. Phys.* 59.8 (2018), pp. 081706, 42.

[MS23] Amine Marrakchi and Mikael de la Salle. *Almost synchronous correlations and Tomita-Takesaki theory*. 2023. arXiv: 2307.08129 [math.OA].

[Mur90] Gerard J. Murphy. *C*-algebras and operator theory*. English. Boston: Academic Press, 1990.

[MvD98] Ann Maes and Alfons van Daele. *Notes on Compact Quantum Groups*. 1998. arXiv: math/9803122 [math.FA].

[MY04] Dominic Mayers and Andrew Yao. “Self Testing Quantum Apparatus”. In: *Quantum Info. Comput.* 4.4 (2004), pp. 273–286.

[NC10] Michael A. Nielsen and Isaac L. Chuang. *Quantum computation and quantum information*. English. 10th anniversary edition. Cambridge: Cambridge University Press, 2010.

[Nee22] Jan van Neerven. *Functional Analysis*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2022.

[Nes96] Yuri Nesterenko. “Modular functions and transcendence problems”. In: *Comptes Rendus- Academie Des Sciences Paris Serie 1* 322.10 (1996), pp. 909–914.

[Nob] Nobel Prize Outreach 2025. *Nobel Prize in Physics 2022*. URL: https://www.nobelprize.org/prizes/physics/2022/summary/. (accessed: 31-05-2025).

[NT13] Sergey Neshveyev and Lars Tuset. *Compact quantum groups and their representation categories*. Vol. 20. Cours Spécialisés [Specialized Courses]. Société Mathématique de France, Paris, 2013, pp. vi+169.

[NZ23] Anand Natarajan and Tina Zhang. “Quantum Free Games”. In: *Proceedings of the 55th Annual ACM Symposium on Theory of Computing (STOC)* (2023).

[OW23] Maris Ozols and Michael Walter. *Lectures Notes on Quantum Information Theory*. 2023. URL: https://qi.rub.de/courses/qit21/qit22.pdf.

[Pad25] Connor Paddock. *Rounding near-optimal quantum strategies for nonlocal games to strategies using maximally entangled states*. 2025. arXiv: 2203.02525 [quant-ph].

[Pau+16] Vern I. Paulsen, Simone Severini, Daniel Stahlke, Ivan G. Todorov and Andreas Winter. “Estimating quantum chromatic numbers”. In: *Journal of Functional Analysis* 270.6 (2016), pp. 2188–2222.

[Pau02] Vern I. Paulsen. *Completely bounded maps and operator algebras*. English. Cambridge studies in advanced mathematics; 78. Cambridge: Cambridge University Press, 2002.

[Pet09a] Jesse Peterson. “L^2-rigidity in von Neumann algebras”. In: *Invent. Math.* 175.2 (2009), pp. 417–433.

[Pet09b] Jesse Peterson. “A 1-cohomology characterization of property (T) in von Neumann algebras”. In: *Pacific J. Math.* 243.1 (2009), pp. 181–199.

[Phi55] Ralph S. Phillips. “The adjoint semi-group”. In: *Pacific Journal of Mathematics* 5.2 (1955), pp. 269–283.

[Pis20] Gilles Pisier. *Tensor products of C^*-algebras and operator spaces—the Connes-Kirchberg problem*. Vol. 96. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2020, pp. x+484.

[PR92] Sandu Popescu and Daniel Rohrlich. “Which states violate Bell’s inequality maximally?” In: *Physics Letters A* 169 (1992), pp. 411–414.

[PSZZ24] Connor Paddock, William Slofstra, Yuming Zhao and Yangchen Zhou. “An operator-algebraic formulation of self-testing”. In: *Annales Henri Poincaré* 25 (2024), pp. 4283–4319.

[QY21] Minglong Qin and Penghui Yao. “Nonlocal Games with Noisy Maximally Entangled States are Decidable”. In: *SIAM Journal on Computing* 50.6 (2021), pp. 1800–1891.

[QY23] Minglong Qin and Penghui Yao. “Decidability of Fully Quantum Nonlocal Games with Noisy Maximally Entangled States”. In: *50th International Colloquium on Automata, Languages, and Programming (ICALP 2023)*. Ed. by Kousha Etessami, Uriel Feige and Gabriele Puppis. Vol. 261. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2023, 97:1–97:20.

[RUV13] Ben W. Reichardt, Falk Unger and Umesh Vazirani. “Classical command of quantum systems”. In: *Nature* 496 (2013), pp. 456–460.

[RV24] Lukas Rollier and Stefaan Vaes. “Quantum automorphism groups of connected locally finite graphs and quantizations of discrete groups”. In: *Int. Math. Res. Not. IMRN* 3 (2024), pp. 2219–2297.

[RW98] Iain Raeburn and Dana P. Williams. *Morita equivalence and continuous-trace C*-algebras*. English. Mathematical surveys and monographs; no. 60; v. 60. Providence, R.I.: American Mathematical Society, 1998.

[Sau96] Jean-Luc Sauvageot. “Semi-groupe de la chaleur transverse sur la C^*-algèbre d’un feuilletage riemannien”. In: *J. Funct. Anal.* 142.2 (1996), pp. 511–538.

[ŠB20] Ivan Šupić and Joseph Bowles. “Self-testing of quantum systems: a review”. In: *Quantum* 4 (2020), p. 337.

[Sch69] Martin Schechter. “On the spectra of operators on tensor products”. In: *Journal of Functional Analysis* 4.1 (1969), pp. 95–99.

[Sha99] Yehuda Shalom. “Bounded generation and Kazhdan’s property (T)”. In: *Publications Mathématiques de l’IHÉS* 90 (1999), pp. 145–168.

[Soł05] Piotr M. Sołtan. “Quantum Bohr compactification”. In: *Illinois J. Math.* 49.4 (2005), pp. 1245–1270.

[SV19] Adam Skalski and Ami Viselter. “Convolution semigroups on locally compact quantum groups and noncommutative Dirichlet forms”. In: *Journal de Mathématiques Pures et Appliquées* 124 (2019), pp. 59–105.

[SW87] Stephen J. Summers and Reinhard F. Werner. “Maximal violation of Bell’s inequalities is generic in quantum field theory”. In: *Communications in Mathematical Physics* 110 (1987), pp. 247–259.

[SW88] Stephen J. Summers and Reinhard F. Werner. “Maximal violation of Bell’s inequalities for algebras of observables in tangent spacetime regions”. In: *Annales De L Institut Henri Poincare-physique Theorique* 49 (1988), pp. 215–243.

[SZ19] Șerban Strătilă and László Zsidó. *Lectures on von Neumann algebras*. English. Cambridge-IISc series. Cambridge: Cambridge University Press, 2019.

[Tak02] Masamichi Takesaki. *Theory of operator algebras I*. English. Encyclopaedia of mathematical sciences; 124; 5. Berlin: Springer, 2002.

[Tak03a] Masamichi Takesaki. *Theory of operator algebras II*. English. Encyclopaedia of mathematical sciences; v. 125, 6. Berlin: Springer, 2003.

[Tak03b] Masamichi Takesaki. *Theory of operator algebras III*. English. Encyclopaedia of mathematical sciences; 8; v. 127. Berlin: Springer, 2003.

[Tsi93] Boris S. Tsirelson. “Some results and problems on quantum Bell-type inequalities”. In: *Hadronic Journal Supplement* 8 (1993), pp. 329–345.

[Vae01] Stefaan Vaes. “The unitary implementation of a locally compact quantum group action”. In: *J. Funct. Anal.* 180.2 (2001), pp. 426–480.

[VDa96] Alfons Van Daele. “Discrete quantum groups”. In: *Journal of Algebra* 180.2 (1996), pp. 431–444.

[Ver05] Roland Vergnioux. “Orientation of quantum Cayley trees and applications”. In: *J. Reine Angew. Math.* 580 (2005), pp. 101–138.

[Ver07] Roland Vergnioux. “The property of rapid decay for discrete quantum groups”. In: *J. Operator Theory* 57.2 (2007), pp. 303–324.

[Ver22] Dominic Verdon. “A covariant Stinespring theorem”. In: *J. Math. Phys.* 63.9 (2022), Paper No. 091705, 49.

[Ver24] Matthijs Vernooij. “On the existence of derivations as square roots of generators of state-symmetric quantum Markov semigroups”. In: *Infinite Dimensional Analysis, Quantum Probability and Related Topics* 27.01 (2024), p. 2350003.

[Vid22] Thomas Vidick. “Almost synchronous quantum correlations”. In: *Journal of Mathematical Physics* 63.2 (2022), p. 022201.

[Vis17] Ami Viselter. “Weak mixing for locally compact quantum groups”. In: *Ergodic Theory Dynam. Systems* 37.5 (2017), pp. 1657–1680.

[VV03] Stefaan Vaes and Leonid Vainerman. “Extensions of locally compact quantum groups and the bicrossed product construction”. In: *Adv. Math.* 175.1 (2003), pp. 1–101.

[VV19] Stefaan Vaes and Matthias Valvekens. “Property (T) discrete quantum groups and subfactors with triangle presentations”. In: *Adv. Math.* 345 (2019), pp. 382–428.

[VW23] Matthijs Vernooij and Melchior Wirth. “Derivations and KMS-Symmetric Quantum Markov Semigroups”. In: *Communications in Mathematical Physics* 403.1 (2023), pp. 381–416.

[VZ25] Matthijs Vernooij and Yuming Zhao. *Lifting the maximally-entangledness assumption in robust self-testing for synchronous games*. 2025. arXiv: 2505.05994 [quant-ph].

[Was23] Mateusz Wasilewski. *On quantum Cayley graphs*. 2023. arXiv: 2306.15315 [math.OA].

[Wat18] John Watrous. *The Theory of Quantum Information*. Cambridge University Press, 2018.

[Wir21] Melchior Wirth. *A Noncommutative Transport Metric and Symmetric Quantum Markov Semigroups as Gradient Flows of the Entropy*. 2021. arXiv: 1808.05419 [math.OA].

[Wir22] Melchior Wirth. *The Differential Structure of Generators of GNS-symmetric Quantum Markov Semigroups*. 2022. arXiv: 2207.09247 [math.OA].

[Wir24] Melchior Wirth. "Christensen–Evans theorem and extensions of GNS-symmetric quantum Markov semigroups". In: *Journal of Functional Analysis* 287.3 (2024), p. 110475.

[WZ21] Melchior Wirth and Haonan Zhang. "Complete gradient estimates of quantum Markov semigroups". In: *Comm. Math. Phys.* 387.2 (2021), pp. 761–791.

[Zha24] Yuming Zhao. *Robust self-testing for nonlocal games with robust game algebras*. 2024. arXiv: 2411.03259 [quant-ph].

Curriculum Vitæ

Matthijs Niels Albert Vernooij

27-07-1997 Born in Nijmegen, the Netherlands

Education

2008–2014 **Secondary education**
Stedelijk Gymnasium Nijmegen

2014–2017 **Bachelor Wiskunde**
Bachelor Natuur- en Sterrenkunde
Thesis: Using clustering to predict infection behaviour in networks
Supervisor: Prof. dr. E.A. Cator
Radboud Universiteit Nijmegen

2017–2021 **Master Mathematics**
Thesis: The noncommutative harmonic oscillator:
A matrix-valued orthogonal polynomials approach
Supervisor: Prof. dr. H.T. Koelink
Master Physics and Astronomy
Thesis: The topological Hall effect
Supervisor: Dr. M. Titov
Radboud Universiteit Nijmegen

2021–2025 **PhD in Mathematics**
Dissertation: Beyond tracial states in robust self-testing and constructing derivations for quantum Markov semigroups
Promotor: Dr. M.P.T. Caspers
Promotor: Prof. dr. J.M.A.M. van Neerven
Technische Universiteit Delft

List of Publications

Published articles

1. Matthijs Vernooij. “On the existence of derivations as square roots of generators of state-symmetric quantum Markov semigroups”. In: *Infinite Dimensional Analysis, Quantum Probability and Related Topics* 27.01 (2024), p. 2350003

2. Matthijs Vernooij and Melchior Wirth. “Derivations and KMS-Symmetric Quantum Markov Semigroups”. In: *Communications in Mathematical Physics* 403.1 (2023), pp. 381–416

Preprints and accepted articles

3. Michael Brannan, Eric Culf and Matthijs Vernooij. *Quantum expanders and property (T) discrete quantum groups*. 2025. arXiv: 2502.01974 [math.OA]

4. Matthijs Vernooij and Yuming Zhao. *Lifting the maximally-entangledness assumption in robust self-testing for synchronous games*. 2025. arXiv: 2505.05994 [quant-ph]

5. Laura Mančinska, Pieter Spaas, Taro Spirig and Matthijs Vernooij. “Gap-preserving reductions and RE-completeness of Independent Set Games”. In: *2025 IEEE 66th Annual Symposium on Foundations of Computer Science (FOCS)*. To appear

6. Eric Culf, Josse van Dobben de Bruyn, Matthijs Vernooij and Peter Zeman. *Existence and nonexistence of commutativity gadgets for entangled CSPs*. 2025. arXiv: 2509.07835 [quant-ph]