

TR 2013 5

STELLINGEN

behorend bij het proefschrift

*THE HAHN-EXTON  $q$ -BESSEL FUNCTION*

door

RENÉ SWARTTOUW

**STELLING 1.** M. Rahman stelt in [1] dat Jackson's  $q$ -Bessel functies waarschijnlijk mooiere eigenschappen hebben dan die van Exton. De resultaten in dit proefschrift tonen het tegendeel aan.

[1] - M. Rahman, *A note on the orthogonality of Jackson's  $q$ -Bessel functions*, Canadian Mathematical Bulletin 32, 1989, 369-376.

**STELLING 2.** De Hahn-Exton  $q$ -Bessel functies  $J_\nu(x; q)$  en  $J_{\nu+1}(x; q)$ , gedefinieerd in paragraaf 3.2.2 van dit proefschrift, hebben geen gemeenschappelijke nulpunten, behalve eventueel  $x = 0$ .

**STELLING 3.** Het onderzoek naar  $q$ -Bessel functies geeft tevens meer inzicht in de theorie van gewone Bessel functies.

**STELLING 4.** De Wall-polynomen  $p_n(x; a; q)$ , gedefinieerd door

$$p_n(x; a; q) = {}_2\Phi_1 \left( \begin{array}{c} q^{-n}, 0 \\ aq \end{array} \middle| q, xq \right),$$

voldoen aan de symmetrierelatie

$$(aq; q)_n p_n(xq^n; a; q) = (xq; q)_n p_n(aq^n; x; q).$$

Als we de limiet  $n \rightarrow \infty$  nemen in bovenstaande relatie, vinden we de symmetrierelatie voor  ${}_1\Phi_1$ 's van stelling 3.1 in dit proefschrift.

**STELLING 5.** De Wall-polynomen, gedefinieerd in stelling 2, hebben een genererende functie

$$\sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n} p_n(xq^n; q^\alpha; q) = \frac{(q; q)_\infty}{(q^{\alpha+1}; q)_\infty} \frac{(xt)^{-\frac{1}{2}\alpha}}{(t; q)_\infty} J_\alpha(\sqrt{xt}; q), \quad |t| < 1,$$

waarin  $J_\alpha(\sqrt{xt}; q)$  de Hahn-Exton  $q$ -Bessel functie is, gedefinieerd in paragraaf 3.2.2 van dit proefschrift.

Deze formule is een  $q$ -uitbreiding van de volgende genererende functie voor Laguerre polynomen  $L_n(x)$

$$\sum_{n=0}^{\infty} \frac{L_n^\alpha(x)t^n}{\Gamma(n + \alpha + 1)} = (xt)^{-\frac{1}{2}\alpha} e^t J_\alpha(2\sqrt{xt}).$$

**STELLING 6.** Definieer de  $q$ -hypergeometrische reeksen

$$\begin{aligned}\Phi &= {}_r\Phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z) \\ \Phi(a_1+) &= {}_r\Phi_s(a_1q, a_2, \dots, a_r; b_1, \dots, b_s; q, z) \\ \Phi(b_1-) &= {}_r\Phi_s(a_1, \dots, a_r; b_1q^{-1}, b_2, \dots, b_s; q, z)\end{aligned}$$

en voer de volgende notaties in:

$$\begin{aligned}A &= \prod_{m=1}^r a_m, & B &= \prod_{m=1}^s b_m, \\ U_j &= \frac{\prod_{m=1}^r (a_m - b_j)}{(1 - b_j) \prod_{m=1}^{j-1} (b_m - b_j) \prod_{m=j+1}^s (b_m - b_j)}, & W_{j,k} &= \frac{U_j}{(a_k - b_j)}.\end{aligned}$$

Een  $q$ -uitbreiding van de  $2r + s$  contigue relaties voor hypergeometrische reeksen (zie Rainville [1]) wordt gegeven door (zie Swarttouw [2])

$$\begin{aligned}(a_1 - a_k)\Phi &= (1 - a_k)a_1\Phi(a_k+) - (1 - a_1)a_k\Phi(a_1+), \quad k = 2, \dots, r, \\ (a_1 - b_kq^{-1})\Phi &= (1 - b_kq^{-1})a_1\Phi(b_k-) - (1 - a_1)b_kq^{-1}\Phi(a_1+), \quad k = 1, \dots, s, \\ \left\{ (1 - a_1)(1 - zAB^{-1}) + a_1z \left( \delta_{r,s+1} - AB^{-1} - \sum_{j=1}^s \frac{1 - b_j}{b_j^2} U_j \right) \right\} \Phi \\ &= (1 - a_1)(1 - zAB^{-1})\Phi(a_1+) - a_1z \sum_{j=1}^s \frac{U_j}{b_j^2} \Phi(b_j+), \\ (1 - zq^{-1}AB^{-1})\Phi &= \Phi(a_k-) - a_kq^{-1}z \sum_{j=1}^s \frac{W_{j,k}}{b_j} \Phi(b_j+), \quad k = 1, \dots, r.\end{aligned}$$

[1] - E.D. Rainville, *The contiguous function relations for  ${}_pF_q$  with applications to Bateman's  $J_n^{u,v}$  and Rice's  $H_n(z, p, v)$* , Bulletin of the American Mathematical Society 51, 1945, 714-723.

[2] - R.F. Swarttouw, *The contiguous function relations for the basic hypergeometric series*, Journal of Mathematical Analysis and Applications 149, 1990, 151-159.

**STELLING 7.** Aangezien hypergeometrische reeksen in zeer veel onderdelen van de wiskunde voorkomen, dient een cursus hypergeometrische reeksen in het curriculum van iedere wiskunde studie te worden opgenomen.

**STELLING 8.** De wiskunde vakgroepen van de faculteit Technische Wiskunde en Informatica van de Technische Universiteit Delft zouden zich duidelijker moeten profileren als een dienstverlenende instantie waar andere faculteiten terecht kunnen met wiskundige problemen.

**STELLING 9.** Net als voor de spelers dient er bij een voetbalwedstrijd voor de supporters van een bepaalde club een gele en rode kaarten regeling te zijn. Na drie lichte onregelmatigheden, die elk met een gele kaart bestraft dienen te worden, volgt automatisch dat de eerstvolgende thuiswedstrijd van die betreffende club zonder publiek wordt gespeeld. Bij ernstige onregelmatigheden, door een arbitrale commissie met een rode kaart te bestraffen, volgen één of meerdere wedstrijden zonder publiek.

**STELLING 10.** Uit commerciële overwegingen dient Sint Nicolaas zijn verjaardag voorstaan een maand eerder te vieren.

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*THE HAHN-EXTON  
q-BESSEL FUNCTION*

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*THE HAHN-EXTON  
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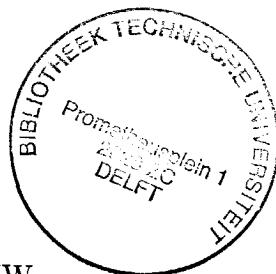
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# Contents

<b>Introduction</b>	<b>ix</b>
<b>Acknowledgements</b>	<b>xi</b>
<b>PART I</b>	<b>1</b>
<b>1 Bessel Functions</b>	<b>3</b>
1.1 Introduction . . . . .	3
1.2 The differential equation . . . . .	3
1.3 Recurrence relations . . . . .	5
1.4 A generating function and an orthogonality relation . . . . .	5
1.5 Addition formulas . . . . .	6
1.6 Integrals and an integral representation . . . . .	6
<b>2 Basic Hypergeometric Series</b>	<b>9</b>
2.1 Introduction . . . . .	9
2.2 Hypergeometric Series . . . . .	9
2.3 Basic Hypergeometric Series . . . . .	12
2.3.1 The history . . . . .	12
2.3.2 Definitions and notations . . . . .	13
2.3.3 Summation and transformation formulas . . . . .	15
2.3.4 The $q$ -Gamma function . . . . .	18
2.3.5 $q$ -Differentiation and $q$ -integration . . . . .	19
2.3.6 The Askey-Wilson integral . . . . .	20
<b>PART II</b>	<b>23</b>
<b>3 The Hahn-Exton <math>q</math>-Bessel function</b>	<b>25</b>
3.1 Jackson's $q$ -analogue of the Bessel function . . . . .	25
3.2 The Hahn-Exton $q$ -Bessel function . . . . .	27

3.2.1	Introduction . . . . .	27
3.2.2	Definition and some elementary relations . . . . .	28
3.2.3	Recurrence and difference-recurrence relations . . . . .	31
3.2.4	Generating functions and orthogonality relations . . . . .	34
3.2.5	A multiplication theory and a related integral . . . . .	36
3.2.6	An integral representation . . . . .	38
3.2.7	Limit transitions with $q$ -orthogonal polynomials . . . . .	38
<b>4</b>	<b>The Hahn-Exton <math>q</math>-Bessel difference equation</b>	<b>45</b>
4.1	Introduction . . . . .	45
4.2	The $q$ -Wronskian . . . . .	46
4.3	The Hahn-Exton $q$ -Bessel difference equation . . . . .	47
<b>5</b>	<b><math>q</math>-Analogues of the Fourier and Hankel transforms</b>	<b>51</b>
5.1	Introduction . . . . .	51
5.2	The $q$ -Hankel transform . . . . .	52
5.3	$q$ -Analogues of the Fourier-sine and the Fourier-cosine transform . . . . .	53
<b>6</b>	<b>Addition formulas</b>	<b>57</b>
6.1	Introduction . . . . .	57
6.2	A $q$ -extension of Graf's addition formula . . . . .	57
6.3	A second $q$ -extension of Graf's addition formula . . . . .	61
6.4	A $q$ -analogue of Gegenbauer's addition formula . . . . .	65
6.4.1	A product formula . . . . .	66
6.4.2	An integral representation . . . . .	68
6.4.3	The addition formula . . . . .	69
<b>Appendix A</b>		<b>75</b>
<b>Appendix B</b>		<b>77</b>
<b>References</b>		<b>81</b>
<b>Summary</b>		<b>85</b>
<b>Samenvatting</b>		<b>87</b>
<b>Curriculum Vitae</b>		<b>89</b>

# Introduction

Bessel functions are probably the most frequently used special functions. Broadly speaking they occur in connection with partial differential equations, usually when the variables are separated, or else in connection with certain definite integrals. In Watson's book [53], which is the standard work on Bessel functions, the history of these functions is traced back to James Bernoulli (about 1700). Since Euler (1764) and Poisson (1823) Bessel functions are associated most commonly with the partial differential equations of the potential, wave motion, or diffusion, in cylindrical or spherical polar coordinates. In chapter 1 of this thesis we will mention some well-known relations involving Bessel functions.

At the beginning of this century F.H. Jackson ([20]-[25]) started the investigation of a generalization of the Bessel function of integer order. This generalization is based on the simple fact that

$$\lim_{q \rightarrow 1} \frac{1 - q^a}{1 - q} = a,$$

where  $\frac{1 - q^a}{1 - q}$  is called *the basic number* of  $a$  and  $q$  is called *the base* (in chapter 2 we will give an extensive introduction to this  $q$ -theory). Roughly speaking Jackson replaced each number and each variable in the Bessel function by its basic number. The resulting function is called a  *$q$ -extension* or a  *$q$ -analogue* of the Bessel function. Jackson also considered a second  $q$ -extension of the Bessel function. He wrote several papers concerning these functions, in which he derived many results, which, of course, tend to the classical results when  $q$  tends to 1.

After Jackson's sequence of papers the subject of  $q$ -Bessel functions remained untouched nearly half a century until W. Hahn [14] started to investigate a more general  $q$ -analogue than Jackson did. He considered two  $q$ -extensions of the Bessel function of general order. He was able to generalize most of the relations that Jackson found to the general order case. He also gave a nice relation between his two  $q$ -Bessel functions.

A few years later Hahn [15] wrote a paper on a  $q$ -extension of a certain second order differential equation. He found his  *$q$ -difference equation* by considering the equations of motion of a swinging rope with infinitely many masspoints in a homogeneous gravitation field. The power series solution appeared to be a  $q$ -extension of the Bessel function of order zero. This  $q$ -Bessel function however differs from the  $q$ -analogues that Jackson and Hahn

investigated before. Hahn neither gave any formulas involving this  $q$ -Bessel function in his paper nor wrote more papers on this subject.

In 1978 H. Exton [10] considered a  $q$ -analogue of the Bessel-Clifford equation. The power series solution of that  $q$ -difference equation is a  $q$ -analogue of  $(\frac{1}{2}x)^{-\alpha}$  times the Bessel function of order  $\alpha$ . The special case  $\alpha = 0$  is the  $q$ -Bessel function that Hahn derived in [15]. Exton found a recurrence relation, some mixed formulas and a generating function for his functions. He also derived a  $q$ -analogue of the Fourier-Bessel orthogonality relations.

Recently Vaksman and Korogodskii [49] gave an interpretation of the  $q$ -Bessel functions that Exton investigated as matrix elements of irreducible representations of the quantum group of plane motions. Their paper, which does not contain proofs, implicitly contains some orthogonality relations for these functions.

In part II of this thesis we will study a slightly modified form of the  $q$ -Bessel function which was introduced by Hahn (in a special case) and by Exton (in full), and which we will call *the Hahn-Exton  $q$ -Bessel function*.

In chapter 3 we will give some general properties and derive several recurrence and difference recurrence relations for this function. Some of them are due to Exton. Further we will state the two types of orthogonality relations that Vaksman and Korogodskii found explicitly, we will show that the first type is immediately implied by the generating function of the Hahn-Exton  $q$ -Bessel function and that the second type can be obtained from the first one by a simple transformation. These results are also published in [35]. Several other new relations are included and some limit transitions with  $q$ -orthogonal polynomials are discussed.

In chapter 4 we will consider a  $q$ -extension of the Bessel differential equation. We will discuss a second solution and we will introduce a  $q$ -analogue of the Wronskian.

In chapter 5 we will show that the second orthogonality relation which Vaksman and Korogodskii gave, yields a  $q$ -analogue of Hankel's Fourier-Bessel integral. As a special case we will discuss  $q$ -analogues of the Fourier-cosine and sine transforms. Most of its contents was published before in [35].

In chapter 6 three  $q$ -analogues of addition formulas for the Bessel functions are discussed. First we will generalize the considerations which led to the orthogonality relations in chapter 3. The resulting formula will turn out to be a  $q$ -analogue of Graf's addition formula and, at the same time, of the discontinuous integral of Weber and Schafheitlin. The second addition formula has originally been discovered by H.T. Koelink [30] using the interpretation of the Hahn-Exton  $q$ -Bessel functions on the quantum group of plane motions as established by Vaksman and Korogodskii. Koelink's proof is formal, so a rigorous proof is needed, which is established in section 6.3. Finally a  $q$ -extension of Gegenbauer's addition formula is obtained. As additional results some product formulas and an integral representation are found. This result was published before in [46].

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# **PART I**

# Chapter 1

## Bessel Functions

### 1.1 Introduction

The Bessel function is probably the best known special function, both within pure and applied mathematics. Numerous books have been written on the subject of which Watson's book [53] is often considered as the standard work. The Bessel function needs therefore hardly any introduction.

In part II of this thesis a generalization of the Bessel function is considered. We will derive many new relations for this generalized Bessel function. In order to show that these results are extensions of the classical formulas, we frequently refer to the well-known results. The main objective of this chapter therefore will be to present several formulas, involving Bessel functions, without giving the proof. For a proof of most of the formulas the reader is referred to [53].

### 1.2 The differential equation

In many books on Bessel functions, they are introduced by their differential equation. It is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2)y = 0. \quad (1.2.1)$$

The point  $x = 0$  is a regular-singular point, so we can find solutions of the form

$$y = x^\mu \sum_{k=0}^{\infty} c_k x^k.$$

Substituting this in (1.2.1) we find with  $c_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$  the solution

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{\Gamma(\nu + k + 1) k!}. \quad (1.2.2)$$

Solution (1.2.2) is called the Bessel function of order  $\nu$ . It is easy to verify that  $J_{-\nu}$  is also a solution of the differential equation of Bessel (1.2.1). To determine if this second solution is linearly independent of the solution (1.2.2) we can apply the determinant of Wronski (see [5])

$$W(y_1, y_2) \stackrel{\text{def}}{=} \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx}, \quad (1.2.3)$$

where  $y_1$  and  $y_2$  are solutions of the linear second order differential equation

$$\frac{d^2y}{dx^2} + a(x) \frac{dy}{dx} + b(x)y = 0.$$

A nice general result concerning the Wronskian is Abel's theorem (see [5] §3.2)

$$W'(y_1, y_2) + a(x)W(y_1, y_2) = 0. \quad (1.2.4)$$

Applying the Wronskian (1.2.3) to the solutions  $J_\nu$  and  $J_{-\nu}$  we find

$$W(J_\nu, J_{-\nu}) = \frac{-2}{\pi x} \sin(\pi\nu), \quad (1.2.5)$$

so that  $W(J_\nu, J_{-\nu}) \neq 0$  if  $\nu \notin \mathbb{Z}$  and thus the solutions are linearly independent if  $\nu \notin \mathbb{Z}$ . If  $\nu \in \mathbb{Z}$  however, the solutions are linearly dependent. This fact can also be obtained from the series expansion. We find with (1.2.2) for  $n \in \mathbb{Z}$

$$J_{-n}(x) = (-1)^n J_n(x). \quad (1.2.6)$$

To determine a second solution that is linearly independent of  $J_\nu$  for all  $\nu$  we define for  $\nu \notin \mathbb{Z}$

$$Y_\nu(x) = \frac{\cos(\pi\nu)J_\nu(x) - J_{-\nu}(x)}{\sin(\pi\nu)}. \quad (1.2.7)$$

For  $n \in \mathbb{Z}$  we define

$$Y_n(x) = \lim_{\nu \rightarrow n} Y_\nu(x). \quad (1.2.8)$$

If we compute the Wronskian of  $J_\nu$  and  $Y_\nu$  we find for all  $\nu$

$$W(J_\nu, Y_\nu) = \frac{2}{\pi x}, \quad (1.2.9)$$

so that  $J_\nu$  and  $Y_\nu$  are indeed linearly independent for all  $\nu$ .

In chapter 4 we shall discuss a generalization of the Bessel equation.

### 1.3 Recurrence relations

By writing out the series expansion (1.2.2) we can prove that the Bessel function satisfies the three term recurrence relation

$$\frac{2\nu}{x} J_\nu(x) = J_{\nu+1}(x) + J_{\nu-1}(x). \quad (1.3.1)$$

Another recurrence relation can be found by differentiating (1.2.2) term by term. This yields

$$2J'_\nu(x) = J_{\nu-1}(x) - J_{\nu+1}(x). \quad (1.3.2)$$

By (1.2.6) we find for  $\nu = 0$  the nice relation

$$J'_0(x) = -J_1(x). \quad (1.3.3)$$

By elimination of  $J_{\nu-1}$  and  $J_{\nu+1}$  from (1.3.1) and (1.3.2) respectively, and by iterating the resulting formulas we obtain the generalized formulas

$$\left(\frac{1}{x} \frac{d}{dx}\right)^k (x^{-\nu} J_\nu(x)) = (-1)^k x^{-\nu-k} J_{\nu+k}(x), \quad (1.3.4)$$

and

$$\left(\frac{1}{x} \frac{d}{dx}\right)^k (x^\nu J_\nu(x)) = x^{\nu-k} J_{\nu-k}(x). \quad (1.3.5)$$

### 1.4 A generating function and an orthogonality relation

The generating function for the Bessel function of integer order is

$$e^{\frac{1}{2}x(t-t^{-1})} = \sum_{n=-\infty}^{\infty} t^n J_n(x). \quad (1.4.1)$$

An important formula that can be obtained from the generating function is an orthogonality relation due to Hansen and Lommel. Substitute in (1.4.1)  $t^{-1}$  for  $t$  and multiply the resulting identity with (1.4.1) to obtain

$$1 = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} t^{n-m} J_n(x) J_m(x).$$

Equality of coefficients of equal powers of  $t$  at both sides yields the Hansen-Lommel orthogonality relation

$$\sum_{k=-\infty}^{\infty} J_{k+n}(x) J_{k+m}(x) = \delta_{m,n}, \quad (1.4.2)$$

where the Kronecker delta function  $\delta_{m,n}$  is defined by

$$\delta_{m,n} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \quad \text{for } m, n = 0, 1, 2, \dots \quad (1.4.3)$$

## 1.5 Addition formulas

There are two types of expansions of Bessel functions which are known as addition theorems. The first type, which is due to Graf, is related to the theory of cylindrical waves. Graf's addition formula reads

$$\left( \frac{y - s^{-1}x}{y - sx} \right)^{\frac{1}{2}\nu} J_\nu \left( \sqrt{(y - s^{-1}x)(y - sx)} \right) = \sum_{k=-\infty}^{\infty} s^k J_k(x) J_{\nu+k}(y). \quad (1.5.1)$$

A special case of Graf's addition formula is Neumann's addition formula. For  $\nu = 0$  we have

$$J_0 \left( \sqrt{(y - s^{-1}x)(y - sx)} \right) = \sum_{k=-\infty}^{\infty} s^k J_k(x) J_k(y). \quad (1.5.2)$$

The second type, which was obtained by Gegenbauer, is more connected with the theory of spherical waves (in  $2\nu + 2$  dimensions). It is

$$\begin{aligned} & \left( x^2 + y^2 - 2xy \cos \phi \right)^{-\frac{1}{2}\nu} J_\nu \left( \sqrt{x^2 + y^2 - 2xy \cos \phi} \right) \\ &= 2^\nu \Gamma(\nu) \sum_{k=0}^{\infty} (\nu + k) x^{-\nu} J_{\nu+k}(x) y^{-\nu} J_{\nu+k}(y) C_k^\nu(\cos \phi), \end{aligned} \quad (1.5.3)$$

where  $C_k^\nu(\cos \phi)$  are the ultraspherical polynomials (see [47]).

## 1.6 Integrals and an integral representation

Numerous integrals involving Bessel functions have been found. Watson [53] for example devoted three chapters to this subject. In this thesis we will derive several new generalizations of integrals involving Bessel functions. In this section we will list the integrals which we will extend in the next chapters.

The first integral we will derive is an extension of an integral of Weber and Sonine. Their integral, which is valid for  $-\frac{1}{2} > \Re e(t) > -\Re e(\alpha) - 1$ , reads (see [53] §13.24 (1))

$$\int_0^\infty x^t J_\alpha(x) dx = \frac{2^t \Gamma(\frac{1}{2}(\alpha + 1 + t))}{\Gamma(\frac{1}{2}(\alpha + 1 - t))}. \quad (1.6.1)$$

A second integral involving a Bessel function is due to Sonine. This integral, which is valid for  $\Re(\alpha) > -1$  and  $\Re(\nu) > 0$  is (see [53] §12.11 (1))

$$J_{\alpha+\lambda}(y) = \frac{y^\lambda}{2^{\lambda-1}\Gamma(\lambda)} \int_0^1 x^{\alpha+1} (1-x^2)^{\lambda-1} J_\alpha(xy) dx. \quad (1.6.2)$$

Finally we will extend the discontinuous integral of Weber and Schafheitlin (see [53] §13.4 (2)). It is, for  $\Re(\alpha + \beta - \gamma + 1) > 0$  and  $\Re(\gamma) > -1$ ,

$$2^\gamma \int_0^\infty x^{-\gamma} J_\alpha(ax) J_\beta(bx) dx = \begin{cases} \frac{a^{\gamma-\beta-1} b^\beta \Gamma(\frac{1}{2}(\alpha+\beta-\gamma+1))}{\Gamma(\frac{1}{2}(\alpha-\beta+\gamma+1)) \Gamma(\beta+1)} {}_2F_1 \left( \begin{array}{c} \frac{1}{2}(\beta-\alpha-\gamma+1), \frac{1}{2}(\alpha+\beta-\gamma+1) \\ \beta+1 \end{array} \middle| \frac{b^2}{a^2} \right), & b < a, \\ \frac{a^\alpha b^{\gamma-\alpha-1} \Gamma(\frac{1}{2}(\alpha+\beta-\gamma+1))}{\Gamma(\frac{1}{2}(\beta-\alpha+\gamma+1)) \Gamma(\alpha+1)} {}_2F_1 \left( \begin{array}{c} \frac{1}{2}(\alpha-\beta-\gamma+1), \frac{1}{2}(\alpha+\beta-\gamma+1) \\ \alpha+1 \end{array} \middle| \frac{a^2}{b^2} \right), & a < b. \end{cases} \quad (1.6.3)$$

We will also derive a generalization of an integral representation for the Bessel function of integer order. For  $J_n(x)$ , with  $n \in \mathbb{Z}$ , we have

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix \sin \phi} e^{-in\phi} d\phi. \quad (1.6.4)$$

This integral representation can be derived from the generating function. Substitute in (1.4.1)  $t = e^{i\phi}$ , multiply by  $e^{-in\phi}$ , and integrate both sides from 0 to  $2\pi$ . This yields (1.6.4).

# Chapter 2

## Basic Hypergeometric Series

### 2.1 Introduction

The main objective in this chapter is to introduce the reader into the world of the  $q$ -analogue, and to present most of the definitions and formulas that we will need in the following chapters. We will begin by defining the hypergeometric series and giving some important special cases. Next we will define the basic hypergeometric series which contains an extra parameter  $q$ , called *the base*. Basic hypergeometric series are called  *$q$ -analogues* (or *basic analogues* or  *$q$ -extensions*) of hypergeometric series because a hypergeometric series can be obtained from a basic hypergeometric series as the  $q \uparrow 1$  limit case.

The binomial theorem is the basis of most of the summation formulas for hypergeometric series. Therefore it seems natural to start the derivation of the summation and transformation formulas for the basic hypergeometric series by introducing a  $q$ -analogue of the binomial theorem. Then we will use this theorem to derive Heine's  $q$ -analogue of Euler's transformation formulas and summation formulas that are  $q$ -analogues of those for hypergeometric series due to Chu and Vandermonde, Gauss, Pfaff and Saalschütz.

We will also introduce  $q$ -analogues of the exponential and the gamma functions, as well as the concept of  $q$ -differentiation and  $q$ -integration. Many additional identities are given in appendix A.

### 2.2 Hypergeometric Series

In 1812 it was Gauss [13] who considered the infinite series

$$1 + \frac{ab}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1 \cdot 2 \cdot 3 \cdot c(c+1)(c+2)} z^3 + \dots,$$

as a function of  $a, b, c, z$  where it is assumed that  $c \neq 0, -1, -2, \dots$ , so that no zero factors appear in the denominators of the terms of the series. In view of Gauss' paper, this series

is frequently called Gauss' series. Since the special case  $a = 1, b = c$  yields the geometric series

$$1 + z + z^2 + z^3 + \dots$$

Gauss' series is also called the (ordinary) hypergeometric series or the Gauss' hypergeometric series. An extension of Gauss' series is the generalized hypergeometric series (or generalized hypergeometric function). It has a series representation

$$\sum_{n=0}^{\infty} c_n,$$

with  $c_{n+1}/c_n$  a rational function of  $n$ . The ratio  $c_{n+1}/c_n$  can be factored, and is usually written as

$$\frac{c_{n+1}}{c_n} = \frac{(n + a_1) \cdots (n + a_r)z}{(n + b_1) \cdots (n + b_s)(n + 1)}. \quad (2.2.1)$$

To solve this equation introduce the shifted factorial

$$(a)_n \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } n = 0, \\ a(a+1) \cdots (a+n-1) & \text{if } n = 1, 2, \dots \end{cases} \quad (2.2.2)$$

Then if  $c_0 = 1$ , equation (2.2.1) can be solved for  $c_n$  as

$$c_n = \frac{(a_1)_n \cdots (a_r)_n z^n}{(b_1)_n \cdots (b_s)_n n!}. \quad (2.2.3)$$

The definition of the generalized hypergeometric series  ${}_rF_s$  now reads

$${}_rF_s = {}_rF_s \left( \begin{array}{c} a_1, \dots, a_r \\ b_1, \dots, b_s \end{array} \middle| z \right) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n z^n}{(b_1)_n \cdots (b_s)_n n!}. \quad (2.2.4)$$

Gauss' hypergeometric series is the special case  $r = 2, s = 1$  of (2.2.4). Sometimes the notation  ${}_rF_s(a_1, \dots, a_r; b_1, \dots, b_s; z)$  is used. Notice the “;” that separates the parameters in the numerator, the parameters in the denominator and the argument  $z$ . In (2.2.3) and (2.2.4) it is assumed that the parameters  $b_1, \dots, b_s$  are such that the denominators of the terms of the series are never zero. Since

$$(-m)_n = 0, \quad \text{if } n = m + 1, m + 2, \dots,$$

the generalized hypergeometric series terminates if one of its numerator parameters is zero or a negative integer. By the ratio test of d'Alembert the  ${}_rF_s$  series converges absolutely for all  $z$  if  $r \leq s$ , and for  $|z| < 1$  if  $r = s + 1$ . By an extension of the ratio test (see [6]) it converges absolutely for  $|z| = 1$  if  $r = s + 1$  and  $\Re[b_1 + \dots + b_s - (a_1 + \dots + a_r)] > 0$ . If  $r > s + 1$  and  $z \neq 0$  or  $r = s + 1$  and  $|z| > 1$ , then this series diverges, unless it terminates.

Some important functions which can be expressed by means of generalized hypergeometric series are the binomial theorem

$$(1-z)^{-a} = {}_1F_0 \left( \begin{array}{c} a \\ - \end{array} \middle| z \right), \quad |z| < 1, \quad (2.2.5)$$

the exponential function

$$e^z = {}_0F_0 \left( \begin{array}{c} - \\ - \end{array} \middle| z \right), \quad (2.2.6)$$

and the Bessel function

$$J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu+1)} {}_0F_1 \left( \begin{array}{c} - \\ \nu+1 \end{array} \middle| -\frac{x^2}{4} \right). \quad (2.2.7)$$

When a “-” is put in a hypergeometric series it is there to indicate the absence of either numerator (when  $r = 0$ ) or denominator (when  $s = 0$ ) parameters.

In his paper Gauss proved the summation formula

$${}_2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \middle| 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \Re e(c-a-b) > 0. \quad (2.2.8)$$

Before Gauss, Chu [7] in 1303 and Vandermonde [51] in 1772 had proved the terminating case  $a = -n$  of (2.2.8)

$${}_2F_1 \left( \begin{array}{c} -n, b \\ c \end{array} \middle| 1 \right) = \frac{(c-b)_n}{(c)_n}, \quad n = 0, 1, \dots, \quad (2.2.9)$$

which is now called the Chu-Vandermonde formula. Two other important results are due to Euler [9]. In 1748 he proved the transformation formulas

$${}_2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \middle| z \right) = (1-z)^{-a} {}_2F_1 \left( \begin{array}{c} a, c-a-b \\ c \end{array} \middle| \frac{z}{z-1} \right), \quad (2.2.10)$$

where  $|z| < 1$ ,  $|z/(z-1)| < 1$  and

$${}_2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \middle| z \right) = (1-z)^{c-a-b} {}_2F_1 \left( \begin{array}{c} c-a, c-b \\ c \end{array} \middle| z \right), \quad |z| < 1. \quad (2.2.11)$$

Note that (2.2.11) is an iterated form of (2.2.10).

The first important result for  ${}_pF_q$  with  $p > 2, q > 1$  is probably Pfaff's sum. This result from 1797 (see [36]) reads

$${}_3F_2 \left( \begin{array}{c} -n, a, b \\ c, a+b+1-c-n \end{array} \middle| 1 \right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}, \quad n = 0, 1, \dots \quad (2.2.12)$$

This summation formula was rediscovered by Saalschütz [41] in 1890 and is usually called Saalschütz formula or the Pfaff-Saalschütz formula. Note that if  $n \rightarrow \infty$  we obtain Gauss' summation formula (2.2.8).

Another important formula for the generalized hypergeometric series  ${}_pF_q$  with  $p > 2, q > 1$  is Whipple's transformation formula, which he discovered in 1926 (see [54]). It is

$${}_4F_3 \left( \begin{matrix} -n, a, b, c \\ d, e, f \end{matrix} \middle| 1 \right) = \frac{(e-a)_n (f-a)_n}{(e)_n (f)_n} {}_4F_3 \left( \begin{matrix} -n, a, d-b, d-c \\ d, 1+a-e-n, 1+a-f-n \end{matrix} \middle| 1 \right), \quad (2.2.13)$$

where  $a + b + c + 1 = d + e + f + n$ .

Quadratic transformations for hypergeometric series are very old, going back to Gauss and Kummer. They are usually given in books as consequences of the differential equation for the hypergeometric series. One fundamental formula is (see [8])

$${}_2F_1 \left( \begin{matrix} a, b \\ 1+a-b \end{matrix} \middle| x^2 \right) = (1-x)^{-2a} {}_2F_1 \left( \begin{matrix} a, a-b+\frac{1}{2} \\ 2a-2b+1 \end{matrix} \middle| \frac{-4x}{(1-x)^2} \right). \quad (2.2.14)$$

In the special case  $a = -n$ , with  $n$  a non-negative integer, we can read the sum on the right hand side backwards to obtain

$${}_2F_1 \left( \begin{matrix} -n, b \\ 1-n-b \end{matrix} \middle| x^2 \right) = x^n \frac{(2b)_n}{(b)_n} {}_2F_1 \left( \begin{matrix} -n, n+2b \\ b+\frac{1}{2} \end{matrix} \middle| -\frac{(1-x)^2}{4x} \right). \quad (2.2.15)$$

In the next section we will derive  $q$ -analogues of these formulas.

## 2.3 Basic Hypergeometric Series

### 2.3.1 The history

While Euler, Gauss and Riemann and many other great mathematicians wrote important papers on hypergeometric series, the development of basic hypergeometric series was much slower. Euler and Gauss did important work on basic hypergeometric series, but most of Gauss' work was unpublished until after his death and Euler's work was more influential on the development of number theory and elliptic functions. The study of basic hypergeometric series essentially started in 1748 when Euler considered the infinite product

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}$$

as a generating function for  $p(n)$ , the number of partitions of a positive integer  $n$  into positive integers. But it was not until about a hundred years later that the subject acquired

an independent status. It was Heine who used the simple fact that

$$\lim_{q \rightarrow 1} \frac{1 - q^a}{1 - q} = a, \quad (2.3.1)$$

to develop a systematic theory of basic hypergeometric series parallel to the theory of Gauss'  ${}_2F_1$  hypergeometric series. The number  $\frac{1-q^a}{1-q}$  is often called the *basic number* of  $a$ . Apart from some work by J. Thomae and L.J. Rogers, there was not payed much attention to the subject until in the beginning of this century F.H. Jackson started to develop the theory of basic hypergeometric series in a systematic way. He studied  $q$ -differentiation and  $q$ -integration and derived  $q$ -analogues of many hypergeometric summation and transformation formulas. W.N. Bailey derived many important results on basic hypergeometric series during the 1930's and 1940's. The interest in basic hypergeometric series kept growing during the 1950's and 1960's, witness the fact that D.B. Sears, L. Carlitz, W. Hahn and L.J. Slater spent a lot of their research in developing the theory of the basic hypergeometric series.

But it was not until G.E. Andrews and R. Askey started to collaborate in the mid 1970's that basic hypergeometric series became the active field of research as it is today. Since then many researchers have produced a substantial amount of interesting work on basic hypergeometric series. In fact the interest in this field has grown so much over the last fifteen years, that one already says that these mathematicians are affected by the  $q$ -disease.

An extensive list of publications on  $q$ -series of the authors mentioned above is included in [12].

### 2.3.2 Definitions and notations

The basic hypergeometric series has a series representation of the form

$$\sum_{n=0}^{\infty} c_n,$$

with  $c_{n+1}/c_n$  a rational function of  $q^n$  for a fixed parameter  $q$ , called the base, which usually satisfies  $|q| < 1$  or  $0 < q < 1$ . Here and everywhere else in this thesis it is assumed that  $0 < q < 1$  unless stated otherwise. The ratio  $c_{n+1}/c_n$  can be factored, and is usually written as

$$\frac{c_{n+1}}{c_n} = \frac{(1 - a_1 q^n) \cdots (1 - a_r q^n)}{(1 - b_1 q^n) \cdots (1 - b_s q^n) (1 - q^{n+1})} (-q^n)^{1+s-r} z. \quad (2.3.2)$$

The factor  $(-q^n)^{1+s-r}$  will be explained later. To solve this equation introduce the  $q$ -shifted factorial

$$(a; q)_n \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } n = 0, \\ (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) & \text{if } n = 1, 2, \dots \end{cases} \quad (2.3.3)$$

Then if  $c_0 = 1$ , equation (2.3.2) can be solved for  $c_n$  as

$$c_n = \frac{(a_1; q)_n \cdots (a_r; q)_n}{(b_1; q)_n \cdots (b_s; q)_n} \frac{z^n}{(q; q)_n} \left\{ (-1)^n q^{\binom{n}{2}} \right\}^{1+s-r}. \quad (2.3.4)$$

The definition of the basic hypergeometric series  ${}_r\Phi_s$  now reads

$${}_r\Phi_s = {}_r\Phi_s \left( \begin{array}{c} a_1, \dots, a_r \\ b_1, \dots, b_s \end{array} \middle| q, z \right) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(b_1; q)_n \cdots (b_s; q)_n} \frac{((-1)^n q^{\binom{n}{2}})^{1+s-r} z^n}{(q; q)_n}. \quad (2.3.5)$$

In (2.3.4) and (2.3.5) it is assumed that the parameters  $b_1, \dots, b_s$  are such that the denominators of the terms of the series are never zero. Since

$$(q^{-m}; q)_n = 0, \quad \text{if } n = m + 1, m + 2, \dots,$$

the basic hypergeometric series terminates if one of its numerator parameters is of the form  $q^{-m}$  with  $m = 0, 1, 2, \dots$ . The  ${}_r\Phi_s$  series converges absolutely for all  $z$  if  $r \leq s$  and for  $|z| < 1$  if  $r = s + 1$ . It diverges for  $z \neq 0$  if  $r > s + 1$ , unless it terminates.

In order to see that the basic hypergeometric series (2.3.5) is an extension of the generalized hypergeometric series (2.2.4), replace in (2.3.5)  $a_i$  by  $q^{a_i}$  and  $b_j$  by  $q^{b_j}$  for  $i = 1, \dots, r$  and  $j = 1, \dots, s$  and replace  $z$  by  $(q - 1)^{1+s-r} z$ . Then take the limit  $q \uparrow 1$  and use (2.3.1) to obtain (2.2.4).

Sometimes, when the parameters of the basic hypergeometric series (2.3.5) satisfy certain relations, we give the  ${}_r\Phi_s$  series a special name. We call an  ${}_{r+1}\Phi_r$  series *balanced* (or *Saalschützian*) if  $b_1 b_2 \cdots b_r = q a_1 a_2 \cdots a_{r+1}$  and  $z = q$ . An  ${}_{r+1}\Phi_r$  series is called *well-poised* if its parameters satisfy the relations

$$q a_1 = a_2 b_1 = a_3 b_2 = \dots = a_{r+1} b_r.$$

Balanced and well-poised basic hypergeometric series frequently appear in this thesis. They usually have nicer properties than other  ${}_r\Phi_s$  series.

Now we will explain the factor  $\left\{ (-1)^n q^{\binom{n}{2}} \right\}^{1+s-r}$ . Observe that the basic hypergeometric series (2.3.5) has the property that if we replace  $z$  by  $z/a_r$  and let  $a_r \rightarrow \infty$ , then the resulting series is again of the form (2.3.5) with  $r$  replaced by  $r - 1$ . Because this is not the case for the  ${}_r\Phi_s$  series defined without the factors  $\left\{ (-1)^n q^{\binom{n}{2}} \right\}^{1+s-r}$  in the books of Bailey [4] and Slater [43] and since we want to handle such limit transitions, we have chosen to use the series defined in (2.3.5). There is no loss in generality, since the Bailey and Slater series can be obtained from the  $r = s + 1$  case of (2.3.5) by choosing  $s$  sufficiently large and setting some of the parameters equal to zero.

For negative subscripts the  $q$ -shifted factorial is defined by

$$(a; q)_{-n} \stackrel{\text{def}}{=} \frac{1}{(1 - aq^{-1})(1 - aq^{-2}) \cdots (1 - aq^{-n})} = \frac{1}{(aq^{-n}; q)_n}, \quad n = 1, 2, \dots \quad (2.3.6)$$

We also define

$$(a; q)_{\infty} \stackrel{\text{def}}{=} \prod_{n=0}^{\infty} (1 - aq^n). \quad (2.3.7)$$

In this thesis we will often use identities involving  $q$ -shifted factorials. Most of them can be checked easily by writing out the factors and rearranging them in a different order. We will give an example:

$$\begin{aligned} (a^{-1}q^{1-n}; q)_n &= (1 - a^{-1}q^{1-n})(1 - a^{-1}q^{2-n}) \cdots (1 - a^{-1}) \\ &= a^{-1}q^{1-n}(aq^{n-1} - 1)a^{-1}q^{2-n}(aq^{n-2} - 1) \cdots a^{-1}(a - 1) \\ &= (a; q)_n(-a^{-1})^n q^{-\binom{n}{2}}. \end{aligned}$$

This identity and other similar identities that will be used in this thesis are listed in appendix A. Since products of  $q$ -shifted factorials occur very often, we shall use the more compact notation

$$(a_1, a_2, \dots, a_m; q)_n \stackrel{\text{def}}{=} (a_1; q)_n(a_2; q)_n \cdots (a_m; q)_n. \quad (2.3.8)$$

More properties of the basic hypergeometric series can be found in the book by G. Gasper and M. Rahman [12]. We shall frequently refer to this book because it contains the vast majority of the known formulas and their proofs of basic hypergeometric series. It also contains an extensive reference list and a lot of exercises are included after each chapter. This makes it both very useful for those who want to get acquainted with the  $q$ -theory and an outstanding entrypoint for the vast amount of references.

### 2.3.3 Summation and transformation formulas

As already said in the introduction, the binomial theorem is the basis of most of the summation and transformation formulas for hypergeometric series. Therefore it seems natural to start this section with the  $q$ -extension of (2.2.5) i.e. the  *$q$ -binomial theorem*

$${}_1\Phi_0 \left( \begin{array}{c} a \\ - \end{array} \middle| q, z \right) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_{\infty}}{(z; q)_{\infty}}, \quad |z| < 1, \quad (2.3.9)$$

which was derived by Cauchy and Heine and by other mathematicians. If we replace in (2.3.9)  $a$  by  $q^a$  and let  $q \uparrow 1$  we obtain the binomial theorem (2.2.5)

$$\frac{(q^a z; q)_{\infty}}{(z; q)_{\infty}} = {}_1\Phi_0 \left( \begin{array}{c} q^a \\ - \end{array} \middle| q, z \right) \rightarrow {}_1F_0 \left( \begin{array}{c} a \\ - \end{array} \middle| z \right) = (1 - z)^{-a}. \quad (2.3.10)$$

Like most of the formulas in this chapter, we will not give a proof of (2.3.9). An elegant proof can be found in the book by G. Gasper and M. Rahman [12]. If we set  $a = 0$  in (2.3.9) we obtain

$$e_q(z) \stackrel{\text{def}}{=} {}_1\Phi_0 \left( \begin{array}{c} 0 \\ - \end{array} \middle| q, z \right) = \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_{\infty}}, \quad |z| < 1, \quad (2.3.11)$$

which can be seen as a  $q$ -analogue of the exponential function (2.2.6) since

$$\lim_{q \uparrow 1} e_q((1-q)z) = e^z. \quad (2.3.12)$$

Another important  $q$ -analogue of the exponential function can be obtained from the  $q$ -binomial theorem by setting  $z$  equal to  $-z/a$  in (2.3.9) and letting  $a \rightarrow \infty$ . We find

$$E_q(z) \stackrel{\text{def}}{=} {}_0\Phi_0 \left( \begin{array}{c} - \\ - \end{array} \middle| q, -z \right) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q; q)_n} z^n = (-z; q)_{\infty}, \quad (2.3.13)$$

with the limit relation

$$\lim_{q \uparrow 1} E_q((1-q)z) = e^z. \quad (2.3.14)$$

Observe that  $e_q(z)E_q(-z) = 1$ .

A terminating case of the  $q$ -binomial theorem is

$$\frac{(z; q)_n}{(q; q)_n} = \sum_{k=0}^n \frac{(-z)^k q^{\binom{k}{2}}}{(q; q)_k (q; q)_{n-k}}. \quad (2.3.15)$$

It can be proved by using the identity (A.9) and the  $q$ -binomial theorem (2.3.9). Because the sum (2.3.15) is finite,  $z$  can be any complex number.

The  $q$ -binomial theorem can be used to prove Heine's  $q$ -analogue of Gauss' summation formula (2.2.8). Heine's summation formula reads

$${}_2\Phi_1 \left( \begin{array}{c} a, b \\ c \end{array} \middle| q, \frac{c}{ab} \right) = \frac{(c/a, c/b; q)_{\infty}}{(c, c/ab; q)_{\infty}}, \quad \left| \frac{c}{ab} \right| < 1. \quad (2.3.16)$$

As an example of the importance of the  $q$ -binomial theorem in deriving summation and transformation formulas we will give the proof of (2.3.16).

**Proof.**

$$\begin{aligned} {}_2\Phi_1 \left( \begin{array}{c} a, b \\ c \end{array} \middle| q, \frac{c}{ab} \right) &= \frac{(b; q)_{\infty}}{(c; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(a; q)_n (cq^n; q)_{\infty}}{(q; q)_n (bq^n; q)_{\infty}} \left( \frac{c}{ab} \right)^n \\ &= \frac{(b; q)_{\infty}}{(c; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} \left( \frac{c}{ab} \right)^n \sum_{k=0}^{\infty} \frac{(c/b; q)_k}{(q; q)_k} (bq^n)^k \end{aligned}$$

$$\begin{aligned}
&= \frac{(b; q)_\infty}{(c; q)_\infty} \sum_{k=0}^{\infty} \frac{(c/b; q)_k}{(q; q)_k} b^k \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} \left(\frac{cq^k}{ab}\right)^n \\
&= \frac{(b; q)_\infty}{(c; q)_\infty} \sum_{k=0}^{\infty} \frac{(c/b; q)_k}{(q; q)_k} b^k \frac{(cq^k/b; q)_\infty}{(cq^k/ab; q)_\infty} \\
&= \frac{(b, c/b; q)_\infty}{(c, c/ab; q)_\infty} \sum_{k=0}^{\infty} \frac{(c/ab; q)_k}{(q; q)_k} b^k = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}.
\end{aligned}$$

Here the  $q$ -binomial theorem is used three times.  $\square$

The terminating case  $a = q^{-n}$  of (2.3.16) becomes

$${}_2\Phi_1 \left( \begin{matrix} q^{-n}, b \\ c \end{matrix} \middle| q, \frac{cq^n}{b} \right) = \frac{(c/b; q)_n}{(c; q)_n}. \quad (2.3.17)$$

By changing the order of summation it follows from (2.3.17) that

$${}_2\Phi_1 \left( \begin{matrix} q^{-n}, b \\ c \end{matrix} \middle| q, q \right) = \frac{(c/b; q)_n}{(c; q)_n} b^n. \quad (2.3.18)$$

Both (2.3.17) and (2.3.18) are  $q$ -analogues of Chu-Vandermonde's formula (2.2.9).

In 1847 Heine [16] derived the following transformation formula

$${}_2\Phi_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| q, z \right) = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} {}_2\Phi_1 \left( \begin{matrix} z, c/b \\ az \end{matrix} \middle| q, b \right), \quad (2.3.19)$$

where  $|z| < 1$  and  $|b| < 1$ . The proof is similar of that of (2.3.16). Note that the argument  $z$  has moved into the parameters and that the parameter  $b$  has become the new argument. Applying (2.3.19) twice to the right hand side, we find a  $q$ -analogue of Euler's transformation formula (2.2.11)

$${}_2\Phi_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| q, z \right) = \frac{(abz/c; q)_\infty}{(z; q)_\infty} {}_2\Phi_1 \left( \begin{matrix} c/a, c/b \\ c \end{matrix} \middle| q, \frac{abz}{c} \right), \quad (2.3.20)$$

where  $|z| < 1$  and  $|abz/c| < 1$ .

Besides (2.3.19) and (2.3.20) we will also use in the next chapters some limit cases, which we shall list here. First replace  $z$  by  $z/a$  in (2.3.19) and let  $a \rightarrow \infty$  to obtain

$${}_1\Phi_1 \left( \begin{matrix} b \\ c \end{matrix} \middle| q, z \right) = \frac{(b, z; q)_\infty}{(c; q)_\infty} {}_2\Phi_1 \left( \begin{matrix} c/b, 0 \\ z \end{matrix} \middle| q, b \right), \quad |b| < 1. \quad (2.3.21)$$

Then let  $b \rightarrow 0$  in (2.3.21) to obtain an important symmetry relation for the  ${}_1\Phi_1$

$$(c; q)_\infty {}_1\Phi_1 \left( \begin{matrix} 0 \\ c \end{matrix} \middle| q, z \right) = (z; q)_\infty {}_1\Phi_1 \left( \begin{matrix} 0 \\ z \end{matrix} \middle| q, c \right). \quad (2.3.22)$$

A straightforward proof of (2.3.22) will be given in chapter 3.

Another important relation follows from (2.3.20). Replace in (2.3.20)  $z$  by  $z/ab$  and let  $a \rightarrow \infty$  and  $b \rightarrow \infty$ . We obtain

$${}_0\Phi_1 \left( \begin{array}{c} - \\ c \end{array} \middle| q, z \right) = (z/c; q)_{\infty} {}_2\Phi_1 \left( \begin{array}{c} 0, 0 \\ c \end{array} \middle| q, \frac{z}{c} \right) |z/c| < 1. \quad (2.3.23)$$

A  $q$ -analogue of (2.2.10), with  $a = -n$  ( $n \in \mathbb{N}$ ), was found by Jackson [23]. His transformation formula is

$${}_2\Phi_1 \left( \begin{array}{c} q^{-n}, b \\ c \end{array} \middle| q, z \right) = (cq/bz; q)_n (-1)^n \left( \frac{bz}{cq} \right)^n q^{-\binom{n}{2}} {}_3\Phi_2 \left( \begin{array}{c} q^{-n}, c/b, 0 \\ c, cq/bz \end{array} \middle| q, q \right). \quad (2.3.24)$$

A  $q$ -analogue of the Pfaff-Saalschütz summation formula (2.2.12) was first discovered by Jackson [27] in 1910. He derived the following sum of a terminating balanced  ${}_3\Phi_2$  series:

$${}_3\Phi_2 \left( \begin{array}{c} q^{-n}, a, b \\ c, abc^{-1}q^{1-n} \end{array} \middle| q, q \right) = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}, \quad n = 0, 1, \dots \quad (2.3.25)$$

A  $q$ -analogue of Whipple's transformation formula (2.2.13) was first derived by Sears [42] in 1951, and hence is called Sears' transformation formula. It reads:

$${}_4\Phi_3 \left( \begin{array}{c} q^{-n}, a, b, c \\ d, e, f \end{array} \middle| q, q \right) = \frac{(e/a, f/a; q)_n}{(e, f; q)_n} a^n {}_4\Phi_3 \left( \begin{array}{c} q^{-n}, a, d/b, d/c \\ d, aq^{1-n}/e, aq^{1-n}/f \end{array} \middle| q, q \right), \quad (2.3.26)$$

where  $abc = defq^{n-1}$ .

The last basic transformation formula we shall consider is a  $q$ -extension of the quadratic transformation (2.2.15). It is (see [2])

$${}_2\Phi_1 \left( \begin{array}{c} q^{-n}, b \\ q^{1-n}b^{-1} \end{array} \middle| q, \frac{x^2q}{b} \right) = \frac{x^n(b^2; q)_n}{b^{\frac{1}{2}n}(b; q)_n} {}_4\Phi_3 \left( \begin{array}{c} q^{-n}, q^n b^2, b^{\frac{1}{2}}x, b^{\frac{1}{2}}x^{-1} \\ bq^{\frac{1}{2}}, -bq^{\frac{1}{2}}, -b \end{array} \middle| q, q \right). \quad (2.3.27)$$

Note that this relation transforms a well-poised  ${}_2\Phi_1$  into a balanced  ${}_4\Phi_3$ . If we replace  $b$  by  $q^b$  and let  $q \uparrow 1$  we find (2.2.15).

### 2.3.4 The $q$ -Gamma function

The  $q$ -gamma function is defined by

$$\Gamma_q(x) \stackrel{\text{def}}{=} \frac{(q; q)_{\infty}}{(q^x; q)_{\infty}} (1 - q)^{1-x}, \quad 0 < q < 1, \quad (2.3.28)$$

and was introduced by Thomae [48]. When  $x = n + 1$  with  $n$  a nonnegative integer, this definition reduces to

$$\Gamma_q(n+1) = 1(1+q)(1+q+q^2)\dots(1+q+\dots+q^{n-1}),$$

which clearly tends to  $n! = \Gamma(n+1)$  as  $q \uparrow 1$ . To show that

$$\lim_{q \uparrow 1} \Gamma_q(x) = \Gamma(x) \quad (2.3.29)$$

for complex  $x$ , we refer to [12] where a simple proof due to Wm. Gosper is given. For a rigorous justification of that proof the reader is referred to Koornwinder [33].

### 2.3.5 $q$ -Differentiation and $q$ -integration

A  $q$ -extension of the derivative operator  $\frac{d}{dx}$  was introduced by Jackson [26]. The  $q$ -difference operator  $D_q$  is defined by

$$D_q f(x) \stackrel{\text{def}}{=} \begin{cases} \frac{f(x) - f(qx)}{(1-q)x} & \text{if } x \neq 0, \\ f'(0) & \text{if } x = 0, \end{cases} \quad (2.3.30)$$

and

$$D_q^n f(x) = D_q \left( D_q^{n-1} f(x) \right), \quad (2.3.31)$$

where the function  $f$  is ( $n$  times) differentiable in a neighbourhood of  $x = 0$ . By l'Hôpital's rule it is easy to see that

$$\lim_{q \uparrow 1} D_q f(x) = f'(x). \quad (2.3.32)$$

As an example we will show how  $D_q$  acts on a power of  $x$ :

$$D_q x^n = \frac{x^n - x^n q^n}{(1-q)x} = \frac{1 - q^n}{1 - q} x^{n-1},$$

which is the basic number of  $n$  times  $x^{n-1}$ .

Two easy consequences of definition (2.3.30) are

$$c(D_q f)(cx) = D_q f(cx), \quad (2.3.33)$$

and the  $q$ -analogue of the product rule for differentiation

$$\begin{aligned} D_q[f(x)g(x)] &= f(qx)D_q g(x) + g(x)D_q f(x) = \\ &= f(x)D_q g(x) + g(x)D_q f(x) - x(1-q)D_q f(x)D_q g(x). \end{aligned} \quad (2.3.34)$$

It is also possible to define a  $q$ -integral. J. Thomae [48] and F.H. Jackson [28] introduced

$$\int_0^1 f(t) d_q t = (1-q) \sum_{n=0}^{\infty} f(q^n) q^n = \sum_{n=0}^{\infty} f(q^n) (q^n - q^{n+1}). \quad (2.3.35)$$

Again as an example we show how this  $q$ -integral acts on a power of  $x$ . Choose  $a > -1$ , then

$$\int_0^1 x^a d_q x = (1-q) \sum_{n=0}^{\infty} (q^n)^a q^n = (1-q) \sum_{n=0}^{\infty} (q^{a+1})^n = \frac{1-q}{1-q^{a+1}},$$

which tends to  $1/(a+1)$  if  $q \uparrow 1$ .

Jackson gave also a more general definition than (2.3.35):

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t, \quad (2.3.36)$$

where

$$\int_0^a f(t) d_q t = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n. \quad (2.3.37)$$

If we replace  $a$  by  $q^{-k}$  in (2.3.37), then shift the sum on the right hand side and finally let  $k \rightarrow \infty$  we obtain a  $q$ -integral defined on  $(0, \infty)$

$$\int_0^{\infty} f(t) d_q t = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n. \quad (2.3.38)$$

For suitably restricted functions  $f$  it can be shown that

$$\lim_{q \uparrow 1} \int_0^{\infty} f(t) d_q t = \int_0^{\infty} f(t) dt. \quad (2.3.39)$$

A similar limit holds for (2.3.37). Although  $q$ -integrals are in fact sums, the  $q$ -integral notation is quite useful in simplifying various formulas and, of course, necessary if we want to take the limit  $q \uparrow 1$ .

### 2.3.6 The Askey-Wilson integral

An important extension of the Beta integral was found by Askey and Wilson [3]. Since it has five degrees of freedom, four free parameters and the parameter  $q$  from basic hypergeometric functions, it has enough flexibility to be useful in many situations. The integral is

$$\int_0^{\pi} \frac{h(\cos \phi; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}; q)}{h(\cos \phi; a, b, c, d; q)} d\phi = \frac{2\pi(abcd; q)_{\infty}}{(q, ab, ac, ad, bc, bd, cd; q)_{\infty}}, \quad (2.3.40)$$

where

$$\max(|a|, |b|, |c|, |d|, |q|) < 1, \quad (2.3.41)$$

and

$$h(\cos \phi; a_1, a_2, \dots, a_k; q) = h(\cos \phi; a_1; q)h(\cos \phi; a_2; q) \cdots h(\cos \phi; a_k; q),$$

$$h(\cos \phi; a; q) = \prod_{n=0}^{\infty} (1 - 2aq^n \cos \phi + a^2 q^{2n}) = (ae^{i\phi}, ae^{-i\phi}; q)_{\infty}. \quad (2.3.42)$$

Askey and Wilson deduced (2.3.40) from a contour integral. Simpler proofs were found by Ismail and Stanton [19] and Rahman [37]. In [12] the proof of Rahman is given and also the limit relation from (2.3.40) to the Beta integral is discussed. Gasper and Rahman also considered the case when the absolute value of one of the parameters is greater than one. We shall however only need the Askey-Wilson integral with the restrictions (2.3.41) and will therefore not discuss the other cases.

## **PART II**

# Chapter 3

## The Hahn-Exton $q$ -Bessel function

### 3.1 Jackson's $q$ -analogue of the Bessel function

At the beginning of the century F.H. Jackson, at the time chaplain in the British Royal Navy, introduced in a series of papers basic analogues of the Bessel function (see [20]-[25]). In our modern notation the most general form reads

$$J_{[n]}(x, \lambda) \stackrel{\text{def}}{=} \sum_{r=0}^{\infty} \frac{(-1)^r \lambda^{n+2r} x^{[n+2r]} (1-q)^{n+2r}}{(q; q)_r (q; q)_{n+r} (-q; q)_r (-q; q)_{n+r}}, \quad (3.1.1)$$

where  $[n] = \frac{1-q^n}{1-q}$  and  $n$  is a non-negative integer. It is easy to verify that (3.1.1) tends to the Bessel function of integer order if  $q \uparrow 1$ . We have the limit relation

$$\lim_{q \uparrow 1} J_{[n]}(x, \lambda) = J_n(\lambda x),$$

where  $J_n(x)$  is the Bessel function (1.2.2) with  $\nu = n \in \mathbb{N}$ .

Notice that (3.1.1) has two variables,  $x$  and  $\lambda$ , and is thus more general than one would expect it to be. This fact is probably explained as follows. It is likely that Jackson was not satisfied with the fact that his  $q$ -analogue has no longer a series expansion in (integer) powers of  $x$  as it has in the  $q = 1$  case, but in “basic” powers of  $x$ . Now that he had introduced an extra parameter  $\lambda$ , (3.1.1) still has a series representation in powers of one of its variables, i.e.  $\lambda$ . The fact that Jackson has chosen the powers of  $x$  to be basic numbers, is understandable if one looks at the derivative. One can easily verify that

$$\frac{d}{dx} \{ J_{[0]}(x, \lambda) \} = -\lambda J_{[1]}(x^q, \lambda),$$

which tends for  $\lambda = 1$  to (1.3.3) if  $q \uparrow 1$ . This relation will not hold anymore if we would replace  $x^{[n+2r]}$  by  $x^{n+2r}$ . Jackson found a lot of results for these functions including a recurrence relation, a differential equation and a generating function.

The concept of a series expansion in basic powers of  $x$  turned out to be not very useful. The introduction of the basic difference operator (2.3.30) however, was a big step forward. Now Jackson could define his  $q$ -analogue of the Bessel function with only one variable by

$$J_n(x; q) \stackrel{\text{def}}{=} \sum_{r=0}^{\infty} \frac{(-1)^r x^{n+2r} (1-q)^{n+2r}}{(q; q)_r (q; q)_{n+r} (-q; q)_r (-q; q)_{n+r}}, \quad (3.1.2)$$

with  $n = 0, 1, \dots$ , and he could find an even nicer  $q$ -analogue of (1.3.3), i.e.

$$D_q J_0(x; q) = -J_1(x; q).$$

Jackson found many results for this  $q$ -Bessel function, including recurrence relations, a differential equation and a generating function. A lot of his results could of course be obtained by the results for (3.1.1) by replacing  $x$  and  $\lambda$  by 1 and  $x$  respectively.

Jackson also introduced a second  $q$ -analogue of the Bessel function with only one variable. If we replace in (3.1.2)  $q$  by  $q^{-1}$  and use (A.4) we find

$$q^{n^2} \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r+n} q^{2r^2+2nr} (1-q)^{n+2r}}{(q; q)_r (q; q)_{n+r} (-q; q)_r (-q; q)_{n+r}} \stackrel{\text{def}}{=} q^{n^2} \mathcal{J}_n(x; q). \quad (3.1.3)$$

There is however an easy relation between  $J_n(x; q)$  and  $\mathcal{J}_n(x; q)$ , which we will give later when a more modern definition of Jackson's  $q$ -Bessel functions is given.

After Jackson's sequence of papers the subject of  $q$ -Bessel functions remained untouched nearly half a century until W. Hahn [14] started to investigate a more general  $q$ -analogue of Jackson's  $q$ -Bessel functions. He defined for complex  $\nu$  (in the modern notation of M.E.H. Ismail)

$$J_{\nu}^{(1)}(x; q) \stackrel{\text{def}}{=} \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \left(\frac{x}{2}\right)^{\nu} {}_2\Phi_1 \left( \begin{array}{c} 0, 0 \\ q^{\nu+1} \end{array} \middle| q, -\frac{x^2}{4} \right), \quad (3.1.4)$$

and

$$J_{\nu}^{(2)}(x; q) \stackrel{\text{def}}{=} \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \left(\frac{x}{2}\right)^{\nu} {}_0\Phi_1 \left( \begin{array}{c} - \\ q^{\nu+1} \end{array} \middle| q, -\frac{q^{\nu+1} x^2}{4} \right). \quad (3.1.5)$$

Notice that (3.1.4) converges only if  $|x| < 2$ . Hahn also proved a relation between the two  $q$ -Bessel functions (3.1.4) and (3.1.5). It reads

$$J_{\nu}^{(2)}(x; q) = (-x^2/4; q)_{\infty} J_{\nu}^{(1)}(x; q), \quad |x| < 2. \quad (3.1.6)$$

It can be proved by using the transformation formula (2.3.23).

Jackson's  $q$ -Bessel functions (3.1.2) and (3.1.3) do not differ very much from the functions that Hahn defined. If we replace in (3.1.4)  $\nu$  by  $n$  and replace  $q$  by  $q^2$  we find with (A.6), (A.13) and (A.16)

$$J_n^{(1)}(x; q^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+n}}{(q; q)_k (q; q)_{n+k} (-q; q)_k (-q; q)_{n+k}},$$

so that the relation

$$J_n^{(1)}(2x(1-q); q^2) = J_n(x; q)$$

holds. A similar relation holds between (3.1.5) and (3.1.3). Although Hahn introduced (3.1.4) and (3.1.5) these functions are now known as the Jackson  $q$ -Bessel functions. Hahn derived a recurrence relation and a  $q$ -difference equation for the  $J_\nu^{(1)}$  and  $J_\nu^{(2)}$  and some other results that are mostly generalizations of Jackson's results, where  $\nu = n \in \mathbb{N}$ .

After Hahn's paper again the subject remained untouched for a few decades until M.E.H. Ismail restarted the investigation of the Jackson  $q$ -Bessel functions in the 1980's ([17], [18]). Ismail found a basic analogue of the Lommel polynomials, could prove several theorems about the zeros of the  $J_\nu^{(2)}$  and derived some results on modified  $q$ -Bessel functions. A few years later M. Rahman started his investigation on the subject of  $q$ -Bessel functions. He derived an addition formula, found some integral representations and computed some infinite integrals for the functions (3.1.4) and (3.1.5) (see [38], [39] and [40]). Recently H.T. Koelink [29] found two  $q$ -analogues of the Hansen-Lommel orthogonality relations for the Jackson  $q$ -Bessel functions.

## 3.2 The Hahn-Exton $q$ -Bessel function

### 3.2.1 Introduction

In 1953 W. Hahn [15] wrote a paper on a certain second order  $q$ -difference equation. He found this equation by considering the equations of motion of a swinging rope with infinitely many masspoints in a homogeneous gravitation field. The power series solution  $V(x)$  is of the form

$$V(x) = \sum_{r=0}^{\infty} \frac{(-1)^r q^{\binom{r}{2}} x^r}{(q; q)_r (q; q)_r}. \quad (3.2.1)$$

Hahn observed that if we replace in (3.2.1)  $x$  by  $\frac{x^2(1-q)^2}{4}$  and let  $q \uparrow 1$  that we obtain the Bessel function of order zero. The function (3.2.1) is therefore a  $q$ -extension of the Bessel function of order zero. However it differs from the  $q$ -analogues that Jackson investigated. Hahn neither mentioned this fact nor wrote more papers about this  $q$ -Bessel function.

In 1978 H. Exton [10] considered a basic analogue of the Bessel-Clifford equation. The power series solution  $C_\alpha(x; q)$  of that  $q$ -difference equation is of the form

$$C_\alpha(x; q) = \frac{(q^{\alpha+1}; q)_\infty}{(q; q)_\infty} \sum_{r=0}^{\infty} \frac{(-1)^r q^{\binom{r}{2}} x^r (1-q)^{2r+\alpha}}{(q^{\alpha+1}; q)_r (q; q)_r}. \quad (3.2.2)$$

Exton observed that if we replace in (3.2.2)  $x$  by  $\frac{1}{4}x^2$  and let  $q \uparrow 1$  we obtain  $\left(\frac{1}{2}x\right)^{-\alpha}$  times the Bessel function of order  $\alpha$ . So  $x^\alpha C_\alpha(x; q)$  is a  $q$ -extension of the Bessel function of

order  $\alpha$  (1.2.2). The case  $\alpha = 0$  leads us to the  $q$ -Bessel function that Hahn investigated in [15]. Exton found a recurrence formula, a generating function and derived a  $q$ -analogue of the Fourier-Bessel orthogonality relations:

$$\int_0^1 x^\alpha C_\alpha(\mu_i q x; q) C_\alpha(\mu_j q x; q) d_q x = 0, \quad i \neq j, \quad \alpha > -1, \quad (3.2.3)$$

where  $\mu_1, \mu_2, \dots$  are the roots of the equation

$$C_\alpha(\mu; q) = 0.$$

Later Exton considered in his book [11] a slightly different form of a  $q$ -Bessel analogue than (3.2.2). Again he obtained his  $q$ -analogue as a solution of a basic Sturm-Liouville equation. He derived a recurrence relation, a generating function, some mixed relations and a  $q$ -analogue of the Fourier-Bessel orthogonality relations, similar to (3.2.3).

In the rest of this thesis we will investigate a slightly modified form of the  $q$ -Bessel functions that Exton (in full) and Hahn (in a special case) investigated. Some results in the next subsections are due to Exton, but most of them are new. The next chapters will give new results for this, what we have called, *Hahn-Exton  $q$ -Bessel function*.

### 3.2.2 Definition and some elementary relations

We define the Hahn-Exton  $q$ -Bessel function by

$$J_\nu(x; q) \stackrel{\text{def}}{=} \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} x^{2k}}{(q^{\nu+1}; q)_k (q; q)_k} = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} x^\nu \Phi_1 \left( \begin{array}{c|c} 0 & \\ q^{\nu+1} & \end{array} \middle| q, x^2 q \right). \quad (3.2.4)$$

This function is well-defined for  $x, \nu \in \mathbb{C}$  ( $x = 0$  being excepted if  $\Re(\nu) < 0$  or if  $\Re(\nu) = 0$  and  $\nu \neq 0$ ) since the factor  $(q^{\nu+1}; q)_\infty$  removes the zeros that will occur in the denominators of the terms of the series (3.2.4) when  $\nu = -1, -2, \dots$ . It is easy to see that the Hahn-Exton  $q$ -Bessel function tends to the ordinary Bessel function when  $q \uparrow 1$ . With (2.3.28) and (2.3.29) we have the (formal) limit relation

$$\lim_{q \uparrow 1} J_\nu(x(1-q)/2; q) = J_\nu(x). \quad (3.2.5)$$

In order to find the classical analogue corresponding to a  $q$ -formula, we will sometimes replace the base  $q$  by  $q^2$ , to make the notations easier. In that case we will use the limit relation

$$\lim_{q \uparrow 1} J_\nu(x(1-q); q^2) = J_\nu(x). \quad (3.2.6)$$

For a rigorous proof of these limit relations see theorem B.1.

We will now consider an estimate and a symmetry relation for the  ${}_1\Phi_1$  basic hypergeometric series, which will lead us to some important properties of the Hahn-Exton  $q$ -Bessel function.

**Theorem 3.1.** *The series*

$$(w; q)_{\infty} {}_1\Phi_1 \left( \begin{matrix} 0 \\ w \end{matrix} \middle| q, z \right) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (wq^k; q)_{\infty} z^k}{(q; q)_k} \quad (3.2.7)$$

defines an entire analytic function in  $z, w$ , which is also symmetric in  $z, w$ :

$$(w; q)_{\infty} {}_1\Phi_1 \left( \begin{matrix} 0 \\ w \end{matrix} \middle| q, z \right) = (z; q)_{\infty} {}_1\Phi_1 \left( \begin{matrix} 0 \\ z \end{matrix} \middle| q, w \right). \quad (3.2.8)$$

Both sides can be majorized by

$$(-|z|; q)_{\infty} (-|w|; q)_{\infty}. \quad (3.2.9)$$

**Proof.** Substitute for  $(wq^k; q)_{\infty}$  in (3.2.7) the  ${}_0\Phi_0$  series given by (2.3.13) to obtain

$$(w; q)_{\infty} {}_1\Phi_1 \left( \begin{matrix} 0 \\ w \end{matrix} \middle| q, z \right) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} z^k}{(q; q)_k} \frac{(-1)^m q^{\binom{m}{2}} w^m}{(q; q)_m} q^{km}.$$

The summand of the double series can be majorized by

$$\frac{q^{\binom{k}{2}} |z|^k}{(q; q)_k} \frac{q^{\binom{m}{2}} |w|^m}{(q; q)_m}.$$

Thus the double sum converges absolutely, uniformly for  $z, w$  on compacta, and is symmetric in  $z, w$ . Using again (2.3.13) it proves that (3.2.7) can be majorized by

$$(-|z|; q)_{\infty} (-|w|; q)_{\infty}.$$

□

**Corollary 1.** The Hahn-Exton  $q$ -Bessel function (3.2.4) is an analytic function in  $x$  for all values of  $x$  ( $x = 0$  being excepted if  $\nu \notin \mathbb{N}$ ) and it is an analytic function in  $\nu$  for all values of  $\nu$ .

**Corollary 2.** If we substitute  $w = q^{\alpha+1}$  and  $z = q^{\beta+1}$  in (3.2.8) we can obtain a symmetry relation for the Hahn-Exton  $q$ -Bessel function from the resulting formula. It is

$$J_{\alpha}(q^{\frac{1}{2}\beta}; q) = J_{\beta}(q^{\frac{1}{2}\alpha}; q). \quad (3.2.10)$$

If  $\nu = n \in \mathbb{Z}$  we can obtain a relation between  $J_{-n}$  and  $J_n$ . In order to derive that relation, we will first give a formula connecting two  ${}_1\Phi_1$  basic hypergeometric series of a special form.

**Theorem 3.2.** *For  $n \in \mathbb{Z}$  the basic hypergeometric series  ${}_1\Phi_1$  satisfies the relation*

$$(q^{1-n}; q)_{\infty} {}_1\Phi_1 \left( \begin{matrix} 0 \\ q^{1-n} \end{matrix} \middle| q, z \right) = (-1)^n q^{\frac{1}{2}n(n-1)} z^n (q^{1+n}; q)_{\infty} {}_1\Phi_1 \left( \begin{matrix} 0 \\ q^{1+n} \end{matrix} \middle| q, q^n z \right). \quad (3.2.11)$$

**Proof.** We assume that  $n \geq 0$ . Since

$$\frac{(q^{1-n}; q)_{\infty}}{(q^{1-n}; q)_k} = (q^{1-n+k}; q)_{\infty} = 0 \text{ if } k \leq n-1,$$

the first  $n$  terms of the series vanish, so the summation starts with  $k = n$ . We obtain with (A.13)

$$\begin{aligned} (q^{1-n}; q)_{\infty} {}_1\Phi_1 \left( \begin{matrix} 0 \\ q^{1-n} \end{matrix} \middle| q, z \right) &= \sum_{k=n}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (q^{1-n+k}; q)_{\infty} z^k}{(q; q)_k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{n+k} q^{\binom{n+k}{2}} (q^{1+k}; q)_{\infty} z^{n+k}}{(q; q)_{k+n}} \\ &= (q^{1+n}; q)_{\infty} (-1)^n z^n q^{\frac{1}{2}n(n-1)} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} z^k q^{nk}}{(q^{1+n}; q)_k (q; q)_k} \\ &= (-1)^n q^{\frac{1}{2}n(n-1)} z^n (q^{1+n}; q)_{\infty} {}_1\Phi_1 \left( \begin{matrix} 0 \\ q^{1+n} \end{matrix} \middle| q, q^n z \right). \end{aligned}$$

The case  $n < 0$  follows from the case  $n > 0$  of (3.2.11) by changing  $z$  into  $zq^{-n}$ .  $\square$

**Corollary.** For  $n \in \mathbb{Z}$  the Hahn-Exton  $q$ -Bessel function satisfies the relation

$$J_{-n}(x; q) = (-1)^n q^{\frac{1}{2}n} J_n(xq^{\frac{1}{2}n}; q). \quad (3.2.12)$$

For  $q \uparrow 1$  this relation tends to (1.2.6).

**Remark 1.** Because of (3.2.11), the behaviour of the two equal sides of (3.2.8) as  $|w| \rightarrow \infty$  drastically improves when  $w$  runs over the values  $q^{1-n}$ ,  $n = 1, 2, \dots$ . For such  $w$  we have, using (3.2.9)

$$\begin{aligned} \left| (q^{1-n}; q)_{\infty} {}_1\Phi_1 \left( \begin{matrix} 0 \\ q^{1-n} \end{matrix} \middle| q, z \right) \right| &= \left| (-1)^n q^{\binom{n}{2}} z^n (q^{1+n}; q)_{\infty} {}_1\Phi_1 \left( \begin{matrix} 0 \\ q^{1+n} \end{matrix} \middle| q, q^n z \right) \right| \\ &\leq q^{\binom{n}{2}} |z|^n (-|z|q^n; q)_{\infty} (-q^{1+n}; q)_{\infty} \\ &\leq q^{\binom{n}{2}} |z|^n (-|z|; q)_{\infty} (-q; q)_{\infty}. \end{aligned} \quad (3.2.13)$$

**Remark 2.** In terms of the Hahn-Exton  $q$ -Bessel function  $J_n(x; q)$  the estimates (3.2.9) and (3.2.13) can be rewritten as

$$|J_n(x; q)| \leq \begin{cases} \frac{(-q|x|^2; q)_\infty(-q; q)_\infty}{(q; q)_\infty} |x|^n & \text{if } n \geq 0, \\ \frac{(-q|x|^2; q)_\infty(-q; q)_\infty}{(q; q)_\infty} |x|^{-n} q^{\frac{1}{2}n(n-1)} & \text{if } n \leq 0. \end{cases} \quad (3.2.14)$$

The estimates (3.2.14) play an important role in the next chapters where we frequently change the order of summation in double and triple sums.

### 3.2.3 Recurrence and difference-recurrence relations

**Theorem 3.3.** *The Hahn-Exton  $q$ -Bessel function satisfies the three term recurrence relation:*

$$(1 - q^\nu + x^2)J_\nu(x; q) = x \{J_{\nu-1}(x; q) + J_{\nu+1}(x; q)\}. \quad (3.2.15)$$

**Proof.** It is possible to prove this theorem by writing out the series expansions of  $J_\nu$ ,  $J_{\nu-1}$  and  $J_{\nu+1}$ , but one can also use the contiguous relations for the basic hypergeometric series (see [45]). We will use the first method. Starting with the right hand side we have

$$\begin{aligned} & x \{J_{\nu-1}(x; q) + J_{\nu+1}(x; q)\} \\ &= \frac{(q^\nu; q)_\infty}{(q; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} x^{2k+\nu}}{(q^\nu; q)_k (q; q)_k} + \frac{(q^{\nu+2}; q)_\infty}{(q; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} x^{2k+\nu+2}}{(q^{\nu+2}; q)_k (q; q)_k} \\ &= \frac{(q^\nu; q)_\infty}{(q; q)_\infty} x^\nu + \sum_{k=1}^{\infty} \left( \frac{(q^\nu; q)_\infty}{(q; q)_\infty} \frac{q^{\binom{k+1}{2}}}{(q^\nu; q)_k (q; q)_k} - \frac{(q^{\nu+2}; q)_\infty}{(q; q)_\infty} \frac{q^{\binom{k}{2}}}{(q^{\nu+2}; q)_{k-1} (q; q)_{k-1}} \right) (-1)^k x^{2k+\nu} \\ &= \frac{(q^\nu; q)_\infty}{(q; q)_\infty} x^\nu + \sum_{k=1}^{\infty} \left( \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \frac{q^{\binom{k+1}{2}} (1 - q^{\nu+k})}{(q^{\nu+1}; q)_k (q; q)_k} - \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \frac{q^{\binom{k}{2}} (1 - q^k)}{(q^{\nu+1}; q)_k (q; q)_k} \right) (-1)^k x^{2k+\nu}. \end{aligned}$$

Now since

$$(1 - q^{\nu+k})q^{\binom{k+1}{2}} - (1 - q^k)q^{\binom{k}{2}} = (1 - q^\nu)q^{\binom{k+1}{2}} - (1 - q^{\nu+k})(1 - q^k)q^{\binom{k}{2}},$$

we have

$$x \{J_{\nu-1}(x; q) + J_{\nu+1}(x; q)\} = \frac{(q^\nu; q)_\infty}{(q; q)_\infty} x^\nu +$$

$$\begin{aligned}
& + \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \sum_{k=1}^{\infty} \left( \frac{q^{\binom{k+1}{2}}(1-q^{\nu})}{(q^{\nu+1}; q)_k(q; q)_k} - \frac{q^{\binom{k}{2}}(1-q^{\nu+k})(1-q^k)}{(q^{\nu+1}; q)_k(q; q)_k} \right) (-1)^k x^{2k+\nu} \\
& = (1-q^{\nu}) \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} x^{2k+\nu}}{(q^{\nu+1}; q)_k(q; q)_k} + \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} q^{\binom{k}{2}} x^{2k+\nu}}{(q^{\nu+1}; q)_{k-1}(q; q)_{k-1}} \\
& = (1-q^{\nu} + x^2) J_{\nu}(x; q).
\end{aligned}$$

□

The recurrence relation (3.2.15) tends to (1.3.1) if we replace  $x$  by  $\frac{1}{2}x(1-q)$  and let  $q \uparrow 1$ .

By using the basic difference operator (2.3.30) we can derive a  $q$ -analogue of the recurrence relation (1.3.2).

**Theorem 3.4.** *The Hahn-Exton  $q$ -Bessel function satisfies the mixed relation*

$$(1-q) D_q J_{\nu}(x; q) = q^{\frac{1}{2}(1-\nu)} \left\{ J_{\nu-1}(xq^{\frac{1}{2}}; q) - J_{\nu+1}(xq^{\frac{1}{2}}; q) \right\}. \quad (3.2.16)$$

**Proof.** Using the fact that  $1 - q^{2k+\nu} = q^k(1 - q^{k+\nu}) + 1 - q^k$  we obtain

$$\begin{aligned}
x(1-q) D_q J_{\nu}(x; q) &= J_{\nu}(x; q) - J_{\nu}(xq; q) \\
&= \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} x^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} x^{2k} (1 - q^{2k+\nu})}{(q^{\nu+1}; q)_k (q; q)_k} \\
&= \frac{(q^{\nu}; q)_{\infty}}{(q; q)_{\infty}} x^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} x^{2k} q^k}{(q^{\nu}; q)_k (q; q)_k} + \\
&\quad - \frac{(q^{\nu+2}; q)_{\infty}}{(q; q)_{\infty}} q x^{\nu+2} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} x^{2k} q^k}{(q^{\nu+2}; q)_k (q; q)_k} \\
&= x q^{\frac{1}{2}(1-\nu)} \left\{ J_{\nu-1}(xq^{\frac{1}{2}}; q) - J_{\nu+1}(xq^{\frac{1}{2}}; q) \right\}.
\end{aligned}$$

Here we have shifted the index of the second sum. □

Using relations (2.3.33) and (2.3.32) we can easily see that (3.2.16) tends to (1.3.2) if we replace  $x$  by  $\frac{1}{2}x(1-q)$  and let  $q \uparrow 1$ .

Again by using the basic difference operator (2.3.30) we can derive some mixed relations for the Hahn-Exton  $q$ -Bessel function. This time however, we will use the operator  $D_{q^{\frac{1}{2}}}$ , which we define, for  $x \neq 0$ , analogous to (2.3.30), by

$$D_{q^{\frac{1}{2}}} f(x) = \frac{f(x) - f(xq^{\frac{1}{2}})}{x(1 - q^{\frac{1}{2}})}. \quad (3.2.17)$$

Using (3.2.17) and the fact that  $1 - q^{k+\frac{1}{2}\nu} = 1 - q^{\frac{1}{2}\nu} + q^{\frac{1}{2}\nu}(1 - q^k)$ , we obtain

$$\begin{aligned} x(1 - q^{\frac{1}{2}})D_{q^{\frac{1}{2}}}J_\nu(x; q) &= J_\nu(x; q) - J_\nu(xq^{\frac{1}{2}}; q) \\ &= \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} x^{2k} (1 - q^{k+\frac{1}{2}\nu})}{(q^{\nu+1}; q)_k (q; q)_k} \\ &= \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} x^\nu (1 - q^{\frac{1}{2}\nu}) \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} x^{2k}}{(q^{\nu+1}; q)_k (q; q)_k} + \\ &\quad - \frac{(q^{\nu+2}; q)_\infty}{(q; q)_\infty} x^{\nu+2} q^{1+\frac{1}{2}\nu} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} x^{2k} q^k}{(q^{\nu+2}; q)_k (q; q)_k}. \end{aligned}$$

In the last step, we have shifted the summation index of the second sum. We thus have

$$x(1 - q^{\frac{1}{2}})D_{q^{\frac{1}{2}}}J_\nu(x; q) = (1 - q^{\frac{1}{2}\nu})J_\nu(x; q) - xq^{\frac{1}{2}}J_{\nu+1}(xq^{\frac{1}{2}}; q). \quad (3.2.18)$$

Now since by (2.3.34) we have

$$D_{q^{\frac{1}{2}}}f(x)g(x) = f(xq^{\frac{1}{2}})D_{q^{\frac{1}{2}}}g(x) + g(x)D_{q^{\frac{1}{2}}}f(x),$$

we can rewrite (3.2.18) as

$$\left\{ x^{-1} D_{q^{\frac{1}{2}}} \right\} \left\{ x^{-\nu} J_\nu(x; q) \right\} = - \frac{q^{\frac{1}{2}(1-\nu)} x^{-\nu-1}}{1 - q^{\frac{1}{2}}} J_{\nu+1}(xq^{\frac{1}{2}}; q). \quad (3.2.19)$$

Iterating (3.2.19) we have proved a  $q$ -extension of formula (1.3.4):

**Theorem 3.5.** *The Hahn-Exton  $q$ -Bessel function satisfies the following basic difference-recurrence relation:*

$$\left\{ x^{-1} D_{q^{\frac{1}{2}}} \right\}^k \left\{ x^{-\nu} J_\nu(x; q) \right\} = \frac{(-1)^k x^{-\nu-k} q^{\frac{1}{2}k(1-\nu)}}{(1 - q^{\frac{1}{2}})^k} J_{\nu+k}(xq^{\frac{1}{2}k}; q). \quad (3.2.20)$$

A  $q$ -analogue of (1.3.5) can be obtained in a similar way. Again we start with the operator  $D_{q^{\frac{1}{2}}}$  acting on  $J_\nu(x; q)$ . This time however, we use the fact that  $1 - q^{k+\frac{1}{2}\nu} = q^{-\frac{1}{2}\nu}(1 - q^{\nu+k}) + 1 - q^{-\frac{1}{2}\nu}$ . We find

$$\begin{aligned} x(1 - q^{\frac{1}{2}})D_{q^{\frac{1}{2}}}J_\nu(x; q) &= J_\nu(x; q) - J_\nu(xq^{\frac{1}{2}}; q) \\ &= \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} x^{2k} (1 - q^{k+\frac{1}{2}\nu})}{(q^{\nu+1}; q)_k (q; q)_k} \\ &= \frac{(q^\nu; q)_\infty}{(q; q)_\infty} x^\nu q^{-\frac{1}{2}\nu} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} x^{2k}}{(q^\nu; q)_k (q; q)_k} + \\ &\quad + \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} (1 - q^{-\frac{1}{2}\nu}) x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} x^{2k}}{(q^{\nu+1}; q)_k (q; q)_k}. \end{aligned}$$

This gives

$$x(1 - q^{\frac{1}{2}})D_{q^{\frac{1}{2}}}J_{\nu}(x; q) = xq^{-\frac{1}{2}\nu}J_{\nu-1}(x; q) + (1 - q^{-\frac{1}{2}\nu})J_{\nu}(x; q). \quad (3.2.21)$$

We can rewrite (3.2.21) as

$$\left\{ x^{-1}D_{q^{\frac{1}{2}}} \right\} \{x^{\nu}J_{\nu}(x; q)\} = \frac{x^{\nu-1}}{1 - q^{\frac{1}{2}}}J_{\nu-1}(x; q). \quad (3.2.22)$$

Iterating (3.2.22) gives us a  $q$ -analogue of (1.3.5):

**Theorem 3.6.** *The Hahn-Exton  $q$ -Bessel function satisfies the following basic difference-recurrence relation:*

$$\left\{ x^{-1}D_{q^{\frac{1}{2}}} \right\}^k \{x^{\nu}J_{\nu}(x; q)\} = \frac{x^{\nu-k}}{(1 - q^{\frac{1}{2}})^k}J_{\nu-k}(x; q). \quad (3.2.23)$$

### 3.2.4 Generating functions and orthogonality relations

Exton [10] found a generating function for his  $q$ -analogue of the Bessel function (3.2.2) by expanding a product of two  $q$ -analogues of the exponential function. We will use this method to derive the following generating function:

**Theorem 3.7.** *For  $x, t \in \mathbb{C}$  such that  $0 < |t| < |x|^{-1}$  there is the absolutely convergent expansion*

$$e_q(xt)E_q(-qxt^{-1}) = \frac{(qxt^{-1}; q)_{\infty}}{(xt; q)_{\infty}} = \sum_{n=-\infty}^{\infty} t^n J_n(x; q). \quad (3.2.24)$$

**Proof.** Using (2.3.13) and (2.3.11) we expand the left hand side of (3.2.24) to obtain

$$\begin{aligned} e_q(xt)E_q(-qxt^{-1}) &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} t^{m-k} x^{m+k} q^k}{(q; q)_k (q; q)_m} \\ &= \sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} t^{m-k} x^{m+k} q^k}{(q; q)_k (q; q)_m}, \end{aligned}$$

since  $(q; q)_m^{-1} = 0$  if  $m < 0$ . This is an absolutely convergent double sum for  $0 < |t| < |x|^{-1}$ . Now introduce the new summation variables  $k, n$  by substituting  $m = k + n$ . This yields

$$e_q(xt)E_q(-qxt^{-1}) = \sum_{n=-\infty}^{\infty} t^n x^n \frac{(q^{n+1}; q)_{\infty}}{(q; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} x^{2k}}{(q^{n+1}; q)_k (q; q)_k} = \sum_{n=-\infty}^{\infty} t^n J_n(x; q).$$

□

If we replace in (3.2.24)  $x$  by  $x(1-q)/2$  and let  $q \uparrow 1$  we find with (2.3.12), (2.3.14) and (3.2.5) the generating function (1.4.1)

$$e^{\frac{1}{2}x(t-t^{-1})} = \sum_{n=-\infty}^{\infty} t^n J_n(x).$$

Replace in (3.2.24)  $t$  by  $t^{-1}q$  and multiply the new identity with the original identity. The resulting formula

$$1 = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} t^{n-m} q^m J_n(x; q) J_m(x; q)$$

is absolutely convergent for  $x, t \in \mathbb{C}$  such that  $t \neq 0$  and  $|xq| < |t| < |x|^{-1}$ . So equality of coefficients of equal powers of  $t$  at both sides yields a  $q$ -analogue of the Hansen-Lommel orthogonality relation (1.4.2):

**Theorem 3.8.** *For  $|x| < q^{-\frac{1}{2}}$  and  $n, m \in \mathbb{Z}$  we have*

$$\sum_{k=-\infty}^{\infty} q^k q^{\frac{1}{2}(n+m)} J_{n+k}(x; q) J_{m+k}(x; q) = \delta_{n,m}, \quad (3.2.25)$$

where the sum at the left hand side is absolutely convergent, uniformly on compact subsets of the open disk  $|x| < q^{-\frac{1}{2}}$ .

**Remark.** Recently Vaksman and Korogodskii [49] gave an interpretation of the Hahn-Exton  $q$ -Bessel functions as matrix elements of irreducible representations of the quantum group of plane motions. Their paper which does not contain proofs, implicitly contains the orthogonality relation (3.2.25).

Because of the symmetry relation (3.2.8) we can find a second generating function and a second orthogonality relation. In terms of the  ${}_1\Phi_1$  basic hypergeometric series we have

**Theorem 3.9.** *For  $x, t \in \mathbb{C}$  such that  $0 < |t| < |x|^{-1}$  there is the absolute convergent expansion*

$$e_q(xt) E_q(-qxt^{-1}) = \frac{(qxt^{-1}; q)_{\infty}}{(xt; q)_{\infty}} = \sum_{n=-\infty}^{\infty} t^n x^n \frac{(x^2q; q)_{\infty}}{(q; q)_{\infty}} {}_1\Phi_1 \left( \begin{array}{c} 0 \\ x^2q \end{array} \middle| q, q^{n+1} \right). \quad (3.2.26)$$

**Theorem 3.10.** *For  $|x| < q^{-\frac{1}{2}}$  and  $n, m \in \mathbb{Z}$  we have*

$$\begin{aligned} & \sum_{k=-\infty}^{\infty} x^{k+n} q^{\frac{1}{2}(k+n)} \frac{(x^2q; q)_{\infty}}{(q; q)_{\infty}} {}_1\Phi_1 \left( \begin{array}{c} 0 \\ x^2q \end{array} \middle| q, q^{n+k+1} \right) \times \\ & \quad \times x^{k+m} q^{\frac{1}{2}(k+m)} \frac{(x^2q; q)_{\infty}}{(q; q)_{\infty}} {}_1\Phi_1 \left( \begin{array}{c} 0 \\ x^2q \end{array} \middle| q, q^{m+k+1} \right) = \delta_{m,n}, \end{aligned} \quad (3.2.27)$$

where the sum at the left hand side is absolutely convergent, uniformly on compact subsets of the open disk  $|x| < q^{-\frac{1}{2}}$ .

The proofs of theorems 3.9 and 3.10 follow easily from (3.2.24) and (3.2.25) by the symmetry relation (3.2.8). In chapter 5 we will show that the orthogonality relation (3.2.27) is a  $q$ -version of Hankel's Fourier-Bessel integral.

In order to find the classical formula corresponding to (3.2.26), we replace in (3.2.26)  $x^2$  by  $q^\alpha$ , and rewrite (3.2.26) in terms of the Hahn-Exton  $q$ -Bessel function. We obtain

$$\frac{(t^{-1}q^{1+\frac{1}{2}\alpha}; q)_\infty}{(tq^{\frac{1}{2}\alpha}; q)_\infty} = \sum_{n=-\infty}^{\infty} t^n J_\alpha(q^{\frac{1}{2}n}; q), \quad 0 < |t| < |q^{\frac{1}{2}\alpha}|^{-1}$$

Now replace  $q$  by  $q^2$  to make the notations easier and then replace  $t$  by  $q^{t+1}$ . This yields

$$\frac{(q^{\alpha+1-t}; q^2)_\infty}{(q^{\alpha+1+t}; q^2)_\infty} = \sum_{n=-\infty}^{\infty} q^n q^{nt} J_\alpha(q^n; q^2), \quad \Re(t) > -\Re(\alpha) - 1.$$

Under the side condition that  $\frac{\log(1-q)}{\log q} \in \mathbb{Z}$  we can replace  $q^n$  by  $(1-q)q^n$  and we can rewrite the relation above with (2.3.38) and (2.3.28) in the basic integral notation

$$\frac{(1+q)^t \Gamma_{q^2}(\frac{1}{2}(\alpha+1+t))}{\Gamma_{q^2}(\frac{1}{2}(\alpha+1-t))} = \int_0^\infty x^t J_\alpha((1-q)x; q^2) d_q x.$$

Formally, as  $q \uparrow 1$ , we find with (3.2.6) the integral of Weber and Sonine (1.6.1)

$$\frac{2^t \Gamma(\frac{1}{2}(\alpha+1+t))}{\Gamma(\frac{1}{2}(\alpha+1-t))} = \int_0^\infty x^t J_\alpha(x) dx,$$

which is valid for  $-\frac{1}{2} > \Re(t) > -\Re(\alpha) - 1$ .

### 3.2.5 A multiplication theory and a related integral

The generating function provides us with a  $q$ -extension of the multiplication theorem of the Bessel function:

**Theorem 3.11.** For  $|x| < q^{-\frac{1}{2}}$  and  $n \in \mathbb{Z}$  the Hahn-Exton  $q$ -Bessel function satisfies the multiplication theorem:

$$J_n(xq^\lambda; q) = q^{\lambda n} \sum_{k=0}^{\infty} x^k q^k \frac{(q^{2\lambda}; q)_k}{(q; q)_k} J_{n+k}(x; q). \quad (3.2.28)$$

**Proof.** Replace in the generating function (3.2.24)  $x$  and  $t$  by  $xq^\lambda$  and  $tq^{-\lambda}$  respectively. Then for  $q|x| < |t| < |x|^{-1}$  we have

$$\sum_{n=-\infty}^{\infty} t^n q^{-\lambda n} J_n(xq^\lambda; q) = \frac{(xt^{-1}q^{2\lambda+1}; q)_\infty}{(xt; q)_\infty} = \frac{(xt^{-1}q^{2\lambda+1}; q)_\infty}{(xt^{-1}q; q)_\infty} \cdot \frac{(xt^{-1}q; q)_\infty}{(xt; q)_\infty}.$$

Now use the generating function (3.2.24) and the  $q$ -binomial theorem (2.3.9) to expand the right hand side of the equation above. This yields:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} t^n q^{-\lambda n} J_n(xq^\lambda; q) &= \sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} x^k q^k t^{m-k} \frac{(q^{2\lambda}; q)_k}{(q; q)_k} J_m(x; q) \\ &= \sum_{n=-\infty}^{\infty} t^n \sum_{k=0}^{\infty} x^k q^k \frac{(q^{2\lambda}; q)_k}{(q; q)_k} J_{n+k}(x; q). \end{aligned}$$

So equality of coefficients of equal powers of  $t$  at both sides yields (3.2.28).  $\square$

**Corollary.** When we let  $\lambda \rightarrow \infty$  in (3.2.28) we obtain for  $|x| < q^{-\frac{1}{2}}$  the expansion

$$\frac{x^n}{(q; q)_n} = \sum_{k=0}^{\infty} \frac{x^k q^k}{(q; q)_k} J_{n+k}(x; q). \quad (3.2.29)$$

When we replace in (3.2.28)  $x$  by  $q^{\frac{1}{2}\alpha}$ ,  $2\lambda$  by  $\lambda$  and apply the symmetry relation (3.2.8) to the resulting formula, we find a  $q$ -analogue of an integral of Sonine:

**Theorem 3.12.** For  $\Re(\alpha) > -1$  and  $n \in \mathbb{Z}$  the Hahn-Exton  $q$ -Bessel function satisfies the relation

$$J_{\alpha+\lambda}(q^{\frac{1}{2}n}; q) = q^{\frac{1}{2}n\lambda} \sum_{k=0}^{\infty} q^{k(1+\frac{1}{2}\alpha)} \frac{(q^\lambda; q)_k}{(q; q)_k} J_\alpha(q^{\frac{1}{2}(n+k)}; q). \quad (3.2.30)$$

In order to find the classical formula corresponding to (3.2.30), we replace  $q$  by  $q^2$  to make the notations easier and then rewrite (3.2.30) in terms of the basic integral (2.3.35). With (A.6) we find

$$J_{\alpha+\lambda}(q^n; q^2) = \frac{q^{n\lambda}}{1-q} \int_0^1 \frac{x^{\alpha+1}(q^{2\lambda}; q^2)_\infty (x^2 q^2; q^2)_\infty}{(x^2 q^{2\lambda}; q^2)_\infty (q^2; q^2)_\infty} J_\alpha(xq^n; q^2) d_q x. \quad (3.2.31)$$

Under the side condition that  $\frac{\log(1-q)}{\log q} \in \mathbb{Z}$  we can replace  $q^n$  by  $(1-q)q^n$ . Further we use the facts that as  $q \uparrow 1$

$$\frac{(1-q^2)^{\lambda-1} (q^{2\lambda}; q^2)_\infty}{(q^2; q^2)_\infty} = \frac{1}{\Gamma_{q^2}(\lambda)} \rightarrow \frac{1}{\Gamma(\lambda)},$$

and that by the  $q$ -binomial theorem (2.3.9) and the limit (2.3.10) we have

$$\frac{(x^2q^2; q^2)_\infty}{(x^2q^{2\lambda}; q^2)_\infty} = {}_1\Phi_0\left(\begin{array}{c} q^{2-2\lambda} \\ x^2, x^2q^{2\lambda} \end{array} \middle| q^2\right) \rightarrow {}_1F_0\left(\begin{array}{c} 1-\lambda \\ x^2 \end{array} \middle| x^2\right) = (1-x^2)^{\lambda-1}.$$

Substituting this results in (3.2.31) and letting  $n$  depend on  $q$  in such a way that  $q^n$  tends to  $y$  if  $q \uparrow 1$ , we obtain at least formally (1.6.2)

$$J_{\alpha+\lambda}(y) = \frac{y^\lambda}{2^{\lambda-1}\Gamma(\lambda)} \int_0^1 x^{\alpha+1}(1-x^2)^{\lambda-1} J_\alpha(xy) dx, \quad (3.2.32)$$

which is valid for  $\Re(\alpha) > -1$  and  $\Re(\lambda) > 0$ .

### 3.2.6 An integral representation

In this subsection we will derive an integral representation for the Hahn-Exton  $q$ -Bessel function of integer order. It is a  $q$ -extension of (1.6.4) and can be obtained in a similar way as the derivation of (1.6.4).

**Theorem 3.13.** *The Hahn-Exton  $q$ -Bessel function  $J_n(x; q)$ , with  $n \in \mathbb{Z}$ , satisfies the integral representation*

$$J_n(x; q) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(qxe^{-i\phi}; q)_\infty}{(xe^{i\phi}; q)_\infty} e^{-in\phi} d\phi, \quad |x| < 1. \quad (3.2.33)$$

**Proof.** Replace in the generating function (3.2.24)  $t$  by  $e^{i\phi}$  to obtain

$$\frac{(qxe^{-i\phi}; q)_\infty}{(xe^{i\phi}; q)_\infty} = \sum_{n=-\infty}^{\infty} e^{in\phi} J_n(x; q).$$

Multiply both sides by  $e^{-im\phi}/2\pi$  and integrate from 0 to  $2\pi$ . Using the fact that

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\phi} d\phi = \delta_{m,n},$$

we obtain (3.2.33). □

### 3.2.7 Limit transitions with $q$ -orthogonal polynomials

Bessel functions are limit cases of some orthogonal polynomials. A well known limit transition is the one between the Legendre polynomial and the Bessel function  $J_0(z)$ . It is (see [53] §5.71)

$$\lim_{n \rightarrow \infty} P_n\left(\cos\left(\frac{z}{n}\right)\right) = J_0(z), \quad (3.2.34)$$

where the Legendre polynomial  $P_n(x)$  is defined by

$$P_n(x) = {}_2F_1 \left( \begin{array}{c} -n, n+1 \\ 1 \end{array} \middle| \frac{1-x}{2} \right).$$

A more general limit transition than (3.2.34) is the relation between the Jacobi polynomial and the Bessel function of order  $\alpha$ . The normalized Jacobi polynomial is defined by

$$P_n^{(\alpha, \beta)}(x) = {}_2F_1 \left( \begin{array}{c} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{array} \middle| \frac{1-x}{2} \right).$$

So we have the limit

$$\lim_{N \rightarrow \infty} P_{nN}^{(\alpha, \beta)}(1 - \frac{x^2}{2N^2}) = {}_0F_1 \left( \begin{array}{c} - \\ \alpha+1 \end{array} \middle| -(\frac{\lambda x}{2})^2 \right) = (\frac{\lambda x}{2})^{-\alpha} \Gamma(\alpha+1) J_\alpha(x\lambda), \quad (3.2.35)$$

where  $n_N/N$  tends to  $\lambda$  for  $N \rightarrow \infty$ .

The  $q$ -analogue of limit transition (3.2.35) starts with the little  $q$ -Jacobi polynomials, which are defined by

$$p_n(x; a, b; q) = {}_2\Phi_1 \left( \begin{array}{c} q^{-n}, abq^{n+1} \\ aq \end{array} \middle| q, qx \right), \quad (3.2.36)$$

and which satisfy the orthogonality relation

$$\begin{aligned} \frac{(aq, bq; q)_\infty}{(abq^2, q; q)_\infty} \sum_{k=0}^{\infty} p_m(q^k; a, b; q) p_n(q^k; a, b; q) (aq)^k \frac{(q^{k+1}; q)_\infty}{(bq^{k+1}; q)_\infty} \\ = \frac{(aq)^n (1 - abq) (bq, q; q)_n}{(1 - abq^{2n+1}) (aq, abq; q)_n} \delta_{m,n}, \end{aligned} \quad (3.2.37)$$

where  $0 < a < q^{-1}$  and  $b < q^{-1}$ . See Andrews and Askey [1].

It is clear that we formally have the termwise limit

$$\lim_{N \rightarrow \infty} p_{N-n}(xq^N; a, b; q) = {}_1\Phi_1 \left( \begin{array}{c} 0 \\ aq \end{array} \middle| q, xq^{n+1} \right). \quad (3.2.38)$$

See theorem B.2 for a rigorous proof of this limit result. Also, when we replace in the orthogonality relation (3.2.37)  $n, m, k$  by  $N - n, N - m, N + k$ , respectively (so the sum runs from  $-N$  to  $\infty$ ), and when we let  $N \rightarrow \infty$ , we obtain as a formal (termwise) limit the orthogonality relations (3.2.27).

Another limit transition from orthogonal polynomials to the Bessel function, starts with the Krawtchouk polynomials. These polynomials are defined by

$$K_n(x; p, N) = {}_2F_1 \left( \begin{array}{c} -n, -x \\ -N \end{array} \middle| \frac{1}{p} \right),$$

with  $0 < p < 1$  and  $N \in \mathbb{N}$ . For  $z \in \mathbb{R}$  they satisfy the limit relation (see [31])

$$\lim_{N \rightarrow \infty} \frac{(2N)!(-1)^{n+k}}{\sqrt{(k+N)!(-k+N)!(n+N)!(-n+N)!}} \left(1 - \frac{z^2}{4N^2}\right)^{2N+\frac{1}{2}(n+k)} \times$$

$$\times \left(\frac{z^2}{4N^2}\right)^{-\frac{1}{2}(k+n)} K_{n+N}(k+N; 1 - \frac{z^2}{4N^2}, 2N) = J_{k+n}(z).$$

Several  $q$ -analogues of the Krawtchouk polynomials have been studied. We will consider two  $q$ -extensions of the Krawtchouk polynomials, which have a limit relation with the Hahn-Exton  $q$ -Bessel function. The first one is defined by (see [32])

$$K_n(q^{-x}; p, N; q) = {}_2\Phi_1 \left( \begin{array}{c} q^{-n}, q^{-x} \\ q^{-N} \end{array} \middle| q, pq^{n+1} \right), \quad (3.2.39)$$

where  $N \in \mathbb{N}$ , and is known as the quantum  $q$ -Krawtchouk polynomial. It satisfies the orthogonality relation

$$\sum_{x=0}^N \frac{(pq; q)_{N-x}}{(q; q)_x (q; q)_{N-x}} q^{\binom{x}{2}} (-1)^{N-x} K_n(q^{-x}; p, N; q) K_m(q^{-x}; p, N; q) =$$

$$= \frac{(q; q)_{N-n} (q, pq; q)_n}{(q, q; q)_N} (-1)^n p^N q^{nN} q^{\binom{N+1}{2}} q^{-\binom{n+1}{2}} \delta_{m,n}. \quad (3.2.40)$$

In order to find the limit relation with the Hahn-Exton  $q$ -Bessel function, we read the  ${}_2\Phi_1$  in (3.2.39) backwards. We have with (A.10) and (A.7)

$$\begin{aligned} K_n(q^{-x}; p, N; q) &= {}_2\Phi_1 \left( \begin{array}{c} q^{-n}, q^{-x} \\ q^{-N} \end{array} \middle| q, pq^{n+1} \right) \\ &= \sum_{k=0}^n \frac{(q^{-n}; q)_k (q^{-x}; q)_k}{(q^{-N}; q)_k (q; q)_k} p^k q^{k(n+1)} \\ &= \sum_{k=0}^n \frac{(q^{-n}; q)_{n-k} (q^{-x}; q)_{n-k}}{(q^{-N}; q)_{n-k} (q; q)_{n-k}} p^{n-k} q^{(n-k)(n+1)} \\ &= \frac{(q^{-x}; q)_n (q^{-n}; q)_n}{(q^{-N}; q)_n (q; q)_n} p^n q^{n(n+1)} \sum_{k=0}^n \frac{(q^{-n}; q)_k (q^{1+N-n}; q)_k}{(q^{1-n+x}; q)_k (q; q)_k} p^{-k} q^{k(x-N)}. \\ &= \frac{(q^{-x}; q)_n}{(q^{-N}; q)_n} (-p)^n q^{\binom{n+1}{2}} {}_2\Phi_1 \left( \begin{array}{c} q^{-n}, q^{1+N-n} \\ q^{1-n+x} \end{array} \middle| q, p^{-1} q^{x-N} \right). \end{aligned}$$

Now replace  $x, N, n, p$  by  $L+k, 2L, L-n, x^{-2}q^{-2L-1}$  respectively. With (A.10), (A.7) and (A.12) we find

$$K_{L-n}(q^{-L-k}; x^{-2}q^{-2L-1}, 2L; q) = \frac{(q; q)_{L+k}(q; q)_{L+n}}{(q; q)_{n+k}(q; q)_{2L}} (-x^2)^{n-L} q^{\binom{n}{2}} q^{(k+1)(n-L)} q^{-\binom{L}{2}} \times \\ \times {}_2\Phi_1 \left( \begin{array}{c} q^{n-L}, q^{1+L+n} \\ q^{n+k+1} \end{array} \middle| q, x^2 q^{L+k+1} \right).$$

It is easy to see that we formally have the termwise limit

$$\lim_{L \rightarrow \infty} (-x^2)^L q^{\binom{L+1}{2}} q^{Lk} K_{L-n}(q^{-L-k}; x^{-2}q^{-2L-1}, 2L; q) \quad (3.2.41) \\ = (q^{n+k+1}; q)_\infty (-x^2)^n q^{\binom{n+1}{2}} q^{nk} {}_1\Phi_1 \left( \begin{array}{c} 0 \\ q^{n+k+1} \end{array} \middle| q, x^2 q^{n+k+1} \right).$$

See theorem B.3 for a rigorous proof of this limit. The right hand side can be seen as a factor times the Hahn-Exton  $q$ -Bessel function  $J_{n+k}(xq^{\frac{1}{2}(n+k)}; q)$ .

When we use the same substitutions in the orthogonality relation (3.2.40), and in addition replace  $m$  by  $L - m$ , we obtain

$$\sum_{k=-L}^L \frac{(x^{-2}q^{-2L}; q)_{L-k}(-1)^{n-k} q^{L(L-1)} q^{\binom{k}{2}} q^{kL}}{(x^{-2}q^{-2L}; q)_{L-n}(q; q)_{L+k}(q; q)_{L-k}} K_{L-n}(q^{-L-k}; x^{-2}q^{-2L-1}, 2L; q) \times \\ \times K_{L-m}(q^{-L-k}; x^{-2}q^{-2L-1}, 2L; q) = \frac{(q; q)_{L+n}(q; q)_{L-n}}{(q; q; q)_{2L}} x^{-4L} q^{-\binom{n}{2}} q^{-L(n+2)} \delta_{m,n}.$$

Since

$$\frac{(x^{-2}q^{-2L}; q)_{L-k}}{(x^{-2}q^{-2L}; q)_{L-n}} = \frac{(x^2q^{L+1}; q)_n}{(x^2q^{L+1}; q)_k} (-x^2)^{k-n} q^{(L+1)(k-n)} q^{\binom{k}{2}} q^{-\binom{n}{2}},$$

we find

$$\sum_{k=-L}^L \frac{(x^2q^{L+1}; q)_n x^{2(k-n)} q^{k-n} q^{L(L+1)} q^{k(k-1)} q^{2kL} x^{4L}}{(x^2q^{L+1}; q)_k (q; q)_{L+k} (q; q)_{L-k}} K_{L-n}(q^{-L-k}; x^{-2}q^{-2L-1}, 2L; q) \times \\ \times K_{L-m}(q^{-L-k}; x^{-2}q^{-2L-1}, 2L; q) = \frac{(q; q)_{L+n}(q; q)_{L-n}}{(q; q; q)_{2L}} \delta_{m,n}. \quad (3.2.42)$$

When we let  $L \rightarrow \infty$  we obtain with (3.2.41) as a formal (termwise) limit the orthogonality relation

$$\sum_{k=-\infty}^{\infty} x^{2(k-n)} q^{k(k-1)} q^{k+m} \frac{(q^{n+k+1}, q^{m+k+1}; q)_\infty}{(q, q; q)_\infty} (-x^2)^n q^{\binom{n}{2}} q^{nk} {}_1\Phi_1 \left( \begin{array}{c} 0 \\ q^{n+k+1} \end{array} \middle| q, x^2 q^{n+k+1} \right) \times \\ \times (-x^2)^m q^{\binom{m}{2}} q^{mk} {}_1\Phi_1 \left( \begin{array}{c} 0 \\ q^{m+k+1} \end{array} \middle| q, x^2 q^{m+k+1} \right) = \delta_{m,n}.$$

With (3.2.4) this yields

$$\sum_{k=-\infty}^{\infty} (-x)^{m-n} q^{\frac{1}{2}(m-n)} J_{n+k}(xq^{\frac{1}{2}(n+k)}; q) J_{m+k}(xq^{\frac{1}{2}(m+k)}; q) = \delta_{m,n}.$$

Note that the factor  $x^{m-n}$  can be removed. When we replace  $k$  by  $-k$ , use (3.2.12) and next replace  $m, n$  by  $-m, -n$  we have found the Hansen-Lommel orthogonality relations (3.2.25).

A second  $q$ -analogue of the Krawtchouk polynomial that tends to the Hahn-Exton  $q$ -Bessel function, is the Affine  $q$ -Krawtchouk polynomial. It is defined by (see [12] ex. 7.11)

$$K_n^{Aff}(q^{-x}; p, N; q) = {}_3\Phi_2 \left( \begin{array}{c} q^{-n}, q^{-x}, 0 \\ pq, q^{-N} \end{array} \middle| q, q \right), \quad (3.2.43)$$

where  $0 < pq < 1$ . It satisfies the orthogonality relation

$$\begin{aligned} \sum_{x=0}^N \frac{(pq; q)_x (q; q)_N}{(q; q)_x (q; q)_{N-x}} (pq)^{-x} K_n^{Aff}(q^{-x}; p, N; q) K_m^{Aff}(q^{-x}; p, N; q) \\ = (pq)^{n-N} \frac{(q; q)_n (q; q)_{N-n}}{(pq; q)_n (q; q)_N} \delta_{m,n}. \end{aligned} \quad (3.2.44)$$

In order to find the limit relation with the Hahn-Exton  $q$ -Bessel function, we transform (3.2.43) to a  ${}_2\Phi_1$  with (2.3.24). This gives

$$K_n^{Aff}(q^{-x}; p, N; q) = \frac{(-1)^n q^{-nN} q^{\binom{n}{2}}}{(q^{-N}; q)_n} {}_2\Phi_1 \left( \begin{array}{c} q^{-n}, pq^{x+1} \\ pq \end{array} \middle| q, q^{N-x+1} \right). \quad (3.2.45)$$

Now replace in (3.2.45)  $n, N, x, p$  by  $L - n, 2L, k + L, x^2$  respectively, we obtain with identities (A.9) and (A.12)

$$K_{L-n}^{Aff}(q^{-k-L}; x^2, 2L; q) = \frac{(q; q)_{n+L}}{(q; q)_{2L}} {}_2\Phi_1 \left( \begin{array}{c} q^{-L+n}, x^2 q^{k+L+1} \\ x^2 q \end{array} \middle| q, q^{L-k+1} \right).$$

The formal (termwise) limit is now easily taken:

$$\begin{aligned} \lim_{L \rightarrow \infty} K_{L-n}^{Aff}(q^{-k-L}; x^2, 2L; q) &= {}_1\Phi_1 \left( \begin{array}{c} 0 \\ x^2 q \end{array} \middle| q, q^{n-k+1} \right) \\ &= \frac{(q^{n-k+1}; q)_\infty}{(x^2 q; q)_\infty} {}_1\Phi_1 \left( \begin{array}{c} 0 \\ q^{n-k+1} \end{array} \middle| q, x^2 q \right) = \frac{(q; q)_\infty}{(x^2 q; q)_\infty} x^{k-n} J_{n-k}(x; q). \end{aligned} \quad (3.2.46)$$

Here the symmetry relation (3.2.8) is used. A rigorous proof is given in theorem B.4. When we choose the same substitutions in the orthogonality relation (3.2.44), and when

we replace in addition  $m$  by  $L - m$ , we obtain

$$\begin{aligned} \sum_{k=-L}^L \frac{(x^2 q; q)_{k+L} (q; q)_{2L}}{(q; q)_{L+k} (q; q)_{L-k}} (x^2 q)^{-k-L} K_{L-n}^{Aff}(q^{-k-L}; x^2, 2L; q) K_{L-m}^{Aff}(q^{-k-L}; x^2, 2L; q) \\ = (x^2 q)^{-L-n} \frac{(q; q)_{L-n} (q; q)_{L+n}}{(x^2 q; q)_{L-n} (q; q)_{2L}} \delta_{m,n}. \end{aligned}$$

The formal (termwise) limit is now easily taken. We obtain the (slightly rewritten) Hansen-Lommel orthogonality relation (3.2.25):

$$\sum_{k=-\infty}^{\infty} q^{n-k} x^{n-m} J_{n-k}(x; q) J_{m-k}(x; q) = \delta_{m,n}.$$

**Remark.** Since we have the symmetry relation (3.2.8) it is obvious that the orthogonality relation (3.2.44) also tends to the orthogonality relation (3.2.27).

# Chapter 4

## The Hahn-Exton $q$ -Bessel difference equation

### 4.1 Introduction

In mathematics very much attention is paid to the subject of differential equations. However, the theory of  $q$ -extensions of differential equations has not yet been developed to a great extent. This can partially be explained by the fact that one is not very familiar with  $q$ -theory and the fact that basic differential equations do not occur frequently in physics. But the most important fact is probably the close relationship with difference equations instead of differential equations.  $q$ -Differential (or  $q$ -difference) equations may even properly be regarded as a part of the field of difference equations. Results on difference equations may sometimes be transformed into results on  $q$ -difference equations and vice versa. However each subject has his own specific problems and it would therefore be desirable if a book on basic difference equations would be available.

It is by no means our aim to give in this thesis such a general theory of  $q$ -difference equations. We will restrict ourselves to the necessary theory in order to give  $q$ -extensions of the results of section 1.2. Therefore in this chapter we will consider only a  $q$ -extension of the second order linear differential equation of Bessel (1.2.1). We will obtain a second solution, which is a  $q$  extension of (1.2.7). In order to obtain a  $q$ -extension of the relations (1.2.5) and (1.2.9) we have to derive a  $q$ -extension of the Wronskian (1.2.3). To my best knowledge, this  $q$ -Wronskian has not been stated elsewhere. So it seems natural to give some general results on the  $q$ -Wronskian first, before we will discuss the Hahn-Exton  $q$ -Bessel difference equation.

## 4.2 The $q$ -Wronskian

We will consider the second order linear  $q$ -difference equation

$$D_q^2 y(x) + a(x)D_q y(x) + b(x)y(xq) = 0, \quad (4.2.1)$$

where  $D_q$  and  $D_q^2$  are defined by (2.3.30) and (2.3.31). The fact that the last term of the  $q$ -difference equation is  $y(xq)$  instead of  $y(x)$  is a result of the definition of the  $q$ -difference operator  $D_q$ . It also agrees with the second order difference equation as it appears in the literature

$$\Delta^2 y(t) + a(t)\Delta y(t) + b(t)y(t+1) = 0, \quad (4.2.2)$$

where  $\Delta y(t) = y(t+1) - y(t)$ .

**Definition.** If  $f_1$  and  $f_2$  are two solutions of the linear  $q$ -difference equation (4.2.1), we define the  $q$ -Wronskian by

$$W_q(f_1(x), f_2(x)) \stackrel{def}{=} f_1(x)D_q f_2(x) - f_2(x)D_q f_1(x). \quad (4.2.3)$$

It is easy to see that the  $q$ -Wronskian tends to the ordinary Wronskian (1.2.3) if  $q \uparrow 1$ .

**Remark 1.** When we apply the definition of the  $q$ -difference operator (2.3.30) to the  $q$ -Wronskian, (4.2.3) can be rewritten in the form

$$W_q(f_1(x), f_2(x)) = \frac{f_1(xq)f_2(x) - f_1(x)f_2(xq)}{x(1-q)} \quad (4.2.4)$$

**Remark 2.** By writing out the definition it is easy to see that the  $q$ -Wronskian has the properties

$$W_q(f(x), c_1g_1(x) + c_2g_2(x)) = c_1W_q(f(x), g_1(x)) + c_2W_q(f(x), g_2(x)), \quad (4.2.5)$$

and

$$W_q(f(x), f(x)) = 0. \quad (4.2.6)$$

A  $q$ -analogue of Abel's theorem (1.2.4) is as follows.

**Theorem 4.1.** *Let  $f_1$  and  $f_2$  be solutions of the  $q$ -difference equation (4.2.1) and let the  $q$ -Wronskian be defined by (4.2.3), then it satisfies the  $q$ -difference equation*

$$D_q W_q(f_1(x), f_2(x)) + a(x)W_q(f_1(x), f_2(x)) = 0. \quad (4.2.7)$$

**Proof.** By the  $q$ -product rule (2.3.34) and by (4.2.3) we have

$$\begin{aligned}
 D_q W_q(f_1(x), f_2(x)) &= D_q(f_1(x)D_q f_2(x) - f_2(x)D_q f_1(x)) \\
 &= f_1(xq)D_q^2 f_2(x) + D_q f_2(x)D_q f_1(x) - f_2(xq)D_q^2 f_1(x) - D_q f_1(x)D_q f_2(x) \\
 &= f_1(xq)(-a(x)D_q f_2(x) - b(x)f_2(xq)) - f_2(xq)(-a(x)D_q f_1(x) - b(x)f_1(xq)) \\
 &= -a(x)(f_1(xq)D_q f_2(x) - f_2(xq)D_q f_1(x)) \\
 &= -a(x)(f_1(x)D_q f_2(x) - f_2(x)D_q f_1(x)) \\
 &= -a(x)W_q(f_1(x), f_2(x)).
 \end{aligned}$$

□

A theory concerning the linear independence of two solutions is as follows.

**Theorem 4.2.** *Let  $f_1$  and  $f_2$  be solutions of the  $q$ -difference equation (4.2.1) and let the  $q$ -Wronskian be defined by (4.2.3). If  $W_q(f_1(x), f_2(x))$  is not identically zero, then the solutions are linearly independent.*

**Proof.** We will prove that the Wronskian is identically zero if the solutions are linearly dependent. Assume that  $f_1(x) = cf_2(x)$ , with  $c \neq 0$ . Then with properties (4.2.5) and (4.2.6) we have

$$W_q(f_1(x), f_2(x)) = W_q(cf_2(x), f_2(x)) = cW_q(f_2(x), f_2(x)) = 0.$$

□

### 4.3 The Hahn-Exton $q$ -Bessel difference equation

**Theorem 4.3.** *The Hahn-Exton  $q$ -Bessel function satisfies the basic difference relation*

$$J_\nu(xq; q) + q^{-\frac{1}{2}\nu}(x^2q - 1 - q^\nu)J_\nu(xq^{\frac{1}{2}}; q) + J_\nu(x; q) = 0. \quad (4.3.1)$$

**Proof.** We start with the recurrence relations (3.2.15), with  $x$  replaced by  $xq^{\frac{1}{2}}$ , and (3.2.16). When we eliminate  $J_{\nu+1}(xq^{\frac{1}{2}}; q)$  we obtain

$$q^{-\frac{1}{2}\nu}(x^2q + 1 - q^\nu)J_\nu(xq^{\frac{1}{2}}; q) + J_\nu(x; q) - J_\nu(xq; q) = 2xq^{\frac{1}{2}(1-\nu)}J_{\nu-1}(xq^{\frac{1}{2}}; q). \quad (4.3.2)$$

Next we use (3.2.21) with  $x$  replaced by  $xq^{\frac{1}{2}}$

$$xq^{\frac{1}{2}(1-\nu)}J_{\nu-1}(xq^{\frac{1}{2}}; q) = q^{-\frac{1}{2}\nu}J_\nu(xq^{\frac{1}{2}}; q) - J_\nu(xq; q). \quad (4.3.3)$$

Substituting (4.3.3) in (4.3.2) we obtain (4.3.1).  $\square$

When we use the  $q$ -difference operator  $D_{q^{\frac{1}{2}}}$ , we can rewrite (4.3.1) as

$$\begin{aligned} q^{\frac{1}{2}}x^2(1-q^{\frac{1}{2}})^2D_{q^{\frac{1}{2}}}^2J_{\nu}(x;q) + x(1-q^{\frac{1}{2}})^2D_{q^{\frac{1}{2}}}J_{\nu}(x;q) + \\ + \left(x^2q^{1-\frac{1}{2}\nu} + (1-q^{\frac{1}{2}\nu})(1-q^{-\frac{1}{2}\nu})\right)J_{\nu}(xq^{\frac{1}{2}};q) = 0. \end{aligned} \quad (4.3.4)$$

So a  $q$ -extension of the Bessel equation (1.2.1) is

$$\begin{aligned} q^{\frac{1}{2}}x^2(1-q^{\frac{1}{2}})^2D_{q^{\frac{1}{2}}}^2y(x) + x(1-q^{\frac{1}{2}})^2D_{q^{\frac{1}{2}}}y(x) + \\ + \left(x^2q^{1-\frac{1}{2}\nu} + (1-q^{\frac{1}{2}\nu})(1-q^{-\frac{1}{2}\nu})\right)y(xq^{\frac{1}{2}}) = 0. \end{aligned} \quad (4.3.5)$$

If in (4.3.5)  $x$  is replaced by  $x(1-q)/2$  and if  $q \uparrow 1$  we find with (3.2.5) the differential equation of Bessel (1.2.1).

A closer look at (4.3.5) shows that  $J_{-\nu}(xq^{-\frac{1}{2}\nu};q)$  also satisfies this  $q$ -difference equation. Using the  $q$ -difference operator  $D_{q^{\frac{1}{2}}}$  we find for  $J_{-\nu}(xq^{-\frac{1}{2}\nu};q)$  the relation

$$J_{-\nu}(xq^{\frac{1}{2}(2-\nu)};q) + q^{-\frac{1}{2}\nu}(x^2q - 1 - q^{\nu})J_{-\nu}(xq^{\frac{1}{2}(1-\nu)};q) + J_{-\nu}(xq^{-\frac{1}{2}\nu};q) = 0. \quad (4.3.6)$$

Now return to the rewritten  $q$ -Wronskian (4.2.4) to obtain a  $q$ -analogue of (1.2.5). We have

$$\begin{aligned} W_{q^{\frac{1}{2}}}\left(J_{\nu}(x;q), J_{-\nu}(xq^{-\frac{1}{2}\nu};q)\right) \\ = \frac{J_{\nu}(xq^{\frac{1}{2}};q)J_{-\nu}(xq^{-\frac{1}{2}\nu};q) - J_{\nu}(x;q)J_{-\nu}(xq^{\frac{1}{2}(1-\nu)};q)}{x(1-q^{\frac{1}{2}})} = \frac{f_{\nu}(x)}{x(1-q^{\frac{1}{2}})}. \end{aligned} \quad (4.3.7)$$

Next consider  $f_{\nu}(xq^{\frac{1}{2}})$ . With (4.3.1) and (4.3.6) we obtain

$$\begin{aligned} f_{\nu}(xq^{\frac{1}{2}}) &= J_{\nu}(xq;q)J_{-\nu}(xq^{\frac{1}{2}(1-\nu)};q) - J_{\nu}(xq^{\frac{1}{2}};q)J_{-\nu}(xq^{\frac{1}{2}(2-\nu)};q) \\ &= \left(\left(q^{-\frac{1}{2}\nu}(1+q^{\nu}-x^2q)\right)J_{\nu}(xq^{\frac{1}{2}};q)\right) - J_{\nu}(x;q)J_{-\nu}(xq^{\frac{1}{2}(1-\nu)};q) + \\ &\quad - \left(\left(q^{-\frac{1}{2}\nu}(1+q^{\nu}-x^2q)\right)J_{-\nu}(xq^{\frac{1}{2}(1-\nu)};q)\right) - J_{-\nu}(xq^{-\frac{1}{2}\nu};q)J_{\nu}(xq^{\frac{1}{2}};q) \\ &= J_{\nu}(xq^{\frac{1}{2}};q)J_{-\nu}(xq^{-\frac{1}{2}\nu};q) - J_{\nu}(x;q)J_{-\nu}(xq^{\frac{1}{2}(1-\nu)};q) = f_{\nu}(x). \end{aligned}$$

Iterating this result gives  $f_{\nu}(x) = f_{\nu}(xq^{\frac{1}{2}}) = \dots = f_{\nu}(xq^{\frac{1}{2}k})$ . The limit  $\lim_{k \rightarrow \infty} f_{\nu}(xq^{\frac{1}{2}k})$  can be calculated by using the series expansions for  $J_{\nu}$  and  $J_{-\nu}$ . We find

$$\lim_{k \rightarrow \infty} f_{\nu}(xq^{\frac{1}{2}k}) = \frac{(q^{\nu+1}, q^{-\nu+1}; q)_{\infty}}{(q, q; q)_{\infty}} \left(q^{\frac{1}{2}\nu(\nu+1)} - q^{\frac{1}{2}\nu(\nu-1)}\right) = -q^{\frac{1}{2}\nu(\nu-1)} \frac{(q^{\nu}, q^{-\nu+1}; q)_{\infty}}{(q, q; q)_{\infty}}.$$

It follows that

$$W_{q^{\frac{1}{2}}} \left( J_{\nu}(x; q), J_{-\nu}(xq^{-\frac{1}{2}\nu}; q) \right) = \frac{-q^{\frac{1}{2}\nu(\nu-1)}(q^{\nu}, q^{-\nu+1}; q)_{\infty}}{x(1-q^{\frac{1}{2}})(q, q; q)_{\infty}} = \frac{-q^{\frac{1}{2}\nu(\nu-1)}(1+q^{\frac{1}{2}})}{x\Gamma_q(\nu)\Gamma_q(1-\nu)}. \quad (4.3.8)$$

Note that the  $q$ -Wronskian is never zero if  $\nu \notin \mathbb{Z}$ . By theorem 4.2 this proves that the solutions  $J_{\nu}(x; q)$  and  $J_{-\nu}(xq^{-\frac{1}{2}\nu}; q)$  are linearly independent. If  $\nu \in \mathbb{Z}$  however, the  $q$ -Wronskian is identically zero. In that case we find with (3.2.12) a clear linear dependence. These properties match with the  $q = 1$  case.

If we replace in (4.3.8)  $x$  by  $x(1-q)/2$  and if we let  $q \uparrow 1$  and if we use the well known relation

$$\frac{1}{\Gamma(\nu)\Gamma(1-\nu)} = \frac{\sin(\pi\nu)}{\pi}, \quad (4.3.9)$$

we obtain with the limits (2.3.29) and (3.2.5) the Wronskian (1.2.5).

Our next aim is to look for a second solution of the Hahn-Exton  $q$ -Bessel equation that is a  $q$ -extension of  $Y_{\nu}(x)$  given by (1.2.7). Let us consider the function  $Y_{\nu}(x; q)$  defined for  $\nu \notin \mathbb{Z}$  by

$$Y_{\nu}(x; q) = \frac{\Gamma_q(\nu)\Gamma_q(1-\nu)q^{\frac{1}{2}\nu(\nu+1)}}{\Gamma_q(-\nu+\frac{1}{2})\Gamma_q(\nu+\frac{1}{2})} J_{\nu}(x; q) - \frac{\Gamma_q(\nu)\Gamma_q(1-\nu)}{\Gamma_q(\frac{1}{2})\Gamma_q(\frac{1}{2})} J_{-\nu}(xq^{-\frac{1}{2}\nu}; q). \quad (4.3.10)$$

For  $n \in \mathbb{Z}$  we define

$$Y_n(x; q) = \lim_{\nu \rightarrow n} Y_{\nu}(x; q). \quad (4.3.11)$$

It is easy to verify that (4.3.10) is a  $q$ -analogue of (1.2.7). The limits (3.2.5) and (2.3.29) give

$$\lim_{q \uparrow 1} Y_{\nu}(x(1-q)/2; q) = \frac{\Gamma(\nu)\Gamma(1-\nu)}{\Gamma(-\nu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})} J_{\nu}(x) - \frac{\Gamma(\nu)\Gamma(1-\nu)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})} J_{-\nu}(x).$$

Using relation (4.3.9) and the identity  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$  we find (1.2.7).

The  $q$ -extension of (1.2.9) can now be derived in an easy way. With the properties (4.2.5) and (4.2.6) we find with the  $q$ -Wronskian (4.3.8)

$$\begin{aligned} & W_{q^{\frac{1}{2}}} (J_{\nu}(x; q), Y_{\nu}(x; q)) \\ &= W_{q^{\frac{1}{2}}} \left( J_{\nu}(x; q), \frac{\Gamma_q(\nu)\Gamma_q(1-\nu)q^{\frac{1}{2}\nu(\nu+1)}}{\Gamma_q(-\nu+\frac{1}{2})\Gamma_q(\nu+\frac{1}{2})} J_{\nu}(x; q) - \frac{\Gamma_q(\nu)\Gamma_q(1-\nu)}{\Gamma_q(\frac{1}{2})\Gamma_q(\frac{1}{2})} J_{-\nu}(xq^{-\frac{1}{2}\nu}; q) \right) \\ &= -\frac{\Gamma_q(\nu)\Gamma_q(1-\nu)}{\Gamma_q(\frac{1}{2})\Gamma_q(\frac{1}{2})} W_{q^{\frac{1}{2}}} (J_{\nu}(x; q), J_{-\nu}(xq^{-\frac{1}{2}\nu}; q)) = \frac{q^{\frac{1}{2}\nu(\nu-1)}(1+q^{\frac{1}{2}})}{x\Gamma_q(\frac{1}{2})\Gamma_q(\frac{1}{2})}. \end{aligned} \quad (4.3.12)$$

Note that this  $q$ -Wronskian is never zero for all  $\nu$ . By theorem 4.2 this means that  $J_{\nu}(x; q)$  and  $Y_{\nu}(x; q)$  are linearly independent for all  $\nu$ . The limit  $q \uparrow 1$  gives with (2.3.29) the classical result (1.2.9).

# Chapter 5

## q-Analogues of the Fourier and Hankel transforms

### 5.1 Introduction

In this chapter we shall discuss some  $q$ -extensions of integral transforms. First we will derive a  $q$ -analogue of the Hankel transform. As a special case we will discuss  $q$ -extensions of the Fourier-cosine and the Fourier-sine transforms.

The Hankel transform of order  $\alpha$  of a function  $f$ , denoted by  $\bar{f}$ , is defined by

$$\bar{f}(t) = \int_0^\infty J_\alpha(xt) f(x) x dx. \quad (5.1.1)$$

If we multiply both sides of (5.1.1) by  $J_\alpha(yt)t$  and integrate from  $t = 0$  to  $\infty$  we obtain

$$\int_0^\infty J_\alpha(yt) \bar{f}(t) t dt = \int_0^\infty J_\alpha(yt) t \int_0^\infty J_\alpha(xt) f(x) x dx dt. \quad (5.1.2)$$

The integral on the left hand side of (5.1.2) is (for suitable functions  $f$ ) equal to  $f(y)$  by the Hankel inversion theorem (see [44]). The resulting double integral is called the Hankel Fourier-Bessel integral:

$$f(y) = \int_0^\infty J_\alpha(yt) \left( \int_0^\infty J_\alpha(xt) f(x) x dx \right) t dt. \quad (5.1.3)$$

It is also common to write the integral (5.1.3) as the transform pair

$$\begin{cases} g(t) = \int_0^\infty J_\alpha(yt) f(y) y dy, \\ f(y) = \int_0^\infty J_\alpha(yt) g(t) t dt. \end{cases} \quad (5.1.4)$$

In the next section we will give a  $q$ -extension of the Hankel Fourier-Bessel integral.

## 5.2 The $q$ -Hankel transform

In this section we will show that the orthogonality relation (3.2.27) is a  $q$ -analogue of Hankel's Fourier-Bessel integral (5.1.3). First replace in (3.2.27)  $q$  by  $q^2$  to symplify the notations and then replace  $x$  by  $q^\alpha$ . This gives for  $\Re(\alpha) > -1$

$$\sum_{k=-\infty}^{\infty} q^{(\alpha+1)(k+n)} \frac{(q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty} {}_1\Phi_1 \left( \begin{matrix} 0 \\ q^{2\alpha+2} \end{matrix} \middle| q^2, q^{2n+2k+2} \right) \times \\ \times q^{(\alpha+1)(k+m)} \frac{(q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty} {}_1\Phi_1 \left( \begin{matrix} 0 \\ q^{2\alpha+2} \end{matrix} \middle| q^2, q^{2m+2k+2} \right) = \delta_{m,n}. \quad (5.2.1)$$

We can rewrite (5.2.1) as the transform pair

$$\begin{cases} g(q^n) = \sum_{k=-\infty}^{\infty} q^{(\alpha+1)(k+n)} \frac{(q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty} {}_1\Phi_1 \left( \begin{matrix} 0 \\ q^{2\alpha+2} \end{matrix} \middle| q^2, q^{2n+2k+2} \right) f(q^k), \\ f(q^k) = \sum_{n=-\infty}^{\infty} q^{(\alpha+1)(k+n)} \frac{(q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty} {}_1\Phi_1 \left( \begin{matrix} 0 \\ q^{2\alpha+2} \end{matrix} \middle| q^2, q^{2n+2k+2} \right) g(q^n), \end{cases}$$

where  $f, g$  are  $L^2$ -functions on the set  $\{q^k | k \in \mathbb{Z}\}$  with respect to the counting measure. When we insert the  $J_\alpha(x; q)$  notation for the Hahn-Exton  $q$ -Bessel function and when we replace  $f(q^k)$  and  $g(q^n)$  by  $q^k f(q^k)$  and  $q^n g(q^n)$  (this implies that  $xf(x)$  and  $xy(x)$  also have to be  $L^2$ -functions on the set  $\{q^k | k \in \mathbb{Z}\}$ ) respectively, this becomes

$$\begin{cases} g(q^n) = \sum_{k=-\infty}^{\infty} q^{2k} J_\alpha(q^{k+n}; q^2) f(q^k), \\ f(q^k) = \sum_{n=-\infty}^{\infty} q^{2n} J_\alpha(q^{k+n}; q^2) g(q^n). \end{cases} \quad (5.2.2)$$

When we let  $q \uparrow 1$  under the side condition that  $\frac{\log(1-q)}{\log q} \in 2\mathbb{Z}$  we can replace  $q^k$  and  $q^n$  in (5.2.2) by  $(1-q)^{\frac{1}{2}} q^k$  and  $(1-q)^{\frac{1}{2}} q^n$  respectively, and next  $f((1-q)^{\frac{1}{2}} q^k)$  and  $g((1-q)^{\frac{1}{2}} q^n)$  by  $f(q^k)$  and  $g(q^n)$  respectively. With the  $q$ -integral notation (2.3.38), (5.2.2) takes the form

$$\begin{cases} g(\lambda) = \int_0^\infty f(x) J_\alpha((1-q)\lambda x; q^2) x \, d_q x, \\ f(x) = \int_0^\infty g(\lambda) J_\alpha((1-q)\lambda x; q^2) \lambda \, d_q \lambda, \end{cases}$$

where  $\lambda$  in the first identity and  $x$  in the second identity take the values  $q^n, n \in \mathbb{Z}$ . For  $q \uparrow 1$  we obtain, at least formally, the Hankel transform pair (5.1.4)

$$\begin{cases} g(\lambda) = \int_0^\infty f(x) J_\alpha(\lambda x) x \, dx, \\ f(x) = \int_0^\infty g(\lambda) J_\alpha(\lambda x) \lambda \, d\lambda. \end{cases}$$

### 5.3 $q$ -Analogues of the Fourier-sine and the Fourier-cosine transform

It is well-known that specialization of the parameter  $\nu$  in the Bessel function (1.2.2) to  $\nu = \frac{1}{2}$  or  $\nu = -\frac{1}{2}$ , leads to a factor times the sine-function and the cosine-function respectively. We have

$$\begin{cases} J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \\ J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x. \end{cases}$$

When we take the same specialization in the Hahn-Exton  $q$ -Bessel function (3.2.4) we obtain  $q$ -analogues of the sine-function and cosine-function. First we replace in (3.2.4)  $q$  by  $q^2$  to simplify the notations. Then we specialize  $\nu$  to  $\pm\frac{1}{2}$ . We obtain with (A.15)

$$\begin{aligned} \cos(x; q^2) &\stackrel{def}{=} {}_1\Phi_1 \left( \begin{array}{c|cc} 0 \\ q \end{array} \middle| q^2, x^2 q^2 \right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)} x^{2k}}{(q; q)_{2k}} \end{aligned} \quad (5.3.1)$$

$$= \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} x^{\frac{1}{2}} J_{-\frac{1}{2}}(x; q^2) \quad (5.3.2)$$

and

$$\begin{aligned} \sin(x; q^2) &\stackrel{def}{=} \frac{x}{1-q} {}_1\Phi_1 \left( \begin{array}{c|cc} 0 \\ q^3 \end{array} \middle| q^2, x^2 q^2 \right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)} x^{2k+1}}{(q; q)_{2k+1}} \end{aligned} \quad (5.3.3)$$

$$= \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} x^{\frac{1}{2}} J_{\frac{1}{2}}(x; q^2) \quad (5.3.4)$$

The functions introduced above should not be confused with the functions  $\cos_q(x)$  and  $\sin_q(x)$  considered in [12]. Clearly we have the formal termwise limits

$$\lim_{q \uparrow 1} \cos((1-q)x; q^2) = \cos x, \quad (5.3.5)$$

and

$$\lim_{q \uparrow 1} \sin((1-q)x; q^2) = \sin x. \quad (5.3.6)$$

Now we can derive  $q$ -analogues of the Fourier-cosine and Fourier-sine transform. When we substitute (5.3.2) and (5.3.4) in (5.2.2) and replace  $f(q^k)$  and  $g(q^n)$  by  $q^{-\frac{1}{2}k} f(q^k)$  and  $q^{-\frac{1}{2}n} g(q^n)$  respectively, we obtain

**Theorem 5.1.** A  $q$ -analogue of the Fourier-cosine and the Fourier-sine transform and their inverse transformations can be represented by the transform pairs

$$\begin{cases} g(q^n) = \frac{(q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{k=-\infty}^{\infty} q^k \cos(q^{k+n}; q^2) f(q^k), \\ f(q^k) = \frac{(q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} q^n \cos(q^{k+n}; q^2) g(q^n), \end{cases} \quad (5.3.7)$$

and

$$\begin{cases} g(q^n) = \frac{(q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{k=-\infty}^{\infty} q^k \sin(q^{k+n}; q^2) f(q^k), \\ f(q^k) = \frac{(q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} q^n \sin(q^{k+n}; q^2) g(q^n), \end{cases} \quad (5.3.8)$$

respectively.

We will show that (5.3.7) and (5.3.8) are  $q$ -analogues of the Fourier-cosine and Fourier-sine transform. Under the side condition that  $\frac{\log(1-q)}{\log q} \in 2\mathbb{Z}$ , we replace  $q^k$  and  $q^n$  by  $(1-q)^{\frac{1}{2}}q^k$  and  $(1-q)^{\frac{1}{2}}q^n$  respectively, and next  $f((1-q)^{\frac{1}{2}}q^k)$  and  $g((1-q)^{\frac{1}{2}}q^n)$  by  $f(q^k)$  and  $g(q^n)$  respectively. Then, with the  $q$ -integral notation (2.3.38) and the  $q$ -Gamma notation (2.3.28), (5.3.7) and (5.3.8) take the form

$$\begin{cases} g(\lambda) = \frac{(1+q)^{\frac{1}{2}}}{\Gamma_{q^2}(\frac{1}{2})} \int_0^\infty f(x) \cos((1-q)\lambda x; q^2) d_q x, \\ f(x) = \frac{(1+q)^{\frac{1}{2}}}{\Gamma_{q^2}(\frac{1}{2})} \int_0^\infty g(\lambda) \cos((1-q)\lambda x; q^2) d_q \lambda, \end{cases}$$

and

$$\begin{cases} g(\lambda) = \frac{(1+q)^{\frac{1}{2}}}{\Gamma_{q^2}(\frac{1}{2})} \int_0^\infty f(x) \sin((1-q)\lambda x; q^2) d_q x, \\ f(x) = \frac{(1+q)^{\frac{1}{2}}}{\Gamma_{q^2}(\frac{1}{2})} \int_0^\infty g(\lambda) \sin((1-q)\lambda x; q^2) d_q \lambda, \end{cases}$$

respectively, where  $\lambda$  and  $x$  take the values  $q^n, n \in \mathbb{Z}$ . Formally, as  $q \uparrow 1$ , we obtain the

classical Fourier pairs

$$\begin{cases} g(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\lambda x) dx, \\ f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(\lambda) \cos(\lambda x) d\lambda, \end{cases}$$

and

$$\begin{cases} g(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\lambda x) dx, \\ f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(\lambda) \sin(\lambda x) d\lambda, \end{cases}$$

respectively.

With the  $q$ -difference operator (2.3.30) we obtain from (5.3.1) and (5.3.3) that

$$\begin{aligned} (1-q)D_q \cos(x; q^2) &= -q \sin(qx; q^2) \\ (1-q)D_q \sin(x; q^2) &= \cos(x; q^2) \end{aligned}$$

Hence we can find the second order  $q$ -difference equations

$$(1-q)^2(D_q^2 f)(q^{-1}x) = \begin{cases} -q^2 \lambda^2 f(x) & \text{if } f(x) = \cos(\lambda x; q^2), \\ -q \lambda^2 f(x) & \text{if } f(x) = \sin(\lambda x; q^2), \end{cases}$$

So the two systems of functions  $x \mapsto \cos(q^n x)$  ( $n \in \mathbb{Z}$ ) and  $x \mapsto \sin(q^n x)$  ( $n \in \mathbb{Z}$ ) have different eigenvalues with respect to the operator which sends  $f$  to the function

$$x \mapsto (1-q)^2(D_q^2 f)(q^{-1}x) = qx^{-2} (qf(q^{-1}x) - (1+q)f(x) + f(qx)).$$

Observe that the  $q$ -deformation of  $d^2/dx^2$  considered above yields a symmetry breaking. The two-dimensional eigenspaces of  $d^2/dx^2$  are broken into one-dimensional eigenspaces associated with different eigenvalues. Therefore it does not seem very useful to consider a  $q$ -exponential built from the functions defined by (5.3.1) and (5.3.3). A linear combination  $f(x)$  of  $\cos(\lambda x; q^2)$  and  $\sin(\lambda x; q^2)$  will no longer satisfy an eigenfunction equation

$$(1-q)^2(D_q^2 f)(q^{-1}x) = \mu f(x).$$

# Chapter 6

## Addition formulas

### 6.1 Introduction

In this chapter we will derive several addition formulas for the Hahn-Exton  $q$ -Bessel function. First we find a  $q$ -analogue of Graf's addition formula (1.5.1) by expanding a quotient of  $q$ -shifted factorials as a Laurent series in two different ways. A special case of this addition formula is a  $q$ -analogue of Neumann's addition formula for Bessel functions  $J_0$ .

The second addition formula that we will derive is also a  $q$ -extension of Graf's addition formula. This formula has been originally discovered by a quantum group theoretical approach. In this chapter a straightforward and analytic proof is given.

The last addition formula that we will consider in this chapter is a  $q$ -analogue of Gegenbauer's addition formula (1.5.3).

We will also obtain some product formulas and integrals involving the Hahn-Exton  $q$ -Bessel function as additional results during the derivation of the addition formulas.

### 6.2 A $q$ -extension of Graf's addition formula

In this section we will generalize the considerations that led to the orthogonality relations (3.2.25) and (3.2.27). The resulting formula will turn out to be a  $q$ -analogue of Graf's addition formula (1.5.1) and, at the same time, of the discontinuous integral of Weber and Schafheitlin (1.6.3).

We will start by expanding the expression

$$\frac{(xs^{-1}t; q)_\infty (yt^{-1}; q)_\infty}{(yt; q)_\infty (xst^{-1}; q)_\infty} \quad (6.2.1)$$

as a Laurent series in  $t$  with ( $|sx| < |t| < |y|^{-1}$ ) in two different ways. On the one hand, (6.2.1) can be expanded by twofold substitution of the  $q$ -binomial formula (2.3.9) as

$$\begin{aligned} {}_1\Phi_0 \left( \begin{matrix} s^{-1}xy^{-1} \\ - \end{matrix} \middle| q, yt \right) {}_1\Phi_0 \left( \begin{matrix} s^{-1}yx^{-1} \\ - \end{matrix} \middle| q, xst^{-1} \right) \\ = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(s^{-1}yx^{-1}; q)_k (s^{-1}xy^{-1}; q)_m}{(q; q)_k (q; q)_m} s^k x^k y^m t^{m-k} \\ = \sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(s^{-1}yx^{-1}; q)_k (s^{-1}xy^{-1}; q)_m (q^{m+1}; q)_{\infty}}{(q; q)_k (q; q)_{\infty}} s^k x^k y^m t^{m-k} \\ = \sum_{n=-\infty}^{\infty} t^n y^n \sum_{k=0}^{\infty} \frac{(s^{-1}yx^{-1}; q)_k (s^{-1}xy^{-1}; q)_{n+k} (q^{n+k+1}; q)_{\infty}}{(q; q)_k (q; q)_{\infty}} (sxy)^k, \end{aligned}$$

where we used identity (A.6), substituted  $m = n + k$  and changed the order of summation. Since the inner sum in the last part can be expressed in terms of a  ${}_2\Phi_1$  series after applying identities (A.6) and (A.13), we obtain the identity

$$\begin{aligned} \frac{(xs^{-1}t; q)_{\infty} (yt^{-1}; q)_{\infty}}{(yt; q)_{\infty} (xst^{-1}; q)_{\infty}} = \sum_{n=-\infty}^{\infty} t^n y^n \frac{(s^{-1}xy^{-1}, q^{n+1}; q)_{\infty}}{(q^n s^{-1}xy^{-1}, q; q)_{\infty}} \times \\ {}_2\Phi_1 \left( \begin{matrix} q^n s^{-1}xy^{-1}, s^{-1}yx^{-1} \\ q^{n+1} \end{matrix} \middle| q, sxy \right), \quad |sx| < |t| < |y|^{-1}. \quad (6.2.2) \end{aligned}$$

**Remark 1.** This identity reduces to (3.2.24), with  $x$  and  $t$  replaced by  $yq^{-\frac{1}{2}}$  and  $tq^{\frac{1}{2}}$  respectively, in the special case  $x = 0$ . It also reduces to (3.2.24), with  $x$  and  $t$  replaced by  $xq^{-\frac{1}{2}}$  and  $st^{-1}q^{\frac{1}{2}}$  respectively, in the special case  $y = 0$ . However in the latter case we have to apply (3.2.11) after we have taken the limit  $y \rightarrow 0$ .

**Remark 2.** Formula (6.2.2) is a  $q$ -analogue of (1.4.1) with  $x$  and  $t$  replaced by  $2(y - s^{-1}x)^{\frac{1}{2}}(y - sx)^{\frac{1}{2}}$  and  $t(y - s^{-1}x)^{\frac{1}{2}}(y - sx)^{-\frac{1}{2}}$  respectively.

**Remark 3.** For  $n < 0$  we use an interpretation of the  ${}_2\Phi_1$  similar to our convention in subsection 3.2.2. The analogue of (3.2.11) becomes

$$\begin{aligned} \frac{(s^{-1}yx^{-1}, q^{1-n}; q)_{\infty}}{(q^{-n}s^{-1}yx^{-1}, q; q)_{\infty}} {}_2\Phi_1 \left( \begin{matrix} q^{-n}s^{-1}yx^{-1}, s^{-1}xy^{-1} \\ q^{1-n} \end{matrix} \middle| q, sxy \right) \\ = (sxy)^n \frac{(s^{-1}xy^{-1}, q^{1+n}; q)_{\infty}}{(q^n s^{-1}xy^{-1}, q; q)_{\infty}} {}_2\Phi_1 \left( \begin{matrix} q^n s^{-1}xy^{-1}, s^{-1}yx^{-1} \\ q^{1+n} \end{matrix} \middle| q, sxy \right). \quad (6.2.3) \end{aligned}$$

**Proof.** Assume that  $n > 0$ . Since

$$\frac{(q^{1-n}, q)_{\infty}}{(q^{1-n}; q)_k} = (q^{1-n+k}; q)_{\infty} = 0 \text{ if } k \leq n - 1,$$

the first  $n$  terms of the series vanish, so the summation starts with  $k = n$ . We obtain with (A.13)

$$\begin{aligned}
& \frac{(s^{-1}yx^{-1}, q^{1-n}; q)_\infty}{(q^{-n}s^{-1}yx^{-1}, q; q)_\infty} {}_2\Phi_1 \left( \begin{array}{c} q^{-n}s^{-1}yx^{-1}, s^{-1}xy^{-1} \\ q^{1-n} \end{array} \middle| q, sxy \right) \\
&= \frac{(s^{-1}yx^{-1}; q)_\infty}{(q^{-n}s^{-1}yx^{-1}, q; q)_\infty} \sum_{k=n}^{\infty} \frac{(q^{-n}s^{-1}yx^{-1}; q)_k (s^{-1}xy^{-1}; q)_k (q^{1-n+k}; q)_\infty (sxy)^k}{(q; q)_k} \\
&= \frac{(s^{-1}yx^{-1}; q)_\infty}{(q^{-n}s^{-1}yx^{-1}, q; q)_\infty} \sum_{k=0}^{\infty} \frac{(q^{-n}s^{-1}yx^{-1}; q)_{k+n} (s^{-1}xy^{-1}; q)_{k+n} (q^{1+k}; q)_\infty (sxy)^{k+n}}{(q; q)_{k+n}} \\
&= \frac{(s^{-1}yx^{-1}, q^{1+n}; q)_\infty (q^{-n}s^{-1}yx^{-1}, s^{-1}xy^{-1}; q)_n}{(q^{-n}s^{-1}yx^{-1}, q; q)_\infty} (sxy)^n \times \\
&\quad \times \sum_{k=0}^{\infty} \frac{(s^{-1}yx^{-1}; q)_k (s^{-1}xy^{-1}q^n; q)_k (sxy)^k}{(q^{1+n}; q)_k (q; q)_k}.
\end{aligned}$$

When we apply identity (A.6) we find (6.2.3) The case  $n < 0$  follows easily from the case  $n > 0$  of (6.2.3).  $\square$

On the other hand we expand (6.2.1) by twofold substitution of the generating function (3.2.24). The first time  $x$  and  $t$  are replaced by  $xq^{-\frac{1}{2}}$  and  $st^{-1}q^{\frac{1}{2}}$  respectively, and the second time  $x$  and  $t$  are replaced by  $yq^{-\frac{1}{2}}$  and  $tq^{\frac{1}{2}}$  respectively. This yields

$$\begin{aligned}
& \frac{(xs^{-1}t; q)_\infty (yt^{-1}; q)_\infty}{(yt; q)_\infty (xst^{-1}; q)_\infty} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} t^n s^k \times \\
& \quad \times y^{n+k} \frac{(q^{n+k+1}; q)_\infty}{(q; q)_\infty} {}_1\Phi_1 \left( \begin{array}{c} 0 \\ q^{n+k+1} \end{array} \middle| q, y^2 \right) x^k \frac{(q^{k+1}; q)_\infty}{(q; q)_\infty} {}_1\Phi_1 \left( \begin{array}{c} 0 \\ q^{k+1} \end{array} \middle| q, x^2 \right).
\end{aligned} \tag{6.2.4}$$

When we compare coefficients of equal powers of  $t$  in (6.2.2) and (6.2.4), we obtain

**Theorem 6.1.** For  $|sxy| < 1$  a  $q$ -analogue of Graf's addition formula is

$$\begin{aligned}
& y^n \frac{(s^{-1}xy^{-1}, q^{n+1}; q)_\infty}{(q^n s^{-1}xy^{-1}, q; q)_\infty} {}_2\Phi_1 \left( \begin{array}{c} q^n s^{-1}xy^{-1}, s^{-1}yx^{-1} \\ q^{n+1} \end{array} \middle| q, sxy \right) \\
&= \sum_{k=-\infty}^{\infty} s^k y^{n+k} \frac{(q^{n+k+1}; q)_\infty}{(q; q)_\infty} {}_1\Phi_1 \left( \begin{array}{c} 0 \\ q^{n+k+1} \end{array} \middle| q, y^2 \right) x^k \frac{(q^{k+1}; q)_\infty}{(q; q)_\infty} {}_1\Phi_1 \left( \begin{array}{c} 0 \\ q^{k+1} \end{array} \middle| q, x^2 \right) \\
&= \sum_{k=-\infty}^{\infty} s^k q^{\frac{1}{2}(n+k)} J_{n+k}(yq^{-\frac{1}{2}}; q) q^{\frac{1}{2}k} J_k(xq^{-\frac{1}{2}}; q).
\end{aligned} \tag{6.2.5}$$

To show that (6.2.5) is a  $q$ -analogue of Graf's addition formula we replace  $x$  and  $y$  by  $x(1-q)/2$  and  $y(1-q)/2$  respectively and let  $q \uparrow 1$ . We obtain

$$\left(\frac{y-s^{-1}x}{y-sx}\right)^{\frac{1}{2}n} J_n\left(\sqrt{(y-s^{-1}x)(y-sx)}\right) = \sum_{k=-\infty}^{\infty} s^k J_{n+k}(y) J_k(x). \quad (6.2.6)$$

The special case  $n = 0$  of (6.2.5) is a  $q$ -analogue of Neumann's addition formula (1.5.2) for Bessel functions  $J_0$ . In the special case  $x = y$  and  $s = 1$  the left hand side of (6.2.5) becomes  $y^n \delta_{n,0}$  so then (6.2.5) reduces to the orthogonality relation (3.2.25).

When we apply the transformation formulas (6.2.3) and (2.3.19) to the left hand side and the symmetry relation (2.3.22) to the right hand side of (6.2.5), and replace  $n$  by  $n-m$  and next  $k$  by  $k+m$  then we obtain an equivalent identity:

**Theorem 6.2.** For  $|sxy| < 1$  we have

$$\begin{aligned} & s^{-m} y^{n-m} \frac{(s^{-1}xy^{-1}, y^2; q)_\infty}{(sxy, q; q)_\infty} {}_2\Phi_1\left(\begin{array}{c} qsx^{-1}y, sxy \\ y^2 \end{array} \middle| q, q^{n-m}s^{-1}xy^{-1}\right) \\ &= s^{-n} x^{m-n} \frac{(s^{-1}yx^{-1}, x^2; q)_\infty}{(sxy, q; q)_\infty} {}_2\Phi_1\left(\begin{array}{c} qsxy^{-1}, sxy \\ x^2 \end{array} \middle| q, q^{m-n}s^{-1}yx^{-1}\right) \\ &= \sum_{k=-\infty}^{\infty} s^k y^{n+k} \frac{(y^2; q)_\infty}{(q; q)_\infty} {}_1\Phi_1\left(\begin{array}{c} 0 \\ y^2 \end{array} \middle| q, q^{n+k+1}\right) x^{m+k} \frac{(x^2; q)_\infty}{(q; q)_\infty} {}_1\Phi_1\left(\begin{array}{c} 0 \\ x^2 \end{array} \middle| q, q^{m+k+1}\right). \end{aligned} \quad (6.2.7)$$

Let us look for the classical analogue of formula (6.2.7). Replace in (6.2.7) first  $q$  by  $q^2$  to make the notations easier. Next replace  $x, y, s$  by  $q^{\alpha+1}, q^{\beta+1}, q^{-\gamma-1}$  respectively. Then formula (6.2.7) can be rewritten as

$$\begin{aligned} & \sum_{k=-\infty}^{\infty} q^{-k(\gamma+1)} q^{(\beta+1)(n+k)} \frac{(q^{2\beta+2}; q^2)_\infty}{(q^2; q^2)_\infty} {}_1\Phi_1\left(\begin{array}{c} 0 \\ q^{2\beta+2} \end{array} \middle| q^2, q^{2n+2k+2}\right) \times \\ & \quad \times q^{(\alpha+1)(m+k)} \frac{(q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty} {}_1\Phi_1\left(\begin{array}{c} 0 \\ q^{2\alpha+2} \end{array} \middle| q^2, q^{2m+2k+2}\right) \\ &= \sum_{k=-\infty}^{\infty} q^{k(1-\gamma)} q^{n+m} J_\alpha(q^{m+k}; q^2) J_\beta(q^{n+k}; q^2) = \\ &= \begin{cases} q^{n(\beta+1)} q^{m(\gamma-\beta)} \frac{(q^{\alpha-\beta+\gamma+1}, q^{2\beta+2}; q^2)_\infty}{(q^{\alpha+\beta-\gamma+1}, q^2; q^2)_\infty} {}_2\Phi_1\left(\begin{array}{c} q^{\beta-\alpha-\gamma+1}, q^{\alpha+\beta-\gamma+1} \\ q^{2\beta+2} \end{array} \middle| q^2, q^{2n-2m+\alpha-\beta+\gamma+1}\right), \\ q^{m(\alpha+1)} q^{n(\gamma-\alpha)} \frac{(q^{\beta-\alpha+\gamma+1}, q^{2\alpha+2}; q^2)_\infty}{(q^{\alpha+\beta-\gamma+1}, q^2; q^2)_\infty} {}_2\Phi_1\left(\begin{array}{c} q^{\alpha-\beta-\gamma+1}, q^{\alpha+\beta-\gamma+1} \\ q^{2\alpha+2} \end{array} \middle| q^2, q^{2m-2n+\beta-\alpha+\gamma+1}\right), \end{cases} \end{aligned} \quad (6.2.8)$$

where  $\Re(\alpha + \beta - \gamma + 1) > 0$ . Now replace  $q^k$  by  $q^k(1-q)$  (with  $\frac{\log(1-q)}{\log q} \in \mathbb{Z}$ ) and let  $m, n$  depend on  $q$  such that, as  $q \uparrow 1$ ,  $q^m$  tends to  $a$  and  $q^n$  tends to  $b$ . The left hand side of

(6.2.8) can then be rewritten in the  $q$ -integral notation (see (2.3.38))

$$\frac{q^{n+m}}{(1-q)^\gamma} \int_0^\infty x^{-\gamma} J_\alpha(x(1-q)q^m; q^2) J_\beta(x(1-q)q^n; q^2) d_q x.$$

Depending on whether  $b < a$  or  $b > a$  we make the formal limit transition  $q \uparrow 1$ . Then, for  $\Re(\alpha + \beta - \gamma + 1) > 0, \Re(\gamma) > -1$ , we obtain the discontinuous integral of Weber and Schafheitlin (1.6.3)

$$\begin{aligned} & 2^\gamma \int_0^\infty x^{-\gamma} J_\alpha(ax) J_\beta(bx) dx \\ = & \begin{cases} \frac{a^{\gamma-\beta-1} b^\beta \Gamma(\frac{1}{2}(\alpha + \beta - \gamma + 1))}{\Gamma(\frac{1}{2}(\alpha - \beta + \gamma + 1)) \Gamma(\beta + 1)} {}_2F_1 \left( \begin{array}{c} \frac{1}{2}(\beta - \alpha - \gamma + 1), \frac{1}{2}(\alpha + \beta - \gamma + 1) \\ \beta + 1 \end{array} \middle| \frac{b^2}{a^2} \right), & b < a, \\ \frac{a^\alpha b^{\gamma-\alpha-1} \Gamma(\frac{1}{2}(\alpha + \beta - \gamma + 1))}{\Gamma(\frac{1}{2}(\beta - \alpha + \gamma + 1)) \Gamma(\alpha + 1)} {}_2F_1 \left( \begin{array}{c} \frac{1}{2}(\alpha - \beta - \gamma + 1), \frac{1}{2}(\alpha + \beta - \gamma + 1) \\ \alpha + 1 \end{array} \middle| \frac{a^2}{b^2} \right), & a < b. \end{cases} \end{aligned} \quad (6.2.9)$$

Note that the two analytic expressions at the right hand side of (6.2.9) are no longer equal, as they were in the  $q$ -case.

### 6.3 A second $q$ -extension of Graf's addition formula

The main objective in this section is to prove the following theorem.

**Theorem 6.3.** *For  $x, y, z \in \mathbb{N}$ ,  $n \in \mathbb{Z}$  and  $R > 0$  we have the addition formula*

$$\begin{aligned} & (-1)^n q^{\frac{1}{2}n} J_{x-n}(q^{\frac{1}{2}z}; q) J_n(Rq^{\frac{1}{2}(z+y+n)}; q) \\ = & \sum_{k=-\infty}^{\infty} q^k J_k(Rq^{\frac{1}{2}(x+y)}; q) J_{k-n}(Rq^{\frac{1}{2}y}; q) J_x(q^{\frac{1}{2}(z+k)}; q). \end{aligned} \quad (6.3.1)$$

This addition formula has originally been discovered by Koelink [30] using the interpretation of the Hahn-Exton  $q$ -Bessel functions on the quantum group of plane motions as established by Vaksman and Korogodskii [49]. The derivation of (6.3.1) is similar to the group theoretic derivation of Graf's addition formula for the Bessel function (see [52]). So this addition formula should be a  $q$ -analogue of Graf's addition formula (1.5.1), but it is not clear how to obtain (1.5.1) from (6.3.1) as  $q \uparrow 1$ . Van Assche and Koornwinder [50] solved a similar problem for the addition formula for the little  $q$ -Legendre polynomials.

Koelink's method of proof is analogous to the quantum group theoretic proof of the addition formula for little  $q$ -Legendre polynomials given by Koornwinder [34], but the proof

is formal. The special case  $n = 0, R = q^{-\frac{1}{2}}$  of (6.3.1) can be found at least formally from Koornwinder's addition formula for little  $q$ -Legendre polynomials [34] by taking limits term by term.

In this section we will give a straightforward and analytic proof of theorem 6.3. However, in order to derive the addition formula (6.3.1), we shall use neither limit transitions nor quantum group techniques, but we shall only use summation and transformation formulas for the basic hypergeometric series. The first step in the derivation of (6.3.1) is the proof of the following theorem

**Theorem 6.4.** *For  $m, n \in \mathbb{Z}$ ,  $x, y \in \mathbb{N}$  and  $R > 0$  we have the product formula*

$$\begin{aligned} & (-1)^n q^{-\frac{1}{2}n} J_{m-n}(Rq^{\frac{1}{2}y}; q) J_m(Rq^{\frac{1}{2}(x+y)}; q) \\ &= \sum_{z=-\infty}^{\infty} q^z J_x(q^{\frac{1}{2}(z+m)}; q) J_{x-n}(q^{\frac{1}{2}z}; q) J_n(Rq^{\frac{1}{2}(z+n+y)}; q). \end{aligned} \quad (6.3.2)$$

**Proof.** Starting with a constant times the left hand side of (6.3.2) we have with (3.2.12)

$$\begin{aligned} & (-1)^n q^{-\frac{1}{2}n(n+1)} J_{m-n}(Rq^{\frac{1}{2}y}; q) J_m(Rq^{\frac{1}{2}(x+y)}; q) \\ &= (-1)^m q^{-\frac{1}{2}(m+n^2)} J_{n-m}(Rq^{\frac{1}{2}(y+n-m)}; q) J_m(Rq^{\frac{1}{2}(x+y)}; q) \\ &= (-1)^m R^n q^{\frac{1}{2}(yn+xm)} q^{-nm} q^{\binom{m}{2}} \frac{(q^{n-m+1}; q)_{\infty} (q^{m+1}; q)_{\infty}}{(q; q)_{\infty} (q; q)_{\infty}} \times \\ & \quad \times {}_1\Phi_1 \left( \begin{array}{c} 0 \\ q^{n-m+1} \end{array} \middle| q, R^2 q^{y+n-m+1} \right) {}_1\Phi_1 \left( \begin{array}{c} 0 \\ q^{m+1} \end{array} \middle| q, R^2 q^{x+y+1} \right) \\ &= \frac{R^n q^{\frac{1}{2}(yn+xm)} q^{\binom{m}{2}} q^{-nm}}{(-1)^m (q; q)_{n-m} (q; q)_m} \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{(-1)^{k+h} q^{\binom{k+1}{2}} q^{\binom{h+1}{2}} q^{k(x+y)} q^{h(y+n-m)} R^{2(h+k)}}{(q^{m+1}; q)_k (q; q)_k (q^{n-m+1}; q)_h (q; q)_h} \\ &= \frac{R^n q^{\frac{1}{2}(yn+xm)} q^{\binom{m}{2}} q^{-nm}}{(-1)^m (q; q)_{n-m} (q; q)_m} \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \frac{(-1)^j q^{\binom{k+1}{2}} q^{xk} q^{\binom{j-k+1}{2}} q^{yj} q^{(j-k)(n-m)} R^{2j}}{(q^{m+1}; q)_k (q; q)_k (q; q)_k (q^{n-m+1}; q)_{j-k} (q; q)_{j-k}} \\ &= \frac{R^n q^{\frac{1}{2}(yn+xm)} q^{\binom{m}{2}} q^{-nm}}{(-1)^m (q; q)_{n-m} (q; q)_m} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j+1}{2}} q^{j(y+n-m)} R^{2j}}{(q^{n-m+1}; q)_j (q; q)_j} \times \\ & \quad \times \sum_{k=0}^j \frac{(q^{-j}; q)_k (q^{-j-n+m}; q)_k}{(q^{m+1}; q)_k (q; q)_k} q^{k(x+j+1)} =: P(x, y|q). \end{aligned}$$

In the last step identity (A.9) was used. Now since by (A.9)

$$(q^{-j-n+m}; q)_k = \frac{(q^{-j-n}; q)_{k+m}}{(q^{-j-n}; q)_m} = \frac{(q^{-j-n}; q)_{k+m} (q; q)_{j+n-m} (-1)^m}{(q; q)_{j+n} q^{\binom{m}{2}} q^{-m(j+n)}}, \quad (6.3.3)$$

and since by (A.13)

$$(q; q)_{n-m}(q^{n-m+1}; q)_j = (q; q)_{j+n-m} \quad (6.3.4)$$

and

$$(q; q)_m(q^{m+1}; q)_k = (q; q)_{m+k} \quad (6.3.5)$$

we find with the terminating  $q$ -binomial formula (2.3.15)

$$\begin{aligned} P(x, y|q) &= R^n q^{\frac{1}{2}(yn+xm)} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j+1}{2}} q^{j(y+n)} R^{2j}}{(q; q)_{j+n}(q; q)_j} \times \\ &\quad \times \sum_{k=0}^j \frac{(q^{-j}; q)_k q^{k(x+j+1)} \sum_{h=0}^{k+m} (-1)^h q^{\binom{h}{2}} q^{-h(j+n)}}{(q; q)_k (q; q)_{k+m-h}(q; q)_h}. \end{aligned}$$

Since by (A.6)

$$\frac{1}{(q; q)_{k+m-h}} = \frac{(q^{k+m-h+1}; q)_{\infty}}{(q; q)_{\infty}} = 0 \text{ if } h > k + m, \quad (6.3.6)$$

we can sum from  $h = 0$  to  $\infty$ . Changing the order of summation we obtain

$$\begin{aligned} P(x, y|q) &= R^n q^{\frac{1}{2}(yn+xm)} \sum_{h=0}^{\infty} \frac{(-1)^h q^{\binom{h}{2}} q^{-hn}}{(q; q)_h} \times \\ &\quad \times \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j+1}{2}} q^{j(y+n-h)} R^{2j}}{(q; q)_{j+n}(q; q)_j} \sum_{k=0}^j \frac{(q^{-j}; q)_k q^{k(x+j+1)} (q^{k+m-h+1}; q)_{\infty}}{(q; q)_k (q; q)_{\infty}} \\ &= R^n q^{\frac{1}{2}(yn+xm)} \sum_{h=0}^{\infty} \frac{(-1)^h q^{\binom{h}{2}} q^{-hn}}{(q; q)_h} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j+1}{2}} q^{j(y+n-h)} R^{2j}}{(q; q)_{j+n}(q; q)_j} \times \\ &\quad \times \frac{(q^{m-h+1}; q)_{\infty}}{(q; q)_{\infty}} {}_2\Phi_1 \left( \begin{matrix} q^{-j}, 0 \\ q^{m-h+1} \end{matrix} \middle| q, q^{x+j+1} \right) = \frac{R^n q^{\frac{1}{2}(yn+xm)}}{(q; q)_{\infty}} \sum_{h=0}^{\infty} \frac{(-1)^h q^{\binom{h}{2}} q^{-hn}}{(q; q)_h} \times \\ &\quad \times \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j+1}{2}} q^{j(y+n-h)} R^{2j}}{(q; q)_{j+n}(q; q)_j} \frac{(q^{x+1}; q)_{\infty}}{(q^{x+j+1}; q)_{\infty}} {}_1\Phi_1 \left( \begin{matrix} q^{x+j+1} \\ q^{x+1} \end{matrix} \middle| q, q^{m-h+1} \right) \\ &= R^n q^{\frac{1}{2}(yn+xm)} \frac{(q^{x+1}; q)_{\infty}}{(q; q)_{\infty}} \sum_{h=0}^{\infty} \frac{(-1)^h q^{\binom{h}{2}} q^{-hn}}{(q; q)_h} \times \\ &\quad \times \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j+1}{2}} q^{j(y+n-h)} R^{2j}}{(q; q)_{j+n}(q; q)_j (q^{x+j+1}; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(q^{x+j+1}; q)_k (-1)^k q^{\binom{k+1}{2}} q^{k(m-h)}}{(q^{x+1}; q)_k (q; q)_k}. \end{aligned}$$

Here we have used (2.3.21). Now by formula (2.3.11) we have

$$\frac{(q^{x+j+1}; q)_k}{(q^{x+j+1}; q)_{\infty}} = \frac{1}{(q^{x+j+k+1}; q)_{\infty}} = \sum_{z=0}^{\infty} \frac{q^{z(x+j+k+1)}}{(q; q)_z}$$

$$= \sum_{z=-h}^{\infty} \frac{q^{(z+h)(x+j+k+1)}}{(q; q)_{z+h}} = \sum_{z=-\infty}^{\infty} \frac{q^{(z+h)(x+j+k+1)}}{(q; q)_{z+h}},$$

since by (A.6)

$$\frac{1}{(q; q)_{z+h}} = \frac{(q^{z+h+1}; q)_{\infty}}{(q; q)_{\infty}} = 0 \text{ if } z < -h. \quad (6.3.7)$$

Substituting this result and changing the order of summation we obtain

$$\begin{aligned} P(x, y|q) &= \frac{R^n q^{\frac{1}{2}(xm+yn)} (q^{x+1}; q)_{\infty}}{(q; q)_n (q; q)_{\infty} (q; q)_{\infty}} \sum_{z=-\infty}^{\infty} q^{z(x+1)} (q^{z+1}; q)_{\infty} \times \\ &\quad \times \sum_{h=0}^{\infty} \frac{(-1)^h q^{\binom{h}{2}} q^{h(x-n+1)}}{(q^{z+1}; q)_h (q; q)_h} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j+1}{2}} q^{j(z+n+y)} R^{2j}}{(q^{n+1}; q)_j (q; q)_j} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} q^{k(m+z)}}{(q^{x+1}; q)_k (q; q)_k} \\ &= \frac{R^n q^{\frac{1}{2}(xm+yn)} (q^{x+1}; q)_{\infty}}{(q; q)_n (q; q)_{\infty} (q; q)_{\infty}} \sum_{z=-\infty}^{\infty} q^{z(x+1)} (q^{z+1}; q)_{\infty} \times \\ &\quad \times {}_1\Phi_1 \left( \begin{matrix} 0 \\ q^{z+1} \end{matrix} \middle| q, q^{x-n+1} \right) {}_1\Phi_1 \left( \begin{matrix} 0 \\ q^{n+1} \end{matrix} \middle| q, R^2 q^{z+n+y+1} \right) {}_1\Phi_1 \left( \begin{matrix} 0 \\ q^{x+1} \end{matrix} \middle| q, q^{m+z+1} \right). \end{aligned}$$

When we apply the symmetry relation (2.3.22) to the first  ${}_1\Phi_1$  we have

$$\begin{aligned} P(x, y|q) &= \frac{R^n q^{\frac{1}{2}(xm+yn)} (q^{x+1}; q)_{\infty}}{(q; q)_n (q; q)_{\infty} (q; q)_{\infty}} \sum_{z=-\infty}^{\infty} q^{z(x+1)} (q^{x-n+1}; q)_{\infty} \times \\ &\quad \times {}_1\Phi_1 \left( \begin{matrix} 0 \\ q^{x-n+1} \end{matrix} \middle| q, q^{z+1} \right) {}_1\Phi_1 \left( \begin{matrix} 0 \\ q^{n+1} \end{matrix} \middle| q, R^2 q^{z+n+y+1} \right) {}_1\Phi_1 \left( \begin{matrix} 0 \\ q^{x+1} \end{matrix} \middle| q, q^{m+z+1} \right) \\ &= q^{-\frac{1}{2}n^2} \sum_{z=-\infty}^{\infty} q^z J_x(q^{\frac{1}{2}(z+m)}; q) J_{x-n}(q^{\frac{1}{2}z}; q) J_n(Rq^{\frac{1}{2}(z+n+y)}; q). \end{aligned}$$

This completes the proof of theorem 6.4.  $\square$

**Remark 1.** L.L.Vaksman and L.I. Korogodskii [49] mentioned product formula (6.3.2) in the special case  $m = n = 0$  and  $R = q^{-\frac{1}{2}}$  without a proof. They found this result by using quantum group techniques.

**Remark 2.** The product formula (6.3.2) can be regarded as a  $q$ -Hankel transform. When we replace in (6.3.2)  $q$  by  $q^2$ , we find with (5.2.2) that  $(-1)^n q^{-n} J_{m-n}(Rq^y; q^2) J_m(Rq^{x+y}; q^2)$  is the  $q$ -Hankel transform of  $J_{x-n}(q^z; q^2) J_n(Rq^{z+n+y}; q^2)$ .

Now the proof of theorem 6.3 follows easily.

**Proof.** Since we have the (rewritten) orthogonality relation (3.2.27)

$$\sum_{z=-\infty}^{\infty} q^{z+m} J_x(q^{\frac{1}{2}(z+k)}; q) J_x(q^{\frac{1}{2}(z+m)}; q) = \delta_{k,m}, \quad (6.3.8)$$

and the product formula (6.3.2), it is natural to look for a series expansion of the form

$$J_{x-n}(q^{\frac{1}{2}z}; q) J_n(Rq^{\frac{1}{2}(z+n+y)}; q) = \sum_{k=-\infty}^{\infty} A_k(x, y|q) J_x(q^{\frac{1}{2}(z+k)}; q). \quad (6.3.9)$$

If we multiply both sides of (6.3.9) by  $q^{z+m} J_x(q^{\frac{1}{2}(z+m)}; q)$  and if we sum both sides over  $z$  from  $-\infty$  to  $\infty$  we obtain with (6.3.8)

$$A_m(x, y|q) = \sum_{z=-\infty}^{\infty} q^{z+m} J_x(q^{\frac{1}{2}(z+m)}; q) J_{x-n}(q^{\frac{1}{2}z}; q) J_n(Rq^{\frac{1}{2}(z+y+n)}; q). \quad (6.3.10)$$

Now the right hand side of (6.3.10) is a constant times the right hand side of the product formula (6.3.2). Using (6.3.2) we find

$$A_m(x, y|q) = (-1)^n q^{m-\frac{1}{2}n} J_{m-n}(Rq^{\frac{1}{2}y}; q) J_m(Rq^{\frac{1}{2}(x+y)}; q). \quad (6.3.11)$$

This proves theorem 6.3.  $\square$

**Remark.** This derivation is in fact the search for the inverse  $q$ -Hankel transform. With this observation, we could have derived (6.3.1) immediately from (6.3.2) with the theory of chapter 5, instead of going through the derivation above.

## 6.4 A $q$ -analogue of Gegenbauer's addition formula

The main objective in this section is the proof of the following theorem.

**Theorem 6.5.** For  $a, b > 0, x > 0, \Re(\nu) > 0$  and  $0 \leq \phi \leq \pi$  we have a  $q$ -analogue of Gegenbauer's addition formula

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(a^{-1}be^{i\phi}q^{\frac{1}{2}(1-n)}, a^{-1}be^{-i\phi}q^{\frac{1}{2}(1-n)}; q)_n}{(q^{\nu+1}, q; q)_n} (-a^2x^2)^n q^{\binom{n+1}{2}} \\ &= \frac{(q, q; q)_{\infty}}{(q^{\nu+1}, q^{\nu+1}; q)_{\infty}} (abx^2)^{-\nu} \sum_{k=0}^{\infty} \frac{1 - q^{\nu+k}}{1 - q^{\nu}} q^{\frac{1}{2}k(1-\nu)} \times \\ & \quad \times J_{\nu+k}(axq^{\frac{1}{4}k}; q) J_{\nu+k}(bxq^{\frac{1}{4}k}; q) C_k(\cos \phi; q^{\nu}|q). \end{aligned} \quad (6.4.1)$$

Here  $C_k(\cos \phi; \beta|q)$  is Rogers'  $q$ -ultraspherical polynomial defined by (see [2])

$$C_n(\cos \phi; \beta|q) = \sum_{k=0}^n \frac{(\beta; q)_k (\beta; q)_{n-k}}{(q; q)_k (q; q)_{n-k}} \cos(n-2k)\phi. \quad (6.4.2)$$

Since the  $q$ -ultraspherical polynomial  $C_k(\cos \phi; q^\nu | q)$  tends to the ultraspherical polynomial  $C_k^\nu(\cos \phi)$  if  $q \uparrow 1$ , it can be calculated that (6.4.1), with  $a, b$  replaced by  $\frac{1}{2}a(1-q), \frac{1}{2}b(1-q)$ , tends to (1.5.3) if  $q \uparrow 1$ . M. Rahman [39] has derived an addition formula for the Jackson  $q$ -Bessel functions  $J_\nu^{(1)}(x; q)$  and  $J_\nu^{(2)}(x; q)$  similar to (6.4.1). The left hand side of (6.4.1) is a differentiable function in  $\cos \phi$  (see [46]) and can not be expressed as an  ${}_r\Phi_s$  series in an easy way.

In order to prove theorem 6.5 we shall first derive a product formula in the next subsection. Then we will give an integral representation for this product. Finally the results from subsections 6.4.1 and 6.4.2 can be combined to derive the addition formula (6.4.1).

### 6.4.1 A product formula

In this subsection we will derive a product formula for the Hahn-Exton  $q$ -Bessel functions. Assume that  $a, b > 0, x > 0, \Re e(\mu) > -1, \Re e(\nu) > -1$ . Then

$$\begin{aligned} J_\nu(ax; q)J_\mu(bx; q) &= \\ &= \frac{(ax)^\nu(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{k=0}^{\infty} \frac{(-a^2x^2)^k q^{\binom{k+1}{2}}}{(q^{\nu+1}; q)_k (q; q)_k} \frac{(bx)^\mu(q^{\mu+1}; q)_\infty}{(q; q)_\infty} \sum_{m=0}^{\infty} \frac{(-b^2x^2)^m q^{\binom{m+1}{2}}}{(q^{\mu+1}; q)_m (q; q)_m} \\ &= K_{\nu, \mu}(x; q) \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{(-1)^n a^{2k} b^{2(n-k)} x^{2n} q^{\binom{k+1}{2}} q^{\binom{n-k+1}{2}}}{(q^{\nu+1}; q)_k (q; q)_k (q^{\mu+1}; q)_{n-k} (q; q)_{n-k}} \\ &= K_{\nu, \mu}(x; q) \sum_{n=0}^{\infty} \frac{(-1)^n (bx)^{2n} q^{\binom{n+1}{2}}}{(q^{\mu+1}; q)_n (q; q)_n} \sum_{k=0}^n \frac{(q^{-n}; q)_k (q^{-n-\mu}; q)_k}{(q^{\nu+1}; q)_k (q; q)_k} \left( \frac{a^2}{b^2} q^{\mu+n+1} \right)^k. \end{aligned}$$

Here we have substituted  $m = n - k$ , changed the order of summation and used identity (A.9).  $K_{\nu, \mu}(x; q)$  is defined by

$$K_{\nu, \mu}(x; q) \stackrel{\text{def}}{=} \frac{(q^{\nu+1}, q^{\mu+1}; q)_\infty}{(q, q; q)_\infty} a^\nu b^\mu x^{\nu+\mu}. \quad (6.4.3)$$

So in terms of basic hypergeometric series the general product formula for the Hahn-Exton  $q$ -Bessel functions reads

$$\begin{aligned} J_\nu(ax; q)J_\mu(bx; q) &= \frac{(q^{\nu+1}, q^{\mu+1}; q)_\infty}{(q, q; q)_\infty} a^\nu b^\mu x^{\nu+\mu} \times \\ &\times \sum_{n=0}^{\infty} \frac{(-1)^n (bx)^{2n} q^{\binom{n+1}{2}}}{(q^{\mu+1}; q)_n (q; q)_n} {}_2\Phi_1 \left( \begin{array}{c} q^{-n}, q^{-n-\mu} \\ q^{\nu+1} \end{array} \middle| q, \frac{a^2}{b^2} q^{\mu+n+1} \right). \quad (6.4.4) \end{aligned}$$

If we replace in (6.4.4)  $x$  and  $y$  by  $\frac{1}{2}x(1-q)$  and  $\frac{1}{2}y(1-q)$  respectively, and let  $q \uparrow 1$ , we find with (3.2.5) and (2.3.29) the product formula (see [53])

$$J_\nu(ax)J_\mu(bx) = \frac{(ax)^\nu(bx)^\mu}{2^{\nu+\mu}\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(-1)^n(bx)^{2n}4^{-n}}{\Gamma(\mu+n+1)n!} {}_2F_1 \left( \begin{matrix} -n, -n-\mu \\ \nu+1 \end{matrix} \middle| \frac{a^2}{b^2} \right). \quad (6.4.5)$$

Now let us choose  $\mu = \nu$  in (6.4.4). The  ${}_2\Phi_1$  on the right hand side becomes well-poised and can be transformed into a balanced  ${}_4\Phi_3$  by formula (2.3.27). Then we apply Sears' transformation formula (2.3.26) twice : in the first time  $a$  and  $d$  in (2.3.26) are chosen as  $ab^{-1}q^{-\frac{1}{2}(n+\nu)}$  and  $-q^{-n-\nu}$  respectively, in the second time  $a$  and  $d$  in (2.3.26) are chosen as  $-q^\nu$  and  $ab^{-1}q^{\frac{1}{2}(1-n+\nu)}$  respectively. This yields

$$\begin{aligned} J_\nu(ax; q)J_\nu(bx; q) &= \frac{(q^{\nu+1}, q^{\nu+1}; q)_\infty}{(q, q; q)_\infty} (abx^2)^\nu \times \\ &\quad \times \sum_{n=0}^{\infty} \frac{(-1)^n(bx)^{2n}q^{\binom{n+1}{2}}}{(q^{\nu+1}; q)_n(q; q)_n} {}_2\Phi_1 \left( \begin{matrix} q^{-n}, q^{-n-\nu} \\ q^{\nu+1} \end{matrix} \middle| q, \frac{a^2}{b^2} q^{\nu+n+1} \right) \\ &= \frac{(q^{\nu+1}, q^{\nu+1}; q)_\infty}{(q, q; q)_\infty} (abx^2)^\nu \sum_{n=0}^{\infty} \frac{(q^{-2n-2\nu}; q)_n(-1)^n(abx^2)^n q^{\frac{1}{2}n(2n+\nu+1)}}{(q^{\nu+1}; q)_n(q; q)_n(q^{-n-\nu}; q)_n} \times \\ &\quad \times {}_4\Phi_3 \left( \begin{matrix} q^{-n}, q^{-n-2\nu}, ab^{-1}q^{-\frac{1}{2}(n+\nu)}, a^{-1}bq^{-\frac{1}{2}(n+\nu)} \\ q^{-n-\nu+\frac{1}{2}}, -q^{-n-\nu+\frac{1}{2}}, -q^{-n-\nu} \end{matrix} \middle| q, q \right) \\ &= \frac{(q^{\nu+1}, q^{\nu+1}; q)_\infty}{(q, q; q)_\infty} (abx^2)^\nu \sum_{n=0}^{\infty} \frac{(q^{-2n-2\nu}, a^{-1}bq^{\frac{1}{2}(1-n-\nu)}, -a^{-1}bq^{\frac{1}{2}(1-n-\nu)}; q)_n}{(q^{\nu+1}, q^{-n-\nu}, q^{-n-\nu+\frac{1}{2}}, -q^{-n-\nu+\frac{1}{2}}, q; q)_n} \times \\ &\quad \times (-a^2x^2)^n q^{\binom{n+1}{2}} {}_4\Phi_3 \left( \begin{matrix} q^{-n}, -q^\nu, ab^{-1}q^{-\frac{1}{2}(n+\nu)}, -ab^{-1}q^{-\frac{1}{2}(n+\nu)} \\ ab^{-1}q^{\frac{1}{2}(1-n+\nu)}, -ab^{-1}q^{\frac{1}{2}(1-n+\nu)}, -q^{-n-\nu} \end{matrix} \middle| q, q \right) \\ &= \frac{(q^{\nu+1}, q^{\nu+1}; q)_\infty}{(q, q; q)_\infty} (abx^2)^\nu \sum_{n=0}^{\infty} \frac{(q^{-2n-2\nu}, a^{-1}bq^{\frac{1}{2}(1-n-\nu)}, -a^{-1}bq^{\frac{1}{2}(1-n-\nu)}; q)_n}{(q^{\nu+1}, q^{-n-\nu}, q^{-n-\nu+\frac{1}{2}}, -q^{-n-\nu+\frac{1}{2}}, q; q)_n} (-a^2x^2)^n q^{\binom{n+1}{2}} \times \\ &\quad \times \frac{(-q^\nu)^n(q^{-n-2\nu}, ab^{-1}q^{\frac{1}{2}(1-n-\nu)}; q)_n}{(-q^{-n-\nu}, -ab^{-1}q^{\frac{1}{2}(1-n+\nu)}; q)_n} {}_4\Phi_3 \left( \begin{matrix} q^{-n}, -q^\nu, q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}} \\ ab^{-1}q^{\frac{1}{2}(1-n+\nu)}, a^{-1}bq^{\frac{1}{2}(1-n+\nu)}, q^{2\nu+1} \end{matrix} \middle| q, q \right). \end{aligned}$$

When we apply identities (A.7) and (A.17) we have found the following product formula for the Hahn-Exton  $q$ -Bessel functions of order  $\nu$  in terms of balanced  ${}_4\Phi_3$ 's

$$\begin{aligned} J_\nu(ax; q)J_\nu(bx; q) &= \frac{(q^{\nu+1}, q^{\nu+1}; q)_\infty}{(q, q; q)_\infty} (abx^2)^\nu \times \\ &\quad \times \sum_{n=0}^{\infty} \frac{(a^{-1}bq^{\frac{1}{2}(1-n-\nu)}, a^{-1}bq^{\frac{1}{2}(1-n+\nu)}; q)_n}{(q^{\nu+1}, q; q)_n} (-1)^n(ax)^{2n} q^{\binom{n+1}{2}} \times \end{aligned}$$

$$\times {}_4\Phi_3 \left( \begin{array}{c} q^{-n}, -q^\nu, q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}} \\ ab^{-1}q^{\frac{1}{2}(1-n+\nu)}, a^{-1}bq^{\frac{1}{2}(1-n+\nu)}, q^{2\nu+1} \end{array} \middle| q, q \right). \quad (6.4.6)$$

### 6.4.2 An integral representation

With the aid of the Askey-Wilson integral (2.3.40) we can find an integral representation for a product of two Hahn-Exton  $q$ -Bessel functions. Since by Saalschütz's summation formula (2.3.25)

$${}_3\Phi_2 \left( \begin{array}{c} q^{-n}, q^{\frac{1}{2}\nu}e^{i\phi}, q^{\frac{1}{2}\nu}e^{-i\phi} \\ a^{-1}bq^{\frac{1}{2}(1+\nu-n)}, ab^{-1}q^{\frac{1}{2}(1+\nu-n)} \end{array} \middle| q, q \right) = \frac{(a^{-1}bq^{\frac{1}{2}(1-n)}e^{i\phi}, a^{-1}bq^{\frac{1}{2}(1-n)}e^{-i\phi}; q)_n}{(a^{-1}bq^{\frac{1}{2}(1-n+\nu)}, a^{-1}bq^{\frac{1}{2}(1-n-\nu)}; q)_n}, \quad (6.4.7)$$

and since by (2.3.42) and (A.6)

$$\frac{(q^{\frac{1}{2}\nu}e^{i\phi}, q^{\frac{1}{2}\nu}e^{-i\phi}; q)_k}{h(\cos \phi; q^{\frac{1}{2}\nu}; q)} = \frac{1}{h(\cos \phi; q^{\frac{1}{2}\nu+k}; q)}, \quad (6.4.8)$$

we have for  $\Re e(\nu) > 0$

$$\begin{aligned} & \int_0^\pi \frac{h(\cos \phi; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}; q)}{h(\cos \phi; q^{\frac{1}{2}\nu}, -q^{\frac{1}{2}\nu}, q^{\frac{1}{2}(\nu+1)}, -q^{\frac{1}{2}(\nu+1)}; q)} (a^{-1}bq^{\frac{1}{2}(1-n)}e^{i\phi}, a^{-1}bq^{\frac{1}{2}(1-n)}e^{-i\phi}; q)_n d\phi \\ &= \sum_{k=0}^n \frac{(q^{-n}; q)_k (a^{-1}bq^{\frac{1}{2}(1-n+\nu)}, a^{-1}bq^{\frac{1}{2}(1-n-\nu)}; q)_n q^k}{(a^{-1}bq^{\frac{1}{2}(1+\nu-n)}, ab^{-1}q^{\frac{1}{2}(1+\nu-n)}, q; q)_k} \times \\ & \quad \times \int_0^\pi \frac{h(\cos \phi; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}; q)}{h(\cos \phi; q^{\frac{1}{2}\nu+k}, -q^{\frac{1}{2}\nu}, q^{\frac{1}{2}(\nu+1)}, -q^{\frac{1}{2}(\nu+1)}; q)} d\phi \\ &= \sum_{k=0}^n \frac{(q^{-n}; q)_k (a^{-1}bq^{\frac{1}{2}(1-n+\nu)}, a^{-1}bq^{\frac{1}{2}(1-n-\nu)}; q)_n q^k}{(a^{-1}bq^{\frac{1}{2}(1+\nu-n)}, ab^{-1}q^{\frac{1}{2}(1+\nu-n)}, q; q)_k} \times \\ & \quad \times \frac{2\pi (q^{k+2\nu+1}; q)_\infty}{(q, -q^{\nu+k}, q^{\nu+k+\frac{1}{2}}, -q^{\nu+k+\frac{1}{2}}, q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, -q^{\nu+1}; q)_\infty} \\ &= \frac{2\pi (q^{2\nu+1}; q)_\infty (a^{-1}bq^{\frac{1}{2}(1-n+\nu)}, a^{-1}bq^{\frac{1}{2}(1-n-\nu)}; q)_n}{(q, -q^\nu, q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, -q^{\nu+1}; q)_\infty} \times \\ & \quad \times {}_4\Phi_3 \left( \begin{array}{c} q^{-n}, -q^\nu, q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}} \\ ab^{-1}q^{\frac{1}{2}(1-n+\nu)}, a^{-1}bq^{\frac{1}{2}(1-n+\nu)}, q^{2\nu+1} \end{array} \middle| q, q \right). \end{aligned}$$

In the last step identity (A.6) was used. After applying (A.17) we find

$$\int_0^\pi \frac{h(\cos \phi; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}; q)}{h(\cos \phi; q^{\frac{1}{2}\nu}, -q^{\frac{1}{2}\nu}, q^{\frac{1}{2}(\nu+1)}, -q^{\frac{1}{2}(\nu+1)}; q)} (a^{-1}bq^{\frac{1}{2}(1-n)}e^{i\phi}, a^{-1}bq^{\frac{1}{2}(1-n)}e^{-i\phi}; q)_n d\phi =$$

$$\begin{aligned}
&= \frac{2\pi(q^{\nu+1}, q^{\nu}; q)_{\infty}}{(q^{2\nu}, q; q)_{\infty}} (a^{-1}bq^{\frac{1}{2}(1-n+\nu)}, a^{-1}bq^{\frac{1}{2}(1-n-\nu)}; q)_n \times \\
&\quad \times {}_4\Phi_3 \left( \begin{matrix} q^{-n}, -q^{\nu}, q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}} \\ ab^{-1}q^{\frac{1}{2}(1-n+\nu)}, a^{-1}bq^{\frac{1}{2}(1-n+\nu)}, q^{2\nu+1} \end{matrix} \middle| q, q \right). \tag{6.4.9}
\end{aligned}$$

When we compare (6.4.9) with (6.4.6) we see that we have found an integral representation for a product of Hahn-Exton  $q$ -Bessel functions. Since by (A.17)

$$\frac{h(\cos \phi; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}; q)}{h(\cos \phi; q^{\frac{1}{2}\nu}, -q^{\frac{1}{2}\nu}, q^{\frac{1}{2}(\nu+1)}, -q^{\frac{1}{2}(\nu+1)}; q)} = \frac{(e^{2i\phi}, e^{-2i\phi}; q)_{\infty}}{(q^{\nu}e^{2i\phi}, q^{\nu}e^{-2i\phi}; q)_{\infty}} \tag{6.4.10}$$

the integral representation for  $\Re e(\nu) > 0$  is:

$$\begin{aligned}
J_{\nu}(ax; q)J_{\nu}(bx; q) &= \frac{(abx^2)^{\nu}(q^{\nu+1}, q^{2\nu}; q)_{\infty}}{2\pi(q^{\nu}, q; q)_{\infty}} \int_0^{\pi} \frac{(e^{2i\phi}, e^{-2i\phi}; q)_{\infty}}{(q^{\nu}e^{2i\phi}, q^{\nu}e^{-2i\phi}; q)_{\infty}} \times \\
&\quad \times \sum_{n=0}^{\infty} \frac{(a^{-1}bq^{\frac{1}{2}(1-n)}e^{i\phi}, a^{-1}bq^{\frac{1}{2}(1-n)}e^{-i\phi}; q)_n}{(q^{\nu+1}, q; q)_n} (-1)^n (ax)^{2n} q^{\binom{n+1}{2}} d\phi. \tag{6.4.11}
\end{aligned}$$

### 6.4.3 The addition formula

R. Askey and M.E.H. Ismail [2] proved the orthogonality relation for the  $q$ -ultraspherical polynomials of Rogers. For  $\Re e(\nu) > 0$  they found

$$\begin{aligned}
&\int_0^{\pi} \frac{(e^{2i\phi}, e^{-2i\phi}; q)_{\infty}}{(q^{\nu}e^{2i\phi}, q^{\nu}e^{-2i\phi}; q)_{\infty}} C_m(\cos \phi; q^{\nu}|q) C_n(\cos \phi; q^{\nu}|q) d\phi \\
&= \frac{2\pi\Gamma_q(2\nu)}{\Gamma_q(\nu)\Gamma_q(\nu+1)} \frac{1-q^{\nu}}{1-q^{\nu+n}} \frac{(q^{2\nu}; q)_n}{(q; q)_n} \delta_{m,n}. \tag{6.4.12}
\end{aligned}$$

The  $q$ -ultraspherical polynomial of Rogers (6.4.2) can be expressed as a well-poised  ${}_2\Phi_1$ . Using (A.9) we find

$$C_n(\cos \phi; \beta|q) = \frac{(\beta; q)_n}{(q; q)_n} e^{in\phi} {}_2\Phi_1 \left( \begin{matrix} q^{-n}, \beta \\ \beta^{-1}q^{1-n} \end{matrix} \middle| q, \beta^{-1}qe^{-2i\phi} \right). \tag{6.4.13}$$

By (2.3.27) this can be transformed into a balanced  ${}_4\Phi_3$ :

$$C_n(\cos \phi; \beta|q) = \frac{(\beta^2; q)_n}{(q; q)_n} \beta^{-\frac{1}{2}n} {}_4\Phi_3 \left( \begin{matrix} q^{-n}, \beta^2q^n, \beta^{\frac{1}{2}}e^{i\phi}, \beta^{\frac{1}{2}}e^{-i\phi} \\ \beta q^{\frac{1}{2}}, -\beta q^{\frac{1}{2}}, -\beta \end{matrix} \middle| q, q \right). \tag{6.4.14}$$

With the orthogonality relation (6.4.12) and the integral representation (6.4.11) it is natural to look for a series expansion of the form

$$\sum_{n=0}^{\infty} \frac{(a^{-1}bq^{\frac{1}{2}(1-n)}e^{i\phi}, a^{-1}bq^{\frac{1}{2}(1-n)}e^{-i\phi}; q)_n}{(q^{\nu+1}, q; q)_n} (-1)^n (ax)^{2n} q^{\binom{n+1}{2}} = \sum_{m=0}^{\infty} A_m(x) C_m(\cos \phi; q^{\nu}|q). \quad (6.4.15)$$

The series on the left hand side of (6.4.15) is a differentiable function in  $\cos \phi$  (see [46]). Hence the Fourier series on the right hand side converges pointwise to the function on the left hand side of (6.4.15). If we multiply both sides of (6.4.15) with

$$\frac{(e^{2i\phi}, e^{-2i\phi}; q)_{\infty}}{(q^{\nu}e^{2i\phi}, q^{\nu}e^{-2i\phi}; q)_{\infty}} C_k(\cos \phi; q^{\nu}|q)$$

and integrate from  $\phi = 0$  to  $\pi$  we find with the orthogonality relation (6.4.12) an expression for  $A_k(x)$

$$A_k(x) = \frac{\Gamma_q(\nu)\Gamma_q(\nu+1)}{2\pi\Gamma_q(2\nu)} \frac{1-q^{\nu+k}}{1-q^{\nu}} \frac{(q; q)_k}{(q^{2\nu}; q)_k} \int_0^{\pi} \frac{(e^{2i\phi}, e^{-2i\phi}; q)_{\infty}}{(q^{\nu}e^{2i\phi}, q^{\nu}e^{-2i\phi}; q)_{\infty}} C_k(\cos \phi; q^{\nu}|q) \times \sum_{n=0}^{\infty} \frac{(a^{-1}bq^{\frac{1}{2}(1-n)}e^{i\phi}, a^{-1}bq^{\frac{1}{2}(1-n)}e^{-i\phi}; q)_n}{(q^{\nu+1}, q; q)_n} (-1)^n (ax)^{2n} q^{\binom{n+1}{2}} d\phi. \quad (6.4.16)$$

For fixed non-negative integers  $n$  and  $k$ , let us first compute the integral

$$I_{n,k} = \int_0^{\pi} \frac{(e^{2i\phi}, e^{-2i\phi}; q)_{\infty}}{(q^{\nu}e^{2i\phi}, q^{\nu}e^{-2i\phi}; q)_{\infty}} (a^{-1}bq^{\frac{1}{2}(1-n)}e^{i\phi}, a^{-1}bq^{\frac{1}{2}(1-n)}e^{-i\phi}; q)_n \times \times {}_4\Phi_3 \left( \begin{matrix} q^{-k}, q^{2\nu+k}, q^{\frac{1}{2}\nu}e^{i\phi}, q^{\frac{1}{2}\nu}e^{-i\phi} \\ q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, -q^{\nu} \end{matrix} \middle| q, q \right) d\phi. \quad (6.4.17)$$

Using Saalschütz's summation formula (2.3.25) we have

$$\begin{aligned} & \frac{(a^{-1}bq^{\frac{1}{2}(1-n)}e^{i\phi}, a^{-1}bq^{\frac{1}{2}(1-n)}e^{-i\phi}; q)_n}{(a^{-1}bq^{\frac{1}{2}(2+\nu-n)}, a^{-1}bq^{-\frac{1}{2}(n+\nu)}; q)_n} \\ &= \sum_{j=0}^n \frac{(q^{-n}, q^{\frac{1}{2}(\nu+1)}e^{i\phi}, q^{\frac{1}{2}(\nu+1)}e^{-i\phi}; q)_j q^j}{(a^{-1}bq^{\frac{1}{2}(2+\nu-n)}, ab^{-1}q^{\frac{1}{2}(2+\nu-n)}, q; q)_j}. \end{aligned}$$

Insert this in the integral (6.4.17) and use (6.4.8), (6.4.10) and (2.3.40). We find

$$\begin{aligned} I_{n,k} &= \sum_{j=0}^n \frac{(q^{-n}; q)_j q^j (a^{-1}bq^{\frac{1}{2}(2-n+\nu)}, a^{-1}bq^{-\frac{1}{2}(n+\nu)}; q)_n}{(a^{-1}bq^{\frac{1}{2}(2-n+\nu)}, ab^{-1}q^{\frac{1}{2}(2-n+\nu)}, q; q)_j} \times \\ & \times \sum_{m=0}^k \frac{(q^{-k}, q^{2\nu+k}; q)_m q^m}{(q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, -q^{\nu}, q; q)_m} \int_0^{\pi} \frac{h(\cos \phi; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}; q)}{h(\cos \phi; q^{\frac{1}{2}\nu+m}, -q^{\frac{1}{2}\nu}, q^{\frac{1}{2}(\nu+1)+j}, -q^{\frac{1}{2}(\nu+1)}; q)} d\phi \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^n \frac{(q^{-n}; q)_j q^j (a^{-1} b q^{\frac{1}{2}(2-n+\nu)}, a^{-1} b q^{-\frac{1}{2}(n+\nu)}; q)_n}{(a^{-1} b q^{\frac{1}{2}(2-n+\nu)}, a b^{-1} q^{\frac{1}{2}(2-n+\nu)}, q; q)_j} \sum_{m=0}^k \frac{(q^{-k}, q^{2\nu+k}; q)_m q^m}{(q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, -q^{\nu}, q; q)_m} \times \\
&\quad \times \frac{2\pi (q^{2\nu+m+j+1}; q)_\infty}{(q, -q^{\nu+m}, q^{\nu+m+j+\frac{1}{2}}, -q^{\nu+m+\frac{1}{2}}, -q^{\nu+j+\frac{1}{2}}, q^{\nu+\frac{1}{2}}, -q^{\nu+j+1}; q)_\infty}.
\end{aligned}$$

Now by identities (A.6) and (A.17) we can write the last factor as

$$\begin{aligned}
&\frac{2\pi (q^{2\nu+1}; q)_\infty (-q^\nu, q^{\nu+j+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}; q)_m (q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, -q^{\nu+1}; q)_j}{(q, -q^\nu, q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, -q^{\nu+1}; q)_\infty (q^{2\nu+1}; q)_j (q^{2\nu+j+1}; q)_m} \\
&= \frac{2\pi (q^{\nu+1}, q^\nu; q)_\infty (q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, -q^{\nu+1}; q)_j}{(q^{2\nu}; q)_\infty (q^{2\nu+1}; q)_j} \frac{(-q^\nu, -q^{\nu+\frac{1}{2}}, q^{\nu+j+\frac{1}{2}}; q)_m}{(q^{2\nu+j+1}; q)_m}.
\end{aligned}$$

When we also use the  $q$ -Gamma notation, the integral  $I_{n,k}$  reads

$$\begin{aligned}
I_{n,k} &= \frac{2\pi \Gamma_q(2\nu)}{\Gamma_q(\nu) \Gamma_q(\nu+1)} (a^{-1} b q^{\frac{1}{2}(2-n+\nu)}, a^{-1} b q^{-\frac{1}{2}(n+\nu)}; q)_n \times \\
&\quad \times \sum_{j=0}^n \frac{(q^{-n}, q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, -q^{\nu+1}; q)_j q^j}{(a^{-1} b q^{\frac{1}{2}(2-n+\nu)}, a b^{-1} q^{\frac{1}{2}(2-n+\nu)}, q^{2\nu+1}, q; q)_j} \sum_{m=0}^k \frac{(q^{-k}, q^{2\nu+k}, q^{\nu+j+\frac{1}{2}}; q)_m q^m}{(q^{\nu+\frac{1}{2}}, q^{2\nu+j+1}, q; q)_m}.
\end{aligned}$$

The sum over  $m$  is a balanced  ${}_3\Phi_2$  and can thus be summed by Saalschütz's summation formula (2.3.25). With identity (A.7) this yields

$${}_3\Phi_2 \left( \begin{matrix} q^{-k}, q^{2\nu+k}, q^{\nu+j+\frac{1}{2}} \\ q^{\nu+\frac{1}{2}}, q^{2\nu+j+1} \end{matrix} \middle| q, q \right) = \frac{(q^{1+j-k}, q^{\nu+\frac{1}{2}}; q)_k}{(q^{2\nu+1+j}, q^{-\nu-k+\frac{1}{2}}; q)_k} = \frac{(q^{-j}; q)_k}{(q^{2\nu+j+1}; q)_k} q^{k(\nu+j+\frac{1}{2})}. \quad (6.4.18)$$

This gives

$$\begin{aligned}
I_{n,k} &= \frac{2\pi \Gamma_q(2\nu)}{\Gamma_q(\nu) \Gamma_q(\nu+1)} (a^{-1} b q^{\frac{1}{2}(2-n+\nu)}, a^{-1} b q^{-\frac{1}{2}(n+\nu)}; q)_n \times \\
&\quad \times \sum_{j=0}^n \frac{(q^{-n}, q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, -q^{\nu+1}; q)_j q^j}{(a^{-1} b q^{\frac{1}{2}(2-n+\nu)}, a b^{-1} q^{\frac{1}{2}(2-n+\nu)}, q^{2\nu+1}, q; q)_j} \frac{(q^{-j}; q)_k}{(q^{2\nu+j+1}; q)_k} q^{k(\nu+j+\frac{1}{2})}.
\end{aligned}$$

Because  $(q^{-j}; q)_k = 0$  if  $j < k$ , the integral vanishes unless  $n \geq k$ . So we can start summing at  $j = k$ . After a shift in the summation index and after applying identities (A.7) and (A.13), the integral becomes

$$I_{n,k} = \frac{2\pi \Gamma_q(2\nu)}{\Gamma_q(\nu) \Gamma_q(\nu+1)} (a^{-1} b q^{\frac{1}{2}(2-n+\nu)}, a^{-1} b q^{-\frac{1}{2}(n+\nu)}; q)_n \times$$

$$\begin{aligned}
& \times \sum_{j=0}^{n-k} \frac{(q^{-n}, q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, -q^{\nu+1}; q)_{j+k} q^{j+k}}{(a^{-1}bq^{\frac{1}{2}(2-n+\nu)}, ab^{-1}q^{\frac{1}{2}(2-n+\nu)}, q^{2\nu+1}, q; q)_{j+k}} \frac{(q^{-j-k}; q)_k}{(q^{2\nu+j+k+1}; q)_k} q^{k(\nu+j+k+\frac{1}{2})} \\
& = \frac{2\pi\Gamma_q(2\nu)}{\Gamma_q(\nu)\Gamma_q(\nu+1)} \frac{(q^{-n}, q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, -q^{\nu+1}; q)_k (a^{-1}bq^{\frac{1}{2}(2-n+\nu)}, a^{-1}bq^{-\frac{1}{2}(n+\nu)}; q)_n}{(q^{2\nu+1}, a^{-1}bq^{\frac{1}{2}(2-n+\nu)}, ab^{-1}q^{\frac{1}{2}(2-n+\nu)}; q)_k} \times \\
& \quad \times \frac{(-1)^k q^{\frac{1}{2}k(k+2\nu+2)}}{(q^{2\nu+k+1}; q)_k} \sum_{j=0}^{n-k} \frac{(q^{-n+k}, q^{\nu+k+\frac{1}{2}}, -q^{\nu+k+\frac{1}{2}}, -q^{\nu+k+1}; q)_j q^j}{(a^{-1}bq^{k+\frac{1}{2}(2-n+\nu)}, ab^{-1}q^{k+\frac{1}{2}(2-n+\nu)}, q^{2\nu+2k+1}, q; q)_j} \\
& = \frac{2\pi\Gamma_q(2\nu)}{\Gamma_q(\nu)\Gamma_q(\nu+1)} \frac{(q^{-n}, q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, -q^{\nu+1}; q)_k (a^{-1}bq^{\frac{1}{2}(2-n+\nu)}, a^{-1}bq^{-\frac{1}{2}(n+\nu)}; q)_n}{(q^{2\nu+1}, a^{-1}bq^{\frac{1}{2}(2-n+\nu)}, ab^{-1}q^{\frac{1}{2}(2-n+\nu)}; q)_k} \times \\
& \quad \times \frac{(-1)^k q^{\frac{1}{2}k(k+2\nu+2)}}{(q^{2\nu+k+1}; q)_k} {}_4\Phi_3 \left( \begin{matrix} q^{-n+k}, q^{\nu+k+\frac{1}{2}}, -q^{\nu+k+\frac{1}{2}}, -q^{\nu+k+1} \\ q^{2\nu+2k+1}, a^{-1}bq^{k+\frac{1}{2}(2-n+\nu)}, ab^{-1}q^{k+\frac{1}{2}(2-n+\nu)} \end{matrix} \middle| q, q \right).
\end{aligned}$$

When we substitute this result in (6.4.16), this leads to

$$\begin{aligned}
A_k(x) &= \frac{1 - q^{k+\nu}}{1 - q^\nu} \frac{(q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, -q^{\nu+1}; q)_k}{(q^{2\nu+1}, q^{2\nu+k+1}; q)_k} (-1)^k q^{\frac{1}{2}k(k+\nu+2)} \times \\
&\quad \times \sum_{n=0}^{\infty} \frac{(a^{-1}bq^{\frac{1}{2}(2-n+\nu)}, a^{-1}bq^{-\frac{1}{2}(n+\nu)}; q)_n (q^{-n}; q)_k}{(q^{\nu+1}, q; q)_n (a^{-1}bq^{\frac{1}{2}(2-n+\nu)}, ab^{-1}q^{\frac{1}{2}(2-n+\nu)}; q)_k} (-1)^n (ax)^{2n} q^{\binom{n+1}{2}} \times \\
&\quad \times {}_4\Phi_3 \left( \begin{matrix} q^{-n+k}, q^{\nu+k+\frac{1}{2}}, -q^{\nu+k+\frac{1}{2}}, -q^{\nu+k+1} \\ q^{2\nu+2k+1}, a^{-1}bq^{k+\frac{1}{2}(2-n+\nu)}, ab^{-1}q^{k+\frac{1}{2}(2-n+\nu)} \end{matrix} \middle| q, q \right).
\end{aligned}$$

Because  $(q^{-n}; q)_k = 0$  if  $k > n$ , we can start summing at  $n = k$ . Then we shift the summation index and apply identities (A.7) and (A.13). This yields

$$\begin{aligned}
A_k(x) &= \frac{1 - q^{k+\nu}}{1 - q^\nu} \frac{(q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, -q^{\nu+1}; q)_k}{(q^{\nu+1}, q^{2\nu+1}, q^{2\nu+k+1}; q)_k} (-1)^k (ax)^{2k} q^{\frac{1}{2}k(k+\nu+2)} \times \\
&\quad \times \sum_{n=0}^{\infty} \frac{(a^{-1}bq^{\frac{1}{2}(2-n-k+\nu)}, a^{-1}bq^{-\frac{1}{2}(n+k+\nu)}; q)_{n+k}}{(q^{\nu+k+1}, q; q)_n (a^{-1}bq^{\frac{1}{2}(2-n-k+\nu)}, ab^{-1}q^{\frac{1}{2}(2-n-k+\nu)}; q)_k} (-1)^n (ax)^{2n} q^{\binom{n+1}{2}} \times \\
&\quad \times {}_4\Phi_3 \left( \begin{matrix} q^{-n}, q^{\nu+k+\frac{1}{2}}, -q^{\nu+k+\frac{1}{2}}, -q^{\nu+k+1} \\ q^{2\nu+2k+1}, a^{-1}bq^{\frac{1}{2}(2-n+k+\nu)}, ab^{-1}q^{\frac{1}{2}(2-n+k+\nu)} \end{matrix} \middle| q, q \right).
\end{aligned}$$

Now we apply Sears' transformation formula (2.3.26), where in (2.3.26)  $a$  and  $d$  are chosen as  $q^{\nu+k+\frac{1}{2}}$  and  $q^{2\nu+2k+1}$  respectively. Then we have

$$\begin{aligned}
A_k(x) &= \frac{1 - q^{k+\nu}}{1 - q^\nu} \frac{(q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, -q^{\nu+1}; q)_k}{(q^{\nu+1}, q^{2\nu+1}, q^{2\nu+k+1}; q)_k} (-1)^k (ax)^{2k} q^{\frac{1}{2}k(k+\nu+2)} \times \\
&\times \sum_{n=0}^{\infty} \frac{(-1)^n (ax)^{2n} q^{n(1+k+\nu+\frac{1}{2}n)}}{(q^{\nu+k+1}, q; q)_n} {}_4\Phi_3 \left( \begin{matrix} q^{-n}, q^{\nu+k+\frac{1}{2}}, -q^{\nu+k+\frac{1}{2}}, -q^{\nu+k} \\ q^{2\nu+2k+1}, a^{-1}bq^{\frac{1}{2}(1-n+k+\nu)}, ab^{-1}q^{\frac{1}{2}(1-n+k+\nu)} \end{matrix} \middle| q, q \right) \times \\
&\times \frac{(a^{-1}bq^{\frac{1}{2}(2-n-k+\nu)}, a^{-1}bq^{-\frac{1}{2}(n+k+\nu)}; q)_{n+k} (a^{-1}bq^{\frac{1}{2}(1-n-k-\nu)}, ab^{-1}q^{\frac{1}{2}(1-n-k-\nu)}; q)_n}{(a^{-1}bq^{\frac{1}{2}(2-n-k+\nu)}, ab^{-1}q^{\frac{1}{2}(2-n+k+\nu)}; q)_k (a^{-1}bq^{\frac{1}{2}(2-n+k+\nu)}, ab^{-1}q^{\frac{1}{2}(2-n+k+\nu)}; q)_n}.
\end{aligned}$$

The last factor in the formula above can be simplified by the following identities (here (A.7) and (A.13) are frequently used):

$$\begin{aligned}
a) \quad & \frac{(a^{-1}bq^{\frac{1}{2}(2-n-k+\nu)}; q)_{n+k}}{(a^{-1}bq^{\frac{1}{2}(2-n-k+\nu)}; q)_k (a^{-1}bq^{\frac{1}{2}(2-n+k+\nu)}; q)_n} = 1, \\
b) \quad & \frac{(a^{-1}bq^{-\frac{1}{2}(n+k+\nu)}; q)_{n+k}}{(ab^{-1}q^{\frac{1}{2}(2-n-k+\nu)}; q)_k} = \frac{(a^{-1}bq^{-\frac{1}{2}(n+k+\nu)}; q)_n (a^{-1}bq^{\frac{1}{2}(n-k-\nu)}; q)_k}{(ab^{-1}q^{\frac{1}{2}(2-n-k+\nu)}; q)_k} \\
& = (-a^{-1}b)^k q^{\frac{1}{2}k(n-\nu-1)} (a^{-1}bq^{-\frac{1}{2}(n+k+\nu)}; q)_n, \\
c) \quad & \frac{(a^{-1}bq^{-\frac{1}{2}(n+k+\nu)}; q)_n}{(ab^{-1}q^{\frac{1}{2}(2-n+k+\nu)}; q)_n} = (-a^{-1}b)^n q^{-\frac{1}{2}n(1+k+\nu)}, \\
d) \quad & (ab^{-1}q^{\frac{1}{2}(1-n-k-\nu)}; q)_n = (-ab^{-1})^n q^{-\frac{1}{2}n(k+\nu)} (a^{-1}bq^{\frac{1}{2}(1-n+k+\nu)}; q)_n.
\end{aligned}$$

Moreover by identity (A.17) we have

$$\frac{(q^{\nu+\frac{1}{2}}, -q^{\nu+\frac{1}{2}}, -q^{\nu+1}; q)_k}{(q^{\nu+1}, q^{2\nu+1}, q^{2\nu+k+1}; q)_k} = \frac{1}{(q^{\nu+1}, q^{\nu+1}; q)_k}. \quad (6.4.19)$$

Applying identities *a*) t/m *d*) and (6.4.19) we finally have

$$\begin{aligned}
A_k(x) &= \frac{1 - q^{k+\nu}}{1 - q^\nu} \frac{(abx^2)^k q^{\binom{k+1}{2}}}{(q^{\nu+1}, q^{\nu+1}; q)_k} \times \\
&\times \sum_{n=0}^{\infty} \frac{(a^{-1}bq^{\frac{1}{2}(1-n+k+\nu)}, a^{-1}bq^{\frac{1}{2}(1-n-k-\nu)}; q)_n}{(q^{\nu+k+1}, q; q)_n} (-1)^n (ax)^{2n} q^{\frac{1}{2}nk} q^{\binom{n+1}{2}} \times \\
&\times {}_4\Phi_3 \left( \begin{matrix} q^{-n}, -q^{\nu+k}, q^{\nu+k+\frac{1}{2}}, -q^{\nu+k+\frac{1}{2}} \\ q^{2\nu+2k+1}, ab^{-1}q^{\frac{1}{2}(1-n+\nu+k)}, a^{-1}bq^{\frac{1}{2}(1-n+\nu+k)} \end{matrix} \middle| q, q \right). \quad (6.4.20)
\end{aligned}$$

When we compare the last sum with the sum in the product formula (6.4.6), we see that they are the same if we replace in (6.4.6)  $\nu$  and  $x$  by  $\nu + k$  and  $xq^{\frac{1}{4}k}$  respectively. Substituting this result into (6.4.15) we have proved theorem 6.5:

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(a^{-1}be^{i\phi}q^{\frac{1}{2}(1-n)}, a^{-1}be^{-i\phi}q^{\frac{1}{2}(1-n)}; q)_n}{(q^{\nu+1}, q; q)_n} (-a^2x^2)^n q^{\binom{n+1}{2}} \\ &= \frac{(q, q; q)_{\infty}}{(q^{\nu+1}, q^{\nu+1}; q)_{\infty}} (abx^2)^{-\nu} \sum_{k=0}^{\infty} \frac{1 - q^{\nu+k}}{1 - q^{\nu}} q^{\frac{1}{2}k(1-\nu)} \times \\ & \quad \times J_{\nu+k}(axq^{\frac{1}{4}k}; q) J_{\nu+k}(bxq^{\frac{1}{4}k}; q) C_k(\cos \phi; q^{\nu}|q), \end{aligned}$$

where  $a, b > 0, x > 0, \Re e(\nu) > 0$  and  $0 \leq \phi \leq \pi$ .

# Appendix A

**Definitions and identities involving  $q$ -shifted factorials:**

Assume that  $n \in \mathbb{N}$ ,  $k \in \mathbb{N}$  and  $k \leq n$ , unless stated otherwise.

$$(a; q)_n \stackrel{\text{def}}{=} \begin{cases} 1 & n = 0 \\ (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) & n = 1, 2, \dots \\ [(1 - aq^{-1})(1 - aq^{-2}) \cdots (1 - aq^n)]^{-1} & n = -1, -2, \dots \end{cases} \quad (\text{A.1})$$

$$(a; q)_\infty \stackrel{\text{def}}{=} \prod_{k=0}^{\infty} (1 - aq^k) \quad (\text{A.2})$$

$$(a_1, a_2, \dots, a_k; q)_n \stackrel{\text{def}}{=} (a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n \quad n \in \mathbb{Z} \text{ or } n = \infty \quad (\text{A.3})$$

$$(a; q^{-1})_n = (a^{-1}; q)_n (-a)^n q^{-\binom{n}{2}} \quad (\text{A.4})$$

$$(a; q)_{-n} = \frac{1}{(aq^{-n}; q)_n} \quad (\text{A.5})$$

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty} \quad n \in \mathbb{Z} \quad (\text{A.6})$$

$$(aq^{-n}; q)_n = (a^{-1}q; q)_n (-aq^{-1})^n q^{-\binom{n}{2}} \quad (\text{A.7})$$

$$\frac{(aq^{-n}; q)_n}{(bq^{-n}; q)_n} = \frac{(a^{-1}q; q)_n}{(b^{-1}q; q)_n} \left(\frac{a}{b}\right)^n \quad (\text{A.8})$$

$$(a; q)_{n-k} = \frac{(a; q)_n}{(a^{-1}q^{1-n}; q)_k} \left(-\frac{q}{a}\right)^k q^{\binom{k}{2}-nk} \quad (\text{A.9})$$

$$\frac{(a; q)_{n-k}}{(b; q)_{n-k}} = \frac{(a; q)_n (b^{-1}q^{1-n}; q)_k}{(b; q)_n (a^{-1}q^{1-n}; q)_k} \left(\frac{b}{a}\right)^k \quad (\text{A.10})$$

$$(aq^{-n}; q)_{n-k} = \frac{(a^{-1}q; q)_n}{(a^{-1}q; q)_k} \left(-\frac{a}{q}\right)^{n-k} q^{\binom{k}{2}-\binom{n}{2}} \quad (\text{A.11})$$

$$(aq^{-2n}; q)_n = \frac{(a^{-1}q; q)_{2n}}{(a^{-1}q; q)_n} \left(-\frac{a}{q^2}\right)^n q^{-3\binom{n}{2}} \quad (\text{A.12})$$

$$(a; q)_{n+k} = (a; q)_n (aq^n; q)_k \quad (\text{A.13})$$

$$(aq^k; q)_{n-k} = \frac{(a; q)_n}{(a; q)_k} \quad (\text{A.14})$$

$$(a; q)_{2n} = (a, aq; q^2)_n \quad n \in \mathbb{N} \text{ or } n = \infty \quad (\text{A.15})$$

$$(a^2; q^2)_n = (a, -a; q)_n \quad n \in \mathbb{N} \text{ or } n = \infty \quad (\text{A.16})$$

$$(a, -a, aq^{\frac{1}{2}}, -aq^{\frac{1}{2}}; q)_n = (a^2; q)_{2n} \quad n \in \mathbb{N} \text{ or } n = \infty \quad (\text{A.17})$$

# Appendix B

**Rigorous proofs of some limit transitions:**

**Theorem B.1.** *For  $\nu > -1$  we have*

$$\lim_{q \uparrow 1} {}_1\Phi_1 \left( \begin{array}{c} 0 \\ q^{\nu+1} \end{array} \middle| q, (1-q)^2 z \right) = {}_0F_1 \left( \begin{array}{c} - \\ \nu + 1 \end{array} \middle| -z \right),$$

*uniformly in  $z$  on compact subsets of  $\mathbb{C}$ .*

**Proof.**

$${}_1\Phi_1 \left( \begin{array}{c} 0 \\ q^{\nu+1} \end{array} \middle| q, (1-q)^2 z \right) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (1-q)^{2k} z^k}{(q^{\nu+1}; q)_k (q; q)_k}$$

and the summand of the sum at the right hand side can be majorized by

$$(q^{-\frac{1}{2}\nu} |z|)^k \prod_{j=0}^{k-1} \frac{(q^{\frac{1}{2}(\nu+j)} - q^{\frac{1}{2}(\nu+j)+1})(q^{\frac{1}{2}j} - q^{\frac{1}{2}j+1})}{(1 - q^{\nu+j+1})(1 - q^{j+1})}. \quad (\text{B.1})$$

Now, by [33] Lemma A.1 (with  $-1 \leq \mu - \lambda$  instead of  $0 \leq \mu - \lambda$ ), we see that

$$\frac{q^{\frac{1}{2}(\nu+j)} - q^{\frac{1}{2}(\nu+j)+1}}{1 - q^{\nu+j+1}}$$

increases to  $(\nu + j + 1)^{-1}$  as  $q \uparrow 1$  if  $\nu + j \geq 0$ . So, the expression in (B.1), for  $\frac{1}{2} < q < 1$ , is dominated by

$$\frac{(2^{\frac{1}{2}\nu} |z|)^k}{(\nu + 1)_k k!} \text{ if } \nu \geq 0$$

and by

$$\text{const.} \frac{|z|^k}{(\nu + 1)_k k!} \text{ if } -1 < \nu < 0.$$

So the theorem follows by dominated convergence.  $\square$

**Theorem B.2.** For  $0 < a < q^{-1}$  and  $0 \leq b < q^{-1}$  we have

$$\lim_{n \rightarrow \infty} {}_2\Phi_1 \left( \begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \middle| q, xq^n \right) = {}_1\Phi_1 \left( \begin{matrix} 0 \\ aq \end{matrix} \middle| q, x \right),$$

uniformly on compact subsets of  $\mathbb{C}$ .

**Proof.** Put

$$\begin{aligned} R_n(x) &= {}_2\Phi_1 \left( \begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \middle| q, xq^n \right) - {}_1\Phi_1 \left( \begin{matrix} 0 \\ aq \end{matrix} \middle| q, x \right) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} x^k}{(aq; q)_k (q; q)_k} \left( -1 + \prod_{j=1}^k (1 - q^{n-j+1})(1 - abq^{n+j}) \right). \end{aligned}$$

Here we have used (A.9). Since

$$\prod_{j=1}^k (1 - x_j) \geq 1 - \sum_{j=1}^k x_j \text{ if } 0 \leq x_j \leq 1, \quad j = 1, \dots, k, \quad (\text{B.2})$$

we have

$$\begin{aligned} \left| -1 + \prod_{j=1}^k (1 - q^{n-j+1})(1 - abq^{n+j}) \right| &\leq \sum_{j=1}^k q^{n-j+1} + \sum_{j=1}^k abq^{n+j} \\ &= \frac{q^{n+1}}{1-q} \left( (1-q^k)ab - 1 + q^{-k} \right). \end{aligned}$$

Thus, for  $|x| \leq M$ :

$$|R_n(x)| \leq \frac{q^{n+1}}{1-q} \sum_{k=1}^{\infty} \frac{q^{\binom{k}{2}} M^k}{(aq; q)_k (q; q)_k} \left( (1-q^k)ab - 1 + q^{-k} \right),$$

where the infinite sum converges for all  $M > 0$  by d'Alembert's ratio test.  $\square$

**Theorem B.3.** Define the quantum  $q$ -Krawtchouk polynomial  $K_n(q^{-x}; p, N; q)$  by (3.2.39). Then for  $n, k \in \mathbb{Z}$  we have

$$\begin{aligned} \lim_{L \rightarrow \infty} (-x^2)^L q^{\binom{L+1}{2}} q^{Lk} K_{L-n}(q^{-L-k}; x^{-2}q^{-2L-1}, 2L; q) \\ = (q^{n+k+1}; q)_{\infty} (-x^2)^n q^{\binom{n+1}{2}} q^{nk} {}_1\Phi_1 \left( \begin{matrix} 0 \\ q^{n+k+1} \end{matrix} \middle| q, x^2 q^{n+k+1} \right), \end{aligned}$$

uniformly on compact subsets of  $\mathbb{C}$ .

**Proof.** In subsection 3.2.7 we have shown that the quantum  $q$ -Krawtchouk polynomial  $K_{L-n}(q^{-L-k}; x^{-2}q^{-2L-1}, 2L; q)$  can be rewritten as

$$\frac{(q; q)_{k+L}(q; q)_{n+L}}{(q; q)_{k+n}(q; q)_{2L}} (-x^2)^{n-L} q^{\binom{n}{2}} q^{(k+1)(n-L)} q^{-\binom{L}{2}} {}_2\Phi_1 \left( \begin{array}{c} q^{n-L}, q^{1+L+n} \\ q^{n+k+1} \end{array} \middle| q, x^2 q^{L+k+1} \right).$$

Thus it suffices to show that

$$\lim_{L \rightarrow \infty} \frac{(q; q)_{k+L}(q; q)_{n+L}}{(q; q)_{k+n}(q; q)_{2L}} {}_2\Phi_1 \left( \begin{array}{c} q^{n-L}, q^{1+n+L} \\ q^{n+k+1} \end{array} \middle| q, x^2 q^{L+k+1} \right) = (q^{n+k+1}; q)_{\infty} {}_1\Phi_1 \left( \begin{array}{c} 0 \\ q^{n+k+1} \end{array} \middle| q, x^2 q^{n+k+1} \right)$$

Since we have

$$\frac{(q; q)_{k+L}(q; q)_{n+L}}{(q; q)_{k+n}(q; q)_{2L}} \rightarrow (q^{n+k+1}; q)_{\infty},$$

if  $L \rightarrow \infty$ , we consider

$$\begin{aligned} R_L(x) &= {}_2\Phi_1 \left( \begin{array}{c} q^{n-L}, q^{1+n+L} \\ q^{n+k+1} \end{array} \middle| q, x^2 q^{L+k+1} \right) - {}_1\Phi_1 \left( \begin{array}{c} 0 \\ q^{n+k+1} \end{array} \middle| q, x^2 q^{n+k+1} \right) \\ &= \sum_{m=1}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} q^{m(n+k+1)} x^{2m}}{(q^{n+k+1}; q)_m (q; q)_m} \left( -1 + \prod_{j=1}^m (1 - q^{L-n-j})(1 - q^{1+n+L+j}) \right). \end{aligned}$$

Now by (B.2) we have

$$\begin{aligned} \left| -1 + \prod_{j=1}^m (1 - q^{L-n-j})(1 - q^{1+n+L+j}) \right| &\leq \sum_{j=1}^m q^{L-n-j} + \sum_{j=1}^m q^{1+n+L+j} \\ &= \frac{q^L}{1-q} (q^{n+2}(1-q^m) + q^{-n}(q^{-m}-q)). \end{aligned}$$

Thus, for  $|x| < M$ :

$$|R_L(x)| \leq \frac{q^L}{1-q} \sum_{m=1}^{\infty} \frac{q^{\binom{m}{2}} q^{m(n+k+1)} M^{2m}}{(q^{n+k+1}; q)_m (q; q)_m} (q^{n+2}(1-q^m) + q^{-n}(q^{-m}-q)),$$

where the infinite sum converges for all  $M > 0$  by d'Alembert's ratio test.  $\square$

**Theorem B.4.** Define the Affine  $q$ -Krawtchouk polynomial  $K_n^{Aff}(q^{-x}; p, N; q)$  by (3.2.43). Then for  $0 < x^2 q < 1$  we have

$$\lim_{L \rightarrow \infty} K_{L-n}^{Aff}(q^{-L-k}; x^2, 2L; q) = {}_1\Phi_1 \left( \begin{array}{c} 0 \\ x^2 q \end{array} \middle| q, q^{n-k+1} \right).$$

uniformly in  $x$ .

**Proof.** In subsection 3.2.7 we have shown that the Affine  $q$ -Krawtchouk polynomial  $K_{L-n}^{Aff}(q^{-L-k}; x^2, 2L; q)$  can be rewritten as

$$K_{L-n}^{Aff}(q^{-L-k}; x^2, 2L; q) = \frac{(q; q)_{n+L}}{(q; q)_{2L}} {}_2\Phi_1 \left( \begin{array}{c} q^{-L+n}, x^2 q^{k+L+1} \\ x^2 q \end{array} \middle| q, q^{L-k+1} \right).$$

Since

$$\frac{(q; q)_{n+L}}{(q; q)_{2L}} \rightarrow 1,$$

if  $L \rightarrow \infty$ , we have proved theorem B.4 by applying theorem B.2.  $\square$

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# Summary

In this thesis we study functions  $J_\nu(x; q)$  which are defined by

$$J_\nu(x; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{1}{2}k(k+1)} x^{2k}}{(q^{\nu+1}; q)_k (q; q)_k},$$

where  $q$ , called the base, satisfies  $0 < q < 1$ , and

$$(a; q)_n = \begin{cases} 1 & \text{if } n = 0, \\ (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) & \text{if } n = 1, 2, \dots, \end{cases}$$

and

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$$

This function is a generalization of the Bessel function  $J_\nu(x)$  since we have the limit relation

$$\lim_{q \uparrow 1} J_\nu(x(1 - q)/2; q) = J_\nu(x).$$

In view of the base  $q$ , this function is called a basic analogue (or a  $q$ -analogue) of the Bessel function.

This  $q$ -Bessel function was introduced by W. Hahn (in the special case  $\nu = 0$ ) in 1953 and by H. Exton (in full) in 1978, so we called this function the Hahn-Exton  $q$ -Bessel function. They obtained this function as the solution of a special basic Sturm-Liouville equation. Exton also derived a generating function and some recurrence and difference-recurrence relations for this function. Vaksman and Korogodskii implicitly gave some orthogonality relations for this function in 1989.

In part I of this thesis we mention the classical results concerning the Bessel function, which we generalize in part II, in chapter 1. Further, in chapter 2 we give a survey on  $q$ -theory and state some known results that we need in the second part.

In part II of this thesis we study the Hahn-Exton  $q$ -Bessel function. In chapter 3 we will give the results that Exton found and we derive some new results. We will also state

the two orthogonality relations explicitly and prove them. In chapter 4 we study the  $q$ -analogue of the Bessel differential equation. We also discuss a second solution, which is a  $q$ -analogue of the classical  $Y_\nu$ . In chapter 5 basic analogues of some integral transforms are considered. We derive  $q$ -analogues of the Hankel transform, the Fourier-sine and Fourier-cosine transform. In the last chapter three  $q$ -extensions of addition formulas are discussed. Two of them are a generalization of Graf's addition theorem. The third one is an extension of Gegenbauers addition formula.

# Samenvatting

In dit proefschrift bestuderen we de functie  $J_\nu(x; q)$  die wordt gedefinieerd door

$$J_\nu(x; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{1}{2}k(k+1)} x^{2k}}{(q^{\nu+1}; q)_k (q; q)_k},$$

waarin  $q$ , de basis genoemd, voldoet aan  $0 < q < 1$ , en

$$(a; q)_n = \begin{cases} 1 & \text{als } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{als } n = 1, 2, \dots, \end{cases}$$

en

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$$

Deze functie is een generalisatie van de Bessel functie  $J_\nu(x)$ , omdat de limietrelatie

$$\lim_{q \uparrow 1} J_\nu(x(1-q)/2; q) = J_\nu(x)$$

geldt. Vanwege de basis  $q$ , wordt deze functie een  $q$ -analogon van de Bessel functie genoemd.

Deze  $q$ -Bessel functie werd geïntroduceerd door W. Hahn (in het speciale geval  $\nu = 0$ ) in 1953 en door H. Exton (voor alle  $\nu$ ) in 1978, en wordt daarom Hahn-Exton  $q$ -Bessel functie genoemd. Zij vonden hun functie als oplossing van een speciale  $q$ -uitbreiding van een Sturm-Liouville vergelijking. Exton vond ook een genererende functie en enkele recurrente en differentie-recurrente betrekkingen voor deze functie. Vaksman en Korogodskii gaven impliciet twee orthogonaliteitsrelaties voor de Hahn-Exton  $q$ -Bessel functie in 1989.

In deel I van dit proefschrift geven we de klassieke resultaten betreffende de Bessel functie, die we in deel II zullen generaliseren, in hoofdstuk 1. In hoofdstuk 2 geven we een beknopt overzicht van de  $q$ -theorie en noemen we enkele bekende resultaten die we in het tweede deel zullen gebruiken.

In deel II van dit proefschrift bestuderen we de Hahn-Exton  $q$ -Bessel functie. In hoofdstuk 3 geven we de resultaten die Exton heeft gevonden en we leiden enkele nieuwe resultaten af. We geven tevens de orthogonaliteitsrelaties van Vaksman en Korogodskii

expliciet en bewijzen deze relaties. In hoofdstuk 4 beschouwen we het  $q$ -analogon van de differentiaalvergelijking van Bessel. We geven een tweede oplossing die een  $q$ -uitbreiding is van de klassieke  $Y_\nu$ . In hoofdstuk 5 worden  $q$ -analoga van enkele integraaltransformaties beschouwd. We leiden een  $q$ -analogon van de Hankel transformatie af en geven als een speciaal geval de  $q$ -Fourier-cosinus en  $q$ -Fourier-sinus transformaties. In het laatste hoofdstuk worden drie generalisaties van additieformules bewezen. Hiervan zijn er twee generalisaties van Graf's additie formule. De derde is een uitbreiding van Gegenbauers additieformule.

# Curriculum Vitae

De schrijver van dit proefschrift werd geboren op 29 maart 1964 te 's-Gravenhage. In 1982 behaalde hij het VWO-diploma aan de Dalton Scholengemeenschap te 's-Gravenhage.

In hetzelfde jaar begon hij de studie Wiskunde aan de Technische Universiteit te Delft. Hij schreef zijn afstudeerscriptie onder leiding van Prof. dr. H.G. Meijer en behaalde op 27 november 1987 zijn ingenieursdiploma alsmede de onderwijsbevoegdheid Wiskunde.

Van 1 januari 1988 tot 1 januari 1992 was hij werkzaam als Assistent in Opleiding bij de faculteit Technische Wiskunde en Informatica van de Technische Universiteit te Delft. In deze functie heeft hij het onderzoek verricht dat heeft geleid tot dit proefschrift.