Technische Universiteit Delft
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Calculating the reactive power using the Kramers-Kronig relations

(Nederlandse titel: Bepalen van het reactief vermogen met behulp van de Kramers-Kronig relaties)

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"Calculating the reactive power using the Kramers-Kronig relations" (Nederlandse titel: "Bepalen van het reactief vermogen met behulp van de Kramers-Kronig relaties")

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Abstract

There different methods to calculate the reactive power of a system. However, all of them are based on measuring values for one case and then calculating the reactive power for that single case. With the use of the Kramers-Kronig relations there's a way more efficient method to calculate the reactive power.

When actually knowing the active power formula of a system it would be possible to get the exact formula of the reactive power. A more interesting method is; to use a finite number of active power measurements to approximate the reactive power on the whole measurement domain. For these measurements you don't need advanced measurement equipment, just a simple watt meter.

This project is based on [1].

Contents

Chapter 1

Preface

Power is something that we use every day, but do we use it efficiently? Look for instance on the TU campus, where everybody is working on their laptop. You would assume that these laptops are all optimal in terms of power usages when charging, but are they really?

To be able to say something about the efficient or inefficient use of power, the power has to be measured. This can be done with different measurement equipment, in this bachelor thesis the focus will be on measuring using only a watt meter.

After taking these measurements there has to be a way to compare different systems. Therefore some different values for electrical systems will be defined in chapter 2. With the measurements taken with a watt meter not all those values can be determined. To be able to get more information out of these measurements the Kramers-Kronig relations will be used. These relations will be explained and proofed in chapter 3.

The numerical use of the Kramers-Kronig relations can be found first in chapter 4 and in a different way in chapter 5. To actually make calculations an example is needed and explained in chapter 6. The results of all calculations will be displayed in chapter 7 and the final conclusions in chapter 8.

Chapter 2

The power factor

Within an electrical system there are a few known values. The first is the voltage given by $\tilde{U}(t)$, this represents the difference in load between two points. The second is the current given by $\tilde{I}(t)$, which represents the current that runs through the system. Both the voltage and the current are periodic in respect to t, with a period of T.
With these two values we can define some different powers that then can be used to calculate the power factor. The power factor is a measure for the usefulness of the energy consumption.

The first power we define is the total energy that is supplied by the energy company, this is called the apparent power and noted by an S. It can be calculated by the following formula

$$S = \sqrt{\frac{1}{T}\int_0^T \tilde{U}^2(t)\ dt} \cdot \sqrt{\frac{1}{T}\int_0^T \tilde{I}^2(t)\ dt}.$$

Another power is that part of the energy that is actually used, which is called the active power and will be noted by a P. The formula that can be used to calculate the power is

$$P = \frac{1}{T}\int_0^T \tilde{U}(t) \cdot \tilde{I}(t)\ dt.$$

The final power that we will define is the energy that is lost because $\tilde{I}$ is perpendicular to $\tilde{U}$. This is called the reactive power and will be noted by a Q. Different from the other two this isn't a power that can be measured easily, but it is possible to calculate the power. To be able to define the power as an integral simular to the previous two powers, we will use the current shifted over $\frac{T}{4}$. This gives a 'current' perpendicular to the real current, with a rms-vallue equal to the current. This means that the reactive power can now be difined as

$$Q = \frac{1}{T}\int_0^T \tilde{U}(t) \cdot \tilde{I}\left(t - \frac{T}{4}\right)\ dt.$$

With the first two powers we can define the power factor, which is noted by λ and can be calculated by the following formula

$$\lambda = \frac{P}{S}.$$

What would be best is if all the energy that is supplied, is also used, that would indicate that for the power factor there would be $\lambda = 1$. In fact the power factor would always be lower than 1, we'll search for the reason that the power factor is lower than 1 and if possible how to increase the power factor.

2.1 Sinusoid case

When we look at the case where we have one sinusoid for the voltage, then the current is also given by one sinusoid with the same frequency. It is possible that the current is shifted with respect to the voltage, assume that this shift is of an angle ϕ. This means we can write

$$\begin{aligned}\tilde{U}(t) &= \sqrt{2}\hat{U} \cdot \sin(\omega t) \\ \tilde{I}(t) &= \sqrt{2}\hat{I} \cdot \sin(\omega t - \phi).\end{aligned}$$

Using that the different powers can be written as

$$\begin{aligned}P &= \hat{U}\hat{I} \cdot \cos(\phi) \\ Q &= \hat{U}\hat{I} \cdot \sin(\phi) \\ S &= \hat{U}\hat{I}.\end{aligned}$$

Which means that for the single sinusoid case the following equation is true

$$S^2 = P^2 + Q^2.$$

Then it can be seen easily that the way to optimize the power factor is to minimize the reactive power.

$$\lambda = \frac{P}{\sqrt{P^2 + Q^2}}$$

For the case where there are more than one sinusoid this isn't always true, because then S exists out of more elements than P and Q.

2.2 Ohm's law

Ohm's law gives a relationship between the voltage and the current. When $R(\omega)$ is the resistance of the system for voltage with a relative frequency of w , then the law can be written as

$$U(\omega) = R(\omega) \cdot I(\omega).$$

Because the voltage is the value that is the easiest manipulated (you can set the power supply to the voltage you want), we will use the form

$$\begin{aligned}I(\omega) &= \frac{1}{R(\omega)} \cdot U(\omega) \\ &= Y(\omega) \cdot U(\omega).\end{aligned}$$

This Y stands for the admittance of the system, it is a complex function that is dependent on the relative frequency ω of the voltage U.

2.3 Reactive power meter

There are different methods used by the current power meters, to calculate the reactive power. The two most used methods will be discussed:

- The first method to calculate the reactive power (and most easy method to implement) is by saying that $Q = \sqrt{S^2 - P^2}$. This would be correct if there was a pure sinusoid case, but when we have more than one sinusoid then $S^2 = P^2 + Q^2$ isn't true anymore. This also means that the definition for the apparent power this way is incorrect. It still does say something. Only now it is just a measure for the energy that is supplied, but isn't used.
- Another method is using the formulla $\tilde{I}(t + \frac{T}{4})$ within the definition of the reactive power. A problem when using this method is that it does use a delay, so it isn't a direct method.

Something both methods have in common is that they only calculate the reactive power for one frequency. When you need to know the reactive power for multiple frequencies you need to start from scratch every time, shouldn't it be possible to reuse the information you already have?

Chapter 3

Kramers-Kronig relations

The Kramers-Kronig relations give a way how the real and the imaginary part of a function are related to each other. Of course you can't use the relations for all function, there are a few conditions that have to be met.
When the relations are used on the function $X(\omega)$ which is the Fourier transform of the function $\tilde{X}(t)$, then the conditions can be written as:

- The function $X(\omega)$ has to be analytic in the right half plane.
- If $\|\omega\|$ goes to ∞ than it is needed that $X(\omega)$ goes to 0.

The second condition can also be relaxed, it is enough to have that when $\|\omega\|$ goes to ∞ the function $X(\omega)$ goes to A where A is a complex (finite) number, then we can use the function $X'(\omega) = X(\omega) - A$. This function $X'(\omega)$ does satisfy all the conditions of the Kramers-Kronig relations.

Another way to prove the Kramers-Kronig relations is by using that the function in the time domain, $\tilde{X}(t)$ is a causal function.

The relations date from 1926/1927. The first one to publish one of the relations is Kramers, in 1926. A year later, in 1927, published Kronig both of the relations. The two did know about each others work, but they didn't work on it together. [2]

The reason for finding the relations comes from the spectroscopy, they both found the relations in this research area. In proving the relations they used the specific properties for the spectroscopy instead of using a purely mathematical proof.

The most general form of the relations can be written in this way.

$$\mathrm{Re}(X(i\omega)) = \frac{-1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}\left(X(ix)\right)}{x-\omega}dx \tag{3.1}$$

$$\mathrm{Im}(X(i\omega)) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\left(X(ix)\right)}{x-\omega}dx \tag{3.2}$$

Most of the times the Kramers-Kronig relations are mentioned the sign of them is exactly opposite, this comes from one of the conditions. Because of the appllication in electrical systems we will use the version where $X(\omega)$ is analytical in the right half plane. This also coincides with the Kramer-Kronig relations that can be proofed out of causality. To get the relations with the opposite sign the analyticality would have to be in the upper half plane.

In this formula $\mathcal{P}$ gives that the Cauchy principle value[1] of should be taken of the integral. The equations are exactly that of a Hilbert transform pair[2], displayed by $\mathcal{H}$, so it could be written as.

$$\begin{aligned}\mathrm{Re}\left(X(i\omega)\right) &= -\mathcal{H}\left\{\mathrm{Im}\left(X(ix)\right)\right\}\\ \mathrm{Im}\left(X(i\omega)\right) &= \mathcal{H}\left\{\mathrm{Re}\left(X(ix)\right)\right\}\end{aligned}$$

3.1 Proof of the KK relations

There are multiple ways to prove the Kramers-Kronig relations, two will be discussed here. First a proof using causality, an intuitive proof, the second more abstract using complex analysis, specifically using the residue theorem.

3.1.1 From causality

Causality means that a reaction always follows the cause, the Kramers-Kronig relations can be proved by using this in the time domain. When we assume that the cause takes place at $t = 0$ then according to causality we need to have $\tilde{X}(t) = 0$ for $t < 0$. Every function can be split in an even and odd part, the even part is displayed by $\tilde{X}_e$ and the odd part is displayed by $\tilde{X}_o$.

$$\begin{aligned}\tilde{X}_e(-t) &= \tilde{X}_e(t)\\ \tilde{X}_o(-t) &= -\tilde{X}_o(t)\\ \tilde{X}(t) &= \tilde{X}_e(t) + \tilde{X}_o(t)\end{aligned}$$

Because of the causality it is also needed that $\tilde{X}(t) = 0$ for $t < 0$ which means that it is necessary that $\tilde{X}_o(t) = -\tilde{X}_e(t)$ for $t < 0$. The relation between $\tilde{X}_e$ and $\tilde{X}_o$ can be completed by saying $\tilde{X}_e(t) = sign(t) \cdot \tilde{X}_o(t)$, where $sign(t)$ is the sign function defined as $sign(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$. When using rule (B.2) from appendix B on this relation between $\tilde{X}_o(t)$ and $\tilde{X}_e(t)$ we get

$$\mathcal{F}(\tilde{X}_e(t)) = \mathcal{F}(sign(t) \cdot \tilde{X}_o(t)) = \mathcal{F}(sign(t)) \otimes \mathcal{F}(\tilde{X}_o(t)). \tag{3.3}$$

In appendix B there is an elaboration of the Fourier transform of the *sign* function.

$$\mathcal{F}(sign(t)) = \frac{1}{\pi i \omega}$$

With that the relation between $\tilde{X}_o(t)$ and $\tilde{X}_e(t)$ can be written as

$$\mathcal{F}(\tilde{X}_e(t)) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\mathcal{F}(\tilde{X}_o)(x)}{\omega - x}\, dx.$$

The Fourier transform of an odd function is pure imaginary and the Fourier transform of an even function is purely real, this means

$$\begin{aligned}X(i\omega) = \mathcal{F}(\tilde{X}(t)) &= \mathcal{F}(\tilde{X}_o(t)) + \mathcal{F}(\tilde{X}_e(t))\\ \mathrm{Re}(X(i\omega)) &= \mathcal{F}(\tilde{X}_e(t))\\ i \cdot \mathrm{Im}(X(i\omega)) &= \mathcal{F}(\tilde{X}_o(t)).\end{aligned}$$

[1] For the definition of the Cauchy principal value see section 3.1.2.
[2] For the definition of the Hilbert transform see the list of used symbols in appendix A.

That means that we can get

$$\begin{aligned}\mathrm{Re}(X(i\omega)) = \mathcal{F}(\tilde{X}_e(t)) &= \frac{1}{i\pi}\int_{-\infty}^{\infty}\frac{\mathcal{F}(\tilde{X}_o(t))}{\omega - x}\,dx \\ &= \frac{1}{i\pi}\int_{-\infty}^{\infty}\frac{i\cdot\mathrm{Im}(X(ix))}{\omega - x}\,dx = \frac{-1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Im}(X(ix))}{x-\omega}\,dx.\end{aligned}$$

This way we have proven the Kramers-Kronig relations with only one condition:

- The function in the time domain $\tilde{X}(t)$ is a causal function, then the Kramers-Kronig relations applies to the function in the frequency domain $X(i\omega)$.

3.1.2 From complex analysis

When proving the Kramers-Kronig relations from complex analysis, you use the residue theorem. To use this we take two conditions:

- The function $X(\omega)$ needs to be analytical on the right half plane.
- When $\|\omega\|$ goes to ∞ then it is needed that $X(\omega)$ goes to 0 as $\frac{1}{\|\omega\|}$ or faster.

We will use the residue theorem not on the function $X(i\omega)$ but on the function $\frac{X(x)}{x-i\omega}$ over the contour displayed in figure 3.1, going clockwise.

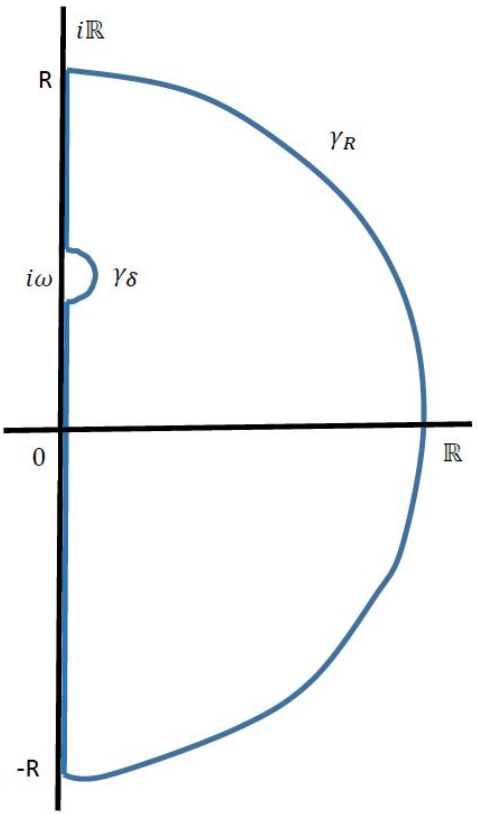

Figure 3.1: The contour used to prove the Kramers-Kronig relations

The integral can be split in different parts, the two half circles, γ_R and γ_δ, and the part over the imaginary axis $\gamma_{i\mathbb{R}}$. The outer contour γ_R doesn't contribute when R goes to ∞, because of the second condition. The function goes to 0 faster than the length of the half circle goes to ∞ when R goes to ∞.

The inner half circle does give a contribution to the value of the integral.

$$\begin{aligned}
\lim_{\delta\to 0}\int_{\gamma_\delta}\frac{X(x)}{x-\omega}dx &= \lim_{\delta\to 0}\int_{-\pi/2}^{\pi/2}\frac{X(i\omega+\delta e^{it})}{i\omega+\delta e^{it}-i\omega}i\delta e^{it}\ dt \\
&= \lim_{\delta\to 0}\int_{-\pi/2}^{\pi/2}\frac{i\delta e^{it}}{\delta e^{it}}X(i\omega+\delta e^{it})dt \\
&= \lim_{\delta\to 0}\int_{-\pi/2}^{\pi/2} iX(i\omega+\delta e^{it})dt \\
&= i\pi X(i\omega)
\end{aligned}$$

The integral over the imaginary axis can be written as

$$\begin{aligned}
\int_{\gamma_{i\mathbb{R}}}\frac{X(x)}{x-i\omega}\ dx &= \mathcal{P}\int_{-i\infty}^{i\infty}\frac{X(y)}{y-i\omega}\ dy \\
&= \mathcal{P}\int_{-\infty}^{\infty}\frac{X(ix)}{ix-i\omega}i\ dx \\
&= \mathcal{P}\int_{-\infty}^{\infty}\frac{X(ix)}{x-\omega}\ dx
\end{aligned}$$

in which the definition of the Cauchy principal value $\mathcal{P}$ can be found in section 3.1.2. Because there are no singularities within the contour, the total integral over the contour is 0.[3]

$$\begin{aligned}
\mathcal{P}\int_{-\infty}^{\infty}\frac{X(ix)}{x-\omega}dx + i\pi X(i\omega) &= 0 \\
\Rightarrow X(i\omega) &= \frac{-\mathcal{P}}{i\pi}\int_{-\infty}^{\infty}\frac{X(ix)}{x-\omega}dx \\
\mathrm{Re}\left(X(i\omega)\right) + i\cdot\mathrm{Im}\left(X(i\omega)\right) &= \frac{\mathcal{P}}{i\pi}\int_{-\infty}^{\infty}\frac{-\mathrm{Re}\left(X(i\omega)\right) - i\cdot\mathrm{Im}\left(X(i\omega)\right)}{x-\omega}dx \\
\mathrm{Re}\left(X(i\omega)\right) + i\cdot\mathrm{Im}\left(X(i\omega)\right) &= \frac{\mathcal{P}}{\pi}\int_{-\infty}^{\infty}\frac{i\cdot\mathrm{Re}\left(X(i\omega)\right) - \mathrm{Im}\left(X(i\omega)\right)}{x-\omega}dx
\end{aligned}$$

Two complex functions are equal to each other when both their real and imaginary part are equal. Hence, we get here

$$\begin{aligned}
\mathrm{Re}\left(X(i\omega)\right) &= \frac{-\mathcal{P}}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Im}\left(X(i\omega)\right)}{x-\omega}dx \\
\mathrm{Im}\left(X(i\omega)\right) &= \frac{\mathcal{P}}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\left(X(i\omega)\right)}{x-\omega}dx.
\end{aligned}$$

[3]For more information about the Residue theorem see appendix C

The Cauchy principal value

The Cauchy principal value gives a value to an integral that doesn't converge. That is done by avoiding the different singularities of a function. Suppose the function that is being integrated has a singularity at $x = \omega$, then the Cauchy principal value of the integral over $[-\infty, \infty]$ can be defined as.

$$\mathcal{P}\int_{-\infty}^{\infty} = \lim_{R\to\infty}\lim_{\delta\downarrow 0}\left\{\int_{-R}^{\omega-\delta} + \int_{\omega+\delta}^{R}\right\}$$

This gives exactly the form of the integral that we were left with in the residuem theorem.

When the integral over $[-\infty, \infty]$ does converge, it will take the same value as the Cauchy principal value of the integral.

$$\begin{aligned}\int_{-\infty}^{\infty} &= \lim_{R_1\to\infty}\lim_{R_2\to\infty}\int_{-R_1}^{R_2} &&= \lim_{R_1\to\infty}\lim_{R_2\to\infty}\int_{-R_1}^{R_2}\\ &= \lim_{R_1\to\infty}\lim_{R_2\to\infty}\lim_{\delta_1\downarrow 0}\lim_{\delta_2\downarrow 0}\left\{\int_{-R_1}^{\omega-\delta_1} + \int_{\omega+\delta_2}^{R_2}\right\}\\ &= \lim_{R\to\infty}\lim_{\delta\downarrow 0}\left\{\int_{-R}^{\omega-\delta} + \int_{\omega+\delta}^{R}\right\} &&= \mathcal{P}\int_{-\infty}^{\infty}\end{aligned}$$

3.2 Another form of the KK-relations

The version of the Kramers-Kronig that is now proved, is not really a useful one. The reason it is not that useful is because we have to know the function value for the negative frequencies as well. But naturally we won't have negative frequency, which means we don't know anything about that part of the function. Also taking a Cauchy principle value integral is not that easy if you don't know the exact function. It is an integral that cannot be calculated easily numerical. Thankfully the Kramers-Kronig relations can also be written in a different way, in which both problems are solved.

3.2.1 Using positive frequency

To lose the negative input we use the information that a natural response is always real. If the function in the time domain $\tilde{X}(t)$ is real, then for the Fourier transform of the function goes $X(-\omega) = \overline{X(\omega)}$, which means

$$\mathrm{Re}\left(X(-\omega)\right) + i\cdot\mathrm{Im}\left(X(-\omega)\right) = \mathrm{Re}\left(X(\omega)\right) - i\cdot\mathrm{Im}\left(X(\omega)\right).$$

This means that $\mathrm{Re}(X(\omega))$ is even and $\mathrm{Im}(X(\omega))$ is odd.

Using this, we can rewrite both relations starting with the relation for $\mathrm{Re}(X(i\omega))$.

$$\begin{aligned}
\mathrm{Re}\left(X(i\omega)\right) &= \frac{-1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}\left(X(ix)\right)}{x-\omega}\,dx \\
&= \frac{-1}{\pi}\mathcal{P}\int_{-\infty}^{0}\frac{\mathrm{Im}\left(Xi(x)\right)}{x-\omega}dx + \frac{-1}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\mathrm{Im}\left(X(ix)\right)}{x-\omega}dx \\
&= \frac{-1}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\mathrm{Im}\left(X(-ix)\right)}{-x-\omega}dx + \frac{-1}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\mathrm{Im}\left(X(ix)\right)}{x-\omega}dx \\
&= \frac{-1}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\mathrm{Im}\left(X(ix)\right)}{x+\omega}dx + \frac{-1}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\mathrm{Im}\left(X(ix)\right)}{x-\omega}dx \\
&= \frac{-1}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\mathrm{Im}\left(X(ix)\right)}{x+\omega} + \frac{\mathrm{Im}\left(X(ix)\right)}{x-\omega}dx \\
&= \frac{-1}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{(x-\omega+x+\omega)\cdot\mathrm{Im}\left(X(ix)\right)}{x^2-\omega^2}dx \\
&= \frac{-2}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{x\cdot\mathrm{Im}\left(X(ix)\right)}{x^2-\omega^2}dx
\end{aligned}$$

The same can be done with the relation for $\mathrm{Im}(X(i\omega))$.

$$\begin{aligned}
\mathrm{Im}\left(Y(i\omega)\right) &= \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\left(X(ix)\right)}{x-\omega}\,dx \\
&= \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{0}\frac{\mathrm{Re}\left(X(ix)\right)}{x-\omega}dx + \frac{1}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\mathrm{Re}\left(X(ix)\right)}{x-\omega}dx \\
&= \frac{1}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\mathrm{Re}\left(X(-ix)\right)}{-x-\omega}dx + \frac{1}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\mathrm{Re}\left(X(ix)\right)}{x-\omega}dx \\
&= \frac{1}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{-\mathrm{Re}\left(X(ix)\right)}{x+\omega}dx + \frac{1}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\mathrm{Re}\left(X(ix)\right)}{x-\omega}dx \\
&= \frac{1}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\mathrm{Re}\left(X(ix)\right)}{x-\omega} - \frac{Re\left(X(ix)\right)}{x+\omega}dx \\
&= \frac{1}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{(x+\omega-x+\omega)\cdot\mathrm{Re}\left(X(ix)\right)}{x^2-\omega^2}dx \\
&= \frac{2\cdot\omega}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\mathrm{Re}\left(X(ix)\right)}{x^2-\omega^2}dx
\end{aligned}$$

This means that there is another version of the Kramers-Kronig relations in which there is no use of the negative frequencies.

$$\begin{aligned}
\mathrm{Re}\left(X(i\omega)\right) &= \frac{-2}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{x\cdot\mathrm{Im}\left(X(ix)\right)}{x^2-\omega^2}dx \\
\mathrm{Im}\left(X(i\omega)\right) &= \frac{2\cdot\omega}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\mathrm{Re}\left(X(ix)\right)}{x^2-\omega^2}dx
\end{aligned}$$

3.2.2 To a normal integral

Even though we now have a integral that consists only of positive frequencies, it is still needed to take the Cauchy principal value. This means that we can't easily evaluate the integral numerically, the solution to this is to add an integral that is equal to 0.

$$\begin{aligned}
\mathcal{P}\int_0^\infty \frac{1}{x^2-\omega^2}\,dx &= \frac{\mathcal{P}}{2\omega}\int_0^\infty \frac{1}{x-\omega}-\frac{1}{x+\omega}\,dx \\
&= \frac{1}{2\omega}\lim_{h\downarrow 0}\left\{\int_0^{\omega-h}\frac{1}{x-\omega}-\frac{1}{x+\omega}\,dx+\int_{\omega+h}^\infty \frac{1}{x-\omega}-\frac{1}{x+\omega}\,dx\right\} \\
&= \frac{1}{2\omega}\lim_{h\downarrow 0}\left\{\Big[\ln|x-\omega|-\ln|x+\omega|\Big]_0^{\omega-h}+\Big[\ln|x-\omega|-\ln|x+\omega|\Big]_{\omega+h}^\infty\right\} \\
&= \frac{1}{2\omega}\lim_{h\downarrow 0}\Big[\ln|x-\omega|-\ln|x+\omega|\Big]_{\omega+h}^{\omega-h}+\frac{1}{2\omega}\Big[\ln|x-\omega|-\ln|x+\omega|\Big]_0^\infty \\
&= \frac{1}{2\omega}\Big[\ln|x-\omega|-\ln|x+\omega|\Big]_0^\infty \\
&= \frac{1}{2\omega}\left(\lim_{x\to\infty}\ln\left|\frac{x-\omega}{x+\omega}\right|-\Big(\ln|-\omega|-\ln|\omega|\Big)\right) \\
&= 0-0=0
\end{aligned}$$

If we multiply the integral by a constant, then it is still is equal to 0, so it is also true that

$$\begin{aligned}
\frac{-2}{\pi}\mathcal{P}\int_0^\infty \frac{-\omega\cdot\mathrm{Im}\left(X(i\omega)\right)}{x^2-\omega^2}dx &= 0 \\
\frac{2\cdot\omega}{\pi}\mathcal{P}\int_0^\infty \frac{-\mathrm{Re}\left(X(i\omega)\right)}{x^2-\omega^2}dx &= 0.
\end{aligned}$$

When that is added to the equations in the previous section we get the equations

$$\begin{aligned}
\mathrm{Re}\left(X(i\omega)\right) &= \frac{-2}{\pi}\mathcal{P}\int_0^\infty \frac{x\cdot\mathrm{Im}\left(X(ix)\right)-\omega\cdot\mathrm{Im}\left(X(i\omega)\right)}{x^2-\omega^2}dx \\
\mathrm{Im}\left(X(i\omega)\right) &= \frac{2\cdot\omega}{\pi}\mathcal{P}\int_0^\infty \frac{\mathrm{Re}\left(X(ix)\right)-\mathrm{Re}\left(X(i\omega)\right)}{x^2-\omega^2}dx.
\end{aligned}$$

The singularity at $x=\omega$ is no longer a pole, this is because with L'Hôpital's rule the value for x goed to ω can still be determined. By doing this the pole is changed into a removable singularity, what means that the Cauchy principal value will give the same result as a normal integral.

$$\mathrm{Re}\left(X(i\omega)\right) = \frac{-2}{\pi}\int_0^\infty \frac{x\cdot\mathrm{Im}\left(X(ix)\right)-\omega\cdot\mathrm{Im}\left(X(i\omega)\right)}{x^2-\omega^2}dx \tag{3.4}$$

$$\mathrm{Im}\left(X(i\omega)\right) = \frac{2\cdot\omega}{\pi}\int_0^\infty \frac{\mathrm{Re}\left(X(ix)\right)-\mathrm{Re}\left(X(i\omega)\right)}{x^2-\omega^2}dx \tag{3.5}$$

The equations (3.4) and (3.5) are written in a way that can be easily implemented in a numerical program, so this will be the relations that will be used in the rest of the project.

Chapter 4

Reactive power

The admittance is an imaginary function depending on the frequency of the voltage.

$$Y(i\omega) = G(\omega) + i \cdot B(\omega)$$

The real part of the admittance $G(\omega)$, is called the conductance and is easily measured by using basic measurement tools. The imaginary part of the admittance $B(\omega)$, the susceptance, on the other hand is not that easily measured. Therefore we will seek a method to get the susceptance by measuring only the conductance of the system.[1] For that we will use the Kramers-Kronig relations which have been explained in the previous chapter.
If we have found the admittance of the system then we can get the active and reactive power of the system by

$$P(\omega) = G(\omega) \cdot \|\tilde{u}(t)\|^2$$
$$Q(\omega) = B(\omega) \cdot \|\tilde{u}(t)\|^2.$$

Throughout this Bachelor Thesis the calculations will be done with the conductance and the susceptance, instead of using the active and reactive power. As shown above the only difference is a factor $\|\tilde{u}(t)\|^2$.

4.1 With the Kramers-Kronig relations

The admittance of a system is an analytical function on the right half plane that has a real limit for ω goes to ∞. That means that the function $Y^*(i\omega) = Y(i\omega) - Y(\infty) = Y(i\omega) - G(\infty)$ has a limit 0 for ω goes to ∞. This means that $Y^*(i\omega)$ satisfies the conditions of the Kramers-Kronig relations from the previous chapter. The only relation we will use is the one to calculate the imaginary part, because generally the real part is known and the imaginary part not.

$$\begin{aligned}
\mathrm{Im}(Y^*(i\omega)) &= \frac{2\omega}{\pi} \int_0^\infty \frac{\mathrm{Re}(Y^*(ix)) - \mathrm{Re}(Y^*(i\omega))}{x^2 - \omega^2} dx \\
&= \frac{2\omega}{\pi} \int_0^\infty \frac{(G(x) - G(\infty)) - (G(\omega) - G(\infty))}{x^2 - \omega^2} dx \\
B(\omega) &= \frac{2\omega}{\pi} \int_0^\infty \frac{G(x) - G(\omega)}{x^2 - \omega^2} dx
\end{aligned}$$

If we multiply this equation with $\|u(t)\|^2$ then we get the equation

$$Q(\omega) = \frac{2\omega}{\pi} \int_0^\infty \frac{P(x) - P(\omega)}{x^2 - \omega^2} dx.$$

This means that if we know the active power for every frequency, then we can determine the reactive power for every frequency.
But most of the time we only know the active power for a finite number of frequencies. Can we use these to make a good estimate of the reactive power?
Because we can only approach an integral over a finite domain with finite number of samples, we will restrict the domain to be $[0, b]$ instead of being $[0, \infty)$.

4.1.1 Trapezoidal rule

The simplest way to approximate a function is by linking the measured values with linear functions. Integrating this piecewise linear function gives a summation over the measured values. Because we have a continuous function, this method should ultimately come close to the true function values. This still is a method which has an error of order h^2 where h is the grid spacing between measure points, so it isn't that fast to converge.

4.1.2 Spline integration

A bit more complicated is to approximate a function by using splines, which are cubic polynomials. The polynomials are defined in such a way that the known function values are assumed, the sample points are also the transition points from one cubic polynomial to the other. The extra necessary conditions come from the fact that the first and second derivatives of the functions should be continuous on all points.

Then are still two extra conditions needed to be able to determine all the constants in the functions. The method used in the spline function integrated in matlab is the not-a-knot method. This means that on the first and last transition point we also demand that the third derivative is continuous. This means that the first two and last two polynomials are exactly the same. It seems that there is no point in between, so the name not-a-knot is born.

4.2 Fast Fourier Transform

When using this method, the original form of the Kramers-Kronig relations, the Hilbert transform, is used.[7] The Hilbert transform is defined as $\mathcal{H}(f)(t) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{t-x}\, dx$, this means that the susceptance can be written as

$$B(\omega) = -\mathcal{H}(G(\omega)). \tag{4.1}$$

In appendix B there is an elaboration of how to get

$$B(\omega) = \mathcal{F}^{-1}\left(i \cdot sign(\omega) \cdot \mathcal{F}(G(\omega))\right)$$

out of equation (4.1).

Fourier transforms are used more than Hilbert transforms, and there has been developed a method to approximate the Fourier transform of a sample. This method is called the Fast Fourier Transform (or fft) and has already been implemented in Matlab, more information can be found in appendix B.

The basic idea with using the fft method, is that you use the fft function on a sample instead of using the continuous Fourier transform on the function. When using this method you have to remember you are using this as a method to approximate an integral over $[-\infty, \infty]$, so it is best to take a sample centered around 0.

Chapter 5

Smaller interval

In the previous chapter it was always assumed that the conductance is known in a large enough area that starts at $\omega = 0$. But what can be done if that is not the case? If the conductance is only known on the area $[a, b]$ then that is not enough information most of the time to just get the susceptance. This is because around 0 there is a lot of information that is needed to use the Kramers-Kronig relations. How could we get the susceptance out of this information anyway?

5.1 Using functions

To get around the lack of information the best way is by taking a function format as a guess for the conductance outside the area $[a, b]$. For the conductance and susceptance it is best to use a logarithmic scale power function.[5] The way to get the constants used in the functions is done by forcing the function to comply with the Kramers-Kronig relations and the known values.

$$G(\omega) = \sum_{k=0}^{K} a_k^{(1)} \cdot (\log \omega)^k \qquad \text{for } \omega \neq [a, b]$$

$$B(\omega) = \sum_{m=0}^{M} a_m^{(2)} \cdot (\log \omega)^m \qquad \text{for } \omega \neq [a, b]$$

The Kramers-Kronig relations are used, but instead of taking the interval $[0, \infty)$ the interval $[c, b]$ is taken. There is known information on the interval $[a, b]$ and the functions defined above are taken on the interval $[c, a)$. The ω used is also taken within the interval $[c, a)$.

$$\begin{aligned}
B(\omega) =& \frac{2\omega}{\pi} \int_0^\infty \frac{G(x) - G(\omega)}{x^2 - \omega^2} dx \\
\approx& \frac{2\omega}{\pi} \int_c^b \frac{G(x) - G(\omega)}{x^2 - \omega^2} dx \\
=& \frac{2\omega}{\pi} \int_c^a \frac{G(x) - G(\omega)}{x^2 - \omega^2} dx + \frac{2\omega}{\pi} \int_a^b \frac{G(x) - G(\omega)}{x^2 - \omega^2} dx \\
\sum_{m=0}^{M} a_m^{(2)} \cdot (\log \omega)^m =& \frac{2\omega}{\pi} \int_c^a \frac{\sum_{k=0}^{K} a_k^{(1)} \cdot (\log x)^k - \sum_{k=0}^{K} a_k^{(1)} \cdot (\log \omega)^k}{x^2 - \omega^2} dx \\
&+ \frac{2\omega}{\pi} \int_a^b \frac{G(x) - \sum_{k=0}^{K} a_k^{(1)} \cdot (\log \omega)^k}{x^2 - \omega^2} dx
\end{aligned}$$

The different integrals are calculated in several ways. The one over the interval $[c, a)$ is approximated by a summation. The integral over $[a, b]$ can be split in two without getting problems, of the two integrals, one can be calculated precisely, the other can be calculated by using the conductance you know.

$$\begin{aligned}
\sum_{m=0}^{M} a_m^{(2)} \cdot (\log\omega)^m =& \frac{2\omega}{\pi} \sum_{k=0}^{K} a_k^{(1)} \int_c^a \frac{(\log x)^k - (\log\omega)^k}{x^2-\omega^2} dx + \frac{2\omega}{\pi} \int_a^b \frac{G(x)}{x^2-\omega^2} dx \\
&+ \frac{2\omega}{\pi} \sum_{k=0}^{K} a_k^{(1)} \cdot (\log\omega)^k \int_a^b \frac{1}{x^2-\omega^2} dx \\
\sum_{m=0}^{M} a_m^{(2)} \cdot (\log\omega)^m =& \frac{2\omega}{\pi} \sum_{k=0}^{K} a_k^{(1)} \sum_{i=1}^{N} \frac{(\log x_i)^k - (\log\omega)^k}{{x_i}^2-\omega^2} d_i + \frac{2\omega}{\pi} \int_a^b \frac{G(x)}{x^2-\omega^2} dx \\
&+ \frac{2\omega}{\pi} \sum_{k=0}^{K} a_k^{(1)} \cdot (\log\omega)^k \frac{1}{2\omega} \log\left(\frac{b-\omega}{b+\omega}\frac{a+\omega}{a-\omega}\right) \\
\sum_{m=0}^{M} a_m^{(2)} \cdot (\log\omega)^m + \sum_{k=0}^{K} a_k^{(1)} & \left\{ \frac{-2\omega}{\pi} \sum_{i=1}^{N} \frac{(\log x_i)^k - (\log\omega)^k}{{x_i}^2-\omega^2} d_i - \frac{(\log\omega)^k}{\pi} \log\left(\frac{b-\omega}{b+\omega}\frac{a+\omega}{a-\omega}\right) \right\} \\
&= \frac{2\omega}{\pi} \int_a^b \frac{G(x)}{x^2-\omega^2} dx \qquad (5.1)
\end{aligned}$$

Equation (5.1) now gives a linear equation with the unknown constants $a_m^{(2)}$ and $a_k^{(1)}$. If we have $K+M+2$ of these equations then we could calculate the $a_m^{(2)}$ and $a_k^{(1)}$ with those equations. That can be accomplished by taking as many different ω's within the interval $[c, a)$ as the number of equations needed.

5.1.1 Using extra information

When using this it is also possible to use some extra information, for instance you could measure the susceptance as well as the conductance. The measurements of the susceptance can be done anywhere within the interval $[c, b]$, these measurements would be the ω that the Kramers-Kronig relations are calculated for. By getting this extra information, only the formula

$$G(\omega) = \sum_{k=0}^{K} a_k^{(1)} \cdot (\log\omega)^k \qquad \text{for } \omega \neq [a, b]$$

is needed to know all the function values in the interval $[c, b]$.

This might give extra information, but if the information that is used is of a too small interval, then the resulting function for $G(\omega)$ will never come close to the real value.

5.2 Measuring the susceptance

Another option is not only taking measurements of the susceptance but even use this instead of the functions. The measurements can be in the interval $[a, b]$ like the conductance, but could also be measurements within the interval $[c, a]$ so that it supplements the information the conductance generates.

The best option is to measure within $[c, a]$, the extra information that is generated that way is important to get a good result. That is something that could be predicted. When there is only information used form a small interval, it is reasonable that the results aren't that good.

Chapter 6

An example

To compare the different methods, we will use an example of a network. The example that will be used it that of a RCL-network with R, C and L connected in a series. Within such a network there are three different electrical elements

- The R represents the resistor of the system, this element has an impedance of R.
- The C represents the capacitor of the system, this element has an impedance of $\frac{1}{i\omega C}$.
- The L represents the inductor of the system, this element has an impedance of $i\omega L$.

The impedance of a series connected system is true the sum of the different impedances. This gives a total impedance for the system of

$$Z(i\omega) = R + \frac{1}{i\omega C} + i\omega L = \frac{i\omega RC + 1 - \omega^2 LC}{i\omega C}.$$

Which means that the admittance is given by

$$\begin{aligned} Y(i\omega) &= \frac{1}{Z(i\omega)} \\ &= \frac{i\omega C}{i\omega RC + 1 - \omega^2 LC} \\ &= \frac{i\omega C}{1 - \omega^2 LC + i\omega RC} \frac{1 - \omega^2 LC - i\omega RC}{1 - \omega^2 LC - i\omega RC} \\ &= \frac{\omega^2 RC^2 + i\omega C - i\omega^3 LC^2}{1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2}. \end{aligned}$$

The admittance can be split in a conductance G and a susceptance B as follows

$$\begin{aligned} Y(\omega) &= G(\omega) + iB(\omega) \\ &= \frac{\omega^2 RC^2}{1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2} + i \frac{\omega C - \omega^3 LC^2}{1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2}. \end{aligned}$$

If you look at the Kramers-Kronig relations that should be valid, then it can be calculated[1] that these two functions for $B(\omega)$ and $G(\omega)$ do satisfy the relations.

[1] For instance with the use of Maple.

6.1 A real system

To actually use the example it is needed to know the values of the different values of the system, the system used for the different graphs is with the values:

L	20	mH
C	10	μF
R	1	Ω

The functions that will be calculated with look like:

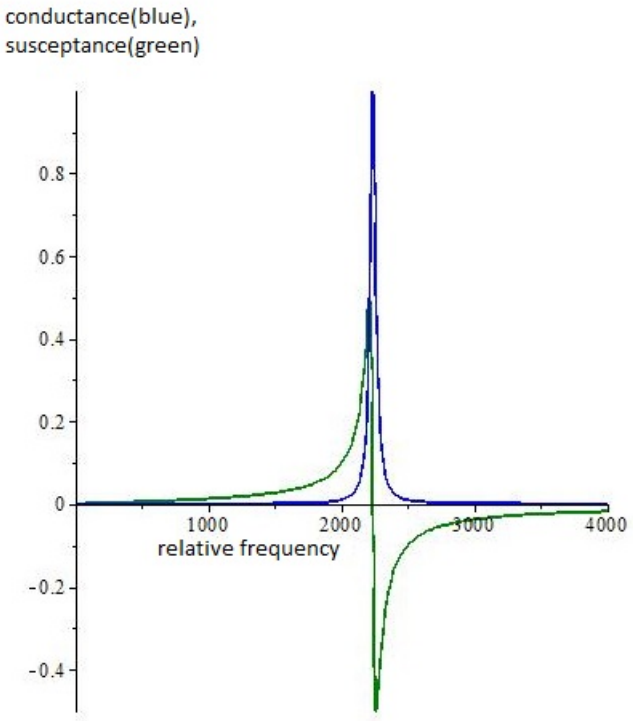

Figure 6.1: The conductance and susceptance for the example

Chapter 7

Results

In the previous chapters there have been defined several methods and they have been implemented in matlab, the codes can be found in appendix E. To compare the different methods the example of chapter 6 will be used. In the different figues the susceptance will be plotted. This figure would be the same as a figure of the the reactive power, when the volatage and current have a rms-value of 1.

Starting with the trapezium rule and spline integration, in figure 7.1 both are plotted along with the real value of the susceptance. These both work in a similar way and are plotted with the same nomber of measurement points taken. As can be expected the spline integration comes

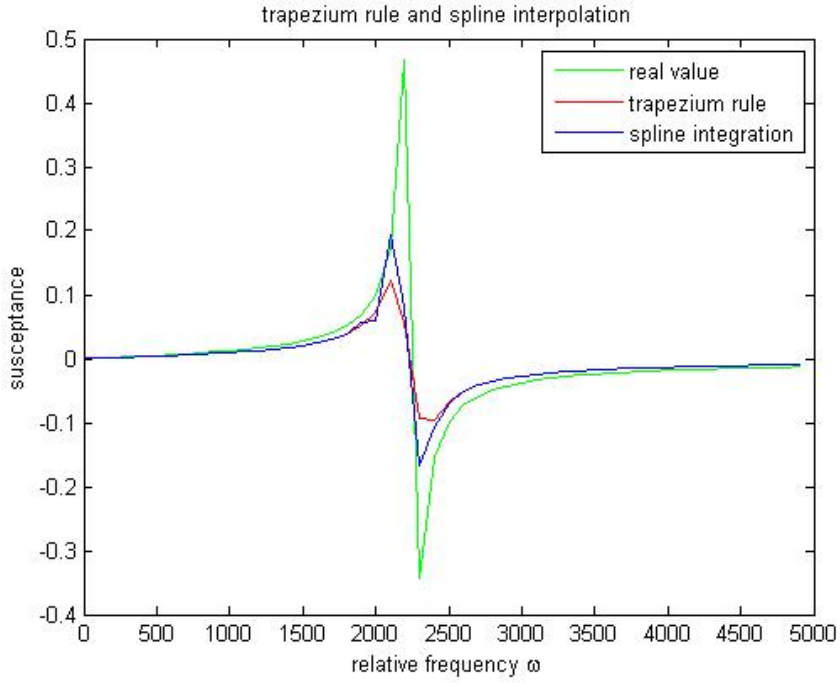

Figure 7.1: Graph of trapezium rule and spline integration

closer to the real value than the trapezium rule. This is because the order of the error is better. This means that with exactly the same number of measurements the spline integration method will always give a better result than the trapezium method.

The spline method does take more time to run than the trapezium method, but it is still a fast method. Also, because it is more precise, it takes less measurements to get the same result.

That means that between the trapezium method and the spline method, using splines would be the best option.

The Fast Fourier Transform (fft) method has been made so that calculations can be done fast, but is it also precise? This method is plotted in figure 7.2 along with the real value for the susceptance.

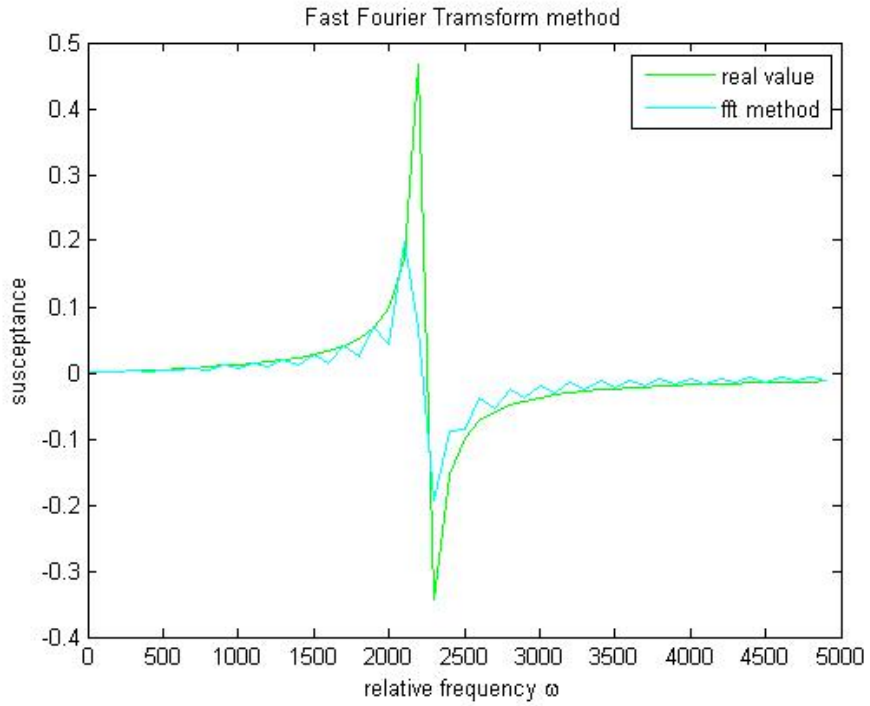

Figure 7.2: Graph of the fft method

In the figure it can be seen that the fft method gives an oscillating function. This comes from the fact that the fft method uses complex exponentials, which are periodic functions. Therefore this might be a fast method but to actually get a precise result, it is needed to take a lot of measurements.

This means that of the three methods, the best method to use would be the spline interpolation method. This is the method that converges the fastest, although it also is the method with the longest running time.

7.1 5% range

We've seen that the spline method works best when there is information on a range starting from 0, but how many measurements would have to be taken to get a good result? To answer that question you have to know what a good result is, the definition that will be used is that all values have to lie within a 5% range of the real value.

In appendix E there is a matlab code to calculate how many measurements are needed to get within a 5% range, the measurement points are uniformly distributed over the interval. To see of all values lie within 5% of the real values a smaller number of point are also uniformly distributed over the interval. For these values it is checked wether or not the result of the spline interpolation lies close enough to the real value.

When applying this code to the example from chapter 6, the result is that there would be 10002 needed to get within a 5% range of the real value.

7.2 Error analysis

With the spline method there are two kinds of errors, one because the last part of the integral isn't taken into account and one for using the spline interpolation. To get a better impression of how good the approximation is, we will look at the different errors, for the RCL-system as it was defined in chapter 6.

The first error comes from the fact that that the integral over $[0, \infty)$ is only calculated over the interval $[0, b]$. This means that the error could be expressed exactly by the other part of the integral,

$$E_1 = \left| \frac{2\omega}{\pi} \int_b^\infty \frac{G(x) - G(\omega)}{x^2 - \omega^2} \, dx \right|.$$

Because this integral is over an interval where x is high, that would mean that $G(x)$ can be approximated.

$$G(x) = \frac{x^2 RC^2}{1 - 2x^2 LC + x^4 L^2 C^2 + x^2 R^2 C^2} \approx \frac{x^2 RC^2}{x^4 L^2 C^2} = \frac{R}{\text{L}^2} \frac{1}{x^2}$$

This means that the error can also be approximated by

$$\begin{aligned}
E_1 &\approx \left| \frac{2\omega}{\pi} \int_b^\infty \frac{\frac{R}{L^2 x^2} - G(\omega)}{x^2 - \omega^2} \, dx \right| \\
&= \left| \frac{2R\omega}{L^2 \pi} \int_b^\infty \frac{\frac{1}{x^2}}{x^2 - \omega^2} \, dx - \frac{2\omega}{\pi} G(\omega) \int_b^\infty \frac{1}{x^2 - \omega^2} \, dx \right| \\
&= \left| \frac{R}{\pi L^2 \omega^2} \ln\left(\frac{b+\omega}{b-\omega}\right) - \frac{2R}{b\omega\pi L^2} + \frac{G(\omega)}{\pi} \ln\left(\frac{b-\omega}{b+\omega}\right) \right| \\
&= \left| \left(\frac{G(\omega)}{\pi} - \frac{R}{\pi L^2 \omega^2}\right) \cdot \ln\left(\frac{b-\omega}{b+\omega}\right) - \frac{2R}{b\omega\pi L^2} \right|.
\end{aligned}$$

In that equation it can be seen that if b goes to ∞ both terms will get smaller, which is logical because that would shorten the interval over which will be integrated. Another way to get a lower error would be to have a high ω, but than it would also be needed that $G(\omega)$ is small. Another reason that would not work is because that way the error would not be lower over the whole interval but only for the first part of the interval.

The second error is caused by using the spline interpolation instead of the real function value. For the spline interpolation the error is known as

$$E \leq \frac{5}{384} \cdot h^4 \cdot \max \left| G^{(4)}(x) \right|.$$

This error can be lowered by taking more measure point end thereby making h smaller. This error also depends on the value of $\max \left| G^{(4)}(x) \right|$, which depends on the system.

7.3 Smaller interval

When the interval over which the conductance (or active power) is known is not over an interval $[0, b]$ but on a smaller interval $[a, b]$. Within appendix E.3 the programs that are described in chapter 5 are implemented in matlab code.

To see if these methods are reliable they are applied on the example of chapter 6 but also on two extra examples which are explained in appendix D. When applying the programs to these examples it seems that the programs don't work as well as the example of chapter 6 would let on. The order of the errors that the programs gets between the real value of the conductance and the return value is:

	Example	Example 2	example 3
Program 1	10^{-6}	10^{2}	10^{7}
Program 2	10^{16}	10^{21}	10^{11}
Program 3	10^{-13}	10^{4}	10^{0}
Program 4	10^{23}	10^{26}	10^{20}

Of the four programs the third would be the best, but even that program is not exactly reliable. The third program is that in which the conductance is measured within $[a, b]$ and the susceptance within $[c, a)$. The second best is the first program that uses both functions. [5]

The other two programs are the ones that use only information on the interval $[a, b]$, the high order of the error shows that this is not enough information to get a good result.

Chapter 8

Conclusion

It is possible to calculate the reactive power when the active power is known, by using the Kramers-Kronig relations. To be able to use the Kramers-Kronig relations the active power should be known over the whole frequency domain. But it is possible to make a good approximation with only a finite number of measurements on a finite domain $[0, b]$.

$$Q(\omega) \approx \frac{2\omega}{\pi} \int_0^b \frac{P_s(x) - P_s(\omega)}{x^2 - \omega^2}$$

Where P_s defines the spline interpolation polynomial taken with the measurements of the active power P that are taken. This way it is possible to calculate the reactive power of a system with the use of only a watt meter. This method is more effective than other methods to calculate the reactive power because it return is a function over the domain where the measurements are taken and not just the reactive power for one frequency.

Chapter 9

Discussion

The spline interpolation is a method that needs quite a lot of measurements, for the example from chapter 6 there are 20002 measurements needed. But with those measurements the reactive power can be determined for all the values in the same range as the measurements. That makes this a good method if there are a lot of frequencies that the reactive power needs to be known for.

If you only need to know a few values there are easier methods, like those explained in section 2.3, for instance by using

$$S^2 = P^2 + Q^2.$$

That means that for one frequency the reactive power measure can be calculated when the rms-values of the voltage and the current along with the active power for the same frequency are known.

With the rms-values

That does mean that there are now tree values, and tree different meters, needed to calculate the reactive power for only one frequency. The method with the splines is more efficient.

9.1 Future work

- For future work it would be useful to get a program like the first in appendix E.3. According to the article [5] it should be possible to implement the method. With this program it would be possible to calculate the susceptance using the Kramers-Kronig relations without have to measure the conductance for the lower frequencies.
- Another question that comes up is if there is a way to determine over which interval there would be taken measurements without knowing the system. When the system is known the interval is determined by making sure that outside the interval the conductance is (practically) equal to 0. When the system is not known, that tactic would be a lot harder.

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Appendix A

List of used symbols

For every function X we have tree different important values defined that are noted as.

$\tilde{X}(t)$	notates the function for X in the time domain.
$\hat{X}$	gives the root-mean-square value of X.
$X(\omega)$	gives the function in the frequency domain.

Different parameters, values and functions.

U	The voltage of the system.
I	The current that goes trough the system.
S	The apperent power of a system, all the energy supplied.
P	The active power of a system, the energy actually used.
Q	The reactive power, the energy lost because U and I are perpendicular.
λ	The power factor, the part of the energy supplied that is used.
R	The resistance of a system.
Y	The admittance of a system.
$\mathcal{P}$	The Cauchy principal value has to be taken, see section 3.1.2 for definition.
$\mathcal{H}$	Taking the Hilbert transform. $\mathcal{H}(f(t)) = \frac{\mathcal{P}}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{t-x}dx$
$\mathcal{F}$	Taking the Fourier transform, see appendix B for definition.
$\delta(t)$	The delta dirac function, defined by $\delta(t) = \begin{cases}\infty & \text{for } t = 0 \\ 0 & \text{for } t \neq 0\end{cases}$.
rms	Taking the Root Mean Square value.

Appendix B

Fourier transforms

Within this Bachelor Thesis there are different Fourier transforms used, all calculated by using ordenary unitary fourier transforms.

$$\mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(x)e^{-2\pi itx}\ dx$$
$$\mathcal{F}^{-1}(f(t)) = \int_{-\infty}^{\infty} f(x)e^{2\pi itx}\ dx.$$

In calculating the Fourier transforms there are different rules used, allong with some basic known Fourier transforms.

$$\mathcal{F}\left(\frac{d^n f(x)}{dx^n}\right) = (2\pi i\omega)^n \mathcal{F}(f(x)) \tag{B.1}$$
$$\begin{aligned}\mathcal{F}\left(f(x)\cdot g(x)\right) &= \mathcal{F}(g(x)) \otimes \mathcal{F}(f(x)) \\ &= \int_{-\infty}^{\infty} \mathcal{F}(g)(x)\cdot\mathcal{F}(f)(\omega - x)\ dx \end{aligned} \tag{B.2}$$
$$\mathcal{F}(\delta(t)) = 1 \tag{B.3}$$

The *sign* function is constant in every point exept for at $t = 0$, there the function has a jump of 2 (from -1 to 1). That means that the derivative of the *sign* function is ∞ for $t = 0$ and 0 everywhere else. This looks a lot like the delta dirac function. Because the total change in value is 2 we get

$$\frac{d}{dt}sign(t) = 2\cdot\delta(t) = \begin{cases}\infty & \text{for } t = 0 \\ 0 & \text{for } t \neq 0\end{cases}.$$

Using this relation along with (B.1) and (B.3) the Fourier transform of the *sign* function can be determined as

$$\mathcal{F}(sign(t)) = \frac{2\cdot\mathcal{F}(\delta(t))}{2\pi i\omega} = \frac{1}{\pi i\omega}.$$

Fast Fourier Transform

For the fft method[7] it is ussed that

$$\begin{aligned} B(\omega) &= \mathcal{H}(G(\omega)) \\ &= \mathcal{P}\left\{\frac{-1}{\pi t} \otimes G(\omega)\right\}. \end{aligned}$$

When taking the Fourier transform of this relation that gives

$$\mathcal{F}(B(\omega)) = \mathcal{P}\left\{\mathcal{F}\left(\frac{-1}{\pi t}\right) \cdot \mathcal{F}(G(\omega))\right\}.$$

To calculate the Fourier transform of $\frac{-1}{\pi t}$ it is needed to take the Cauchy principal value[7], which is included in the formulla, so that

$$\mathcal{F}\left(\frac{-1}{\pi t}\right) = i \cdot sign(t),$$

using this we can write

$$\mathcal{F}(B(\omega)) = i \cdot sign(t) \cdot G(\omega).$$

This gives a different way to calculate the susceptance by using

$$B(\omega) = \mathcal{F}^{-1}\left(i \cdot sign(\omega) \cdot \mathcal{F}(G(\omega))\right).$$

If there are only a finite number of sample points for a function, a way to approximate the continuous Fourier transform is by using the discrete Fourier transform, which transforms $x_0, x_1, \ldots, x_{N-1}$ to $X_1, X_2, \ldots, X_{N-1}$ and back by using

$$\begin{aligned} X_k &= \sum_{n=0}^{N-1} x_n e^{-2\pi i k n/N} \quad \text{and} \\ x_k &= \frac{1}{N} \sum_{n=0}^{N-1} X_n e^{2\pi i k n/N}. \end{aligned}$$

For this transformation there have been developed some algorithms to calculate this faster, this is done by rewriting the transformation it till it is a problem with several smaller transformations. By doing this you reduce the normally $N \times N$ steps, to $N \log N$ steps. This algorithm is called the Fast Fourier Transform, and it is an algorithm already programmed in within matlab.

Appendix C

Residue theorem

The Residue theorem gives a way to calculate a countourintegral. To define the theorem it is needed to make a few definitions. First the winding number $\chi(\gamma; a) \coloneqq \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$ this is always a integer and is equal to the number of times that γ encircles a. The seccond definition that is needed is the residue of a point $\text{Res}(f; a) \coloneqq \frac{1}{2\pi i}\oint_{|z-a|=\epsilon} f(z)\ dz$ where ϵ is small enough that there is no other singularity inside $\{z \in \mathbb{C} : |z-a| \leq \epsilon\}$ that a. With those two definitions the Residue theorem can be defined.

Residue theorem. *Suppose that $D \subseteq \mathbb{C}$ and $z_1, \cdots, z_k \in D$ (pairswise disjoint). If $f : D\backslash\{z_1, \cdots, z_k\} \to \mathbb{C}$ is a holomorphic function and γ is a closed curve over $D\backslash\{z_1, \cdots, z_k\}$ and the interior of γ is a subset of D then*

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{m=1}^{k} Res(f; z_m)\chi(\gamma; z_j).$$

The contour integral that is used in the proof of the Kramers-Kronig relations (see figure 3.1) is in the right halfplane, where the function $X(\omega)$ is analytic end thus also holomorphic. This means that $f(z) = \frac{X(z)}{z-\omega}$ has only one singularity within the right halfplane at $z = \omega$. For $z = \omega$ the winding number $\chi(\gamma; \omega)$ is 0 because the contour does not encircle ω at all. This means that the sum for the residue theorem is 0 which means that

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = 0 \Rightarrow \int_\gamma f(z)dz = 0,$$

which is what is used in the proof of section 3.1.2.

Appendix D

Examples

Two extra examples to help determine wether or not the methods do really work. Example 2 will consist of a RL-network as a series conected. The third example comes from one or the articles used.[5]

D.1 Example 2

For this example a network a RL-network will be used in which a resistor of 1Ω and an inductor of $2H$ are series conected.
This network has a impendance of $Z(i\omega) = 1 + 2i\omega$ which gieves a admittance of

$$Y(i\omega) = \frac{1}{Z(i\omega)} = \frac{1}{1+2i\omega} = \frac{1}{1+2i\omega}\frac{1-2i\omega}{1-2i\omega} = \frac{1}{1+4\omega^2} + i\frac{-2\omega}{1+4\omega^2}.$$

As a check; the functions $\frac{1}{1+4\omega^2}$ and $\frac{-2\omega}{1+4\omega^2}$ are functions that are each others hilbert transform. They can be found in most tables with hilbert transform pairs.

D.2 Example 3

This a an example that can be found in an article, it is example 2a of this article.[5] In the article the impendance is alredy given as

$$Z(i\omega) = R_s + \frac{R_1}{1+i\omega R_1 C_1} = \frac{R_1 + R_s + i\omega R_s R_1 C_1}{1+i\omega R_1 C_1}.$$

This means that the admittance is given by

$$\begin{aligned} Y(i\omega) &= \frac{1}{Z(i\omega)} \\ &= \frac{1+i\omega R_1 C_1}{R_1 + R_s + i\omega R_s R_1 C_1} \\ &= \frac{1+i\omega R_1 C_1}{R_1 + R_s + i\omega R_s R_1 C_1}\frac{R_1 + R_s - i\omega R_s R_1 C_1}{R_1 + R_S - i\omega R_s R_1 C_1} \\ &= \frac{R_1 + R_s + \omega^2 {C_1}^2 {R_1}^2 R_s}{{R_1}^2 + 2R_1 R_s + {R_s}^2 + \omega^2 {C_1}^2 {R_1}^2 {R_s}^2} + i\omega\frac{C_1 {R_1}^2 + C_1 R_1 R_s - {C_1}^2 {R_1}^2 {R_s}^2}{{R_1}^2 + 2R_1 R_s + {R_s}^2 + \omega^2 {C_1}^2 {R_1}^2 {R_s}^2}. \end{aligned}$$

Appendix E

Matlab programs

E.1 Integration methods

```
%spline interpolation, trapezium rule and fft method

%the integration is done over [a,b] instead of [0,\infty)
a=0;
b=20000;
%n gives the number of measurements are taken (n+1), h the distance between
%the measurements
n=200;
h=(b-a)/n;

x=a:h:b;
y=zeros(n+1,4);
Gx=G_RCL(x);
y(:,1)=B_RCL(x);
t=zeros(n+1,n+1);

%using the trapezium rule to integrate
for k=1:n+1
    %i geeft de x_j
    p=zeros(n+1,1);
    for i=1:n+1
        %k geeft de w
        if i==k
            %if this is the case, you have '0/0', officially you have to
            %use L'Hopital, but G' is not known. Therefore take this term 0
        else
            p(i)=(Gx(i)-Gx(k))/(x(i)^2-x(k)^2);
        end
    end
    y(k,2)=(2/pi)*x(k)*trapz(x,p);
end

%first spline interpolation and after that integration
```

```
pp=spline(x,Gx);
 for k=1:n+1
     yw=ppval(pp,x(k));
     y(k,3)=(2/pi)*x(k)*integral(@(t) testfunctie3(x(k),yw,t,ppval(pp,t)),0,b)
 end

 %for the fft method it is needed to center the measurements around 0
 %this can be done because G is even
l2=length(x);
x2=zeros(2*l2-1,1);
Gx2=zeros(2*l2-1,1);
for i=0:l2-1
    x2(i+1)=-x(l2-i);
    Gx2(i+1)=Gx(l2-i);
    x2(l2+i)=x(1+i);
    Gx2(l2+i)=Gx(1+i);
end
x3=x2(1:length(x2)-1);
Gx3=Gx2(1:length(x2)-1);
%the actual implementation of the fft method
F=fft(Gx3);
F2=sign(x3).*F;
F3=ifft(F2);
F4=imag(F3);
%only the positive results are plotted, there are only positive frequencies
F5=F4(length(x):length(F4));
x4=x(1:length(x)-1);

%taking only the first, interesting, part of the function to plot
xp=x(1:50);
yr=y(1:50,1);
ytrpz=y(1:50,2);
yspl=y(1:50,3);
yfft=F5(1:50);

%plotting the real values for B along with the values that the different
%methods have given
figure
plot(xp,yr,'g')
hold on
plot(xp,ytrpz,'r')
plot(xp,yspl,'b')
plot(xp,yfft,'c')
legend('real value','trapezium rule','spline integration','fft method')
title('Different methods')
xlabel('relative frequency \omega')
ylabel('susceptance')
```

```
function [ r ] = testfunctie3( w,yw,t,yt )
%gives the vallue of the function that is being integrated for the
%Kramers-Kronig relations
n=length(t);
r = (yt-yw)./(t.^2-w.^2);
for i=1:n
    if t(i)==w
        %removing possible NaN vallues by replacing them with 0
        r(i)=0;
    end
end
end
```

E.2 5% test

```
%Testing how manny measurements are needed to get within p% range of the
%true value, the measurements are uniformly spread within [0,b]
b=20000;
p=5;
n1=b/200;
n=b;
h=b/n1;
%the points of xt are used to check if the values lie within 5%
xt=0:h:b;
y=zeros(n1+1,3);
Bx=B_RCL(xt);
Gxt=G_RCL(xt);

k=p+1;
while k>p
    h=b/n;
    x=0:h:b;
    Gx=G_RCL(x);
    pp=spline(x,Gx);
    for i=1:n1+1
        yw=ppval(pp,xt(i));
        y(i,1)=(2/pi)*xt(i)*integral(@(t) testfunctie3(xt(i),yw,t,ppval(pp,t))
    end
    kt=abs((Bx-y(:,1))./Bx)*100;
    k=max(kt);
    %first find an overestimation of n
    n=n*2;
end
n=n/4+1;
k=p+1;
while k>p
    h=b/n;
    x=0:h:b;
    Gx=G_RCL(x);
    pp=spline(x,Gx);
    for i=1:n1+1
        yw=ppval(pp,xt(i));
        y(i,1)=(2/pi)*xt(i)*integral(@(t) testfunctie3(xt(i),yw,t,ppval(pp,t))
    end
    kt=abs((Bx-y(:,1))./Bx)*100;
    k=max(kt);
    %then the precise number for n is determind
    n=n+1;
end
n;
```

E.3 Smaller interval

Program 1

```
%Determine B and G with the useage of functions
format long
%number of measurements on the interval [b,a]and on the interval [c,a]
N1=1000;
N2=1000;
%highest powers of the functions for G, given by K, and B, given by M
K=4;
M=4;
T=K+M+2;
%the 10-log of the a,b and c
a2=-2;
b2=4;
c2=-6;
%then we get
a=10^a2;
b=10^b2;
c=10^c2;

%point/measurements on the interval [a,b]
h1=(b-a)/N1;
x1=a:h1:b;
x1t=log10(x1);
h1=h1*ones(N1,1);
Gx=G_RCL(x1);

%points on the interval [c,a]
h2=(a-c)/N2;
x2=c:h2:a;
x2t=log10(x2);
d2=h2*ones(N2,1);

%making the w's
hw=(a-c)/T;
w=c:hw:a;
wt=log10(w);


A=zeros(T,T);
C=zeros(N2,1);
for n1=1:T
    %n1 gives the w that is used, so the first variable of A
    for n2=1:T
        if n2<=K+1
            %this indicates that we are in the part that defines G
            k=n2-1;
```

```
            if k==0
                %if k=0 the summation is always 0
                C=zeros(N2,1);
            else
                %now goes k>=1
                for n3=1:N2
                    if x2(n3)==w(n1)
                        %exceptions, application of 'l hôpital because '0/0'
                        %C(n3)=(k*x2t(n3)^(k-1))/(2*x2(n3)^2*log(10))*d2(n3);
                        %for simplification it is used that;
                        C(n3)=0;
                        %this doesn't influence the results
                    else
                        C(n3)=(x2t(n3)^k-wt(n1)^k)/(x2(n3)^2-w(n1)^2)*d2;
                    end
                end
            end
            A(n1,n2)=((-2*w(n1))/pi)*sum(C)+(1/pi)*wt(n1)^k*log((a+w(n1))/(a-w
        else
            %we zitten in het gedeelte van B
            m=n2-K-2;
            A(n1,n2)=wt(n1)^m;
        end
    end
end

B=zeros(T,1);
C2=zeros(N1,1);
for n1=1:T
    for j=1:N1
        C2(j)=Gx(j)/(x1(j)^2-w(n1)^2)*h1(j);
    end
    B(n1)=(2*w(n1)/pi)*sum(C2);
end

as=A\B;

x3=c:0.000001:a;
x4=c:0.0001:b;
Temp=f1(K,M,as,x3);
%the Gg and Bg are the guesses of G and B
Gg=Temp(:,1);
Bg=Temp(:,2);
G=G_RCL(x4);
B=B_RCL(x4);
plot(log10(x4),G,'g')
hold on
%plot(log10(x3),B,'b')
%plot(log10(x3),Bg,'c')
```

```
plot(log10(x3),Gg,'r')
hold off
```

In this program the $f1$ function used is defined as:

```
function [ f ] = f1(K,M,as,x)
f=zeros(length(x),2);
% f1 gives the vallues for G and f2 the vallues for B
for k=0:K
    f(:,1)=f(:,1)+(as(1+k)*log10(x).^k)';
end
for m=0:M
    f(:,2)=f(:,2)+(as(K+2+m)*log10(x).^m)';
end
end
```

Program 2

```
%Using the vallues of the susceptance as well as the vallues of the
%conductance, also with the use of the function
format long
%number of measurements on the interval [b,a]and on the interval [c,a]
N1=1000;
N2=1000;
%highest power of the function for G
K=4;
%the 10-log of the a,b and c
a2=-2;
b2=4;
c2=-6;
%then we get
a=10^a2;
b=10^b2;
c=10^c2;

%point/measurements on the interval [a,b]
h1=(b-a)/N1;
x1=a:h1:b;
Gx1=Gtest(x1);
ppg=spline(x1,Gx1);
Bx1=Btest(x1);
ppb=spline(x1,Bx1);

%points on the interval [c,a]
h2=(a-c)/N2;
x2=c:h2:a;

%making the w's
hw=(b-a)/(K+2);
wt=a:hw:b;

A=zeros(K+1,K+1);
B=zeros(K+1,1);
C=zeros(N2+1,1);

for n1=2:K+2
    %n1 gives the w
    for n2=0:K
        %gives the power
        for i=1:N2+1
            if x2(i)==wt(n1)
                C(i)=0;
            else
                C(i)=h2*(log(x2(i))^n2)/(x2(i)^2-wt(n1)^2);
            end
```

```
        end
        A(n1-1,n2+1)=(2*wt(n1)/pi)*sum(C);
    end
    gw=ppval(ppg,wt(n1));
    bw=ppval(ppb,wt(n1));
    temp=(-2*wt(n1)/pi)*integral(@(t) testfunctie3(wt(n1),gw,t,ppval(ppg,t)),a
    B(n1-1)=bw+temp+(gw/pi)*log((b-wt(n1))/(b+wt(n1))*(a+wt(n1))/(a-wt(n1)));
end


as=A\B;

x3=c:0.000001:a;
x4=c:0.001:b;
Gg=f12(as,x3);
G=Gtest(x4);

plot(log10(x4),G,'g')
hold on
plot(log10(x3),Gg,'r')
```

In this program the $f12$ function used is defined as:

```
function [ f ] = f12(as,x)
%gives the vallue of G
l=length(as);
f=zeros(length(x),1);
for i=0:l-1
      f=f+(as(1+i)*log10(x).^i)';
end

end
```

Program 3

```
%measuring G on [a,b] and B on [c,a]
format long
%number of measurements on the interval [b,a]and on the interval [c,a]
N1=1000;
N2=1000;
%the 10-log of the a,b and c
a2=-2;
b2=4;
c2=-6;
%then we get
a=10^a2;
b=10^b2;
c=10^c2;

%point/measurements on the interval [a,b]
h1=(b-a)/N1;
x1=a:h1:b;
Gx=G_RCL(x1);
pp=spline(x1,Gx);

%points on the interval [0,a]
h2=a/N2;
x2=0:h2:a-h2;

%making the w's on the interval [c,a]
hw=(a-c)/N2;
w=c:hw:a-hw;
Bw=B_RCL(w);

A=zeros(N2,N2);
B=zeros(N2,1);
for n1=1:N2
    %n1 gives the w that is used, so the first variable of A
    for n2=1:N2
        %and n2 gives the x that is used, so the seccond variable of A
        if x2(n2)==w(n1)
        else
            A(n1,n2)=h2/(x2(n2)^2-w(n1)^2);
        end
    end
    int=integral(@(t) testfunctie3(w(n1),0,t,ppval(pp,t)),a,b);
    B(n1)=(pi*Bw(n1))/(2*w(n1))-int;
end

C=A\B;
pp2=spline(x2,C);
```

```
x3=0:0.0001:a;
x4=0:0.0001:b;
%the Gg and Bg are the guesses of G and B
Gg=ppval(pp2,x3);
G=G_RCL(x4);
plot(log10(x4),G,'g')
hold on
plot(log10(x3),Gg,'r')
hold off
```

Program 4

```
%measure both G and B on [a,b]
format long
%number of measurements on the interval [b,a]and on the interval [c,a]
N1=1000;
N2=1000;
%the 10-log of the a,b and c
a2=-2;
b2=4;
c2=-6;
%then we get
a=10^a2;
b=10^b2;
c=10^c2;

%point/measurements on the interval [a,b]
h1=(b-a)/N1;
x1=a:h1:b;
Gx=G_RCL(x1);
pp=spline(x1,Gx);

%points on the interval [c,a]
h2=(a-c)/N2;
x2=c:h2:a-h2;

%making the w's on the interval [a,b]
hw=(b-a)/N2;
w=a+hw:hw:b;
Bw=B_RCL(w);

A=zeros(N2,N2);
B=zeros(N2,1);
for n1=1:N2
        %n1 gives the w that is used, so the first variable of A
    for n2=1:N2
        %and n2 gives the x that is used, so the seccond variable of A
        if x2(n2)==w(n1)
        else
            A(n1,n2)=h2/(x2(n2)^2-w(n1)^2);
        end
    end
    gw=ppval(pp,w(n1));
    int=integral(@(t) testfunctie3(w(n1),gw,t,ppval(pp,t)),a,b);
    B(n1)=int-(pi*Bw(n1))/(2*w(n1))-(gw/(2*w(n1)))*log((w(n1)-a)/(w(n1)+a));
end

C=A\B;
pp2=spline(x2,C);
```

```
x3=c:0.0001:a;
x4=c:0.0001:b;
%the Gg and Bg are the guesses of G and B
Gg=ppval(pp2,x3);
G=G_RCL(x4);
plot(log10(x4),G,'g')
hold on
plot(log10(x3),Gg,'r')
hold off
```