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Hölder Regularity of the Multifractional Stable Motion

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Abstract

In this master thesis, we introduce a new multifractional stable motion, which we refer to as the Itô multifractional stable motion. The definition of the Itô multifractional stable motion is inspired by a relatively recently proposed alternative to the multifractional Brownian motion. The Itô multifractional stable motion is defined as

$$Y(t) = \int_{\mathbb{R}} (t-x)_+^{H(x)-\frac{1}{\alpha}} - (-x)_+^{H(x)-\frac{1}{\alpha}} \, dL(x).$$

Here $(x)_+ = \max(x,0), \alpha \in (0,2), L$ is a standard symmetric α-stable Lévy process and finally, the multifractional parameter H is a jointly measurable stochastic process, adapted to the natural filtration generated by L, taking values in $[\underline{H}, \overline{H}] \subseteq (0,1)$. Under the assumption that H admits a deterministic modulus of continuity w and that H is strictly bounded from below by $\frac{1}{\alpha}$, it is proven that the uniform Hölder exponent $\rho_Y^{\text{unif}}([a,b])$ over a compact interval satisfies

$$\rho_Y^{\text{unif}}([a,b]) \geq \min_{t\in[a,b]} H(t) - \frac{1}{\alpha}.$$

Under the further assumption that $w(h)\log h \to 0$ as $h \downarrow 0$, it is shown that Y is locally self-similar and that the pointwise Hölder exponent $\rho_Y(t)$ satisfies

$$\rho_Y(t) \leq H(t).$$

Contents

1 Introduction

The fractional Brownian motion is a generalization of the Brownian motion, parameterized by a constant $0 < H < 1$, that was introduced by Mandelbrot and Van Ness [MN68]. It is given by

$$\int_{\mathbb{R}} (t-x)_+^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}} \, dW(x) \quad t \in \mathbb{R},$$

where W is a Brownian motion and $(x)_+ = \max(x, 0)$. They showed that the process is H-self-similar and has stationary increments. These properties are characteristic in the sense that any Gaussian process satisfying them must be equal in distribution to a constant multiple of the fractional Brownian motion. Unlike the Brownian motion, the fractional Brownian motion can have dependent increments, which gives it many applications in areas such as network traffic, finance, autoregressive modeling and hydrology [DOT02].

Having said that, the homogeneity of the Hölder exponent (which is equal to H) of the fractional Brownian motion along its paths makes it unfit for modeling non-stationary phenomena. For this reason, Peltier and Lévy Véhel introduced the multifractional Brownian motion [PV95], where the constant H is replaced by a function $H(t)$. They show that the pointwise Hölder exponent of this new process is $H(t)$ so that it changes along its path. However, there are two theoretical drawbacks to Peltier and Lévy Véhel's multifractional Brownian motion. Namely that their formulation does not provide a natural framework for modeling a multifractional Brownian motion with a random multifractional parameter (even though it is possible [AT05]). Secondly, the Hölder regularity of the resulting process is conditional on a Hölder condition on the multifractional parameter H.

The first drawback was addressed by Ayache, Esser and Hamonier [AEH18], who introduce a new multifractional Brownian motion where the multifractional parameter $H(x)$ is a random function of the integration parameter x. Note that the kernels (indexed by t) are adapted to the natural filtration generated by the Brownian motion W, which means the resulting process may be understood as a collection of Itô integrals. To distinguish between the two types of multifractional Brownian motions, we will call Peltier and Lévy Véhel's process the classical multifractional Brownian motion. In contrast, we will refer to Ayache et al's process as the Itô multifractional Brownian motion. Ayache et al prove that the pointwise Hölder exponent of the Itô multifractional Brownian motion at t is at least $H(t)$, but still have to assume a Hölder condition on the multifractional parameter $H(x)$. This last drawback is solved by Loboda, Mies and Steland [LMS21], who show that the pointwise Hölder exponent of the Itô multifractional Brownian motion at t is equal to $H(t)$, irrespective of the Hölder regularity of $H(x)$.

Up to this point, the considered processes have all been Gaussian. However, comparable processes have been considered in the stable regime. Stable distributions are a generalization of normal distributions that share the ubiquity of normal distributions in the sense that they serve as limit laws in a central limit theorem. However, unlike normal distributions, the tails of stable distributions are heavy, which makes them more suitable for modeling certain phenomena. The fractional Brownian motion has its analogue in the stable regime: The linear stable fractional motion. This process is given by

$$\int_{\mathbb{R}} (t-x)_+^{H-\frac{1}{\alpha}} - (-x)_+^{H-\frac{1}{\alpha}} \, dL(x) \quad t \in \mathbb{R},$$

where L is a symmetric α-stable Lévy process. Just like in the Gaussian case, a multifractional variant has been suggested [ST04], where the fractional parameter H is replaced by a deterministic function $H(t)$. The pointwise Hölder regularity is a bit more nuanced in the stable case. Indeed, Ayache and Hamonier find that, under a Hölder condition on the multifractional parameter $H(t)$, the pointwise Hölder exponent is $H(t)$, and the *uniform* pointwise Hölder exponent is $H(t) - \frac{1}{\alpha}$ [AH14]. The same two theoretical drawbacks of the classical multifractional Brownian motion arise in the stable case: The formulation does not provide a natural framework for considering random multifractional parameters, and the Hölder regularity of the resulting process depends on the Hölder regularity of the multifractional parameter. In the Gaussian case, these drawbacks were resolved by considering an Itô multifractional Brownian motion, but no such attempts have been made in the stable regime.

The goal of this thesis is to introduce an Itô multifractional stable motion, where the multifractional parameter $H(x)$ is a random function of the integration variable. This allows us to naturally interpret the resulting stochastic process as an Itô integral. Moreover, we will attempt to compute the (uniform) pointwise Hölder regularity of this process, without imposing any conditions on the Hölder regularity of the multifractional parameter.

In Section 2, some preliminary topics will be laid out. We will cover (uniform) (pointwise) Hölder exponents, quasinormed spaces, the Kolmogorov extension and continuity theorem, weak convergence and (weak) Lebesgue spaces. In Section 3, we will cover the multifractional Brownian motion in detail. First, the fractional Brownian motion will be properly defined. Then, we will cover the previously stated results on the Hölder regularity of the different types of multifractional Brownian motions in more detail. Sections 4 and 5 will be about the theory of stable distributions and Lévy processes respectively. Stable Lévy processes are the stable counterpart to Brownian motion and will be used in defining the (multi)fractional stable motion as a collection of Itô integrals. Finally, in Section 6 we will develop an Itô calculus for stable Lévy processes and consider the linear stable fractional motion and its classical multifractional variant. The previously stated results will be covered in detail here too. Finally, we introduce the Itô multifractional stable motion, and compute its pointwise Hölder regularity.

2 Preliminaries

Before heading into the main topics of this thesis, we will cover some preliminary topics needed to create a foundation for the rest of the document. Our main result is a bound on the Hölder exponent of a stochastic process, so will of course introduce Hölder exponents. Next, we will cover a bit of functional analysis which will allow us to develop the Itô calculus through which the stochastic process is defined. Finally, we will introduce some concepts from stochastics and fix some notation.

2.1 Hölder exponents

Hölder exponents are a way to measure the smoothness or regularity of a continuous function in the absence of differentiability. This can be done globally over a set of points, or pointwise. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous functions and let $\rho \geq 0$. Let $I \subseteq \mathbb{R}$ be a non-empty interval and let $t \in \mathbb{R}$ be a point. Consider the following three Hölder conditions.

1. $$\sup_{\substack{t_1,t_2\in I\\ t_1\neq t_2}} \frac{|f(t_2)-f(t_1)|}{|t_2-t_1|^\rho} < \infty,$$
2. $$\limsup_{h\to 0} \frac{|f(t+h)-f(t)|}{|h|^\rho} = 0,$$
3. $$\limsup_{h\downarrow 0} \sup_{\substack{t_1,t_2\in [t-h,t+h]\\ t_1\neq t_2}} \frac{|f(t_2)-f(t_1)|}{|t_2-t_1|^\rho} = 0.$$

If condition 1 is satisfied then f is **uniformly ρ-Hölder continuous** on I, under condition 2 f is called **pointwise ρ-Hölder continuous** at t and under condition 3 f is **uniformly pointwise ρ-Hölder continuous** at t. If f is uniformly ρ-Hölder continuous on all non-empty compact intervals then f is said to be **locally ρ-Hölder continuous**. The Hölder exponents $\rho_f^{\text{unif}}(I), \rho_f(t)$ and $\rho_f^{\text{unif}}(t)$ are then defined as the supremum of all ρ such that the Hölder condition is satisfied for ρ.

Definition 2.1. The **uniform Hölder exponent**, the **pointwise Hölder exponent** and the **uniform pointwise Hölder exponent** are respectively defined as the quantities

$$\rho_f^{\text{unif}}(I) = \sup\left\{\rho \geq 0 : \sup_{\substack{t_1,t_2\in I\\ t_1\neq t_2}} \frac{|f(t_2)-f(t_1)|}{|t_2-t_1|^\rho} < \infty\right\},$$
$$\rho_f(t) = \sup\left\{\rho \geq 0 : \limsup_{h\to 0} \frac{|f(t+h)-f(t)|}{|h|^\rho} = 0\right\},$$
$$\rho_f^{\text{unif}}(t) = \sup\left\{\rho \geq 0 : \limsup_{h\downarrow 0} \sup_{\substack{t_1,t_2\in [t-h,t+h]\\ t_1\neq t_2}} \frac{|f(t_2)-f(t_1)|}{|t_2-t_1|^\rho} = 0\right\}.$$

From these definitions it readily follows that

$$\rho_f^{\text{unif}}(I) \leq \rho_f^{\text{unif}}(t) \leq \rho_f(t)$$

whenever $t \in I$. The difference between the pointwise Hölder exponent and the uniform pointwise Hölder exponent is a bit subtle but they are not equal in general: Consider the function $f(t) = t\sin(1/t)$ (with $f(0) = 0$). Then, whenever $\rho < 1$ we have $|h|^{-\rho}|f(h)| = |h|^{1-\rho}|\sin(1/h)| \to 0$ as $h \to 0$, so $\rho_f(t) \geq 1$. However, letting $1/t_n^1 = \pi/2 + 2\pi n$ and $1/t_n^2 = 3\pi/2 + 2\pi n$, we have

$$\frac{|f(t_n^2) - f(t_n^1)|}{|t_n^2 - t_n^1|^{\frac{1}{2}}} = \frac{t_n^1 + t_n^2}{(\pi t_n^1 t_n^2)^{\frac{1}{2}}} \overset{n\to\infty}{\to} 2\pi^{-\frac{1}{2}}.$$

Thus, $|t_n^2 - t_n^1|^{-\rho}\,|f(t_n^2) - f(t_n^1)| \to \infty$ whenever $\rho > \frac{1}{2}$ and $\rho_f^{\text{unif}}(t) \leq \frac{1}{2} < 1 \leq \rho_f(t)$.

Note that being ρ-Hölder continuous on I can be restated to the existence of a constant $C > 0$ such that $|f(t_2) - f(t_1)| \leq C|t_2 - t_1|^\rho$ for all $t_1, t_2 \in I$. Measuring the behavior of increments like this can be done more precisely by considering a general modulus of continuity.

Definition 2.2. An increasing function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $w(0) = 0$ and w is continuous at 0 is a **modulus of continuity**. A function f **admits** w **as a modulus of continuity** on $I \subseteq \mathbb{R}$ if, for all $t_1, t_2 \in I$,

$$|f(t_2) - f(t_1)| \leq w(|t_2 - t_1|).$$

2.2 Quasinormed spaces

This section will cover quasinormed spaces, which are needed to develop the Itô calculus through which we will define stochastic processes of interest. Indeed: These Itô integrals will be defined as continuous dense extensions of continuous linear operators between spaces that almost satisfy the axioms for a normed space, but the triangle inequality is weakened.

Definition 2.3. Let X be a real vector space. The functional $\|\cdot\| : X \to \mathbb{R}$ is a **quasinorm** if it satisfies

- Positivity: $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$,
- Absolute homogeneity: $\|cx\| = |c|\|x\|$ for all $c \in \mathbb{R}$ and $x \in X$,
- The weakened triangle inequality: There is a constant $C \geq 1$ such that $\|x + y\| \leq C(\|x\| + \|y\|)$ for all $x, y \in X$.

If, instead of the weakened triangle inequality, it holds that $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for some $p \in (0, 1]$, then $\|\cdot\|$ is called a ***p*-norm**.

Open balls (i.e. sets of the form $\{x \in X : \|x - x_0\| < r\}$ do not form a basis for a topology when $\|\cdot\|$ is a quasinorm, but they do if $\|\cdot\|$ is a p-norm. In fact, in this case $d(x, y) = \|x - y\|^p$ is a metric and if this metric is complete then the p-normed space $(X, \|\cdot\|)$ is called a ***p*-Banach space**. To define and reason about a topology on a quasinormed space, then, the following result is crucial because it allows a quasinormed space to be renormed to a p-norm.

Theorem 2.4 (Aoki-Rolewicz theorem)**.** *Let $(X, \|\cdot\|)$ be a quasinormed space. Then there are $p \in (0, 1]$, a p-norm $[\cdot]$ on X and constants $c > 0$ and $C > 0$ such that*

$$c[x] \leq \|x\| \leq C[x] \quad \textit{for all } x \in X.$$

Proof. [Aok42; Rol57], see also [Kal03]. □

Now a quasinormed space $(X, \|\cdot\|)$ may be topologized by attributing to it the metric topology generated by an equivalent p-norm (this topology does not depend on such a p-norm). If this metric topology is complete (this property is also independent of the chosen p-norm), then $(X, \|\cdot\|)$ is a **quasi-Banach space**. Itô integrals will be constructed in quasi-Banach spaces, for which we will need the following two lemmas.

Lemma 2.5. *Let $(X, \|\cdot\|_X)$ be a p-normed space and let $(Y, \|\cdot\|_Y)$ be a q-Banach space with $p, q \in (0, 1]$. Suppose $f : D \to Y$ is a linear isometry on the subspace $D \subseteq X$. Then f uniquely extends to a linear isometry $\overline{f} : \overline{D} \to Y$ on the closure.*

Proof. Let $x \in \overline{D}$ and choose a sequence $(x_n)_{n\in\mathbb{N}}$ in D such that $x_n \to X$ in $(X, \|\cdot\|_X)$. Then $\|f(x_n) - f(x_m)\|_Y = \|x_n - x_m\|_X$ for all $n, m \in \mathbb{N}$, so $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy in $(Y, \|\cdot\|_Y)$. Define $\overline{f}(x)$ to be the unique limit of $(f(x_n))_{n\in\mathbb{N}}$ in $(Y, \|\cdot\|_Y)$.

To show that this is well defined, let $(x'_n)_{n\in\mathbb{N}}$ be another sequence in D such that $x'_n \to x$ in $(X, \|\cdot\|_X)$. Then $\|f(x_n) - f(x'_n)\|_Y = \|x_n - x'_n\|_X \leq (\|x_n - x\|_X + \|x'_n - x\|)^{1/p} \to 0$, so $f(x_n) - f(x'_n) \to 0$ in $(Y, \|\cdot\|_Y)$. It follows that $(f(x_n))_{n\in\mathbb{N}}$ and $(f(x'_n))_{n\in\mathbb{N}}$ converge to the same limit in $(Y, \|\cdot\|_Y)$.

Next we show that $\overline{f} : \overline{D} \to Y$ is linear. Let $x, x' \in \overline{D}$ and $a, b \in \mathbb{R}$ and take sequences $(x_n)_{n\in\mathbb{N}}$ and $(x'_n)_{n\in\mathbb{N}}$ in D such that $x_n \to x$ and $x'_n \to x'$ in $(X, \|\cdot\|_X)$. Then $ax_n + bx'_n \to ax + bx'$ in $(X, \|\cdot\|_X)$, so $f(ax_n + bx'_n) \to \overline{f}(ax + bx')$ in $(Y, \|\cdot\|_Y)$ by definition. But also, $f(ax_n + bx'_n) = af(x_n) + bf(x'_n) \to a\overline{f}(x) + b\overline{f}(x')$ in $(Y, \|\cdot\|_Y)$, so $\overline{f}(ax + bx') + a\overline{f}(x) + b\overline{f}(x')$.

Finally, $\overline{f} : \overline{D} \to Y$ is an isomtery by continuity of p-norms. Indeed: If $x \in \overline{D}$ and $(x_n)_{n\in\mathbb{N}}$ is a sequence in D such that $x_n \to x$ in $(X, \|\cdot\|_X)$, then $\|f(x_n)\|_Y = \|x_n\|_X$ for all $n \in \mathbb{N}$. Because the p-norm $\|\cdot\|_X$ and the q-norm $\|\cdot\|_Y$ are continuous with respect to the topology they generate, taking $n \to \infty$ reveals that $\|\overline{f}(x)\|_Y = \|x\|_X$. The extension is unique because $(Y, \|\cdot\|_Y)$ is Hausdorff. □

Lemma 2.6. *Let $\{(X_i, \|\cdot\|_{X_i}) : i \in \mathcal{I}\}$ be a collection of p_i-normed spaces with each $p_i \in (0, 1]$, let $(Y, \|\cdot\|_Y)$ be a quasi-Banach space. Suppose $f_i : D_i \to Y$ with $i \in \mathcal{I}$ are linear operators on the subspaces $D_i \subseteq X_i$ that are uniformly bounded (i.e. there is a constant $A > 0$ such that $\|f_i(x)\|_Y \leq A\|x\|_{X_i}$ for all $i \in \mathcal{I}$ and $x \in X_i$). Then these linear operators uniquely extend to a collection of uniformly bounded linear operators $\overline{f_i} : \overline{D_i} \to Y$ on the closures.*

Proof. Let $[\cdot]_Y$ be an equivalent norm on Y making $(Y, [\cdot]_Y)$ a p-Banach space ($0 < p \leq 1$) obtained from the Aoki-Rolewicz theorem and let $c > 0$ and $C > 0$ be such that $c[y]_Y \leq \|y\|_Y \leq C[y]_Y$ for all $y \in Y$. Fix $i \in \mathcal{I}$, let $x \in \overline{D_i}$ and choose a sequence $(x_n)_{n\in\mathbb{N}}$ in D_i such that $x_n \to x$ in $(X_i, \|\cdot\|_{X_i})$. Then $[f(x_n) - f(x_m)]_Y \leq \frac{A}{c}\|x_n - x_m\|_{X_i}$. Thus, $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $(Y, [\cdot]_Y)$. Define $\overline{f_i}(x)$ to be the unique limit of $(f(x_n))_{n\in\mathbb{N}}$ in $(Y, [\cdot]_Y)$. The function $\overline{f_i} : \overline{D_i} \to Y$ is well defined and linear by the same arguments as in Lemma 2.5.

We prove that the collection $\{\overline{f_i} : i \in \mathcal{I}\}$ is uniformly bounded. Let $i \in \mathcal{I}$ and $x \in \overline{D_i}$ and choose a sequence $(x_n)_{n\in\mathbb{N}}$ in D_i such that $x_n \to x$ in $(X_i, \|\cdot\|_{X_i})$. Then for each $n \in \mathbb{N}$, $[f_i(x_n)]_Y \leq \frac{A}{c}\|x_n\|_{X_i}$. From continuity of p-norms it follows that $[\overline{f_i}(x)]_Y \leq \frac{A}{c}\|x\|_{X_i}$, so $\|\overline{f_i}(x)\|_Y \leq \frac{AC}{c}\|x\|_{X_i}$. Again, the extensions are unique due to the Hausdorff property. □

2.3 Stochastics

Finally, some topics from probability theory that will be used later will be treated. We will cover the definition of random variables and stochastic processes, the Kolmogorov extension and continuity theorems, the Brownian motion, weak convergence, the Cramér-wold theorem and (weak) Lebesgue spaces.

Definition 2.7. A measurable function $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (E, \mathcal{G})$ from a probability space to a measurable space is called an **E-valued random variable**. Letting T be an indexing set, a collection $(X(t))_{t\in T}$ of E-valued random variables is called an **E-valued stochastic process**.

A random variable induces a probability measure $X^*\mathbb{P}$ on E, given by

$$X^*\mathbb{P}(G) = \mathbb{P}(X^{-1}[G]) \quad G \in \mathcal{G}.$$

This measure is called the **pushforward measure** or the **law** of X. If two random variables X and Y (not necessarily defined on the same probability space) have the same law they are **equal in distribution**, this will be denoted by $X \overset{d}{=} Y$. If two random variables are independent this will be denoted with the symbol $\perp$.

Note that, by the universal property of the product, an E-valued stochastic process $(X(t))_{t\in T}$ can be viewed as an E^T-valued random variable, it can also be viewed as a joint map $X : \Omega \times T \to E$, but in this case, the notion measurability only makes sense if T attains the structure of a measurable space. If T has the structure of a measurable space and the joint map $\Omega \times T \to E$ is measurable, then the stochastic process is called **jointly measurable**. For fixed $\omega \in \Omega$ the map $X(\omega, \cdot) : T \to E$ is called a **sample path** of the process $(X(t))_{t\in T}$. If $(Y(t))_{t\in T}$ is another stochastic process and $\mathbb{P}(X(t) = Y(t)) = 1$ for all $t \in T$ then Y is a **modification** of X. If $\mathbb{P}(\forall t \in T\ X(t) = Y(t)) = 1$ then X and Y are **indistinguishable**. If $E = \mathbb{R}$ then the law of a stochastic process (on $\mathbb{R}^T$) is completely characterized by its finite-dimensional distributions. This is an implication of the Kolmogorov extension theorem.

Theorem 2.8 (Kolmogorov extension theorem)**.** *Let T be an index set, suppose each finite subset $F \subseteq T$ is attributed a probability measure $\mathscr{P}_F$ on $\mathbb{R}^F$ such that the **Kolmogorov consistency criterion** is met: For all pairs of finite subsets $F \subseteq T$ and $G \subseteq T$ such that $F \subseteq G$, we have*

$$(\restriction^G_F)^* \mathscr{P}_G = \mathscr{P}_F,$$

where $\restriction^G_F\colon \mathbb{R}^G \to \mathbb{R}^F$ restricts a function $G \to \mathbb{R}$ to F. Then there is a unique probability measure $\mathscr{P}_T$ on $\mathbb{R}^T$ such that $(\restriction^T_F)^* \mathscr{P}_T = \mathscr{P}_F$ for all finite $F \subseteq T$.

If $T \subseteq \mathbb{R}$ is a subset of the real line then the Kolmogorov extension theorem can be restated to the following: Suppose that for all $t_0 < \ldots < t_n$ in T there is a probability measure $\mathscr{P}_{t_0,\ldots,t_n}$ on $\mathbb{R}^n$ such that

$$\mathscr{P}_{t_0,\ldots,t_n}(A_0 \times \ldots \times A_n) = \mathscr{P}_{t_0,\ldots,t_{k-1},t_{k+1},\ldots,t_n}(A_0 \times \ldots \times A_{k-1} \times A_{k+1} \times \ldots \times A_n)$$

whenever $A_0 \ldots A_n$ are Borel measurable with $A_k = \mathbb{R}$. Then there is a unique probability measure on $\mathbb{R}^T$ that has $\mathscr{P}_{t_1\ldots t_n}$ as its finite-dimensional distributions.

The uniqueness part of this theorem implies that the law of a stochastic process is characterized by its finite-dimensional distributions, and the existence part allows us to find stochastic processes by prescribing its finite-dimensional distributions. However, no properties are placed on the sample paths of a stochastic process obtained in this way. For this, another theorem due to Kolmogorov is needed.

Theorem 2.9 (Kolmogorov-Chentsov continuity theorem)**.** *Let $T \subseteq \mathbb{R}$ be a (possibly unbounded) interval. Suppose $(X(t))_{t\in T}$ is a real-valued stochastic process and suppose there are positive constants α, β and C such that*

$$\mathbb{E}\,|X(t) - X(s)|^\alpha \le C|t - s|^{1+\beta} \quad s, t \in T.$$

Then there is a modification $(\tilde{X}(t))_{t\in T}$ of $(X(t))_{t\in T}$ that such that its sample paths are locally ρ-Hölder continuous for all $0 < \rho < \frac{\beta}{\alpha}$.

One of the most well known stochastic processes is the Brownian motion. Often this process is only considered on the half line $[0, \infty)$, but we will also need to consider this and other processes on the whole real line $\mathbb{R}$.

Definition 2.10. Let T be either $\mathbb{R}$ or $[0, \infty)$. The Brownian motion $(W(t))_{t \in T}$ is a real-valued stochastic process such that the following properties are satisfied.

1. $W(0) = 0$ almost surely.
2. Normally distributed increments: There is a constant $\sigma > 0$ such that $W(t) - W(s) \sim N(0, \sigma^2(t - s))$ for all $t > s$ in T.
3. Independent increments: For any $t_0 < t_1 < \ldots < t_n$ in T, the random variables $W(t_1) - W(t_0), W(t_2) - W(t_1) \ldots W(t_n) - W(t_{n-1})$ are independent.

Usually it is also required that the sample paths of the Brownian motion are continuous, and it may be more appropriate to call a process satisfying the conditions above a Brownian motion *in law*. This would follow the convention that Sato uses for Lévy processes [Sat99]. However, for the sake of synergy with the rest of the document, we will define the Brownian motion by its finite-dimensional distributional properties and demand no properties on the sample paths. Existence of the Brownian motion can then be shown by appealing to the Kolmogorov extension theorem. With the Kolmogorov-Chentsov continuity theorem it can then be shown that the Brownian motion admits a modification with continuous sample paths. The next topic that will be covered is weak convergence and convergence in distribution.

Definition 2.11. Let E be a metric space (equipped with the Borel sigma algebra). Then the sequence $(P_n)_{n \in \mathbb{N}}$ of probability measures on E **converges weakly** to the probability measure P if, for all bounded continuous functions $f : E \to \mathbb{R}$,

$$\int_E f \, dP_n \to \int_E f \, dP.$$

This defines a topology on the set of probability measures on E. If the laws of a sequence $(X_n)_{n \in \mathbb{N}}$ of E-valued random variables converge weakly to the law of the E-valued random variable X then we say that X_n **converges** to X **in distribution** (X_n and X may all be defined on different probability spaces).

To show convergence in distribution of random vectors, the following result, known as the Cramér-Wold theorem, is quite useful.

Theorem 2.12 (Cramér-Wold theorem)**.** *Let $\mathbf{X}_n$ and $\mathbf{X}$ be $\mathbb{R}^d$-valued random variables. Then $\mathbf{X}_n$ converges to $\mathbf{X}$ in distribution if and only if $\langle \mathbf{t}, \mathbf{X}_n \rangle$ converges in distribution to $\langle \mathbf{t}, \mathbf{X} \rangle$ for all $\mathbf{t} \in \mathbb{R}^d$, here $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^d$.*

Next we will discuss weak convergence in the metric space $C([a, b])$ of continuous real valued functions on a compact interval $[a, b] \subseteq \mathbb{R}$, equipped uniform convergence. This topic is covered extensively in Chapter 2 of Billingsley's book [Bil99]. The reason that weak convergence in this space is of interest to us is that we wish to speak about a form of distributional convergence of stochastic processes with continuous sample paths that is stronger than convergence in finite-dimensional distributions. To this end, due to the next lemma, we may understand a stochastic process $(X(t))_{t \in [a,b]}$ with continuous sample paths as a $C([a, b])$-valued random variable so that we can talk about convergence in distribution in the metric space $C([a, b])$.

Lemma 2.13. *The Borel sigma algebra on $C([a, b])$ is equal to the sigma algebra generated by the projections for $t \in [a, b]$, given by*

$$\begin{aligned} \pi_t : C([a, b]) &\to \mathbb{R} \\ x &\mapsto x(t). \end{aligned}$$

.

Proof. Write $\mathcal{B}$ and $\mathcal{P}$ for the Borel sigma algebra on $C([a,b])$ and the sigma algebra generated by the projections respectively. Then $\mathcal{P} \subseteq \mathcal{B}$ is immediate, because the projections are continuous (uniform convergence implies pointwise convergence). We will show that closed balls of the form

$$\overline{B}_r(x) = \left\{ y \in C([a,b]) : \sup_{t\in[a,b]} |y(t) - x(t)| \leq r \right\} \qquad x \in C([a,b]), r > 0$$

are in $\mathcal{P}$. It then follows that open balls are in $\mathcal{P}$, because these can be written as a countable union of closed balls. Finally, because $C([a,b])$ is separable and therefore second countable, any open set can be written as a countable union of open balls so it follows that all open sets are in $\mathcal{P}$ and that $\mathcal{B} \subseteq \mathcal{P}$. Thus it remains to be shown that $\overline{B}_r(x) \in \mathcal{P}$ for $x \in C([a,b])$ and $r > 0$. To this end, simply note that

$$\overline{B}_r(x) = \bigcap_{t\in\mathbb{Q}\cap[a,b]} \pi_t^{-1} \left[[x(t) - r, x(t) + r] \right].$$

□

It is thus relevant to have criteria for weak convergence of probability measures on $C([a,b])$. These criteria are given by convergence in finite-dimensional distributions, in the presence of a property known as tightness. These criteria are an immediate consequence of Prohorov's theorem.

Definition 2.14. Let Π be a set of probability measures on E. Then Π is **tight** if, for every $\epsilon > 0$ there is a compact subset $K \subseteq E$ such that $P(K) > 1 - \epsilon$ for all $P \in \Pi$. Moreover, Π is **relatively compact** if every sequence $(P_n)_{n\in\mathbb{N}}$ in Π contains a subsequence $(P_{n_k})_{k\in\mathbb{N}}$ that converges weakly to some probability measure Q (not necessarily a member of Π).

Theorem 2.15 (Prohorov's theorem)**.** *Let E be a metric space and let Π be a set of probability measures on E. Then Π being tight implies that Π is relatively compact. If E is separable and complete, then Π being relatively compact also implies that Π is tight.*

Proof. [Bil99, Theorems 5.1 and 5.2]. □

In section 7 of [Bil99], Billingsley derives sufficient conditions for tightness of a sequence of probability measures on $C([a,b])$ and these lead to the following criteria for convergence in distribution of $C([a,b])$-valued random variables (which are stochastic processes on $[a,b]$ with continuous sample paths).

Theorem 2.16. *Let X and $(X_n)_{n\in\mathbb{N}}$ be $C([a,b])$-valued random variables. Suppose $X_n(t_1,\ldots,t_k)$ converges to $X(t_1,\ldots,t_k)$ in distribution (in the metric space $\mathbb{R}^k$) for each $t_1 \ldots t_k \in [a,b]$. Moreover, suppose that for each $\epsilon > 0$,*

$$\lim_{h\to 0} \limsup_{n\to\infty} \mathbb{P}(w(X_n,h) \geq \epsilon) = 0, \tag{2.1}$$

where $w(x,h) = \sup_{|t-s|\leq h} |x(t) - x(s)|$ is the canonical modulus of continuity of a continuous function $x \in C([a,b])$. Then X_n converges to X in distribution.

Proof. [Bil99, Theorem 7.5] □

Finally, we show that the condition (2.1) is implied by a criterion similar to that of the Kolmogorov-Chentsov continuity theorem.

Proposition 2.17. *Suppose there are positive constants α, β and C such that for all $n \in \mathbb{N}$ and all $s, t \in [a, b]$,*

$$\mathbb{E}|X_n(t) - X_n(s)|^\alpha \leq C|t-s|^{1+\beta}.$$

Then condition (2.1) *is satisfied.*

Proof. Without loss of generality suppose $[a, b] = [0, 1]$. Let $D_k = \{j2^{-k} : j = 0 \dots 2^k\}$ and $D = \bigcup_{k=0}^\infty D_k$ be the dyadic rationals and set

$$\xi_{n,k} = \max_{1 \leq j \leq 2^k} |X_n(j2^{-k}) - X_n((j-1)2^{-k})|.$$

Then, by induction over $l \geq k$ it may be shown that $|X_n(t) - X_n(s)| \leq \xi_{n,k} + 2\sum_{j=k+1}^l \xi_{n,j}$ whenever $s, t \in D_l$ with $|t - s| \leq 2^{-k}$. It then follows that, if $s, t \in D$ and $|t-s| \leq 2^{-k}$, we have $|X_n(t) - X_n(s)| \leq 2\sum_{j \geq k} \xi_{n,j}$. Thus, for all $n \in \mathbb{N}$ and $k \in \mathbb{N}$,

$$w(X_n, 2^{-k}) \leq 2\sum_{j \geq k} \xi_{n,j}.$$

Now note that

$$\mathbb{E}[\xi_{n,k}^\alpha] \leq \sum_{j=1}^{2^k} \mathbb{E}|X_n(j2^{-k}) - X_n((j-1)2^{-k})|^\alpha \leq 2^k C 2^{-k(1+\beta)} = C2^{-k\beta}.$$

We will distinguish between the cases $\alpha \geq 1$ and $\alpha < 1$ because in the first case the Minkowski inequality will be used and in the second case subadditivity of $t \mapsto t^\alpha$ will be used. First suppose that $\alpha \geq 1$. Then it follows that for all $n \in \mathbb{N}$,

$$\begin{aligned}
\left(\mathbb{E}|w(X_n, 2^{-k})|^\alpha\right)^{\frac{1}{\alpha}} &\leq 2\sum_{j \geq k} \left(\mathbb{E}[\xi_{n,j}^\alpha]\right)^{\frac{1}{\alpha}} \\
&\leq 2C^{\frac{1}{\alpha}} \sum_{j \geq k} 2^{-j\frac{\beta}{\alpha}} \\
&= 2C^{\frac{1}{\alpha}} \left(1 - 2^{-\frac{\beta}{\alpha}}\right)^{-1} 2^{-k\frac{\beta}{\alpha}}.
\end{aligned}$$

If $\alpha < 1$, then for all $n \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{E}|w(X_n, 2^{-k})|^\alpha &\leq 2^\alpha \sum_{j \geq k} \mathbb{E}[\xi_{n,j}^\alpha] \\
&\leq 2^\alpha C \sum_{j \geq k} 2^{-j\beta} \\
&= 2^\alpha C \left(1 - 2^{-\beta}\right)^{-1} 2^{-k\beta}.
\end{aligned}$$

In both cases, condition (2.1) is satisfied because of Markov's inequality. ☐

Remark. If $C((a, b))$ is the space of continuous functions on an *open* interval (a, b). This space is (metrizably) topologized by uniform convergence on compacts, i.e. $x_n \to x$ whenever $\sup_{t \in [a', b']} |x_n(t) - x(t)| \to 0$ for all $[a', b'] \subseteq (a, b)$. In this regime, (slightly modified versions of) Theorem 2.16 and Proposition 2.17 still hold (see [Kal21, Chapter 23]).

The final topics that will be discussed are Lebesgue spaces $\mathbb{L}^p(S, \mathcal{F}, \mu)$ and weak Lebesgue spaces $\Lambda^p(S, \mathcal{F}, \mu)$, where $(S, \mathcal{F}, \mu)$ is a measure space. These spaces are indexed by a number $p > 0$ and are given respectively by the measurable functions $f : S \to \mathbb{R}$, modulo equality almost everywhere, such that

$$\|f\|_{\mathbb{L}^p(S,\mathcal{F},\mu)} = \left(\int_S |f|^p \, d\mu\right)^{\frac{1}{p}} < \infty,$$

$$\|f\|_{\Lambda^p(S,\mathcal{F},\mu)} = \left(\sup_{\lambda>0} \lambda^p \mu(|f| > \lambda)\right)^{\frac{1}{p}} < \infty.$$

$\mathbb{L}^0(S, \mathcal{F}, \mu)$ will denote the space of all measurable functions $S \to \mathbb{R}$, modulo equality almost everywhere, metrizably topologized by convergence in measure. Usually the sigma algebra $\mathcal{F}$ and the measure μ are unambiguous and we will simply write $\mathbb{L}^p(S)$ and $\Lambda^p(S)$. The space $\mathbb{L}^p(S)$ is a Banach space for $p \geq 1$ and a p-Banach space for $p \leq 1$. The space $\Lambda^p(S)$ is a quasi-Banach space for all $p > 0$. Indeed: Positivity and absolute homogeneity are clear. For the weakened triangle inequality, if $f, g \in \Lambda^p(S)$ then, for any $\lambda > 0$, $\{|f + g| > \lambda\} \subseteq \{|f| > \lambda/2\} \cup \{|g| > \lambda/2\}$. It follows that

$$\lambda^p \mu(|f + g| > \lambda) \leq 2^p \left[(\lambda/2)^p \mu(|f| > \lambda/2) + (\lambda/2)^p \mu(|g| > \lambda/2)\right] \leq 2^p \left(\|f\|^p_{\Lambda^p(S)} + \|g\|^p_{\Lambda^p(S)}\right).$$

Taking supremum over $\lambda > 0$ and p'th root reveals that

$$\|f + g\|_{\Lambda^p(S)} \leq 2 \left(\|f\|^p_{\Lambda^p(S)} + \|g\|^p_{\Lambda^p(S)}\right)^{\frac{1}{p}} \leq 2^{1\vee 1/p} \left(\|f\|^p_{\Lambda^p(S)} + \|g\|^p_{\Lambda^p(S)}\right).$$

The proof of completeness is omitted (see e.g. [Gra14]). Note that, for $\lambda > 0$,

$$\lambda^p \mu(|f| > \lambda) \leq \int_{\{|f|>\lambda\}} |f|^p \, d\mu \leq \|f\|^p_{\mathbb{L}^p(S)}.$$

It follows that $\|f\|_{\Lambda^p(S)} \leq \|f\|_{\mathbb{L}^p(S)}$ and that $\mathbb{L}^p(S) \subseteq \Lambda^p(S)$. The following and final lemma of the section shows that the property that Lebesgue spaces over a probability space are closed downwards (with respect to p) extends to weak Lebesgue spaces.

Lemma 2.18. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $0 < p < q$ and suppose $X \in \Lambda^q(\Omega)$, then*

$$\|X\|_{\mathbb{L}^p(\Omega)} \leq \left(\frac{q}{q-p}\right)^{\frac{1}{p}} \|X\|_{\Lambda^q(\Omega)}.$$

Proof. If $X = 0$ the result is trivial, so assume $X \neq 0$. For any $t > 0$, we have

$$\begin{aligned}
\mathbb{E}|X|^p &= p \int_0^\infty \lambda^{p-1} \mathbb{P}(|X| > \lambda) \, d\lambda \\
&\leq p \left[\int_0^t \lambda^{p-1} \, d\lambda + \|X\|^q_{\Lambda^q(\Omega)} \int_t^\infty \lambda^{p-q-1} \, d\lambda\right] \\
&= t^p + \frac{p}{q-p} \|X\|^q_{\Lambda^q(\Omega)} t^{p-q}.
\end{aligned}$$

The result follows by setting $t = \|X\|_{\Lambda^q(\Omega)} > 0$ (which minimizes the right hand side) and taking p'th root. □

3 Multifractional Brownian Motion

The first real part of this thesis is dedicated to covering the fractional Brownian motion and its development into two of its multifractional variants. The purpose of this section is to contextualize and motivate the primary focus of this thesis, which is the (multi)fractional stable motion. The fractional stable motion is a direct analogue to the fractional Brownian motion, but Gaussian distributions are replaced with so-called stable distributions, which will be covered in Section 4. In section 6 we will propose a new type of multifractional stable motion which is directly inspired by recent developments in the theory of the multifractional Brownian motion.

3.1 The Fractional Brownian Motion

Throughout this section fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The first appearance of the fractional Brownian motion is in an article by Mandelbrot and Van Ness [MN68]. They define the fractional Brownian motion as its moving average representation which we will get to later. However, we will follow Section 7.2 of Samorodnitsky and Taqqu's book [ST94] and define the fractional Brownian motion in the context of H-self-similar Gaussian processes with stationary increments.

Definition 3.1. Let T be either $\mathbb{R}$ or $[0,\infty)$. A stochastic process $(X(t))_{t\in T}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is ***H*-self-similar** with $H > 0$ if, for every $a > 0$,

$$(a^H X(t))_{t\in T} \stackrel{d}{=} (X(at))_{t\in T}.$$

The process has **stationary increments** if, for every $h > 0$,

$$(X(t+h) - X(h))_{t\in T} \stackrel{d}{=} (X(t) - X(0))_{t\in T}.$$

If a stochastic process is H-self-similar and has stationary increments, then this will be abbreviated to ***H*-sssi**.

Note that an H-self-similar process always obeys $X(0) = 0$ almost surely. Indeed: For any $a > 0$ we have $X(0) = X(a0) \stackrel{d}{=} a^H X(0))$. It then follows that any H-sssi process $(X(t))_{t\in\mathbb{R}}$ satisfies $X(-t) \stackrel{d}{=} -X(t)$ for all $t \in \mathbb{R}$. The fractional Brownian motion is essentially the only Gaussian process which is H-self-similar and has stationary increments, by which we mean that two Gaussian H-self-similar processes with stationary increments are equal to teacher in finite-dimensional distributions, up to a multiplicative factor. In order to show this, it will first be proven that the value of the self-similarity index H is limited by existence of absolute moments. These next two results are Lemma 7.1.9 and Proposition 7.1.10 from [ST94].

Lemma 3.2. *Let $(X(t))_{t\in\mathbb{R}}$ be H-sssi. Then, for $s \neq 0$ and $t \neq 0$,*

$$\mathbb{P}(X(s) = 0 \wedge X(t) = 0) = \mathbb{P}(X(1) = 0),$$
$$\mathbb{P}(X(s) \neq 0 \wedge X(t) \neq 0) = \mathbb{P}(X(1) \neq 0).$$

Proof. For any $s \neq 0$, by self-similarity and $X(-1) \stackrel{d}{=} -X(1)$, we have

$$\mathbb{P}(X(s) = 0) = \mathbb{P}(|s|^H X(\text{sign}(s)) = 0) = \mathbb{P}(X(1) = 0). \tag{3.1}$$

Note that the statement of the lemma follows under the hypothesis that for all $s \neq 0$ and $t \neq 0$,

$$\mathbb{P}(X(s) = 0 \wedge X(t) \neq 0) = 0. \tag{3.2}$$

Indeed: Assuming (3.2), writing R for either the relation $=$ or $\neq$ on $\mathbb{R}$, for $s \neq 0$ and $t \neq 0$,

$$\mathbb{P}(X(s)\,R\,0 \wedge X(t)\,R\,0) = \mathbb{P}(X(s)\,R\,0) - \mathbb{P}(X(s)\,R\,0 \wedge (\neg X(t)\,R\,0)) = \mathbb{P}(X(s)\,R\,0) = \mathbb{P}(X(1)\,R\,0).$$

Thus, (3.2) will be established. By stationary increments, $X(0) = 0$ and (3.1), for any $s \neq t$ it holds that

$$\mathbb{P}(X(s) = X(t)) = \mathbb{P}(X(s-t) = 0) = \mathbb{P}(X(1) = 0). \tag{3.3}$$

Now fix $t \neq 0$, for any $u > 0$ and $M > 0$ it holds that

$$\begin{aligned}\mathbb{P}(X(t) = X(u) \neq 0) &\leq \mathbb{P}(|X(t)| \geq M) + \mathbb{P}(0 < |X(u)| \leq M) \\ &= \mathbb{P}(|X(t)| \geq M) + \mathbb{P}(0 < |X(1)| \leq Mu^{-H}).\end{aligned}$$

Taking $u \to \infty$ and $M \to \infty$ such that $Mu^{-H} \to 0$ and using right-continuity of cumulative distribution functions reveals that

$$\lim_{u\to\infty} \mathbb{P}(X(t) = X(u) \neq 0) = 0. \tag{3.4}$$

Moreover, for $s \neq 0$ and $u \neq s$,

$$\begin{aligned}\mathbb{P}(X(s) = X(u)) &= \mathbb{P}(X(s) = X(u) \neq 0) + \mathbb{P}(X(s) = 0 \wedge X(u) = 0) \\ &= \mathbb{P}(X(s) = X(u) \neq 0) + \mathbb{P}(X(s) = 0) - \mathbb{P}(X(s) = 0 \wedge X(u) \neq 0).\end{aligned}$$

From (3.1) and (3.3) it then follows that

$$\mathbb{P}(X(s) = X(u) \neq 0) = \mathbb{P}(X(s) \wedge X(u) \neq 0). \tag{3.5}$$

Now fix $s \neq 0$ and $t \neq 0$ and let $u > s \vee t \vee 0$. Then

$$\begin{aligned}\mathbb{P}(X(s) = 0 \wedge X(t) \neq 0) &\leq \mathbb{P}(X(s) = 0 \wedge X(u) \neq 0) + \mathbb{P}(X(u) = 0 \wedge X(t) \neq 0) \\ &= \mathbb{P}(X(s) = X(u) \neq 0) + \mathbb{P}(X(u) = X(t) \neq 0).\end{aligned}$$

Letting $u \to \infty$ and using (3.4) shows (3.2). □

Lemma 3.3. *Suppose $(X(t))_{t\in\mathbb{R}}$ is H-sssi and $\mathbb{P}(X(1) \neq 0) > 0$. Then the relation*

$$\mathbb{E}|X(1)|^\gamma < \infty$$

implies

$$\begin{cases} 0 < H < \frac{1}{\gamma} & \text{if } 0 < \gamma < 1 \\ 0 < H \leq 1 & \text{if } \gamma \geq 1 \end{cases}$$

Proof. First suppose $0 < \gamma < 1$, then $(x+y)^\gamma < x^\gamma + y^\gamma$ for $x, y > 0$. Thus, $|X(2)|^\gamma < |X(2) - X(1)|^\gamma + |X(1)|^\gamma$ on the set $\{X(1) \neq 0 \wedge X(2) - X(1) \neq 0\}$. By Lemma 3.2 and the assumption $\mathbb{P}(X(1) \neq 0) > 0$, this set has positive probability. Using H-self-similarity and stationary increments, it follows that

$$2^{\gamma H}\mathbb{E}|X(1)|^\gamma = \mathbb{E}|X(2)|^\gamma < \mathbb{E}|X(2) - X(1)|^\gamma + \mathbb{E}|X(1)|^\gamma = 2\mathbb{E}|X(1)|^\gamma.$$

Note that $\mathbb{E}|X(1)|^\gamma > 0$ because $\mathbb{P}(X(1) \neq 0) > 0$. It follows that $H < \frac{1}{\gamma}$.

If $\gamma \geq 1$ then $\mathbb{E}|X(1)|^\rho < \infty$ for all $0 < \rho < 1$ and therefore $H < \frac{1}{\rho}$ for all $0 < \rho < 1$, implying $H \leq 1$. □

We conclude that the self-similarity index H of a non-degenerate Gaussian H-sssi stochastic process $(X(t))_{t\in\mathbb{R}}$ is limited to the values $0 < H \le 1$, degenerate in this context means that $(X(t))_{t\in\mathbb{R}}$ is indistinguishable from 0. Moreover, the finite-dimensional distributions of a non-degenerate Gaussian H-sssi stochastic process are locked down by these characteristics. The next result corresponds to Lemma 7.2.1 in [ST94].

Proposition 3.4. *Let $(X(t))_{t\in\mathbb{R}}$ be a non-degenerate Gaussian H-sssi stochastic process. Then*

$$\operatorname{Cov}(X(s),X(t)) = \frac{1}{2}\operatorname{Var}(X(1))\left(|s|^{2H}+|t|^{2H}-|t-s|^{2H}\right) \quad s,t\in\mathbb{R}.$$

Moreover,

$$\begin{cases}\mathbb{E}[X(t)] = 0 & \text{if } 0 < H < 1,\\ X(t) = tX(1) \text{ a.s.} & \text{if } H = 1.\end{cases}$$

Proof. Using the fact that $(X(t))_{t\in\mathbb{R}}$ is H-sssi we find that

$$\begin{aligned}\mathbb{E}[X(s)X(t)] &= \frac{1}{2}\left(\mathbb{E}[X(s)^2]+\mathbb{E}[X(t)^2]-\mathbb{E}[(X(t)-X(s))^2]\right)\\ &= \frac{1}{2}\left(\mathbb{E}[X(s)^2]+\mathbb{E}[X(t)^2]-\mathbb{E}[X(t-s)^2]\right)\\ &= \frac{1}{2}\mathbb{E}[X(1)^2]\left(|s|^{2H}+|t|^{2H}-|t-s|^{2H}\right).\end{aligned}$$

Consider first the case $0 < H < 1$. Then

$$\mathbb{E}[X(1)] = \mathbb{E}[X(2)] - \mathbb{E}[X(1)] = (2^H-1)\mathbb{E}[X(1)],$$

showing that $\mathbb{E}[X(1)] = 0$ and therefore that $\mathbb{E}[X(t)] = 0$ for all $t \neq 0$. Now consider the case $H = 1$. Then $\mathbb{E}[X(s)X(t)] = ts\mathbb{E}[X(1)^2]$ and we find, for $t \neq 0$,

$$\mathbb{E}[(X(t)-tX(1))^2] = \mathbb{E}[X(t)^2] - 2t\mathbb{E}[X(t)X(1)] + t^2\mathbb{E}[X(1)^2] = (t^2-2t^2+t^2)\mathbb{E}[X(1)^2] = 0.$$

Thus, $X(t) = tX(1)$ almost surely. Of course these are the only two cases to consider by Lemma 3.3. □

We will consider the case $H = 1$ where the process is simply a drifting normally distributed random variable as uninteresting and unworthy of the name fractional Brownian motion. However, for $0 < H < 1$, Proposition 3.4 identifies non-degenerate Gaussian H-sssi as centered and locks down the covariance function up to a constant multiplicative factor. In turn, the finite-dimensional distributions of non-degenerate Gaussian H-sssi stochastic processes are uniquely determined up to a multiplicative factor.

Definition 3.5. Let $0 < H < 1$. The name **fractional Brownian motion** with **fractional parameter** H shall refer to any non-degenerate Gaussian H-sssi stochastic process $(X(t))_{t\in\mathbb{R}}$. In turn, its mean and covariance functions are given by

$$\mathbb{E}[X(t)] = 0 \qquad t\in\mathbb{R}, \tag{3.6}$$

$$\operatorname{Cov}(X(s),X(t)) = \frac{1}{2}C\left(|s|^{2H}+|t|^{2H}-|t-s|^{2H}\right) \qquad s,t\in\mathbb{R}. \tag{3.7}$$

Note that if $(X(t))_{t\in\mathbb{R}}$ is a fractional Brownian motion with $H = \frac{1}{2}$, then for $s \neq 0$ and $t \neq 0$ we have

$$\operatorname{Cov}(X(s),X(t)) = \begin{cases}C(|s|\wedge|t|) & \operatorname{sign}(s) = \operatorname{sign}(t),\\ 0 & \operatorname{sign}(s) \neq \operatorname{sign}(t).\end{cases}$$

This identifies $(X(t))_{t\in\mathbb{R}}$ as a Brownian motion and explains why the fractional Brownian motion is called the way it is. Determining the Hölder regularity of fractional Brownian motion can be done by using the Kolmogorov-Chentsov continuity theorem and the fact that the covariance function is known.

Theorem 3.6. *Let $(X(t))_{t\in\mathbb{R}}$ be the fractional Brownian motion with fractional parameter $H \in (0,1)$. Then, there is a modification $(\tilde{X}(t))_{t\in\mathbb{R}}$ of $(X(t))_{t\in\mathbb{R}}$ such that, with probability one, $\rho^{unif}_{\tilde{X}}([S,T]) \geq H$ for all $S < T$. Moreover, for all $t \in \mathbb{R}$, $\rho_{\tilde{X}}(t) \leq H$ almost surely.*

Proof. Let the covariance function of $(X(t))_{t\in\mathbb{R}}$ be described by (3.7). Let $s,t \in \mathbb{R}$, then $X(t)-X(s)$ is normally distributed with mean 0 and variance

$$\begin{aligned}\mathrm{Var}(X(t)-X(s)) &= \mathrm{Var}(X(t)) - 2\mathrm{Cov}(X(s),X(t)) + \mathrm{Var}(X(s))\\ &= C\left[|t|^{2H} - (|s|^{2H} + |t|^{2H} - |t-s|^{2H}) + |s|^{2H}\right]\\ &= C|t-s|^{2H}.\end{aligned}$$

Letting $p > 0$ and Z be standard normal, it follows that

$$\mathbb{E}[|X(t)-X(s)|^p] = \mathbb{E}\left|\sqrt{C}|t-s|^H Z\right|^p = \mathbb{E}|Z|^p C^{p/2}|t-s|^{pH}.$$

If $p > \frac{1}{H}$ then the Kolmogorov-Chentsov continuity theorem provides a continuous modification $(\tilde{X}_p(t))_{t\in\mathbb{R}}$ that is locally ρ-Hölder continuous for any $0 < \rho < H - \frac{1}{p}$ and letting $p \to \infty$ yields a modification $(\tilde{X}(t))_{t\in\mathbb{R}}$ satisfying $\rho^{\mathrm{unif}}_{\tilde{X}}([S,T]) \geq H$. To be more precise: Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $p_n \to \infty$ as $n \to \infty$. Then the processes in the collection $(\tilde{X}_{p_n})_{n\in\mathbb{N}}$ are pairwise indistinguishable because $(\tilde{X}_{p_n}(t))_{t\in\mathbb{R}}$ and $(\tilde{X}_{p_m}(t))_{t\in\mathbb{R}}$ are modifications of each other with continuous sample paths. Because $\mathbb{N}\times\mathbb{N}$ is countable, they are uniformly indistinguishable in the sense that there is a probability one set $\Omega^* \subseteq \Omega$ such that $\tilde{X}_{p_n}(t) = \tilde{X}_{p_m}(t)$ for all $\omega \in \Omega^*$, $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Fixing $n_0 \in \mathbb{N}$ we find that, with probability one, $(\tilde{X}_{p_{n_0}}(t))_{t\in\mathbb{R}}$ is locally ρ-Hölder continuous for any $0 < \rho < H - \frac{1}{p_n}$, for every $n \in \mathbb{N}$. Since $\frac{1}{p_n} \to 0$ it follows that, with probability one, $(\tilde{X}_{p_{n_0}}(t))_{t\in\mathbb{R}}$ is locally ρ-Hölder continuous for any $0 < \rho < H$ and thus, with probability one, $\rho^{\mathrm{unif}}_{\tilde{X}_{p_{n_0}}}([S,T]) \geq H$ for all $S < T$.

To show the upper bound on the pointwise Hölder exponent, let $\gamma > 0, t \in \mathbb{R}$ and $h > 0$. Then

$$\begin{aligned}\mathbb{E}\left[\left(\frac{\tilde{X}(t+h)-\tilde{X}(t)}{h^{H+\gamma}}\right)^2\right] &= h^{-2H-2\gamma}\left(\mathbb{E}[(\tilde{X}(t+h)^2] + \mathbb{E}[\tilde{X}(t))^2] - 2\mathbb{E}[\tilde{X}(t+h)X(t)]\right)\\ &= Ch^{-2H-2\gamma}\left(|t+h|^{2H} + |t|^{2H} - (|t+h|^{2H} + |t|^{2H} - h^{2H})\right)\\ &= Ch^{-2\gamma}.\end{aligned}$$

We see that $h^{-H-\gamma}|\tilde{X}(t+h)-\tilde{X}(t)| \to \infty$ in probability as $h \downarrow 0$ and thus there is a sequence $h_n \downarrow 0$ such that $h_n^{-H-\gamma}|\tilde{X}(t+h_n)-\tilde{X}(t)| \to \infty$ almost surely as $n \to \infty$. Thus, almost surely,

$$\limsup_{h\to 0}\frac{|\tilde{X}(t+h)-\tilde{X}(t)|}{h^{H+\gamma}} = \infty$$

and $\rho_{\tilde{X}}(t) \leq H$. □

Regarding the existence of fractional Brownian motion, it is relatively simple to show that a Gaussian process obeying (3.6) and (3.7) is H-sssi, and an easy way of showing there is some probability space supporting a Gaussian process with (3.6) and (3.7) as its mean and covariance function comes down to showing the expression on the right hand side of (3.7) is non-negative definite and appealing to the Kolmogorov extension theorem. However, we will show that, for all $0 < H < 1$, a fractional Brownian motion with fractional parameter H exists, provided the probability space can support Brownian motion, by giving an explicit representation as an Itô integral against a Brownian motion.

3.2 Moving Average Representation of the Fractional Brownian Motion

The moving average representation of fractional Brownian motion is how the fractional Brownian motion was first introduced [MN68]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(W(x))_{x\in\mathbb{R}}$ be a Brownian motion on Ω, then the moving average representation of fractional Brownian motion with fractional parameter $H \in (0,1)$ is given by the Itô integral

$$W^H(t) = \int_{\mathbb{R}} (t-x)_+^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}} \, dW(x) \quad t \in \mathbb{R}. \tag{3.8}$$

Here the quantity $(x)_+^r$ is defined for $x, r \in \mathbb{R}$ as

$$(x)_+^r = \begin{cases} x^r & x > 0, \\ 0 & x \le 0. \end{cases}$$

Of course we ought to make sense of this improper Itô integral over the entire real line. Writing $f_t(x) = (t-x)_+^{H-1/2} - (-x)_+^{H-1/2}$ for the collections of kernels, by the Itô isometry, using that the kernels are deterministic, for all real numbers $a < b$,

$$\mathbb{E}\left[\left(\int_a^b f_t \, dW\right)^2\right] = \int_a^b (f_t(x))^2 \, dx. \tag{3.9}$$

Lemma 3.7. *For any $H \in (0,1)$ and $t \in \mathbb{R}$,*

$$\int_{\mathbb{R}} (f_t(x))^2 \, dx < \infty.$$

Proof. Note that $f_t(x) = 0$ for $x > t \vee 0$ so the integral is only improper around $-\infty$ and at the possible singularities $x = 0$ and $x = t$. If $H = \frac{1}{2}$ then $f_t = \mathbb{1}_{[0,t)}$ (which should be understood as $-\mathbb{1}_{(-t,0]}$ for $t < 0$) and the claim is clear. Now suppose $H \neq \frac{1}{2}$, then $f_t(x) \sim (H - \frac{1}{2})(-x)^{H-3/2}$ as $x \to -\infty$ which is square integrable around $-\infty$. Moreover, $f_t(x) \sim (t-x)_+^{H-1/2}$ as $x \to t$ which is square integrable around $x = t$. Finally, $f_t(x) \sim (-x)_+^{H-1/2}$ as $x \to 0$ which is square integrable around $x = 0$. $\square$

Using Equation (3.9) we conclude that, for fixed $t \in \mathbb{R}$, the net

$$\left(\int_a^b f_t \, dW\right)_{[a,b]\in\mathcal{I}}$$

is Cauchy in $\mathbb{L}^2(\Omega)$. Here $\mathcal{I}$ is the directed set of non-empty compact intervals $[a,b] \subseteq \mathbb{R}$, partially ordered by set inclusion. We define (3.8) as the limit in $\mathbb{L}^2(\Omega)$ of this net. Note that, due to continuity of the functional $\|\cdot\|_{\mathbb{L}^2(\Omega)}$ and Equation (3.9), we find

$$\mathbb{E}[(W^H(t))^2] = \int_{\mathbb{R}} (f_t(x))^2\, dx.$$

Proposition 3.8. *Let $(W^H(t))_{t\in\mathbb{R}}$ be defined in (3.8), then $(W^H(t))_{t\in\mathbb{R}}$ is a fractional Brownian motion with fractional parameter H.*

Proof. First we show that $(W^H(t))_{t\in\mathbb{R}}$ is a centered Gaussian process. Let $c_1 \dots c_n \in \mathbb{R}$ and $t_1 \dots t_n \in \mathbb{R}$. Then

$$\sum_{k=1}^n c_k W^H(t_k) = \int_{\mathbb{R}} \sum_{k=1}^n c_k f_{t_k}\, dW.$$

Since the kernel $\sum_{k=1}^n c_k f_{t_k}$ is deterministic, the random variable above is normally distributed with mean 0. Next, we show that $(W^H(t))_{t\in\mathbb{R}}$ is H-self-similar, so let $a > 0$, $c_1 \dots c_n \in \mathbb{R}$ and $t_1 \dots t_n \in \mathbb{R}$. Then, using the Itô isometry,

$$\begin{aligned}
\operatorname{Var}\left(\sum_{k=1}^n c_k W^H(at_k)\right) &= \mathbb{E}\left[\left(\int_{\mathbb{R}} \sum_{k=1}^n c_k f_{at_k}\, dW\right)^2\right] \\
&= \int_{\mathbb{R}} \left(\sum_{k=1}^n c_k \left[(at_k - x)_+^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}}\right]\right)^2 dx \\
&\overset{x=a\hat{x}}{=} a^{2H} \int_{\mathbb{R}} \left(\sum_{k=1}^n c_k \left[(t_k - \hat{x})_+^{H-\frac{1}{2}} - (-\hat{x})_+^{H-\frac{1}{2}}\right]\right)^2 d\hat{x} \\
&= a^{2H}\mathbb{E}\left[\left(\int_{\mathbb{R}} \sum_{k=1}^n c_k f_{t_k}\, dW\right)^2\right] \\
&= \operatorname{Var}\left(\sum_{k=1}^n c_k a^H W^H(t_k)\right).
\end{aligned}$$

We conclude that $\sum_{k=1}^n c_k W^H(at_k) \overset{d}{=} \sum_{k=1}^n c_k a^H W^H(t_k)$. Since $c_1 \dots c_n$ were arbitrary and the distribution of a random vector is determined by its linear combinations, it follows that $(W^H(at_1) \dots W^H(at_n)) \overset{d}{=} (a^H W(t_1) \dots a^H W(t_n))$ and that $(W(at))_{t\in\mathbb{R}} \overset{d}{=} (a^H W(t))_{t\in\mathbb{R}}$. Finally, we prove that $(W^H(t))_{t\in\mathbb{R}}$ has stationary increments. Let $h \in \mathbb{R}$, $c_1 \dots c_n \in \mathbb{R}$ and $t_1 \dots t_n \in \mathbb{R}$. Then

$$\begin{aligned}
\operatorname{Var}\left(\sum_{k=1}^{n} c_k (W^H(t_k+h) - W^H(h))\right) &= \mathbb{E}\left[\left(\int_{\mathbb{R}} \sum_{k=1}^{n} c_k (f_{t_k+h} - f_h)\, dW\right)^2\right] \\
&= \int_{\mathbb{R}} \left(\sum_{k=1}^{n} c_k \left[(t_k+h-x)_+^{H-\frac{1}{2}} - (h-x)_+^{H-\frac{1}{2}}\right]\right)^2 dx \\
&\overset{x=\hat{x}-h}{=} \int_{\mathbb{R}} \left(\sum_{k=1}^{n} c_k \left[(t_k-\hat{x})_+^{H-\frac{1}{2}} - (-\hat{x})_+^{H-\frac{1}{2}}\right]\right)^2 d\hat{x} \\
&= \mathbb{E}\left[\left(\int_{\mathbb{R}} \sum_{k=1}^{n} c_k f_{t_k}\, dW\right)^2\right] \\
&= \operatorname{Var}\left(\sum_{k=1}^{n} c_k W^H(t_k)\right).
\end{aligned}$$

By the same reasoning as before, it follows that $(W^H(t+h) - W^H(h))_{t\in\mathbb{R}} \overset{d}{=} (W^H(t))_{t\in\mathbb{R}}$. □

The representation (3.8) is not unique: Writing $(x)_-^r = (-x)_+^r$ and letting $(a_+, a_-) \in \mathbb{R}^2 \setminus \{(0,0)\}$, the representation

$$W^H_{a_+,a_-}(t) = \int_{\mathbb{R}} a_+ \left((t-x)_+^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}}\right) + a_- \left((t-x)_-^{H-\frac{1}{2}} - (-x)_-^{H-\frac{1}{2}}\right) dW(x) \quad t \in \mathbb{R}$$

also results in fractional Brownian motion with fractional parameter H, due to the very same arguments we have just witnessed. Of course, by Proposition 3.4, the finite-dimensional distributions may only differ from (3.8) by a multiplicative factor.

3.3 Multifractional Brownian Motion

In this section we will cover the developments of the fractional Brownian motion into two of its multifractional variants. Both of these variants are constructed from (3.8) by replacing the fractional parameter H with a function. The first multifractional Brownian motion that appeared in the literature replaces the fractional parameter H with a deterministic function $H(t)$ depending on the variable t indexing the stochastic process [PV95]. Later the case where $H(t)$ is a random process is considered [AT05]. However, as pointed out by Ayache, Esser and Hamonier [AEH18], letting $H(t)$ depend on the variable t indexing the stochastic process makes the kernels in (3.8) unadapted, so this process needs to be defined through wavelet methods. They suggest to instead let $H = H(x)$ depend on the integration variable, so that the kernels are adapted and the process may be defined as an Itô integral. The case where $H = H(t)$ depends on the variable indexing the process will be referred to as the classical multifractional Brownian motion, and the case where $H = H(x)$ depends on the integration variable will be referred to as the Itô multifractional Brownian motion. The Itô multifractional Brownian motion is later studied by Loboda, Mies and Steland [LMS21], where they simplify the analysis of the pointwise Hölder exponent. In this article, they manage to compute the pointwise Hölder exponent of the Itô multifractional Brownian motion, without any assumptions on the Hölder regularity of the function H. This stands in stark contrast with the previous articles [PV95; AT05; AEH18], where the Hölder regularity of the resulting process depended on the Hölder regularity of the functional multifractional parameter H.

The reason we cover the development of the fractional Brownian motion into two of its multifractional variants is to motivate our study of the multifractional stable motion in Section 6. Classical multifractional stable motions, where the functional multifractional parameter depends on the variable t indexing the stochastic process, have been introduced [ST04; ST05; AH14]. However, only with with deterministic multifractional parameter, and the Hölder regularity of the resulting process depends on the Hölder regularity of the multifractional parameter. We intend to define an Itô multifractional stable motion where the multifractional parameter depends on the integration parameter, analogous to the process suggested in [AEH18], so that the process with random multifractional parameter can be defined through an Itô calculus, and the Hölder regularity of the resulting process is independent of the Hölder regularity of the multifractional parameter.

Again we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Brownian motion $(W(x))_{x\in\mathbb{R}}$ on Ω. Multifractional Brownian motion is a class of stochastic processes $(X(t))_{t\in\mathbb{R}}$ such that, around time $t \in \mathbb{R}$, the process locally behaves like a fractional Brownian motion with fractional parameter $H(t)$, where $H : \mathbb{R} \to (0, 1)$ is a function. Making this intuitive explanation rigorous requires a formal definition of stochastic processes locally behaving like other stochastic processes.

Definition 3.9. Let $H > 0$ and $T \subseteq \mathbb{R}$ be an interval containing 0, a stochastic process $(X(t))_{t\in\mathbb{R}}$ is H-**localizable at** $t \in T$ with **local form** $(X'_t(r))_{r\in T}$ if, in terms of finite-dimensional distributions:

$$\left(\frac{X(t+hr) - X(t)}{h^H}\right)_{r\in T} \overset{h\downarrow 0}{\to} (X'_t(r))_{r\in T}.$$

If the processes above have continuous sample paths and the convergence above is weak convergence of push-forward measures in the space $C(T)$ of continuous functions $T \to \mathbb{R}$, then we say that $(X(t))_{t\in\mathbb{R}}$ is **strongly** H-**localizable** at t.

Constructing a multifractional Brownian motion thus comes down to constructing a stochastic process that is, at each t, H-localizable for some H such that its local form is a fractional Brownian motion with fractional parameter $H(t)$. This was first achieved by Peltier and Lévy Véhel [PV95] where they simply replaced the fractional parameter H by a function $H(t)$ in the moving average representation of the fractional Brownian motion (3.8). Processes obtained from (3.8) by replacing H with a function $H(t)$ will be referred to as **classical multifractional Brownian motion** with **multifractional parameter** $H(\cdot)$. In [PV95] the authors consider the classical multifractional Brownian motion given by

$$Y(t) = \frac{1}{\Gamma(H(t) + 1/2)} \int_{\mathbb{R}} (t-x)_+^{H(t)-\frac{1}{2}} - (-x)_+^{H(t)-\frac{1}{2}} \, dW(x) \quad t \in \mathbb{R}_{\geq 0}.$$

Here, the multifractional parameter $H : \mathbb{R}_{\geq 0} \to (0, 1)$ is assumed to be uniformly ρ-Hölder continuous for some $\rho > 0$. Under the assumption that $\rho > H(t)$ for each $t \geq 0$ they show that, for each $t \geq 0$, $\rho_Y(t) = H(t)$ almost surely. Note that a Hölder condition on the multifractional parameter is necessary to obtain the pointwise Hölder exponent of the resulting process.

The authors from [AT05] define a classical multifractional Brownian motion on $t \in [0, 1]$ with a stochastic process $(H(t))_{t\in[0,1]}$ taking values in $[a, b] \subseteq (0, 1)$ as multifractional parameter. Note that, even under the assumption that $(H(t))_{t\in[0,1]}$ is adapted to the natural filtrated generated by $(W(x))_{x\in[0,1]}$, the kernels

$$\Omega \times [0, 1] \to \mathbb{R} : (\omega, t) \mapsto (t-x)_+^{H(\omega,t)-\frac{1}{2}} - (-x)_+^{H(\omega,t)-\frac{1}{2}}$$

are not necessarily adapted to the natural filtration generated by $(W(x))_{x\in[0,1]}$. Thus, the classical multifractional Brownian motion cannot be interpreted as an Itô integral. Instead, the authors consider the moving average representation of the fractional Brownian motion (3.8) as a random field $(W^H(t))_{(t,H)\in[0,1]\times[a,b]}$. Then they define the multifractional Brownian motion with random multifractional parameter as the composition of $(\omega,t)\mapsto(t,H(\omega,t))$ with this random field. This is quite a tricky definition to work with, so to make progress, a (random) wavelet representation of the random field $(W^H(t))_{(t,H)\in[0,1]\times[a,b]}$ is obtained. This wavelet representation is used to show that, under the condition that $\rho_H^{\text{unif}}([0,1]) > \sup_{t\in[0,1]} H(t)$ almost surely, we have

- for all $t\in(0,1)$, $\rho_Y(t) = H(t)$ almost surely,
- For all non-empty intervals $I\subseteq[0,1]$, $\rho_Y^{\text{unif}}(I) = \inf_{t\in I} H(t)$ almost surely.

Note that again, the Hölder regularity of the resulting process is dependent on the Hölder regularity of its multifractional parameter. Later, Ayache, Esser and Hamonier get the idea to replace the fractional parameter H in the moving average representation of the multifractional Brownian motion (3.8) by a random process $(H(x))_{x\in[0,1]}$ depending on the integration variable [AEH18]. Processes obtained like this will be named **Itô multifractional Brownian motion** with **multifractional parameter** $H(\cdot)$. The article focuses on the "high-frequency" part of the Itô multifractional Brownian motion because this part dominates the regularity of the paths. It is given by

$$Y(t) = \int_0^1 (t-x)_+^{H(x)-\frac{1}{2}}\,dW(x) \quad t\in[0,1]. \tag{3.10}$$

It is assumed that $(H(x))_{x\in[0,1]}$ is a continuous stochastic process adapted to the natural filtration generated by $(W(x))_{x\in[0,1]}$, taking values in $[a,b]\subseteq(\frac{1}{2},1)$. Then, writing $F_t(\omega,x) = (t-x)_+^{H(\omega,x)-\frac{1}{2}}$ for the collection of kernels, for each $t\in[0,1]$ we have

- $F_t : \Omega\times[0,1]\to\mathbb{R}$ is jointly measurable (by continuity),
- $(F_t(x))_{x\in[0,1]}$ is adapted to the natural filtration generated by $(W(x))_{x\in[0,1]}$,
- $\mathbb{E}\left[\int_0^1 F_t(x)^2\,dx\right] < \infty$.

Thus, (3.10) is well-defined as an Itô integral. Through a very involved wavelet analysis, the authors prove in Theorem 3.1 the following result: Suppose there are constants $c>0$ and $\rho\in(0,1]$ such that

$$\mathbb{E}\left[(H(y)-H(x))^2\right] \le c|y-x|^{2\rho}$$

for all $x,y\in[0,1]$. Moreover, suppose there is some $\gamma\in(\frac{1}{2},1)$ such that, with probability one, $(H(x))_{x\in[0,1]}$ is uniformly γ-Hölder continuous. Then, with probability one, for all compact subintervals $I\subseteq[0,1]$,

$$\rho_Y^{\text{unif}}(I) \ge \min_{x\in I} H(x).$$

The Hölder regularity of the resulting process is still dependent on the Hölder regularity of the multifractional parameter. This condition was removed by Loboda, Mies and Steland in [LMS21]. In this article, the authors consider a general Itô multifractional Brownian motion of the form

$$Y(t) = \int_{\mathbb{R}} g_t(x)\,dW(x) \quad t\in\mathbb{R}_{\ge 0}. \tag{3.11}$$

For each $t \geq 0$, the kernel $(g_t(x))_{x\in\mathbb{R}}$ is assumed to be an $(\mathcal{F}_x)_{x\in\mathbb{R}}$-adapted (they work in a filtered probability space) stochastic processes such that $g_t(x) = 0$ for $x > t$, and

$$\int_{\mathbb{R}} (g_t(x))^2 \, dx < \infty.$$

Moreover, the following three assumptions are made.

Condition (A): The function $t \mapsto g_t(x)$ is differentiable in $t > x$ for all $x \in \mathbb{R}$. There exist $(\mathcal{F}_x)_{x\in\mathbb{R}}$-adapted processes $(H(x))_{x\in\mathbb{R}}, (L(x))_{x\in\mathbb{R}}$ and $(R(x))_{x\in\mathbb{R}}$ such that

$$\begin{aligned}
|g_t(x)| &\leq L(x)|t-x|^{H(x)-\frac{1}{2}} && x \in (t-1, t), \\
|\partial_t g_t(x)| &\leq L(x)|t-x|^{H(x)-\frac{3}{2}} && t \in (t-1, t), \\
|\partial_T g_t(x)| &\leq L(x)|t-x|^{-R(x)} && x \in (-\infty, t-1].
\end{aligned}$$

Condition (B): There are deterministic constants $\underline{H} \in (0,1), \overline{H} \in (0,1), \overline{L} > 0$ and $\underline{R} > \frac{1}{2}$ such that for all $t \geq 0$,

$$\underline{H} \leq H(t) \leq \overline{H} \qquad |L(t)| \leq \overline{L} \qquad R(t) \geq \underline{R}$$

Condition (C): There exists a continuous, increasing function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $w(0) = 0$ such that, for all $t \geq 0$ and $h > 0$,

$$|H(t+h) - H(t)| \leq w(h).$$

Under these conditions they show that, fixing a time horizon $T > 0$, the process $(Y(t))_{t\in(0,T)}$ given by (3.11) admits a continuous modification $(\tilde{Y}(t))_{t\in(0,T)}$ such that, with probability one, for all $t \in (0,T)$ we have $\rho_{\tilde{Y}}(t) \geq H(t)$. To obtain an upper bound on the pointwise Hölder exponent the following assumption is also made.

Condition (A*): There is a continuous $(\mathcal{F}_x)_{x\in\mathbb{R}}$-adapted process $(\sigma(x))_{x\in\mathbb{R}}$ satisfying $|\sigma(x)| \leq L(x)$, and a $\rho > 0$ such that, for all $t \geq 0$,

$$\begin{aligned}
\left| g_t(x) - \sigma(x)(t-x)^{H(x)-\frac{1}{2}} \right| &\leq L(x)|t-x|^{H(x)-\frac{1}{2}+\rho}, \\
\left| \partial_t g_t(x) - \partial_t \sigma(x)(t-x)^{H(x)-\frac{1}{2}} \right| &\leq L(x)|t-x|^{H(x)-\frac{3}{2}+\rho}.
\end{aligned}$$

Under these four assumptions and the assumption that the modulus of continuity satisfies $w(h)\log h \to 0$ as $h \downarrow 0$ it is shown that the continuous modification $(\tilde{Y}(t))_{t\in(0,T)}$ of (3.11) is strongly $H(t)$-localizable at every t and this in turn is used to show that, for all $t > 0$, $\rho_{\tilde{Y}}(t) = H(t)$ almost surely and that, with probability one, for all non-empty $[a,b] \subseteq [0,T)$, $\rho_{\tilde{Y}}^{\text{unif}}([a,b]) = \min_{t\in[a,b]} H(t)$.

These results on the Hölder regularity are not dependent anymore on the Hölder regularity of the multifractional parameter $H(\cdot)$. Indeed: For the lower bound on the pointwise Hölder exponent, any modulus of continuity will suffice and for the upper bound it is only needed that $w(h)\log h \to 0$ as $h \downarrow 0$, which is a weaker assumption that one on the Hölder regularity. Moreover, the proofs in this article are of a stochastic nature, based on the Kolmogorov-Chentsov continuity theorem, in stark contrast to the more analytically minded wavelet techniques from [AEH18]. In [LMS21], the following improvement of the Kolmogorov-Chentsov continuity theorem is proved (only the one-dimensional version is stated here).

Theorem 3.10. *Let $(Y(t))_{t\in(S,T)}$ and $(a(t))_{t\in(S,T)}$ be stochastic processes on $(\Omega,\mathcal{F},\mathbb{P})$ taking values in $\mathbb{R}$ and $(0,1)$ respectively. Denote $\underline{a}_I = \inf_{t\in I} a(t)$, where $I \subseteq (S,T)$ is an interval. Suppose that $\underline{a}_{(S,T)} > 0$ and that for some $p > 0$ there exist $\epsilon > 0$ and $C > 0$ such that for all $s,t \in (S,T)$ with $|t-s| \leq \epsilon$,*

$$\mathbb{E}\left|\frac{Y(t)-Y(s)}{|t-s|^{\underline{a}_{[s\wedge t, s\vee t]}}}\right|^p \leq C|t-s|.$$

Then $(Y(t))_{t\in(S,T)}$ has a modification $(\tilde{Y}(t))_{t\in(S,T)}$ such that, with probability one, for all $\gamma > 0$ and all $S < a < b < T$, it holds that

$$\sup_{\substack{s,t\in[a,b]\\ s\neq t}} \frac{|\tilde{Y}(t)-\tilde{Y}(s)|}{|t-s|^{\underline{a}_{[a,b]}-\gamma}} < \infty.$$

This theorem improves over the classic Kolmogorov-Chentsov continuity theorem in two important ways, namely that the exponent $(a(t))_{t\in(S,T)}$ may be time-dependent and random. Favoring a stochastic attitude over an analytic one results in much simpler proofs compared to the proofs in [AEH18], and removes the need for a Hölder condition on the multifractional parameter.

As mentioned before, we cover these developments in the study into multifractional Brownian motions because these developments inspire our research into multifractional stable motions. In the stable setting, fractional stable motion inspired by the moving average representation of the fractional Brownian motion has been introduced by Taqqu and Wolpert [TW83] (they call it fractional Lévy motion). Multifractional variants in the classical sense, with the fractional parameter H replaced by a deterministic function $H(t)$ of the parameter indexing the stochastic process, have also been considered [ST04; ST05; AH14] and results on the Hölder regularity have been obtained (these results will be covered in detail in Section 6). However, no multifractional stable motion with random multifractional parameter has been considered, and the Hölder regularity of the multifractional stable processes that have been considered in the three cited articles are all dependent on assumptions on the Hölder regularity of the multifractional parameter. We intend to define an Itô multifractional stable motion, where the multifractional parameter $H(x)$ depends on the integration variable. This allows us to consider random values of $H(x)$ and to find bounds on the Hölder exponent of the resulting process, irrespective of the Hölder regularity of the multifractional parameter. We will adapt the arguments from [LMS21] to the stable case and will even directly make use of Theorem 3.10.

4 Stable Distributions

Now that we have covered the multifractional Brownian motion, it is time to introduce the tools to generalize this process to the (multi)fractional stable motion: Stable distributions. Stable distributions were originally formulated by P. Lévy in the early 1920s when he was investigating sums of independent random variables in his quest to stretch the Central Limit Theorem to its limit [Lév22; Lév23; Lév24; Lév25]. During this period, weakening the conditions of the Central Limit Theorem was a popular occupation among probability theorists: in 1901, Lyapunov formulated his famous condition, removing the need for the random variables to be identically distributed [Lya01]. In 1922, Lindeberg formulated an even weaker condition [Lin22]. However, both of these conditions required the existence of variances. For a Central Limit Theorem in the absence of variances and even first moments, it is necessary to consider non-Gaussian distributions as the limit law.

Lévy realized that there was a crucial property of the Gaussian distribution making it suitable as the limit law for the Central Limit Theorem: Its stability under independent positive linear combinations. Consider two independent standard normal random variables X and Y. Then $aX + bY \sim N(0, a^2 + b^2)$ whenever $a > 0$ and $b > 0$ are positive constants. It follows that $aX + bY \stackrel{d}{=} (a^2 + b^2)^{\frac{1}{2}} Z$, with Z another standard normal random variable. This stability property is generalizable and leads to the definition of the class of stable distributions.

Definition 4.1. Let $\mathscr{P}$ be a non-degenerate (i.e. not of the form δ_x) probability measure on $\mathbb{R}$. Then $\mathscr{P}$ is a **stable distribution** if, whenever X, Y and Z are random variables distributed according to $\mathscr{P}$ such that $X \perp Y$, and $a > 0$ and $b > 0$ are positive constants, we have

$$aX + bY \stackrel{d}{=} cZ + d, \tag{4.1}$$

for some constants $c > 0$ and $d \in \mathbb{R}$. If (4.1) holds with $d = 0$ then $\mathscr{P}$ is **strictly stable**. We will say that a real-valued random variable is stable if its law is stable.

As we have seen, the standard normal distribution is strictly stable, and the constant c in Equation (4.1) follows the predictable relation $a^2 + b^2 = c^2$. It turns out that, for any stable distribution, the constants c and d in Equation (4.1) are uniquely determined by a and b. Moreover, c follows the relation $a^\alpha + b^\alpha = c^\alpha$ for some $\alpha \in (0, 2]$.

Lemma 4.2. *Let X be a non-degenerate real-valued random variable. Suppose $cX + d \stackrel{d}{=} c'X + d'$ with $c, c' > 0$ and $d, d' \in \mathbb{R}$. Then $c = c'$ and $d = d'$. As a consequence, the constants c and d in Equation (4.1) are uniquely determined by a and b.*

Proof. Let Y be an independent copy of X. First, we show that $X - Y$ is not almost surely equal to zero. Suppose, to provoke a contradiction, that $Y = X$ almost surely for some. Then, writing F for the shared cumulative distribution function of X and Y, we have, for all $x \in \mathbb{R}$,

$$F(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X \leq x \wedge Y \leq x) = \mathbb{P}(X \leq x)\mathbb{P}(Y \leq x) = (F(x))^2.$$

So $F(x) \in \{0, 1\}$ and it follows that X is degenerate, a contradiction. Now, we have

$$c(X - Y) = cX + d - (cY + d) \stackrel{d}{=} c'X + d' - (c'Y + d') = c'(X - Y).$$

Thus, the characteristic function of $X - Y$ satisfies $\phi_{X-Y}(c\theta) = \phi_{X-Y}(c'\theta)$ and therefore $\phi_{X-Y}(\theta) = \phi_{X-Y}(c'/c\,\theta)$. From induction it then follows that $\phi_{X-Y}(\theta) = \phi((c'/c)^n\,\theta)$ for all $n \in \mathbb{N}$. Suppose that $c' \neq c$ and assume, without loss of generality (due to the symmetry in c and c'), that $c' < c$. Choose $\theta \in \mathbb{R}$ such that $\phi_{X-Y}(\theta) \neq 1$ (which exists because $X - Y$ is not almost surely equal to zero). Then $1 = \phi_{X-Y}(0) = \lim_{n\to\infty} \phi((c'/c)^n\,\theta) = \phi(\theta) \neq 1$, a contradiction, so $c' = c$. Now, the characteristic function of X satisfies $\phi_X(c\theta)e^{id\theta} = \phi_X(c'\theta)e^{id'\theta} = \phi_X(c\theta)e^{id'\theta}$. Because $\phi_X(c\theta)$ is non-zero in a neighborhood of 0 it follows that $d = d'$. $\square$

Proposition 4.3. *If $\mathscr{P}$ is a stable distribution, then the constant c in Equation* (4.1) *follows the relation $a^\alpha + b^\alpha = c^\alpha$, for some constant $\alpha \in (0, 2]$. This constant α is called the* ***index of stability*** *of $\mathscr{P}$ and $\mathscr{P}$ is said to be* ***α-stable****.*

To prove Proposition 4.3, we will be taking Feller's approach [Fel71, Section VI.1]. He uses a slightly different, but equivalent notion of stability. In his formulation, a stable distribution is required to be stable under n-ary sums for any $n > 0$, instead of under binary positive linear combinations.

Definition 4.4 (Equivalent to Definition 4.1)**.** Let $\mathscr{P}$ be a non-degenerate probability measure on $\mathbb{R}$. Then $\mathscr{P}$ is a **stable distribution** if, whenever $X_1 \dots X_n$ are independent random variables distributed according to $\mathscr{P}$, we have

$$\sum_{k=1}^{n} X_k \overset{d}{=} p_n X + q_n, \tag{4.2}$$

for some constants $p_n > 0$ and $q_n \in \mathbb{R}$, and X a random variable distributed according to $\mathscr{P}$. If (4.2) holds with $q_n = 0$ then $\mathscr{P}$ is **strictly stable**.

By Lemma 4.2, p_n and q_n are uniquely determined by n and writing them suggestively as functions of n is appropriate. From an inductive argument, it is easy to see that Definition 4.1 implies Definition 4.4. The converse will be shown along the way to proving Proposition 4.3: We will do so by showing that the constants $(p_n)_{n=1}^\infty$ in Equation (4.2) follow the relation $p_n = n^{1/\alpha}$. As a corollary we will obtain Proposition 4.3 and the equivalence between Definitions 4.1 and 4.4. First, the relation $p_n = n^{1/\alpha}$ will be shown under the further assumption that $\mathscr{P}$ is symmetric, the non-symmetric case immediately follows.

Lemma 4.5. *Let $\mathscr{P}$ be a symmetric (i.e. $\mathscr{P}(A) = \mathscr{P}(-A)$ for all Borel measurable sets $A \subseteq \mathbb{R}$) distribution satisfying Definition 4.4. Then there exists a constant $\alpha \in (0, 2]$ such that the numbers $(p_n)_{n=1}^\infty$ in Equation* (4.2) *follow the relation*

$$p_n = n^{\frac{1}{\alpha}}.$$

Moreover, the characteristic function attributed to $\mathscr{P}$ is of the form

$$\phi(\theta) = \int_{\mathbb{R}} e^{ix\theta}\, d\mathscr{P}(x) = e^{-\sigma^\alpha |\theta|^\alpha} \tag{4.3}$$

for some $\sigma > 0$.

Proof. First, note that symmetric stable distributions are necessarily strictly stable. It is impossible for p_n to equal 1 whenever $n > 1$. Indeed: If $p_n = 1$ with $n > 1$, then the stability property implies that the characteristic function of $\mathscr{P}$ satisfies $\phi^n = \phi$. Since ϕ is continuous and $\phi(0) = 1$, it follows that $\phi = 1$ and that $\mathscr{P} = \delta_0$ is degenerate. Thus, it is possible to define $\alpha(n) = \frac{\log n}{\log p_n}$ for $n > 1$ so that $p_n = n^{1/\alpha(n)}$. We shall prove that $\alpha(n) = \alpha(m)$ whenever $n > 1$ and $m > 1$. To this end, we establish the following auxiliary results about $(p_n)_{n=1}^\infty$.

1. $(p_n)_{n=1}^\infty$ is multiplicative, that is, $p_{n \cdot m} = p_n \cdot p_m$. As a result, $p_\nu = p_n^k = \nu^{1/\alpha(n)}$ whenever $\nu = n^k$.
2. The set $\{p_n/p_m : 0 < n < m\}$ is bounded.

1. Let n and m be positive integers and let $X_1 \dots X_{nm}$ be independent random variables distributed according to $\mathscr{P}$. Consider the block sum

$$\sum_{k=1}^{nm} X_k = \sum_{l=1}^{m} \left(\sum_{k=(l-1)n+1}^{ln} X_k \right).$$

We obtain the multiplicative property by applying stability to the entire sum at once and blockwise. Let X be distributed according to $\mathscr{P}$. On the one hand, $\sum_{k=1}^{nm} X_k \stackrel{d}{=} p_{nm} X$. On the other hand, for each $l = 1 \dots m$ it holds that $\sum_{k=(l-1)n+1}^{ln} X_k \stackrel{d}{=} p_n \widehat{X}_l$, where $\widehat{X}_1 \dots \widehat{X}_m$ are independently distributed according to $\mathscr{P}$. Since the blocks are independent, and by applying stability once more, it follows that

$$\sum_{l=1}^{m} \left(\sum_{k=(l-1)n+1}^{ln} X_k \right) \stackrel{d}{=} \sum_{l=1}^{m} p_n \widehat{X}_l = p_n \sum_{l=1}^{m} \widehat{X}_l \stackrel{d}{=} p_n p_m X.$$

Thus, $p_{n \cdot m} X \stackrel{d}{=} p_n \cdot p_m X$ and, by Lemma 4.2, $p_{n \cdot m} = p_n \cdot p_m$.

2. Let $X_1 \dots X_{n+m}$ be independent and distributed according to $\mathscr{P}$. By applying the stability property to the two sides in the equality

$$\sum_{k=1}^{n+m} X_k = \sum_{k=1}^{n} X_k + \sum_{k=n+1}^{n+m} X_k,$$

it follows that $p_{n+m} X \stackrel{d}{=} p_n Y + p_m Z$, where X, Y and Z are distributed according to $\mathscr{P}$ and $Y \perp Z$. Let $\overline{F}(t) = \mathscr{P}((t, \infty))$ denote the survival function of $\mathscr{P}$. Then

$$\overline{F}(t) = \mathbb{P}(X > t) = \mathbb{P}(p_{n+m} X > p_{n+m} t) = \mathbb{P}(p_n Y + p_m Z > p_{n+m} t).$$

The event $(p_n Y > p_{n+m} t) \wedge (Z \geq 0)$ is contained in the event $(p_n Y + p_m Z > p_{n+m} t)$. Moreover, since we are working under the assumption that $\mathscr{P}$ is symmetric, $\mathbb{P}(Z \geq 0) \geq \frac{1}{2}$. It follows that

$$\overline{F}(t) \geq \frac{1}{2} \mathbb{P}(p_n Y > p_{n+m} t) = \frac{1}{2} \overline{F} \left(\frac{p_{n+m}}{p_n} t \right).$$

From this it follows that the set $\{p_n / p_{n+m} : n, m > 0\} = \{p_n / p_m : 0 < n < m\}$ is bounded, because if this set were unbounded, from right-continuity of the survival function it would follow that $\overline{F}(t) \geq \frac{1}{2} \overline{F}(0)$ for all t. Taking the limit $t \to \infty$ reveals that $\overline{F}(0) = 0$ which is not possible for a survival function coming from a non-degenerate symmetric distribution (if $\mathscr{P}((0, \infty)) = 0$ then $\mathscr{P}((-\infty, 0)) = 0$ as well so that $\mathscr{P}(\{0\}) = 1$ and $\mathscr{P} = \delta_0$ is degenerate).

Now that the auxiliary results have been established, it is time to prove that $\alpha(n) = \alpha(m)$ whenever $n > 1$ and $m > 1$. First note that properties 1 and 2 imply that $p_n > 1$ for all $n > 1$. Indeed: If $p_n < 1$ for some $n > 1$, then by property 1, $p_1 / p_{n^k} = p_n^{-k} \to \infty$ as $k \to \infty$ which contradicts property 2. It follows that $\alpha(n) > 0$ for all $n > 1$. Now fix $n > 1$ and $m > 1$. We will consider indices of the form $\nu(k) = n^k$ and $\mu(l) = m^l$. For each integer $k > 0$ there exists a unique integer $l = l(k) \geq 0$ such that $\mu(l) < \nu(k) \leq \mu(l+1) = m \cdot \mu(l)$ (the dependence of l on k will be suppressed for readability). Then, using property 1 to find that $p_{\nu(k)} = \nu(k)^{1/\alpha(n)}$ and $\mu(l)^{1/\alpha(m)} = p_{\mu(l)}$, we obtain

$$p_{\nu(k)} = \nu(k)^{\frac{1}{\alpha(n)}} \leq (m \cdot \mu(l))^{\frac{1}{\alpha(n)}} = m^{\frac{1}{\alpha(n)}} \cdot \left(\mu(l)^{\frac{1}{\alpha(m)}} \right)^{\frac{\alpha(m)}{\alpha(n)}} = m^{\frac{1}{\alpha(n)}} \cdot \left(p_{\mu(l)} \right)^{\frac{\alpha(m)}{\alpha(n)}}.$$

Thus,

$$\frac{p_{\mu(l)}}{p_{\nu(k)}} \geq m^{-\frac{1}{\alpha(n)}} \cdot \left(p_{\mu(l)} \right)^{1 - \frac{\alpha(m)}{\alpha(n)}}.$$

As a consequence, by applying property 2, the set $\{p_{\mu(l(k))}^{1-\frac{\alpha(m)}{\alpha(n)}} : k > 0\}$ is bounded. Now $p_{\mu(l)} \to \infty$ as $l \to \infty$ (by property 1 and $p_m > 1$). Moreover, $l = l(k) \to \infty$ as $k \to \infty$. It follows that $1 - \alpha(m)/\alpha(n) \leq 0$ and that $\alpha(n) \leq \alpha(m)$. From symmetry in n and m it follows that $\alpha(n) = \alpha(m)$.

Finally, the characteristic function having the form of Equation (4.3) implies that $\alpha \leq 2$: If the characteristic function has this form and $\alpha > 2$, then the characteristic function is twice differentiable so that the variance of $\mathscr{P}$ exists. In this case, taking variance of Equation (4.2) yields $n\mathrm{Var}(\mathscr{P}) = p_n^2 \,\mathrm{Var}(\mathscr{P})$. Since $\mathscr{P}$ is non-degenerate, $\mathrm{Var}(\mathscr{P}) \neq 0$ and we find $p_n = n^{1/2}$ and $\alpha = 2$, a contradiction.

It remains to be shown that the characteristic function ϕ has the form of Equation (4.3). Since $\mathscr{P}$ is symmetric, the characteristic function is real. Note that Equation (4.2) implies that the characteristic function satisfies $\phi(\theta)^n = \phi(p_n\theta) = \phi(n^{1/\alpha}\theta)$ for every $n > 0$ (recall that $\mathscr{P}$ is strictly stable due to symmetry). By substituting $\theta \hookleftarrow n^{-1/\alpha}\theta$ we find that the characteristic function satisfies $\phi(\theta) = \phi(n^{-1/\alpha}\theta)^n$. Taking $n = 2$ shows that the characteristic function is nonnegative. If $\phi(\theta) = 0$ for some θ, then $0 = \phi(\theta) = \phi(n^{-1/\alpha}\theta)^n$ for all $n > 0$. So $\phi(n^{-1/\alpha}\theta) = 0$ for all $n > 0$ and taking $n \to \infty$ would reveal that $\phi(0) = 0$. Thus, the characteristic function is positive and satisfies $\phi(\theta)^{1/n} = \phi(n^{-1/\alpha}\theta)$. For $n > 0$ and $m > 0$ it holds that

$$\phi((n/m)^{1/\alpha}) = \phi(n^{1/\alpha}\, m^{-1/\alpha}) = \phi(m^{-1/\alpha})^n = \phi(1)^{n/m}.$$

Consequently, $\phi(\theta) = \phi(1)^{\theta^\alpha}$ for $\theta \in \mathbb{Q}_{>0}$. Since ϕ is symmetric and continuous, $\phi(\theta) = \phi(1)^{|\theta|^\alpha}$ for $\theta \in \mathbb{R}$. Because ϕ is not constantly equal to 1, $\phi(1) < 1$. The form of Equation (4.3) is obtained by writing $\sigma = (-\log\phi(1))^{1/\alpha} > 0$. □

Corollary 4.6. *If $\mathscr{P}$ satisfies Definition 4.4 then then there is some $\alpha \in (0, 2]$ such that the numbers $(p_n)_{n=1}^\infty$ in Equation (4.2) satisfy $p_n = n^{1/\alpha}$.*

Proof. Let $\mathscr{P}_S = \mathscr{P} * \mathscr{P}(-\cdot)$ be the symmetrization of $\mathscr{P}$ (i.e. the law of $X - Y$, where X and Y are independently distributed according to $\mathscr{P}$). Then $\mathscr{P}_S$ is symmetric. Moreover, $\mathscr{P}_S$ satisfies Definition 4.4, and the numbers $(p_n)_{n=1}^\infty$ in Equation (4.2) are the same as those from $\mathscr{P}$. Indeed: Let $X_1 \dots X_n, Y_1 \dots Y_n$ be independent random variables distributed according to $\mathscr{P}$. Then, since $\mathscr{P}$ satisfies Definition 4.4, there are $p_n > 0$ and $q_n \in \mathbb{R}$ such that $\sum_{k=1}^n X_k \overset{d}{=} p_n X + q_n$ and $\sum_{k=1}^n Y_k \overset{d}{=} p_n Y + q_n$. Since $\sum_{k=1}^n X_k \perp \sum_{k=1}^n Y_k$ and $X \perp Y$, it follows that

$$\sum_{k=1}^n (X_k - Y_k) = \sum_{k=1}^n X_k - \sum_{k=1}^n Y_k \overset{d}{=} p_n X + q_n - (p_n Y + q_n) = p_n(X - Y).$$

□

Having determined the constants $(p_n)_{n=1}^\infty$ from Equation (4.2), it is now time to prove Proposition 4.3. Of course, the α in this proposition is the same as the one from Corollary 4.6. The equivalence between Definition 4.1 and 4.4 will also be shown.

Lemma 4.7. *Suppose $\mathscr{P}$ satisfies Definition 4.4 with $p_n = n^{1/\alpha}$, $\alpha \in (0, 2]$. Then there is a constant $C \in \mathbb{R}$ such that*

$$q_n = \begin{cases} C(n^{1/\alpha} - n) & \alpha \neq 1 \\ Cn\log n & \alpha = 1 \end{cases}$$

Proof. [Nol20, Lemma 3.3 and 3.4] □

Proof of Proposition 4.3, and the equivalence between Definition 4.1 and 4.4. As mentioned before, Definition 4.1 implies Definition 4.4 by induction. Suppose $\mathscr{P}$ satisfies Definition 4.4, with $p_n = n^{1/\alpha}$, $\alpha \in (0,2]$. Let n and m be positive integers and let $X_1 \ldots X_{n+m}$ be independently distributed according to $\mathscr{P}$. By applying the stability property to the left and right hand side of the equality

$$\sum_{k=1}^{n+m} X_k = \sum_{k=1}^{n} X_k + \sum_{k=n+1}^{n+m} X_k,$$

we find that $p_{n+m}Z + q_{n+m} \overset{d}{=} p_n X + q_n + p_m Y + q_m$, where X, Y and Z are distributed according to $\mathscr{P}$, and $X \perp Y$. Rearranging and using that $p_n = n^{1/\alpha}$, it follows that

$$\left(\frac{n}{m}+1\right)^{\frac{1}{\alpha}} Z + Q_{n,m} \overset{d}{=} \left(\frac{n}{m}\right)^{\frac{1}{\alpha}} X + Y$$

for all $n > 0$ and $m > 0$, where $Q_{n,m} = m^{-1/\alpha}(q_{n+m} - q_n - q_m)$. Now let $a > 0$ and $b > 0$, and choose sequences of positive integers $(n_k)_{k=1}^{\infty}$ and $(m_k)_{k=1}^{\infty}$ such that $n_k/m_k \to a/b$ as $k \to \infty$. Then from Lemma 4.7,

$$Q_{n_k,m_k} \overset{k\to\infty}{\to} \begin{cases} C\left(\left(\frac{a}{b}+1\right)^{\frac{1}{\alpha}} - \left(\frac{a}{b}\right)^{\frac{1}{\alpha}} - 1\right) & \alpha \neq 1, \\ C\left(\left(\frac{a}{b}+1\right)\log(a+b) - \frac{a}{b}\log a - \log b\right) & \alpha = 1, \end{cases}$$

for some $C \in \mathbb{R}$. By taking limit in distribution and rearranging, we obtain $(a+b)^{1/\alpha}Z + d(a,b) \overset{d}{=} a^{1/\alpha}X + b^{1/\alpha}Y$, with

$$d(a,b) = \begin{cases} C\left((a+b)^{\frac{1}{\alpha}} - a^{\frac{1}{\alpha}} - b^{\frac{1}{\alpha}}\right) & \alpha \neq 1, \\ C\left((a+b)\log(a+b) - a\log a - b\log b\right) & \alpha = 1. \end{cases}$$

Thus, the implication of Definition 4.4 to 4.1 and Proposition 4.3 follow by substituting $a \hookleftarrow a^{\alpha}$ and $b \hookleftarrow b^{\alpha}$. □

Having established the existence of the index of stability, note that Lemma 4.5 identifies symmetric 2-stable distributions as centered normal distributions. In the absence of symmetry, there is still a characterization of α-stable distributions in terms of their characteristic function which identifies all 2-stable distributions as Gaussian. Because we will focus on symmetric α-stable distributions, this representation of the characteristic function of general α-stable distributions will not be proven. Instead, we mention that the standard version of the proof retrieves this as a special case of the Lévy-Khintchine representation for infinitely divisible distributions [Sat99; ST94], however, it is also possible to prove the statement without referring to infinitely divisible distributions [Nol20].

Proposition 4.8. *A distribution $\mathscr{P}$ is α-stable if and only if its characteristic function is of the following form,*

$$\phi(\theta) = \int_{\mathbb{R}} e^{ix\theta}\, d\mathscr{P}(x) = \begin{cases} \exp\left(-\sigma^{\alpha}|\theta|^{\alpha}(1 - i\beta\,\mathrm{sign}(\theta)\tan(\frac{\pi\alpha}{2})) + i\mu\theta\right) & \alpha \neq 1, \\ \exp\left(-\sigma|\theta|(1 + i\beta\frac{2}{\pi}\mathrm{sign}(\theta)\ln|\theta|) + i\mu\theta\right) & \alpha = 1. \end{cases}$$

Here $\sigma > 0$ and $\mu \in \mathbb{R}$ are uniquely determined by $\mathscr{P}$, and $-1 \leq \beta \leq 1$ is uniquely determined by $\mathscr{P}$ if $\alpha < 2$ (if $\alpha = 2$ then β is irrelevant).

We see that the characteristic functions of stable distributions are absolutely integrable and therefore absolutely continuous with respect to the Lebesgue measure [Fel71, Section XV.3, Theorem 3]. In particular, stable distributions are atomless. In the Gaussian case $\alpha = 2$ the distribution has squared exponentially decaying tails. However, for $\alpha < 2$, the tails of α-stable distributions are much heavier: If $\mathscr{P}$ is an α-stable distribution with $\alpha < 2$ then the tails $0 < t \mapsto \mathscr{P}((t,\infty))$ and $0 < t \mapsto \mathscr{P}((-\infty,-t))$ are both asymptotically equivalent to a constant times $t^{-\alpha}$ [ST94, Property 1.2.15]. This entails that α-stable distributions have finite absolute moments of all orders up to but not including α, and infinite absolute moments of order all orders greater than or equal to α. Moreover, α-stable random variables $X : \Omega \to \mathbb{R}$ are elements of the weak Lebesgue space $\Lambda^\alpha(\Omega)$.

4.1 Symmetric α-stable Random Variables

Recall that a (real-valued) random variable X is stable if its law is stable. If X is also symmetric (i.e. $X \overset{d}{=} -X$), then X is called **symmetric α-stable**, with $\alpha \in (0,2]$ the index of stability from the law of X. If we also allow 0 to count as a symmetric α-stable random variable, by Lemma 4.5, the characteristic function of a symmetric α-stable random variable X is of the form

$$\phi_X(\theta) = e^{-\sigma^\alpha|\theta|^\alpha},$$

for some $\sigma \geq 0$. For fixed α, the number σ is of course uniquely determined by X so we may write $\sigma = \|X\|_\alpha$. Note that $\|Y\|_\alpha X \overset{d}{=} \|X\|_\alpha Y$ for any pair of symmetric α-stable random variables X and Y. This property identifies $\|X\|_\alpha$ as the **scale parameter** of X. If Ψ is a symmetric α-stable random variable with $\|\Psi\|_\alpha = 1$, then Ψ is deemed **standard symmetric α-stable**. The suggestive norm-like notation $\|\cdot\|_\alpha$ is not meant to fool you. Indeed: If S_α is a linear set of symmetric α-stable random variables on $(\Omega,\mathcal{F},\mathbb{P})$, modulo almost sure equality, then $(S_\alpha, \|\cdot\|_\alpha)$ is a normed space if $\alpha \geq 1$, and an α-normed space if $\alpha \leq 1$. To prove the triangle inequality the notion of symmetric α-stable random vectors and a representation of their characteristic functions are required.

Definition 4.9. A random vector $\mathbf{X} = (X_1 \dots X_d)$ in $\mathbb{R}^d$ is **(jointly) symmetric α-stable** if all its linear combinations are symmetric α-stable. Similarly, a stochastic process $(X_t)_{t\in T}$ is **symmetric α-stable** if all its finite linear combinations are symmetric α-stable.

Let $S_d = \{\mathbf{x} \in \mathbb{R}^d : \|x\| = 1\}$ denote the unit sphere in $\mathbb{R}^d$ (with respect to the standard Euclidean norm).

Lemma 4.10. *Let $0 < \alpha < 2$. A random vector $\mathbf{X} = (X_1 \dots X_d)$ in $\mathbb{R}^d$ is symmetric α-stable if and only if there exists a finite symmetric measure Γ on S_d such that its characteristic function is given by*

$$\phi_{\mathbf{X}}(\boldsymbol{\theta}) = \mathbb{E}\left[e^{i\langle\boldsymbol{\theta},\mathbf{X}\rangle}\right] = \exp\left(-\int_{S_d} |\langle\boldsymbol{\theta},\mathbf{x}\rangle|^\alpha \, d\Gamma(\mathbf{x})\right)$$

*The measure Γ is referred to as the **spectral measure** of $\mathbf{X}$.*

Proof. [ST94, Theorem 2.4.3] □

Proposition 4.11. *Let S_α be a linear set of symmetric α-stable random variables on the probability space $(\Omega,\mathcal{F},\mathbb{P})$, modulo almost sure equality. Then $(S_\alpha, \|\cdot\|_\alpha)$ is a normed space for $\alpha \geq 1$, and an α-normed space for $\alpha \leq 1$.*

Proof. Note that $X = 0$ almost surely if and only if its law is δ_0 which is the case if and only if $\phi_X = 1$ which happens precisely when $\|X\|_\alpha = 0$. Thus, $\|\cdot\|_\alpha$ is positive on S_α. Moreover, note that for $X \in S_\alpha$ and $c \in \mathbb{R}$ we have

$$\phi_{cX}(\theta) = \phi_X(c\theta) = e^{-\|X\|_\alpha^\alpha|c\theta|^\alpha} = e^{-(|c|\|X\|_\alpha)^\alpha|\theta|^\alpha},$$

so $\|cX\|_\alpha = |c|\|X\|_\alpha$. Finally, we show that the triangle inequality is satisfied for $\alpha \geq 1$, and the α-triangle inequality for $\alpha \leq 1$. Let $X, Y \in S_\alpha$ and note that (X, Y) is jointly symmetric α-stable (because S_α is a linear set). Let Γ be the spectral measure of (X, Y). Suppose $\alpha \geq 1$, then, applying the Minkowski inequality to the coordinate projections on the unit circle yields

$$\begin{aligned}\|X+Y\|_\alpha &= \left(\int_{S_2} |x+y|^\alpha \, d\Gamma(x,y)\right)^{\frac{1}{\alpha}} \\ &\leq \left(\int_{S_2} |x|^\alpha \, d\Gamma(x,y)\right)^{\frac{1}{\alpha}} + \left(\int_{S_2} |y|^\alpha \, d\Gamma(x,y)\right)^{\frac{1}{\alpha}} \\ &= \|X\|_\alpha + \|Y\|_\alpha.\end{aligned}$$

If $\alpha \leq 1$, by subbaditivity of $|\cdot|^\alpha$,

$$\begin{aligned}\|X+Y\|_\alpha^\alpha &= \int_{S_d} |x+y|^\alpha \, d\Gamma(x,y) \\ &\leq \int_{S_d} |x|^\alpha \, d\Gamma(x,y) + \int_{S_d} |y|^\alpha \, d\Gamma(x,y) \\ &= \|X\|_\alpha^\alpha + \|Y\|_\alpha^\alpha.\end{aligned}$$

□

Lemma 4.12. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of symmetric α-stable random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $0 < p < \alpha$. Then the following are equivalent.*

1. $\|X_n\|_\alpha \to 0$,
2. $X_n \to 0$ *in probability,*
3. $X_n \to 0$ *in* $\mathbb{L}^p(\Omega)$,
4. $X_n \to 0$ *in* $\Lambda^\alpha(\Omega)$.

Proof. To see why 1 implies 4, let Ψ be a standard symmetric α-stable random variable. Then

$$\|X_n\|_{\Lambda^\alpha(\Omega)} = \|\|X_n\|_\alpha \, \Psi\|_{\Lambda^\alpha(\Omega)} = \|X_n\|_\alpha \, \|\Psi\|_{\Lambda^\alpha(\Omega)} \overset{n\to\infty}{\to} 0.$$

Now 4 implies 3 by Lemma 2.18, and 3 implies 2 by Markov's inequality. Finally, we prove that 2 implies 1: If $X_n \to 0$ in probability, then $X_n \to 0$ in distribution so that $\phi_{X_n} \to 1$ pointwise. Thus, $\|X_n\|_\alpha \to 0$. □

Lemma 4.13. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of symmetric α-stable random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $\|X_n\|_\alpha \to \infty$, then $|X_n| \to \infty$ in probability.*

Proof. Let Ψ be a standard symmetric α-stable random variable and let $M > 0$ be arbitrary. Then

$$\mathbb{P}(|X_n| \leq M) = \mathbb{P}\left(|\Psi| \leq \frac{M}{\|X_n\|_\alpha}\right) \overset{n\to\infty}{\to} 0.$$

□

Lemma 4.14. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of symmetric α-stable random variables and suppose $X_n \to X$ in distribution. Then X is symmetric α-stable.*

Proof. If $X_n \to X$ in distribution, then $\phi_{X_n} \to \phi_X$ pointwise. Thus, ϕ_X is real and $0 \le \phi_X(\theta) \le 1$ for all $\theta \in \mathbb{R}$. Take $0 \ne \theta \in \mathbb{R}$ such that $0 < \phi_X(\theta) \le 1$ (which exists because ϕ is continuous at $\theta = 0$ and equal to 1 at $\theta = 0$). Then $\|X_n\|_\alpha = |\theta|^{-1} (-\log \phi_{X_n}(\theta))^{\frac{1}{\alpha}} \overset{n\to\infty}{\to} |\theta|^{-1} (-\log \phi_X(\theta))^{\frac{1}{\alpha}}$. Thus, $(\|X_n\|_\alpha)_{n\in\mathbb{N}}$ converges, and X is symmetric α-stable with $\|X\|_\alpha = \lim_{n\to\infty} \|X_n\|_\alpha$. □

Let S_α be a linear set of symmetric α-stable random variables. Due to Lemma 4.12, $(S_\alpha, \|\cdot\|_\alpha)$ may be understood as a subspace (as a topological vector space) of $\Lambda^\alpha(\Omega)$, $\mathbb{L}^p(\Omega)$ for $0 < p < \alpha$ or $\mathbb{L}^0(\Omega)$. Combining Lemmas 4.12 and 4.14 reveals that the topological closure $\overline{S_\alpha}$ in any of these spaces are equal as sets, moreover, $\overline{S_\alpha}$ is again a linear set of symmetric α-stable random variables, and $(\overline{S_\alpha}, \|\cdot\|_\alpha)$ is a Banach space for $\alpha \ge 1$ and an α-Banach space for $\alpha \le 1$.

5 Infinitely Divisible Distributions and Lévy Processes

This section is on Lévy processes and their intimate connection to infinitely divisible distributions, which is a class of probability measures on $\mathbb{R}$ that includes the stable distributions. The notion of a (symmetric α-stable) Lévy process is needed to talk about a stable analogue of the (multi)fractional Brownian motion. Indeed: Normal distributions alone are not enough to develop a stochastic calculus. One must organize these into a stochastic process with a dependence structure, the Brownian motion, to make sense of Itô integrals. The analogue to Brownian motion in the absence of normal distributions is a Lévy process.

Definition 5.1. Let T be $\mathbb{R}$ or $[0,\infty)$ and let $(L(t))_{t\in T}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. Then $(L(t))_{t\in T}$ is a **Lévy process** if the following properties are satisfied.

1. $L(0) = 0$ almost surely.
2. Stochastic continuity: $L(s) \to L(t)$ in probability whenever $s \to t$.
3. Independent increments: For any $t_0 < t_1 < \ldots < t_n$ in T, the random variables $L(t_1) - L(t_0), L(t_2) - L(t_1) \ldots L(t_n) - L(t_{n-1})$ are independent.
4. stationary increments: $(L(t+h) - L(h))_{t\in T} \overset{d}{=} (L(t))_{t\in T}$ for all $h \in T$.

Since we demand no path properties, it may be more appropriate to follow Sato [Sat99] in his convention to call the objects defined as above Lévy processes *in law*, and use the name Lévy process only under the further assumption that almost all paths are càdlàg (it can be shown that such a modification exists [Sat99, Theorem 11.5]). However, since we are strictly interested in the distributional properties of Lévy processes, we will refrain from doing so. The distribution of a Lévy process is completely determined by its law at $t = 1$. Moreover, the possible laws at $t = 1$ for a Lévy process are precisely the infinitely divisible distributions.

Definition 5.2. Let $\mathscr{P}$ be a probability measure on $\mathbb{R}$. Then $\mathscr{P}$ is an **infinitely divisible distribution** if one of the following three equivalent conditions is satisfied

1. For every integer $n > 0$ there are i.i.d random variables $X_1 \ldots X_n$ such that $\sum_{k=1}^{n} X_k$ is distributed according to $\mathscr{P}$.
2. For every integer $n > 0$ there exists a probability measure $\mathscr{P}_n$ on $\mathbb{R}$ such that $\mathscr{P}_n^{*n} = \mathscr{P}$, where $\mathscr{P}^{*n} = \underbrace{\mathscr{P} * \mathscr{P} * \ldots * \mathscr{P}}_{n \text{ times}}$.
3. For every integer $n > 0$ there exists a characteristic function ϕ_n such that $\phi_n^n = \phi$, where ϕ is the characteristic function attributed to $\mathscr{P}$.

The goal is now to show that, in distribution, Lévy processes on $[0,\infty)$ correspond precisely to the class of infinitely divisible distributions. This will done by following Section 7 in [Sat99].

Theorem 5.3. *There is a bijection between the set of Lévy processes $(L(t))_{t\in[0,\infty)}$, modulo equality in distribution, and the set of infinitely divisible distribution. This bijection sends a Lévy process to the law of $L(1)$.*

First it will be shown that an infinitely divisible distribution $\mathscr{P}$ can be interpolated to a weakly continuous convolution semigroup $(\mathscr{P}^t)_{t\in[0,\infty)}$ such that $\mathscr{P}^1 = \mathscr{P}$ (the term weakly continuous convolution semigroup is not used by Sato and comes from Applebaum's book [App09]). Next, we will apply the Kolmogorov extension theorem to show that weakly continuous convolution semigroup $(\mathscr{P}^t)_{t\in[0,\infty)}$ allows a Lévy process $(L(t))_{t\in[0,\infty)}$ such that $\mathscr{P}^t$ is the law of $L(t)$.

Definition 5.4. A **weakly continuous convolution semigroup** is a collection of probability measures $(\mathscr{P}^t)_{t\in[0,\infty)}$ on $\mathbb{R}$ such that

- $\mathscr{P}^0 = \delta_0$,
- $\mathscr{P}^{s+t} = \mathscr{P}^s * \mathscr{P}^t$ for $s, t \in [0,\infty)$,
- $\mathscr{P}^t$ converges weakly to δ_0 as $t \downarrow 0$.

We use the fact that the characteristic function ϕ of an infinitely divisible distribution admits a unique continuous function $\psi : \mathbb{R} \to \mathbb{C}$ such that $\psi(0) = 0$ and $\phi(\theta) = \exp(\psi(\theta))$. This allows us to interpolate the characteristic function to a family of characteristic functions $(\phi^t)_{t\in[0,\infty)}$ such that their corresponding probability measures form a weakly continuous convolution semigroup and $\phi^1 = \phi$.

Lemma 5.5. *Let $\phi : \mathbb{R} \to \mathbb{C}$ be a continuous function such that $\phi(0) = 1$ and $\phi(\theta) \neq 0$ for all $\theta \in \mathbb{R}$. Then there is a unique continuous function $\psi : \mathbb{R} \to \mathbb{C}$ such that $\psi(0) = 0$ and $\phi(\theta) = \exp(\psi(\theta))$.*

Proof. [Sat99, Lemma 7.6] □

Lemma 5.6. *Let $\mathscr{P}$ be an infinitely divisible distribution and let ϕ be its characteristic function. Then $\phi(\theta) \neq 0$ for all $\theta \in \mathbb{R}$.*

Proof. For each integer $n > 0$ let ϕ_n be a characteristic function such that $\phi_n^n = \phi$. Define

$$\varphi(\theta) = \lim_{n\to\infty} |\phi_n(\theta)|^2 = \lim_{n\to\infty} |\phi(\theta)|^{2/n} = \begin{cases} 1 & \text{if } \phi(\theta) \neq 0, \\ 0 & \text{if } \phi(\theta) = 0. \end{cases}$$

Since ϕ is a characteristic function it is non-zero in a neighborhood of 0, so φ is constantly equal to 1 in a neighborhood of 0 and therefore continuous at 0. Thus, from Lévy's continuity theorem and the fact that $|\phi_n|^2 = \phi_n \cdot \overline{\phi_n}$ is a characteristic function for each n, it follows that φ is a characteristic function. Hence, it is continuous, therefore constantly equal to 1 and it follows that $\phi(\theta) \neq 0$ for all $\theta \in \mathbb{R}$. □

Proposition 5.7. *Let $\mathscr{P}$ be an infinitely divisible distribution. Then there is a unique weakly continuous convolution semigroup $(\mathscr{P}^t)_{t\in[0,\infty)}$ such that $\mathscr{P}^1 = \mathscr{P}$.*

Proof. Let ϕ be the characteristic function attributed to $\mathscr{P}$ and for $n > 0$ let ϕ_n be a characteristic function such that $\phi_n^n = \phi$. Then $\phi(\theta) \neq 0$ for all $\theta \in \mathbb{R}$ by Lemma 5.6 and thus $\phi_n(\theta) \neq 0$ for all $\theta \in \mathbb{R}$. By virtue of Lemma 5.5 there are continuous functions $\psi : \mathbb{R} \to \mathbb{C}$ and $\psi_n : \mathbb{R} \to \mathbb{C}$ such that $\psi(0) = \psi_n(0) = 0$, $\phi(\theta) = \exp(\psi(\theta))$ and $\phi_n(\theta) = \exp(\psi_n(\theta))$. Then $n\psi_n : \mathbb{R} \to \mathbb{C}$ is a continuous function satisfying $n\psi_n(0) = 0$ and $\phi(\theta) = (\phi_n(\theta))^n = (\exp(\psi_n(\theta)))^n = \exp(n\psi_n(\theta))$. Since $\psi : \mathbb{R} \to \mathbb{C}$ is the *unique* function satisfying these properties it follows that $\psi_n(\theta) = \frac{1}{n}\psi(\theta)$. Now for $t \geq 0$ define

$$\phi^t(\theta) = \exp(t\psi(\theta)). \tag{5.1}$$

We will show that (5.1) is a characteristic function for all $t \geq 0$. Firstly, note that, for an integer $n > 0$,

$$\phi^{1/n}(\theta) = \exp\left(\frac{1}{n}\psi(\theta)\right) = \exp\left(\psi_n(\theta)\right) = \phi_n(\theta).$$

Thus, $\phi^{1/n}$ is a characteristic function. Letting $m \geq 0$ be another integer, it follows that $\phi^{m/n} = (\phi^{1/n})^m$ is a characteristic function. We see that ϕ^t is a characteristic function for all $t \in \mathbb{Q}_{\geq 0}$. Then, because ϕ^t is continuous at 0 for all $t \in [0,\infty)$, it follows from Lévy's continuity theorem that ϕ^t is a characteristic function for all $t \in [0,\infty)$. From (5.1) it is clear that the characteristic functions $(\phi^t)_{t\in[0,\infty)}$ satisfy

- $\phi^0 = 1$,
- $\phi^{s+t} = \phi^s \cdot \phi^t$ for $s, t \in [0, \infty)$,
- $\phi^t \to 1$ pointwise as $t \downarrow 0$.

These properties translate precisely to the assertion that the corresponding probability measures $(\mathscr{P}^t)_{t\in[0,\infty)}$ form a weakly continuous convolution semigroup. Finally, we have $\phi^1(\theta) = \exp(\psi(\theta)) = \phi(\theta)$ so that $\mathscr{P}^1 = \mathscr{P}$. To show that $(\mathscr{P}^t)_{t\in[0,\infty)}$ is unique, suppose $(\tilde{\phi}^t)_{t\in[0,\infty)}$ is another collection of characteristic functions satisfying $\tilde{\phi}^0 = 1, \tilde{\phi}^{s+t} = \tilde{\phi}^s \cdot \tilde{\phi}^t, \tilde{\phi}^t \to 1$ pointwise as $t \downarrow 0$ and $\tilde{\phi}^1 = \phi = \exp \circ \psi$. Fixing $\theta \in \mathbb{R}$, the function $t \mapsto \tilde{\phi}^t(\theta)$ is continuous and satisfies $\tilde{\phi}^0(\theta) = 1, \tilde{\phi}^1(\theta) = \exp(\psi(\theta))$ and $\tilde{\phi}^{s+t}(\theta) = \tilde{\phi}^s(\theta)\tilde{\phi}^t(\theta)$. From these properties it follows that $\tilde{\phi}^t(\theta) = \exp(t\psi(\theta)) = \phi^t(\theta)$. □

Proposition 5.8. *Let $(\mathscr{P}^t)_{t\in[0,\infty)}$ be a weakly continuous convolution semigroup. Then there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Lévy process $(L(t))_{t\in[0,\infty)}$ such that $\mathscr{P}^t$ is the law of $L(t)$ for all $t \geq 0$.*

Proof. For $0 \leq t_0 < t_1 < \ldots < t_n$ define the finite-dimensional distribution $\mathscr{P}_{t_0\ldots t_n}$ on $\mathbb{R}^{n+1}$ as

$$\begin{aligned}\mathscr{P}_{t_0\ldots t_n}(A_0 \times \ldots \times A_n) &= \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} \mathbb{1}_{A_0}(x_0)\mathbb{1}_{A_1}(x_0+x_1)\ldots\mathbb{1}_{A_n}(x_0+\ldots+x_n)\, d\mathscr{P}^{t_0}(x_0)d\mathscr{P}^{t_1-t_2}(x_1)\ldots d\mathscr{P}^{t_n-t_{n-1}}(x_n)\\ &= \int_{A_0} d\mathscr{P}^{t_0}(y_0) \int_{A_1} d\mathscr{P}^{t_1-t_0}(y_1-y_0)\ldots\int_{A_n} d\mathscr{P}^{t_n-t_{n-1}}(y_n-y_{n-1}).\end{aligned}$$

We verify that the these finite-dimensional distributions satisfy the Kolmogorov consistency criterion. So suppose $A_k = \mathbb{R}$. Then, fixing y_{k-1},

$$\begin{aligned}\int_{A_{k+1}} d\mathscr{P}^{t_{k+1}-t_{k-1}}(y_{k+1}-y_{k-1}) &= \int_{A_{k+1}} d\left(\mathscr{P}^{t_k-t_{k-1}} * \mathscr{P}^{t_{k+1}-t_k}\right)(y_k - y_{k-1} + y_{k+1} - y_k)\\ &= \int_{A_k} d\mathscr{P}^{t_k-t_{k-1}}(y_k-y_{k-1}) \int_{A_{k+1}} d\mathscr{P}^{t_{k+1}-t_k}(y_{k+1}-y_k).\end{aligned}$$

It follows that

$$\begin{aligned}\mathscr{P}_{t_0\ldots t_n}(A_0 \times \ldots \times A_n) = &\int_{A_0} d\mathscr{P}^{t_0}(y_0)\ldots\int_{A_{k-1}} d\mathscr{P}^{t_{k-1}-t_{k-2}}(y_{k-1}-y_{k-2})\\ &\int_{A_{k+1}} d\mathscr{P}^{t_{k+1}-t_{k-1}}(y_{k+1}-y_{k-1})\ldots\int_{A_n} d\mathscr{P}^{t_n-t_{n-1}}(y_n-y_{n-1})\\ =&\mathscr{P}_{t_0\ldots t_{k-1},t_{k+1}\ldots t_n}(A_0 \times \ldots \times A_{k-1} \times A_{k+1} \times \ldots \times A_n).\end{aligned}$$

By the Kolmogorov extension theorem, there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a stochastic process $(L(t))_{t\in[0,\infty)}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ obeying these finite-dimensional distributions. In particular, the law of $L(t)$ is $\mathscr{P}^t$. We will show that $(L(t))_{t\in[0,\infty)}$ is a Lévy process. Firstly, note that the law of $L(0)$ is δ_0 so that $L(0) = 0$ almost surely. Next, for all $0 \leq t_0 < \ldots < t_n$ and all bounded measurable functions $f : \mathbb{R}^{n+1} \to \mathbb{C}$, we have

$$\mathbb{E}[f(L(t_0), \ldots, L(t_n))] = \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} f(x_0, x_0+x_1, \ldots, x_0+\ldots x_n)\, d\mathscr{P}^{t_0}(x_0)d\mathscr{P}^{t_1-t_0}(x_1)\ldots d\mathscr{P}^{t_n-t_{n-1}}(x_n). \quad (5.2)$$

Fixing $(\theta_1 \ldots \theta_n) \in \mathbb{R}^n$ and taking in (5.2)

$$f(x_0, \ldots, x_n) = \exp\left(i\sum_{j=1}^{n}\theta_j(x_j - x_{j-1})\right),$$

it follows that the characteristic function of the random vector $(L(t_1) - L(t_0), \dots, L(t_n) - L(t_{n-1}))$ is given by

$$\begin{aligned}\mathbb{E}\left[\exp\left(i\sum_{j=1}^{n-1}\theta_j(L(t_{j+1}) - L(t_j))\right)\right] &= \int_{\mathbb{R}}\dots\int_{\mathbb{R}}\exp\left(i\sum_{j=1}^{n}\theta_j x_j\right)\, d\mathscr{P}^{t_0}(x_0)d\mathscr{P}^{t_1-t_0}(x_1)\dots d\mathscr{P}^{t_n-t_{n-1}}(x_n)\\ &= \prod_{j=1}^{n}\int_{\mathbb{R}} e^{i\theta_j x_j}\, d\mathscr{P}^{t_j-t_{j-1}}(x_j)\\ &= \prod_{j=1}^{n}\mathbb{E}\left[e^{i\theta_j(L(t_j)-L(t_{j-1}))}\right],\end{aligned}$$

which is the product of the characteristic functions of its components. In the final step we use (5.2) with $f(x_0, \dots, x_n) = e^{i\theta_j(x_j - x_{j-1})}$. Thus, the random vector $(L(t_1) - L(t_0), \dots, L(t_n) - L(t_{n-1}))$ is independent and $(L(t))_{t\in[0,\infty)}$ has independent increments. Now fix $h \geq 0$ and let $0 \leq t_1 < \dots < t_n$. Then, using (5.2), the characteristic function of $(L(t_1 + h) - L(h), \dots, L(t_n + h) - L(h))$ is given by

$$\begin{aligned}\mathbb{E}\left[\exp\left(i\sum_{j=1}^{n}\theta_j(L(t_j+h) - L(h))\right)\right] &= \int_{\mathbb{R}}\dots\int_{\mathbb{R}}\exp\left(i\sum_{j=1}^{n}\theta_j x_j\right)\, d\mathscr{P}^{h}(x_0)d\mathscr{P}^{t_1+h-h}(x_1)\dots d\mathscr{P}^{t_n+h-h}(x_n)\\ &= \int_{\mathbb{R}}\dots\int_{\mathbb{R}}\exp\left(i\sum_{j=1}^{n}\theta_j x_j\right)\, d\mathscr{P}^{0}(x_0)d\mathscr{P}^{t_1}(x_1)\dots d\mathscr{P}^{t_n}(x_n)\\ &= \mathbb{E}\left[\exp\left(i\sum_{j=1}^{n}\theta_j L(t_j)\right)\right],\end{aligned}$$

which is the characteristic function of $(L(t_1), \dots, L(t_n))$. So $(L(t))_{t\in[0,\infty)}$ has stationary increments. Finally, we show stochastic continuity. Note that $L(t) \to 0$ in distribution as $t \downarrow 0$ so that $L(t) \to 0$ in probability as $t \downarrow 0$. Then, for $\epsilon > 0$,

$$\mathbb{P}(|L(t) - L(s)| > \epsilon) = \mathbb{P}(L(|t-s|) > \epsilon) \to 0$$

as $s \to t$. □

These two propositions provide a way to assign a Lévy process with $\mathscr{P}$ as law at $t = 1$ to any infinitely divisible distribution $\mathscr{P}$, allowing us to prove Theorem 5.3.

Proof of Theorem 5.3. Let $(L(t))_{t\in[0,\infty)}$ be a Lévy process. Then the law of $L(1)$ is infinitely divisible, because for any $n > 0$,

$$L(1) = \sum_{k=1}^{n} L\left(\frac{k}{n}\right) - L\left(\frac{k-1}{n}\right),$$

and the summands are i.i.d. Conversely, if $\mathscr{P}$ is infinitely divisible, then there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Lévy process $(L(t))_{t\in[0,\infty)}$ such that $\mathscr{P}$ is the law of $L(1)$ by Propositions 5.7 and 5.8. It remains to be shown that $(L(t))_{t\in[0,\infty)} \overset{d}{=} (\tilde{L}(t))_{t\in[0,\infty)}$ whenever $L(1) \overset{d}{=} \tilde{L}(1)$. So suppose $L(1) \overset{d}{=} \tilde{L}(1)$, then, by uniqueness of the weakly continuous convolution semigroup in Proposition 5.7, it follows that $L(t) \overset{d}{=} \tilde{L}(t)$ for all $t \geq 0$. By independence, it follows that

$$(L(t_1), L(t_2) - L(t_1), \dots, L(t_n) - L(t_{n-1})) \overset{d}{=} (\tilde{L}(t_1), \tilde{L}(t_2) - \tilde{L}(t_1), \dots, \tilde{L}(t_n) - \tilde{L}(t_{n-1}))$$

for any $0 \le t_1 < \ldots < t_n$. Since $(x_1, \ldots, x_n)$ is a (measurable) function of $(x_1, x_2 - x_1, \ldots, x_n - x_{n-1})$, it follows that
$$(L(t_1), \ldots, L(t_n)) \stackrel{d}{=} (\tilde{L}(t_1), \ldots, \tilde{L}(t_n)).$$
□

Having established that the distribution of a Lévy process is determined by its law at $t = 1$, it is time to connect Lévy processes to stable distributions. Note that from (4.2) it readily follows that all stable distributions are infinitely divisible, so any stable distribution $\mathscr{P}$ has a Lévy process with $\mathscr{P}$ as its law at $t = 1$. Lévy processes coming from a stable distribution are intimately related to self-similar processes. Indeed: Stable Lévy processes precisely coincides with self-similar Lévy processes (Proposition 13.5 in [Sat99]).

Definition 5.9. Let T be $\mathbb{R}$ or $[0, \infty)$ and let $\alpha \in (0, 2]$. A Lévy process $(L(t))_{t \in T}$ is called **((standard) symmetric) (strictly) α-stable** if $L(1)$ is ((standard) symmetric) (strictly) α-stable.

Proposition 5.10. *A Lévy process $(L(t))_{t\in[0,\infty)}$ is strictly α-stable if and only if it is H-self-similar, with $H = \frac{1}{\alpha}$.*

Proof. Let $L(t)$ be a Lévy process and let ϕ^t be the characteristic function of $L(t)$. As in the proof of Proposition 5.7, write
$$\phi^t(\theta) = \exp(t\psi(\theta)),$$
where $\psi : \mathbb{R} \to \mathbb{C}$ is the unique continuous function such that $\psi(0) = 0$ and $\phi^1 = \exp \circ \psi$. Suppose $(L(t))_{t\in[0,\infty)}$ is strictly α-stable. Then, from Definition 4.4 and $p_n = n^{1/\alpha}$ it follows that ϕ^1 satisfies, for integers $n > 0$,
$$\exp(n\psi(\theta)) = (\phi^1(\theta))^n = \phi^1(n^{\frac{1}{\alpha}}\theta).$$
Substituting $\theta \hookleftarrow n^{-1/\alpha}\theta$ reveals that $\theta \mapsto n\psi(n^{-1/\alpha}\theta)$ is continuous, satisfies $n\psi(n^{-1/\alpha} \cdot 0) = 0$, and
$$\exp(n\psi(n^{-\frac{1}{\alpha}}\theta)) = \phi^1(\theta).$$
Since ψ is the unique function satisfying these properties, it follows that $\frac{1}{n}\psi(\theta) = \psi(n^{-1/\alpha}\theta)$ and that $\phi^{1/n}(\theta) = \phi^1(n^{-1/\alpha}\theta)$. Letting $m \ge 0$ be another integer, it follows that
$$\phi^{\frac{m}{n}}(\theta) = (\phi^{\frac{1}{n}}(\theta))^m = \left(\phi^1(n^{-1/\alpha}\theta)\right)^m = \phi^1\left((m/n)^{\frac{1}{\alpha}}\theta\right).$$
Thus, $\phi^a(\theta) = \phi^1(a^{1/\alpha}\theta)$ for all $a \in \mathbb{Q}_{\ge 0}$ and by continuity for all $a \in [0, \infty)$. We see that $L(a) \stackrel{d}{=} a^{1/\alpha}L(1)$ for all $a > 0$. Now, because $(L(at))_{t\in[0,\infty)}$ and $(a^{1/\alpha}L(t))_{t\in[0,\infty)}$ are both Lévy processes, their distributions are determined by their laws at $t = 1$ and it follows that $(L(at))_{t\in[0,\infty)} \stackrel{d}{=} (a^{1/\alpha}L(t))_{t\in[0,\infty)}$. Conversely, if $(L(at))_{t\in[0,\infty)} \stackrel{d}{=} (a^{1/\alpha}L(t))_{t\in[0,\infty)}$ for all $a > 0$, then in particular $L(a) \stackrel{d}{=} a^{1/\alpha}L(1)$ so that $\phi^a(\theta) = \phi^1(a^{1/\alpha}\theta)$. Taking $a = n$ a positive integer reveals that $(\phi^1(\theta))^n = \phi^1(n^{1/\alpha}\theta)$ so that $L(1)$ is strictly α-stable. □

Of course up to this point we have only considered Lévy processes on the half line $[0, \infty)$. However, the story for a Lévy process on $\mathbb{R}$ is not any different. Indeed: If $(L(t))_{t\in\mathbb{R}}$ is a Lévy process, then, because $L(0) = 0$ and by independent increments, $(L(t))_{t\le 0}$ and $(L(t))_{t\ge 0}$ are independent. Moreover, due to stationary increments, $L(-1) \stackrel{d}{=} -L(1)$. Thus, the sets of Lévy processes on $[0, \infty)$ and on $\mathbb{R}$, modulo equality in distribution are in bijection: A Lévy process on $\mathbb{R}$ may simply be restricted to $[0, \infty)$, and if $(L_1(t))_{t\in[0,\infty)}$ is a Lévy process on the half line, then taking an independent copy $(L_2(t))_{t\in[0,\infty)}$, we may define a Lévy process $(L(t))_{t\in\mathbb{R}}$ by setting
$$L(t) = \begin{cases} L_1(t) & t \ge 0, \\ -L_2(-t) & t \le 0. \end{cases}$$

6 Multifractional Stable Motion

In this final section, the multifractional stable motion will be covered. The definition of this process is inspired by the moving average representation of the multifractional Brownian motion, but Gaussian distributions are replaced by stable distributions. The main goals are to define the multifractional stable motion and to determine its Hölder regularity. In view of Definition 3.5, one might assume that a reasonable definition for the fractional stable motion is a stable process that is H-self-similar and has stationary increments. However, unlike in the Gaussian case, this does not lock down the finite-dimensional distributions [ST94, Theorem 7.4.5]. Instead, we will define the fractional stable motion by replacing the Gaussian distributions in the moving average representation of the fractional Brownian motion (3.8) by stable distributions. For this it will be necessary to develop an Itô calculus against the symmetric α-stable Lévy motion, which is the stable counterpart to the Brownian motion. To differentiate the fractional stable motion inspired by the moving average representation of fractional Brownian motion (3.8) from other stable H-sssi processes, it is often referred to as the *linear* stable fractional motion. However, because we will not be considering other stable H-sssi processes, we will simply refer to it as the fractional stable motion.

First we will consider deterministic integrands to construct an Itô calculus against symmetric α-stable random measures. In this case, a linear set of symmetric α-stable random variables, equipped with the scale parameter norm, can be used to perform the functional analysis. This allows us to define the (multi)fractional stable motion with a deterministic multifractional parameter. We will prove an upper bound and a lower bound for the pointwise Hölder exponent of the multifractional stable motion, without placing assumptions on the Hölder regularity of the multifractional parameter.

Next, the multifractional stable motion with random multifractional parameter will be defined. In order to facilitate this, an Itô calculus against symmetric α-stable random measures that allows for random integrands will be constructed. In this case, it will not be possible to work in a space of symmetric α-stable random variables, and we will work in a weak Lebesgue space instead. Finally, upper and lower bounds for the pointwise Hölder regularity of the multifractional stable motion are obtained. Again these bounds are irrespective of the Hölder regularity of the multifractional parameter.

6.1 Stable Itô Calculus: Deterministic Integrands

Here we will construct an Itô calculus for deterministic integrands against symmetric α-stable random measures on an abstract measure space. We follow Section 3.3 and 3.4 of [ST94]. First, we will define symmetric α-stable random measures and show that these objects naturally correspond to standard symmetric α-stable Lévy processes. Then we will define integrals of a class of deterministic functions against symmetric α-stable random measures and show that this construction provides a natural way to construct symmetric α-stable stochastic stochastic processes. Finally, we state the theorem that symmetric α-stable stochastic processes that result from integrating against a symmetric α-stable random measure are representative in distribution.

Definition 6.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $(E, \mathcal{E}, m)$ be a measure space and set $\mathcal{E}_0 = \{A \in \mathcal{E} : m(A) < \infty\}$. A **symmetric α-stable** random measure on the measurable space $(E, \mathcal{E})$ with **control measure** m is a map $M : \mathcal{E}_0 \to \mathbb{L}^0(\Omega)$ that is

- σ-additive: If $(A_n)_{n=0}^{\infty}$ is a disjoint sequence in $\mathcal{E}_0$ such that $\bigcup_{n=0}^{\infty} A_n \in \mathcal{E}_0$, then

$$M\left(\bigcup_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} M(A_n) \qquad \text{a.s.}$$

- Independently scattered: If $A_0 \dots A_n$ are disjoint sets in $\mathcal{E}_0$, then the random variables $M(A_0) \dots M(A_n)$ are independent.
- Symmetric α-stable: If $A \in \mathcal{E}_0$ then $M(A)$ is a symmetric α-stable random variable with scale parameter $\|M(A)\|_\alpha = m(A)^{\frac{1}{\alpha}}$.

It is always implied that the control measure of a real valued symmetric α-stable random measure is the Lebesgue measure. If M is a real valued symmetric α-stable random measure, then $M([s,t))$ denotes $-M((t,s])$ whenever $s > t$. This next proposition ties real valued symmetric α-stable random measures to symmetric α-stable Lévy processes. The proof follows Sato's argument in the more general setting of additive processes [Sat04, Theorem 3.2]. Let $\mathcal{B}_0$ denote all Borel measurable subsets of the real line with finite Lebesgue measure and view a symmetric α-stable random measure $(M(B))_{B\in\mathcal{B}_0}$ as a stochastic process indexed on $\mathcal{B}_0$.

Proposition 6.2. *The set of real valued symmetric α-stable random measures $(M(B))_{B\in\mathcal{B}_0}$, modulo modifications, is in bijection with the set of standard symmetric α-stable Lévy processes, modulo modifications. The bijection sends a symmetric α-stable random measure M to the Lévy process $L(t) = M([0,t))$.*

Proof. First we show that $L(t) = M([0,t))$ is a standard symmetric α-stable Lévy process whenever M is a real valued symmetric α-stable measure. Note that $L(0) = M(\emptyset)$ is a symmetric α-stable random variable with scale parameter $\|L(0)\|_\alpha = 0$, so $L(0) = 0$ almost surely. Moreover, $(L(t))_{t\in\mathbb{R}}$ has independent increments due to the fact that M is independently scattered. Furthermore, for any $h \in \mathbb{R}$, $c_1 \dots c_n \in \mathbb{R}$ and $t_1 \dots t_n \in \mathbb{R}$, assuming without loss of generality that $t_1 < t_2 < \dots < t_n$ and taking $t_0 = 0$, we have

$$\sum_{k=1}^{n} c_k(L(t_k+h) - L(h)) = \sum_{k=1}^{n}\left(\sum_{l=k}^{n} c_l\right)(L(t_k+h) - L(t_{k-1}+h)).$$

Moreover,

$$\sum_{k=1}^{n} c_k L(t_k) = \sum_{k=1}^{n}\left(\sum_{l=k}^{n} c_l\right)(L(t_k) - L(t_{k-1})).$$

Since both $L(t_k+h) - L(t_{k-1}+h) = M([t_{k-1}+h, t_k+h))$ and $L(t_k) - L(t_{k-1}) = M([t_{k-1}, t_k))$ are symmetric α-stable random variables with scale parameter $|t_k - t_{k-1}|^{\frac{1}{\alpha}}$, they are equal in distribution. It follows that $\sum_{k=1}^{n} c_k(L(t_k+h) - L(h))$ and $\sum_{k=1}^{n} c_k L(t_k)$ are independent sums of summands that are equal in distribution, so $\sum_{k=1}^{n} c_k(L(t_k+h) - L(h)) \overset{d}{=} \sum_{k=1}^{n} c_k L(t_k)$ and $(L(t))_{t\in\mathbb{R}}$ has stationary increments. Next, suppose that $s \neq t$, then $L(t) - L(s) = M([s,t))$ is symmetric α-stable with scale parameter $|t-s|^{\frac{1}{\alpha}} \to 0$ as $s \to t$. Thus, $L(t) - L(s) \to 0$ in probability as $s \to t$ and $(L(t))_{t\in\mathbb{R}}$ is stochastically continuous. Finally, $L(1) = M([0,1))$ is symmetric α-stable with scale parameter 1. It is clear that $(M([0,t)))_{t\in\mathbb{R}}$ and $(M'([0,t)))_{t\in\mathbb{R}}$ are modifications whenever $(M(B))_{B\in\mathcal{B}_0}$ and $(M'(B))_{B\in\mathcal{B}_0}$ are modifications so the map is well defined between the quotients.

Next, we will show that the map is surjective, so let $(L(t))_{t\in\mathbb{R}}$ be a standard symmetric α-stable Lévy process. We will construct a real valued symmetric α-stable random measure M such that $L(t) = M([0,t))$.

Step 1: if I is empty or a one point set, define $M(I) = 0$, if I is an interval of the form $(s,t), [s,t), (s,t]$ or $[s,t]$ with $s < t$, define $M(I) = L(t) - L(s)$. Finally, if $J = \bigcup_{k=1}^{n} I_k$ is a finite union of intervals, define $M(J) = \sum_{k=1}^{n} M(I_k)$. From the fact that the Lévy process L is standard symmetric α-stable, it follows that $\|M(J)\|_\alpha = \lambda(J)^{1/\alpha}$ (where λ denotes the Lebesgue measure). It is also clear that M is finitely additive and independently scattered for sets of this class.

Step 2: Suppose $G \subseteq \mathbb{R}$ is a non-empty bounded open set. Then G is expressed uniquely (up to the order) as a countable union of disjoint non-empty open intervals $G = \bigcup_k (s_k, t_k)$ (countable because $\mathbb{R}$ is second countable and the union can be made disjoint by identifying connected components). If the union is finite, then it was dealt with in step 1. Suppose $G = \bigcup_{k=0}^{\infty}(s_k, t_k)$ is a countably infinite union. Let $S_n = \sum_{k=0}^{n} L(t_k) - L(s_k)$ denote the partial sums, and let Ψ be a standard symmetric α-stable random variable. Then for all $n > m$ and $\epsilon > 0$,

$$\mathbb{P}(|S_n - S_m| \geq \epsilon) \leq \epsilon^{-\alpha}\|S_n - S_m\|_{\Lambda^\alpha(\Omega)}^\alpha = \epsilon^{-\alpha}\|\Psi\|_{\Lambda^\alpha(\Omega)}^\alpha \|S_n - S_m\|_\alpha^\alpha = \epsilon^{-\alpha}\|\Psi\|_{\Lambda^\alpha(\Omega)}^\alpha \sum_{k=m+1}^{n}(t_k - s_k) \overset{n,m\to\infty}{\to} 0.$$

Thus, $(S_n)_{n=0}^{\infty}$ converges in probability and because the summands are independent it converges almost surely. Define $M(G)$ as this limit (which is independent of the order of the summands).

Step 3: Let $K \subseteq \mathbb{R}$ be a compact set and choose $s < t$ such that $K \subseteq (s, t)$. Let $G = (s, t) \setminus K$ which is a nonempty open bounded set. Define $M(K) = M((s, t)) - M(G)$, which is independent of the choice of s and t. Write $G = \bigcup_k (s_k, t_k)$ as a countable disjoint union of open intervals. If $G = \bigcup_{k=0}^{n}(s_k, t_k)$ is a finite union then we may suppose that $t_{k-1} \leq s_k$ for $k = 1 \dots n$, so that $K = \bigcup_{k=1}^{n}[t_{k-1}, s_k]$. It follows that $M(K) = \sum_{k=1}^{n} L(s_k) - L(t_{k-1})$. Now suppose $G = \bigcup_{k=0}^{\infty}(s_k, t_k)$ is an infinite union and write $G_n = \bigcup_{k=0}^{n}(s_k, t_k)$. Then, by definition of $M(G)$ it follows that, almost surely,

$$M(K) = M((s,t)) - M(G) = \lim_{n\to\infty} M((s,t)) - M(G_n) = \lim_{n\to\infty} M((s,t) \setminus G_n).$$

Now, without loss of generality, suppose that $(s_0, t_0) = (s, \min K)$ and $(s_1, t_1) = (\max K, t)$. Then, for $n \geq 1$, we have an expression $(s, t) \setminus G_n = \bigcup_{k=1}^{n}[\tilde{s}_{n,k}, \tilde{t}_{n,k}]$ as a finite disjoint union of compact intervals. It follows that, almost surely,

$$M(K) = \lim_{n\to\infty} \sum_{k=1}^{n} L(\tilde{t}_{n,k}) - L(\tilde{s}_{n,k}).$$

From this representation the following three things readily follow:

- If K_1 and K_2 are compact and $K_1 \subseteq K_2$, then $\|M(K_2) - M(K_1)\|_\alpha = \lambda(K_2 \setminus K_1)^{\frac{1}{\alpha}}$.
- If K is compact, then $\|M(K)\|_\alpha = \lambda(K)^{\frac{1}{\alpha}}$.
- If $K_1 \dots K_n$ are disjoint compact sets then $M(K_1) \dots M(K_n)$ are independent and $M\left(\bigcup_{j=1}^{n} K_j\right) = \sum_{j=1}^{n} M(K_j)$ almost surely.

Step 4: Let $B \in \mathcal{B}_0$ be an arbitrary Borel measurable set of finite Lebesgue measure. By (inner) regularity of the Lebesgue measure, there is an increasing sequence of compact sets $K_n \subseteq B$ such that $\lambda(K_n) \to \lambda(B)$. Then, from the first property established in step 3, if $m \geq n$,

$$\|M(K_m) - M(K_n)\|_\alpha = \lambda(K_n \setminus K_m)^{\frac{1}{\alpha}} \overset{m,n\to\infty}{\to} 0.$$

It follows that $(M(K_n))_{n\in\mathbb{N}}$ converges in probability and we define $M(B)$ as this limit, which is independent of the chosen sequence $(K_n)_{n\in\mathbb{N}}$. From the second and third properties established in step 3, it follows that $\|M(B)\|_\alpha = \lambda(B)^{\frac{1}{\alpha}}$, M is σ-additive and that M is independently scattered.

Finally, we show that the map is injective. So let M and M' be real valued symmetric α-stable random measures, and suppose that $M([0,t)) = M'([0,t))$ almost surely for all $t \in \mathbb{R}$. We will show that $M(B) = M'(B)$ almost surely for all $B \in \mathcal{B}_0$. Fix $s < t$ and let $\mathcal{D}$ denote the class of Borel measurable subsets $D \subseteq (s, t]$ such that $M(D) = M'(D)$ almost surely. Then $\mathcal{D}$ is a Dynkin system of subsets of $(s, t]$ containing the half open intervals

$(a, b]$ in $(s, t]$. By Dynkin's $\pi - \lambda$ theorem, it follows that $\mathcal{D}$ contains all Borel measurable subsets of $(s, t]$. Now let $B \in \mathcal{B}_0$, then, almost surely,

$$M(B) = M\left(\bigcup_{n\in\mathbb{Z}} B \cap (n, n+1]\right) = \sum_{n\in\mathbb{Z}} M(B\cap(n,n+1]) = \sum_{n\in\mathbb{Z}} M'(B\cap(n,n+1]) = M'\left(\bigcup_{n\in\mathbb{Z}} B \cap (n, n+1]\right) = M'(B)$$

□

Now that we have connected real valued symmetric α-stable random measures to Lévy processes, it is time to define stochastic integrals against these objects. We do this in the setting of a general measurable space with a general control measure. Let M be a symmetric α-stable random measure on $(E, \mathcal{E})$ with control measure m. We will define the (random) integrals $\int_E f dM$ for the deterministic functions $f \in \mathbb{L}^\alpha(E, \mathcal{E}, m)$ by taking the obvious choice for simple functions, showing that this defines a linear isometry and extending by functional analytical means. For the next part, functions $f \in \mathbb{L}^\alpha(E, \mathcal{E}, m)$ will be treated as representatives of their equivalence class under almost everywhere equality.

Definition 6.3. Let $f \in \mathbb{L}^\alpha(E, \mathcal{E}, m)$. Then f is **simple** if it admits a representation

$$f(x) = \sum_{i=1}^{n} c_i \mathbb{1}_{A_i}(x),$$

with each $c_i \in \mathbb{R}$ and $A_1 \dots A_n$ a disjoint sequence in $\mathcal{E}_0$. The space of simple functions is denoted $\mathcal{S}(E)$. For simple functions with a representation as above the stochastic integral is defined as

$$\int_E f\, dM = \sum_{i=1}^{n} c_i M(A_i).$$

Note that the integral is linear on simple functions. Furthermore, it is well defined. That is, independent of the chosen representation of the simple function (up to almost sure equality, so well defined in $\mathbb{L}^0(\Omega)$).

Lemma 6.4. *If $f \in \mathcal{S}(E)$ is a simple function, then $\int_E f\, dM$ is a symmetric α-stable random variable with scale parameter $\| \int_E f\, dM\|_\alpha = \|f\|_{\mathbb{L}^\alpha(E,\mathcal{E},m)}$.*

Proof. Write $f = \sum_{j=1}^n c_j \mathbb{1}_{A_j}$, with $A_1 \dots A_n \in \mathcal{E}_0$ disjoint, then $M(A_1) \dots M(A_n)$ are independent and symmetric α-stable with scale parameters $m(A_1)^{\frac{1}{\alpha}} \dots m(A_n)^{\frac{1}{\alpha}}$ respectively, so

$$\mathbb{E}\left[e^{i\theta \int_E f\, dM}\right] = \prod_{j=1}^{n} \mathbb{E}\left[e^{i\theta c_j M(A_j)}\right] = \prod_{j=1}^{n} e^{-m(A_j)|\theta c_j|^\alpha} = \exp\left(-\left(\sum_{j=1}^{n} |c_j|^\alpha m(A_j)\right)|\theta|^\alpha\right).$$

We see that $\int_E f\, dM$ is symmetric α-stable with scale parameter

$$\left\|\int_E f\, dM\right\|_\alpha = \left(\sum_{j=1}^{n} |c_j|^\alpha m(A_j)\right)^{\frac{1}{\alpha}} = \left(\int_E |f|^\alpha dm\right)^{\frac{1}{\alpha}} = \|f\|_{\mathbb{L}^\alpha(E,\mathcal{E},m)}.$$

□

Lemma 6.5. *$S(\mathcal{E})$ is dense in $\mathbb{L}^\alpha(E, \mathcal{E}, m)$ (with respect to the norm $\| \cdot \|_{\mathbb{L}^\alpha(E,\mathcal{E},m)}$).*

Proof. Let $f \in \mathbb{L}^\alpha(E,\mathcal{E},m)$ and define

$$f_n = \sum_{i=1}^{n^2-1} \left(\frac{i}{n}\right) \mathbb{1}_{f^{-1}\left[\frac{i}{n},\frac{i+1}{n}\right)} + \sum_{i=1}^{n^2-1} \left(-\frac{i}{n}\right) \mathbb{1}_{f^{-1}\left(-\frac{i+1}{n},-\frac{i}{n}\right]}.$$

Then $f_n(x) \to f(x)$ whenever $|f(x)| < \infty$, which is almost everywhere because $f \in \mathbb{L}^\alpha(E,\mathcal{E},m)$. Furthermore, $|f_n| \le |f|$ so that $|f_n - f| \le 2|f|$. Lastly, f_n is simple. Indeed: From the fact that $f \in \mathbb{L}^\alpha(E,\mathcal{E},m)$ it readily follows that $m(f^{-1}[\mathbb{R}\setminus(-\delta,\delta)]) < \infty$ for any $\delta > 0$. Since the sets $[\frac{i}{n},\frac{i+1}{n})$ and $(-\frac{i+1}{n},-\frac{i}{n}]$ are all contained in $\mathbb{R}\setminus(-\delta,\delta)$ for some $\delta > 0$ it follows that f_n is simple. From the dominated convergence theorem it follows that $f_n \to f$ in $\mathbb{L}^\alpha(E,\mathcal{E},m)$. □

Let S_α be the image of $\mathcal{S}(E)$ under the linear operator $\int_E \cdot\, dM : \mathcal{S}(E) \to \mathbb{L}^0(\Omega)$. Then by Lemma 6.4, S_α is a linear set of symmetric α-stable random variables. Letting $\overline{S_\alpha}$ denote its topological closure in $\mathbb{L}^0(\Omega)$, by the discussion at the end of Section 4.1 and Lemma 6.4, the operator $\int_E \cdot dM : (\mathcal{S}(E), \|\cdot\|_{\mathbb{L}^\alpha(E,\mathcal{E},m)}) \to (\overline{S_\alpha}, \|\cdot\|_\alpha)$ is a linear isometry into a Banach space if $\alpha \ge 1$, and into an α-Banach space if $\alpha \le 1$. Thus, by virtue of Lemmas 2.5 and 6.5, the operator extends uniquely to a linear isometry

$$\int_E \cdot\, dM : \left(\mathbb{L}^\alpha(E,\mathcal{E},m), \|\cdot\|_{\mathbb{L}^\alpha(E,\mathcal{E},m)}\right) \to (\overline{S_\alpha}, \|\cdot\|_\alpha).$$

This integral operator provides a natural method for constructing symmetric α-stable random processes. Indeed: If $(f_t)_{t\in T}$ is a collection of functions in $\mathbb{L}^\alpha(E,\mathcal{E},m)$, then $\left(\int_E f_t\, dM\right)_{t\in T}$ is a symmetric α-stable random process. This follows immediately from the fact that the integral is a linear operator into a linear set of symmetric α-stable random variables. The following theorem states that, in distribution, *all* symmetric α-stable random processes are of this form.

Theorem 6.6. *Let $(X_t)_{t\in T}$ be a symmetric α-stable process with $0 < \alpha < 2$. Then there is a measure space $(E,\mathcal{E},m)$, a symmetric α-stable random measure M on $(E,\mathcal{E})$ with control measure m and a collection of functions $(f_t)_{t\in T}$ in $\mathbb{L}^\alpha(E,\mathcal{E},m)$ such that*

$$(X(t))_{t\in T} \overset{d}{=} \left(\int_E f_t\, dM\right)_{t\in T}.$$

Proof. [ST94, Theorem 13.2.2]. □

If a symmetric α-stable process is represented as a collection of integrals against a symmetric α-stable random measure, then localizability may be related to the collection of kernels through the following lemma, which is a specification of Theorem 3.2 in [FL12] to the case where α is constant.

Lemma 6.7. *Let M be a real symmetric α-stable random measure and $H > 0$. Let $Y(t) = \int_{\mathbb{R}} f_t\, dM, t \in \mathbb{R}$ with each $f_t \in \mathbb{L}^\alpha(\mathbb{R})$ and fix $t \in \mathbb{R}$. Let $Y'_t(r) = \int_{\mathbb{R}} g^t_r\, dM, r \in \mathbb{R}$ with each $g^t_r \in \mathbb{L}^\alpha(\mathbb{R})$. Suppose that for all $r \in \mathbb{R}$,*

$$\int_{\mathbb{R}} \left| \frac{f_{t+hr}(x) - f_t(x)}{h^H} - h^{-\frac{1}{\alpha}} g^t_r\left(\frac{x-t}{h}\right)\right|^\alpha dx \overset{x=t+h\hat{x}}{=} \int_{\mathbb{R}} \left| \frac{f_{t+hr}(t+h\hat{x}) - f_t(t+h\hat{x})}{h^{H-1/\alpha}} - g^t_r(\hat{x})\right|^\alpha d\hat{x} \overset{h\downarrow 0}{\to} 0$$

Then, in terms of finite-dimensional distributions, as $h \downarrow 0$,

$$\left(\frac{Y(t+hr) - Y(t)}{h^H}\right)_{r\in\mathbb{R}} \to (Y'_t(r))_{r\in\mathbb{R}}.$$

Proof. Let $c_1 \dots c_n \in \mathbb{R}$ and $r_1 \dots r_n \in \mathbb{R}$. By the Cramér-Wold Theorem, it suffices to show that

$$\sum_{i=1}^{n} c_i \frac{Y(t+hr_i) - Y(t)}{h^H} \overset{h\downarrow 0}{\to} \sum_{i=1}^{n} c_i Y_t'(r_i) \quad \text{in distribution.}$$

This will be done by showing that the scale parameters of the left side converge to the scale parameter of the right side as $h \downarrow 0$. Note that

$$\left\| \sum_{i=1}^{n} c_i \frac{Y(t+hr_i) - Y(t)}{h^H} \right\|_\alpha^\alpha = \int_{\mathbb{R}} \left| \sum_{i=1}^{n} c_i \frac{f_{t+hr_i}(x) - f_t(x)}{h^H} \right|^\alpha dx = \int_{\mathbb{R}} \left| \sum_{i=1}^{n} c_i \frac{f_{t+hr_i}(t+h\hat{x}) - f_t(t+h\hat{x})}{h^{H-1/\alpha}} \right|^\alpha d\hat{x}.$$

By the assumption,

$$\int_{\mathbb{R}} \left| \sum_{i=1}^{n} c_i \frac{f_{t+hr_i}(t+h\hat{x}) - f_t(t+h\hat{x})}{h^{H-1/\alpha}} - \sum_{i=1}^{n} c_i g_{r_i}^t(\hat{x}) \right|^\alpha d\hat{x} \leq n^\alpha \sum_{i=1}^{n} |c_i|^\alpha \int_{\mathbb{R}} \left| \frac{f_{t+hr_i}(t+h\hat{x}) - f_t(t+h\hat{x})}{h^{H-1/\alpha}} - g_{r_i}^t(\hat{x}) \right|^\alpha d\hat{x} \to 0$$

as $h \downarrow 0$. By continuity of the functional $\| \cdot \|_{\mathbb{L}^\alpha(\mathbb{R})}$ (which is a true norm for $\alpha \geq 1$ and an α-norm for $\alpha \leq 1$), it follows that

$$\int_{\mathbb{R}} \left| \sum_{i=1}^{n} c_i \frac{f_{t+hr_i}(t+h\hat{x}) - f_t(t+h\hat{x})}{h^{H-1/\alpha}} \right|^\alpha d\hat{x} \overset{h\downarrow 0}{\to} \int_{\mathbb{R}} \left| \sum_{i=1}^{n} c_i g_{r_i}^t(\hat{x}) \right|^\alpha d\hat{x} = \left\| \sum_{i=1}^{n} c_i Y_t'(r_i) \right\|_\alpha^\alpha.$$

□

6.2 Multifractional Stable Motion with Deterministic Multifractional Parameter

Throughout this section, M will denote a symmetric α-stable random measure on $\mathbb{R}$ with $\alpha \in (0,2)$. Now that an Itô calculus against symmetric α-stable random measures has been developed we are in a position to define the stochastic process that this project is focused on: The fractional stable motion. This process is the α-stable counterpart to the fractional Brownian motion based on the moving average integral representation, and is given by

$$X(t) = \int_{\mathbb{R}} (t-x)_+^{H-\frac{1}{\alpha}} - (-x)_+^{H-\frac{1}{\alpha}} \, dM(x) \qquad t \in \mathbb{R}.$$

Just as in the Gaussian case, this process will be generalized to a multifractional processes where the fractional parameter H varies such that the local form of the multifractional process is a fractional stable motion. The most obvious way to achieve this is to let $H = H(t)$ vary with the parameter indexing the stochastic process (thought of as time). However, the same problems plaguing the multifractional Brownian motion arise in the stable setting. Namely, the Hölder regularity of the resulting process is dependent on the Hölder regularity of $H(t)$, and if we intend to make $H(t)$ random, then the resulting kernel will not be adapted, which means that an Itô calculus cannot be used to define this process. Thus, we will instead opt to let $H = H(x)$ vary with the integration variable, and consider a process of the form

$$Y(t) = \int_{\mathbb{R}} (t-x)_+^{H(x)-\frac{1}{\alpha}} - (-x)_+^{H(x)-\frac{1}{\alpha}} \, dM(x).$$

However, before covering the multifractional stable motion, we will first study the fractional stable motion. This next lemma shows that the definition from earlier is well-defined.

Lemma 6.8. *Fix $\alpha \in (0,2)$. For any $t \in \mathbb{R}$ and $H \in (0,1)$ the function $\mathbb{R} \to \mathbb{R} : x \mapsto (t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha}$ is in $\mathbb{L}^\alpha(\mathbb{R})$.*

Proof. If $t = 0$ then the statement is trivial, so assume $t \neq 0$. Integrability issues may arrive at $-\infty$, and possibly at the singularities $x = 0$ and $x = t$ if $H < \frac{1}{\alpha}$. For the limit to $-\infty$, fix $x < 0 \wedge t$. By the mean value theorem, there is a point $\xi_{t,x}$ between 0 and t such that

$$(t-x)^{H-\frac{1}{\alpha}} - (-x)^{H-\frac{1}{\alpha}} = \left(H - \frac{1}{\alpha}\right)(\xi_{t,x} - x)^{H-\frac{1}{\alpha}-1}\,|t|.$$

It follows that, for all $x < 0 \wedge t$,

$$\left|(t-x)^{H-\frac{1}{\alpha}} - (-x)^{H-\frac{1}{\alpha}}\right|^\alpha \leq \left|H - \frac{1}{\alpha}\right|^\alpha |t|^\alpha\,((0 \wedge t) - x)^{\alpha(H-1)-1}.$$

Since $\alpha(H-1) - 1 < -1$, the integral converges in the limit to $-\infty$. The possible singularities at $x = 0$ and $x = t$ are integrable because $\alpha(H - \frac{1}{\alpha}) > -1$. □

Definition 6.9. Let $a_+, a_- \in \mathbb{R}$. The **generating field for the fractional stable motion** with **scaling coefficients** a_+ and a_- is the stochastic field $(\mathfrak{F}(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$, given by

$$\mathfrak{F}(t,H) = \int_{\mathbb{R}} a_+\left((t-x)_+^{H-\frac{1}{\alpha}} - (-x)_+^{H-\frac{1}{\alpha}}\right) + a_-\left((t-x)_-^{H-\frac{1}{\alpha}} - (-x)_-^{H-\frac{1}{\alpha}}\right)dM(x).$$

If $H \in (0,1)$ is fixed, then the stochastic process $(X(t))_{t\in\mathbb{R}} = (\mathfrak{F}(t,H))_{t\in\mathbb{R}}$ is called the **fractional stable motion** with **fractional parameter** H and **scaling coefficients** a_+ and a_-.

To make plausible the claim that the fractional stable motion is the α-stable counterpart to the fractional Brownian motion, we will show that it is an H-self-similar α-stable process with stationary increments. However, unlike in the Gaussian case, it is not in distribution the only process with this property. Indeed, different values of a_+ and a_- result in different finite-dimensional distributions for the fractional stable motion [ST94, Theorem 7.4.5].

Proposition 6.10. *The fractional stable motion with fractional parameter H is H-self-similar and has stationary increments.*

Proof. For simplicity we take $a_+ = 1$ and $a_- = 0$, for general values of a_+ and a_- the proof is the same. Let $X(t) = \int_{\mathbb{R}}(t-x)_+^{H-\frac{1}{\alpha}} - (-x)_+^{H-\frac{1}{\alpha}}\,dM(x)$ be the fractional stable motion with fractional parameter H and write $f_t(x) = (t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha}$ for its kernel. Let $a > 0$, to show that $(a^H X(t))_{t\in\mathbb{R}} \stackrel{d}{=} (X(at))_{t\in\mathbb{R}}$, let

$c_1 \dots c_n \in \mathbb{R}$ and $t_1 \dots t_n \in \mathbb{R}$. Then, because $\int_{\mathbb{R}} \cdot \, dM$ is a linear isometry,

$$\begin{aligned}
\left\| \sum_{k=1}^{n} c_k X(at_k) \right\|_{\alpha}^{\alpha} &= \left\| \sum_{k=1}^{n} c_k f_{at_k} \right\|_{\mathbb{L}^{\alpha}(\mathbb{R})}^{\alpha} \\
&= \int_{\mathbb{R}} \left| \sum_{k=1}^{n} c_k \left((at_k - x)_+^{H-\frac{1}{\alpha}} - (-x)_+^{H-\frac{1}{\alpha}} \right) \right|^{\alpha} dx \\
&\overset{x=a\hat{x}}{=} a^{\alpha H} \int_{\mathbb{R}} \left| \sum_{k=1}^{n} c_k \left((t_k - \hat{x})_+^{H-\frac{1}{\alpha}} - (-\hat{x})_+^{H-\frac{1}{\alpha}} \right) \right|^{\alpha} d\hat{x} \\
&= a^{\alpha H} \left\| \sum_{k=1}^{n} c_k f_{t_k} \right\|_{\mathbb{L}^{\alpha}(\mathbb{R})}^{\alpha} \\
&= \left\| \sum_{k=1}^{n} c_k a^H X(t_k) \right\|_{\alpha}^{\alpha}.
\end{aligned}$$

Because the distribution of a symmetric α-stable random variable is determined by its scale parameter, and the distribution of a random vector is determined by the distributions of its linear combinations, it follows that $(a^H X(t))_{t\in\mathbb{R}} \overset{d}{=} (X(at))_{t\in\mathbb{R}}$. Next, let $h \in \mathbb{R}$. To show that $(X(t+h) - X(h))_{t\in\mathbb{R}} \overset{d}{=} (X(t))_{t\in\mathbb{R}}$), let $c_1 \dots c_n \in \mathbb{R}$ and $t_1 \dots t_n \in \mathbb{R}$. Then

$$\begin{aligned}
\left\| \sum_{k=1}^{n} c_k (X(t_k - h) - X(h)) \right\|_{\alpha}^{\alpha} &= \left\| \sum_{k=1}^{n} c_k (f_{t_k+h} - f_h) \right\|_{\mathbb{L}^{\alpha}(\mathbb{R})}^{\alpha} \\
&= \int_{\mathbb{R}} \left| \sum_{k=1}^{n} c_k \left((t_k + h - x)_+^{H-\frac{1}{\alpha}} - (h - x)_+^{H-\frac{1}{\alpha}} \right) \right|^{\alpha} dx \\
&\overset{\hat{x}=x-h}{=} \int_{\mathbb{R}} \left| \sum_{k=1}^{n} c_k \left((t_k - \hat{x})_+^{H-\frac{1}{\alpha}} - (-\hat{x})_+^{H-\frac{1}{\alpha}} \right) \right|^{\alpha} d\hat{x} \\
&= \left\| \sum_{k=1}^{n} c_k f_{t_k} \right\|_{\mathbb{L}^{\alpha}(\mathbb{R})}^{\alpha} \\
&= \left\| \sum_{k=1}^{n} c_k X(t_k) \right\|_{\alpha}^{\alpha}.
\end{aligned}$$

□

The next objective is to determine the Hölder regularity of the fractional stable motion. If $H < \frac{1}{\alpha}$ then the paths of any modification of the fractional stable motion are unbounded on all intervals [ST05]. If $H = \frac{1}{\alpha}$ then the fractional stable motion reduces to a symmetric α-stable Lévy motion (and recall that $H = \frac{1}{\alpha}$ is necessary for an H-self-similar symmetric α-stable Lévy process by Proposition 5.10). For this reason, we will restrict our attention to the case $H > \frac{1}{\alpha}$. Note that this restricts the parameters to $\alpha \in (1, 2)$ and $H \in (\frac{1}{2}, 1)$. To obtain a modification with Hölder continuous sample paths, we will employ the Kolmogorov-Chentsov continuity theorem, so we need upper bounds on the moments $\mathbb{E}|X(t) - X(s)|^p$. Note that for $p \geq \alpha$, these moments are infinite, because $X(t) - X(s)$ is a symmetric α-stable random variable, thus we will have to bound moments of order $p < \alpha$. Under that restriction, it holds that $\mathbb{E}|X(t) - X(s)|^p = \|X(t) - X(s)\|_\alpha^p \mathbb{E}|\Psi|^p$, where Ψ is a standard symmetric α-stable random variable, so it suffices to bound the scale parameter of $X(t) - X(s)$ which we can relate to the kernel through the isometry $\int_{\mathbb{R}} \cdot \, dM$.

Lemma 6.11. *Let $(X(t))_{t\in\mathbb{R}}$ be a fractional stable motion with fractional parameter $H > \frac{1}{\alpha}$. Let $t \in \mathbb{R}$ and $h > 0$. Then there are constants C_1 and C_2, independent of t and h, such that*

$$\|X(t+h) - X(t)\|_\alpha^\alpha \leq C_1 h^\alpha + C_2 h^{H\alpha}.$$

Proof. Decompose

$$\begin{aligned}\|X(t+h) - X(t)\|_\alpha^\alpha = &\underbrace{\int_{-\infty}^{t-1} \left[(t+h-x)^{H-\frac{1}{\alpha}} - (t-x)^{H-\frac{1}{\alpha}}\right]^\alpha dx}_{D(t,h)} \\ &+ \underbrace{\int_{t-1}^{t} \left[(t+h-x)^{H-\frac{1}{\alpha}} - (t-x)^{H-\frac{1}{\alpha}}\right]^\alpha dx}_{E(t,h)} \\ &+ \underbrace{\int_{t}^{t+h} (t+h-x)^{H\alpha-1} \, dx}_{F(t,h)}.\end{aligned}$$

We will bound all three terms. For the first term, we apply the mean value theorem to obtain $\xi_{t,h,x} \in [0, h]$ such that

$$\begin{aligned}D(t,h) &= \left(H - \frac{1}{\alpha}\right)^\alpha h^\alpha \int_{-\infty}^{t-1} (t + \xi_{t,h,x} - x)^{\alpha(H-1)-1} \, dx \\ &\leq \left(H - \frac{1}{\alpha}\right)^\alpha h^\alpha \int_{-\infty}^{t-1} (t - x)^{\alpha(H-1)-1} \, dx \\ &= \frac{(H - 1/\alpha)^\alpha}{\alpha(1-H)} h^\alpha.\end{aligned}$$

In the second term we substitute $x = t + h\hat{x}$ to obtain

$$E(t,h) = h^{H\alpha} \int_{-h^{-1}}^{0} \left[(1-\hat{x})^{H-\frac{1}{\alpha}} - (-\hat{x})^{H-\frac{1}{\alpha}}\right]^\alpha d\hat{x} \leq h^{H\alpha} \int_{-\infty}^{0} \left[(1-\hat{x})^{H-\frac{1}{\alpha}} - (-\hat{x})^{H-\frac{1}{\alpha}}\right]^\alpha d\hat{x}.$$

The integral is a finite constant independent of h by Lemma 6.8. Finally, the third term may simply be computed to be $F(s,t) = (\alpha H)^{-1} h^{H\alpha}$. □

The growth of absolute moments of increments is different for small h than for large h. Thus, to be able to globalize the constants in the conditions for the Kolmogorov-Chentsov continuity theorem, we will restrict the fractional stable motion to a bounded interval of the real line.

Corollary 6.12. *there is a modification $(X(t))_{t\in[S,T]}$ of the fractional stable motion such that almost surely, its paths are ρ-Hölder continuous for all $0 < \rho < H - \frac{1}{\alpha}$.*

Proof. Setting $K = (C_1 + C_2) \vee (C_1(T-S)^{\alpha(1-H)} + C_2)$ we find that $\|X(t) - X(s)\|_\alpha^\alpha \leq K|t-s|^{\alpha H}$ for all $s, t \in [S,T]$, so $\mathbb{E}|X(t) - X(s)|^p \leq \mathbb{E}|\Psi|^p K^{p/\alpha}|t-s|^{pH}$ for all $1/H < p < \alpha$, where Ψ is a standard symmetric α-stable random variable. By the Kolmogorov-Chentsov continuity theorem, the fractional stable motion has a modification such that almost surely, its paths are ρ-Hölder continuous for all $0 < \rho < H - \frac{1}{p}$. The result follows by letting $p \uparrow \alpha$. □

Proposition 6.13. *Let $(X(t))_{t\in[S,T]}$ be the modification of the fractional stable motion from Corollary 6.12 and let $t \in (S,T)$. Then, almost surely,*

$$H - \frac{1}{\alpha} \leq \rho_X^{unif}([S,T]) \leq \rho_X(t) \leq H.$$

Proof. The lower bound $H - \frac{1}{\alpha} \leq \rho_X^{\text{unif}}([S,T])$ follows from Corollary 6.12, the intermediate bound $\rho_X^{\text{unif}}([S,T]) \leq \rho_X(t)$ is generally true for Hölder exponents. We prove the upper bound $\rho_X(t) \leq H$. Let $\rho > 0$, then, for $h > 0$,

$$\begin{aligned}
\left\| \frac{X(t+h) - X(t)}{h^{H+\rho}} \right\|_\alpha^\alpha &= h^{-\alpha H - \alpha\rho} \int_{\mathbb{R}} \left| (t+h-x)_+^{H-\frac{1}{\alpha}} - (t-x)_+^{H-\frac{1}{\alpha}} \right|^\alpha dx \\
&\geq h^{-\alpha H - \alpha\rho} \int_t^{t+h} (t+h-x)^{\alpha H - 1}\, dx \\
&= (\alpha H)^{-1} h^{-\alpha\rho} \overset{h\downarrow 0}{\to} \infty.
\end{aligned}$$

By Lemma 4.13, there is a sequence $h_n \downarrow 0$ such that $h_n^{-H-\rho}|X(t+h_n) - X(t)| \to \infty$ in probability as $n \to \infty$. So there is a subsequence $h_{n_k} \downarrow 0$ such that $h_{n_k}^{-H-\rho}(X(t+h_{n_k}) - X(t)) \to \infty$ almost surely as $k \to \infty$. We conclude that

$$\limsup_{h\to 0} \frac{|X(t+h) - X(t)|}{|h|^{H+\rho}} = \infty,$$

so that $\rho_X(t) \leq H$. □

Remark. The bounds in Proposition 6.13 are sharp: From Theorem 6.1 (and Remark 6.1) in [AH14] it follows that $\rho_X^{\text{unif}}([S,T]) = H - \frac{1}{\alpha}$ almost surely. From Corollary 5.2 in the same article it follows that for all $t \in (S,T)$, almost surely: $\rho_X(t) = H$.

Thus concludes the examination of the fractional stable motion. The most natural way to generalize the fractional stable motion by making the fractional parameter time-dependent is by diagonalizing the generating field from Definition 6.9. That is, by considering the following stochastic process.

Definition 6.14. Let $H : \mathbb{R} \to (0,1)$ be a function and let $a_+, a_- \in \mathbb{R}$, then the stochastic process given by

$$Y(t) = \int_{\mathbb{R}} a_+ \left((t-x)_+^{H(t)-\frac{1}{\alpha}} - (-x)_+^{H(t)-\frac{1}{\alpha}} \right) + a_- \left((t-x)_-^{H(t)-\frac{1}{\alpha}} - (-x)_-^{H(t)-\frac{1}{\alpha}} \right) dM(x) \qquad t \in \mathbb{R},$$

is called the **classical multifractional stable motion** with **multifractional parameter** H and **scaling coefficients** a_+ and a_-.

This is the process that Stoev and Taqqu consider in [ST04]. In this article, Stoev and Taqqu establish stochastic properties of $\partial_H^n \mathfrak{F}(t,H)$, the derivatives with respect to H of the generating field of the fractional stable motion. They use these properties to show, among other things, that, under conditions on the Hölder regularity of H, this process is $H(t)$-localizable at t with a fractional stable motion as local form. Indeed: Theorem 5.1 in this article states that, if $H(r) - H(t)$ is $o\left(|r-t|^{H(t)}\right)$ as $r \to t$, then, in terms of finite-dimensional distributions:

$$\left(\frac{Y(t+hr) - Y(t)}{h^{H(t)}}\right)_{t\in\mathbb{R}} \stackrel{h\downarrow 0}{\to} \left(Y_t'(r)\right)_{r\in\mathbb{R}},$$

where $(Y_t'(r))_{r\in\mathbb{R}}$ is a fractional stable motion with fractional parameter $H(t)$ and scaling coefficients a_+ and a_-. The same authors also published an article on the path properties of this process [ST05]. Theorem 3.2 in this article establishes a bound on the uniform Hölder exponents over compact intervals and states the following.

Theorem 6.15. *Let $(Y(t))_{t\in\mathbb{R}}$ be the classical multifractional stable motion and let $S < T$. Suppose the multifractional parameter $H : \mathbb{R} \to (0,1)$ is continuous on (S,T), and $H(t) > \frac{1}{\alpha}$ for all $t \in (S,T)$. Finally, suppose that there are $C > 0$ and $\rho > \frac{1}{\alpha}$ such that for all $t_1, t_2 \in (S,T)$,*

$$|H(t_2) - H(t_1)| \leq C|t_2 - t_1|^{\rho}.$$

Then $(Y(t))_{t\in\mathbb{R}}$ admits a modification $(\tilde{Y}(t))_{t\in\mathbb{R}}$ whose paths are continuous on (a,b) and such that, for all $[a',b'] \subseteq (a,b)$, almost surely,

$$\rho_{\tilde{Y}}^{unif}([a',b']) \geq \left(\rho \wedge \min_{t\in[a',b']} H(t)\right) - \frac{1}{\alpha}.$$

Moreover, Theorem 4.1 in [ST05] provides a bound on the pointwise Hölder exponent and reads as follows.

Theorem 6.16. *Let $(Y(t))_{t\in\mathbb{R}}$ be the classical multifractional stable motion. Suppose the multifractional parameter $H : \mathbb{R} \to (0,1)$ satisfies $H(t) > \frac{1}{\alpha}$ and $\rho_H^{unif}(t) > \frac{1}{\alpha}$ for all $t \in \mathbb{R}$. Then $(Y(t))_{t\in\mathbb{R}}$ admits a continuous modification $(\tilde{Y}(t))_{t\in\mathbb{R}}$ such that, with probability one, $\rho_{\tilde{Y}}^{unif}(t) \geq (\rho_H^{unif}(t) \wedge H(t)) - \frac{1}{\alpha}$ for all $0 \neq t \in \mathbb{R}$. Moreover, for all $0 \neq t \in \mathbb{R}$, $\rho_{\tilde{Y}}(t) \leq \rho_H(t) \wedge H(t)$ almost surely.*

The path properties of the classical multifractional stable motion are later refined by Ayache and Hamonier using wavelet analysis [AH14]. These authors focus on the special case where the scaling coefficients are given by $(a_+, a_-) = (1,0)$. Moreover, they assume that the multifractional parameter $H : \mathbb{R} \to (0,1)$ takes values in a compact subset $[\underline{H}, \overline{H}] \subseteq (1/\alpha, 1)$. From Corollary 5.1 (ii), Theorem 6.1 and Remark 6.1 in this article, it follows that if $\rho_H^{\text{unif}}([S,T]) > \frac{1}{\alpha}$, then one has almost surely

$$\rho_Y^{\text{unif}}([S,T]) = \min_{t\in[S,T]} H(t) - \frac{1}{\alpha}. \tag{6.1}$$

In Theorem 8.1, Ayache and Hamonier use this property to establish that, with probability one, $\rho_Y^{\text{unif}}(t) = H(t) - \frac{1}{\alpha}$ for all $t \in \mathbb{R}$ satisfying $\rho_H^{\text{unif}}(t) > \frac{1}{\alpha}$. Finally, Corollary 5.2 and Theorem 7.2 imply that if $t \in \mathbb{R}$ is such that there is a constant $C > 0$ satisfying

$$|H(t) - H(s)| \leq C|t-s|^{H(t)} \left(1 + |\log|t-s||\right)^{\frac{1}{\alpha}}$$

for all $s \in \mathbb{R}$, then it holds almost surely that $\rho_Y(t) = H(t)$.

Remark. Even though we only stated the Hölder exponents here, the results in [AH14] on the path regularity of the classical multifractional stable motion determine the modulus of continuity to a finer degree than Hölder exponents (i.e. they include logarithmic terms).

Firstly note that, unlike in the Gaussian case, the uniform pointwise Hölder exponent $\rho_Y^{\text{unif}}(t) = H(t) - \frac{1}{\alpha}$ and the pointwise Hölder exponent $\rho_Y(t) = H(t)$ are not equal, and the uniform pointwise Hölder exponent is strictly lower. Intuitively, this difference may be explained by the fact that the driving motion of the classical multifractional stable motion, which is a symmetric α-stable Lévy motion is a pure jump process, whereas the driving motion in the Gaussian case is a Brownian motion, which has continuous sample paths. At any fixed $t \in \mathbb{R}$, the probability that the driving Lévy motion has a jump at t is equal to zero, but with probability one it has infinitely many jumps on any interval containing t, which at least intuitively explains the difference between the uniform pointwise Hölder exponent and the pointwise Hölder exponent of the classical multifractional stable motion.

Also note that in all the results obtained in [ST05] and [AH14], the Hölder regularity of the classical multifractional stable motion is dependent on the Hölder regularity of $H(\cdot)$, as was the case for the classical multifractional Brownian motion. Just as in the Gaussian case, we will overcome this drawback by letting $H = H(x)$ vary with the integration variable, instead of with the variable indexing the stochastic process. This will allow us to obtain results on the Hölder regularity of the paths of the process without imposing conditions on the Hölder regularity of the function H. Moreover, it will allow us to consider random multifractional parameters. For integrability reasons, we must restrict the range of the function H to a compact subset of $(0, 1)$. Under this requirement, the arguments from Lemma 6.8 still apply to the function $x \mapsto (t-x)_+^{H(x)-1/\alpha} - (-x)_+^{H(x)-1/\alpha}$.

Lemma 6.17. *Fix $\alpha \in (0, 2)$, let $t \in \mathbb{R}$ and let $0 < \underline{H} < \overline{H} < 1$. Then there is a constant $C(\alpha, t, \underline{H}, \overline{H})$ such that for all functions $H : \mathbb{R} \to [\underline{H}, \overline{H}]$,*

$$\int_{\mathbb{R}} \left|(t-x)_+^{H(x)-\frac{1}{\alpha}} - (-x)_+^{H(x)-\frac{1}{\alpha}}\right|^\alpha dx \leq C(\alpha, t, \underline{H}, \overline{H}).$$

Proof. Without loss of generality assume $t > 0$ (otherwise, reverse the roles of 0 and t). Write $L = (\overline{H} - 1/\alpha) \vee (1/\alpha - \underline{H})$ so that $|H(x) - 1/\alpha| \leq L$ for all $x \in \mathbb{R}$. For $x \leq -1$, apply the mean value theorem to obtain $\xi_{x,t} \in [0, t]$ such that

$$(t-x)^{H(x)-1/\alpha} - (-x)^{H(x)-1/\alpha} = (H(x) - 1/\alpha)(\xi_{x,t} - x)^{H(x)-1/\alpha-1} t.$$

Then

$$\int_{-\infty}^{-1} \left|(t-x)^{H(x)-\frac{1}{\alpha}} - (-x)^{H(x)-\frac{1}{\alpha}}\right|^\alpha dx \leq (Lt)^\alpha \int_{-\infty}^{-1} (-x)^{\alpha(\overline{H}-1)-1} dx = \frac{(Lt)^\alpha}{\alpha(1-\overline{H})}.$$

Next,

$$\begin{aligned}
\int_{-1}^{0} \left|(t-x)^{H(x)-\frac{1}{\alpha}} - (-x)^{H(x)-\frac{1}{\alpha}}\right|^\alpha dx &\leq 2^\alpha \left(\int_{-1}^{0} (t-x)^{\alpha H(x)-1} dx + \int_{-1}^{0} (-x)^{\alpha H(x)-1} dx\right) \\
&\leq 2^\alpha \left(\int_{-1}^{0} (t-x)^{\alpha\underline{H}-1} \mathbb{1}_{\{t-x\leq 1\}} + (t-x)^{\alpha\overline{H}-1} \mathbb{1}_{\{t-x\geq 1\}} dx + \int_{-1}^{0} (-x)^{\alpha\underline{H}-1} dx\right) \\
&\leq 2^\alpha \left(\frac{1-t^{\alpha\underline{H}}}{\alpha\underline{H}} \mathbb{1}_{\{t\leq 1\}} + \frac{(t+1)^{\alpha\overline{H}} - 1}{\alpha\overline{H}} + \frac{1}{\alpha\underline{H}}\right)
\end{aligned}$$

Finally,

$$\begin{aligned}
\int_0^t (t-x)^{\alpha H(x)-1} dx &\leq \int_0^t (t-x)^{\alpha\underline{H}-1} \mathbb{1}_{\{t-x\leq 1\}} + (t-x)^{\alpha\overline{H}-1} \mathbb{1}_{\{t-x\geq 1\}} dx \\
&\leq \frac{1}{\alpha\underline{H}} + \frac{t^{\alpha\overline{H}} - 1}{\alpha\overline{H}} \mathbb{1}_{\{t\geq 1\}}.
\end{aligned}$$

□

Definition 6.18. Let $H : \mathbb{R} \to [\underline{H}, \overline{H}] \subseteq (0,1)$ be a function. The **Itô multifractional stable motion** with **multifractional parameter** H is the stochastic process $(Y(t))_{t\in\mathbb{R}}$, given by

$$Y(t) = \int_{\mathbb{R}} (t-x)_+^{H(x)-\frac{1}{\alpha}} - (-x)_+^{H(x)-\frac{1}{\alpha}} \, dM(x).$$

First, we will obtain a continuous modification of the Itô multifractional stable motion $(Y(t))_{t\in(S,T)}$ restricted to an open subset (S,T) of the real line with a lower bound on the uniform Hölder exponents over compact subsets of (S,T). This will be done by mimicking the arguments from [LMS21] and employing Theorem 3.10. It will be assumed that the multifractional parameter $H : \mathbb{R} \to [\underline{H}, \overline{H}] \subseteq (0,1)$ satisfies a continuity condition which allows us to bound the norms of difference of kernels defining the Itô multifractional stable motion.

Lemma 6.19. *Fix $\alpha \in (0,2)$, $S < T$, $0 < \underline{H} < \overline{H} < 1$ and a modulus of continuity $w : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. There exists a constant $C(\alpha, \underline{H}, \overline{H})$ such that for all $\epsilon \in (0,1)$, all functions $H : \mathbb{R} \to [\underline{H}, \overline{H}]$ admitting w as a modulus of continuity on $(S-\epsilon, T)$, all $t \in (S,T)$ and all $h \in (0, (T-t) \wedge \frac{1}{2})$,*

$$\int_{-\infty}^{t-\epsilon} \left|(t+h-x)^{H(x)-\frac{1}{\alpha}} - (t-x)^{H(x)-\frac{1}{\alpha}}\right|^{\alpha} dx \leq C(\alpha, \underline{H}, \overline{H})\, \epsilon^{\alpha(\underline{H}-1)} h^{\alpha}, \tag{6.2}$$

$$\int_{t-\epsilon}^{t} \left|(t+h-x)^{H(x)-\frac{1}{\alpha}} - (t-x)^{H(x)-\frac{1}{\alpha}}\right|^{\alpha} dx \leq C(\alpha, \underline{H}, \overline{H})\, h^{\alpha(H(t)-w(\epsilon))}, \tag{6.3}$$

$$\int_{t}^{t+h} (t+h-x)^{\alpha H(x)-1}\, dx \leq C(\alpha, \underline{H}, \overline{H})\, h^{\alpha(H(t)-w(h))}. \tag{6.4}$$

Proof. Write $L = (\overline{H} - 1/\alpha) \vee (1/\alpha - \underline{H})$ so that $|H(x) - 1/\alpha| \leq L$ for all $x \in \mathbb{R}$. Apply the mean value theorem to obtain $\xi_{t,h,x} \in [0,h]$ so that the left hand side of Equation (6.2) equals

$$\begin{aligned}
& h^{\alpha} \int_{-\infty}^{t-\epsilon} \left|H(x) - \frac{1}{\alpha}\right|^{\alpha} (t+\xi_{t,h,x}-x)^{\alpha(H(x)-1)-1}\, dx \\
\leq & L^{\alpha} h^{\alpha} \int_{-\infty}^{t-\epsilon} (t-x)^{\alpha(\underline{H}-1)-1} \mathbb{1}_{\{t+\xi_{t,h,x}-x\leq 1\}} + (t-x)^{\alpha(\overline{H}-1)-1} \mathbb{1}_{\{t+\xi_{t,h,x}-x>1\}}\, dx \\
\leq & L^{\alpha} h^{\alpha} \left(\int_{-\infty}^{t-\epsilon} (t-x)^{\alpha(\underline{H}-1)-1}\, dx + \int_{-\infty}^{t-\frac{1}{2}} (t-x)^{\alpha(\overline{H}-1)-1}\, dx \right) \\
\leq & C(\alpha, \underline{H}, \overline{H}) \epsilon^{\alpha(\underline{H}-1)}\, h^{\alpha}.
\end{aligned}$$

For the second term, substitute $x = t + h\hat{x}$ and use the fact that $H(t+h\hat{x}) \geq H(t) - w(\epsilon)$ whenever $t + h\hat{x} \in [t-\epsilon, t] \subseteq (S-\epsilon, T)$ to find that the left hand side of (6.3) equals

$$\begin{aligned}
& \int_{-\epsilon h^{-1}}^{0} h^{\alpha H(t+h\hat{x})} \left|(1-\hat{x})^{H(t+h\hat{x})-\frac{1}{\alpha}} - (-\hat{x})^{H(t+h\hat{x})-\frac{1}{\alpha}}\right|^{\alpha} d\hat{x} \\
\leq & h^{\alpha(H(t)-w(\epsilon))} \int_{-\infty}^{0} \left|(1-\hat{x})^{H(t+h\hat{x})-\frac{1}{\alpha}} - (-\hat{x})^{H(t+h\hat{x})-\frac{1}{\alpha}}\right|^{\alpha} d\hat{x} \\
\leq & C(\alpha, \underline{H}, \overline{H}) h^{\alpha(H(t)-w(\epsilon))},
\end{aligned}$$

where we use Lemma 6.17 for the final inequality. Finally,

$$\int_t^{t+h} (t+h-x)^{\alpha H(x)-1}\,dx \le \int_t^{t+h} (t+h-x)^{\alpha \underline{H}_{[t,t+h]}-1}\,dx \le \frac{1}{\alpha \underline{H}} h^{\alpha(H(t)-w(h))}$$

□

Corollary 6.20. *Let $(Y(t))_{t\in(S,T)}$ be the Itô multifractional stable motion with multifractional parameter $H : \mathbb{R} \to [\underline{H}, \overline{H}] \subseteq (0,1)$. Assume that H admits a modulus of continuity $w : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ on (S', T) for some $S' < S$ and that $\underline{H}_{(S,T)} > \frac{1}{\alpha}$. Then $(Y(t))_{t\in(S,T)}$ has a modification $(\tilde{Y}(t))_{t\in(S,T)}$ such that, with probability one, for all $\gamma > 0$ and all $S < a < b < T$,*

$$\sup_{\substack{s,t\in[a,b]\\ s\neq t}} \frac{|\tilde{Y}(t)-\tilde{Y}(s)|}{|t-s|^{\underline{H}_{[a,b]}-1/\alpha-\gamma}} < \infty$$

Proof. Choose $\epsilon \in (0, \frac{1}{2})$ small enough such that $(S-\epsilon, T) \subseteq (S', T)$ and $w(\epsilon) < \underline{H}_{(S,T)} - \frac{1}{\alpha}$. Then, by Lemma 6.19, there is a constant $C = C(\alpha, \underline{H}, \overline{H}, \epsilon)$ such that, for all $s, t \in (S,T)$ obeying $|t-s| \le \epsilon$,

$$\|Y(t)-Y(s)\|_\alpha \le C|t-s|^{H(s\wedge t)-w(\epsilon)} \le C|t-s|^{\underline{H}_{[s\wedge t, s\vee t]}-w(\epsilon)}.$$

Now take $\frac{1}{\underline{H}_{(S,T)}-w(\epsilon)} < p < \alpha$ and let Ψ be standard symmetric α-stable, then for all $s,t \in (S,T)$ with $|t-s| \le \epsilon$,

$$\mathbb{E}\left|\frac{Y(t)-Y(s)}{|t-s|^{\underline{H}_{[s\wedge t,s\vee t]}-w(\epsilon)-1/p}}\right|^p = \frac{\mathbb{E}|\Psi|^p \|Y(t)-Y(s)\|_\alpha^p}{|t-s|^{p(\underline{H}_{[s\wedge t,s\vee t]}-w(\epsilon))-1}} \le \mathbb{E}|\Psi|^p C^p |t-s|.$$

By Theorem 3.10, $(Y(t))_{t\in(S,T)}$ admits a modification $(\tilde{Y}_{\epsilon,p}(t))_{t\in(S,T)}$ such that, with probability one, for all $\gamma > 0$ and all $S < a < b < T$,

$$\sup_{\substack{s,t\in[a,b]\\ s\neq t}} \frac{|\tilde{Y}_{\epsilon,p}(t)-\tilde{Y}_{\epsilon,p}(s)|}{|t-s|^{\underline{H}_{[a,b]}-w(\epsilon)-1/p-\gamma}} < \infty.$$

The result follows by letting $\epsilon \downarrow 0$ and $p \uparrow \alpha$. □

Now that the existence of a continuous modification has been established, we will show the property that legitimizes the naming of the Itô multifractional stable motion. Namely, that this process is $H(t)$-localizable at t with a fractional stable motion as its local form. This will be shown by using Lemma 6.7, and to bound the appropriate integrals we will need the following lemma, which corresponds to Lemma 4.2 in [LMS21].

Lemma 6.21. *Let $\alpha \in (0,2)$ and $0 < \underline{c} < \overline{c} < 1$. Then there is a constant $C(\alpha, \underline{c}, \overline{c})$ such that for all functions $a : \mathbb{R} \to [\underline{c}, \overline{c}]$ and $b : \mathbb{R} \to [\underline{c}, \overline{c}]$ satisfying $|a(x) - b(x)| \le \Delta$ for all $x \in \mathbb{R}$, and for all $h \in (0, 1/e)$,*

$$\int_{\mathbb{R}} \left| \left((h-x)_+^{a(x)-\frac{1}{\alpha}} - (-x)_+^{a(x)-\frac{1}{\alpha}}\right) - \left((h-x)_+^{b(x)-\frac{1}{\alpha}} - (-x)_+^{b(x)-\frac{1}{\alpha}}\right) \right|^\alpha dx \le C(\alpha, \underline{c}, \overline{c})\, \Delta^\alpha h^{\alpha\, \underline{a\wedge b}}\, |\log h|^\alpha,$$

where $\underline{a \wedge b} = \inf_{x\in\mathbb{R}}(a(x) \wedge b(x))$.

Proof. Split up the integral into

$$\int_{-\infty}^{0}\left|\left((h-x)^{a(x)-\frac{1}{\alpha}}-(-x)^{a(x)-\frac{1}{\alpha}}\right)-\left((h-x)^{b(x)-\frac{1}{\alpha}}-(-x)^{b(x)-\frac{1}{\alpha}}\right)\right|^{\alpha}dx \tag{6.5}$$

$$+\int_{0}^{h}\left|(h-x)^{a(x)-\frac{1}{\alpha}}-(h-x)^{b(x)-\frac{1}{\alpha}}\right|^{\alpha}dx \tag{6.6}$$

To bound (6.5), substitute $x=h\hat{x}$ and apply the mean value theorem to obtain $\xi_{\hat{x},h}$ between $a(h\hat{x})$ and $b(h\hat{x})$ such that

$$\begin{aligned}
&\int_{-\infty}^{0}\left|\left((h-x)^{a(x)-\frac{1}{\alpha}}-(-x)^{a(x)-\frac{1}{\alpha}}\right)-\left((h-x)^{b(x)-\frac{1}{\alpha}}-(-x)^{b(x)-\frac{1}{\alpha}}\right)\right|^{\alpha}dx\\
&=\int_{-\infty}^{0}\left|h^{a(h\hat{x})}\left((1-\hat{x})^{a(h\hat{x})-\frac{1}{\alpha}}-(-\hat{x})^{a(h\hat{x})-\frac{1}{\alpha}}\right)-h^{b(h\hat{x})}\left((1-\hat{x})^{b(h\hat{x})-\frac{1}{\alpha}}-(-\hat{x})^{b(h\hat{x})-\frac{1}{\alpha}}\right)\right|^{\alpha}d\hat{x}\\
&=\int_{-\infty}^{0}|b(h\hat{x})-a(h\hat{x})|^{\alpha}h^{\alpha\xi_{\hat{x},h}}\left|\log h\left((1-\hat{x})^{\xi_{\hat{x},h}-\frac{1}{\alpha}}-(-\hat{x})^{\xi_{\hat{x},h}-\frac{1}{\alpha}}\right)\right.\\
&\quad\left.+\left((1-\hat{x})^{\xi_{\hat{x},h}-\frac{1}{\alpha}}\log(1-\hat{x})-(-\hat{x})^{\xi_{\hat{x},h}-\frac{1}{\alpha}}\log(-\hat{x})\right)\right|^{\alpha}d\hat{x}\\
&\leq 2^{\alpha}\Delta^{\alpha}h^{\alpha\,\underline{a\wedge b}}|\log h|^{\alpha}\left[\int_{-\infty}^{0}\left|(1-\hat{x})^{\xi_{\hat{x},h}-\frac{1}{\alpha}}-(-\hat{x})^{\xi_{\hat{x},h}-\frac{1}{\alpha}}\right|^{\alpha}d\hat{x}+\int_{-\infty}^{0}\left|(1-\hat{x})^{\xi_{\hat{x},h}-\frac{1}{\alpha}}\log(1-\hat{x})-(-\hat{x})^{\xi_{\hat{x},h}-\frac{1}{\alpha}}\log(-\hat{x})\right|^{\alpha}d\hat{x}\right]
\end{aligned}$$

Both of the integrals between the square brackets are bounded above, independently from h and the functions a and b: For the first integral, this follows from Lemma 6.17. Similarly, applying the mean value theorem for $x\leq -1$,

$$\begin{aligned}
&\int_{-\infty}^{0}\left|(1-\hat{x})^{\xi_{\hat{x},h}-\frac{1}{\alpha}}\log(1-\hat{x})-(-\hat{x})^{\xi_{\hat{x},h}-\frac{1}{\alpha}}\log(-\hat{x})\right|^{\alpha}d\hat{x}\\
&\leq 2^{\alpha}\int_{-\infty}^{-1}[1+|\log(-\hat{x})|^{\alpha}](-\hat{x})^{\alpha(\overline{c}-1)-1}\,d\hat{x}+2^{\alpha}\int_{-1}^{0}(1-\hat{x})^{\alpha\overline{c}-1}|\log(1-\hat{x})|^{\alpha}+(-\hat{x})^{\alpha\underline{c}-1}|\log(-\hat{x})|^{\alpha}\,d\hat{x}\\
&<\infty.
\end{aligned}$$

To bound (6.6), the mean value theorem implies that

$$\begin{aligned}
\int_{0}^{h}\left|(h-x)^{a(x)-\frac{1}{\alpha}}-(h-x)^{b(x)-\frac{1}{\alpha}}\right|^{\alpha}dx &\leq \Delta^{\alpha}\int_{0}^{h}(h-x)^{\alpha\underline{a\wedge b}-1}|\log(h-x)|^{\alpha}\,dx\\
&\overset{x=h-h\hat{x}}{\leq}\Delta^{\alpha}h^{\alpha\underline{a\wedge b}}\int_{0}^{1}\hat{x}^{\alpha\underline{c}-1}|\log\hat{x}+\log h|^{\alpha}\,d\hat{x}\\
&\leq 2^{\alpha}\Delta^{\alpha}h^{\alpha\underline{a\wedge b}}|\log h|^{\alpha}\int_{0}^{1}\hat{x}^{\alpha\underline{c}-1}(|\log\hat{x}|^{\alpha}+1)\,d\hat{x}.
\end{aligned}$$

□

Proposition 6.22. *Let $(Y(t))_{t\in(S,T)}$ be the continuous modification of the Itô multifractional stable motion from Corollary 6.20. Moreover, assume that $w(h)\log(h)\to 0$ as $h\downarrow 0$. Fix $t\in(S,T)$ and $a<0<b$. Then, as $h\downarrow 0$,*

$$\left(\frac{Y(t+hr)-Y(t)}{h^{H(t)}}\right)_{r\in(a,b)}\to\left(Y_t'(r)\right)_{r\in(a,b)},$$

where $(Y_t'(r))_{r\in(a,b)}$ *is a continuous fractional stable motion with fractional parameter* $H(t)$. *The convergence here is distributional in the space* $C((a,b))$.

Proof. By Lemma 6.7, for convergence in finite-dimensionl distribution, it suffices to show that for all $r \in (a,b)$,

$$A(h) = h^{-\alpha H(t)} \int_{\mathbb{R}} \left|(t+hr-x)_+^{H(x)-\frac{1}{\alpha}} - (t-x)_+^{H(x)-\frac{1}{\alpha}} - \left((t+hr-x)_+^{H(t)-\frac{1}{\alpha}} - (t-x)_+^{H(t)-\frac{1}{\alpha}}\right)\right|^\alpha dx \overset{h\downarrow 0}{\to} 0.$$

Let $q \in (0,1)$ and split up the integral

$$\begin{aligned}
A(h) &= h^{-\alpha H(t)} \int_{-\infty}^{t-h^q} \left|\left((t+hr-x)^{H(x)-\frac{1}{\alpha}} - (t-x)^{H(x)-\frac{1}{\alpha}}\right) - \left((t+hr-x)^{H(t)-\frac{1}{\alpha}} - (t-x)^{H(t)-\frac{1}{\alpha}}\right)\right|^\alpha dx \\
&\quad + h^{-\alpha H(t)} \int_{t-h^q}^{t+hr} \left|\left((t+hr-x)_+^{H(x)-\frac{1}{\alpha}} - (t-x)_+^{H(x)-\frac{1}{\alpha}}\right) - \left((t+hr-x)_+^{H(t)-\frac{1}{\alpha}} - (t-x)_+^{H(t)-\frac{1}{\alpha}}\right)\right|^\alpha dx \\
&\le 2^\alpha h^{-\alpha H(t)} \int_{-\infty}^{t-h^q} \left|(t+hr-x)^{H(x)-\frac{1}{\alpha}} - (t-x)^{H(x)-\frac{1}{\alpha}}\right|^\alpha dx \\
&\quad + 2^\alpha h^{-\alpha H(t)} \int_{-\infty}^{t-h^q} \left|(t+hr-x)^{H(t)-\frac{1}{\alpha}} - (t-x)^{H(t)-\frac{1}{\alpha}}\right|^\alpha dx \\
&\quad + h^{-\alpha H(t)} \int_{t-h^q}^{t+hr} \left|(t+hr-x)_+^{H(x)-\frac{1}{\alpha}} - (t-x)_+^{H(x)-\frac{1}{\alpha}} - \left((t+hr-x)_+^{H(t)-\frac{1}{\alpha}} - (t-x)_+^{H(t)-\frac{1}{\alpha}}\right)\right|^\alpha dx.
\end{aligned}$$

We show that, for $q \in (0,1)$ sufficiently small, each of the three addends converge to 0 as $h \downarrow 0$. For the first term, assuming $r > 0$ and taking $\epsilon = h^q$ in (6.2) in Lemma 6.19, it follows that

$$\begin{aligned}
h^{-\alpha H(t)} \int_{-\infty}^{t-h^q} \left|(t+hr-x)^{H(x)-\frac{1}{\alpha}} - (t-x)^{H(x)-\frac{1}{\alpha}}\right|^\alpha dx &\le C(\alpha, \underline{H}, \overline{H})\, h^{-\alpha H(t)} h^{q\alpha(\underline{H}-1)} (hr)^\alpha \\
&\le C(\alpha, \underline{H}, \overline{H}, r)\, h^{\alpha[1-H(t)-q(1-\underline{H})]}
\end{aligned}$$

which does converge to 0 for $q < \frac{1-H(t)}{1-\underline{H}}$. If $r < 0$ then take $\epsilon = hr + h^q$ (which is positive for h small enough) in (6.2) in Lemma 6.19 to conclude that

$$\begin{aligned}
h^{-\alpha H(t)} \int_{-\infty}^{t-h^q} \left|(t+hr-x)^{H(x)-\frac{1}{\alpha}} - (t-x)^{H(x)-\frac{1}{\alpha}}\right|^\alpha dx &= h^{-\alpha H(t)} \int_{-\infty}^{t+hr-\epsilon} \left|(t-x)^{H(x)-\frac{1}{\alpha}} - (t+hr-x)^{H(x)-\frac{1}{\alpha}}\right|^\alpha dx \\
&\le C(\alpha, \underline{H}, \overline{H})\, h^{-\alpha H(t)} (hr+h^q)^{\alpha(\overline{H}-1)} (hr)^\alpha \\
&\le C(\alpha, \underline{H}, \overline{H}, r)\, h^{\alpha[1-H(t)-q(1-\underline{H})]} (1+rh^{1-q})^{\alpha(\underline{H}-1)}.
\end{aligned}$$

Again we see that the first term converges to zero for $q < \frac{1-H(t)}{1-\underline{H}}$. For the second term, note that

$$h^{-\alpha H(t)} \int_{-\infty}^{t-h^q} \left|(t+hr-x)^{H(t)-\frac{1}{\alpha}} - (t-x)^{H(t)-\frac{1}{\alpha}}\right|^\alpha dx \overset{x=t+h\hat{x}}{=} \int_{-\infty}^{-h^{q-1}} \left|(r-\hat{x})^{H(t)-\frac{1}{\alpha}} - (-\hat{x})^{H(t)-\frac{1}{\alpha}}\right|^\alpha d\hat{x} \overset{h\downarrow 0}{\to} 0.$$

For the final term, by Lemma 6.21, for h sufficiently small,

$$\begin{aligned}
&h^{-\alpha H(t)}\int_{t-h^q}^{(t+hr)\vee t}\left|\left((t+hr-x)_+^{H(x)-\frac{1}{\alpha}}-(t-x)_+^{H(x)-\frac{1}{\alpha}}\right)-\left((t+hr-x)_+^{H(t)-\frac{1}{\alpha}}-(t-x)_+^{H(t)-\frac{1}{\alpha}}\right)\right|^\alpha dx\\
&=h^{-\alpha H(t)}\int_{-h^q}^{hr\vee 0}\left|\left((hr-x)_+^{H(t+x)-\frac{1}{\alpha}}-(-x)_+^{H(t+x)-\frac{1}{\alpha}}\right)-\left((hr-x)_+^{H(t)-\frac{1}{\alpha}}-(-x)_+^{H(t)-\frac{1}{\alpha}}\right)\right|^\alpha dx\\
&\leq h^{-\alpha H(t)}C(\alpha,\underline{H},\overline{H})\sup_{x\in[t-h^q,(t+hr)\vee t]}|H(x)-H(t)|^\alpha(hr)^{\alpha(H(t)-w(h^q\vee hr))}|\log(h|r|)|^\alpha\\
&\leq C(\alpha,\underline{H},\overline{H},r)\,w(h^q\vee hr)^\alpha\,h^{-\alpha w(h^q\vee hr)}|\log h|^\alpha\\
&=C(\alpha,\underline{H},\overline{H},r)|w(h^q)\log h|^\alpha h^{-\alpha w(h^q)}.
\end{aligned}$$

Now, from the assumption that $w(h)\log h\to 0$ as $h\downarrow 0$ it follows that $w(h^q)\log h\to 0$ and therefore that $h^{-\alpha w(h^q)}\to 1$ as $h\downarrow 0$. Thus, the final addend converges to 0 as $h\downarrow 0$ as well.

To show that the convergence is functional, by Proposition 2.17, it suffices to show that for $\frac{1}{\alpha}<\frac{1}{p}<\frac{1}{p'}<\underline{H}_{(S,T)}$ there are constants $C>0$ and $\rho>1$, independent of h,s and r, such that for all $s,r\in(a,b)$,

$$\mathbb{E}\left|\frac{Y(t+hr)-Y(t+hs)}{h^{H(t)}|r-s|^{\underline{H}_{(S,T)}-1/p'}}\right|^p\leq C|r-s|^\rho.$$

Note that, assuming without loss of generality that $0\leq r-s\leq 1$,

$$\mathbb{E}\left|\frac{Y(t+hr)-Y(t+hs)}{h^{H(t)}|r-s|^{\underline{H}_{(S,T)}-1/p'}}\right|^p\leq|r-s|^{p/p'}h^{p(H(t)-H(t+hs))}|hr-hs|^{-pH(t+hs)}\mathbb{E}|Y(t+hr)-Y(t+hs)|^p$$

By applying Lemma 6.19 again with $\epsilon=h^q$, $q\in(0,1)$, we see that

$$\|Y(t+hr)-Y(t+hs)\|_\alpha\leq C(\alpha,\underline{H},\overline{H})\left[h^{q(\underline{H}-1)}|hr-hs|+|hr-hs|^{H(t+hs)-w(h^q)}+|hr-hs|^{H(t+hs)-w(|hr-hs|)}\right].$$

Applying this to the inequality from before, it follows that

$$\begin{aligned}
\mathbb{E}\left|\frac{Y(t+hr)-Y(t+hs)}{h^{H(t)}|r-s|^{\underline{H}_{(S,T)}-1/p'}}\right|^p\leq C(\alpha,\underline{H},\overline{H},p)|r-s|^{p/p'}\,h^{-pw(h(b-a))}\,\Big[&h^{p(q(\underline{H}-1)+1-\overline{H})}|r-s|^{p(1-\overline{H})}\\
&+h^{-pw(h^q)}|r-s|^{-pw(h^q)}\\
&+h^{-pw(h(b-a))}|r-s|^{-pw(h(b-a))}\Big]
\end{aligned}$$

Now take q small enough such that $h^{p(q(\underline{H}-1)+1-\overline{H})}\to 0$ as $h\downarrow 0$. Due to the asymptotic assumption placed on the modulus of continuity w, it follows that $h^{-pw(h(b-a))}\to 1$ and $h^{-pw(h^q)}\to 1$. Thus, in a neighborhood $h\in(0,\delta)$, with $\delta>0$ small enough so that $pw(\delta^q)\vee pw(\delta(b-a))<p/p'-1$ it holds that, for all $s,t\in(a,b)$,

$$\mathbb{E}\left|\frac{Y(t+hr)-Y(t+hs)}{h^{H(t)}|r-s|^{\underline{H}_{(S,T)}-1/p'}}\right|^p\leq C(\alpha,\underline{H},\overline{H},p,\delta)|r-s|^{p/p'-(pw(\delta^q)\vee pw(\delta(b-a)))}.$$

□

Theorem 6.23. *Let $(Y(t))_{t\in(S,T)}$ be the continuous modification of the Itô multifractional stable motion from Corollary 6.20. Then, with probability one, $\rho_Y^{unif}(t) \geq H(t) - \frac{1}{\alpha}$ for all $t \in (S,T)$. Moreover, if the modulus of continuity satisfies $w(h)\log h \to 0$ as $h \downarrow 0$, then for all $t \in (S,T)$, $\rho_Y(t) \leq H(t)$ almost surely.*

Proof. For the lower bound, note that by Corollary 6.20 we have with probability one, for all $S < a < t < b < T$,

$$\rho_Y^{\text{unif}}(t) \geq \rho_Y^{\text{unif}}([a,b]) \geq \underline{H}_{[a,b]} - \frac{1}{\alpha}.$$

From continuity of H it follows that $\rho_Y^{\text{unif}}(t) \geq H(t) - \frac{1}{\alpha}$. Now fix $t \in (S,T)$, from Proposition 6.22, it follows that

$$\frac{Y(t+h) - Y(t)}{h^{H(t)}} \overset{h\downarrow 0}{\to} X_{H(t)}(1)$$

in distribution, where $(X_{H(t)}(r))_{r\in(a,b)}$ is a fractional stable motion with fractional parameter $H(t)$. So, whenever $\rho > 0$ there is a sequence $h_n \downarrow 0$ such that

$$\frac{|Y(t+h_n) - Y(t)|}{h_n^{H(t)+\rho}} \to \infty$$

in probability and we can find a subsequence that diverges almost surely. Thus, almost surely,

$$\limsup_{h\to 0} \frac{|Y(t+h) - Y(t)|}{|h|^{H(t)+\rho}} = \infty,$$

and $\rho_Y(t) \leq H(t)$ almost surely. □

6.3 Stable Itô Calculus: Random Integrands

As mentioned, one of the advantages of considering a multifractional stable motion where the multifractional parameter depends on the integration variable, instead of the variable indexing the process, is that the resulting kernel is adapted if the multifractional parameter is made random. To exploit this advantage, this section is dedicated to constructing an Itô calculus against symmetric an α-stable Lévy motion. That is, we will define the stochastic integrals

$$\int_{\mathbb{R}} F\,dL, \tag{6.7}$$

where $(L(x))_{x\in\mathbb{R}}$ is a standard symmetric α-stable Lévy motion, and $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a jointly measurable stochastic process, adapted to the natural filtration generated by $(L(x))_{x\in\mathbb{R}}$, and such that

$$\mathbb{E}\int_{\mathbb{R}} |F(x)|^\alpha\,dx < \infty.$$

Throughout this section we will fix $\alpha \in (0,2)$ and a standard symmetric α-stable Lévy motion $(L(x))_{x\in\mathbb{R}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and write $(\mathcal{F}_x)_{x\in\mathbb{R}}$ for the natural filtration generated by $(L(x))_{x\in\mathbb{R}}$, i.e. $\mathcal{F}_x = \sigma(L(y) : y \leq x)$. In order to define the stochastic integrals (6.7), we will first define stochastic integrals over a bounded interval $[a,b] \subseteq \mathbb{R}$. Then, the stochastic integral over the entire real line will be obtained as a limit by letting $a \to -\infty$ and $b \to \infty$. The Itô calculus over bounded intervals is a simplified variant of the one developed in [GM83] and [RW86]. These authors envision the stochastic integral as an operator mapping stochastic processes to stochastic processes, but for our purposes it suffices to consider only the random variable $\int_a^b F\,dL$, simplifying the analysis.

Definition 6.24. If $I \subseteq \mathbb{R}$ is a (possibly unbounded) interval, let $\mathbb{L}^\alpha(\Omega \times I; (\mathcal{F}_x)_{x\in I})$ denote the space of jointly measurable stochastic processes $F : \Omega \times I \to \mathbb{R}$ adapted to the filtration $(\mathcal{F}_x)_{x\in I}$, such that

$$\|F\|_{\mathbb{L}^\alpha(\Omega\times I)} = \left(\mathbb{E}\int_I |F(x)|^\alpha \, dx\right)^{\frac{1}{\alpha}} < \infty.$$

Of course, the integral will be defined first for simple processes.

Definition 6.25. Let $a < b$. A process $F \in \mathbb{L}^\alpha(\Omega \times [a, b]; (\mathcal{F}_x)_{x\in[a,b]})$ is **simple** if there is a partition $a = x_0 < \ldots < x_n = b$ and random variables $\xi_1 \ldots \xi_n \in \mathbb{L}^\alpha(\Omega)$ such that ξ_k is $\mathcal{F}_{x_{k-1}}$-measurable, and

$$F(\omega, x) = \sum_{k=1}^n \xi_k(\omega)\mathbb{1}_{(x_{k-1},x_k]}(x).$$

The space of simple functions is denoted $\mathcal{S}([a, b])$. For simple functions with a representation as above the stochastic integral is defined as

$$\int_a^b F\, dL = \sum_{k=1}^n \xi_k(L(x_k) - L(x_{k-1})).$$

This definition yields a linear operator $\int_a^b \cdot\, dL : \mathcal{S}([a, b]) \to \mathbb{L}^0(\Omega)$. However, unlike the case of deterministic integrands, this operator does not map into a linear set of symmetric α-stable random variables. Instead, the operator maps into the weak Lebesgue space $\Lambda^\alpha(\Omega)$. Moreover, the operators are uniformly bounded in a and b. This next lemma corresponds to Lemma 3.3 in [GM83].

Lemma 6.26. *The linear operators $\int_a^b \cdot\, dL : \mathcal{S}([a, b]) \to \Lambda^\alpha(\Omega)$ with $a < b$ are uniformly bounded. That is, there is a constant C, not depending on a and b, such that for all $a < b$ and all simple processes $F \in \mathcal{S}([a, b])$, it holds that*

$$\left\|\int_a^b F dL\right\|_{\Lambda^\alpha(\Omega)} \le C \|F\|_{\mathbb{L}^\alpha(\Omega\times[a,b])}.$$

Proof. Let Ψ be a standard symmetric α-stable random variable. Let $F = \sum_{k=1}^n \xi_k \mathbb{1}_{(x_{k-1},x_k]} \in \mathcal{S}([a, b])$ be a simple process. Write $\Delta x_k = x_k - x_{k-1}$ and $\Delta L_k = L(x_k) - L(x_{k-1})$. Then, for $\lambda > 0$,

$$\begin{aligned}
\mathbb{P}\left(\left|\int_a^b F\, dL\right| > \lambda\right) &= \mathbb{P}\left(\left|\sum_{k=1}^n \xi_k \Delta L_k\right| > \lambda\right) \\
&= \mathbb{P}\left(\left|\sum_{k=1}^n \xi_k \Delta L_k\right| > \lambda \wedge (\exists k)|\xi_k \Delta L_k| > \lambda\right) + \mathbb{P}\left(\left|\sum_{k=1}^n \xi_k \Delta L_k\right| > \lambda \wedge (\forall k)|\xi_k \Delta L_k| \le \lambda\right) \\
&\le \sum_{k=1}^n \mathbb{P}(|\xi_k \Delta L_k| > \lambda) + \mathbb{P}\left(\left|\sum_{k=1}^n \xi_k \Delta L_k \mathbb{1}_{\{|\xi_k\Delta L_k|\le\lambda\}}\right| > \lambda\right).
\end{aligned}$$

For $k = 1 \ldots n$, letting $\mathscr{P}_{|\xi_k|}$ denote the law of $|\xi_k|$, by independence we have

$$\mathbb{P}(|\xi_k \Delta L_k| > \lambda) = \int_0^\infty \mathbb{P}(|\Delta L_k| > \lambda/x)\, d\mathscr{P}_{|\xi_k|}(x) \le \lambda^{-\alpha}\|\Delta L_k\|^\alpha_{\Lambda^\alpha(\Omega)} \int_0^\infty x^\alpha d\mathscr{P}_{|\xi_k|}(x) = \lambda^{-\alpha}\Delta x_k \|\Psi\|^\alpha_{\Lambda^\alpha(\Omega)}\mathbb{E}|\xi_k|^\alpha.$$

It follows that

$$\sum_{k=1}^{n} \mathbb{P}(|\xi_k \Delta L_k| > \lambda) \le \lambda^{-\alpha} \|\Psi\|^{\alpha}_{\Lambda^\alpha(\Omega)} \sum_{k=1}^{n} \mathbb{E}|\xi_k|^\alpha \Delta x_k = \lambda^{-\alpha} \|\Psi\|^{\alpha}_{\Lambda^\alpha(\Omega)} \|F\|^{\alpha}_{\mathbb{L}^\alpha(\Omega\times[a,b])}.$$

Moreover, using independence to reason that the cross terms are zero,

$$\begin{aligned}\mathbb{P}\left(\left|\sum_{k=1}^{n} \xi_k \Delta L_k \mathbb{1}_{\{|\xi_k \Delta L_k| \le \lambda\}}\right| > \lambda\right) &\le \lambda^{-2}\, \mathbb{E}\left[\left(\sum_{k=1}^{n} \xi_k \Delta L_k \mathbb{1}_{\{|\xi_k \Delta L_k| \le \lambda\}}\right)^2\right] \\ &= \lambda^{-2} \sum_{k=1}^{n} \mathbb{E}[\xi_k^2 \Delta L_k^2 \mathbb{1}_{\{|\xi_k \Delta L_k| \le \lambda\}}].\end{aligned}$$

For $k = 1 \ldots n$ also write $\mathscr{P}_{|\Delta L_k|}$ for the law $|\Delta L_k|$, by independence it holds that

$$\begin{aligned}\mathbb{E}[\xi_k^2 \Delta L_k^2 \mathbb{1}_{\{|\xi_k \Delta L_k| \le \lambda\}}] &= \int_0^\infty \int_0^\infty x^2 y^2 \mathbb{1}_{\{xy \le \lambda\}}\, d\mathscr{P}_{|\Delta L_k|}(y)\, d\mathscr{P}_{|\xi_k|}(x) \\ &= \int_0^\infty x^2 \int_0^{\lambda/x} y^2\, d\mathscr{P}_{|\Delta L_k|}(y)\, d\mathscr{P}_{|\xi_k|}(x) \\ &\le \int_0^\infty x^2 \int_0^{\lambda/x} 2y \mathbb{P}(|\Delta L_k| > y)\, dy\, d\mathscr{P}_{|\xi_k|}(x) \\ &\le 2\|\Delta L_k\|^{\alpha}_{\Lambda^\alpha(\Omega)} \int_0^\infty x^2 \int_0^{\lambda/x} y^{1-\alpha}\, dy\, d\mathscr{P}_{|\xi_k|}(x) \\ &= 2\Delta x_k \|\Psi\|^{\alpha}_{\Lambda^\alpha(\Omega)} \int_0^\infty \frac{1}{2-\alpha} x^2\, (\lambda/x)^{2-\alpha}\, d\mathscr{P}_{|\xi_k|}(x) \\ &= \frac{2}{2-\alpha} \Delta x_k \|\Psi\|^{\alpha}_{\Lambda^\alpha(\Omega)} \lambda^{2-\alpha}\, \mathbb{E}|\xi_k|^\alpha.\end{aligned}$$

Thus,

$$\mathbb{P}\left(\left|\sum_{k=1}^{n} \xi_k \Delta L_k \mathbb{1}_{\{|\xi_k \Delta L_k| \le \lambda\}}\right| > \lambda\right) \le \lambda^{-\alpha} \frac{2}{2-\alpha} \|\Psi\|^{\alpha}_{\Lambda^\alpha(\Omega)} \sum_{k=1}^{n} \mathbb{E}|\xi_k|^\alpha \Delta x_k = \lambda^{-\alpha} \frac{2}{2-\alpha} \|\Psi\|^{\alpha}_{\Lambda^\alpha(\Omega)} \|F\|^{\alpha}_{\mathbb{L}^\alpha(\Omega\times[a,b])}.$$

In conclusion, for all $\lambda > 0$, the following inequality holds,

$$\lambda^\alpha\, \mathbb{P}\left(\left|\int_a^b F\, dL\right| > \lambda\right) \le \left(1 + \frac{2}{2-\alpha}\right) \|\Psi\|^{\alpha}_{\Lambda^\alpha(\Omega)} \|F\|^{\alpha}_{\mathbb{L}^\alpha(\Omega\times[a,b])}.$$

The statement of the lemma follows by taking supremum over $\lambda > 0$ and taking α'th root. □

Lemma 6.27. *$\mathcal{S}([a,b])$ is dense in $\mathbb{L}^\alpha(\Omega \times [a,b]; (\mathcal{F}_x)_{x\in[a,b]})$ with respect to the norm $\|\cdot\|_{\mathbb{L}^\alpha(\Omega\times[a,b])}$.*

Proof. [GM83, Remark 3.2] □

Combining Lemmas 6.26 and 6.27 with Lemma 2.6 extends the integral operators defined in Definition 6.25 into a uniformly bounded set of linear operators

$$\int_a^b \cdot\, dL : \mathbb{L}^\alpha(\Omega \times [a,b]; (\mathcal{F}_x)_{x\in[a,b]}) \to \Lambda^\alpha(\Omega) \qquad a < b.$$

Some elementary properties of these integral operators will be proven before defining the integrals (6.7) over the entire real line.

Lemma 6.28 (Additivity of the integral)**.** *Let $a < b < c$ and let $F \in \mathbb{L}^\alpha(\Omega \times [a,c]; (\mathcal{F}_x)_{x\in[a,c]})$. Then $F_{\restriction \Omega\times[a,b]} \in \mathbb{L}^\alpha(\Omega \times [a,b]; (\mathcal{F}_x)_{x\in[a,b]})$, $F_{\restriction \Omega\times[b,c]} \in \mathbb{L}^\alpha(\Omega \times [b,c]; (\mathcal{F}_x)_{x\in[b,c]})$ and*

$$\int_a^c F\,dL = \int_a^b F\,dL + \int_b^c F\,dL \qquad \textit{almost surely.}$$

Proof. The first part of the claim is obvious. If $F = \sum_{k=1}^n \xi_k \mathbb{1}_{(x_{k-1},x_k]} \in \mathcal{S}([a,c])$ is a simple process, then the identity is a simple computation. By adding b to the partition we may assume that $x_m = b$. Then

$$\int_a^c F\,dL = \sum_{k=1}^n \xi_k(L(x_k) - L(x_{k-1})) = \sum_{k=1}^m \xi_k(L(x_k) - L(x_{k-1})) + \sum_{k=m+1}^n \xi_k(L(x_k) - L(x_{k-1})) = \int_a^b F\,dL + \int_b^c F\,dL.$$

Now suppose $F \in \mathbb{L}^\alpha(\Omega \times [a,c]; (\mathcal{F}_x)_{x\in[a,c]})$ and take a sequence of simple processes $F_n \in \mathcal{S}([a,c])$ such that $\|F_n - F\|_{\mathbb{L}^\alpha(\Omega\times[a,c])} \to 0$ as $n \to \infty$. Then the same holds for on the subintervals $\|F_n - F\|_{\mathbb{L}^\alpha(\Omega\times[a,b])} \to 0$ and $\|F_n - F\|_{\mathbb{L}^\alpha(\Omega\times[a,b])} \to 0$. Thus, $\int_a^b F_n\,dL \to \int_a^b F\,dL$ and $\int_b^c F_n\,dL \to \int_b^c F\,dL$ in $\Lambda^\alpha(\Omega)$. Because the addition operator in the topological vector space $\Lambda^\alpha(\Omega)$ is continuous, it follows that

$$\int_a^c F_n\,dL = \int_a^b F_n\,dL + \int_b^c F_n\,dL \to \int_a^b F\,dL + \int_b^c F\,dL.$$

The claimed identity now follows from uniqueness of limits. □

Lemma 6.29. *Let $a < b$, suppose that $F \in \mathbb{L}^\alpha(\Omega \times [a,b]; (\mathcal{F}_x)_{x\in[a,b]}$ and that ζ is a real-valued bounded $\mathcal{F}_a$-measurable random variable on Ω. Then $\zeta F \in \mathbb{L}^\alpha(\Omega \times [a,b]; (\mathcal{F}_x)_{x\in[a,b]})$, and*

$$\zeta \int_a^b F\,dL = \int_a^b \zeta F\,dL \qquad \textit{almost surely.}$$

Proof. Again, the measurability claim is clear. First suppose $F = \sum_{k=1}^n \xi_k \mathbb{1}_{(x_{k-1},x_k]} \in \mathcal{S}([a,b])$ is a simple process. Then once again the identity is a matter of computation:

$$\zeta \int_a^b F\,dL = \zeta \sum_{k=1}^n \xi_k(L(x_k) - L(x_{k-1})) = \sum_{k=1}^n \zeta\xi_k(L(x_k) - L(x_{k-1})) = \int_a^b \zeta F\,dL.$$

Of course, in that last equality we use that $\zeta F = \sum_{k=1}^n \zeta\xi_k \mathbb{1}_{(x_{k-1},x_k]}$ and that $\zeta\xi_k \in \mathbb{L}^\alpha(\Omega)$ is $\mathcal{F}_{x_{k-1}}$-measurable because ζ is $\mathcal{F}_a$-measurable, making ζF a simple process. Now let $F \in \mathbb{L}^\alpha(\Omega \times [a,b]; (\mathcal{F}_x)_{x\in[a,b]})$ and choose a sequence of simple processes $(F_n)_{n\in\mathbb{N}}$ such that $\|F_n - F\|_{\mathbb{L}^\alpha(\Omega\times[a,b])} \to 0$ as $n \to \infty$. Take $K > 0$ such that $|\zeta(\omega)| \le K$ for all $\omega \in \Omega$. Then

$$\|\zeta F_n - \zeta F\|_{\mathbb{L}^\alpha(\Omega\times[a,b])} = \left(\mathbb{E}\int_a^b |\zeta F_n(x) - \zeta F(x)|^\alpha\,dx\right)^{\frac{1}{\alpha}} \le K\left(\mathbb{E}\int_a^b |F_n(x) - F(x)|^\alpha\,dx\right)^{\frac{1}{\alpha}} = K\|F_n - F\|_{\mathbb{L}^\alpha(\Omega\times[a,b])} \to 0.$$

It follows that

$$\zeta \int_a^b F_n\,dL = \int_a^b \zeta F_n\,dL \to \int_a^b \zeta F\,dL$$

in $\Lambda^\alpha(\Omega)$. Now let $\lambda > 0$, then

$$\mathbb{P}\left(\left|\zeta\int_a^b F_n\,dL - \zeta\int_a^b F\,dL\right| > \lambda\right) \le \mathbb{P}\left(K\left|\int_a^b F_n\,dL - \int_a^b F\,dL\right| > \lambda\right) = \mathbb{P}\left(\left|\int_a^b F_n\,dL - \int_a^b F\,dL\right| > \frac{\lambda}{K}\right).$$

Thus,

$$\lambda^\alpha\,\mathbb{P}\left(\left|\zeta\int_a^b F_n\,dL - \zeta\int_a^b F\,dL\right| > \lambda\right) \le K^\alpha\left(\frac{\lambda}{K}\right)^\alpha\mathbb{P}\left(\left|\int_a^b F_n\,dL - \int_a^b F\,dL\right| > \frac{\lambda}{K}\right) \le K^\alpha\left\|\int_a^b F_n\,dL - \int_a^b F\,dL\right\|_{\Lambda^\alpha(\Omega)}^\alpha.$$

Taking supremum over $\lambda > 0$ and α'th root reveals that

$$\left\|\zeta\int_a^b F_n\,dL - \zeta\int_a^b F\,dL\right\|_{\Lambda^\alpha(\Omega)} \le K\left\|\int_a^b F_n\,dL - \int_a^b F\,dL\right\|_{\Lambda^\alpha(\Omega)} \to 0.$$

We conclude that $\zeta\int_a^b F_n\,dL \to \zeta\int_a^b F\,dL$ in $\Lambda^\alpha(\Omega)$ and from uniqueness of limits it now follows that

$$\zeta\int_a^b F\,dL = \int_a^b \zeta F\,dL \qquad \text{almost surely.}$$

□

Finally, the stochastic integral (6.7) over the entire real line will be obtained as a limit. Let $\mathcal{I}$ denote the directed set of non-empty compact intervals $[a,b] \subseteq \mathbb{R}$, partially ordered by set-inclusion. Whenever $F \in \mathbb{L}^\alpha(\Omega\times\mathbb{R};(\mathcal{F}_x)_{x\in\mathbb{R}})$ it holds that $F_{\restriction\Omega\times[a,b]} \in \mathbb{L}^\alpha(\Omega\times[a,b];(\mathcal{F}_x)_{x\in[a,b]})$ for all $[a,b]\in\mathcal{I}$. We will define the integral (6.7) as the limit in $\Lambda^\alpha(\Omega)$ of the Cauchy net

$$\left(\int_a^b F\,dL\right)_{[a,b]\in\mathcal{I}}. \tag{6.8}$$

Lemma 6.30. *Let $F \in \mathbb{L}^\alpha(\Omega\times\mathbb{R};(\mathcal{F}_x)_{x\in\mathbb{R}})$. Then the net from* (6.8) *is Cauchy in* $\Lambda^\alpha(\Omega)$.

Proof. Let $K \ge 1$ be a constant such that $\|X+Y\|_{\Lambda^\alpha(\Omega)} \le K(\|X\|_{\Lambda^\alpha(\Omega)} + \|Y\|_{\Lambda^\alpha(\Omega)})$ and let $C > 0$ be a constant making the integral operators uniformly bounded. That is, for all $[a,b]\in\mathcal{I}$ and for all $F \in \mathbb{L}^\alpha(\Omega\times[a,b];(\mathcal{F}_x)_{x\in[a,b]})$, we have

$$\left\|\int_a^b F\,dL\right\|_{\Lambda^\alpha(\Omega)} \le C\,\|F\|_{\mathbb{L}^\alpha(\Omega\times[a,b])}.$$

Let $\epsilon > 0$ be arbitrary, choose $[a_0,b_0]\in\mathcal{I}$ such that

$$\mathbb{E}\int_{-\infty}^{a_0} |F(x)|^\alpha\,dx + \mathbb{E}\int_{b_0}^{\infty} |F(x)|^\alpha\,dx < (KC)^{-1}\epsilon.$$

Suppose $[a_0, b_0] \subseteq [a, b]$ and $[a_0, b_0] \subseteq [a', b']$. Then, using additivity of the integral and the fact that $a \vee a' \le a_0 < b_0 \le b \wedge b'$, it follows that

$$\begin{aligned}\left\| \int_a^b F\,dL - \int_{a'}^{b'} F\,dL \right\|_{\Lambda^\alpha(\Omega)} &= \left\| (-1)^{\mathbb{1}_{a>a'}} \int_{a\wedge a'}^{a\vee a'} F\,dL + (-1)^{\mathbb{1}_{b<b'}} \int_{b\wedge b'}^{b\vee b'} F\,dL \right\|_{\Lambda^\alpha(\Omega)} \\ &\le KC \left(\|F\|_{\mathbb{L}^\alpha(\Omega\times[a\wedge a', a\vee a'])} + \|F\|_{\mathbb{L}^\alpha(\Omega\times[b\wedge b', b\vee b'])} \right) \\ &\le KC \left(\mathbb{E} \int_{-\infty}^{a_0} |F(x)|^\alpha\,dx + \mathbb{E} \int_{b_0}^{\infty} |F(x)|^\alpha\,dx \right) \\ &< \epsilon.\end{aligned}$$

□

Definition 6.31. Let $F \in \mathbb{L}^\alpha(\Omega \times \mathbb{R}; (\mathcal{F}_x)_{x\in\mathbb{R}}$, then the stochastic integral

$$\int_{\mathbb{R}} F\,dL$$

is defined as the limit in $\Lambda^\alpha(\Omega)$ of the net in (6.8).

Similarly, if $a \in \mathbb{R}$ is fixed and $F \in \mathbb{L}^\alpha(\Omega \times [a, \infty); (\mathcal{F}_x)_{x\in[a,\infty)})$, then the following net is Cauchy in $\Lambda^\alpha(\Omega)$,

$$\left(\int_a^b F\,dL \right)_{b\in[a,\infty)}.$$

Dually, if $b \in \mathbb{R}$ is fixed and $G \in \mathbb{L}^\alpha(\Omega \times (-\infty, b]; (\mathcal{F}_x)_{x\in(-\infty,b]})$, then the net

$$\left(\int_a^b G\,dL \right)_{a\in(-\infty,b]}$$

is Cauchy in $\Lambda^\alpha(\Omega)$ (with respect to the dual order $\ge$ on $(-\infty, b]$). Taking limit in $\Lambda^\alpha(\Omega)$ defines the integrals $\int_a^\infty F\,dL$ and $\int_{-\infty}^b G\,dL$ for $F \in \mathbb{L}^\alpha(\Omega \times [a, \infty); (\mathcal{F}_x)_{x\in[a,\infty)})$ and $G \in \mathbb{L}^\alpha(\Omega \times (-\infty, b]; (\mathcal{F}_x)_{x\in(-\infty,b]})$. From these definitions and Lemma 6.28, it becomes clear that for all $F \in \mathbb{L}^\alpha(\Omega \times \mathbb{R}; (\mathcal{F}_x)_{x\in\mathbb{R}})$ and all $a < b$, we have

$$\int_{\mathbb{R}} F\,dL = \int_{-\infty}^a F\,dL + \int_a^b F\,dL + \int_b^\infty F\,dL \qquad \text{almost surely.} \tag{6.9}$$

Finally, the uniformly bounded nature of the stochastic integral operators extends to the case of unbounded integrals.

Proposition 6.32. *There is a constant $C > 0$ such that*

1. *For all $a < b$ and all $F \in \mathbb{L}^\alpha(\Omega \times [a, b]; (\mathcal{F}_x)_{x\in[a,b]})$,*

$$\left\| \int_a^b F\,dL \right\|_{\Lambda^\alpha(\Omega)} \le C \|F\|_{\mathbb{L}^\alpha(\Omega\times[a,b])}.$$

2. *For all* $a \in \mathbb{R}$ *and all* $F \in \mathbb{L}^\alpha(\Omega \times [a,\infty); (\mathcal{F}_x)_{x\in[a,\infty)})$,

$$\left\| \int_a^\infty F\, dL \right\|_{\Lambda^\alpha(\Omega)} \leq C\|F\|_{\mathbb{L}^\alpha(\Omega\times[a,\infty))}.$$

3. *For all* $b \in \mathbb{R}$ *and all* $F \in \mathbb{L}^\alpha(\Omega \times (-\infty,b]; (\mathcal{F}_x)_{x\in(-\infty,b]})$,

$$\left\| \int_{-\infty}^b F\, dL \right\|_{\Lambda^\alpha(\Omega)} \leq C\|F\|_{\mathbb{L}^\alpha(\Omega\times(-\infty,b])}.$$

4. *For all* $F \in \mathbb{L}^\alpha(\Omega \times \mathbb{R}; (\mathcal{F}_x)_{x\in\mathbb{R}})$,

$$\left\| \int_{\mathbb{R}} F\, dL \right\|_{\Lambda^\alpha(\Omega)} \leq C\|F\|_{\mathbb{L}^\alpha(\Omega\times\mathbb{R})}.$$

Proof. It has already been established that there is a consant $C' > 0$ such that 1 holds. However, we cannot simply take limit $b \to \infty$ to obtain 2, $a \to -\infty$ to obtain 3 and limit along $\mathcal{I}$ to obtain 4: This would require continuity of the quasinorm $\|\cdot\|_{\Lambda^\alpha(\Omega)}$ which is not guaranteed. Instead we apply the Aoki-Rolewicz to pass through a p-norm $[\cdot]_{\Lambda^\alpha(\Omega)}$ on $\Lambda^\alpha(\Omega)$ such that there are constants $k > 0$ and $K > 0$ satisfying $k[X]_{\Lambda^\alpha(\Omega)} \leq \|X\|_{\Lambda^\alpha(\Omega)} \leq K[X]_{\Lambda^\alpha(\Omega)}$ for all $X \in \Lambda^\alpha(\Omega)$. Then, for all $a < b$ and all $F \in \mathbb{L}^\alpha(\Omega \times [a,b]; (\mathcal{F}_x)_{x\in[a,b]})$ it holds that

$$\left[\int_a^b F\, dL \right]_{\Lambda^\alpha(\Omega)} \leq C'k^{-1}\|F\|_{\mathbb{L}^\alpha(\Omega\times[a,b])}.$$

Because the p-norm $[\cdot]_{\Lambda^\alpha(\Omega)}$ is continuous with respect to the topology it gerenates, taking limits and using that $\|X\|_{\Lambda^\alpha(\Omega)} \leq K[X]_{\Lambda^\alpha(\Omega)}$ reveals that the statement holds with $C = C'Kk^{-1}$. □

6.4 Multifractional Stable Motion with Random Multifractional Parameter

In this section, an Itô multifractional stable motion with random multifractional parameter will be considered. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a standard symmetric α-stable Lévy process $(L(x))_{x\in\mathbb{R}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and we denote $(\mathcal{F}_x)_{x\in\mathbb{R}}$ for the natural filtration generated by $(L(x))_{x\in\mathbb{R}}$. Of the multifractional parameter $H : \Omega \times \mathbb{R} \to (0,1)$ we demand that it is jointly measurable, adapted to $(\mathcal{F}_x)_{x\in\mathbb{R}}$ and that it takes values in a compact subset $[\underline{H}, \overline{H}] \subseteq (0,1)$, where $\underline{H}$ and $\overline{H}$ are deterministic bounds, i.e. for all $\omega \in \Omega$ and all $x \in \mathbb{R}$ we have $\underline{H} \leq H(\omega,x) \leq \overline{H}$. Then, due to measurability of the map $\mathbb{R} \times \mathbb{R} \to \mathbb{R} : (u,a) \mapsto (u)_+^a$, it follows that the kernels $(\omega,x) \mapsto (t-x)_+^{H(\omega,x)-1/\alpha} - (-x)_+^{H(\omega,x)-1/\alpha}$ are jointly measurable and $(\mathcal{F}_x)_{x\in\mathbb{R}}$-adapted for any $t \in \mathbb{R}$. Moreover, since the bound from Lemma 6.17 is uniform in the functions $H : \mathbb{R} \to [\underline{H}, \overline{H}]$, we find that for all $\omega \in \Omega$,

$$\int_{\mathbb{R}} \left| (t-x)_+^{H(\omega,x)-\frac{1}{\alpha}} - (-x)_+^{H(\omega,x)-\frac{1}{\alpha}} \right|^\alpha dx \leq C(\alpha, t, \underline{H}, \overline{H}).$$

Taking expectation reveals that, for any $t \in \mathbb{R}$ the kernel $(\omega,x) \mapsto (t-x)_+^{H(\omega,x)-1/\alpha} - (-x)_+^{H(\omega,x)-1\alpha}$ is in $\mathbb{L}^\alpha(\Omega \times \mathbb{R}; (\mathcal{F}_x)_{x\in\mathbb{R}})$.

Definition 6.33. Let $H : \Omega \times \mathbb{R} \to [\underline{H}, \overline{H}] \subseteq (0,1)$ be jointly measurable and $(\mathcal{F}_x)_{x\in\mathbb{R}}$-adapted. The **Itô multifractional stable motion** with **random multifractional parameter** H is the stochastic process $(Y(t))_{t\in\mathbb{R}}$, given by

$$Y(t) = \int_{\mathbb{R}} (t-x)_+^{H(x)-\frac{1}{\alpha}} - (-x)_+^{H(x)-\frac{1}{\alpha}}\, dL(x).$$

Just as in the case of a deterministic multifractional parameter, the two main goals are to show that the Itô multifractional stable motion is localizable, and to find lower and upper bounds for the (pointwise) Hölder exponent. The structure of this section will be similar to Section 6.2: First, we obtain a lower bound for the uniform Hölder exponent over compacts by bounding difference moments and employing Theorem 3.10. Then, we wil prove that the Itô multifractional stable motion with random multifractional parameter is localizable and use this to find an upper bound for the pointwise Hölder exponent. Just as in the deterministic case, we will place an assumption of continuity on the random multifractional parameter $H : \Omega \times \mathbb{R} \to [\underline{H}, \overline{H}]$: It will be required that the modulus of continuity is deterministic and admitted uniformly in $\omega \in \Omega$.

Definition 6.34. Let $X : \Omega \times \mathbb{R} \to \mathbb{R}$ be a stochastic process and let $w : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a modulus of continuity. It will be said that X **deterministically admits** w **as a modulus of continuity** on $I \subseteq \mathbb{R}$ if, for all $s, t \in I$ and all $\omega \in \Omega$,

$$|X(\omega, t) - X(\omega, s)| \leq w(|t - s|).$$

Theorem 6.35. *Fix $S < T$ and let $(Y(t))_{t \in (S,T)}$ be the Itô multifractional stable motion with random multifractional parameter $H : \Omega \times \mathbb{R} \to [\underline{H}, \overline{H}] \subseteq (0,1)$. Suppose that H deterministically admits $w : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as a modulus of continuity on (S', T) for some $S' < S$. Moreover, suppose that there is a deterministic constant $\underline{H}^*_{(S,T)} > \frac{1}{\alpha}$ such that $H(\omega, t) \geq \underline{H}^*_{(S,T)}$ for all $\omega \in \Omega$ and $t \in (S, T)$. Then $(Y(t))_{t \in (S,T)}$ has a modification $(\tilde{Y}(t))_{t \in (S,T)}$ such that, with probability one, for all $\gamma > 0$ and all $S < a < b < T$,*

$$\sup_{\substack{s,t \in [a,b] \\ s \neq t}} \frac{|\tilde{Y}(t) - \tilde{Y}(s)|}{|t - s|^{\underline{H}_{[a,b]} - 1/\alpha - \gamma}} < \infty.$$

Proof. Choose $\epsilon \in (0, \frac{1}{2})$ small enough such that $(S - \epsilon, T) \subseteq (S', T)$ and $3w(\epsilon) < \underline{H}^*_{(S,T)} - \frac{1}{\alpha}$ and choose $\frac{1}{\underline{H}^*_{(S,t)} - 3w(\epsilon)} < p < \alpha$. The goal will be to show that for all $s, t \in (S, T)$ satisfying $|t - s| \leq \epsilon$, we have

$$\mathbb{E}\left|\frac{Y(t) - Y(s)}{|t - s|^{\underline{H}_{[s \wedge t, s \vee t]} - 3w(\epsilon) - \frac{1}{p}}}\right|^p = |t - s|^{3pw(\epsilon)+1}\,\mathbb{E}\left|\frac{Y(t) - Y(s)}{|t - s|^{\underline{H}_{[s \wedge t, s \vee t]}}}\right|^p \leq C(\alpha, \underline{H}, \overline{H}, p, \epsilon)|t - s|, \tag{6.10}$$

so that Theorem 3.10 applies. Note that the denominator is actually random now, meaning that we cannot simply take it out of the expectation like we did in the proof of Corollary 6.20. To account for this, we will use Equation (6.9) to decompose the difference process $Y(t) - Y(s)$ directly, instead of decomposing the norm of difference kernels. For $s, t \in (S, T)$ such that $s < t$ and $t - s \leq \epsilon$, decompose

$$Y(t) - Y(s) = \underbrace{\int_{-\infty}^{s-\epsilon} (t - x)^{H(x) - \frac{1}{\alpha}} - (s - x)^{H(x) - \frac{1}{\alpha}}\, dL(x)}_{D_\epsilon} + \underbrace{\int_{s-\epsilon}^{s} (t - x)^{H(x) - \frac{1}{\alpha}} - (s - x)^{H(x) - \frac{1}{\alpha}}\, dL(x)}_{E_\epsilon} + \underbrace{\int_{s}^{t} (t - x)^{H(x) - \frac{1}{\alpha}}\, dL(x)}_{F}.$$

Note that

$$\mathbb{E}\left|\frac{Y(t) - Y(s)}{(t - s)^{\underline{H}_{[s,t]}}}\right|^p \leq \mathbb{E}\left|\frac{Y(t) - Y(s)}{(t - s)^{H(s)}}\right|^p \leq 3^p\left(\mathbb{E}\left|\frac{D_\epsilon}{(t - s)^{H(s)}}\right|^p + \mathbb{E}\left|\frac{E_\epsilon}{(t - s)^{H(s)}}\right|^p + \mathbb{E}\left|\frac{F}{(t - s)^{H(s)}}\right|^p\right). \tag{6.11}$$

Thus, it suffices to bound the three terms separately. Write $C_1 > 0$ for the constant from Proposition 6.32 showing the integral operators against symmetric α-stable Lévy motion are uniformly bounded, and let $C_2(\alpha, p) > 0$ denote the constant from Lemma 2.18, so that $\|X\|_{\mathbb{L}^p(\Omega)} \leq C_2(\alpha, p)\|X\|_{\Lambda^\alpha(\Omega)}$ for all $X \in \Lambda^\alpha(\Omega)$. We have

$$\mathbb{E}\left|\frac{D_\epsilon}{(t - s)^{H(s)}}\right|^p \leq (t-s)^{-p\overline{H}}\mathbb{E}|D_\epsilon|^p \leq (C_1 C_2(\alpha, p))^p (t-s)^{-p\overline{H}}\left(\mathbb{E}\int_{-\infty}^{s-\epsilon}\left|(t - x)^{H(x) - \frac{1}{\alpha}} - (s - x)^{H(x) - \frac{1}{\alpha}}\right|^\alpha dx\right)^{p/\alpha}.$$

By applying (6.2) from Lemma 6.19, using that this bound is uniform in the functions $H : \mathbb{R} \to [\underline{H}, \overline{H}]$, it follows that for all $\omega \in \Omega$,

$$\int_{-\infty}^{s-\epsilon} \left|(t-x)^{H(\omega,x)-\frac{1}{\alpha}} - (s-x)^{H(\omega,x)-\frac{1}{\alpha}}\right|^{\alpha} dx \leq C(\alpha, \underline{H}, \overline{H})\epsilon^{\alpha(\underline{H}-1)}(t-s)^{\alpha}.$$

Taking expectation and combining with the previous bound yields that

$$\mathbb{E}\left|\frac{D_\epsilon}{(t-s)^{H(s)}}\right|^p \leq C(\alpha, \underline{H}, \overline{H}, p)\epsilon^{p(\underline{H}-1)}(t-s)^{p(1-\overline{H})} \leq C(\alpha, \underline{H}, \overline{H}, p, \epsilon)(t-s)^{-3pw(\epsilon)}. \tag{6.12}$$

For the second term, the term $(t-s)^{H(s)}$ in the denominator will be replaced by $(t-s)^{H(s-\epsilon)}$, making the factor measurable with respect to the lower integration bound so that the factor can be pulled inside of the integral. Use that $H(\omega, s) \leq H(\omega, s-\epsilon) + w(\epsilon)$ for all $\omega \in \Omega$ to conclude that

$$\mathbb{E}\left|\frac{E_\epsilon}{(t-s)^{H(s)}}\right|^p \leq (t-s)^{-pw(\epsilon)}\mathbb{E}\left|\frac{E_\epsilon}{(t-s)^{H(s-\epsilon)}}\right|^p.$$

Now, using Lemma 6.29 and the fact that H is adapted,

$$\frac{E_\epsilon}{(t-s)^{H(s-\epsilon)}} = \int_{s-\epsilon}^{s} (t-s)^{-H(s-\epsilon)} \left[(t-x)^{H(x)-\frac{1}{\alpha}} - (s-x)^{H(x)-\frac{1}{\alpha}}\right] dL(x).$$

Thus,

$$\mathbb{E}\left|\frac{E_\epsilon}{(t-s)^{H(s-\epsilon)}}\right|^p \leq (C_1C_2(\alpha,p))^p \left(\mathbb{E}\,(t-s)^{-\alpha H(s-\epsilon)} \int_{s-\epsilon}^{s} \left|(t-x)^{H(x)-\frac{1}{\alpha}} - (s-x)^{H(x)-\frac{1}{\alpha}}\right|^{\alpha} dx\right)^{p/\alpha}.$$

Applying (6.3) from Lemma 6.19 shows that for all $\omega \in \Omega$,

$$\begin{aligned}(t-s)^{-\alpha H(\omega,s-\epsilon)} \int_{s-\epsilon}^{s} \left|(t-x)^{H(\omega,x)-\frac{1}{\alpha}} - (s-x)^{H(\omega,x)-\frac{1}{\alpha}}\right|^{\alpha} dx &\leq C(\alpha, \underline{H}, \overline{H})(t-s)^{\alpha(H(\omega,s)-H(\omega,s-\epsilon)-w(\epsilon))} \\ &\leq C(\alpha, \underline{H}, \overline{H})(t-s)^{-2\alpha w(\epsilon)}.\end{aligned}$$

The upper bound is now determinstic so that we can take expectation. In conclusion,

$$\mathbb{E}\left|\frac{E_\epsilon}{(t-s)^{H(s)}}\right|^p \leq C(\alpha, \underline{H}, \overline{H}, p)(t-s)^{-3pw(\epsilon)}. \tag{6.13}$$

Finally, from Lemma 6.29 it follows that

$$\begin{aligned}\mathbb{E}\left|\frac{F}{(t-s)^{H(s)}}\right|^p &= \mathbb{E}\left|\int_s^t (t-s)^{-H(s)}\,(t-x)^{H(x)-\frac{1}{\alpha}}\,dL(x)\right|^p \\ &\leq (C_1C_2(\alpha,p))^p \left(\mathbb{E}(t-s)^{-\alpha H(s)} \int_s^t (t-x)^{\alpha H(x)-1}\,dx\right)^{p/\alpha}.\end{aligned}$$

Applying Lemma 6.19 one more time shows that for all $\omega \in \Omega$,

$$(t-s)^{-\alpha H(\omega,s)} \int_s^t (t-x)^{\alpha H(\omega,x)-1}\,dx \leq C(\alpha, \underline{H}, \overline{H})(t-s)^{-\alpha w(t-s)},$$

and from this it follows that

$$\mathbb{E}\left|\frac{F}{(t-s)^{H(s)}}\right|^p \leq C(\alpha,\underline{H},\overline{H},p)(t-s)^{-pw(t-s)} \leq C(\alpha,\underline{H},\overline{H},p)(t-s)^{-3pw(\epsilon)}. \tag{6.14}$$

Combining (6.11) with (6.12), (6.13) and (6.14) and using symmetry in s and t shows that (6.10) does in fact hold. Thus applying Theorem 3.10 shows that there are modifications $(\tilde{Y}_{\epsilon,p}(t))_{t\in(S,T)}$ such that, with probability one, for all $\gamma > 0$ and all $S < a < b < T$,

$$\sup_{\substack{s,t\in[a,b]\\ s\neq t}} \frac{|\tilde{Y}_{\epsilon,p}(t) - \tilde{Y}_{\epsilon,p}(s)|}{|t-s|^{\underline{H}_{[a,b]}-3w(\epsilon)-1/p-\gamma}} < \infty.$$

The result follows by letting $\epsilon \downarrow 0$ and $p \uparrow \alpha$. □

Proposition 6.36. *Let $(Y(t))_{t\in(S,T)}$ be the continuous modification of the Itô multifractional stable motion from the previous proposition. Moreover, assume that $w(h)\log(h) \to 0$ as $h \downarrow 0$. Fix $t \in (S,T)$ and $a < 0 < b$. Then there is a standard symmetric α-stable Lévy motion $(\tilde{L}(x))_{x\in\mathbb{R}}$, independent of $\mathcal{F}_{t-}$, such that, as $h \downarrow 0$:*

$$\left(\frac{Y(t+hr)-Y(t)}{h^{H(t)}}\right)_{r\in(a,b)} \to \left(\int_{\mathbb{R}} (r-x)_+^{H(t)-\frac{1}{\alpha}} - (-x)_+^{H(x)-\frac{1}{\alpha}}\, d\tilde{L}(x)\right)_{r\in(a,b)}.$$

The convergence is distributional in the space $C((a,b))$.

Proof. To show finite-dimensional convergence, let $r \in (a,b)$. First it will be established that, for $q \in (0,1)$ small enough,

$$h^{-H(t)}(Y(t+hr)-Y(t)) = h^{-H(t-h^q)}\int_{t-h^q}^{\infty} (t+hr-x)_+^{H(t-h^q)-\frac{1}{\alpha}} - (t-x)_+^{H(t-h^q)-\frac{1}{\alpha}}\, dL(x) + o_P(1). \tag{6.15}$$

The first step to establishing (6.15) is to show that

$$h^{-H(t)}\int_{-\infty}^{t-h^q} (t+hr-x)^{H(x)-\frac{1}{\alpha}} - (t-x)^{H(x)-\frac{1}{\alpha}}\, dL(x) \to 0 \quad \text{in probability as } h \downarrow 0. \tag{6.16}$$

Firstly note that

$$\begin{aligned}&\left\| h^{-H(t)}\int_{-\infty}^{t-h^q} (t+hr-x)^{H(x)-\frac{1}{\alpha}} - (t-x)^{H(x)-\frac{1}{\alpha}}\, dL(x)\right\|_{\Lambda^\alpha(\Omega)} \\ \leq& Ch^{-\overline{H}}\left(\mathbb{E}\int_{-\infty}^{t-h^q} \left|(t+hr-x)^{H(x)-\frac{1}{\alpha}} - (t-x)^{H(x)-\frac{1}{\alpha}}\right|^\alpha dx\right)^{\frac{1}{\alpha}}\end{aligned}$$

Assume that $r > 0$. Then, using $\epsilon = h^q$ in (6.2) from Lemma 6.19 shows that for all $\omega \in \Omega$,

$$\int_{-\infty}^{t-h^q} \left|(t+hr-x)^{H(\omega,x)-\frac{1}{\alpha}} - (t-x)^{H(\omega,x)-\frac{1}{\alpha}}\right|^\alpha dx \leq C(\alpha,\underline{H},\overline{H})h^{q\alpha(\underline{H}-1)}(hr)^\alpha \leq C(\alpha,\underline{H},\overline{H},r)h^{\alpha[q(\underline{H}-1)+1]}.$$

The upper bound here is deterministic, allowing us to take expectation and combine with the previous bound to conclude that

$$\left\| h^{-H(t)} \int_{-\infty}^{t-h^q} (t+hr-x)^{H(x)-\frac{1}{\alpha}} - (t-x)^{H(x)-\frac{1}{\alpha}} \, dL(x) \right\|_{\Lambda^\alpha(\Omega)} \leq C(\alpha, \underline{H}, \overline{H}, r) h^{1-\overline{H}-q(1-\underline{H})} \overset{h\downarrow 0}{\to} 0$$

whenever $q < \frac{1-\overline{H}}{1-\underline{H}}$. If $r < 0$ then we take $\epsilon = hr + h^q$ (under the assumption that h is close enough to zero so that this is positive) to conclude that for all $\omega \in \Omega$,

$$\begin{aligned}\int_{-\infty}^{t-h^q} \left|(t+hr-x)^{H(\omega,x)-\frac{1}{\alpha}} - (t-x)^{H(\omega,x)-\frac{1}{\alpha}}\right|^\alpha dx &= \int_{-\infty}^{t+hr-\epsilon} \left|(t-x)^{H(x)-\frac{1}{\alpha}} - (t+hr-x)^{H(x)-\frac{1}{\alpha}}\right|^\alpha dx \\ &\leq C(\alpha, \underline{H}, \overline{H})(hr+h^q)^{\alpha(\underline{H}-1)}(hr)^\alpha \\ &\leq C(\alpha, \underline{H}, \overline{H}, r) h^{\alpha[q(\underline{H}-1)+1]}(1+rh^{1-q})^{\alpha(\underline{H}-1)}.\end{aligned}$$

Again, under the assumption that $q < \frac{1-\overline{H}}{1-\underline{H}}$, it follows that

$$\left\| h^{-H(t)} \int_{-\infty}^{t-h^q} (t+hr-x)^{H(x)-\frac{1}{\alpha}} - (t-x)^{H(x)-\frac{1}{\alpha}} \, dL(x) \right\|_{\Lambda^\alpha(\Omega)} \leq C(\alpha, \underline{H}, \overline{H}, r) h^{1-\overline{H}-q(1-\underline{H})}(1+rh^{1-q})^{\alpha(\underline{H}-1)} \overset{h\downarrow 0}{\to} 0.$$

Since convergence in a weak Lebesgue space implies convergence in probability (see Lemma 4.12), (6.16) follows, thus

$$h^{-H(t)}(Y(t+hr) - Y(t)) = h^{-H(t)} \int_{t-h^q}^{\infty} (t+hr-x)_+^{H(x)-\frac{1}{\alpha}} - (t-x)_+^{H(x)-\frac{1}{\alpha}} \, dL(x) + o_P(1).$$

The second step is replacing $H(t)$ in the prefactor $h^{-H(t)}$ by $H(t-h^q)$. By the mean value theorem, for all $\omega \in \Omega$ there is some $\xi_{\omega,h}$ between $H(\omega,t)$ and $H(\omega, t-h^q)$, such that

$$|h^{-H(\omega,t)} - h^{-H(\omega,t-h^q)}| = |H(\omega,t) - H(\omega,t-h^q)| h^{-\xi_{\omega,h}} |\log h| \leq h^{-H(\omega,t-h^q)} w(h^q) |\log h| h^{-w(h^q)}.$$

Thus,

$$\begin{aligned}&\left\| \left(h^{-H(t)} - h^{-H(t-h^q)}\right) \int_{t-h^q}^{\infty} (t+hr-x)_+^{H(x)-\frac{1}{\alpha}} - (t-x)_+^{H(x)-\frac{1}{\alpha}} \, dL(x) \right\|_{\Lambda^\alpha(\Omega)} \\ &\leq w(h^q) |\log h| h^{-w(h^q)} \left\| h^{-H(t-h^q)} \int_{t-h^q}^{\infty} (t+hr-x)_+^{H(x)-\frac{1}{\alpha}} - (t-x)_+^{H(x)-\frac{1}{\alpha}} \, dL(x) \right\|_{\Lambda^\alpha(\Omega)}.\end{aligned}$$

Now, by (6.3) and (6.4) in Lemma 6.19, we have for all $\omega \in \Omega$,

$$\begin{aligned}h^{-\alpha H(\omega,t-h^q)} \int_{t-h^q}^{\infty} \left|(t+hr-x)_+^{H(\omega,x)-\frac{1}{\alpha}} - (t-x)_+^{H(\omega,x)-\frac{1}{\alpha}}\right|^\alpha dx &\leq C(\alpha, \underline{H}, \overline{H}) h^{-\alpha H(\omega,t-h^q)} (hr)^{\alpha(H(t\wedge(t+hr)) - w(h^q+h|r|))} \\ &\leq C(\alpha, \underline{H}, \overline{H}, r) h^{-2\alpha w(h^q+h|r|)}.\end{aligned}$$

Taking expectation and α'th root, and using Lemma 6.29 and Proposition 6.32, it follows that

$$\begin{aligned}&\left\| \left(h^{-H(t)} - h^{-H(t-h^q)}\right) \int_{t-h^q}^{\infty} (t+hr-x)_+^{H(x)-\frac{1}{\alpha}} - (t-x)_+^{H(x)-\frac{1}{\alpha}} \, dL(x) \right\|_{\Lambda^\alpha(\Omega)} \\ &\leq C(\alpha, \underline{H}, \overline{H}, r) w(h^q) |\log h| \, h^{-w(h^q)-2w(h^q+h|r|)} \overset{h\downarrow 0}{\to} 0.\end{aligned}$$

We conclude that

$$h^{-H(t)}(Y(t+hr)-Y(t)) = h^{-H(t-h^q)}\int_{t-h^q}^{\infty}(t+hr-x)_+^{H(x)-\frac{1}{\alpha}} - (t-x)_+^{H(x)-\frac{1}{\alpha}}\,dL(x) + o_P(1).$$

The third and final step is approximating $H(x)$ in the integrand by $H(t-h^q)$, making the integrand measurable with respect to $\mathcal{F}_{t-h^q}$. This will be done using Lemma 6.21, which shows that, for all $\omega\in\Omega$,

$$\begin{aligned}
&h^{-\alpha H(\omega,t-h^q)}\int_{t-h^q}^{(t+hr)\vee t}\Big|\Big((t+hr-x)_+^{H(\omega,x)-\frac{1}{\alpha}}-(t-x)_+^{H(\omega,x)-\frac{1}{\alpha}}\Big)-\\
&\qquad\Big((t+hr-x)_+^{H(\omega,t-h^q)-\frac{1}{\alpha}}-(t-x)_+^{H(\omega,t-h^q)-\frac{1}{\alpha}}\Big)\Big|^{\alpha}\,dx\\
=&h^{-\alpha H(\omega,t-h^q)}\int_{-h^q}^{hr\vee 0}\Big|\Big((hr-x)_+^{H(\omega,x+t)-\frac{1}{\alpha}}-(-x)_+^{H(\omega,x+t)-\frac{1}{\alpha}}\Big)-\\
&\qquad\Big((hr-x)_+^{H(\omega,t-h^q)-\frac{1}{\alpha}}-(-x)_+^{H(\omega,t-h^q)-\frac{1}{\alpha}}\Big)\Big|^{\alpha}\,dx\\
\le&C(\alpha,\underline{H},\overline{H})h^{-\alpha H(\omega,t-h^q)}\left(\sup_{x\in[t-h^q,(t+hr)\vee t]}|H(\omega,x)-H(\omega,t-h^q)|\right)^{\alpha}(hr)^{\alpha(H(\omega,t-h^q)-w(h^q+h|r|))}|\log(h|r|)|^{\alpha}\\
\le&C(\alpha,\underline{H},\overline{H},r)\,|w(h^q+h|r|)\log h|^{\alpha}\,h^{-\alpha w(h^q+h|r|)}.
\end{aligned}$$

The upper bound is deterministic, so taking expectation and α'th root, using Lemma 6.29 and Proposition 6.32, we find that

$$\begin{aligned}
&\Bigg\|h^{-H(t-h^q)}\Bigg(\int_{t-h^q}^{\infty}(t+hr-x)_+^{H(x)-\frac{1}{\alpha}}-(t-x)_+^{H(x)-\frac{1}{\alpha}}\,dL(x)-\\
&\qquad\int_{t-h^q}^{\infty}(t+hr-x)_+^{H(t-h^q)-\frac{1}{\alpha}}-(t-x)_+^{H(t-h^q)-\frac{1}{\alpha}}\,dL(x)\Bigg)\Bigg\|_{\Lambda^{\alpha}(\Omega)}\\
\le&\,C(\alpha,\underline{H},\overline{H},r)w(h^q+h|r|)|\log h|h^{-w(h^q+h|r|)}\overset{h\downarrow 0}{\to}0,
\end{aligned}$$

so that (6.15) holds. It follows that for $r_1\dots r_n\in(a,b)$,

$$\left(\frac{Y(t+hr_k)-Y(t)}{h^{H(t)}}\right)_{k=1\dots n}=\left(h^{-H(t-h^q)}\int_{t-h^q}^{\infty}(t+hr_k-x)_+^{H(t-h^q)}-\frac{1}{\alpha}-(t-x)_+^{H(t-h^q)-\frac{1}{\alpha}}\,dL(x)\right)_{k=1\dots n}+o_P(1).$$

Since the integrand on the right hand side is independent of $(L(x)-L(t-h^q))_{x\geq t-h^q}$, we may introduce a standard symmetric α-stable Lévy process $(\tilde{L}(x))_{x\in\mathbb{R}}$, independent of the processes L and H, such that

$$\begin{aligned}
&\left(h^{-H(t-h^q)}\int_{t-h^q}^{\infty}(t+hr_k-x)_+^{H(t-h^q)}-\frac{1}{\alpha}-(t-x)_+^{H(t-h^q)-\frac{1}{\alpha}}\,dL(x)\right)_{k=1\dots n}\\
\overset{d}{=}&\left(h^{-H(t-h^q)}\int_{t-h^q}^{\infty}(t+hr_k-x)_+^{H(t-h^q)}-\frac{1}{\alpha}-(t-x)_+^{H(t-h^q)-\frac{1}{\alpha}}\,d\tilde{L}(x)\right)_{k=1\dots n}
\end{aligned}$$

By applying the same steps as before, we find that

$$
\begin{aligned}
&\left(\frac{Y(t+hr_k)-Y(t)}{h^{H(t)}}\right)_{k=1\dots n} \\
&\overset{d}{=} \left(h^{-H(t-h^q)}\int_{t-h^q}^{\infty}(t+hr_k-x)_+^{H(t-h^q)-\frac{1}{\alpha}}-(t-x)_+^{H(t-h^q)-\frac{1}{\alpha}}\,d\tilde{L}(x)\right)_{k=1\dots n}+o_P(1) \\
&= \left(h^{-H(t-h^q)}\int_{\mathbb{R}}(t+hr_k-x)_+^{H(t-h^q)-\frac{1}{\alpha}}-(t-x)_+^{H(t-h^q)-\frac{1}{\alpha}}\,d\tilde{L}(x)\right)_{k=1\dots n}+o_P(1) \\
&\overset{d}{=} \left(\int_{\mathbb{R}}(r_k-x)_+^{H(t-h^q)-\frac{1}{\alpha}}-(-x)_+^{H(t-h^q)-\frac{1}{\alpha}}\,d\tilde{L}(x)\right)_{k=1\dots n}+o_P(1) \\
&= \left(\int_{\mathbb{R}}(r_k-x)_+^{H(t)-\frac{1}{\alpha}}-(-x)_+^{H(t)-\frac{1}{\alpha}}\,d\tilde{L}(x)\right)_{k=1\dots n}+o_P(1).
\end{aligned}
$$

In the third step we use that the α-stable Lévy motion is $\frac{1}{\alpha}$-self-similar and has stationary increments. This shows convergence in finite-dimensional distribution. To show that the convergence is functional, by Proposition 2.17, it suffices to show that for $\frac{1}{\alpha}<\frac{1}{p}<\frac{1}{p'}<\underline{H}^*_{(S,T)}$ there are constants $C>0$ and $\rho>1$, independent of h, s and r, such that for all $s,r\in(a,b)$,

$$
\mathbb{E}\left|\frac{Y(t+hr)-Y(t+hs)}{h^{H(t)}|r-s|^{\underline{H}^*_{(S,T)}-1/p'}}\right|^p \le C|r-s|^{\rho}.
$$

Assume that $0\le r-s\le 1$, then

$$
\mathbb{E}\left|\frac{Y(t+hr)-Y(t+hs)}{h^{H(t)}|r-s|^{\underline{H}^*_{(S,T)}-1/p'}}\right|^p \le |r-s|^{p/p'}h^{-pw(h(b-a))}\mathbb{E}\left|\frac{Y(t+hr)-Y(t+hs)}{|hr-hs|^{H(t+hs)}}\right|^p.
$$

Applying (6.12), (6.13) and (6.14) with $\epsilon=h^q$ for $q\in(0,1)$ reveals that

$$
\mathbb{E}\left|\frac{Y(t+hr)-Y(t+hs)}{|hr-hs|^{H(t+hs)}}\right| \le C(\alpha,\underline{H},\overline{H},p)\left[\underbrace{h^{qp(\underline{H}-1)}|hr-hs|^{p(1-\overline{H})}}_{h^{p(1-\overline{H}-q(1-\underline{H}))}|r-s|^{p(1-\underline{H})}}+|hr-hs|^{-3pw(h^q)}+|hr-hs|^{-pw(h(b-a))}\right].
$$

Now, for $q<\frac{1-\overline{H}}{1-\underline{H}}$, $h^{p(1-\overline{H}-q(1-\underline{H}))}\to 0$ as $h\downarrow 0$. Moreover, $h^{-pw(h(b-a))}\to 1$ and $h^{-3pw(h^q)}\to 1$. Thus, for $h\in(0,\delta)$ in a small enough neighborhood so that $3pw(\delta^q)\vee pw(\delta(b-a))<p/p'-1$, for all $r,s\in(a,b)$,

$$
\mathbb{E}\left|\frac{Y(t+hr)-Y(t+hs)}{h^{H(t)}|r-s|^{\underline{H}^*_{(S,T)}-1/p'}}\right|^p \le C(\alpha,\underline{H},\overline{H},p,\delta)|r-s|^{p/p'-(3pw(\delta^q)\vee pw(\delta(b-a)))}.
$$

□

Theorem 6.37. *Let $(Y(t))_{t\in(S,T)}$ be the continuous modification of the Itô multifractional stable motion with random multifractional parameter from Theorem 6.35. Then, with probability one, $\rho_Y^{unif}(t)\ge H(t)-\frac{1}{\alpha}$ for all $t\in(S,T)$. Moreover, if the deterministic modulus of continuity satisfies $w(h)\log h\to 0$ as $h\downarrow 0$, then for all $t\in(S,T)$, $\rho_Y(t)\le H(t)$ almost surely.*

Proof. These bounds for the pointwise Hölder exponents follow from Theorem 6.35 and Proposition 6.36 by following the same reasoning as in Theorem 6.23. □

We will compare these results to results on the classical multifractional stable motion first by Stoev and Taqqu [ST05] and later by Ayache and Hamonier [AH14]. Let Y denote (a continuous modification of) the classical multifractional stable motion and let Z denote (a continuous modification of) the Itô multifractional stable motion. Stoev and Taqqu had shown that

$$\left(\rho_H^{\mathrm{unif}}(t) \wedge H(t)\right) - \frac{1}{\alpha} \leq \rho_Y^{\mathrm{unif}}(t) \leq \rho_Y(t) \leq \rho_H(t) \wedge H(t),$$

under a Hölder condition on H (and of course $H(t) > \frac{1}{\alpha}$)). Compare this result to Theorem 6.37, which finds that

$$H(t) - \frac{1}{\alpha} \leq \rho_Z^{\mathrm{unif}}(t) \leq \rho_Z(t) \leq H(t),$$

under a weaker condition on the modulus of continuity of H (which is only needed for the upper bound). It should be noted that (perhaps unsurprisingly), these inequalities are quite similar. Moreover, both inequalities have a discrepancy of $\frac{1}{\alpha}$ between the lower bound and the upper bound, leaving both the pointwise Hölder exponent and the uniform pointwise Hölder exponent not completely specified. For the classical multifractional stable motion, this discrepancy is removed in [AH14]. Indeed: There it is shown that, under a Hölder condition on H, we have

$$\rho_Y^{\mathrm{unif}}(t) = H(t) - \frac{1}{\alpha} \qquad \rho_Y(t) = H(t).$$

This leads to the suspicion that $\rho_Z^{\mathrm{unif}}(t) = H(t) - \frac{1}{\alpha}$ and $\rho_Z(t) = H(t)$ for the Itô multifractional stable motion too, under a reasonable assumption on H. Of course it would be preferable to avoid placing assumptions on the Hölder regularity of H, since this was one of the main reasons that the Itô multifractional stable motion was introduced. Regretfully, we have not managed to obtain equality for the (uniform) pointwise Hölder exponent of the Itô multifractional stable motion. Ayache and Hamonier obtain the fine path properties of the classical multifractional stable motion by employing analytical wavelet techniques. In the Gaussian case, wavelet techniques have already been applied in the 'Itô' regime (where the multifractional parameter depends on the integration variable) [AEH18]. Wavelet methods may therefore be able to aid in finding equality for the (uniform) pointwise Hölder exponent of the Itô multifractional stable motion.

7 Conclusion

The aim of this thesis was to define a new (Itô) multifractional stable motion and to determine its pointwise Hölder regularity. This process is defined as

$$X(t) = \int_{\mathbb{R}} (t-x)_+^{H(x)-\frac{1}{\alpha}} - (-x)_+^{H(x)-\frac{1}{\alpha}} \, dL(x),$$

where L is a standard symmetric α-stable Lévy process, and $H : \mathbb{R} \times \Omega \to [\underline{H}, \overline{H}] \subseteq (0,1)$ is a jointly measurable stochastic process adapted to the natural filtration generated by L. Previously, researchers had considered a (classical) multifractional stable motion of the form

$$Y(t) = \int_{\mathbb{R}} (t-x)_+^{H(t)-\frac{1}{\alpha}} - (-x)_+^{H(t)-\frac{1}{\alpha}} \, dL(t),$$

where $H : \mathbb{R} \to (0,1)$ is a deterministic function. The two advantages of this new process over the previously defined multifractional stable process are:

- The kernels of the Itô multifractional stable motion are adapted to the natural filtration generated by L, so the process can be defined as an Itô integral. This is not the case for the classical multifractional stable motion.
- The Hölder regularity of the Itô multifractional stable motion is independent of the Hölder regularity of its fractional parameter. This is not the case for the classical multifractional stable motion.

Indeed: We have found that the uniform pointwise Hölder exponent of the Itô multifractional stable motion is at least $H(t) - \frac{1}{\alpha}$ if H admits *any* modulus of continuity w. If it holds that $w(h)log(h) \to 0$ as $h \downarrow 0$, then it is also true that the pointwise Hölder exponent is at most $H(t)$. Sadly we have not managed to prove an equality for these quantities, like for the classical multifractional stable motion under a Hölder condition on the multifractional parameter [AH14]. The wavelet analysis that these authors employ may be of use in proving an equality for the Hölder exponents of the Itô multifractional stable motion.

A topic that has been left outside of consideration in this thesis is the topic of multistable process, where the stability index α is also allowed to vary. Several formulations for multistable processes have been introduced [FGV09; FL12; GV12]. Multistable processes had been considered as a potential research direction for this project, but have been left out due to time constraints. However, using the framework of Falconer and Liu [FL12], it is possible to define an Itô multifractional multistable process with deterministic parameters. Using the norm inequalities proven by these authors it should be possible to formulate conditions on the multifractional and multistability parameters such that similar bounds on the pointwise Hölder exponents can be obtained. Modeling the multifractional and multistability parameters as random functions requires more machinery though. These could be potential avenues to explore for further research.

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