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**”Een reeks autoresonanties in een
versnellend liftkabelsysteem”**

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in an accelerating elevator cable system”)**

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A cascade of autoresonances in an accelerating elevator cable system

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Abstract

In this project the transversal vibrations of an accelerating elevator cable system are studied, with the aim to find the resonance times, the resonance duration and the resonance amplitude.

The elevator cable is modelled as an axially moving string, with length given by $l(t) = l_0 + \frac{1}{2}at^2$, with a the acceleration and t the time. The cable is sinusoidally excited at the top and fixed at the bottom. It is assumed that the axial acceleration is small compared to the transversal acceleration, that the cable mass is small compared to the car mass, and that the excitation amplitude is small compared to the length of the cable. Using these estimations, the solution for the transversal displacement u is approximated up to $\mathcal{O}(\varepsilon)$, with ε a small parameter.

The elevator cable goes through a cascade of autoresonances: the eigenfrequencies of the cable are varying because the cable length is varying, and at several times an eigenfrequency matches the excitation frequency. These are the resonance times, and they have been found as $t^+ = \sqrt{\frac{2}{\varepsilon a_1 l_0}} \arccos\left(\sqrt{\frac{\Omega l_0}{\chi_k}}\right)$, with t^+ a measure of oscillation of t , Ω the angular excitation frequency, l_0 the initial length, χ_k the eigenfrequency of mode k and $\varepsilon a_1 = a$. The duration of the resonances (the timescale) is shown to be $\mathcal{O}\left(\frac{1}{\sqrt[4]{\varepsilon}}\right)$ if $\chi_k \neq \Omega l_0$ and $\mathcal{O}\left(\frac{1}{\sqrt[6]{\varepsilon}}\right)$ if $\chi_k = \Omega l_0$ (a bifurcation of the problem). The amplitude scale is thus $\mathcal{O}\left(\sqrt[4]{\varepsilon^3}\right)$ or $\mathcal{O}\left(\sqrt[6]{\varepsilon^5}\right)$, respectively, and solutions for the amplitude are calculated both outside and inside the resonance zone.

1 Introduction

Many appliances and systems with axially moving elements can be modelled with the wave equation or the beam equation. A few of these are conveyor belts, transport cables and elevator cables. The last two are characterized by time-varying length, space-time-varying tension, and constant or time-varying axial velocity. These models are one-dimensional in space and thus depend only on one spatial coordinate, and on time.

This project is about elevator cable dynamics. There have been several researchers that studied models that can be used for elevator cables: Zhu and Ni [12] have investigated general stability characteristics of horizontally and vertically translating strings and beams, with arbitrary varying length and with various boundary conditions. Vibrations in strings can be longitudinal or transversal. Chi and Shu [2] have calculated the natural frequencies associated with the longitudinal vibration of a stationary elevator system. Zhu and Ren [13] have done this for transversal vibrations. Sandilo and van Horssen [7]-[10] have studied transversal vibrations in a string with time-varying length with constant and time-varying axial velocity. For the case $l(t) = l_0 + \varepsilon\beta_0 \sin(\omega t)$, it has been shown that there are infinitely many values of ω giving rise to internal resonances in the elevator system, and for the case $l(t) = l_0 + \varepsilon t$ it has been shown that a cascade of autoresonances with decreasing amplitudes occurs, on an unexpected timescale of order $\frac{1}{\sqrt{\varepsilon}}$.

In this project we investigate transversal vibrations in an elevator cable with time-varying velocity, so that the elevator car is uniformly accelerating. This situation corresponds to the start of every elevator ride and is thus of practical interest. In order to improve the design of elevators, it is important to develop a better understanding of elevator cable dynamics and to construct new methods to effectively reduce the vibrations, which can lead to wear on the cables, noise and transversal movement of the elevator car. The elevator system is modelled by a vertically hung string which is excited periodically at the upper end by a horizontal displacement of the building from its equilibrium due to the wind. Attached at the lower end of the cable is a rigid point mass m , the elevator car, which we assume to have rigid suspension against the guide rails. We also assume the transversal vibrations to be uncoupled from the longitudinal vibrations because the excitation is very small compared to the height of the building. The length of the cable will be given by $l(t) = l_0 + \frac{at^2}{2}$ and the external excitation at the top by $u(0, t) = \alpha \sin(\Omega t)$. In these formulae l_0 is the initial cable length, a the constant acceleration or deceleration, $u(x, t)$ the horizontal displacement of the cable with x the vertical space coordinate with its origin at the top, α is the amplitude of the excitation at the top, and Ω is its angular frequency. This excitation is modelled to be present only at the top of the building. Gaiko and van Horssen [4] have researched the case of constant velocity taking the effect of the excitation along the entire building into account.

It is assumed that the longitudinal acceleration is small compared to the transversal acceleration ($\ddot{l} \ll u_{tt} < g$), that the cable mass is small compared to the car mass ($\rho L \ll m$), and that the excitation amplitude is small compared to the length of the cable ($\alpha \ll L$). Here ρ is the cable mass density, m the car mass, g the gravitational acceleration and L the maximum length of the cable. The parameters which we found to be small are $\mathcal{O}(\varepsilon)$, with ε a small dimensionless parameter. The solution of the problem will then be approximated up to $\mathcal{O}(\varepsilon)$.

The length of the cable is constantly changing, and so are its natural frequencies. When a natural frequency mode of the cable vibrations gets near the excitation frequency, resonance occurs. A simple equation that describes resonance in general is $y''(t) + \omega^2 y(t) = \sin(f(t))$. The natural frequency is ω , and the excitation frequency is the first coefficient of the power series of $f(t)$. If they near each other the excitation term will be integrated to a term linear in t in the solution, thus going to infinity when $t \rightarrow \infty$. The amplitude thus grows significantly; for constant velocity it was shown in [9] that for an $\mathcal{O}(\varepsilon)$ excitation, the amplitude will grow with $\sqrt{\varepsilon}$ (note that $\sqrt{\varepsilon} > \varepsilon$ as $\varepsilon \ll 1$). After a certain time, the system gets out of the resonance zone, until the next frequency mode matches the excitation frequency and the process repeats itself. The cable is getting into and out of the resonance zone by its own movement, so it is called an autoresonance,

and an elevator cable is thus subdued to a cascade of autoresonances.

The goal of this project is to find the times at which the resonances occur, the amplitudes of the resonances and their duration (the timescale), in the case of varying velocity. We will follow an approach similar to that taken in [9]. This paper consists of the following: in section 2 the equation of motion for a vertically moving string is adapted for general time-varying length: it is made dimensionless, the inhomogeneous boundary condition corresponding to the excitation is made homogeneous, and the order of all the terms is estimated. In section 3 the above-mentioned equation for $l(t)$ is applied and the equation is adapted further: the time-varying domain is converted to a fixed domain, the functions are expanded in Fourier sine series, and a new time variable is introduced. In section 4 an interior layer analysis is applied to the secular terms of the resulting equation. The resonance times and duration are determined and the solution for this simplified equation is calculated outside and inside the resonance region. Finally, in section 5 the results are presented and in section 6 conclusions are drawn, the analysis is summarized, and recommendations are made for future work.

2 The governing equation for general transversal motion

We will first consider the general case of a vertically translating cable with an attached mass. The cable has density ρ [$\frac{\text{kg}}{\text{m}}$] and a time-varying length $l(t)$ [m]. The transversal and longitudinal motions of the cable are assumed to be uncoupled, and the latter are not considered in this project. We are interested in the transversal displacement of a cable particle instantaneously located at spatial position x at time t , with $0 \leq x \leq l(t)$. The equation of motion for an axially moving string with transversal vibrations u , having time-varying length and space-time-varying tension are obtained by modifying the standard wave equation:

$$u_{tt} - c^2 u_{xx} = u_{tt} - \frac{T}{\rho} u_{xx} = 0, \quad (1)$$

with T [N] the tension force and $c = \sqrt{\frac{T}{\rho}}$ [$\frac{\text{m}}{\text{s}}$] the wave speed. In the current situation the cable moves in the longitudinal direction, so instead of the variables x and t on their respective domains we are dealing with $x \in [0, l(t)]$ and t , so the domain of x depends on t . This means we need to replace u_{tt} in the following way:

$$u_{tt} \rightarrow u_{tt} + 2u_{xt}\dot{l} + u_x\ddot{l} + u_{xx}\dot{l}^2. \quad (2)$$

We can also write down the forces that constitute the tension force:

$$T u_{xx} \rightarrow \left((mg + \rho(l(t) - x)g - m\ddot{l} - \rho(l(t) - x)\dot{l}) u_x \right)_x \equiv (P(x, t)u_x)_x, \quad (3)$$

which are the axial force caused by the weight of the car and the cable, and the tension force caused by vertical acceleration of the car and the cable, respectively. Using these expressions, multiplying eq. (1) with ρ and adding the conditions we arrive at:

$$\rho(u_{tt} + 2\dot{l}u_{xt} + \ddot{l}u_x + \dot{l}^2 u_{xx}) - (P(x, t)u_x)_x = 0, \quad t > 0, \quad 0 < x < l(t), \quad (4)$$

$$u(0, t) = \alpha \sin(\Omega t), \quad u(l(t), t) = 0, \quad t > 0, \quad (5)$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = h(x), \quad 0 < x < l(0), \quad (6)$$

where α and Ω are the amplitude and frequency of the excitation at the upper end ($x = 0$), g is the gravitational acceleration, \dot{l} and \ddot{l} are the first and second derivative of the length of the cable, corresponding to the velocity and acceleration of the car, and $f(x)$ and $h(x)$ are the initial displacement and velocity, respectively. Before applying the needed transformations in order to handle the above-mentioned equation of motion, we put them into a non-dimensional form, by transforming all introduced parameters to dimensionless ones:

$$\begin{aligned} u^* &= \frac{u}{L}, & x^* &= \frac{x}{L}, & l^* &= \frac{l}{L}, & l_0^* &= \frac{l_0}{L}, & \alpha^* &= \frac{\alpha}{L}, & f^* &= \frac{f}{L}, \\ i^* &= \dot{l} \sqrt{\frac{\rho}{mg}}, & h^* &= h \sqrt{\frac{\rho}{mg}}, & u_t^* &= u_t \sqrt{\frac{\rho}{mg}}, \\ \ddot{l}^* &= \ddot{l} \frac{L\rho}{mg}, & u_{tt}^* &= u_{tt} \frac{L\rho}{mg}, \\ \Omega^* &= \Omega L \sqrt{\frac{\rho}{mg}}, & u_{xt}^* &= u_{xt} L \sqrt{\frac{\rho}{mg}}, \\ t^* &= \frac{t}{L} \sqrt{\frac{mg}{\rho}}, & \mu &= \frac{\rho L}{m}, & u_{xx}^* &= u_{xx} L, & u_x^* &= u_x, \end{aligned} \quad (7)$$

with L the maximum length of the string. We can now derive the equation of motion in non-dimensional form:

$$\begin{aligned} &\rho \left(\frac{mg}{L\rho} u_{tt}^* + 2i^* \sqrt{\frac{mg}{\rho}} u_{xt}^* \frac{1}{L} \sqrt{\frac{mg}{\rho}} + \ddot{l}^* \frac{mg}{L\rho} u_x^* + \dot{l}^{*2} \frac{mg}{\rho} u_{xx}^* \frac{1}{L} \right) \\ &- mg u_{xx}^* \frac{1}{L} - \rho(l^* L - x^* L) g u_{xx}^* \frac{1}{L} + m \ddot{l}^* \frac{mg}{L\rho} u_{xx}^* \frac{1}{L} + \rho(l^* L - x^* L) \dot{l}^* \frac{mg}{L\rho} u_{xx}^* \frac{1}{L} \\ &+ \rho g u_x^* - \rho \dot{l}^* \frac{mg}{L\rho} u_x^* = 0, \quad 0 < x < l(t), \quad t > 0, \end{aligned} \quad (8)$$

$$u^*(0, t)L = \alpha^* L \sin\left(\Omega^* \frac{1}{L} \sqrt{\frac{mg}{\rho}} t^* L \sqrt{\frac{\rho}{mg}}\right), \quad u^*(l(t), t)L = 0, \quad t^* > 0, \quad (9)$$

$$u^*(x, 0)L = f^*(x)L, \quad u_t^*(x, 0) \sqrt{\frac{mg}{\rho}} = h^*(x) \sqrt{\frac{mg}{\rho}}, \quad 0 < x^*L < l_0^*L, \quad (10)$$

which reduces to the following after dividing the equations by $\frac{mg}{L}$, L , L , L and $\sqrt{\frac{mg}{\rho}}$, respectively, removing the asterisks and rearranging:

$$u_{tt} - u_{xx} = -2\dot{l}u_{xt} - \left(\ddot{l} + \frac{\dot{l}}{\mu} + (\ddot{l} - \mu)(l(t) - x)\right)u_{xx} - \mu u_x, \quad 0 < x < l(t), \quad t > 0, \quad (11)$$

$$u(0, t) = \alpha \sin(\Omega t), \quad u(l(t), t) = 0, \quad t > 0, \quad (12)$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = h(x), \quad 0 < x < l_0, \quad (13)$$

One of the boundary conditions in eq. (12) is inhomogeneous. In order to solve the problem, we have to convert it into one with only homogeneous boundary conditions, using the following transformation:

$$u(x, t) = \bar{u}(x, t) + \left(1 - \frac{x}{l(t)}\right) \alpha \sin(\Omega t), \quad (14)$$

where $\bar{u}(x, t)$ satisfies the homogeneous boundary conditions. This yields:

$$\begin{aligned} \bar{u}_{tt} - \bar{u}_{xx} &= \left(\frac{2x\dot{l}^2}{l^3} - \frac{x\ddot{l}}{l^2} - \frac{2\dot{l}^2}{l^2} + \frac{\mu}{l}\right) \alpha \sin(\Omega t) + \left(-\frac{2x\dot{l}}{l^2} + \frac{2\dot{l}}{l}\right) \alpha \Omega \cos(\Omega t) \\ &- \left(1 - \frac{x}{l}\right) \alpha \Omega^2 \sin(\Omega t) - 2\dot{l}\bar{u}_{xt} - \left(\ddot{l} + \frac{\dot{l}}{\mu} + (\ddot{l} - \mu)(l - x)\right) \bar{u}_{xx} - \mu \bar{u}_x, \quad 0 < x < l(t), \quad t > 0 \end{aligned} \quad (15)$$

$$\bar{u}(0, t) = 0, \quad \bar{u}(l(t), t) = 0, \quad t > 0, \quad (16)$$

$$\bar{u}(x, 0) = f(x), \quad \bar{u}_t(x, 0) + \alpha \Omega \left(1 - \frac{x}{l_0}\right) = h(x), \quad 0 < x < l_0, \quad (17)$$

We will assume the following about the parameters \ddot{l} , \dot{l} , μ and α and the initial conditions $f(x)$ and $h(x)$: the longitudinal acceleration and velocity are small compared to the transversal acceleration and velocity ($\ddot{l} \ll u_{tt} < g, \dot{l} \ll u_t$), the cable mass is small compared to the car mass ($\rho L \ll m$), and the oscillation amplitude is small compared to the length of the cable ($\alpha \ll L$). These assumptions are supported in the Appendix by using realistic numbers in the equations. It is also assumed that both initial conditions are $\mathcal{O}(\varepsilon)$. For these reasons, we can write $\ddot{l} = \varepsilon \ddot{l}_1$, $\dot{l} = \varepsilon \dot{l}_1$, $\mu = \varepsilon \mu_1$, $\alpha = \varepsilon \alpha_1$, $f(x) = \varepsilon \bar{f}(x)$ and $h(x) = \varepsilon \bar{h}(x)$, which means that $\ddot{l}, \dot{l}, \mu, \alpha, f(x)$ and $h(x)$ are assumed to be $\mathcal{O}(\varepsilon)$ (and the newly introduced $\ddot{l}_1, \dot{l}_1, \mu_1, \alpha_1, \bar{f}(x)$ and $\bar{h}(x)$ are $\mathcal{O}(1)$). The new equation of motion for $\bar{u}(x, t; \varepsilon)$ is now:

$$\begin{aligned} \bar{u}_{tt} + \left(\frac{\ddot{l}_1}{\mu_1} - 1\right) \bar{u}_{xx} &= \varepsilon \left(-2\dot{l}_1 \bar{u}_{xt} - (\ddot{l}_1 - \mu_1)(l(t) - x) \bar{u}_{xx}\right. \\ &\left. - \mu_1 \bar{u}_x + \alpha_1 \Omega^2 \left(1 - \frac{x}{l(t)}\right) \sin(\Omega t)\right) + \mathcal{O}(\varepsilon^2), \quad 0 < x < l(t), \quad t > 0, \end{aligned} \quad (18)$$

$$\bar{u}(0, t; \varepsilon) = 0, \quad \bar{u}(l(t), t; \varepsilon) = 0, \quad t > 0, \quad (19)$$

$$\bar{u}(x, 0; \varepsilon) = \varepsilon \bar{f}(x) + \mathcal{O}(\varepsilon^2), \quad \bar{u}_t(x, 0; \varepsilon) = \varepsilon \left(\bar{h}(x) - \alpha_1 \Omega \left(1 - \frac{x}{l_0}\right)\right) + \mathcal{O}(\varepsilon^2), \quad 0 < x < l_0. \quad (20)$$

In the next section we will specify $l(t)$ and then proceed to adapt the equation further.

3 The model for an accelerating cable

In the preceding section, we have made the governing equation of motion dimensionless, removed the inhomogeneous boundary condition, and used assumptions to estimate all terms, yielding a general result for cables with time-varying length. In this project we examine the situation where the car is in uniformly accelerating motion, so:

$$l(t) = l_0 + \frac{at^2}{2}, \quad (21)$$

where $a = \varepsilon a_1 = \ddot{l} = \varepsilon \ddot{l}_1$ is the constant axial downward acceleration and $l_0 = \mathcal{O}(\varepsilon)$. The most general form for uniformly acceleration or deceleration is $l(t) = l_0 + v_0 t + \frac{at^2}{2}$. However, the extra term makes upcoming transformations much more difficult to handle analytically, so eq. (21) has been chosen to proceed with. Note that the problem of uniformly accelerating motion starting with any initial velocity can be reduced to a problem of the presented form by extrapolating back to the moment when the initial velocity was zero, and the same holds for decelerating motion. In this text we will work with accelerating motion, so we take $a > 0$.

This expression can be filled in in the equations in the place of l and its derivatives from now on, however, we will still use the latter in most places for the sake of brevity. Please note at this point that the index 0 is used for an initial situation, and the index 1 for the $\mathcal{O}(1)$ parameters as introduced in the preceding section. An increase in length x corresponds to extension of the cable (downward movement of the car) and a decrease in length to retraction (upward movement of the car).

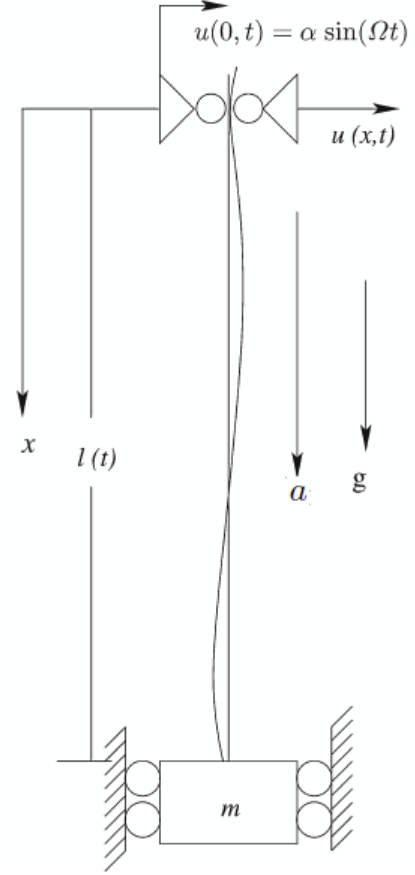


Figure 1: Sketch of the elevator shaft with the used parameters indicated. Modified after [9].

We can now proceed to modify the equation of motion using this specified length function: In the obtained equations the spatial domain is time-varying: $x \in [0, l(t)]$. We would like to convert this to a fixed domain, with a new independent non-dimensional spatial coordinate in the place of x : $\xi \in [0, 1]$. This is done with the transformation $\xi = \frac{x}{l(t)}$, so that the new variable for the transversal deviation is $\hat{u}(\xi, t; \varepsilon) = \bar{u}(x, t; \varepsilon)$. The derivatives of u also transform and are calculated using the rules of total derivatives:

$$\bar{u}_x = \frac{\partial \hat{u}}{\partial \xi} \frac{\partial \xi}{\partial x} = \frac{\hat{u}_\xi}{l(t)}, \quad (22)$$

$$\bar{u}_{xx} = \frac{\partial}{\partial \xi} \left(\frac{\hat{u}_\xi}{l(t)} \right) \frac{\partial \xi}{\partial x} = \frac{\hat{u}_{\xi\xi}}{l^2(t)}, \quad (23)$$

$$\bar{u}_t = \frac{\partial \hat{u}}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial \hat{u}}{\partial t} \frac{\partial t}{\partial t} = -\frac{\hat{u}_\xi \dot{l}(t) \xi}{l(t)} + \hat{u}_t, \quad (24)$$

$$\begin{aligned} \bar{u}_{tt} &= \frac{\partial}{\partial \xi} \left(-\frac{\hat{u}_\xi \dot{l}(t)\xi}{l(t)} + \hat{u}_t \right) \frac{\partial \xi}{\partial t} + \frac{\partial}{\partial t} \left(-\frac{\hat{u}_\xi \dot{l}(t)\xi}{l(t)} + \hat{u}_t \right) \frac{\partial t}{\partial t} = \left(-\frac{\hat{u}_{\xi\xi} \dot{l}(t)\xi}{l(t)} - \frac{\hat{u}_\xi \ddot{l}(t)}{l(t)} + \hat{u}_{t\xi} \right) \frac{-\xi \dot{l}(t)}{l(t)} + \\ &\left(-\frac{\hat{u}_{\xi t} \dot{l}(t)\xi}{l(t)} - \hat{u}_{\xi\xi} \frac{\partial}{\partial t} \left(\frac{\dot{l}(t)}{l(t)} \right) + \hat{u}_{tt} \right) = \frac{\hat{u}_{\xi\xi} \dot{l}^2(t)\xi^2}{l^2(t)} + 2\frac{\hat{u}_\xi \dot{l}^2(t)\xi}{l^2(t)} - 2\frac{\hat{u}_{\xi t} \dot{l}(t)\xi}{l(t)} - \frac{\hat{u}_\xi \ddot{l}\xi}{l(t)} + \hat{u}_{tt}, \end{aligned} \quad (25)$$

$$\bar{u}_{xt} = \frac{\partial}{\partial \xi} \left(\frac{\hat{u}_\xi}{l(t)} \right) \frac{\partial \xi}{\partial t} + \frac{\partial}{\partial t} \left(\frac{\hat{u}_\xi}{l(t)} \right) \frac{\partial t}{\partial t} = -\frac{\hat{u}_{\xi\xi} \dot{l}(t)\xi}{l^2(t)} + \frac{\hat{u}_{\xi t}}{l(t)} - \frac{\hat{u}_\xi \dot{l}(t)}{l^2(t)}. \quad (26)$$

Before we apply these transformations to the equation, we need to determine the timescale and with that the order of the factors $p \equiv \frac{\varepsilon t}{l(t)}$, $q \equiv \frac{\varepsilon t}{l^2(t)}$ and $r \equiv \frac{\varepsilon^2 t^2}{l^2(t)}$, which appear in the transformations when l , \dot{l} and \ddot{l} are filled in. To this end we determine the maximum in t by setting the derivatives to zero. For p and r , the maximum is $t_{max} = \sqrt{\frac{2l_0}{\varepsilon a_1}}$, and for q the maximum is $t_{max} = \sqrt{\frac{2l_0}{3\varepsilon a_1}}$. So we find $t = \mathcal{O}(\frac{1}{\sqrt{\varepsilon}})$, filling this in in p , q and r yields $p = \mathcal{O}(\sqrt{\varepsilon})$, $q = \mathcal{O}(\sqrt{\varepsilon})$ and $r = \mathcal{O}(\varepsilon)$.

Using the transformations and $t = \mathcal{O}(\frac{1}{\sqrt{\varepsilon}})$ (so that $\sqrt{\varepsilon}t = \mathcal{O}(1)$ and $\varepsilon t^2 = \mathcal{O}(1)$), eqs. (18)-(20) transform to:

$$\begin{aligned} \hat{u}_{tt} + \frac{a_1 - \mu_1}{\mu_1 l^2(t)} \hat{u}_{\xi\xi} &= \sqrt{\varepsilon} 2 \frac{\sqrt{\varepsilon} a_1 t (\xi - 2)}{l(t)} \hat{u}_{\xi t} + \varepsilon \left(\left(\frac{\varepsilon t^2 a_1^2 (2\xi - \xi^2)}{l^2(t)} - \frac{(a_1 - \mu_1)(1 - \xi)}{l(t)} \right) \hat{u}_{\xi\xi} \right. \\ &+ \left. \left(2 \frac{\varepsilon t^2 a_1^2 (1 - \xi)}{l^2(t)} + \frac{a_1 \xi - \mu_1}{l(t)} \right) \hat{u}_\xi + \alpha_1 \Omega^2 (1 - \xi) \sin(\Omega t) \right) + \mathcal{O}(\varepsilon^2), \quad 0 < \xi < 1, \quad t > 0, \end{aligned} \quad (27)$$

$$\hat{u}(0, t; \varepsilon) = 0, \quad \hat{u}(1, t; \varepsilon) = 0, \quad t > 0, \quad (28)$$

$$\hat{u}(\xi, 0; \varepsilon) = \varepsilon \hat{f}(\xi) + \mathcal{O}(\varepsilon^2), \quad \hat{u}_t(\xi, 0; \varepsilon) = \varepsilon \left(\hat{h}(\xi) - \alpha_1 \Omega (1 - \xi) \right) + \mathcal{O}(\varepsilon^2), \quad 0 < \xi < 1, \quad (29)$$

where $\hat{f}(\xi) = \bar{f}(x)$ and $\hat{h}(\xi) = \bar{h}(x)$ at $t = 0$. The boundary conditions are now ready to be satisfied, which is done by expanding u in a Fourier sine series in ξ : $\hat{u}(\xi, t; \varepsilon) = \sum_{n=1}^{\infty} u_n(t; \varepsilon) \sin(n\pi\xi)$. This yields the following for the differential equation and its initial condition:

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\ddot{u}_n(t; \varepsilon) - \frac{a_1 - \mu_1}{\mu_1} \left(\frac{n\pi}{l(t)} \right)^2 u_n(t; \varepsilon) \right) \sin(n\pi\xi) &= \sqrt{\varepsilon} \sum_{n=1}^{\infty} 2 \frac{\sqrt{\varepsilon} a_1 t (\xi - 1) n\pi}{l(t)} \dot{u}_n(t; \varepsilon) \cos(n\pi\xi) \\ &+ \varepsilon \sum_{n=1}^{\infty} \left(\left(\left(2 \frac{\varepsilon t^2 a_1^2 (1 - \xi) n\pi}{l^2(t)} + \frac{(a_1 \xi - \mu_1) n\pi}{l(t)} \right) \cos(n\pi\xi) \right. \right. \\ &+ \left. \left(\frac{(a_1 - \mu_1)(1 - \xi) n^2 \pi^2}{l(t)} - \frac{\varepsilon t^2 a_1^2 (2\xi - \xi^2) n^2 \pi^2}{l^2(t)} \right) \sin(n\pi\xi) \right) u_n(t; \varepsilon) \\ &+ \alpha_1 \Omega^2 (1 - \xi) \sin(\Omega t) \Big) + \mathcal{O}(\varepsilon^2), \quad 0 < \xi < 1, \quad t > 0, \end{aligned} \quad (30)$$

$$\sum_{n=1}^{\infty} u_n(0; \varepsilon) \sin(n\pi\xi) = \varepsilon \hat{f}(\xi) + \mathcal{O}(\varepsilon^2), \quad \sum_{n=1}^{\infty} \dot{u}_n(0; \varepsilon) \sin(n\pi\xi) = \varepsilon \hat{q}(\xi) + \mathcal{O}(\varepsilon^2), \quad 0 < \xi < 1, \quad (31)$$

where $\hat{q}(\xi) = \hat{h}(\xi) - \alpha_1 \Omega (1 - \xi)$. We can now remove the ξ -dependency from the equations, by multiplying them with $\sin(k\pi\xi)$, integrating from $\xi = 0$ to $\xi = 1$, using the orthogonality properties of the eigenfunctions,

and multiplying the equation by two. This turns the PDE into an ODE (one for every k):

$$\begin{aligned} \ddot{u}_k(t; \varepsilon) - \frac{a_1 - \mu_1}{\mu_1} \left(\frac{k\pi}{l(t)} \right)^2 u_k(t; \varepsilon) = \sqrt{\varepsilon} \left(-\frac{\sqrt{\varepsilon} a_1 t}{l(t)} \dot{u}_k(t; \varepsilon) + \sum_{n=1, n \neq k}^{\infty} 4 \frac{\sqrt{\varepsilon} a_1 t (f_1(n, k) - f_2(n, k)) n \pi}{l(t)} \dot{u}_n(t; \varepsilon) \right) \\ + \varepsilon \left[\left(-\frac{a_1}{2l(t)} + \left(\frac{a_1 - \mu_1}{2l(t)} - \frac{\varepsilon t^2 a_1^2}{l^2(t)} \left(\frac{2}{3} - \frac{1}{2k^2 \pi^2} \right) \right) k^2 \pi^2 \right) u_k(t; \varepsilon) + \frac{2\alpha_1 \Omega^2}{k\pi} \sin(\Omega t) \right. \\ \left. + \sum_{n=1, n \neq k}^{\infty} \left(n\pi \left(4 \frac{\varepsilon t^2 a_1^2 (f_2(n, k) - f_1(n, k))}{l^2(t)} + 2 \frac{a_1 f_1(n, k) - \mu_1 f_2(n, k)}{l(t)} \right) \right. \right. \\ \left. \left. - n^2 \pi^2 \left(\frac{2(a_1 - \mu_1) f_3(n, k)}{l(t)} + 2 \frac{\varepsilon t^2 a_1^2 (2f_3(n, k) - f_4(n, k))}{l^2(t)} \right) u_n(t; \varepsilon) \right) \right] + \mathcal{O}(\varepsilon^2), \quad t > 0, \end{aligned} \quad (32)$$

$$u_k(0; \varepsilon) = 2\varepsilon \hat{F}(k) + \mathcal{O}(\varepsilon^2), \quad \dot{u}_k(0; \varepsilon) = 2\varepsilon \hat{Q}(k) + \mathcal{O}(\varepsilon^2), \quad (33)$$

where

$$\begin{aligned} \hat{F}(k) &= \int_0^1 \hat{f}(\xi) \sin(k\pi\xi) d\xi, & \hat{Q}(k) &= \int_0^1 \hat{q}(\xi) \sin(k\pi\xi) d\xi, \\ f_1(n, k) &= \int_0^1 \xi \cos(n\pi\xi) \sin(k\pi\xi) d\xi, & f_2(n, k) &= \int_0^1 \cos(n\pi\xi) \sin(k\pi\xi) d\xi, \\ f_3(n, k) &= \int_0^1 \xi \sin(n\pi\xi) \sin(k\pi\xi) d\xi, & f_4(n, k) &= \int_0^1 \xi^2 \sin(n\pi\xi) \sin(k\pi\xi) d\xi. \end{aligned} \quad (34)$$

Note that we can expect oscillatory solutions if the prefactor of $u_k(t; \varepsilon)$ in the left-hand side of eq. (32) is a positive constant: using eq. (7) we have $\mu_1 = \frac{\rho L}{m\varepsilon}$ and $a_1 = \frac{\rho L a}{m g \varepsilon}$ with a the non-dimensionless acceleration. This yields $\frac{a_1}{\mu_1} = \frac{a}{g} < 1$ by assumption. So we have $-\frac{a_1 - \mu_1}{\mu_1} \left(\frac{k\pi}{l(t)} \right)^2 > 0$. To make this into a constant, we introduce a variable to replace t that is a measure of the period of oscillation: $t^+ = \int_0^t \frac{ds}{l(s)} = \int_0^t \frac{ds}{l_0 + \varepsilon a_1 s^2 / 2} = \sqrt{\frac{2}{\varepsilon a_1 l_0}} \arctan \left(\sqrt{\frac{\varepsilon a_1}{2 l_0}} t \right)$. t^+ is monotonically increasing with t . The inverse transformation is $t = \sqrt{\frac{2 l_0}{\varepsilon a_1}} \tan \left(\sqrt{\frac{\varepsilon a_1 l_0}{2}} t^+ \right)$. In the case that $a < 0$ one would obtain exponential and logarithmic functions for this transformation.

We can now substitute $u_k(t; \varepsilon) = \hat{u}_k(t^+; \varepsilon)$. The length function l is transformed as:

$$\begin{aligned} \hat{l}(t^+) &= l_0 + \frac{\varepsilon a_1 t^2}{2} = l_0 + \frac{\varepsilon a_1}{2} \frac{2 l_0}{\varepsilon a_1} \tan^2 \left(\sqrt{\frac{\varepsilon a_1 l_0}{2}} t^+ \right) \\ &= l_0 \left(1 + \tan^2 \left(\sqrt{\frac{\varepsilon a_1 l_0}{2}} t^+ \right) \right) = l_0 \sec^2 \left(\sqrt{\frac{\varepsilon a_1 l_0}{2}} t^+ \right). \end{aligned} \quad (35)$$

The time derivatives of u are transformed as:

$$\dot{u}_k = \frac{d\hat{u}_k}{dt^+} \frac{dt^+}{dt} = \frac{d\hat{u}_k}{dt^+} \frac{1}{l_0} \frac{1}{\frac{\varepsilon a_1}{2 l_0} t^2 + 1} = \frac{d\hat{u}_k}{dt^+} \frac{1}{\hat{l}(t^+)}, \quad (36)$$

$$\ddot{u}_k = \frac{d^2 \hat{u}_k}{dt^{+2}} \left(\frac{dt^+}{dt} \right)^2 + \frac{d\hat{u}_k}{dt^+} \frac{d^2 t^+}{dt^2} = \frac{d^2 \hat{u}_k}{dt^{+2}} \frac{1}{\hat{l}(t^+)^2} - \frac{d\hat{u}_k}{dt^+} \frac{\varepsilon a_1 t}{\hat{l}(t^+)^2}. \quad (37)$$

The new equation for $\hat{u}_k(t^+; \varepsilon)$, coming from eqs. (32) and (33) is, after multiplying by $l^2(t)$:

$$\begin{aligned}
\frac{d^2 \hat{u}_k}{dt^{+2}} + \frac{\mu_1 - a_1}{\mu_1} (k\pi)^2 \hat{u}_k &= \sqrt{\varepsilon} \sum_{n=1, n \neq k}^{\infty} 4 (\sqrt{\varepsilon} a_1 t (f_1(n, k) - f_2(n, k)) n \pi) \frac{d \hat{u}_n}{dt^+} + \varepsilon \left[\left(-\frac{a_1 \hat{l}(t^+)}{2} \right. \right. \\
&+ \left. \left(\frac{(a_1 - \mu_1) \hat{l}(t^+)}{2} - \varepsilon t^2 a_1^2 \left(\frac{2}{3} - \frac{1}{2k^2 \pi^2} \right) \right) k^2 \pi^2 \right) \hat{u}_k + \frac{2\alpha_1 \Omega^2 \hat{l}^2(t^+)}{k\pi} \sin(\Omega t) \\
&+ \sum_{n=1, n \neq k}^{\infty} \left(n\pi \left(4 (\varepsilon t^2 a_1^2 (f_2(n, k) - f_1(n, k))) + 2 (a_1 f_1(n, k) - \mu_1 f_2(n, k)) \hat{l}(t^+) \right) \right. \\
&\left. \left. - n^2 \pi^2 \left(2(a_1 - \mu_1) f_3(n, k) \hat{l}(t^+) + 2\varepsilon t^2 a_1^2 (2f_3(n, k) - f_4(n, k)) \right) \hat{u}_n \right) \right] + \mathcal{O}(\varepsilon^2),
\end{aligned} \tag{38}$$

$$\hat{u}_k(0; \varepsilon) = 2\varepsilon \hat{F}(k) + \mathcal{O}(\varepsilon^2), \quad \frac{d \hat{u}_k(0; \varepsilon)}{dt^+} = 2\varepsilon l_0 \hat{Q}(k) + \mathcal{O}(\varepsilon^2), \tag{39}$$

where t is not substituted everywhere in order to avoid even longer expressions. This is the final differential equation with initial conditions that needs to be solved to obtain the vibration amplitudes u . In the next section, an internal layer analysis will be performed on this equation to obtain the resonance times, the time scale, the resonance angle and a set of differential equations that defines an expression for u .

4 Interior layer analysis on the obtained ODE

In the last section we derived the ODE for u as a function of t^+ . We will now study the behaviour of the secular terms by averaging them. The secular terms of the equation are the terms that lead to a solution with terms that go to infinity when the time goes to infinity. These are the terms that can cause resonance. We will first determine which terms are secular and then apply this to eq. (38).

4.1 Secular terms

We can now take any term on the right hand side of eq. (38) separately and qualitatively (disregarding prefactors) determine to what solutions this leads. We will work out the case where \hat{u}_k itself is the term on the right hand side, and comment on the other cases. Since t^+ is the only variable left, derivatives can be written with accents again:

$$\hat{u}_k'' + \chi_k^2 \hat{u}_k = \varepsilon \hat{u}_k, \quad (40)$$

with $\chi_k = k\pi \sqrt{\frac{\mu_1 - a_1}{\mu_1}}$. If we expand the solution as $\hat{u}_k = \hat{u}_{k,0} + \varepsilon \hat{u}_{k,1} + \mathcal{O}(\varepsilon^2)$, eq. (40) becomes:

$$\hat{u}_{k,0}'' + \chi_k^2 \hat{u}_{k,0} = 0 \rightarrow \hat{u}_{k,0} = c_1 \cos(\chi_k t^+) + c_2 \sin(\chi_k t^+), \quad (41)$$

$$\hat{u}_{k,1}'' + \chi_k^2 \hat{u}_{k,1} = \hat{u}_{k,0} \rightarrow \hat{u}_{k,1} = c_3 \sin(\chi_k t^+) + c_4 \cos(\chi_k t^+) + c_5 \chi_k t^+ \sin(\chi_k t^+) + c_6 \chi_k t^+ \cos(\chi_k t^+). \quad (42)$$

The last two terms go to infinity when $t \rightarrow \infty$, so we see that \hat{u}_k is a secular term. The same can be derived for \hat{u}_k' and \hat{u}_k'' . This derivation depends on the fact that \hat{u}_k and its derivatives have the same eigenfrequency χ_k as is present in the left hand side. For this reason the summation terms $\sum_{n=1, n \neq k}^{\infty} \hat{u}_n$ and $\sum_{n=1, n \neq k}^{\infty} \hat{u}_n'$ are not secular terms, since all \hat{u}_n 's appear but \hat{u}_k . Lastly, the excitation term will also lead to resonance under certain conditions (which will be derived), so it belongs to the secular terms.

4.2 Interior layer analysis using the secular terms

The secular terms of eq. (38) are:

$$\begin{aligned} \hat{u}_k'' + \chi_k^2 \hat{u}_k = \varepsilon \left[\left(-\frac{a_1 \hat{l}(t^+)}{2} + \left(\frac{(a_1 - \mu_1) \hat{l}(t^+)}{2} \right. \right. \right. \\ \left. \left. \left. - \varepsilon t^2 a_1^2 \left(\frac{2}{3} - \frac{1}{2k^2 \pi^2} \right) \right) k^2 \pi^2 \right) \hat{u}_k + \frac{2\alpha_1 \Omega^2 \hat{l}^2(t^+)}{k\pi} \sin(\Omega t) \right] + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (43)$$

On this equation we will perform the averaging method. The first step is to find the form of the solution, and to apply variation of constants:

The homogeneous solution (for $\varepsilon = 0$) of eq. (43) can be written as:

$$\hat{u}_k(t^+) = A_k \cos(\chi_k t^+) + B_k \sin(\chi_k t^+), \quad (44)$$

for certain constants A_k and B_k . This yields the following for \hat{u}_k' :

$$\hat{u}_k'(t^+) = -\chi_k A_k \sin(\chi_k t^+) + \chi_k B_k \cos(\chi_k t^+). \quad (45)$$

Now we apply variation of constants: we assume that eq. (43) for nonzero ε is still given by eq. (44), but now with time-varying A_k and B_k . The problem of finding \hat{u}_k is thus changed into the problem of finding A_k and B_k , so an extra unknown has been introduced. This means we can impose a condition in addition to eq. (44), which will be: eq. (45) still holds for time-varying A_k and B_k . Comparing eq. (45) to the real \hat{u}_k' :

$$\hat{u}_k'(t^+) = -\chi_k A_k(t^+) \sin(\chi_k t^+) + \chi_k B_k(t^+) \cos(\chi_k t^+) + A_k'(t^+) \cos(\chi_k t^+) + B_k'(t^+) \sin(\chi_k t^+), \quad (46)$$

we find the following condition:

$$A'_k(t^+) \cos(\chi_k t^+) + B'_k(t^+) \sin(\chi_k t^+) = 0. \quad (47)$$

This condition will form a system of two equations together with the differential equation. In order to write the DE as function for A_k and B_k we still need the second derivative of \hat{u}_k , which we find by differentiating eq. (45):

$$\hat{u}_k''(t^+) = -\chi_k^2 A_k \cos(\chi_k t^+) - \chi_k^2 B_k \sin(\chi_k t^+) - \chi_k A'_k \sin(\chi_k t^+) + \chi_k B'_k \cos(\chi_k t^+). \quad (48)$$

So we find for the DE:

$$\begin{aligned} \chi_k A'_k \sin(\chi_k t^+) - \chi_k B'_k \cos(\chi_k t^+) = \varepsilon \left[\left(\frac{a_1 \hat{l}(t^+)}{2} - \left(\frac{(a_1 - \mu_1) \hat{l}(t^+)}{2} \right) \right. \right. \\ \left. \left. - \varepsilon t^2 a_1^2 \left(\frac{2}{3} - \frac{1}{2k^2 \pi^2} \right) \right) k^2 \pi^2 \right] (A_k \cos(\chi_k t^+) + B_k \sin(\chi_k t^+)) - \frac{2\alpha_1 \Omega^2 \hat{l}^2(t^+)}{k\pi} \sin(\Omega t). \end{aligned} \quad (49)$$

We can now form expressions for A'_k and B'_k from eqs. (47) and 49):

$$\begin{aligned} A'_k = \frac{1}{2\chi_k} \varepsilon \left[\left(\frac{a_1 \hat{l}(t^+)}{2} - \left(\frac{(a_1 - \mu_1) \hat{l}(t^+)}{2} \right) \right. \right. \\ \left. \left. - \varepsilon t^2 a_1^2 \left(\frac{2}{3} - \frac{1}{2k^2 \pi^2} \right) \right) k^2 \pi^2 \right] \\ \left(A_k \sin(2\chi_k t^+) + B_k (1 - \cos(2\chi_k t^+)) \right) - \frac{2\alpha_1 \Omega^2 \hat{l}^2(t^+)}{k\pi} \sin(\chi_k t^+) \sin(\Omega t), \end{aligned} \quad (50)$$

and

$$\begin{aligned} B'_k = -\frac{1}{2\chi_k} \varepsilon \left[\left(\frac{a_1 \hat{l}(t^+)}{2} - \left(\frac{(a_1 - \mu_1) \hat{l}(t^+)}{2} \right) \right. \right. \\ \left. \left. - \varepsilon t^2 a_1^2 \left(\frac{2}{3} - \frac{1}{2k^2 \pi^2} \right) \right) k^2 \pi^2 \right] \\ \left(A_k (\cos(2\chi_k t^+) + 1) + B_k \sin(2\chi_k t^+) \right) + \frac{2\alpha_1 \Omega^2 \hat{l}^2(t^+)}{k\pi} \cos(\chi_k t^+) \sin(\Omega t). \end{aligned} \quad (51)$$

We can now assess which terms in these equations will average out when an averaging method is applied (the fast-varying terms) and which will not (the slow-varying terms). We will examine all t^+ -dependent terms or factors. All but the last term consist of a multiplication of either $\hat{l}(t^+)$ or t^2 with either a sine, cosine or constant. We know that $\hat{l}(t^+) = l_0 \sec^2 \left(\sqrt{\frac{\varepsilon a_1 l_0}{2}} t^+ \right)$ and $t^2 = \frac{2l_0}{\varepsilon a_1} \tan^2 \left(\sqrt{\frac{\varepsilon a_1 l_0}{2}} t^+ \right)$. Both are slowly oscillating functions because the small $\sqrt{\varepsilon}$ makes the period large. Furthermore, the sine and cosine are fast-varying, and the constant is invariant. Thus the terms containing the sine or cosine are fast-varying and the others are slowly-varying. We will now look at the last term:

Firstly, we rename some of the expressions appearing in the equations: $\tau = \sqrt{\frac{\varepsilon a_1 l_0}{2}} t^+$, $\phi = \chi_k t^+$ and $\psi = \Omega \sqrt{\frac{2l_0}{\varepsilon a_1}} \tan(\tau)$.

The derivatives of these are $\tau' = \sqrt{\frac{\varepsilon a_1 l_0}{2}}$, $\phi' = \chi_k$ and $\psi' = \Omega l_0 \sec^2(\tau)$ and we have $\tau(0) = \phi(0) = \psi(0) = 0$.

Using this and some trigonometrics in eqs. (50) and (51) we find the following five differential equations:

$$\begin{aligned}
A'_k &= \frac{1}{2\chi_k} \varepsilon \left[\left(\frac{a_1 \hat{l}(t^+)}{2} - \left(\frac{(a_1 - \mu_1) \hat{l}(t^+)}{2} - \varepsilon t^2 a_1^2 \left(\frac{2}{3} - \frac{1}{2k^2 \pi^2} \right) \right) k^2 \pi^2 \right) \right. \\
&\quad \left. (A_k \sin(2\chi_k t^+) + B_k (1 - \cos(2\chi_k t^+))) + \frac{\alpha_1 \Omega^2 \hat{l}^2(t^+)}{k\pi} (\cos(\phi + \psi) - \cos(\phi - \psi)) \right], \\
B'_k &= -\frac{1}{2\chi_k} \varepsilon \left[\left(\frac{a_1 \hat{l}(t^+)}{2} - \left(\frac{(a_1 - \mu_1) \hat{l}(t^+)}{2} - \varepsilon t^2 a_1^2 \left(\frac{2}{3} - \frac{1}{2k^2 \pi^2} \right) \right) k^2 \pi^2 \right) \right. \\
&\quad \left. (A_k (\cos(2\chi_k t^+) + 1) + B_k \sin(2\chi_k t^+)) + \frac{\alpha_1 \Omega^2 \hat{l}^2(t^+)}{k\pi} (\sin(\phi + \psi) - \sin(\phi - \psi)) \right]. \tag{52}
\end{aligned}$$

$$\begin{aligned}
\tau' &= \sqrt{\frac{\varepsilon a_1 l_0}{2}}, \quad \tau(0) = 0, \\
\phi' &= \chi_k, \quad \phi(0) = 0, \\
\psi' &= \Omega l_0 \sec^2(\tau), \quad \psi(0) = 0.
\end{aligned}$$

Resonance due to the last term in the first two equations can be expected when the argument of the sine and cosine is (approximately) constant, because then the term will integrate to a linearly increasing contribution. This happens when $\phi' + \psi' = 0$ or $\phi' - \psi' = 0$. Since we know that $\phi' > 0$ and $\psi' > 0$, the only case left is $\phi' - \psi' = 0 \iff \chi_k - \Omega l_0 \sec^2(\tau) = 0$. We know that $\tau, \chi_k, \Omega, l_0 > 0$ so this corresponds to $\sec(\tau) = \sqrt{\frac{\chi_k}{\Omega l_0}} \iff \tau = \arccos\left(\sqrt{\frac{\Omega l_0}{\chi_k}}\right)$. These are the resonance times. The further analysis of the equations above can be divided in the case inside the resonance zones and the case outside the resonance zones. We will both cases in the following subsections.

4.3 Outside the resonance zone

Outside the resonance zone the excitation term is averaged out. Averaging over ϕ and ψ yields the following for A'_k and B'_k :

$$\begin{aligned}
A'_k &= \frac{1}{2\chi_k} \varepsilon \left(\frac{a_1 \hat{l}(t^+)}{2} - \left(\frac{(a_1 - \mu_1) \hat{l}(t^+)}{2} - \varepsilon t^2 a_1^2 \left(\frac{2}{3} - \frac{1}{2k^2 \pi^2} \right) \right) k^2 \pi^2 \right) B_k, \\
B'_k &= -\frac{1}{2\chi_k} \varepsilon \left(\frac{a_1 \hat{l}(t^+)}{2} - \left(\frac{(a_1 - \mu_1) \hat{l}(t^+)}{2} - \varepsilon t^2 a_1^2 \left(\frac{2}{3} - \frac{1}{2k^2 \pi^2} \right) \right) k^2 \pi^2 \right) A_k. \tag{53}
\end{aligned}$$

If we now define $c_k(t^+) \equiv \frac{1}{2\chi_k} \varepsilon \left(\frac{a_1 \hat{l}(t^+)}{2} - \left(\frac{(a_1 - \mu_1) \hat{l}(t^+)}{2} - \varepsilon t^2 a_1^2 \left(\frac{2}{3} - \frac{1}{2k^2 \pi^2} \right) \right) k^2 \pi^2 \right)$, this can be rewritten as

$$\begin{aligned}
A'_k &= c_k(t^+) B_k, \\
B'_k &= -c_k(t^+) A_k. \tag{54}
\end{aligned}$$

In order to solve eq. (54) we need to remove the t^+ -dependency of the prefactors of A_k and B_k . This is done by introducing another time transformation, which transforms $A_k(t^+)$ to $\hat{A}_k(s)$ and $B_k(t^+)$ to $\hat{B}_k(s)$, defined as: $s = \int_0^{t^+} c_k(t^+) dt^+$. This leads to $\frac{dA_k}{dt^+} = \frac{d\hat{A}_k}{ds} \frac{ds}{dt^+} = \frac{d\hat{A}_k}{ds} \hat{c}_k(s) = \hat{c}_k(s) \hat{B}'_k(s)$. Then system (54) reduces to

$$\begin{aligned}
\hat{A}'_k &= \hat{B}_k, \\
\hat{B}'_k &= -\hat{A}_k. \tag{55}
\end{aligned}$$

The solution to this is found by differentiating and substituting the equations, and setting $t^+ = 0$ to find the constants. It is given by:

$$\begin{aligned} A_k(t^+) &= A_k(0) \cos(s) + B_k(0) \sin(s), \\ B_k(t^+) &= -A_k(0) \sin(s) + B_k(0) \cos(s). \end{aligned} \quad (56)$$

4.4 Inside the resonance zone

We will now focus on the resonance zone. We will use $\Phi = \phi - \psi$ for the resonance frequency determined in section 4.2, and rescale the zone with $\tau - \arccos\left(\sqrt{\frac{\Omega l_0}{\chi_k}}\right) = \delta(\varepsilon)\bar{\tau}$, where $\bar{\tau} = \mathcal{O}(1)$ and $\delta(\varepsilon)$ is the rescaling parameter. So we find $\Phi' = \phi' - \psi' = \chi_k - \Omega l_0 \sec^2(\tau) = \chi_k - \Omega l_0 \sec^2\left(\delta(\varepsilon)\bar{\tau} + \arccos\left(\sqrt{\frac{\Omega l_0}{\chi_k}}\right)\right)$.

The Taylor series of $\sec^2(x+b)$ around $x=0$ is $\sec^2(x+b) = \sec^2(b) + 2x \tan(b) \sec^2(b) + \mathcal{O}(x^2)$, so we can further rewrite this as:

$$\Phi' = \chi_k - \Omega l_0 \left(\frac{\chi_k}{\Omega l_0} + 2\delta(\varepsilon)\bar{\tau} \sqrt{\left(\frac{\chi_k}{\Omega l_0}\right)^3 - \left(\frac{\chi_k}{\Omega l_0}\right)^2} \right) = -2\delta(\varepsilon)\bar{\tau} \sqrt{\frac{\chi_k^3}{\Omega l_0} - \chi_k^2} + \mathcal{O}(\delta(\varepsilon)^2). \quad (57)$$

In the case that $\chi_k = \Omega l_0$ this term vanishes, so we have a bifurcation. We need to use the next term in the Taylor series, leading to:

$$\Phi' = -\chi_k \delta(\varepsilon)^2 + \mathcal{O}(\delta(\varepsilon)^3). \quad (58)$$

The system of five DE's (52) can be rewritten as a system of six DE's:

$$\begin{aligned} A'_k &= \frac{1}{2\chi_k} \varepsilon \left[\left(\frac{a_1 \hat{l}(t^+)}{2} - \left(\frac{(a_1 - \mu_1) \hat{l}(t^+)}{2} - \varepsilon t^2 a_1^2 \left(\frac{2}{3} - \frac{1}{2k^2 \pi^2} \right) \right) k^2 \pi^2 \right) \right. \\ &\quad \left. (A_k \sin(2\chi_k t^+) + B_k (1 - \cos(2\chi_k t^+))) + \frac{\alpha_1 \Omega^2 \hat{l}^2(t^+)}{k\pi} (\cos(\phi + \psi) - \cos(\Phi)) \right], \\ B'_k &= -\frac{1}{2\chi_k} \varepsilon \left[\left(\frac{a_1 \hat{l}(t^+)}{2} - \left(\frac{(a_1 - \mu_1) \hat{l}(t^+)}{2} - \varepsilon t^2 a_1^2 \left(\frac{2}{3} - \frac{1}{2k^2 \pi^2} \right) \right) k^2 \pi^2 \right) \right. \\ &\quad \left. (A_k (\cos(2\chi_k t^+) + 1) + B_k \sin(2\chi_k t^+)) + \frac{\alpha_1 \Omega^2 \hat{l}^2(t^+)}{k\pi} (\sin(\phi + \psi) - \sin(\Phi)) \right], \\ \bar{\tau}' &= \sqrt{\frac{\varepsilon a_1 l_0}{2}} \frac{1}{\delta(\varepsilon)}, \quad \bar{\tau}(0) = -\frac{\arccos\left(\sqrt{\frac{\Omega l_0}{\chi_k}}\right)}{\delta(\varepsilon)}, \\ \Phi' &= -2\delta(\varepsilon)\bar{\tau} \sqrt{\frac{\chi_k^3}{\Omega l_0} - \chi_k^2} + \mathcal{O}(\delta(\varepsilon)^2), \quad \Phi(0) = 0, \\ \phi' &= \chi_k, \quad \phi(0) = 0, \\ \psi' &= \chi_k + 2\delta(\varepsilon)\bar{\tau} \sqrt{\frac{\chi_k^3}{\Omega l_0} - \chi_k^2}, \quad \psi(0) = 0. \end{aligned} \quad (59)$$

The resonance region is where Φ varies slowly, so when $\Phi' \approx \bar{\tau}'$. Then averaging over ϕ and ψ will not remove the term with Φ , i.e. resonance occurs. Ignoring prefactors, this balancing condition can be written as $\frac{\sqrt{\varepsilon}}{\delta(\varepsilon)} = \delta(\varepsilon)$ so our rescaling parameter is $\delta(\varepsilon) = \sqrt[4]{\varepsilon}$. This leads to $\bar{\tau}' = \sqrt{\frac{\varepsilon a_1 l_0}{2}}$, $\Phi' = -2\sqrt[4]{\varepsilon} \bar{\tau} \sqrt{\frac{\chi_k^3}{\Omega l_0} - \chi_k^2} + \mathcal{O}(\sqrt{\varepsilon})$ and $\psi' = \chi_k + 2\sqrt[4]{\varepsilon} \bar{\tau} \sqrt{\frac{\chi_k^3}{\Omega l_0} - \chi_k^2}$. Going back to t^+ we find for Φ :

$\Phi' = -2 \left(\sqrt{\frac{\varepsilon a_1 l_0}{2}} t^+ - \arccos\left(\sqrt{\frac{\Omega l_0}{\chi_k}}\right) \right) \sqrt{\frac{\chi_k^3}{\Omega l_0} - \chi_k^2} + \mathcal{O}(\sqrt{\varepsilon})$. So our resonance angle, Φ itself, is:

$$\Phi = -2 \left(\sqrt{\frac{\varepsilon \alpha_1 l_0}{8}} t^{+2} - \arccos \left(\sqrt{\frac{\Omega l_0}{\chi_k}} t^+ \right) \right) \sqrt{\frac{\chi_k^3}{\Omega l_0} - \chi_k^2} + \mathcal{O}(\sqrt{\varepsilon}).$$

In the case that $\chi_k = \Omega l_0$ the balancing condition is $\frac{\sqrt{\varepsilon}}{\delta(\varepsilon)} = \delta^2(\varepsilon)$ so that $\delta(\varepsilon) = \sqrt[6]{\varepsilon}$, $\bar{\tau}' = \sqrt[3]{\varepsilon} \sqrt{\frac{\alpha_1 l_0}{2}}$, $\Phi' = -\sqrt[3]{\varepsilon} \chi_k + \mathcal{O}(\sqrt{\varepsilon})$ and $\Phi = -\sqrt[3]{\varepsilon} \chi_k t^+ + \mathcal{O}(\sqrt{\varepsilon})$.

Now that we know all about our resonance angle, we can integrate the equations for A'_k and B'_k in (59) to obtain A_k and B_k in the resonance region. Averaging over ϕ and ψ yields:

$$\begin{aligned} A'_k &= \frac{1}{2\chi_k} \varepsilon \left[\left(\frac{a_1 \hat{l}(t^+)}{2} - \left(\frac{(a_1 - \mu_1) \hat{l}(t^+)}{2} - \varepsilon t^2 a_1^2 \left(\frac{2}{3} - \frac{1}{2k^2 \pi^2} \right) \right) k^2 \pi^2 \right) B_k - \frac{\alpha_1 \Omega^2 \hat{l}^2(t^+)}{k\pi} \cos(\Phi) \right], \\ B'_k &= -\frac{1}{2\chi_k} \varepsilon \left[\left(\frac{a_1 \hat{l}(t^+)}{2} - \left(\frac{(a_1 - \mu_1) \hat{l}(t^+)}{2} - \varepsilon t^2 a_1^2 \left(\frac{2}{3} - \frac{1}{2k^2 \pi^2} \right) \right) k^2 \pi^2 \right) A_k - \frac{\alpha_1 \Omega^2 \hat{l}^2(t^+)}{k\pi} \sin(\Phi) \right]. \end{aligned} \quad (60)$$

If we define $c_k(t^+)$ as in section 4.3, $a_k(t^+) \equiv \frac{1}{2\chi_k} \varepsilon \frac{\alpha_1 \Omega^2 \hat{l}^2(t^+)}{k\pi} \cos(\Phi)$ and $b_k(t^+) = \frac{1}{2\chi_k} \varepsilon \frac{\alpha_1 \Omega^2 \hat{l}^2(t^+)}{k\pi} \sin(\Phi)$, this reduces to

$$\begin{aligned} A'_k &= c_k(t^+) B_k - a_k(t^+), \\ B'_k &= -c_k(t^+) A_k + b_k(t^+). \end{aligned} \quad (61)$$

In order to solve eq. (61) we need to remove the t^+ -dependent prefactors of A_k and B_k again, so we need to use the transformation from t^+ to s again, as was introduced in section 4.3. The system reduces to:

$$\begin{aligned} \hat{A}'_k &= \hat{B}_k - \frac{\hat{a}_k}{\hat{c}_k}(s), \\ \hat{B}'_k &= -\hat{A}_k + \frac{\hat{b}_k}{\hat{c}_k}(s). \end{aligned} \quad (62)$$

This can be solved using the fundamental matrix method and variation of parameters, as can be found in section 3.12 of [1]. The solution of an initial-value-system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(s)$ with $\mathbf{x}(s_0) = \mathbf{x}^0$ is given by $\mathbf{x}(s) = \mathbf{X}(s)\mathbf{X}^{-1}(s_0)\mathbf{x}^0 + \mathbf{X}(s) \int_{s_0}^s \mathbf{X}^{-1}(s')\mathbf{f}(s')ds'$. If we now set $s_0 = 0$, $\mathbf{x}(s) = \begin{bmatrix} \hat{A}_k(s) \\ \hat{B}_k(s) \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

and $\mathbf{f}(s) = \begin{bmatrix} -\frac{\hat{a}_k}{\hat{c}_k}(s) \\ \frac{\hat{b}_k}{\hat{c}_k}(s) \end{bmatrix}$, then we have written system (62) in this form. It remains to determine $\mathbf{X}(s)$ and \mathbf{x}^0 .

Using eqs. (44) and (39) we find that $\mathbf{x}^0 = \mathbf{x}(0) = \begin{bmatrix} \hat{A}_k(0) \\ \hat{B}_k(0) \end{bmatrix} = \begin{bmatrix} \hat{u}_k(0) \\ \frac{\hat{u}_k}{\hat{d}i^+}(0) \end{bmatrix} = \begin{bmatrix} 2\varepsilon \hat{F}(k) \\ 2\varepsilon l_0 \hat{Q}(k) \end{bmatrix}$.

To find the fundamental matrix $\mathbf{X}(s)$ we need the solutions to the homogeneous problem and write them as $\mathbf{x}(s) = \mathbf{X}(s) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. These were calculated in section 4.3, and we find: $\mathbf{X}(s) = \begin{bmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{bmatrix}$.

The solution for $\mathbf{x}(s)$ is thus:

$$\begin{aligned} \mathbf{x}(s) &= \begin{bmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\varepsilon \hat{F}(k) \\ 2\varepsilon l_0 \hat{Q}(k) \end{bmatrix} + \begin{bmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{bmatrix} \int_0^s \begin{bmatrix} \cos(s') & -\sin(s') \\ \sin(s') & \cos(s') \end{bmatrix} \begin{bmatrix} -\frac{\hat{a}_k}{\hat{c}_k}(s') \\ \frac{\hat{b}_k}{\hat{c}_k}(s') \end{bmatrix} ds' \\ &= \begin{bmatrix} 2\varepsilon \hat{F}(k) \cos(s) + 2\varepsilon \hat{Q}(k) l_0 \sin(s) \\ -2\varepsilon \hat{F}(k) \sin(s) + 2\varepsilon \hat{Q}(k) l_0 \cos(s) \end{bmatrix} + \begin{bmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{bmatrix} \int_0^s \begin{bmatrix} -\frac{\hat{a}_k}{\hat{c}_k}(s') \cos(s') - \frac{\hat{b}_k}{\hat{c}_k}(s') \sin(s') \\ -\frac{\hat{a}_k}{\hat{c}_k}(s') \sin(s') + \frac{\hat{b}_k}{\hat{c}_k}(s') \cos(s') \end{bmatrix} ds'. \end{aligned} \quad (63)$$

Thus, we have found an expression for A_k and B_k .

5 Results

The goal of this project was to determine the resonance times, the resonance amplitudes and the resonance duration for a downwards accelerating elevator cable.

In section 4 we found the resonance times to be $\tau = \arccos\left(\sqrt{\frac{\Omega l_0}{\chi_k}}\right)$, which can be rewritten as

$t^+ = \sqrt{\frac{2}{\varepsilon a_1 l_0}} \arccos\left(\sqrt{\frac{\Omega l_0}{\chi_k}}\right)$. We see that any $k \in \mathbb{Z}^+$ such that $\chi_k > \Omega l_0$ will lead to resonance capture.

If we use the values of the parameters as stated in the Appendix, and set $L = 250$ m, we find that $\chi_k > \Omega l_0$ for $2 \leq k \in \mathbb{Z}$. A graph of the resonance times for $k = 2$ to $k = 10$ is presented in Figure 2.

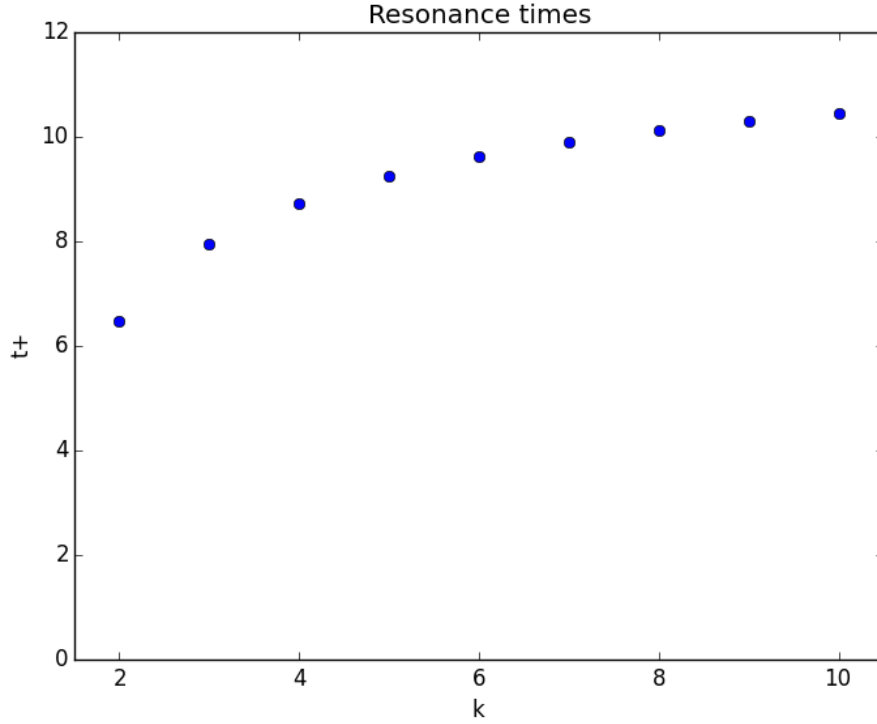


Figure 2: Graph of the first 9 resonance times of the elevator cable.

In the case where $\chi_k = \Omega l_0$ there is a resonance immediately, at $t^+ = 0$. Using the same parameters, we find that this occurs for $k = 1$ when $L = 243$ m.

From the interior layer analysis it can be gathered that for $\chi_k \neq \Omega l_0$ the duration of the resonances (timescale) is $\mathcal{O}(\varepsilon^{-\frac{1}{4}})$. Thus the resonance amplitude is $\varepsilon \cdot \mathcal{O}(\varepsilon^{-\frac{1}{4}}) = \mathcal{O}(\varepsilon^{\frac{3}{4}})$, since the excitation term is integrated to a term linear in t^+ , so that the amplitude is proportional to εt^+ . When $\chi_k = \Omega l_0$ we find $\mathcal{O}(\varepsilon^{-\frac{1}{6}})$ and $\mathcal{O}(\varepsilon^{\frac{5}{6}})$. In both cases the amplitude is larger than the original excitation of $\mathcal{O}(\varepsilon)$ because $\varepsilon < 1$.

The solutions to the system of ODE's for A_k and B_k have been calculated both inside and outside the resonance zone, but using only the secular terms. These solutions can then be used to find an approximation to the solution for u . Outside the resonance zone the solution for A_k and B_k is given by eq. (56), and inside the resonance zone it is given by (63). With these solutions one could find back the solution for u by using eq. (44), then sum all $u_n(t; \varepsilon)$'s in the Fourier sine series to obtain $\hat{u}(\xi, t; \varepsilon)$, and finally use eq. (7) to find u in meters.

6 Conclusions and recommendations

In this project the transversal vibrations of an elevator cable system have been studied. The system was modelled as an initial-boundary value problem for an axially accelerating string; the length of the cable is given by $l(t) = l_0 + \frac{1}{2}at^2$. The upper end of the string is excited sinusoidally, while the lower end is fixed. The length of the cable is changing and so are its eigenfrequencies. When one of the eigenfrequencies fits the excitation frequency the system goes through resonance. After a while the eigenfrequency again fits the excitation frequency, the process repeats itself, and thus we get a cascade of autoresonances. The goal of this project was to determine the resonance times, the resonance amplitudes and their duration (the timescale).

In section 2 the starting point was the general wave equation for the transversal displacement u , which was rewritten for an axially moving string, which applies to the situation of the moving elevator cable, and the boundary conditions and initial condition were added. The equation was made dimensionless through transformations using known parameters, before proceeding with a transformation that made the inhomogeneous boundary condition at $x = 0$ (the excitation) homogeneous. After that, several parameters were estimated to be $\mathcal{O}(\varepsilon)$, and after filling in these parameters the equation could be approximated to $\mathcal{O}(\varepsilon)$, and higher terms were disregarded.

In section 3 the general movement of section 2 was specified with a formula for $l(t)$. The domain of x was t -dependent; $x \in [0, l(t)]$, so x was replaced by $\xi \in [0, 1]$ through transformation, so that u could be expanded as a Fourier sine series. Multiplying the equation with $\sin(k\pi\xi)$ and integrating over the domain removed the ξ -dependency and yielded an ODE for u_k . The ODE could be expected to have oscillatory solutions when the prefactor of $u_k(t; \varepsilon)$ in the left-hand side was independent of t . In order to reach this, a transformation of t to t^+ was introduced, so that the prefactor of $\hat{u}_k(t^+; \varepsilon)$ in the new equation was independent of t^+ , leading to the oscillatory solutions.

In section 4 interior layer analysis has been applied on the secular terms of the ODE for t^+ : using the method of variation of parameters the problem of finding u_k was changed to finding A_k and B_k . After that it was assessed which terms were fast-varying and would average out when using the method of averaging. The excitation term was treated separately; a new time coordinate τ was introduced, as well as the numbers ϕ and ψ , which were used to find the resonance times, which are at $\tau = \arccos(\sqrt{\frac{\Omega l_0}{\chi_k}})$, or $t^+ = \sqrt{\frac{2}{\varepsilon a_1 l_0}} \arccos(\sqrt{\frac{\Omega l_0}{\chi_k}})$. Then the equations for A_k and B_k were ready to be solved both outside and inside the resonance zone. Inside the resonance zone this involved the explicit calculation of $\Phi = \phi - \psi$ and the balance condition $\Phi' = \tau'$. The solution was calculated using the fundamental matrix method. The balancing method yielded a timescale of $\mathcal{O}(\frac{1}{\sqrt[3]{\varepsilon}})$ for the case that $\chi_k \neq \Omega l_0$ and $\mathcal{O}(\frac{1}{\sqrt[5]{\varepsilon}})$ for $\chi_k = \Omega l_0$. The amplitude scales are thus $\mathcal{O}(\sqrt[4]{\varepsilon^3})$ and $\mathcal{O}(\sqrt[6]{\varepsilon^5})$, respectively. The actual expression for the amplitudes A_k and B_k outside and inside the resonance is given by eq. (56) and eq. (63), respectively.

The results in this text are obtained using only the secular terms of eq. (38). Future work could include the calculation of the solution by applying a perturbation method to eq. (38) as was done in [9] for constant velocity. Furthermore, the mathematical model can be improved by taking the excitation displacement into account over the entire cable length. This has been done in [4] for constant velocity. Lastly, one could combine the results for constant velocity and for varying velocity and analyse a full elevator trajectory; accelerating, moving with constant velocity, and decelerating. Elevator manufacturers can then use the results in their investigation of how to reduce the negative effects of the resonances.

Appendix

In this paper, we have assumed \ddot{i}, \dot{i}, μ and α to be $\mathcal{O}(\varepsilon)$. We will now calculate typical values of these parameters to support this claim. A ride with an elevator consist of an accelerating period, a period of constant maximum velocity, and a decelerating period. The acceleration was assumed to be constant and equal to the deceleration in absolute value. The standard acceleration for a comfortable elevator ride according to literature [6] is $4 \frac{ft}{s^2} = 1.22 \frac{m}{s^2}$, but the maximum velocity seems to get higher every time a taller skyscraper is built [3]. This is due to the fact that larger velocities do not make the ride less comfortable, and the taller the building, the longer the car can be accelerated. Using ordinary mechanics formulas on this, we find the following formula for \dot{i} :

$$\dot{i}_{max} = \ddot{i}t_{end} = \ddot{i}\sqrt{\frac{2L}{\ddot{i}}} = \sqrt{2\ddot{i}L}, \quad (64)$$

In order to calculate the other parameters, and their dimensionless values, we will need values for the constants m, ρ, g, α and Ω . Firstly, $g = 9.81 \frac{m}{s^2}$. The maximum load of an elevator car is mostly around 1000 kg, while the car itself weighs around 600 kg. We will use the average value, so $m = 1100$ kg. The elevator cables are virtually always made of steel ($\rho_{steel} = 7900 \frac{kg}{m^3}$), and have a typical diameter of half an inch [11], which sets $\rho = 1.001 \frac{kg}{m}$. Buildings are known to be swaying with periods of around 5 seconds, and the amplitude of the building sway is known to be in the order of decimetres, as seen for example in [5]. We will thus take $\Omega = 2\pi \cdot 0.2 \frac{rad}{s}$ and $\alpha = 1$ m. We can now calculate the dimensionless values of the parameters under consideration using the transformations of eq. 7:

$$\begin{aligned} \ddot{i}^* &= \ddot{i} \frac{L\rho}{mg} = 1.1 \cdot 10^{-4} L, \dot{i}^* = \dot{i} \sqrt{\frac{\rho}{mg}} = \sqrt{2\ddot{i}L} \sqrt{\frac{\rho}{mg}} = 0.015\sqrt{L}, \\ \mu &= \frac{\rho L}{m} = 9.1 \cdot 10^{-4} L, \alpha^* = \frac{\alpha}{L} = \frac{1}{L}, \Omega^* = \Omega L \sqrt{\frac{\rho}{mg}} = 0.012L \end{aligned} \quad (65)$$

We observe that for buildings of around a few hundred meters tall, the variables can be assumed to be $\mathcal{O}(\varepsilon)$.

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