Criticality in the Abelian sandpile model

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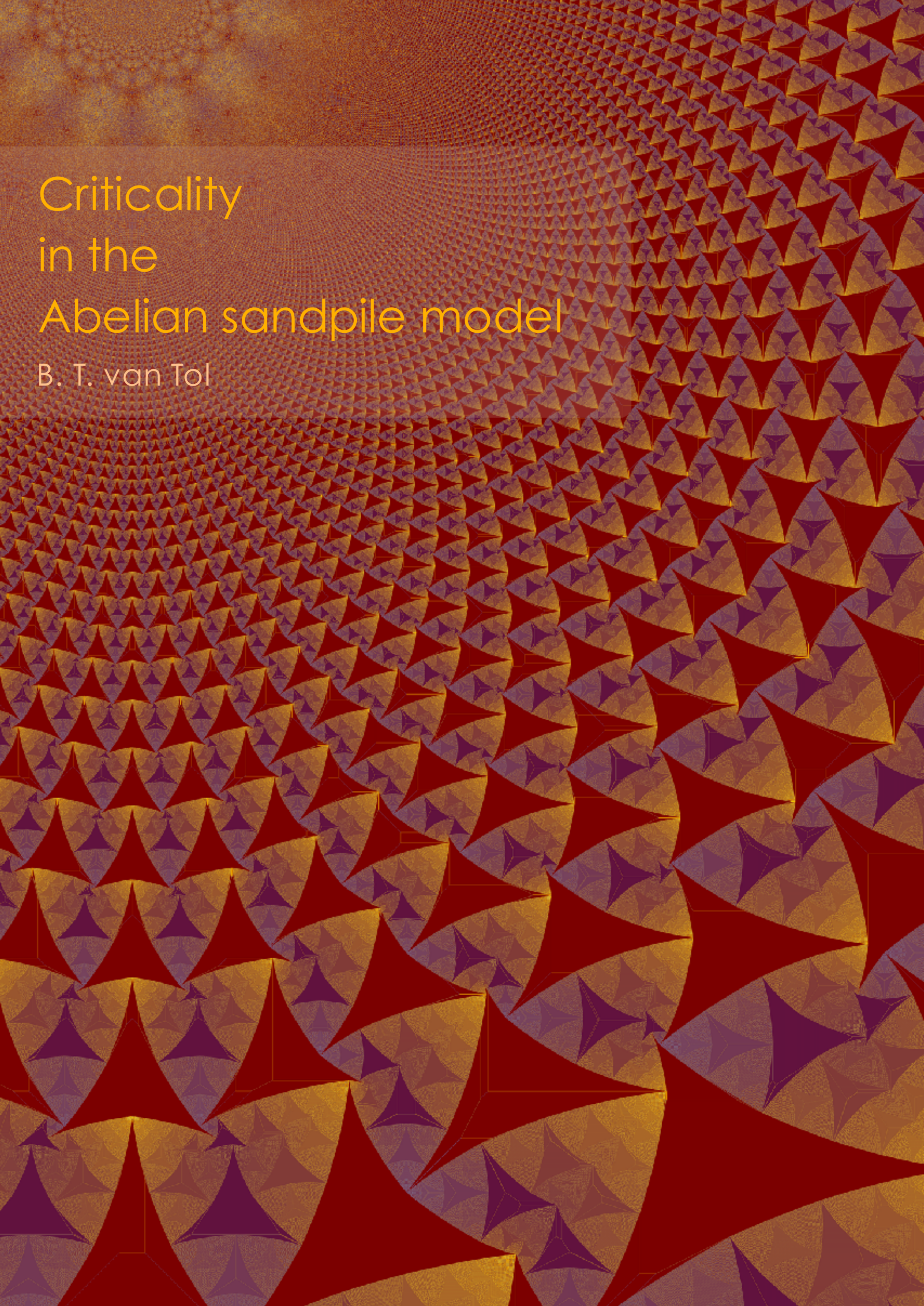

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Abstract

In this thesis we study criticality in the context of the dissipative Abelian sandpile model. The model is linked to a simple trapped random walk, giving a practical method to determine criticality for certain landscapes of dissipative sites. The main results concern the lifetime of the random walk, especially the divergence of its first moment for traps placed on spherical shells. For the one dimensional case the point of divergence is determined with reasonable precision. In higher dimensions the divergence is shown to be possible for an infinite amount of shells. The connection between the sandpile model and a random walk is shown mathematically and further researched via simulation.

Contents

1 Introduction

The Abelian sandpile model hides many fascinating mathematical properties and patterns. One of these can be admired on the front cover of this bachelor thesis, where an instance of the model showing fractal behaviour is displayed [6]. Also graph theory, algebra and random walks can be encountered while studying the sandpile model. It was first introduced by Per Bak, Chao Tang and Kurt Wiessenfeld [1] in 1987 and has been subject to research in mathematical and pysical communities ever since. Bak, Tang and Wiessenfeld originally introduced the model as an example of a system exhibiting so called *self-organised* criticality. They argued that distinct patterns found in complex systems transcending the traditional methods of analysis, like reduction of degrees of freedom or mean field theory, stem from this concept. As a consequence an understanding of these self-organised systems would benefit physics, biology and even social sciences like economics. Self-organised critical means in this context that the system naturally evolves to a critical state without detailed specification of the initial conditions or tuning the system parameters. In physics the state of a system is usually described as critical when its typical correlation length diverges, meaning that small perturbations can cause disproportionate change. This type of behaviour often arises around phase transitions and plays an important role in statistical physics. The Abelian sandpile model, first confirmed by simulation and later by rigorous argument, is indeed self-organised critical. However, the definition of the model can be tweaked until criticality is lost, leading to a quest for borderline cases. In this thesis we explore these cases with the help of random walks. We aim to this in a rigorously justified manner, starting from clear definitions progressing via proof. Along the way we make thankful use of the Abelian group structure discovered by Deepak Dhar [2] to derive the very applicable relation to random walks.

2 The Classical Abelian Sandpile Model

2.1 Two Dimensional Example

Before giving a formal definition of the classical Abelian sandpile model, we will first illustrate the relevant mechanics using a simple two dimensional example. Consider a three by three grid as below. Each of the squares contains either zero, one, two or three grains of sand. Now we add an extra grain to the grid. Say we add it in the upper left corner, then the total number of grains on the square becomes one. Something more interesting happens when we were to add the grain at the middle square, where already three grains are present. Together with the extra grain this makes four grains on a single square, which is not allowed. To solve this we perform an operation called "toppling". Each neighbor (top, bottom, left and right) obtains one of the four grains from the middle square which is now empty.

0	3	1		1	3	1		1	3	1		1	4	1
3	3	2	→	3	3	2	→	3	4	2	→	4	0	3
0	2	3		0	2	3		0	2	3		0	3	3

The toppling, while solving the problem for the middle square, resulted in two other sites with four grains. Now we have two choices for which site we are going to topple next. Suppose we decide to go with the upper middle square first. Again we give a grain to all three neighbors (bottom, left and right). We also throw the away the grain which is supposed to go to the upper neighbor, as if it is falling over the edge. Next we topple the the left middle square in a similar manner. The result is a stable configuration where every site contains three or less grains of sand. One might wonder what would have happened when we went with the middle left square when there were two sites to topple. As demonstrated in the figure below, this would yield the exact same final configuration. We will later find that this is not a coincidence. The order of toppling does not influence the end result, hence the name "Abelian" [1] sandpile.

				2	0	2		3	0	2
			↗	4	1	3	→	0	2	3
1	4	1		0	3	3		1	3	3
4	0	3								
0	3	3		2	4	1		3	0	2
			↘	0	1	3	→	0	2	3
				1	3	3		1	3	3

2.2 Definitions and Notation

We will proceed in a more general setting and define the model above for arbitrary grid size and dimension. Most notation and results below are either from *Mathematical aspects of the abelian sandpile model* by F. Redig [7] or from *Non-criticality criteria for Abelian sandpile models with sources and sinks* by F. Redig, W. Ruszel and E. Saada [8].

The grid will be generalized as the simply connected set $\Lambda_n = [-n, n]^d \cap \mathbb{Z}^d$, which is a cubic grid of size $2n$ in d dimensions.

[1] The Abelian property is also commonly referred to as the commutative property.

We also define a "toppling matrix", which determines the toppling rules. In the classical Abelian sandpile model the toppling matrix, indicated by Δ, is minus the lattice Laplacian[2]:

$$\Delta^{\Lambda_n}_{x,y} = \begin{cases} -1 & \text{for } x, y \in \Lambda_n, x \sim y \\ 2d & \text{for } x = y, \in x \in \Lambda_n \\ 0 & \text{otherwise} \end{cases} . \quad (1)$$

Here we use $x \sim y$ as a notation for "site x is next to site y (diagonal neighbors excluded)".

Bear in mind that this toppling matrix might differ from the one used in the classical model. Later on we will introduce "dissipative sites", which topple slightly different than normal sites. We can model this by modifying Δ.

A stable configuration of grains on the grid Λ_n can now be seen as a map $\eta : \Lambda_n \to \{1, 2, 3, ...\}$ where $\eta(x) \leq \Delta_{xx}$. We will write $\mathcal{H}$ for the set of stable configurations and Ω for the set of all (possibly unstable) configurations. If the map is unstable, i.e. $\exists x \in \Lambda_n : \eta(x) > \Delta_{xx}$, then site x is toppled according to the following definition:

$$T_x(\eta)(y) = \eta(y) - \Delta_{xy}.$$

When we fill in $y = x$, we see that the site that is toppled loses $2d$ grains. The neighboring sites each gain a grain of sand. This is due to Δ_{xy} being -1 whenever $x \sim y$.

Remark 2.1. *The definitions above are consistent with the two dimensional example. In that case we had $d = 2$, which means that sites with four grains or more are unstable and topple, as in the example. One can also easily explain the Abelian behaviour of the topplings now:*

$$T_x T_y(\eta) = \eta - \Delta_x - \Delta_y = T_y T_x(\eta).$$

Here η is written as a column matrix. Δ_x is notation for the column matrix indexed by the sites $z \in \Lambda_n$ and has elements Δ_{xz}.

For a general height configuration we define a possible stabilization by

$$\mathcal{S}(\eta) = T_{x_1} ... T_{x_k}(\eta). \quad (2)$$

Given that every toppling T_{x_i} is legal and the final result $\mathcal{S}(\eta)$ is stable. In a moment we will show that there always exist a stabilization and that this stabilization is unique. First we define the toppling number m_x based on this sequence of topplings as

$$m_x = \sum_{i=1}^{k} I(x_i = x). \quad (3)$$

Here I denotes the indicator function. This definition allows us to write $T_{x_1} ... T_{x_k}(\eta) = \eta - \Delta m$. Here m is a column matrix indexed by $x \in \Lambda_n$, with elements m_x.

It is not directly obvious that $\mathcal{S}(\eta)$ exists for every unstable configuration η, and even if $\mathcal{S}(\eta)$ exists it is not at all trivial that $\mathcal{S}(\eta)$ is well-defined. Nonetheless both statements turn out to be true. To make the existence of $\mathcal{S}(\eta)$ plausible, one might imagine what happens when there would be a site that topples infinitely often. In that case all neighboring sites would receive sand grains every toppling, implying that the neighboring sites themselves should topple

[2] This matrix is well-known from graph theory. $-\Delta = \partial^2$ is a discrete version of the Laplacian with Dirichlet boundary conditions, hence the name "Lattice Laplacian".

infinitely often as well. Repeating the argument for all neighbors of neighbors, we derive that all sites would topple infinitely often, including the sites at the boundary. Now it is clear that such a scenario is not possible, since every time a boundary site topples sand is lost, meaning that after a certain time there would be no grains left on the grid and the configuration would hence be stable [3].

Showing that $\mathcal{S}(\eta)$ is well-defined requires a bit more cleverness. It can be done using lemma 2.2 which states the, perhaps unexpected, fact that the toppling numbers for a stabilizing sequence of topplings is maximal.

Lemma 2.2. *If $\eta \in \mathcal{H}$ is height configuration and $T_{x_1}...T_{x_k}$ a sequence of legal topplings such that the resulting configuration is stable, then the numbers m_x, $x \in \Lambda_n$ are maximal. I.e. for every sequence of legal topplings $T_{y_1}...T_{y_l}$ the toppling numbers n' satisfy $n'_x \leq n_x$ for all $x \in \Lambda_n$.*

Proof. Suppose that we have the identity

$$\xi = \eta - \Delta m$$

with ξ stable and $m_x \geq 0$ for all $x \in \Lambda_n$. Suppose that $x_1, ..., x_k$ is a legal sequence of topplings with with toppling numbers $p_x = \sum_{i=1}^{k} I(x_i = x)$ such that $p_x \leq m_x$. Furthermore we assume that for a site $j \in \Lambda_n$ an extra toppling can be performed i.e. we assume there is at least one unstable site left after this sequence of topplings. We define

$$\zeta = \eta - \Delta p.$$

We know that an extra toppling is allowed at site j in ζ, hence $\zeta_j > \xi_j$. Subtracting the expressions for ξ and ζ we obtain

$$(\xi - \zeta)_j = [\Delta(p - m)]_j = \sum_{i,j}(m_i - p_i)\Delta_{ij} \leq 0$$
$$\Rightarrow (p_j - m_j)\Delta_{jj} < \sum_{i \neq j}(m_i - p_i)\Delta_{ij} \leq 0.$$

In the last inequality we used that $\Delta_{ij} \leq 0$ and $m_i \geq p_i$. The result above implies that $p_j + 1 \leq m_j$. Let p' denote the toppling vector where we legally topple site j once more. We have now shown that we still have $p'_x \leq m_x$ for all $x \in \Lambda_n$. This is enough to conclude via an induction argument: one can start with toppling numbers $p_x = 0$, which is equivalent to not toppling at all, and then keep performing legal topplings. The inequality will remain preserved. □

Theorem 2.3. *$\mathcal{S}$ is well-defined.*

Proof. Suppose there are two legal sequences of topplings leading to a stable configuration, say $T_{x_1}...T_{x_k}$ and $T_{x_1}...T_{x_l}$ with toppling numbers m and p leading to $\mathcal{S}_1(\eta)$ and $\mathcal{S}_2(\eta)$. Then the resulting stable configuration is, by the Abelian property, only a function of the toppling numbers. Now we can apply the previous lemma. By maximality we have $m_x = p_x$ for all $x \in \Lambda_n$. Hence $\mathcal{S}_1(\eta) = \eta - \Delta m = \eta - \Delta p = \mathcal{S}_2(\eta)$. □

[3] This explanation is not necessarily valid for a more general toppling matrix like in definition 3.1, however the statement remains true.

Next we define the addition operator a_x on Ω. This operator gives the stabilization after adding a grain at site x.

$$a_x \eta = \mathcal{S}(\eta + \delta_x) \tag{4}$$

where $\delta_x \in \mathcal{H}$ is the configuration with one grain on site x and zero grains on the other sites. The "+" sign means point-wise addition.

Using the previous definitions we can now describe the dynamics of the Abelian sandpile model. Let $p = p(x)$ be a probability distribution on Λ_n, i.e. $p(x) > 0$ and $\sum_{x \in \Lambda_n} p(x) = 1$. Starting from configuration $\eta_0 = \eta \in \Omega$, the state of the model at time t is given by the random variable

$$\eta_t = \prod_{i=1}^{t} a_{X_i} \eta. \tag{5}$$

In this expression $X_1, ..., X_t$ are i.i.d. with distribution p. One might interpret this process as a grain falling on a random site on the grid at every time step. After the addition of the new grain some toppling might take place until the system is stable again. At that point the next grain is added.

This Process defines a Markov chain on the finite state space Ω. It is easy to see that this Markov chain is *reducible*. Reducible means in this context that it is not generally possible to go from one arbitrary state in the state space to another. Consider for example a configuration with two empty site right next to each other. Such state can never be reached through toppling from a state where either of the two sites does contain grains, since every toppling would give sand from one of the two sites to the other. We see that it is indeed generally impossible to reach a state with two empty sites neighboring each other.
However, restricted to the set of *recurrent* configurations, which we will denote by $\mathcal{R}$, the Markov chain turns out to be irreducible. The recurrent configurations are those stable states which will be revisited by the Markov chain with probability one. To understand why this restriction leads to an irreducible Markov chain we consider $\eta^{max}(x) = 2d$ for all $x \in \Lambda_n$. This state is obviously stable and can be reached from any other state in Ω. Now assume $\eta, \zeta \in \mathcal{R}$ to be recurrent states. Both can with non-zero probability reach η^{max}. Since they were assumed to be recurrent, the probability to reach either η or ζ from η^{max} must be one. As a consequence we can reach ζ from η (or vice versa) with non-zero probability via η^{max}.

2.3 Group Structure

In order to properly define criticality in our model we identify the group behaviour of the addition operators a_x on the set of recurrent states $\mathcal{R}$. We define

$$G = \left\{ \prod_{x \in \Lambda_n} a_x^{k_x}, k_x \in \mathbb{N} \right\} \tag{6}$$

and show that G acting on $\mathcal{R}$ indeed defines an Abelian group.

From the what we derived earlier the Abelian property is clear:

$$a_x a_y \eta = \mathcal{S}(\mathcal{S}(\eta + \delta_y) + \delta_x) = \mathcal{S}(\eta + \delta_x + \delta_y) = a_y a_x \eta. \tag{7}$$

To find the identity element we first use that a configuration η is recurrent when there exist integers $k_x > 0$ such that

$$\prod_{x \in \Lambda_n} a_x^{k_x} \eta = \eta. \tag{8}$$

This product is well-defined due to the Abelian property. The reason why we can choose all $k_x > 0$ might be not directly obvious. To see this, apply a_x for all $x \in \Lambda_n$ on η. This results in some other state η'. By recurrence we must be able to apply some sequence of addition operators again, bringing us back to η. This results in $k_x > 0$. Now we have a candidate for the identity element. We call $e = \prod_{x \in \Lambda_n} a_x^{k_x}$ and consider the set

$$A = \{\zeta \in \mathcal{R} : e\zeta = \zeta\}. \tag{9}$$

We immediately see that A contains η and is hence non-empty. Moreover, if we have $g \in G$ and $\zeta \in \mathcal{R}$ then $g\zeta \in A$:

$$e(g\zeta) = g(e\zeta) = g\zeta. \tag{10}$$

This makes A a *trapping set* for the Markov chain. A trapping set is a subset of state space Ω that the Markov chain cannot leave once entered. Since the Markov chain is irreducible on $\mathcal{R}$ we must have $\mathcal{R} \subseteq A$. One can understand this by a contradiction argument. Imagine $\mathcal{R} \nsubseteq A$, then there must be a recurrent configuration $\eta \notin A$. Suppose the Markov chain enters A, then it will never leave A and as a consequence never reach η again.
Since the definition of A gives $A \subseteq \mathcal{R}$, we have shown $A = \mathcal{R}$. This means that e indeed acts as a identity element for G.

Since we chose exponents k_x in the definition for e strictly larger than zero, we can now easily find the inverse element of an arbitrary addition operator a_i:

$$a_i^{-1} = a_i^{k_i - 1} \prod_{\substack{x \in \Lambda_n \\ x \neq i}} a^{k_x}. \tag{11}$$

This indeed gives $a_i a_i^{-1} = a_i a_i^{k_i - 1} \prod_{\substack{x \in \Lambda_n \\ x \neq i}} a^{k_x} = e$.

2.4 Criticality

In statistical physics the behaviour of a system is often described as critical when the system shows a diverging correlation length. The critical phenomenon in the Abelian sandpile model is the power-law divergence of a quantity which we will refer to as *avalanche size.* The avalanche cluster for $\eta \in \Omega$ at site $i \in \Lambda_n$ is defined as

$$C_{\Lambda_n}(i, \eta) = \{j \in \Lambda_n : m^i_\eta > 0\}. \tag{12}$$

Where m^i_η denotes the vector of toppling numbers after adding a grain to site i:

$$\eta + \delta_i - \Delta m^i_\eta = a_i \eta. \tag{13}$$

One can easily see the analogy with a real avalanche. In a physical sandpile avalanches form when the sandpile locally topples over and the fallen sand grains cause other grains to start rolling. In our model the toppling of a single site can cause it's neighbors to topple and induce toppling at of a whole set of other sites. This set is the above defined avalanche cluster.

So we define the model to be critical based on the power-law divergence of $|C_{\Lambda_n}|$ for an infinite grid (Λ_n as $n \to \infty$). More precisely this means that the number of recurrent configurations[4] resulting in an avalanche cluster of size $|C_{\Lambda_n}| = R$ decays according to a power-law in R. Such power-law behaviour implies that the mean avalanche size over the recurrent states would get infinite. Hence the definition

Definition 2.4 (criticality)**.** *The Abelian sandpile model is non-critical if for all $x \in \mathbb{Z}^d$*

$$\limsup_{n\to\infty} \mathbb{E}_{\mu_n}\left(\left|C_{\Lambda_n}(x, \eta)\right|\right) < \infty. \tag{14}$$

Where μ_n is the simply uniform measure (15) on the set of recurrent states $\mathcal{R}$ for grid size n. The model is critical if it is not non-critical.

The classical Abelian sandpile model, where the toppling matrix is minus the lattice Laplacian, turns out to be critical. In the next section we consider another type of toppling matrix which can make the model non-critical. To rigorously show criticality we will associate a discrete time random walk to the toppling matrix and study the random walk instead. However, this is not an easy task and we will need Dhar's formula (17). This is a very powerful equation linking $\mathbb{E}_{\mu_n}\left(m^x_\eta(y)\right)$ to the toppling matrix. The proof of this result relies on the fact that the simply uniform measure μ is stationary for the Markov chain.

Theorem 2.5 (stationary measure)**.** *The simply uniform measure μ on $\mathcal{R}$*

$$\mu = \frac{1}{|\mathcal{R}|} \sum_{\eta \in \mathcal{R}} \delta_\eta \tag{15}$$

is a stationary measure for the Markov chain.

To prove this theorem it is essential that there exists a bijection $\Psi : \mathcal{R} \to G$. Indeed, the group G acts transitively and freely on $\mathcal{R}$. This means that for all $\eta \in \mathcal{R}$ the orbit $O_\eta = \{g\eta : g \in G\} = \mathcal{R}$ and if $g\eta = g'\eta$ for some $g, g' \in G$, then $g = g'$.
For the first statement we reason as follows: if $\eta \in \mathcal{R}$ and $g \in G$, then $g\eta \in \mathcal{R}$ since we can obtain $g\eta$ via the Markov chain by adding grains to recurrent configuration η. Therefore we

[4] After a certain time we the Markov chain gets trapped in $\mathcal{R}$, hence we are only interested in the avalanche sizes of the recurrent states.

have $O_\eta \subset \mathcal{R}$. Notice that O_η is a trapping set for the Markov chain, implying that $O_\eta \supset \mathcal{R}$. We can conclude that indeed $O_\eta = \mathcal{R}$
For the second statement we assume $g\eta = g'\eta$ and consider the associated set $A = \{\zeta \in \mathcal{R} : g\zeta = g'\zeta\}$. Then $A = \mathcal{R}$ by the same reasoning we used to find the identity element of G.

Proof. There exists a bijection $\Psi : \mathcal{R} \to G$ which maps all $i \in \mathcal{R}$ to some $a_i \in G$. Therefore the image measure $\mu \circ a_i$ is again uniform on $\mathcal{R}$. As a result μ is invariant under the addition operators a_i and thus under the Markov chain. As a matter of fact we even have:

for all functions $f, g : \Omega \to \mathbb{R}$

$$\int f(\eta)g(a_i\eta)\mu(d\eta) = \int f(a_i^{-1}\eta)g(\eta)\mu(d\eta).$$

For the transition operator $Pf(\eta) = \frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n} f(a_i\eta)$ of the Markov chain and $P^*f(\eta) = \frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n} f(a_i^{-1}\eta)$. we find the following relation:

$$\int gPfd\mu = \int fP^*gd\mu. \tag{16}$$

Substituting $g \equiv 1$ gives the stationarity of μ □

Dhar's formula can now be derived by simply integrating the definition of the addition operator a_x over μ. We will call the inverse toppling matrix $(\Delta_{xy})^{-1}$ the *Green's function.*

Theorem 2.6 (Dhar's formula)**.** *For an instance of the abelian sandpile model with toppling matrix $\Delta_{x,y}$ on Λ_n we have*

for all $x, y \in \Lambda_n$:

$$\mathbb{E}_\mu[m_\eta^x(y)] = (\Delta_{x,y})^{-1} \equiv G(x,y). \tag{17}$$

Proof. For $\eta \in \Omega$ a configuration, we can write the stabilization of η as:

$$\eta - \Delta m = \mathcal{S}(\eta).$$

If η is a stable configuration, then after addition at $x \in \Lambda_n$ this is equivalent to:

$$\eta + \delta_x - \Delta m_\eta^x = a_x(\eta).$$

Integrating this equation over μ yields:

$$\begin{aligned}
& \int (\eta + \delta_x - \Delta m_\eta^x)\mu(d\eta) = \int a_x(\eta)\mu(d\eta) \\
\Rightarrow \quad & \int (\delta_x - \Delta m_\eta^x)\mu(d\eta) = 0 \qquad \text{(invariance of } \mu \text{ under } a_x) \\
\Rightarrow \quad & \int \delta_x \mu(d\eta) = \int \Delta m_\eta^x \mu(d\eta) \\
\Rightarrow \quad & \Delta \int m_\eta^x \mu(d\eta) = \delta_x \qquad \text{(linearity of the integral)}
\end{aligned}$$

Which is equivalent to $\mathbb{E}_\mu[m_\eta^x(y)] = (\int m_\eta^x \mu(d\eta))_y = \Delta_{x,y}^{-1}$.

□

3 The Abelian Sandpile Model with Dissipative Sites

3.1 Toppling Matrix

In the previous section we analysed the classical Abelian sandpile model, for which the topping matrix Δ is given by minus the lattice Laplacian. In this case the model was critical. Our next step will be to tweak Δ such that we achieve non-criticality. It turns out that we can still use our results from earlier given that our new toppling matrix satisfies the definition below

Definition 3.1 (general toppling matrices)**.** *The previous results hold for toppling matrix Δ if*

$$1.\textit{For all } x, y \in \Lambda_n\textit{: } \Delta_{xx} \geq 2d,\ \Delta_{xy} \leq 0 \textit{ for } x \neq y \tag{18}$$

$$2.\textit{Symmetry: for all } x, y \in \Lambda_n\textit{: } \Delta_{xy} = \Delta_{yx} \tag{19}$$

$$3.\textit{Dissipativity: for all } x \in \Lambda_n\textit{: } \sum_y \Delta_{xy} \geq 0 \tag{20}$$

$$4.\textit{Strict Dissipativity: } \sum_x \sum_y \Delta_{xy} > 0 \tag{21}$$

A toppling matrix is called irreducible if from every site $x \in V$ there is a path $x_0 = x, \cdots, x_n = y$ where for all $i \in \{1, \cdots, n\}$, $\Delta_{x_{i-1}x_i} < 0$ and y is a dissipative site.

To obtain non-criticality we want to reduce the avalanche sizes such that the mean avalanche cluster is finite whenever the grid size tends to infinity. Our approach is to introduce "sinks" or "dissipative sites". These are sites where an extra grain is allowed, so they can hold at most $2d$ grains instead of just $2d - 1$. However, when such a site topples it also loses $2d + 1$ grains. In that case all $2d$ neighbors receive one grain and the extra grain just disappears. As a consequence the number of grains on this site is not conserved. By reducing the total number of grains in the system avalanches clusters will generally stagnate quicker. We will denote the set of such dissipative sites on $\mathbb{Z}^d$ by D. The perturbed toppling matrix is given below:

Definition 3.2 (Toppling matrix for a system with dissipative sites)**.** *The toppling matrix for the finite box Λ_n with dissipative sites $D_n = D \cap \Lambda_n$ and $D_n^c = D^c \cap \Lambda_n$ is defined as:*

$$\Delta_{x,y}^{D_n} = \begin{cases} -1 & \textit{for } x, y \in \Lambda_n, x \sim y \\ 2d + 1 & \textit{for } x = y, x \in D_n \\ 2d & \textit{for } x = y, \in x \in D_n^c \end{cases} \tag{22}$$

Before we can show the connection between criticality and random walks we need the following characterization of criticality

Theorem 3.3. *Consider a sandpile model on Λ_n with a set $D \subset \mathbb{Z}_d$ of dissipative sites. Then the model is non-critical if*

a) For all $x \in \mathbb{Z}^d$:

$$\limsup_{n\to\infty} \sum_{y\in\Lambda_n} G_n(x, y) < \infty. \tag{23}$$

The model is critical if

b) For all $x \in \mathbb{Z}^d$, $\lim_{n\to\infty} G_n(x, y) = G(x, y)$ is well-defined and there exists a dissipative site z such that

$$\sum_{y\in\mathbb{Z}^d} G(z,y)=\infty. \tag{24}$$

This theorem, in combination with Dhar's formula, shows that we can replace $\left|C_{\Lambda_n}(x,\eta)\right|$ by $\sum_{y\in\Lambda_n} m_\eta^x(y)$ in the definition of criticality. In the case of non-criticality this is not so hard to see. Namely, $\left|C_{\Lambda_n}(x,\eta)\right| \leq \sum_{y\in\Lambda_n} m_\eta^x(y)$ since every site in the avalanche cluster has toppling number one or higher. However, it remains to show that case *b*) in the theorem indeed implies criticality according to definition 2.4. One can use theorem 6.1 *b*) in [4] to show this, but we won't do this here.

3.2 Associated Random Walk

In this section we will associate the Abelian sandpile model with dissipative sites to a trapped random walk. The main motivation to do so is that we will find a, relatively convenient, characterization of criticality in terms of this trapped random walk. These results, theorem 3.4 and 3.5, are adopted from the thesis of J. Zaat [11]. First we introduce the (not trapped) random walk on $\Lambda_n \cup \{*\}$ with increments X_i. $*$ is an additional site where the random walk gets stuck after leaving Λ_n. The transition probability matrix $P_{x,y}$ is given by

$$P_{x,y}=\begin{cases}\frac{1}{2d} & \text{for } x\sim y \text{ and } x,y\in\Lambda_n\\ \frac{2d-\alpha_{\Lambda_n(x)}}{2d} & \text{for } x\in\Lambda_n \text{ and } y=*\\ 1 & \text{for } x=y=*\\ 0 & \text{otherwise}\end{cases} \tag{25}$$

Where $\alpha_{\Lambda_n(x)}$ is the number of neighbors of site x that are in Λ_n . For such a random walk the following theorem holds

Theorem 3.4. *For the Abelian sandpile model extended with dissipation, with Δ as in definition 3.2, the Green's function is given by*

$$G(x,y)=\frac{1}{2d}\mathbb{E}_x^{RW}\left[\sum_{k=0}^{\tau}\left(\frac{2d}{2d+1}\right)^{l_k(D_n)} I(X_k=y)\right]. \tag{26}$$

Where the number of visits to D_n is given by:

$$l_k(D_n)=\sum_{i=0}^{k} I(X_i\in D_n) \tag{27}$$

and $\tau = inf\{k>0 : X_k = *\}$.

Here $\mathbb{E}_x^{RW}[\cdot]$ denotes the expectation of a function of the random walk described above, starting at site x.

Proof. To simplify the notation we call $g(x,y)=\mathbb{E}_x^{RW}\left[\sum_{k=0}^{\tau}\left(\frac{2d}{2d+1}\right)^{l_k(D_n)} I(X_k=y)\right]$. We claim that $\Delta^{-1}_{x,y}=\frac{1}{2d}g(x,y)$

First we split out the contribution from time $k=0$.

$$g(x,y) = \mathbb{E}_x^{RW}\left[\left(\frac{2d}{2d+1}\right)^{I(X_0\in D_n)}\left(I(X_0=y)+\sum_{k=1}^{\tau}\left(\frac{2d}{2d+1}\right)^{\sum_{i=1}^k I(X_i\in D_n)} I(X_k=y)\right)\right]$$

$$= \left(\frac{2d}{2d+1}\right)\delta_{x,y}I(x\in D_n)+\delta_{x,y}I(x\notin D_n)$$

$$+\left(\frac{2d}{2d+1}\right)I(x\in D_n)\mathbb{E}_x^{RW}\left[\sum_{k=1}^{\tau}\left(\frac{2d}{2d+1}\right)^{\sum_{i=1}^k I(X_i\in D_n)} I(X_k=y)\right]$$

$$+I(x\notin D_n)\mathbb{E}_x^{RW}\left[\sum_{k=1}^{\tau}\left(\frac{2d}{2d+1}\right)^{\sum_{i=1}^k I(X_i\in D_n)} I(X_k=y)\right]$$

$$= \left(\frac{2d}{2d+1}\right)\delta_{x,y}I(x\in D_n)+\delta_{x,y}I(x\notin D_n)$$

$$+\left(\frac{2d}{2d+1}\right)I(x\in D_n)\sum_{z\sim y}\frac{-\Delta_{x,z}}{2d}g(z,y)$$

$$+I(x\notin D_n)\sum_{z\sim y}\frac{-\Delta_{x,z}}{2d}g(z,y)$$

We now multiply both sides by $I(x\in D_n)$, and $I(x\notin D_n)$ to obtain

$$(2d+1)I(x\in D_n)g(x,y) = (2d)\delta_{x,y}I(x\in D_n)+I(x\in D_n)\sum_{z\sim x}-\Delta_{x,y}g(z,y)$$

and

$$(2d)I(x\notin D_n)g(x,y) = (2d)\delta_{x,y}I(x\notin D_n)+I(x\notin D_n)\sum_{z\sim x}-\Delta_{x,z}g(z,y).$$

Adding up these two equations yields

$$[(2d)I(x\notin D_n)+(2d+1)I(x\in D_n)]g(x,y) = (2d)\delta_{x,y}+\sum_{z\sim y}-\Delta_{x,z}g(z,y)$$

with the left side equal to $\Delta_{xx}g(x,y)$.

This gives

$$\sum_{z\in\Lambda_n}\Delta_{x,z}g(z,y)(2d)\delta_{x,y} \text{ or } \Delta^{-1}=(2d)^{-1}g.$$

□

We have now successfully associated the Green's function, and hence the criticality of the system, to a random walk. This random walk walks freely on the grid and only dies when it leaves the finite lattice. Equation 26 shows that the existence of dissipative sites indeed influences the criticality of the system. This happens due to the $\left(\frac{2d}{2d+1}\right)^{l_k(D_n)}$ term. It is now quite plausible that we can make the classical Abelian sandpile model non-critical by including dissipative sites. If we, for example, consider a grid with only dissipative sites the system is definitely non-critical. Indeed, in that case

$\forall x \in \mathbb{Z}^d$:

$$\limsup_{n\to\infty} \sum_{y\in\Lambda_n} G(x,y) = \limsup_{n\to\infty} \sum_{y\in\Lambda_n} \frac{1}{2d}\mathbb{E}_x^{RW}\Big[\sum_{k=0}^{\tau}\Big(\frac{2d}{2d+1}\Big)^{l_k(D_n)} I(X_k = y)\Big]$$
$$\leq \frac{1}{2d}\sum_{k=0}^{\infty}\Big(\frac{2d}{2d+1}\Big)^{k} \tag{28}$$

is a converging sum, so Theorem 3.3 gives non-criticality.

We see that Theorem 3.4 already makes for a helpful tool to show criticality. However, we can find an even more convenient expression for the Green's function. It turns out that the expectation in (26) is equal to the expected number of visits to site y of a *trapped* random walk. This trapped random walk is similar to the previous random walk, except now the dissipative sites correspond to *traps*. Traps are sites on which the random walk has a chance to die (i.e. go to the $*$ site). The transition probability matrix is this time given by

$$P_{x,y} = \begin{cases} \frac{1}{2d} & \text{for } x\sim y \text{ and } x,y\in\Lambda_n \\ \frac{2d-\alpha_{\Lambda_n(x)}}{2d} & \text{for } x\in\Lambda_n\setminus D_n \text{ and } y=* \\ \frac{2d-\alpha_{\Lambda_n(x)}+1}{2d+1} & \text{for } x\in\Lambda_n\cap D_n \text{ and } y=* \\ 1 & \text{for } x=y=* \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

Theorem 3.5. *For the Abelian sandpile model extended with dissipation, with Δ as in definition 3.2, the Green's function is given by*

$$G(x,y) = \frac{1}{2d}\mathbb{E}_x^{TRW}\left[\sum_{k=0}^{\tau} I(X_k=y)\right]. \tag{30}$$

Proof. By conditioning on the first step we obtain

$$\mathbb{E}_x^{TRW}\left[\sum_{k=0}^{\tau} I(X_k=y)\right] = \sum_{z\in\Lambda_n}\mathbb{E}_x^{TRW}\left[\sum_{k=0}^{\tau} I(X_k=y)|X_1=z\right]\mathbb{P}(X_1=z|X_0=x)+\delta_{x,y}.$$

Since the killing probability is $1/(2d+1)$ on traps

$$\mathbb{P}(X_1=z|X_0=x) = \begin{cases} \frac{1}{2d+1} & \text{for } z\sim x \text{ and } x\in D_n \\ \frac{1}{2d} & \text{for } z\sim x \text{ and } x\notin D_n \end{cases}$$

this means

$$\mathbb{E}_x^{TRW}\left[\sum_{k=0}^{\tau} I(X_k=y)\right] = I(x\in D_n)\frac{1}{2d+1}\sum_{z\sim x}\mathbb{E}_x^{TRW}\left[\sum_{k=1}^{\tau} I(X_k=y)|X_1=z\right] \tag{31}$$
$$+ I(x\notin D_n)\frac{1}{2d}\sum_{z\sim x}\mathbb{E}_x^{TRW}\left[\sum_{k=1}^{\tau} I(X_k=y)|X_1=z\right]+\delta_{x,y}. \tag{32}$$

we use that $\frac{1}{2d+1} = \frac{2d}{2d+1}\cdot\frac{1}{2d}$ to write

$$\mathbb{E}_x^{TRW}\left[\sum_{k=0}^{\tau} I(X_k=y)\right] = \left(\frac{2d}{2d+1}\right)^{I(x\in D_n)}\sum_{z\sim x}\mathbb{E}_x^{TRW}\left[\sum_{k=1}^{\tau} I(X_k=y)|X_1=z\right]+\delta_{x,y}.$$

Now we use the fact that the random walk is memoryless, meaning

$$\mathbb{E}_x^{TRW}\left[\sum_{k=1}^{\tau} I(X_k = y) | X_1 = z\right] = \mathbb{E}_x^{TRW}\left[\sum_{k=0}^{\tau} I(X_k = y)\right].$$

We obtain

$$\mathbb{E}_x^{TRW}\left[\sum_{k=0}^{\tau} I(X_k = y)\right] = \left(\frac{2d}{2d+1}\right)^{I(x \in D_n)} \frac{1}{2d} \sum_{z \sim x} \mathbb{E}_x^{TRW}\left[\sum_{k=0}^{\tau} I(X_k = y)\right] + \delta_{x,y}.$$

We can now use our previous notation $l_k(D_n) = \sum_{i=0}^{k} I(X_i \in D_n)$

If we iterate the process of conditioning on each step we end up with

$$\mathbb{E}_x^{TRW}\left[\sum_{k=0}^{\tau} I(X_k = y)\right] = \mathbb{E}_x^{RW}\left[\sum_{k=0}^{\tau} \left(\frac{2d}{2d+1}\right)^{l_k(D_n)} I(X_k = y)\right].$$

We can conclude using Theorem 3.4 □

4 Survival Time of Trapped Random Walks

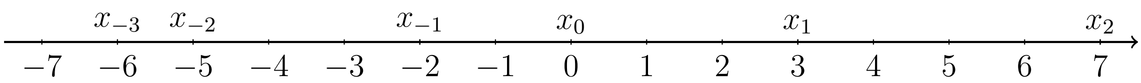
In the following section we consider several cases of trapped random walks. With the aid of Theorem 3.5 we immediately see that the corresponding sandpile model is critical if the expected survival time τ is infinite. Likewise, a finite survival time indicates non-criticality. Indeed, when we sum $G(x,y) = \frac{1}{2d}\mathbb{E}_x^{TRW}\left[\sum_{k=0}^{\tau} I(X_k = y)\right]$ over $y \in \mathbb{Z}^d$ we find $\frac{1}{2d}\mathbb{E}_x^{TRW}(\tau)$. The characterization in Theorem 3.3 gives the relation between survival time and criticality.

4.1 Criticality in One Dimension

In this subsection we aim to develop an intuition for how traps should be distributed in one dimension to have criticality. At this point one might wonder if the model can be non-critical if we have only finitely many traps, or, the other way around, if the model can be critical if there is an infinite amount of traps. The answer to the first question turns out to be no, for a finite number of traps the model will always be critical. The answer to the second question however is yes. We can indeed place infinitely many traps at different positions and still have an infinite expected survival time for our random walk. We prove these claims in the next subsections.

4.1.1 Finitely Many Traps

For $d = 1$ we have $\lim_{n\to\infty} \Lambda_n = \mathbb{Z}$. On the axis below we indicated the traps by x_i, with index $i \in \mathbb{Z}$ chosen zero for the closest trap right of (or at) the origin.



One should realise that if $\mathbb{E}_x^{TRW}(\tau)$ is infinite if and only if $\mathbb{E}_0^{TRW}(\tau)$ infinite. We can see this by by the following argument. Define

$$A = \text{"the random walk walks straight from } x \text{ to zero, surviving all traps"} \quad (33)$$
$$B = \text{"the random walk walks straight from zero to } x\text{, surviving all traps".} \quad (34)$$

Then we use that the random walk is memoryless to see that

$$\mathbb{E}_x^{TRW}(\tau) = \mathbb{P}\{A\}\mathbb{E}_x^{TRW}(\tau|A) + \mathbb{P}\{\neg A\}\mathbb{E}_x^{TRW}(\tau|\neg A) \geq \mathbb{P}\{A\}[\mathbb{E}_0^{TRW}(\tau) + |x|] \quad (35)$$
$$\mathbb{E}_0^{TRW}(\tau) = \mathbb{P}\{B\}\mathbb{E}_0^{TRW}(\tau|B) + \mathbb{P}\{\neg B\}\mathbb{E}_0^{TRW}(\tau|\neg B) \geq \mathbb{P}\{B\}[\mathbb{E}_x^{TRW}(\tau) + |x|]. \quad (36)$$

Since both $P\{A\}$ and $P\{B\}$ are strictly larger than zero, we see that the starting position of our random walk does not influence criticality. Indeed $\mathbb{E}_x^{TRW}(\tau) = \infty \Leftrightarrow \mathbb{E}_0^{TRW}(\tau) = \infty$ and $\mathbb{E}_x^{TRW}(\tau) < \infty \Leftrightarrow \mathbb{E}_0^{TRW}(\tau) < \infty$. From now on we will only consider random walks starting from zero and we will drop subscript x and superscript TRW.

Theorem 4.1. *Let $(S_k)_{k\geq 0}$ be a simple random walk on $\mathbb{Z}$ with traps $D \subset \mathbb{Z}$. If $|D| < \infty$ then $\mathbb{E}(\tau) = \infty$.*

Proof. For any non-negative integer-valued random variable Y we have $\mathbb{E}(Y) = \sum_{n=1}^{\infty}\mathbb{P}\{Y \geq n\}$ [9], hence

$$\mathbb{E}(\tau) = \sum_{n=1}^{\infty}\mathbb{P}\{\tau \geq n\} = \sum_{n=1}^{\infty}\mathbb{E}^{NT}\left[p^{\sum_{k=0}^{n-1} I(S_k \in D)}\right].$$

Here $p = \frac{2}{3}$ is the probability that the random walk survives when hitting a trap. The superscript NT, "not trapped", indicates that we are considering a random walk on a landscape without any traps. Since $|D| \leq \infty$ there must be a trap furthest right from the origin. Call this trap x_i and denote the statement $A = (\forall k \leq |x_i| : S_k < S_{k+1}) \wedge (\forall k > |x_i| : S_k > x_i) \wedge (\tau > |x_i|)$. Then, for $n > |x_i|$, we find a lower bound for the terms in the previous sum

$$\mathbb{E}^{NT}\left[p^{\sum_{k=0}^{n-1} I(S_k \in D)}\right] \geq \mathbb{E}^{NT}\left[p^{\sum_{k=0}^{n-1} I(S_k \in D)} I(A)\right] = \left(\frac{1}{2}\right)^{x_i+1} \frac{1}{\sqrt{n - |x_i|}} p^{i+1} > c\frac{1}{\sqrt{n}}.$$

Here $c > 0$ is some positive constant. The first factor is due to the random walk walking to the right for $x_i + 1$ steps and the third factor is for surviving $i + 1$ traps walking past x_i. The second factor comes from the fact that a one dimensional simple random walk has probability $\frac{1}{\sqrt{t}}$ to not revisit its starting point in t time steps. Since $\sum_{n=x_i+1}^{\infty} \frac{c}{\sqrt{n}}$ diverges, we can conclude $\mathbb{E}(\tau) = \infty$. □

4.1.2 Symmetric Trap Distributions

Having shown that an infinite amount of traps is indispensable for non-criticality, we are now interested in the distribution of these traps. It is intuitively clear that the traps should be placed increasingly further apart to have a critical model. In that case the walk can move away from the origin until the traps are so scarcely distributed that the walk almost never meets them. Before going in more detail about how fast the density of these traps must decay we first make an observation. Whenever the trap landscape is symmetrical around the origin, that is $x_i = -x_{-i}$, the expected survival time of a unilateral random walk is equal to that of a bilateral random walk. With a unilateral random walk we mean a walk which remains at one side of the origin, like $(S_k)_{k\geq 0} \geq 0$. A bilateral random walk is just the usual random walk which can cross the origin.

Theorem 4.2. *For a one dimensional simple random walk $(S_k)_{k\geq 0}$ on a trap landscape with traps $x_i = -x_{-i}$, $i \in \mathbb{Z}$, we have*

$$\mathbb{E}[\tau | \forall k \geq 0 : S_k \geq 0] = \mathbb{E}[\tau]. \tag{37}$$

Proof. We first define

$$T_i = \inf\left\{k : \sum_{j=0}^{k} I(S_j = 0) = i\right\}$$

Let $N+1$ be the number of visits to the origin before getting killed. Then we have, assuming a certain value for $N = n$

$$\mathbb{E}[\tau | N = n] = \mathbb{E}[(T_1 - T_0) + (T_2 - T_1) + ... + (\tau - T_n) | N = n]$$

and

$$\mathbb{E}[\tau | (N = n) \wedge (\forall k \geq 0 : S_k \geq 0)] = \mathbb{E}[(T_1 - T_0) + (T_2 - T_1) + ... + (\tau - T_n) | (N = n) \wedge (\forall k \geq 0 : S_k \geq 0)].$$

Note that $T_0 = 0$ since the random walk starts from the origin and by the symmetry requirement $x_0 = -x_0 = 0$. Due to the symmetry we have termwise equality

$$\mathbb{E}[T_i - T_{i-1} | N = n] = \mathbb{E}[T_i - T_{i-1} | (N = n) \wedge (\forall k \geq 0 : S_k \geq 0)]$$
$$\mathbb{E}[\tau - T_n | N = n] = \mathbb{E}[\tau - T_n | (N = n) \wedge (\forall k \geq 0 : S_k \geq 0)].$$

Hence $\mathbb{E}[\tau|(N=n)] = \mathbb{E}[\tau|(N=n) \wedge (\forall k \geq 0 : S_k \geq 0)]$. Now we only need

$$\mathbb{P}\{n = N\} = \mathbb{P}\{n = N|\forall k \geq 0 : S_k \geq 0)\}.$$

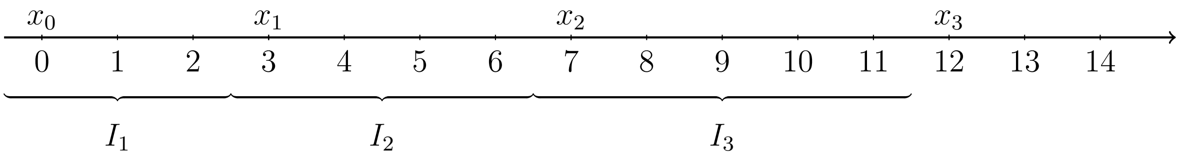
In both the unilateral and the bilateral case N has geometric distribution with parameter $p = \mathbb{P}\{T_1 \geq \tau\}$. Hence the probability that $N = n$ is in both cases the same. This is enough to conclude

$$\mathbb{E}[\tau] = \sum_{n=0}^{\infty} \mathbb{P}\{n = N\}\mathbb{E}[\tau|(N=n)] =$$
$$\sum_{n=0}^{\infty} \mathbb{P}\{n = N|\forall k \geq 0 : S_k \geq 0)\}\mathbb{E}[\tau|(N=n) \wedge (\forall k \geq 0 : S_k \geq 0)] =$$
$$\mathbb{E}[\tau|\forall k \geq 0 : S_k \geq 0].$$

□

4.1.3 Quadratical Recursion of Interval Sizes

The main result of our research will be discussed next. We aim to understand how the distances between the traps should increase such that we are on the borderline between a non-critical and a critical model. It turns out that if we define the lengths of the intervals between succeeding traps recursively as $|x_i - x_{i-1}| = c|x_{i-1} - x_{i-2}|^2$, that the criticality depends on c. More precise, we find an upper bound for c below which the walk is definitely non-critical and a lower bound above which the walk is definitely critical. We start by giving the necessary definition which we use to derive an important lemma. This lemma will allow us to find the upper bound for non-criticality. After that we also give a (less complicated) proof for the lower bound. For the remainder of this subsection we will work with unilateral random walks. However, we can easily apply Theorem 4.2 to see that the results also hold when we take the symmetric bilateral counterpart of any of the unilateral trap landscapes.



We consider a one dimensional simple random walk $(S_k)_{k\geq 0}$ right of the origin, that is $\forall k \geq 0 : s_k \geq 0$ and we assume that $S_0 = 0$. The walk moves through a landscape of traps which are numbered from left to right starting from zero, more precise we use x_i as notation for the location of trap number i and define $x_0 \equiv 0$. The traps now partition the non-negative integers in intervals: $I_i = \{x_{i-1}, x_{i-1}+1, ..., x_i - 2, x_i - 1\}$. Let $D = \cup_{i=0}^{\infty} x_i$ denote the set of trap locations. For our purposes it is interesting to consider another random walk $(\xi_n)_{n\geq 0}$ which is embedded in $(S_k)_{k\geq 0}$. For $k_i = \inf\{k \geq 0 : \sum_{j=1}^{k} I(S_j \in D) = i\}$ and $k_0 = 0$, we define $\xi_i = S_{k_i}$ and $\xi_0 = 0$. One can think of $(\xi_n)_{n\geq 0}$ as the walk moving on the set of trap locations D visiting the traps in the same order as $(S_k)_{k\geq 0}$.

We are eventually interested the survival time τ of $(S_k)_{k\geq 0}$. By partitioning the non-negative integers in intervals I_i we can write the survival time as a sum $\tau = \sum_{i=1}^{N} Y_i$ where $N = \sum_{j=0}^{\infty} I(S_j \in D)$ and $Y_i = k_i - k_{i-1}$. One can interpret Y_i as the time between two hits and N as the total number of hits before getting killed. In order to say something meaningful about Y_i we should know in which interval the walker is after hitting $i-1$ traps. This depends entirely on $(\xi_n)_{n\geq 0}$. We describe the first $i-2$ increments of $(\xi_n)_{n\geq 0}$ by the vector $\vec{\kappa} = (\kappa_1, \kappa_2, ..., \kappa_{i-2}) \in \{-1, 0, 1\}^{i-2}$. If

$$\begin{aligned} \xi_{i+1} &= \xi_i \text{ then we have } \kappa_i = 0 \\ \xi_{i+1} &> \xi_i \text{ then we have } \kappa_i = 1 \\ \xi_{i+1} &< \xi_i \text{ then we have } \kappa_i = -1. \end{aligned}$$

We denote the statement "$(\xi_n)_{n\geq 0}$ walks according to $\vec{\kappa}$" by $\mathcal{A}_{\vec{\kappa}}$. Note that if an instance of $(S_k)_{k\geq 0}$ follows a path in accordance with $\vec{\kappa}$ it hits $i-1$ traps. Indeed, at $k = 0$ we have $S_0 = 0$ which we assumed to be a trap. After this first hit the random walk hits a trap for every element of the vector $\vec{\kappa}$. As a consequence the walk eventually ends up on its $i-1^{th}$ trap, ξ_{i-1}. Remember that we are interested in Y_i which is nothing more than the time until the next hit from this point. Using basic facts about the one dimensional simple random walk we can calculate the expectation of Y_i based of length of the intervals left and right of ξ_{i-1}. In the upcoming calculations we will assume that $(S_k)_{k\geq 0}$ jumps right, $\xi_{i-1} < S_{k_{i-1}+1}$, after following $\vec{\kappa}$. Hence we define $\mathcal{A}^+_{\vec{\kappa}} = (\mathcal{A}_{\vec{\kappa}}) \wedge (\xi_{i-1} < S_{k_{i-1}+1})$. Finally we remark that not every $\vec{\kappa}$ yields a valid walk. This is due to our assumption that $\forall k \geq 0 : s_k \geq 0$. We call the set of valid vectors $\mathcal{K}$.

Lemma 4.3. *For traps distributed such that $|I_{i+1}| = c|I_i|^2$, $i \geq 1$, with $c \geq \frac{1}{|I_1|}$ we have*

$\forall \vec{\kappa} \in \mathcal{K}$:

$$\mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\}\mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\kappa}}] \leq (|I_1|c)\Big(\frac{2}{c}\Big)^l\Big(\frac{2}{|I_1|^2c^2}\Big)^j\mathbb{P}\{\mathcal{A}_{\vec{\mathbb{1}}}\}\mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\mathbb{1}}}]. \tag{38}$$

Where $\vec{\mathbb{1}} = \{1\}^{i-2}$ and l, j denote the number of entries $\kappa_x = 0$ and $\kappa_x = -1$ respectively.

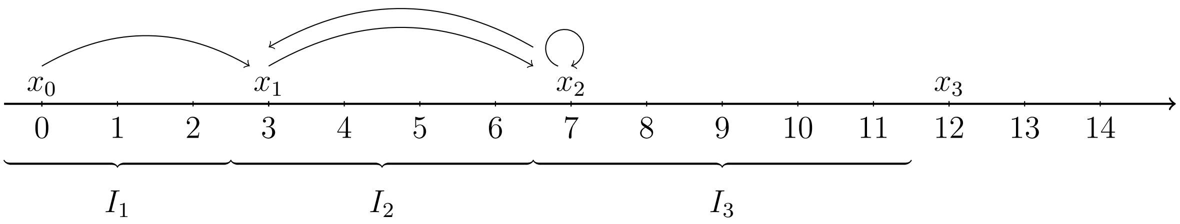
To understand the lemma above we observe how the quantity $\mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\}\mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\kappa}}]$ changes upon swapping an entry $\kappa_x = 1$ for a minus one or zero. We will see that if we start with $\vec{\kappa} = \vec{\mathbb{1}}$ and alter the entries in increasing order (left to right) we obtain the factors in front of the right hand side of (38). For the sake of clarity we first consider a concrete example. Let $\vec{\kappa} = (1, 1, 0, -1, 1, 1, 1)$. In this case first two entries are left at a value of 1, but the third and fourth entry are altered. We now swap out the fifth entry for a zero. We write $\vec{\kappa}^5_0 = (1, 1, 0, -1, 0, 1, 1)$. The notation $\vec{\kappa}^x_y$ is used for the vector $\vec{\kappa}$ with the x^{th} entry set to y. In this particular case we have

$$\mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\} = \frac{1}{|I_1|}\frac{1}{2|I_2|}\Big(1 - \frac{1}{2|I_2|} - \frac{1}{2|I_3|}\Big)\frac{1}{2|I_2|}\frac{1}{2|I_2|}\frac{1}{2|I_3|}\frac{1}{2|I_4|} \tag{39}$$

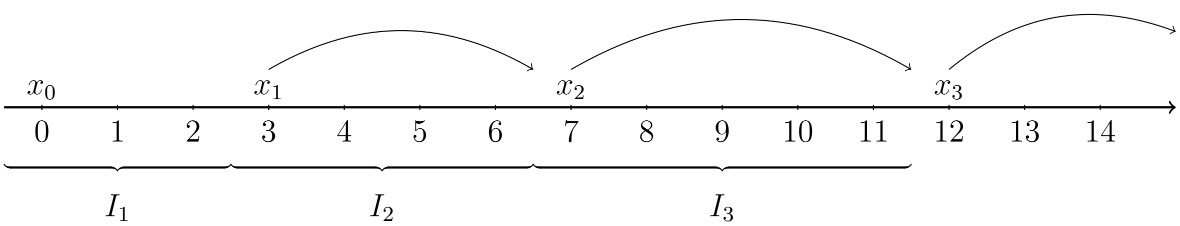
and

$$\mathbb{P}\{\mathcal{A}_{\vec{\kappa}^5_0}\} = \frac{1}{|I_1|}\frac{1}{2|I_2|}\Big(1 - \frac{1}{2|I_2|} - \frac{1}{2|I_3|}\Big)\frac{1}{2|I_2|}\Big(1 - \frac{1}{2|I_2|} - \frac{1}{2|I_1|}\Big)\frac{1}{2|I_2|}\frac{1}{2|I_3|}. \tag{40}$$

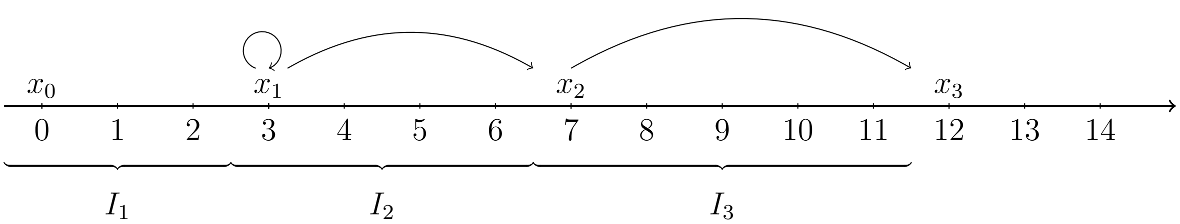
We will go through the factors in (39) one by one.



The first jump of $(S_k)_{k\geq 0}$ is always to the right since we require $\forall k \geq 0 : s_k \geq 0$. Next we use the well known fact that a simple random walk starting at $a+1$ reaches $b > a$ before a with probability $\frac{1}{b-a}$. We see that $(S_k)_{k\geq 0}$ reaches x_1 before x_0, as required by κ_1, with probability $\frac{1}{|I_1|}$. Hence the first factor $\frac{1}{|I_1|}$.
After reaching x_1 the walk $(S_k)_{k\geq 0}$ has to jump right since $\kappa_2 = 1$. This happens with probability $\frac{1}{2}$. We now are in a similar situation as before, the walk is located at 4, one step right of x_1 and has to reach x_2 before bumping into x_1 again. This happens, by the same reasoning we used for κ_1, with probability $\frac{1}{|I_2|}$. We hence get a total factor $\frac{1}{2|I_2|}$ due to κ_2.
$(S_k)_{k\geq 0}$ arrives at x_2 next. This time κ_3 requires the walk to hit x_2 again before reaching either x_1 or x_3. This happens with probability $(1 - \frac{1}{2|I_2|} - \frac{1}{2|I_3|})$. Indeed, the walk can jump left and then reach x_1 before returning to x_2 with probability $\frac{1}{2|I_2|}$ or similarly jump right en reach x_3 with probability $\frac{1}{2|I_3|}$.
At this point $(S_k)_{k\geq 0}$ is still at x_2. $\kappa_4 = -1$, dictates $(S_k)_{k\geq 0}$ to jump left, and reach x_1 without revisiting x_2. Again we are in a situation very similar to what we had for κ_1. The probability is now $\frac{1}{|I_2|}$ since this time $(S_k)_{k\geq 0}$ should transverse interval I_2, giving total factor $\frac{1}{2|I_2|}$.



Now $(S_k)_{k\geq 0}$ is at x_1. The remaining entries κ_5, κ_6 and κ_7 are all one, so $(S_k)_{k\geq 0}$ will jump to the right, reach x_2 before returning to x_1, jump right again, reach x_3 before returning to x_2 and finally reaching x_4 in a similar fashion. This results in the last three factors $\frac{1}{2|I_2|}\frac{1}{2|I_3|}\frac{1}{2|I_4|}$ in (39).



When we calculate $\mathbb{P}\{\mathcal{A}_{\vec{\kappa}_0^5}\}$, we obviously find that the first four factors are the same, since the first four entries of $\vec{\kappa}$ and $\vec{\kappa}_0^5$ are the same. However, the fifth entry of $\vec{\kappa}_0^5$ requires $(\xi_n)_{n\geq 1}$ to remain at x_1. This means we have the factor $(1 - \frac{1}{2|I_1|} - \frac{1}{2|I_2|})$ instead of $\frac{1}{2|I_2|}$ in (40). The extra "delay" at x_1 also means that $(S_k)_{k\geq 0}$ won't transverse interval I_4 in the end. Hence

$$\mathbb{P}\{\mathcal{A}_{\vec{\kappa}_0^5}\} = 2|I_4|\Big(1 - \frac{1}{2|I_1|} - \frac{1}{2|I_2|}\Big) \cdot \mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\} \leq 2|I_4| \cdot \mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\}. \tag{41}$$

For the general case, where vector $\vec{\kappa}$ is obtained by altering entries $\kappa_1, \kappa_2, ..., \kappa_{x-1}$ in $\vec{\mathbb{1}}$, we can formulate a similar inequality for $\vec{\kappa}$ and $\vec{\kappa}_0^x$.

$$\mathbb{P}\{\mathcal{A}_{\vec{\kappa}_0^x}\} = 2|I_{f-1}|\Big(1 - \frac{1}{2|I_{c-1}|} - \frac{1}{2|I_c|}\Big) \cdot \mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\} \leq 2|I_{f-1}| \cdot \mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\}. \tag{42}$$

Here I_f is the interval right of the last trap $(S_k)_{k\geq 0}$ visits when walking in accordance to $\vec{\kappa}$. In the case of $\vec{\kappa} = (1,1,0,-1,1,1,1)$ the walk ends up at x_3, so $I_f = I_4$ since I_4 is the interval right of x_3. I_c is just an unknown interval, it depends on entries $\kappa_1, ..., \kappa_{x-1}$. The factor $2|I_f|$ comes from the delay we described earlier in the case $\vec{\kappa} = (1,1,0,-1,1,1,1)$, which results in the walk not crossing I_{f-1} for $\vec{\kappa}_0^x$. The factor $(1 - \frac{1}{2|I_{c-1}|} - \frac{1}{2|I_c|})$ is the probability that $(S_k)_{k\geq 0}$ hits a certain trap twice in a row, as is dictated by entry x of $\vec{\kappa}_0^x$. Notice that this factor is slightly different if the extra zero in $\vec{\kappa}_0^x$ requires $(S_k)_{k\geq 0}$ to return to x_0. Since we bound the factor $(1 - \frac{1}{2|I_{c-1}|} - \frac{1}{2|I_c|})$ by one, our inequality remains valid.

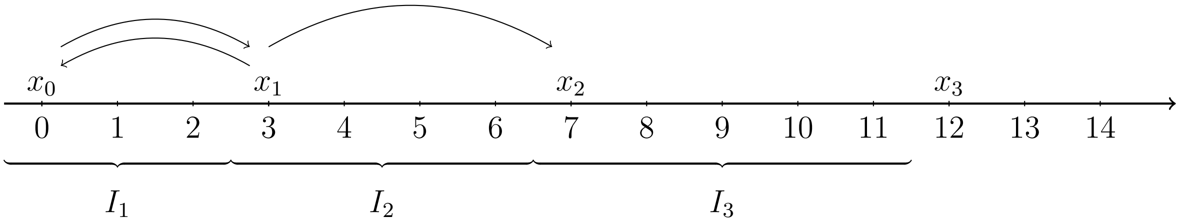
We eventualy want to compare $\mathbb{P}\{\mathcal{A}_{\vec{\kappa}_0^x}\}\mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\kappa}_0^x}]$ with $\mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\}\mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\kappa}}]$. We just studied $\mathbb{P}\{\mathcal{A}_{\vec{\kappa}_0^x}\}$ and $\mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\}$, so the next step is to compare the expectations. We use the well known fact that a simple random walk starting at $a < \bar{x} < b$ reaches either a or b at time T with

$$\mathbb{E}[T] = (\bar{x} - a)(b - \bar{x}) \tag{43}$$

In the example $\vec{\kappa} = (1,1,0,-1,1,1,1)$ leads $(S_k)_{k\geq 0}$ to trap x_4. Then $\mathcal{A}^+_{\vec{\kappa}}$ implies that $(S_k)_{k\geq 0}$ jumps between x_4 and x_5. Via (43) we now immediately see that $\mathbb{E}[k_i - k_{i-1}|\mathcal{A}^+_{\vec{\kappa}}] = \mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\kappa}}] = ((x_4+1) - x_4)(x_5 - (x_4+1)) + 1 = |I_5|$. The extra $+1$ is necessary to compensate for the jump from x_4 to x_4+1. A walk according to $\vec{\kappa}_0^5$ ends up at trap x_3. As a consequence $\mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\kappa}_0^5}] = |I_4|$. In the general case this translates as $\mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\kappa}}] = |I_f|$ and $\mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\kappa}_0^x}] = |I_{f-1}|$. Hence

$$\mathbb{P}\{\mathcal{A}_{\vec{\kappa}_0^x}\}\mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\kappa}_0^x}] \leq 2|I_{f-1}| \cdot \mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\} \cdot \frac{|I_{f-1}|}{|I_f|} \cdot \mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\kappa}}] = \frac{2}{c}\mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\}\mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\kappa}}]. \tag{44}$$

Here we used that $|I_f| = c|I_{f-1}|^2$. The inequality above gives the factor $(\frac{2}{c})^l$ in (38). Indeed, each time we swap out a one in $\vec{\mathbb{1}}$ for a zero we have to add the factor $\frac{2}{c}$.



Now we have an understanding of the influence of swapping a one for a zero in $\vec{\kappa}$, we will reason what happens when we replace a one by a minus one. Keep in mind that we change the entries in $\vec{\mathbb{1}}$ from left to right. Because we alter $\vec{\mathbb{1}}$ this way we can assume that each time we change an entry $\kappa_x = 1$ the entries with index $y > x$ are $\kappa_y = 1$. For the moment we will assume that the entry x in $\vec{\kappa}_{-1}^x$ is not the last entry, i.e. $x \neq i-2$. We will first look at the example $\vec{\kappa} = (1,1,0,-1,1,1,1)$ again. In this case

$$\mathbb{P}\{\mathcal{A}_{\vec{\kappa}_{-1}^5}\} = \frac{1}{|I_1|}\frac{1}{2|I_2|}\Big(1 - \frac{1}{2|I_2|} - \frac{1}{2|I_3|}\Big)\frac{1}{2|I_2|}\frac{1}{2|I_1|}\frac{1}{|I_1|}\frac{1}{2|I_2|}. \tag{45}$$

The first four factors are again the same as in (39). The fifth factor is $\frac{1}{2|I_1|}$ since $(\xi_n)_{n\geq 0}$ jumps from x_1 to x_0 by the extra one in $\vec{\kappa}_{-1}^5$. The last two entries in $\vec{\kappa}_{-1}^5$ are one, so they make $(\xi_n)_{n\geq 0}$

jump forwards twice, from x_0 to x_1 and from x_1 to x_2. Therefore the last two factors are now $\frac{1}{|I_1|}\frac{1}{2|I_2|}$. For general $\vec{\kappa}$, meeting the requirements from earlier, we have the similar result

$$\mathbb{P}\{\mathcal{A}_{\vec{\kappa}^x_{-1}}\} \leq \frac{2|I_{f-1}|2|I_{f-2}|}{2|I_c||I_c|} \cdot \mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\} \leq \frac{2}{|I_1|^2}|I_{f-1}||I_{f-2}| \cdot \mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\}. \tag{46}$$

Just as in the case where we added a zero, we have that the extra minus one causes some form of delay. Only now $(\xi_n)_{n\geq 0}$ spends two jumps, one backward and one forwards, before jumping only forward due to the remaining tail of ones in $\vec{\kappa}^x_{-1}$. Due to these two extra jumps $(S_k)_{k\geq 0}$ strands two intervals earlier than without the perturbation. Hence the factor $2|I_{f-1}|2|I_f|$. $(S_k)_{k\geq 0}$ also transverses a certain interval I_c two times more than in the unperturbed case, giving rise to the $\frac{1}{2|I_c||I_c|}$. One can easily verify that this is exactly what happens in the example $\vec{\kappa} = (1, 1, 0, -1, 1, 1, 1)$.

For $\vec{\kappa}^{i-2}_{-1}$ the situation is slightly different. Because this time it is the last entry of $\vec{\kappa}$ we alter. this means that $|I_c| = |I_{f-2}|$ and that the random walk has to cross $|I_c|$ only once. This means

$$\mathbb{P}\{\mathcal{A}_{\vec{\kappa}^{i-2}_{-1}}\} = \frac{|I_{f-1}|}{|I_{f-2}|} \cdot \mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\}. \tag{47}$$

The expectation $\mathbb{E}[Y_i|\mathcal{A}_{\vec{\kappa}^x_{-1}}]$ is both for $x = i-2$ and $x \neq i-2$ equal to $|I_{f-2}|$, compared to $\mathbb{E}[Y_i|\mathcal{A}_{\vec{\kappa}}] = |I_f|$ for $\vec{\kappa}$ unaltered.

Using our upperbound (46) and $\mathbb{E}[Y_i|\mathcal{A}_{\vec{\kappa}^x_{-1}}] = |I_{f-2}|$ we find

$$\mathbb{P}\{\mathcal{A}_{\vec{\kappa}^x_{-1}}\}\mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\kappa}^x_{-1}}] \leq \frac{2}{|I_1|^2}|I_{f-1}||I_{f-2}| \cdot \mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\} \cdot \frac{|I_{f-2}|}{|I_f|} \cdot \mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\kappa}}] = \frac{2}{(|I_1|c)^2}\mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\}\mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\kappa}}] \tag{48}$$

for the scenario where $x \neq i-2$. Of course we used the relation $|I_i| = c|I_{i-1}|^2$ to cancel the necessary interval lengths. When $x = i-2$ we find the similar result

$$\mathbb{P}\{\mathcal{A}_{\vec{\kappa}^{i-2}_{-1}}\}\mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\kappa}^{i-2}_{-1}}] = \frac{|I_{f-1}|}{|I_{f-2}|} \cdot \mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\} \cdot \frac{|I_{f-2}|}{|I_f|} \cdot \mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\kappa}}] \leq \frac{1}{|I_1|c}\mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\}\mathbb{E}[Y_i|\mathcal{A}^+_{\vec{\kappa}}]. \tag{49}$$

Using inequalities (44) and (48) we now find the statement in Lemma 4.3. For every entry we make zero, we obtain a factor $\frac{2}{c}$ and for every entry we make minus one we obtain $\frac{1}{(|I_1|c)^2}$. If we happen to make our last entry minus one however, we have to add a factor $c|I_1|$ to correct. By our assumption $c \geq \frac{1}{|I_1|}$ we can add this factor without ruining our upper bound, also in the case where we don't change the last entry.

Theorem 4.4. *For traps distributed such that* $|I_{i+1}| = c|I_i|^2$, *with* $\frac{1}{|I_1|} \leq c < \frac{|I_1|+\sqrt{|I_1|^2-8}}{2|I_1|}$ *and* $|I_1| \geq 3$, *we have*

$$\mathbb{E}[\tau] < \infty. \tag{50}$$

Proof. The total lifetime of $(S_k)_{k\geq 0}$ is given by $\tau = \sum_{i=1}^N Y_i$. We write the expectation as a sum over the possible values of N.

$$\mathbb{E}[\tau] = \mathbb{E}\left[\sum_{i=1}^N Y_i\right] = \sum_{n=1}^\infty \left[\mathbb{P}\{N = n\} \sum_{i=1}^n \mathbb{E}[Y_i|N = n]\right].$$

Next we expand the expectations $\mathbb{E}[Y_i|N=n]$ in a sum over all allowed $\vec{\kappa}$

$$\begin{aligned}\mathbb{E}[\tau] &= \sum_{n=1}^{\infty}\left[\mathbb{P}\{N=n\}\sum_{i=1}^{n}\sum_{\vec{\kappa}\in\mathcal{K}}\mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\}\mathbb{E}[Y_i|(N=n)\wedge(\mathcal{A}_{\vec{\kappa}})]\right]\\ &\le \sum_{n=1}^{\infty}\left[\mathbb{P}\{N=n\}\sum_{i=1}^{n}\sum_{\vec{\kappa}\in\mathcal{K}}\mathbb{P}\{\mathcal{A}_{\vec{\kappa}}\}\mathbb{E}[Y_i|\mathcal{A}_{\vec{\kappa}}^{+}]\right]\\ &\le \sum_{n=1}^{\infty}\left[\mathbb{P}\{N=n\}\sum_{i=1}^{n}\sum_{l=0}^{i}\sum_{j=0}^{l}c|I_1|\binom{i}{l}\binom{l}{j}\Big(\frac{2}{c}\Big)^{l-j}\Big(\frac{2}{|I_1|^2c^2}\Big)^{j}\mathbb{P}\{\mathcal{A}_{\vec{\mathbb{1}}}\}\mathbb{E}[Y_i|\mathcal{A}_{\vec{\mathbb{1}}}^{+}]\right].\end{aligned}$$

For the last inequality we filled in our upper bound from Lemma 4.3 and summed over all configurations of $\vec{\kappa}$. Notice that not all terms represent allowed configurations. This is not a problem for the argument since all terms are positive. We can calculate the sums over l and j exactly using the binomial theorem

$$\begin{aligned}\sum_{l=0}^{i}\sum_{j=0}^{l}\binom{i}{l}\binom{l}{j}\Big(\frac{2}{c}\Big)^{l-j}\Big(\frac{2}{|I_1|^2c^2}\Big)^{j} &= \sum_{l=0}^{i}\binom{i}{l}\Big(\frac{2}{c}\Big)^{l}\sum_{j=0}^{l}\binom{l}{j}\Big(\frac{c}{2}\Big)^{j}\Big(\frac{2}{|I_1|^2c^2}\Big)^{j}\\ &= \sum_{l=0}^{i}\binom{i}{l}\Big(\frac{2}{c}\Big)^{l}\Big(1+\frac{1}{|I_1|^2c}\Big)^{l}\\ &= \Big(1+\frac{2}{c}+\frac{2}{|I_1|^2c^2}\Big)^{i}.\end{aligned}$$

Substituting this back in the original inequality we get

$$\mathbb{E}[\tau] \le |I_1|c\sum_{n=1}^{\infty}\left[\mathbb{P}\{N=n\}\sum_{i=1}^{n}\mathbb{P}\{\mathcal{A}_{\vec{\mathbb{1}}}\}\mathbb{E}[Y_i|\mathcal{A}_{\vec{\mathbb{1}}}^{+}]\Big(1+\frac{2}{c}+\frac{2}{|I_1|^2c^2}\Big)^{i}\right].$$

Since the walk has has chance $\frac{1}{3}$ to be killed on each trap, we have $\mathbb{P}\{N=n\}=\frac{1}{3}(\frac{2}{3})^{n-1}$. We can also easily calculate $\mathbb{P}\{\mathcal{A}_{\vec{\mathbb{1}}}\}\mathbb{E}[Y_i|\mathcal{A}_{\vec{\mathbb{1}}}^{+}] = \frac{|I_i|}{\prod_{j=1}^{i-1}2|I_j|} = |I_1|\frac{c^{i-1}}{2^{i-1}}$ as is shown via induction below.

For $i=1$ we have $\frac{|I_1|}{1}=|I_1|$. Assume $\frac{|I_k|}{\prod_{j=1}^{k-1}|I_j|}=|I_1|c^{k-1}$ then

$$\frac{|I_{k+1}|}{\prod_{j=1}^{k}|I_j|} = \frac{c|I_k|^2}{\prod_{j=1}^{k-1}|I_j|\cdot|I_k|} = c\frac{|I_k|}{\prod_{j=1}^{k-1}|I_j|} = |I_1|c^k.$$

Hence by induction we have shown $\frac{|I_i|}{\prod_{j=1}^{i-1}|I_j|}=|I_1|c^{i-1}$.

As a consequence the right hand side converges whenever $(\frac{c}{2})^i(1+\frac{2}{c}+\frac{2}{|I_1|^2c^2})^i<(\frac{3}{2})^i$, or in another form $|I_1|^2c^2-|I_1|^2c+2<0$. This yields $\frac{|I_1|-\sqrt{|I_1|^2-8}}{2|I_1|}<c<\frac{|I_1|+\sqrt{|I_1|^2-8}}{2|I_1|}$. □

Remark 4.5. *On first sight it might seem that $|I_1|$ plays an important role for the bounds on c. For example $|I_1|=1$ or $|I_1|=2$ give non-real solutions in the bounds derived above. However as long as $c\ge\frac{1}{|I_1|}$ the traps become spaced further apart if $|I_1|$ grows. Therefore we would* *expect, for a certain value of c, that the survival time increases with $|I_1|$. If this intuition turns out to be correct, we can just replace $\frac{|I_1|+\sqrt{|I_1|^2-8}}{2|I_1|}$ by one (taking the limit $|I_1| \to \infty$). Also note that $\frac{|I_1|-\sqrt{|I_1|^2-8}}{2|I_1|} < \frac{1}{|I_1|}$ for $|I_1| \geq 3$. Since we need $c > \frac{1}{|I_1|}$ anyway to have increasing interval lengths, the lower bound $\frac{|I_1|-\sqrt{|I_1|^2-8}}{2|I_1|} < c$ can be safely replaced by $\frac{1}{|I_1|} < c$ as is done in the statement of the theorem.*

We can also find, using a similar approach as above, a lower bound for c above which the model is certainly critical. This is in fact an easier task than finding the regime where the model is definitely non-critical since we can bound the expectation of Y_i from below by the contribution (to this expectation) of a single class of walks, in this case $\mathcal{A}_{\vec{\mathbb{1}}}$. We can easily identify for which c this contribution becomes infinite, and we hence have criticality. This is exactly what is done in the proof below.

Theorem 4.6. *For traps distributed such that $|I_{i+1}| = c|I_i|^2$ with $c > 3$ we have*

$$\mathbb{E}[\tau] = \infty \tag{51}$$

Proof. As in the proof for non-criticality we write $\tau = \sum_{i=1}^{N} Y_i$. The expectation is written again as a sum over the possible values of N.

$$\mathbb{E}[\tau] = \mathbb{E}\left[\sum_{i=1}^{N} Y_i\right] = \sum_{n=1}^{\infty}\left[\mathbb{P}\{N=n\}\sum_{i=1}^{n}\mathbb{E}[Y_i|N=n]\right].$$

We use the contribution of instances walking according to $\mathcal{A}^{+}_{\vec{\mathbb{1}}}$ to bound the expectations $\mathbb{E}[Y_i|N=n]$ from below.

$$\mathbb{E}[Y_i|N=n] \geq \mathbb{E}[Y_i \cdot I(\mathcal{A}^{+}_{\vec{\mathbb{1}}})|N=n] = \mathbb{P}\{\mathcal{A}^{+}_{\vec{\mathbb{1}}}\}\mathbb{E}[Y_i|\mathcal{A}^{+}_{\vec{\mathbb{1}}}]$$

Notice that it is easy to calculate both $\mathbb{P}\{\mathcal{A}^{+}_{\vec{\mathbb{1}}}\}$ and $\mathbb{E}[Y_i|\mathcal{A}^{+}_{\vec{\mathbb{1}}}]$

$$\mathbb{P}\{\mathcal{A}^{+}_{\vec{\mathbb{1}}}\} = \prod_{j=1}^{i-2}\frac{1}{2|I_j|}$$

$$\mathbb{E}[Y_i|\mathcal{A}^{+}_{\vec{\mathbb{1}}}] = |I_{i-1}|$$

This yields

$$\mathbb{E}[\tau] \geq \sum_{n=1}^{\infty}\left[\mathbb{P}\{N=n\}\sum_{i=1}^{n}\frac{|I_{i-1}|}{\prod_{j=1}^{i-2}2|I_j|}\right].$$

Since $\mathbb{P}\{N=n\} = (\frac{2}{3})^{n-1}(\frac{1}{3})$ we must have $\sum_{i=1}^{n}\frac{|I_{i-1}|}{\prod_{j=1}^{i-2}2|I_j|} \gtrsim (\frac{3}{2})^n$ to obtain divergence. This is the case when $\frac{|I_i|}{\prod_{j=1}^{i-1}2|I_j|} \geq (\frac{3}{2})^i$ or $\frac{|I_i|}{\prod_{j=1}^{i-1}|I_j|} \geq 3^i$. As a consequence

$$\frac{|I_i|}{|I_{i-1}|} \geq \frac{3^i\prod_{j=1}^{i-1}|I_j|}{3^{i-1}\prod_{j=1}^{i-2}|I_j|} = 3|I_{i-1}|$$

or

$$|I_i| \geq 3|I_{i-1}|^2$$

This concludes the proof. □

Remark 4.7. *For traps distributed with quadratically increasing intervals, $|I_{i+1}| = c|I_i|^2$, we have that function $\Phi(i) = |I_i|$ is given by*

$$\begin{aligned}
|I_1| &= |I_1| \\
|I_2| &= c|I_1|^2 \\
|I_3| &= c \cdot (c|I_1|^2)^2 = c^3|I_1|^4 \\
|I_4| &= c \cdot (c^3|I_1|^4)^2 = c^7|I_1|^8 \\
|I_5| &= c \cdot (c^7|I_1|^8)^2 = c^{15}|I_1|^{16} \\
&\dots \\
|I_i| &= c^{2^{i-1}-1}|I_1|^{2^{i-1}} = \frac{1}{c}\sqrt{c|I_1|}^{2^i}
\end{aligned} \tag{52}$$

For $|I_1| = 4$ and $c = \frac{1}{2} < \frac{|I_1|+\sqrt{|I_1|^2-8}}{2|I_1|} = \frac{4+2\sqrt{2}}{8}$ we have $\Phi(i) = 2^{2^{i-1}+1}$, which is an example of a trap distribution for which the model is non-critical. For $|I_1| = 2$ and $c = 4 > 3$ we find $\Phi(i) = 2^{3\cdot 2^{i-1}-2}$, this trap distribution gives criticality.

4.2 Criticality in Higher Dimensions

In dimensions $d = 2$ and higher the traps do not generally partition the grid in disjoint areas in a way similar to how this happened in the one dimensional case. Hence the results we found are not necessarily easy to extend to higher dimensions. Furthermore in dimensions $d = 3$ and higher the simple random walk is known to be *transient*, while for $d = 1$ or $d = 2$ it is *recurrent* [9], meaning that

$$\mathbb{P}_{\vec{x}}\{\exists k > 0 : S_k = \vec{x}\} = 1 \text{ for } d = 1, 2 \tag{53}$$

$$\mathbb{P}_{\vec{x}}\{\exists k > 0 : S_k = \vec{x}\} < 1 \text{ for } d \geq 3. \tag{54}$$

Here the subscript $\mathbb{P}_{\vec{x}}$ indicates that $S_0 = \vec{x}$. We are not taking traps into account yet, hence we should interpret this result as follows. In one and two dimensions we know for sure that once the random walk leaves a certain site $\vec{x}$ it will return at some later time. We can even say that the random walk will visit every site infinitely often. Indeed, we can always walk from $\vec{x}$ to a certain site $\vec{y}$ with some probability $0 < p < 1$. Since we can visit $\vec{y}$ with non-zero probability infinitely often (every time the walk hits $\vec{x}$), we will certainly visit it at some point. In dimensions three and higher we don't have this property, there we have in fact a non-zero probability that the random walk never visits its starting point again. This implies that the walk will move towards infinity over time, in the sense that $\forall n \geq 0 : \lim_{k\to\infty} \mathbb{P}\{|S_k| > n\} = 1$. One can easily see this by considering a closed ball with radius n. Since there are finitely many sites located inside the ball, the random walk should leave the ball at some point. If it were to keep returning there would be a minimum probability p that it would reach its starting point via some path inside the ball. Hence the walk would visit its starting point infinitely often, which is by definition not the case. This transient property for $d \geq 3$ allows the random walk to avoid certain sites. As a consequence it is possible to distribute the traps so scarce that the walk avoids all of them with non-zero probability. This would instantly yield an infinite expectation of the survival time. This "easy" strategy to find trap landscapes leading to criticality is discussed below. After that we propose a way to arrange the traps in spherical shells around the origin such that the random walk is forced to hit the traps, even in higher dimensions, but is still critical.

4.2.1 Randomly Placed Traps

In [10] the following problem is analysed. For $d \geq 3$ let $p = (p_x)_{x \in \mathbb{Z}^d}$ b a collection of numbers in the interval $[0, 1)$ satisfying

$$\lim_{||x|| \to \infty} p_x = 0. \tag{55}$$

Site x is a trap with probability p_x, independent of all other sites. We want to find necessary and sufficient conditions on p such that the walk never hits a trap and hence survives forever. In [10] F. den Hollander, M.V. Menshikov and S.E.Volkov distinguish two different scenarios. In the first scenario site x is a trap forever with probability p_x. In the second scenario a site x changes it status every time step. The probability to become a trap at a certain time step is then p_x. These scenarios are also commonly referred to as the "quenched" and the "annealed" problem. For us the quenched problem is most interesting, because of its correspondence to the Abelian sandpile model. The most important result in the article by F. den Hollander, M.V. Menshikov and S.E.Volkov is stated below.

Theorem 4.8. *For a simple random walk on $\mathbb{Z}^d$, with $d \geq 3$, we have*
$\forall x \in \mathbb{Z}^d : \pi(x) = 1$*, if* $p_x \geq \alpha/||x||^2$ *for large* $||x||$ *and some* $\alpha > 0$.
$\forall x \in \mathbb{Z}^d : \pi(x) < 1$*, if* $p_x \leq p(||x||)$*, with* $r \to p(r)$ *non-increasing and*

$$\int_0^\infty rp(r)dr < \infty. \tag{56}$$

Here $\pi(x) = \mathbb{P}\{\exists k \geq 0 : S_k \in D | X_0 = x\}$ is the probability of hitting a trap at some point starting from x. For every two sites x and y there is non-zero probability that the traps are such that one can walk from x to y without being trapped. As a consequence we either have $\pi(x) = 1$ for all x or $\pi(x) < 1$ for all x. The proof of this theorem relies on some results of potential theory and is covered in full detail in [10].

If we assume p_x to decay with $||x||$ according to a power law $p_x = \frac{\alpha}{||x||^c}$, $p_0 = 0$ we find for any $0 < \alpha < 1$ and $c \leq 2$ that $\pi(x) = 1$. This means that the walk will hit a trap eventually with absolute certainty. However, this does not mean that the walk is non-critical, in dimensions one and two the walk also hits the traps (if there are any) with absolute certainty, but this doesn't a priori mean that the expected survival time is finite. For $c > 2$, we have $p_x \leq p(||x||)$ with $p(r) = \frac{\alpha}{r^c}$ such that

$$\int_0^\infty rp(r)dr = \int_0^\infty \frac{\alpha}{r^{c-1}}dr = \frac{\alpha}{2-c} < \infty. \tag{57}$$

In this case the walk has non-zero probability to avoid all traps. Trivially this gives an infinite expectation for the survival time and hence criticality.

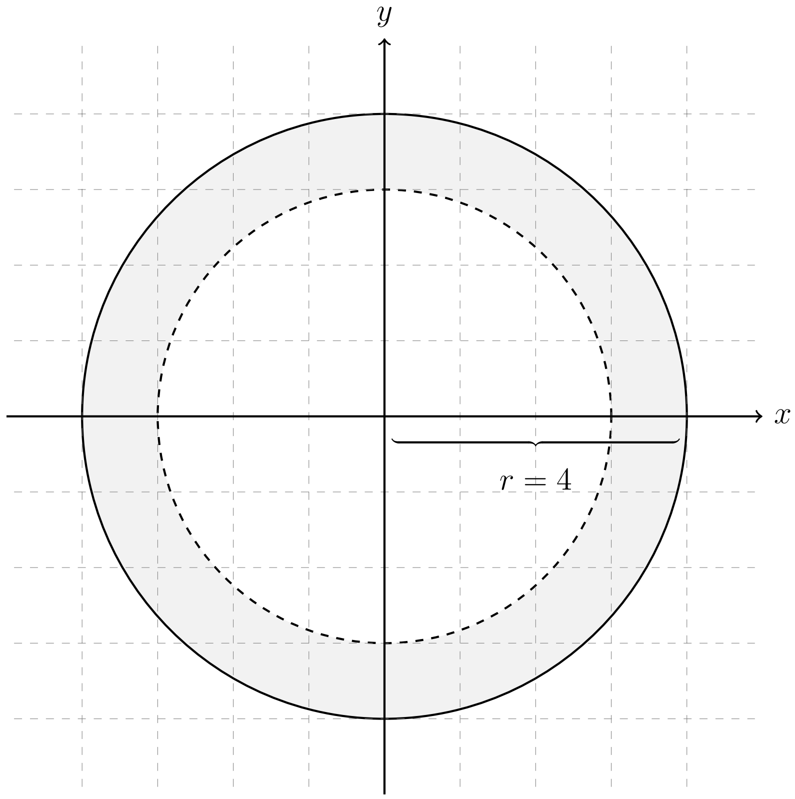
Remark 4.9. *Note that at this point it is unclear how many traps* $p_x = \frac{\alpha}{||x||^c}$, $p_0 = 0$ *gives. If there is a non-zero probability that there are only finitely many traps this result is not very impressive. However, this is not the case. We can use the Borel-Cantelli lemma, which states that if the sum of the probabilities of a series of events is finite, then these events cannot occur simultaniously with non-zero probability. The partial converse of this is also true. If the events are independent and the sum of their probabilities diverges, then infinitely many of these events will occur with probability one. Suppose that* $p_x = \frac{\alpha}{||x||^c}$*. The events we will consider are of the form* $\{$*site* x *is a trap*$\}$*. Then we can find values for* c *such that*

$$\sum_{x\in\mathbb{Z}^d} \frac{1}{||x||^c} = \infty. \tag{58}$$

Hence infinitely many sites will be traps.

4.2.2 Traps on Spherical Shells

Now we have an idea of how the traps should be distributed in order for the random walk to avoid them all with non-zero probability, we are interested in higher dimensional cases where the walk is forced to hit some traps on its way to infinity. Obviously spherical shells $\mathbb{S}_r$ with radius $r > 1$ around the origin are unavoidable when moving towards infinity. We define these as follows



$$x \in \mathbb{S}_r \subset \mathbb{Z}^d \Leftrightarrow r - 1 < ||x|| \leq r. \tag{59}$$

The question is now if we can place infinitely many of these shells with inreasing radii and conserve the criticality of the random walk. It turns out this is indeed possible, as we will demonstrate next. We write the survival time as we did in the proof of Theorem 4.1

$$\mathbb{E}(\tau) = \sum_{n=1}^{\infty} \mathbb{P}\{\tau \geq n\} = \sum_{n=1}^{\infty} \mathbb{E}^{NT}\left[p^{\sum_{k=0}^{n-1} I(S_k\in D)}\right].$$

The p in this expression was the probability of survival on a trap and has value $p = \frac{2d}{2d+1}$. Notice that the quantity $\sum_{k=0}^{n-1} I(S_k \in D)$ is nothing more than the number of traps hit in the first n steps. We assume that the walk follows a certain strategy. First it only walks in one direction, say left, until it passes the point $||S_k|| \geq \gamma(n) = \frac{1}{2}\log_{2d}(n)$. We define the shells to be placed such that there is at most one shell $\mathbb{S}_r$ with $\gamma(n) \leq r \leq n$. Now our strategy continues by first hitting $\mathbb{S}_r$ before returning to $\gamma(n)$. After the walk hits $\mathbb{S}_r$ it moves on to radius n without visiting $\mathbb{S}_r$ again. This strategy is summarized in the statement

$$\mathcal{A}_n = (\forall k \leq \gamma(n) : S_{k+1} = S_k + \hat{e}_1) \wedge (\Theta_{\gamma(n)} > \Theta_{R^{(1)}}) \wedge (\Theta_{R^{(2)}} > \Theta_n) \wedge (\tau \geq n), \tag{60}$$

with

$$\begin{aligned} \Theta_{\gamma(n)} &= \inf\{k > \gamma(n) : ||S_k|| < \gamma(n)\} \\ \Theta_{R^{(1)}} &= \inf\{k > \gamma(n) : S_k \in \mathbb{S}_r\} \\ \Theta_{R^{(2)}} &= \inf\{k > \Theta_{R^{(1)}} : S_k \in \mathbb{S}_r\} \\ \Theta_n &= \inf\{k > \gamma(n) : ||S_k|| > n\}. \end{aligned} \tag{61}$$

Where $\hat{e}_1$ is just a basis vector in the standard basis of the grid. When we say we follow this strategy, we mean that only the contribution of instances of the random walk in accordance to $\mathcal{A}_n$ are taken into account. This give

$$\begin{aligned} \mathbb{E}^{NT}\left[p^{\sum_{k=0}^{n-1} I(S_k \in D)}\right] &\geq \mathbb{E}^{NT}\left[p^{\sum_{k=0}^{n-1} I(S_k \in D)} \cdot I(\mathcal{A}_n)\right] \\ &= \left(\frac{1}{2d}\right)^{\gamma(n)} \cdot \mathbb{P}\{\Theta_{\gamma(n)} > \Theta_{R^{(1)}}\} \cdot \mathbb{P}\{\Theta_{R^{(2)}} > \Theta_n\} \cdot p^{\sum_{k=0}^{r} I(k \cdot e_1 \in D)}. \end{aligned} \tag{62}$$

Our goal is now to bound this inequality from below by something of order $\frac{1}{n}$, such that $\mathbb{E}(\tau)$ diverges. We can easily see that $(\frac{1}{2d})^{\gamma(n)} = \frac{1}{\sqrt{n}}$. The factor $p^{\sum_{k=0}^{r} I(k \cdot \hat{e}_1 \in D)}$ will be almost constant in comparison to the other terms, so we can ignore it. This is due to the fact that we assume rapidly increasing differences in the radii of successive shells by requiring only one shell between $\gamma(n)$ and n. The only terms we still have to deal with are $\mathbb{P}\{\Theta_{\gamma(n)} > \Theta_{R^{(1)}}\}$ and $\mathbb{P}\{\Theta_{R^{(2)}} > \Theta_n\}$. For this we use a result from [5].

Theorem 4.10. *Suppose $\{B(t) : t \geq 0\}$ is a Brownian motion in dimension $d \geq 1$ started in*

$$x \in A := x \in \mathbb{R}^d : r \leq ||x|| \leq R \tag{63}$$

inside an annulus A with radii $0 < r < R < \infty$. Then,

$$\mathbb{P}_x\{T_R > T_r\} = \begin{cases} \frac{R - ||x||}{R - r} & \text{if } d = 1 \\ \frac{logR - log||x||}{logR - log\ r} & \text{if } d = 2 \\ \frac{R^{2-d} - ||x||^{2-d}}{R^{2-d} - r^{2-d}} & \text{if } d \geq 3 \end{cases} \tag{64}$$

Where T_a for $0 < a \in \mathbb{R}$ is defined as

$$T_a = inf\{t > 0 : ||B(t)|| = a\}. \tag{65}$$

This theorem is stated for Brownian motion $\{B(t) : t \geq 0\}$, which is a continuous time stochastic process satisfying certain properties. The exact definition won't be discussed here. It turns out that for large n the behaviour of the simple random walk and Brownian motion are similar. As a consequence we can also use Theorem 4.10 with

$$T_r = \inf\{k > 0 : ||\xi_k|| < r\} \tag{66}$$

and

$$T_R = \inf\{k > 0 : ||\xi_k|| > R\}, \tag{67}$$

where $(\xi_k)_{k\geq 0}$ is a simple random walk starting at x. We can state that

$$\mathbb{P}\{\Theta_{\gamma(n)} > \Theta_{R^{(1)}}\} = 1 - \mathbb{P}\{\Theta_{\gamma(n)} < \Theta_{R^{(1)}}\} = 1 - \mathbb{P}_{\gamma(n)+1}\{T_{\gamma(n)} < T_r\} \tag{68}$$

and

$$\mathbb{P}\{\Theta_{R^{(2)}} > \Theta_n\} = 1 - \mathbb{P}\{\Theta_{R^{(2)}} < \Theta_n\} = 1 - \mathbb{P}_{r+1}\{T_r < T_n\}. \tag{69}$$

Using Theorem 4.10 it is easy to verify that

$$1 - \mathbb{P}_{\gamma(n)+1}\{T_{\gamma(n)} < T_r\} > 1 - \mathbb{P}_{\gamma(n)+1}\{T_{\gamma(n)} < T_n\} \tag{70}$$

and

$$1 - \mathbb{P}_{r+1}\{T_r < T_n\} > 1 - \mathbb{P}_{\gamma(n)+1}\{T_{\gamma(n)} < T_n\}. \tag{71}$$

Therefore we try to bound $1 - \mathbb{P}_{\gamma(n)+1}\{T_{\gamma(n)} < T_n\}$ instead of the original probabilities. In the case d $= 2$ we find for large n

$$\begin{aligned} 1 - \mathbb{P}_{(\gamma(n)+1)\hat{e}_1}\{T_{\gamma(n)} < T_n\} &= 1 - \frac{\log n - \log (\gamma(n)+1)}{\log n - \log \gamma(n)} \\ &= \frac{\log (\gamma(n)+1) - \log \gamma(n)}{\log n - \log \gamma(n)} \\ &= \frac{\log \left(1 + \frac{1}{\gamma(n)}\right)}{\log \left(\frac{n}{\gamma(n)}\right)}. \end{aligned} \tag{72}$$

Notice that for large n, $\log\,(1+\gamma(n)^{-1})$ can be approximated by $\gamma(n)^{-1}$ since $\log\,(1+\epsilon) \approx \epsilon$ for small ϵ. Hence

$$\begin{aligned} 1 - \mathbb{P}_{(\gamma(n)+1)\hat{e}_1}\{T_{\gamma(n)} < T_n\} &= \frac{\log \left(1 + \frac{1}{\gamma(n)}\right)}{\log \left(\frac{n}{\gamma(n)}\right)} \\ &\approx \frac{\gamma(n)^{-1}}{\log\,(n) - \log\,(\gamma(n)^{-1})} \\ &\geq \frac{\gamma(n)^{-1}}{\log\,(n)} \\ &= \frac{2\log\,(2d)}{\log\,(n)^2} \geq \frac{1}{\sqrt[4]{n}}. \end{aligned} \tag{73}$$

For the case $d = 3$ we have the similar situation

$$\begin{aligned} 1 - \mathbb{P}_{(\gamma(n)+1)\hat{e}_1}\{T_{\gamma(n)} < T_n\} &= 1 - \frac{n^{2-d} - (\gamma(n)+1)^{2-d}}{n^{2-d} - \gamma(n)^{2-d}} \\ &= \frac{(\gamma(n)+1)^{2-d} - \gamma(n)^{2-d}}{n^{2-d} - \gamma(n)^{2-d}} \\ &\geq \frac{\gamma(n)^{2-d} - (\gamma(n)+1)^{2-d}}{\gamma(n)^{2-d}} \\ &= 1 - \left(\frac{\gamma(n)+1}{\gamma(n)}\right)^{2-d} \geq \frac{1}{\sqrt[4]{n}}. \end{aligned} \tag{74}$$

We can obtain $\left(\frac{\gamma(n)+1}{\gamma(n)}\right)^{2-d} \approx 1 - \frac{d-2}{\gamma(n)}$ for large n via a Taylor approximation. Indeed, for ϵ small and $a > 0$ we have $(1+\epsilon)^{-a} \approx 1 - a\epsilon$. As a consequence

$$\begin{aligned} 1 - \mathbb{P}_{(\gamma(n)+1)\hat{e}_1}\{T_{\gamma(n)} < T_n\} &= 1 - \Big(\frac{\gamma(n)+1}{\gamma(n)}\Big)^{2-d} \geq \frac{1}{\sqrt[4]{n}} \\ &\approx 1 - (1 - \frac{d-2}{\gamma(n)}) \\ &= \frac{d-2}{\gamma(n)}) \geq \frac{1}{\sqrt[4]{n}}. \end{aligned} \tag{75}$$

We can now bound (62) such that

$$\mathbb{E}^{NT}\big[p^{\sum_{k=0}^{n-1} I(S_k \in D)}\big] \geq \frac{1}{n} \tag{76}$$

Giving the divergence of $\mathbb{E}(\tau)$. Now we have shown that our strategy gives criticality. However, it is not directly clear how these shells should be spaced. We derived the above under the assumption that there is at most one shell with radius between $\gamma(n)$ and n. This implies that for a shell with radius r, the previous shell should have radius $\gamma(r)$. Suppose the location of radii of the traps are given by $f(x)$, x being the shell number counted from the origin. Let g be the inverse of f. Then we can express our assumption as

$$f(g(\gamma(n)) + 1) = n \tag{77}$$

To find the corresponding series of shells we substitute $x = \gamma(n)$ and $y = g(x)$. Then

$$f(y+1) = n = (2d)^{2x} = (4d^2)^{f(y)}. \tag{78}$$

This gives a recursive formula. If we assume $f(0) = 0$ we obtain the following radii

$$\begin{aligned} f(0) &= 0 \\ f(1) &= (4d^2)^{f(0)} = 1 \\ f(2) &= (4d^2)^{f(1)} = (4d^2) \\ f(3) &= (4d^2)^{f(2)} = (4d^2)^{4d^2} \\ f(4) &= (4d^2)^{f(3)} = (4d^2)^{(4d^2)^{4d^2}} \\ &\dots \end{aligned} \tag{79}$$

This gives a tower exponential of increasing length, which is an extremely fast growing function. Since we used very rough bounds to derive these traps, one can likely place the shells a lot denser and still conserve criticality. However, so far this result can be regarded as a proof of concept, showing that it is indeed possible to construct infinitely many unavoidable structures and meanwhile maintaining criticality.

5 Avalanche Size Distribution via Simulation

In chapter three we defined the Abelian sandpile model to be critical whenever the first moment of the avalanche size is infinite. After that we used Theorem 3.3 to express criticality in terms of the Green's function $G(x,y) = \mathbb{E}_\mu[m^x_\eta(y)]$. This theorem essentially stated that the avalanche size can be replaced by the total number of topplings in the definition of criticality. This alternative definition turned out to be very convenient, because of the intimate link between the Green's function and a simple random walk. It is however not at all clear at this point how similar the distribution of the survival time and the avalanche size really are. To investigate this we wrote a C program to simulate the Abelian sandpile model with randomly placed dissipative sites. Via these simulations we estimate the distribution of the avalanche size and compare these to a stretched exponential. A stretched exponential is a function describing the survival time of a simple random walk in a field of randomly placed traps, as is known from arguments and simulations by for example P. Grassberger and I. Procaccia [3].

5.1 Stretched Exponential in Two Dimensions

Before we go further into the simulations we give an explanation of how a stretched exponentials arise as the distribution function of τ, the survival time of a simple random walk. This derivation is meant to give some insight but is certainly not totally rigorous.

We consider a simple random walk $(S_t)_t \geq 0$ on a two dimensional grid with traps at $D = \{\vec{r}_1, \vec{r}_2, ...\}$. The walk has probability U_o to get killed when hitting a trap. We define the probability that the random walk is at position $\vec{r} \in \mathbb{Z}^2$ as

$$W(\vec{r}, t) = \mathbb{P}_0\{S_t = \vec{r}\}. \tag{80}$$

We can easily find an expression for $W(\vec{r}, t+1)$ as the walk leaps to each of its neighbors with probability $\frac{1}{4}$.

$$W(\vec{r}, t+1) = \frac{1}{4}[W(\vec{r}+\hat{e}_1, t) + W(\vec{r}-\hat{e}_1, t) + W(\vec{r}+\hat{e}_2, t) + W(\vec{r}-\hat{e}_2, t))] - \sum_n U_0 \delta(\vec{r}-\vec{r}_n) \tag{81}$$

Of course the last term is due to the traps. We subtract $W(\vec{r}, k)$ from both sides and write this in the continuum limit.

$$W(\vec{r}, t+1) - W(\vec{r}, t) \approx \frac{\partial W}{\partial t}(\vec{r}, t) \tag{82}$$

and

$$\frac{1}{4}[W(\vec{r}+\hat{e}_1, t) + W(\vec{r}-\hat{e}_1, t) + W(\vec{r}+\hat{e}_2, t) + W(\vec{r}-\hat{e}_2, t))] - W(\vec{r}, t) \approx \frac{1}{4}\nabla^2 W(\vec{r}, t). \tag{83}$$

We obtain a diffusion equation

$$\frac{\partial W}{\partial t} = \frac{1}{4}\nabla^2 W - \sum_n U_0 \delta(\vec{r} - \vec{r}_n). \tag{84}$$

We could now solve this equation using a mean field approximation, that is $\sum_n U_0 \delta(\vec{r}-\vec{r}_n) = \bar{U}_0$ with $\bar{U}_0$ a constant. This would lead to solutions such that $\mathbb{P}\{\tau > t\} = \sum_{\vec{r}\in\mathbb{Z}^2} W(\vec{r}, t) \propto e^{-ct}$ with c some positive constant. However, at large t the distribution is dominated by the situation where there are only very little traps near to the origin, where the random walk starts. These instances of the field are not well described by the mean field approximation, in which the traps are distributed homogeneously. To model this we assume that $(S_t)_{t\geq 0}$ starts of in a circular area with radius R free of traps. We will refer to such areas as a "lakes". Instead of the normal exponential we will now find a stretched exponential for $\mathbb{P}\{\tau > t\}$. This is a function of the form

$$e^{-\alpha t^{\beta}} \text{ with } \alpha > 0 \text{ and } 0 < \beta < 1. \tag{85}$$

To find the probability that $(S_t)_{t\geq 0}$ is still in the lake at a certain point in time we solve (84) with boundary condition $W(\vec{r}, t) = 0$ for $||\vec{r}|| = R$. This can be done via seperation of variables $W(\vec{r}, t) = \Phi(\vec{r})\Theta(t)$. The resulting ordinary differential equations are

$$\dot{\Theta}(t) = \lambda\Theta(t) \tag{86}$$

and

$$\nabla^2\Phi(\vec{r}) = \frac{d^2\Phi(r)}{dr^2} + \frac{1}{r}\frac{d\Phi(r)}{dr} = 4\lambda\Phi(r). \tag{87}$$

Here λ is the separation constant. Since we have a circular lake we can assume $\Phi(\vec{r})$ to depend only on $r = ||\vec{r}||$. The next step is to solve the spacial equation with boundary condition $\Phi(R) = 0$. This is not such a difficult task since (87) is very similar to the Bessel equation

$$\frac{d^2J_0(x)}{dx^2} + \frac{1}{x}\frac{dJ_0(x)}{dx} + J_0 = 0. \tag{88}$$

Hence the solutions are of the form $\Phi(r) = J_0(kr)$ with k such that $\lambda = -4k^2$. Notice that for our case we need $\Phi(0)$ to be non-zero since the walk starts from the origin. As a consequence the solution should be a Bessel function of order zero. Keep in mind that the boundary condition dictates $\Phi(R) = J_0(kR) = 0$. This gives a series of values for k. However, over time the probability $W(t, \vec{r})$ is dominated by the solutions with maximal λ. This is due to temporal part $\Theta(t) = e^{\lambda t}$ following from (87). As a consequence we want k to be minimal. The first zero of $J_0(k_{min}R)$ occurs at $k_{min}R = \mu \approx 2.4$, hence $k_{min} = \frac{\mu}{R}$. We can state that

$$\mathbb{P}\{T_R > t\} = \sum_{\vec{r}\in\mathbb{Z}^2} W(\vec{r}, t) = \sum_{\vec{r}\in\mathbb{Z}^2} \Phi(\vec{r})\Theta(t) \propto e^{-\frac{\mu^2 t}{4R^2}}, \tag{89}$$

with $T_R = inf\{t : S_t > R\}$. Now we have the probability that the walk survives in the lake for a certain period of time. However, this is not yet an accurate approximation for the distribution of the lifetime, since we should also consider the probability that a lake of radius R forms. This is approximately

$$\mathbb{P}\{r = R\} \approx (1 - C)^{\pi R^2} = \left(1 - \frac{C\pi R^2}{\pi R^2}\right)^{\pi R^2} \approx e^{-C\pi R^2}, \tag{90}$$

with C the probability that a site is a trap. For the last approximation we assumed large values of R. We can now write down the our approximation for the survival probability

$$\mathbb{P}\{\tau > t\} = \int_0^{\infty} e^{-x}e^{-\frac{\alpha t}{x}}\,dx. \tag{91}$$

Here we made substitutions $x = \pi C R^2$ and $\alpha = \frac{\pi\mu^2 C}{4}$. For this integral we can apply a saddle point approximation. This eventually yields the stretched exponential

$$\mathbb{P}\{\tau > t\} = \int_0^{\infty} e^{-x}e^{-\frac{\alpha t}{x}}\,dx = e^{-2\sqrt{\alpha t}}. \tag{92}$$

5.2 Simulation

In [3] Grassberger and Procaccia argue that a random walk in d dimensions always has a stretched exponential distribution for its survival time. They state that for a d dimensional random walk the stretching exponent is $\beta = \frac{d}{d+2}$, which is consistent with the case $d = 2$ explained above. We are interested in to what extent the avalanche size distribution is similar to a stretched exponential. Our strategy is to simulate the sandpile model via a computer program in the C language.

We first initialise a large grid where each site has a height between zero and $2d - 1$. Then we change each site into a trap with probability p. This is all done via the $rand()$ function in the *stdlib.h* header. After the initialisation of the grid we add one to the height of a random site and topple if necessary. This happens according to the rules discussed in chapter two and three. If we indeed had to topple we check if neighboring sites are still stable and process them similar to the original site. This gives a recursive algorithm, which is generally not so fast. However, in this case we avoid checking an unnecessary amount of sites as we would if we were to loop through all sites when toppling. To account for the grains falling of the edge we don't process the sites at the border of the grid, so they can't topple and bring back the fallen sandgrains into the system. Along the toppling process we track which sites have toppled at least once, giving us the avalanche size. At this point we performed one iteration. We repeat this process until we have sufficiently many avalanche sizes to estimate the true distribution.

Remark 5.1. *One might notice that this is not necessarily the same as dropping a sandgrain on one site for different recurrent configurations μ. However, since we drop a grain every iteration, the Markov chain will get stuck in $\mathcal{R}$ after a certain time. Since each grain is dropped at a different site we have an equivalent situation to where we drop on a single site and change the configuration every time. Since the traps are placed randomly they shouldn't show any kind of pattern, we can assume the situation is similar to when we would choose new traps every single iteration.*

5.3 Results

The classical sandpile model without any traps should be critical, and hence decay as a power law rather than an exponential. Before discussing the stretched exponential behaviour of the model with traps, we check if this is the case. To uncover this behaviour we plot the points $(\ln t, \ln N(t))$, since

$$\begin{aligned} N(t) &= at^{-c} \\ \Rightarrow \ln N(t) &= \ln a - c\ln t. \end{aligned} \tag{93}$$

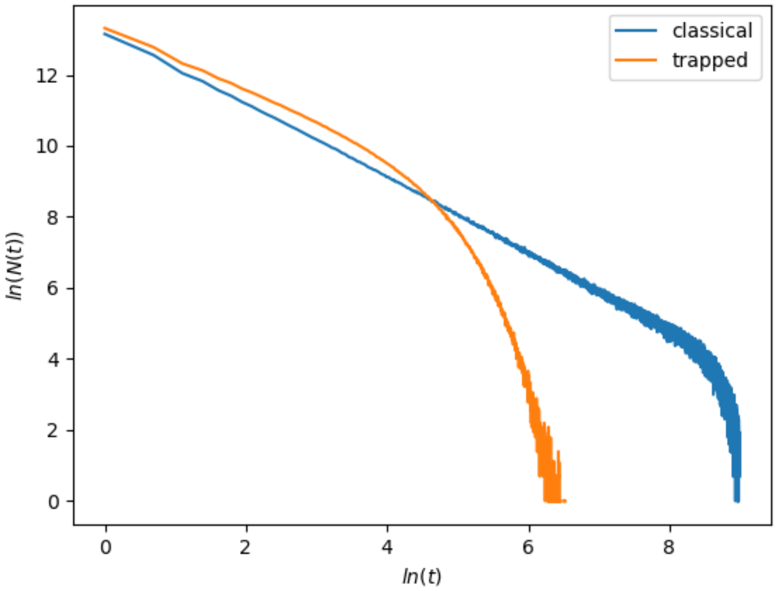


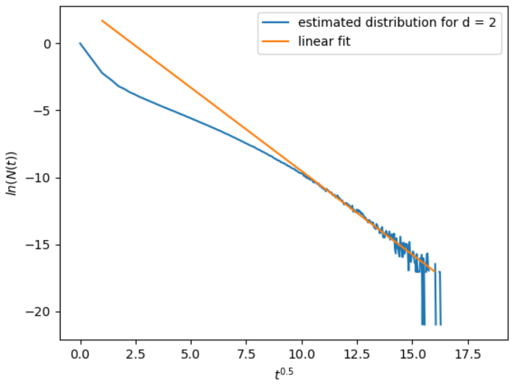
Figure 1: Classical and trapped estimated distributions for $d = 2$

As seen in Figure 1, the classical curve does follow a straight line, as expected. From [1] we know that the slope of this line, the exponent in the powerlaw, should have a value $c = 1.0$. The value we found is 1.09 ± 0.01, so the model seems to behave as it should in the classical case.

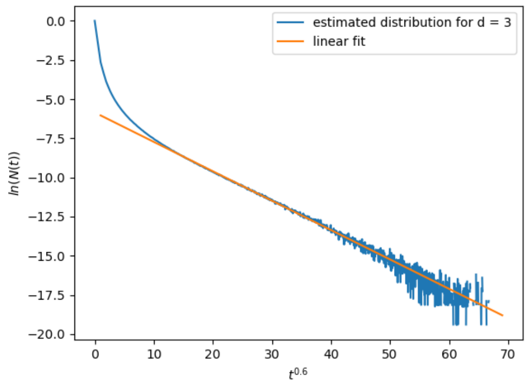
Below we discuss the results for instances of the sandpile model with randomly placed traps in two and three dimensions. In both cases, the probability to be dissipative is 0.1 for each site, If the distribution in the trapped case were to follow stretched exponentials then the points $(t^\beta, \ln N(t))$ should be on a straight line

$$\begin{aligned} N(t) &= e^{-\alpha t^\beta} \\ \Rightarrow \ln N(t) &= -\alpha t^\beta. \end{aligned} \tag{94}$$

In Figure 2 the cases $d = 2$ and $d = 3$ are shown in this fashion. To make these plots we assumed the theoretical values $\beta_{d=2} = 0.5$ and $\beta_{d=2} = 0.6$.
In both cases, especially for $d = 3$ the estimated distributions seemingly follow a straight line, as they would for a stretched exponential with stretch exponent $\beta = \frac{d}{d+2}$. However the results are quite wide spread for the larger avalanche sizes. This makes it difficult to judge if they really follow a straight line, or if they slightly curve. This chaotic behaviour can be reduced



(a) $d = 2$



(b) $d = 3$

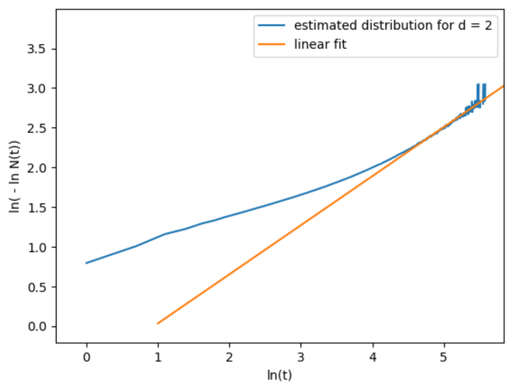
Figure 2: The estimated distributions $N(t)$ of the avalanche sizes plotted as $(t^\beta, \ln N(t))$.

by taking taking more data points into account. This can be achieved by either running more iterations per simulation or by averaging the outcomes of several simulations. The latter option has the advantage that one can easily run multiple instances of the same program on different cores, while performing more iterations per simulations would require the parallelization of the program. Since parallelization is rather difficult for the sandpile model we choose to average the results. For the plots above we used 200 simulations of each 10^7 iterations.

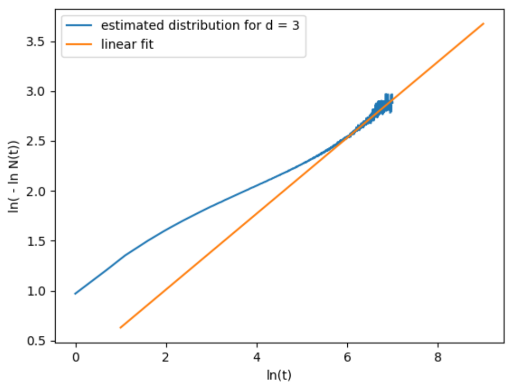
To obtain an empirical estimate of the stretch exponent we plot the points $(\ln\, t, \ln(-\ln\, N(t)))$. This should give a straight line with slope β, since

$$\begin{aligned} N(t) &= e^{-\alpha t^\beta} \\ \Rightarrow \ln\, N(t) &= -\alpha t^\beta. \\ \Rightarrow \ln\,(-\ln\, N(t)) &= \ln\,(\alpha) + \beta\, \ln\, t. \end{aligned} \tag{95}$$

Figure 3 shows the corresponding plots for the same simulations as in Figure 2.



(a) $d = 2$



(b) d=3

Figure 3: The estimated distributions $N(t)$ of the avalanche sizes plotted as $(\ln\, t, \ln(-\ln\, N(t)))$.

The majority of the data points are now mapped very close to each other. This makes it more difficult to identify the linear behaviour we are after. Nonetheless the results seem to be compatible with a stretched exponential distribution. When we try to fit a line to these graphs we find $\beta_{d=2} = 0.61 \pm 0.01$ and $\beta_{d=3} = 0.38 \pm 0.01$. Both of these results are quite far from the values we get from $\beta = \frac{d}{d+2}$. This could be due to an actual difference in distribution between the survival time of the random walk and the avalanche size. Any random variable with stretched exponential distribution has finite expectation. Since we found two stretch exponents between one and zero, we have sandpile models which should be critical. This is consistent with the finite expected survival time of the corresponding walks. However, intuitively we would expect $\beta_{d=2}$ and $\beta_{d=3}$ to be a lot closer to the stretch exponents for the random walks. The fact that $\beta_{d=2} > 0.5$ while $\beta_{d=3} < 0.6$ is also suspicious. There are several reasons why the exponents we found might differ from those of the random walk. The first of which being that we used finite grids in our simulation. As we saw in Figure 1 the grid size might influence the results if not large enough. Especially in the case $d = 3$ we choose a grid with rather small dimensions to combat the rapidly increasing volume. While the volume of the grid remains just as large as in the two dimensional case, the distance to border of the grid is a lot smaller for most points. This might explain the relatively larger difference for $d = 3$. Another reason that these results might not be accurate is a shortage of data points. Even after averaging 200 results the graphs above show a bad statistic for the larger avalanche sizes, making it difficult clearly recognise linear behaviour. However, since we don't have any conclusive evidence that the results we found are actually wrong, we can only conclude that further research is needed. To obtain truly unambiguous results one could try to take a more serious approach, for example by employing a computer cluster to run more or parallelized instances of the code.

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