The Atmoscope: Using a Planet's Atmosphere as a Telescope

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Abstract

The Atmoscope is the idea to use the atmosphere of a planet as a telescopic lens. It uses the refractive properties of an atmosphere to converge light rays to a focal line. It has been hypothesized that using Earth's atmosphere could, under favorable circumstances, yield an amplification of $55,000$ for a detector that has a surface area of $1\,\mathrm{m}^2$. However, the precise effects of the oblateness of Earth in combination with the effects of physical effects such as turbulence inside the atmosphere have not yet been researched. Here we show that incorporating the eccentricity of Earth diminishes the amplification greatly. We found using a ray tracing model for gradient-index media that the amplification for ellipsoidal planets has the following relation with the detector or pixel size: $A \propto D^{-0.66}$. For spherical planets this relation is $A \propto D^{-1}$. Furthermore, we implemented velocity diffusion to simulate physical effects inside the atmosphere, which surprisingly did not affect the amplification significantly. We anticipate this result to be a starting point for more sophisticated ray tracing models. For example, a full map for the refraction in the atmosphere could be used to test the effect of a non-homogeneous atmosphere. Much research still needs to be done, before it can be decided definitively whether the Atmoscope could be a useful telescope.

Contents

1. Introduction to the Atmoscope

1.1. Telescopes:

Humans have always been interested in the stars: what is out there in outer space? It is then no surprise that instruments used to gaze at celestial objects have been used as early as the beginning of the 1600s [1], with most famously Galileo Galilei, who was the first to use glass lenses to observe celestial objects. These type of telescopes that use refractivity (using lenses to bend the light) are called refraction telescopes. Nowadays, refraction telescopes are still common, but the famous telescopes used for scientific research are now often reflective telescopes: these use mirrors to reflect the light from object to observer. Examples of these are Hubble, and the recently deployed James Webb Telescope. These examples are also space-telescopes: they have the advantage that they do not have to collect light that has traveled through the atmosphere. In figure 1 a diagram of both types of telescopes is shown, showing how the light travels through both versions.

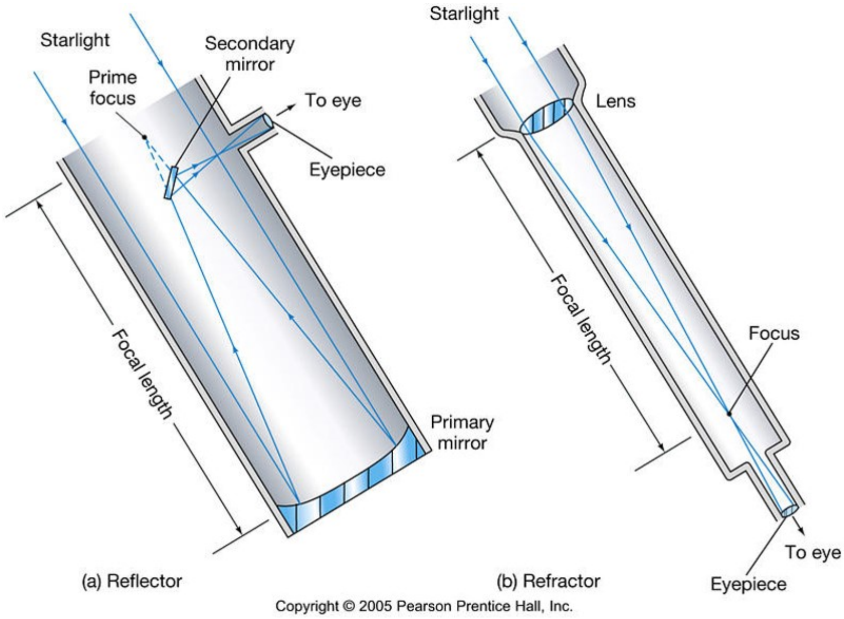

Figure 1: A diagram depicting the two different types of telescopes. Figure a) shows the path of light rays traveling through a reflective telescope, and b) shows the path of light rays traveling through a refractive telescope.

1.2. The Atmoscope

The atmosphere of Earth has been a problem for astronomers since they started using telescopes. To capture starlight, they want a calm, clear atmosphere, since any turbulence or other imperfection can scatter or absorb the starlight. They build their telescopes on top of mountains, so as to have the least amount of atmosphere to look through [2]. The Atmoscope poses the idea of, instead of seeing the atmosphere as an obstacle to be overcome, using it as a gigantic lens for a telescope. The first mention for idea for this was in a paper by W.B. Hubbard [3], who commented on the possible uses of the bending of light through the atmosphere of planets. A more detailed idea and research on the Atmoscope was written by Columbia University astronomer David Kipping in 2019 [4], where he called it the "Terrascope". We use the term Atmoscope in this report, because we want to point out that this idea is not only possible for Earth, but for any planet with an atmosphere. With the atmosphere of Earth being thinner at the top, and thicker the closer to the surface, one can imagine it could work similarly to a convex lens. This causes light rays that enter the atmosphere to converge, and suddenly we have a lens that is the size of the atmosphere.

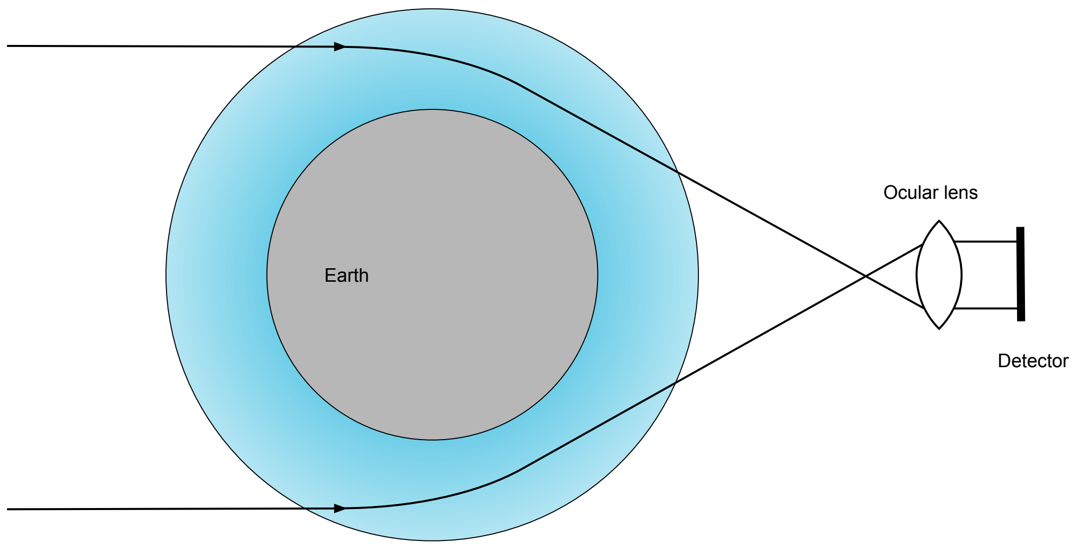

Figure 2: Kippings proposed atmoscope: Light rays travel through the atmosphere which acts like an objective lens, after which these light rays travel through an ocular lens piece and are finally captured by a detector. Not to scale.

A diagram of the Atmoscope is shown in figure 2. It shows that when light hits the atmosphere, it bends and converges behind the earth. Then it goes through a secondary lens (this is optional), and is captured by the detector.

There are a few benefits to the Atmoscope when comparing it to normal telescopes.

- It is relatively cheap to build: only a detector needs to be sent out in space, the rest of the telescope is already in place.
- The amplification of the Atmoscope would be incredible: no other instrument with a lens this large has ever been used.
- We can use other solar system objects with an atmosphere as telescopes as well.

In Kipping's paper, he modeled the atmosphere as concentric circles, each with a different refractive index, and was able from there to construct how light rays travel through the atmosphere.

1.3. Central flash maps

The consequences of the Atmoscope have already been observed a few times, namely as "central flashes". A central flash is an effect that can be observed with stellar occulations: when we measure the starlight captured from a star that is occulted by another celestial object, instead of not measuring any starlight anymore, we observe an intensity peak when the star is right behind the celestial object. This has first been observed by J.L. Elliott in 1977, where they observed this central flash while studying the occultation of a star by Mars [5]. More recently, when studying the two Titan stellar occultations of 14 November 2003, B. Sicardy et. al. looked more into the central flashes that were a result of these occultations [6]. In order to understand the results of this report, we will show the results of this paper, and explain the context briefly.

One of the occultations of Titan was visible in a region, and the study observed the occultation from multiple locations in this region. In the figure below, the intensity of collected light is plotted against time for 5 of these locations. There is a red line plotted over these figures with the expected intensity.

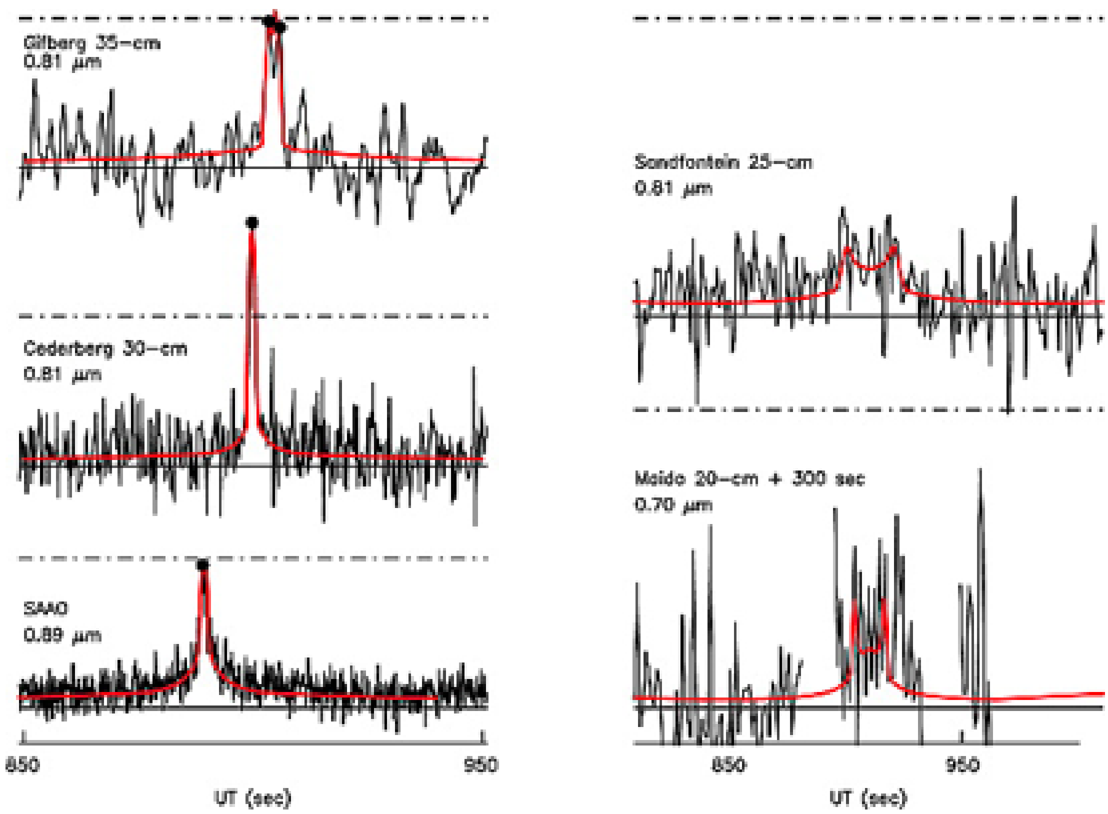

Figure 3: Observations for the occulation of Titan. There are 5 graphs in this figure, each for different locations where the occultation has been observed. These locations are Gifberg, Cederberg, SAAO, Sondfontein and Maïdo (all in South-Africa).

We see in figure 3 that for different locations, the central flashes are different: in Cederberg there is one very high flash, and in Maïdo there are two intensity peaks. This result led the authors to construct a 'central flash map', a figure in which they show the intensity as a function of location.

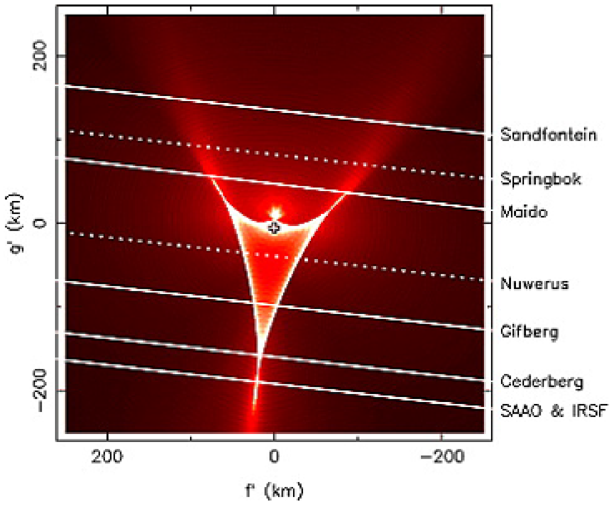

Figure 4: Central flash map of the occultation of Titan [6].

These kinds of flash maps are a large part of the results of this report, and we will show that the shape of the figure depends on the geometry of the planet that occults the star.

1.4. Research questions

In this report, the Atmoscope will be studied in greater detail. It aims to answer the following research questions:

- Could the Atmoscope be a useful telescope?
  - What effect does the shape of the planet have on the maximum amplification achieved by the Atmoscope?
  - How can we account for physical effects inside the atmosphere that may alter the quality of the Atmoscope?

1.5. Methodology

These questions will be answered throughout the next chapters. Chapter 2 explains the model for the Atmoscope used in the rest of the report. Chapter 3 explores the Atmoscope for a spherical planet, while chapter 4 applies the model to an ellipsoidal planet. Chapter 5 details the effects of the atmosphere. Chapter 6 discusses the final results, and chapter 7 answers the research questions and discusses further research questions.

2. Ray tracing for gradient-index media

Throughout this report, we use geometrical optics: we describe light as rays, and consider that photons follow this light ray. We need to find these light rays, to be able to obtain the amplification that results from the atmosphere, and to obtain the flash maps. This is called ray tracing: for every single light ray we will look at the path it has taken. In this section we will talk about how we can describe the path of a light ray, by deriving the equations of motion for gradient-index optics. We also will look at how these equations of motion can then be used in our code: what numerical methods we use to go from a differential expression to getting our path.

2.1. Euler-Lagrange equations

Let us consider the light traveling through a medium with changing refractive index, which influences the speed. Light will bend in the direction of the gradient of the refractive index. Its path can be derived using the Euler-Lagrange equation. These calculations have been made before [7]. It is included below for completeness sake.

2.1.1. Euler-Lagrange equation

According to the principle of Fermat, light traveling between two points P_0 and P takes the path that takes the shortest amount of time. The integral of this path, also known as the optical path length, can be written as in the following equation [8]:

$$S = \int_{P_0}^{P} n(\mathbf{r}) ds. \tag{2.1}$$

Here $n(\mathbf{r})$ is the refractive index, which depends on the position $\mathbf{r}$, and ds an infinitesimal length element of the light path. We can rewrite this by seeing that ds can be rewritten, since $ds = vdt$ where $v(\mathbf{r})$ is the instantaneous velocity, and dt a time element. We introduce $\mathbf{v}$ as the instantaneous the velocity vector:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \dot{\mathbf{r}} \tag{2.2}$$

One can see that v is therefore the length of this vector:

$$v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{2.3}$$

which, in combination with the observation that the refractive index is solely a function of location, gives us the following function for the optical path length:

$$S = \int_{P_0}^{P} nvdt = \int_{P_0}^{P} n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt. \tag{2.4}$$

Furthermore, our integral S can be written as

$$S = \int \mathcal{L}(x, y, z, \dot{x}, \dot{y}, \dot{z}) dt, \tag{2.5}$$

where $\mathcal{L}$ is the Lagrangian of our system.

$$\mathcal{L}(x, y, z, \dot{x}, \dot{y}, \dot{z}) = n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{2.6}$$

From this we see we can also write the Lagrangian of our system as follows:

$$\mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}) = n(\mathbf{r})v \tag{2.7}$$

This we can now use in the Euler-Lagrange equations [9]

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r_i}}\right) = \frac{\partial \mathcal{L}}{\partial r_i}, \tag{2.8}$$

where r_i and $\dot{r_i}$ are the i component of $\mathbf{r}$ and $\dot{\mathbf{r}}$ respectively (so $r_1 = x$, and $\dot{r_1} = v_1$, etc). We start with the right hand side of this equation with, and note that v does not depend on $\mathbf{r}$. The partial derivative with respect to x is

$$\frac{\partial \mathcal{L}}{\partial x} = v\frac{\partial n(\mathbf{r})}{\partial x} \tag{2.9}$$

We get similar results for taking the derivative of $\mathcal{L}$ with respect to y and z, and thus we can write down the following:

$$\begin{pmatrix} \frac{\partial \mathcal{L}}{\partial x} \\ \frac{\partial \mathcal{L}}{\partial y} \\ \frac{\partial \mathcal{L}}{\partial z} \end{pmatrix} = \begin{pmatrix} v\frac{\partial n(\mathbf{r})}{\partial x} \\ v\frac{\partial n(\mathbf{r})}{\partial y} \\ v\frac{\partial n(\mathbf{r})}{\partial z} \end{pmatrix} = v\nabla n(\mathbf{r}) \tag{2.10}$$

Then for the left hand side of the equations 2.8, note that the second term is a derivative to $\dot{r_i}$, which is the same as deriving to velocity. So now we look to find these derivatives. To start with $\frac{\partial \mathcal{L}}{\partial \dot{r_1}}$.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{r}_1} = \frac{\partial \mathcal{L}}{\partial v_1} &= n\frac{\partial v}{\partial v_1} \\ &= n\frac{\partial\sqrt{v_1^2 + v_2^2 + v_3^2}}{\partial v_1} \end{aligned} \tag{2.11}$$

Using the chain rule, we see that this becomes

$$\frac{\partial \mathcal{L}}{\partial \dot{r}_1} = n(\mathbf{r})\frac{v_1}{\sqrt{v_1^2 + v_2^2 + v_3^2}} = n\frac{v_1}{v}, \tag{2.12}$$

and similarly we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{r}_2} &= n(\mathbf{r})\frac{v_2}{v} \\ \frac{\partial \mathcal{L}}{\partial \dot{r}_3} &= n(\mathbf{r})\frac{v_3}{v}. \end{aligned} \tag{2.13}$$

We now define the unit velocity vector as follows:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \tag{2.14}$$

This gives us for our left hand side of equation (2.8):

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i}\right) = \frac{d}{dt}(n(\mathbf{r})u_i) \tag{2.15}$$

Substituting this equation and equation (2.10) into equation (2.8) we find the final gradient-index optics Euler-Lagrange equation:

$$\frac{d}{dt}(n(\mathbf{r})\hat{\mathbf{v}}) = v\nabla n(\mathbf{r}) \tag{2.16}$$

This means that the change in velocity over time is a function of the gradient of the index, hence the name gradient-index optics. A dimension check gives us that the left hand side of equation (2.16) has the following dimension:

$$\left[\frac{d}{dt}(n(\mathbf{r})\hat{\mathbf{v}})\right] = \left[\mathrm{s}^{-1}\right], \tag{2.17}$$

since $n(\mathbf{r})$ is dimensionless, and $\hat{\mathbf{v}}$ is also dimensionless, because it is a unit vector, which are dimensionless by definition. On the right hand side of equation (2.16) we get for our dimension:

$$[v\nabla n(\mathbf{r})] = \left[\mathrm{m\,s^{-1}} \cdot \mathrm{m}^{-1}\right] = \left[\mathrm{s}^{-1}\right], \tag{2.18}$$

which is the same as the left hand side of equation (2.16).

2.1.2. Equations of motion

Now we have the Euler-Lagrange equations of gradient-index optics, but we still need to apply these to find the equations of motion for our model of the Atmoscope. To do that, we start by defining the system of analysis. The coordinate system we have defined as in figure 5.

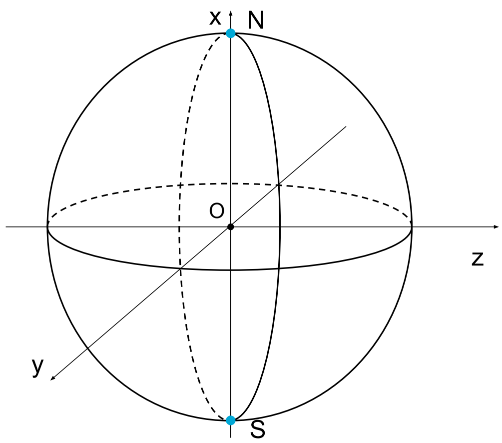

Figure 5: The planet placed onto the coordinate system we use, with the blue points being the North and South Pole.

In this report we talk about 2 types of planets, with spherical and ellipsoidal shape. Now that we have defined our system of analysis, let us continue by looking at the left hand side of equation (2.16). We start by using the chain rule:

$$\begin{aligned}\frac{d}{dt}(n(\mathbf{r})\hat{\mathbf{v}}) &= \hat{\mathbf{v}}\frac{dn(\mathbf{r})}{dt} + n(\mathbf{r})\frac{d\hat{\mathbf{v}}}{dt} \\ &= \hat{\mathbf{v}}\left(\frac{\partial n}{\partial x}\frac{dx}{dt} + \frac{\partial n}{\partial y}\frac{dy}{dt} + \frac{\partial n}{\partial z}\frac{dz}{dt}\right) + n(\mathbf{r})\frac{d\hat{\mathbf{v}}}{dt}\end{aligned} \tag{2.19}$$

Now we note that $\frac{dx}{dt} = v_1$, and similarly for $\frac{dy}{dt}$ and $\frac{dy}{dt}$, so

$$\begin{aligned}\hat{\mathbf{v}}\left(\frac{\partial n}{\partial x}\frac{dx}{dt} + \frac{\partial n}{\partial y}\frac{dy}{dt} + \frac{\partial n}{\partial z}\frac{dz}{dt}\right) &= \hat{\mathbf{v}}\left(v_1\frac{\partial n}{\partial x} + v_2\frac{\partial n}{\partial y} + v_3\frac{\partial n}{\partial z}\right) \\ &= (\nabla n(\mathbf{r}) \cdot \mathbf{v})\hat{\mathbf{v}}.\end{aligned} \tag{2.20}$$

Filling this back into the equation (2.19), we obtain the following:

$$(\nabla n(\mathbf{r}) \cdot \mathbf{v})\hat{\mathbf{v}} + n(\mathbf{r})\frac{d\hat{\mathbf{v}}}{dt} = v\nabla n(\mathbf{r}) \tag{2.21}$$

Bringing the first part of the left hand side to the right we see

$$n(\mathbf{r})\frac{d\hat{\mathbf{v}}}{dt} = v\nabla n(\mathbf{r}) - (\nabla n(\mathbf{r}) \cdot \mathbf{v})\hat{\mathbf{v}} \tag{2.22}$$

Rearranging the terms

$$\begin{aligned}n(\mathbf{r})\frac{d\hat{\mathbf{v}}}{dt} &= v\nabla n(\mathbf{r}) - \hat{\mathbf{v}}(\mathbf{v} \cdot \nabla n(\mathbf{r}) \\ n(\mathbf{r})\frac{d\hat{\mathbf{v}}}{dt} &= v\nabla n(\mathbf{r}) - \hat{\mathbf{v}}(v\hat{\mathbf{v}} \cdot \nabla n(\mathbf{r})) \\ n(\mathbf{r})\frac{d\hat{\mathbf{v}}}{dt} &= v(1 - \hat{\mathbf{v}}\hat{\mathbf{v}}\cdot)\nabla n(\mathbf{r}) \\ \frac{d\hat{\mathbf{v}}}{dt} &= \frac{v}{n(\mathbf{r})}(1 - \hat{\mathbf{v}}\hat{\mathbf{v}}\cdot)\nabla n(\mathbf{r})\end{aligned} \tag{2.23}$$

Now, since we are working with light rays, the speed of the light rays is defined as $v = \frac{c}{n}$, so we define our unit as $c = 1$, which gives us

$$\frac{d\hat{\mathbf{v}}}{dt} = \frac{1}{n(\mathbf{r})^2}(1 - \hat{\mathbf{v}}\hat{\mathbf{v}}\cdot)\nabla n(\mathbf{r}) \tag{2.24}$$

Now we could keep it here, and start to find our equations of motion, but we simplify the model, and say we propagate not in time, but in the z-direction. We can do this only if there is enough symmetry in the models, which in the case of this report there is. The reason for doing this means we only need to solve 4 equations of motion, instead of 6. We obtain this change by first applying the chain rule:

$$\begin{aligned}\frac{d\hat{\mathbf{v}}}{dz} &= \frac{d\hat{\mathbf{v}}}{dt}\frac{dt}{dz} \\ &= \frac{d\hat{\mathbf{v}}}{dt}\frac{1}{v_3} \\ &= \frac{d\hat{\mathbf{v}}}{dt}\frac{1}{vu_3} \\ &= \frac{d\hat{\mathbf{v}}}{dt}\frac{n}{u_3}\end{aligned} \tag{2.25}$$

Note again, v_i are the elements of the instantaneous velocity vector, and u_i are the elements of the unity velocity vector. In the last step we used that $v = \frac{1}{n}$. We fill this into equation (2.24).

$$\frac{d\hat{\mathbf{v}}}{dz} = \frac{1}{n(\mathbf{r})u_3}(1 - \hat{\mathbf{v}}\hat{\mathbf{v}}\cdot)\nabla n(\mathbf{r}) \tag{2.26}$$

Let's now find what this equation is per element of the vector. We start by showing for the x-element.

$$\begin{aligned}\frac{du_1}{dz} &= \frac{1}{n(\mathbf{r})u_3}\left(\frac{\partial n(\mathbf{r})}{\partial x} - u_1\left(u_1\frac{\partial n(\mathbf{r})}{\partial x} + u_2\frac{\partial n(\mathbf{r})}{\partial y} + u_3\frac{\partial n(\mathbf{r})}{\partial z}\right)\right) \\ &= \frac{1}{n(\mathbf{r})u_3}\left((1-u_1^2)\frac{\partial n(\mathbf{r})}{\partial x}\right) - \frac{u_1u_2}{n(\mathbf{r})u_3}\frac{\partial n(\mathbf{r})}{\partial y} - \frac{u_1}{n(\mathbf{r})}\frac{\partial n(\mathbf{r})}{\partial z} \\ &= \frac{1}{n(\mathbf{r})u_3}\left((u_2^2+u_3^2)\frac{\partial n(\mathbf{r})}{\partial x}\right) - \frac{u_1u_2}{n(\mathbf{r})u_3}\frac{\partial n(\mathbf{r})}{\partial y} - \frac{u_1}{n(\mathbf{r})}\frac{\partial n(\mathbf{r})}{\partial z} \\ &= \frac{u_3}{n(\mathbf{r})}\frac{\partial n(\mathbf{r})}{\partial x} + \frac{u_2^2}{n(\mathbf{r})u_3}\frac{\partial n(\mathbf{r})}{\partial x} - \frac{u_1u_2}{n(\mathbf{r})u_3}\frac{\partial n(\mathbf{r})}{\partial y} - \frac{u_1}{n(\mathbf{r})}\frac{\partial n(\mathbf{r})}{\partial z}\end{aligned} \tag{2.27}$$

In the middle steps we used that since $\hat{\mathbf{v}}$ is the unit vector, we can write $1 = u_1^2 + u_2^2 + u_3^2$ We can do a similar approach for $\frac{du_2}{dz}$. Furthermore, looking at the method we used in equation (2.25), we can find expressions for $\frac{dx}{dz}$ and $\frac{dy}{dz}$. Lastly, we can use $u_3 = \sqrt{1-u_1^2-u_2^2}$, and we have found our complete set of equations of motion.

$$\begin{cases} u_3 = \sqrt{1-u_1^2-u_2^2} & (2.28a) \\ \begin{pmatrix}\frac{dx}{dz} \\ \frac{dy}{dz}\end{pmatrix} = \begin{pmatrix}\frac{u_1}{u_3} \\ \frac{u_2}{u_3}\end{pmatrix} & (2.28b) \\ \begin{pmatrix}\frac{du_1}{dz} \\ \frac{du_2}{dz}\end{pmatrix} = \begin{pmatrix}\frac{u_3}{n(\mathbf{r})}\frac{\partial n(\mathbf{r})}{\partial x} + \frac{u_2^2}{n(\mathbf{r})u_3}\frac{\partial n(\mathbf{r})}{\partial x} - \frac{u_1u_2}{n(\mathbf{r})u_3}\frac{\partial n(\mathbf{r})}{\partial y} - \frac{u_1}{n(\mathbf{r})}\frac{\partial n(\mathbf{r})}{\partial z} \\ \frac{u_3}{n(\mathbf{r})}\frac{\partial n(\mathbf{r})}{\partial y} + \frac{u_1^2}{n(\mathbf{r})u_3}\frac{\partial n(\mathbf{r})}{\partial y} - \frac{u_1u_2}{n(\mathbf{r})u_3}\frac{\partial n(\mathbf{r})}{\partial x} - \frac{u_2}{n(\mathbf{r})}\frac{\partial n(\mathbf{r})}{\partial z}\end{pmatrix} & (2.28c)\end{cases}$$

Solving this system of ordinary differential equations will give us the path of the light ray.

2.2. Code

We have found the equations of motion, that can used for a ray on an atmosphere of a planet by substitution of a refractive index of that planet's atmosphere. Now we need to look at how we use these equations of motion to get the path for our light rays. There are many numerical integrators, each with their own pro's and cons: usually you either give up precision and accuracy, or computing time is very long. For this project, we found that Leapfrog works perfectly well.

2.2.1. Numerical method: Leapfrog

The numeric method we use to solve the equations of motion is called *Leapfrog*. Leapfrog is a time-reversible, second order numerical integrator. The way this algorithm works is shown in equation (2.29) [10]. Knowing the position x_i, velovity v_i and acceleration a_i at a certain time t, this method gives the new position and new velocity at a later time $t + \Delta t$.

$$\begin{aligned} x_{i+1} &= x_i + v_i \Delta t + \frac{1}{2} a_i \Delta t^2 \\ v_{i+1} &= v_i + \frac{1}{2}\left(a_i + a_{i+1}\right)\Delta t \end{aligned} \tag{2.29}$$

Note that the velocity is computed using the average acceleration between the two points so it should be known. This is clearly for if your ODE is just dependent on 2 variables, t and x, but its easily changed for our variables x, y and z.

$$\begin{aligned} x_{i+1} &= x_i + v_{1,i} \Delta z + \frac{1}{2} a_{x,i} \Delta z^2 \\ y_{i+1} &= y_i + v_{2,i} \Delta z + \frac{1}{2} a_{x,i} \Delta z^2 \\ v_{1,i+1} &= v_{1,i} + \frac{1}{2}\left(a_{x,i} + a_{x,i+1}\right)\Delta z \\ v_{2,i+1} &= v_{2,i} + \frac{1}{2}\left(a_{y,i} + a_{y,i+1}\right)\Delta z \end{aligned} \tag{2.30}$$

In this case, $a_{j,i} = \frac{dv_{j,i}}{dz}$, which of course can be given by equation 2.28c. The code for this algorithm in `Python` is shown below [11] [1]:

```
def LeapFrogSolve(dvdz, zspan, v0, n, a, b, c):

  z0 = zspan[0]
  zstop = zspan[1]
  dz = ( zstop - z0 ) / n

  #initializing arrays for z, and v
  z = np.zeros( n + 1 )
  v = np.zeros( [ n + 1, 4 ] )

  for i in range ( 0, n + 1 ):

    if ( i == 0 ): #filling in the initial conditions
      z[0]    = z0
      v[0,0] = v0[0]
```

[1] All the code used in this report can be found in the following GitHub repository: `https://github.com/leeuwenella/BEP-Atmoscope`

```
        v[0,1] = v0[1]
        v[0,2] = v0[2]
        v[0,3] = v0[3]
        anew   = dvdz( v[i,:], z[i], a,b,c )
    else: #running leapfrog
        z[i]   = z[i-1] + dz
        aold   = anew
        v[i,0] = v[i-1,0] + dz * ( v[i-1,2] + 0.5 * dz * aold[2] )
        v[i,1] = v[i-1,1] + dz * ( v[i-1,3] + 0.5 * dz * aold[3] )
        anew   = dvdz ( v[i,:], z[i], a,b,c )
        v[i,2] = v[i-1,2] + 0.5 * dz * ( aold[2] + anew[2] )
        v[i,3] = v[i-1,3] + 0.5 * dz * ( aold[3] + anew[3] )

  return v
```

Listing 1: The code used to propagate light rays using the Leapfrog numerical method.

The result of this code is a $(n+1) \times 4$ matrix `v`, where row `i` is an array of the shape $[x_i, y_i, u_{1_i}, u_{2_i}]$ at that z-location. In this code, Leapfrog uses the following variables: `dvdz`, `zspan`, `v0`, `n` `a`, `b` and `c`. Below the meaning of these variables is explained.

- `dvdz`: This is our system of ODE's. This is implemented as a function of x, y, u_1 and u_2 and z, (and of a, b, c when working with an ellipsoidal planet, where those are the axes of the ellipsoid), and it returns an array in the shape of $[\frac{dx}{dz}, \frac{dy}{dz}, \frac{du_1}{dz}, \frac{du_2}{dz}]$
- `zspan`: This is an array with the begin and end values of z over which we want to integrate.
- `v0`: This is an array with our initial conditions, given in the form `[x0, y0, ux0, uy0]`.
- `n`: This is the number of steps the integrator takes to calculate the path of one light ray. This is the variable that decides how big the steps are in the propagation direction. We found that using `n` = 15 gives us enough accuracy while still being low enough that the code doesn't take too long to run.
- `a`, `b`, `c`: These are the principal semi-axis used in describing the shape of the planet. These are only used when working with an ellipsoidal planet.

This piece of code we use to calculate the path of light rays within the atmosphere, since after they leave the atmosphere, the light rays find themselves back in vacuum, and will therefore simply travel in a straight line. This line we parameterized in the following way:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix} + t \begin{pmatrix} u_{1_A} \\ u_{2_A} \\ u_{3_A} \end{pmatrix} \tag{2.31}$$

Here A denotes the location where we exit the atmosphere. We can then decide we want to look at the x-y plane for a certain z, and find t from there. Then we can use this t to find the coordinates of all light rays at this z-plane.

2.2.2. Initial conditions: a thin band

So far we have found the general formula of the equations of motion for the light rays, and how to integrate these. The last thing we have to do before diving into specific models is determining the initial conditions: in which light rays' path are we actually interested? Ideally, we would take a homogeneous division of light rays in a plane that all travel towards the earth. This however will take an a lot of computing power, and is quite unnecessary, since most light rays will then either crash into the earth, or miss the detector. Therefore we decided to only look at a single band of light rays, and we give all of them the same initial direction and velocity: $v_3 = 1$, as shown in figure 6 below.

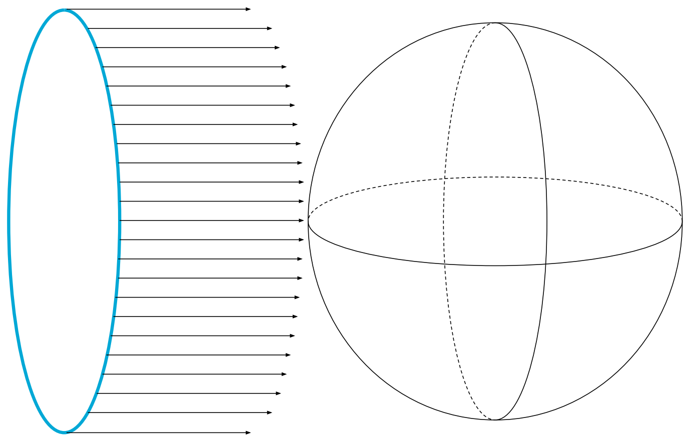

Figure 6: The propagation of light rays, where the initial conditions of the rays mean they are homogeneously divided on a thin ring away from the earth.

The way this band is constructed depends on the geometry of the planet we are propagating our light rays towards, so these will be discussed more in their respective sections of the spherical and ellipsoidal planet models (sections 3.2.2 and 4.3). A final remark: we need to consider the effect the thickness of the band has on our final results. This we've shown in figure 6 below.

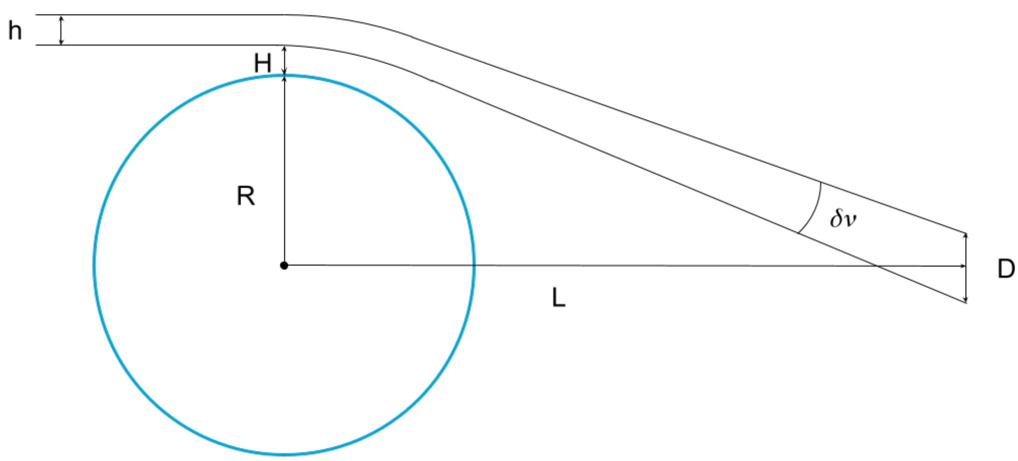

Figure 7: The effect of the width of the band in which we divide our light rays: the size of the detector D is bigger than the width h.

In figure 7 we tried to illustrate that the size of the detector that is needed to catch all the light from a certain band in the atmosphere, is bigger than the band itself. The reasoning behind this is as follows: say light ray A is below light ray B. Then light ray A enters the atmosphere before B does, which means it is in the atmosphere for longer, which means that it will bend more than B. Furthermore, A is also lower in the atmosphere than B. Lower in the atmosphere the refractive index is higher; this also increases the bending. These two effects result in a difference in bending angle between the two light rays, which will mean that they diverge, and thus will the detector have to be bigger than the width h.

This is also an argument why choosing to model the band is a valid idea for our initial conditions. Say we have a detector, with size D, placed somewhere on the z-axis. Any light ray lower than the the lowest ray in the figure will end up below the detector, and any light ray above the highest ray will end up above the detector, so there's no use in using those light rays.

2.2.3. Amplification

Throughout this report we will be talking about amplification, so here we show the definition we will be using throughout the report. Amplification is how much more light rays per unit surface you catch when using the Atmoscope compared to if it was not there. As an equation. First we introduce light ray density as follows:

$$\rho = \frac{\#\textit{light rays}}{\textit{surface}} \tag{2.32}$$

This gives us our function for amplification:

$$A = \frac{\rho}{\rho_0} \tag{2.33}$$

Here ρ is the light ray density in a pixel at the detector, and ρ_0 the light ray density of our initial conditions. To write it out even further, using that we are using a band of homogeneously placed light rays as our initial conditions:

$$A = \frac{P_r}{A_p}\frac{A_{ring}}{P_{total}} \tag{2.34}$$

Here P_p is the number of light rays that end in a certain pixel, A_p is the area of that pixel, and A_{ring} is the total area of the initial band, and P_{total} is the total number of light rays. Note that if you make a flash map with even spaced pixels, we can rewrite this again into:

$$A = C_{amp}P_p, \tag{2.35}$$

where we call C_{amp} the amplification factor: this factor describes the relation between the amplification and the actual number of light rays that end in a certain pixel. This factor can be useful to determine whether the results are viable.

3. Amplification by the atmosphere of a spherical planet

For the first step of making a full model for the effect of light bending in the atmosphere of a planet, we will simply be looking at a perfect spherical planet. This makes everything the model quite a bit simpler, because there is so much symmetry. In this chapter we will first obtain an analytical expression for the path of a light ray traveling through the atmosphere. The refractive index is proportional to the density of the atmosphere, and since the density of the atmosphere is exponentially decaying, the refractive index is also exponentially decaying, which means we can use the following equation for our refractive index:

$$n = 1 + \eta_0 \mathrm{e}^{-\frac{r-R}{H}}. \tag{3.1}$$

Here $1+\eta_0$ is the refraction index of air at sea level with $\eta_0 > 0$, R is the radius of the earth, and H is the scale height of the atmosphere: the height from the sea level after which the density has decreased with a factor e.

3.1. The path of a single light ray

3.1.1. Analytical calculation of the path

If we are able to find an analytical solution for the path of a light-ray, this would save us a lot of computing time, so in this section we are looking to find such a solution.

We start our search with equation 2.5. Different from chapter 2 however, we now change to from Cartesian to spherical coordinates. Keeping in mind that the problem is perfectly spherically symmetric, the ray will stay in a plane through the origin, and we can work in only 2 dimensions, instead of 3. This gives us the following coordinate transformation:

$$\begin{aligned} y =& 0 \\ z =& r\cos\theta \\ x =& r\sin\theta \end{aligned} \tag{3.2}$$

We may now assume the path of the light ray can be described as a function of θ, hence r is only a function of θ. This means we can rewrite ds in the following way, using the chain rule:

$$\begin{aligned} ds &= \left| d \begin{pmatrix} r(\theta)\cos\theta \\ r(\theta)\sin\theta \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} r'\cos\theta - r\sin\theta \\ r'\sin\theta + r\cos\theta \end{pmatrix} \right| d\theta \\ &= \sqrt{(r')^2 + r^2} d\theta \end{aligned} \tag{3.3}$$

Here we introduced the notation r' to mean the derivative of r with respect to θ: $r' = \frac{dr}{d\theta}$. Combining eqiatopm (3.3) with equation (2.1), we see:

$$
\begin{aligned}
S &= \int n(r)ds \\
&= \int n(r)\sqrt{(r')^2 + r^2}d\theta \\
&= \int \mathcal{L}(r,\theta)d\theta
\end{aligned} \tag{3.4}
$$

So the Lagrangian of this system can be written as

$$\mathcal{L}(r, \theta, \dot{r}, \dot{theta} = n(r)\sqrt{(r')^2 + r^2} \tag{3.5}$$

Now we will go from our Lagrangian to the Hamiltonian that fits with this system, using the following relation between the two [12]:

$$\mathcal{H} = p\dot{q} - \mathcal{L} \tag{3.6}$$

where normally with the Hamiltonian we work with parameters time t, coordinate q and momentum p, in our case we replace these by θ, r and $\frac{d\mathcal{L}}{dr'}$, respectively. Taking the derivative of our Lagrangian with respect to r' gives us the momentum variable

$$p = \frac{d\mathcal{L}}{dr'} = n(r)\frac{r'}{\sqrt{(r')^2 + r^2}} \tag{3.7}$$

Substituting all of this into equation (3.6) we get:

$$\mathcal{H}(r, \theta, p) = \frac{n(r)r'^2}{\sqrt{(r')^2 + r^2}} - n(r)\sqrt{(r')^2 + r^2} = -\frac{n(r)r^2}{\sqrt{(r')^2 + r^2}} = -\sqrt{n(r)^2 - p^2}r \tag{3.8}$$

We see that when written to depend on only r, p, and θ, the Hamiltonian is not a function of θ. Therefore the Hamiltonian of motion is constant [12]. We name this constant J.

We can now derive an expression for $r'(\theta)$.

$$J = -\frac{n(r)r^2}{\sqrt{(r')^2 + r^2}} \tag{3.9}$$

Now we square both sides, and multiply both sides by the denominator of the right hand side.

$$\left(r'^2 + r^2\right) J^2 = (-)^2 n(r)^2 r^4 \tag{3.10}$$

Now we can devide by J^2 to isolate r'.

$$r'^2 = \frac{n(r)^2 r^4 - r^2 J^2}{J^2} \tag{3.11}$$

Taking the square root of both sides gives us

$$r' = \pm\frac{\sqrt{r^4 n(r)^2 - r^2 J^2}}{J} \tag{3.12}$$

Now we write down the definition of r'. and plug J back into the square root.

$$\frac{dr}{d\theta} = \pm\sqrt{\frac{r^4 n(r)^2}{J^2} - r^2} \tag{3.13}$$

Then, we take $d\theta$ to one side, and all the others to the other side.

$$d\theta = \pm\frac{dr}{\sqrt{\frac{r^4 n(r)^2}{J^2} - r^2}} \tag{3.14}$$

Integrating this last equation gives us a function for θ:

$$\theta = \pm\int \frac{dr}{\sqrt{\frac{r^4 n(r)^2}{J^2} - r^2}} \tag{3.15}$$

One more thing we need to do with this formula is find an expression for J. To do that we use the expression for J as described in equation (3.9). We know that at the minimal distance a light ray reaches from the origin, $r' = 0$. This minimal distance we can call $R + h$ where R is still the radius of the planet. Filling this into the expression we have for J gives us:

$$J = -\frac{n(R+h)(R+h)^2}{\sqrt{0 + (R+h)^2}} = -(R+h)n(R+h) \tag{3.16}$$

Since J is a constant (since the Hamiltonian is constant) this expression is constant through the entire path of the light ray.

3.1.1.1 Checking the formula

In order to check if equation (3.15) could be the right formula for θ, we will check the integral for the case where $n = 1$, e.g. the light traveling through space. This gives us

$$\theta = \pm\int \frac{dr}{\sqrt{\frac{r^4}{J^2} - r^2}} \tag{3.17}$$

We will use a substitution: $u = \frac{J}{r}$, which gives us $du = -\frac{-J}{r^2}dr$.

$$
\begin{aligned}
\theta &= \pm \int \frac{\frac{-r^2}{J} du}{r\sqrt{\frac{1}{u^2} - 1}} \\
&= \mp \int \frac{\frac{1}{u} du}{\sqrt{\frac{1}{u^2} - 1}} \\
&= \mp \int \frac{du}{u\sqrt{\frac{1}{u^2} - 1}} \\
&= \mp \int \frac{du}{\sqrt{1 - u^2}} \\
&= \mp \arcsin(u) + C \\
&= \mp \arcsin\left(\frac{J}{r}\right) + C
\end{aligned} \tag{3.18}
$$

Rewriting this and reverting to Cartesian coordinates gives us:

$$
\begin{aligned}
&r \sin(\theta - C) = \mp J \Rightarrow \\
&r \sin(\theta) \cos(C) - r \cos(\theta) \sin(C) = \mp J \Rightarrow \\
&y \cos(C) - x \sin(C) = \mp J \Rightarrow \\
&y = \tan(C) x \mp \frac{J}{\cos(C)}
\end{aligned} \tag{3.19}
$$

Here we recognize the equation for a straight line: $y = ax + b$. This corroborates the equation we have found for theta, since we know that in a constant medium, light travels in a straight line.

3.1.1.2 Deflection angle

In the previous sections we found an equation for the angle θ of a light rays' path as a function of its distance r to a planet. In our case, obviously, the refraction index is not unity, which makes the equation we found quite difficult to solve. So what we will try to do instead is use this equation to find the deflection angle instead. The deflection angle ν we have defined as described in figure 8 below.

To give a mathematical description, we say we will define ν as follows:

$$
\nu = -\pi + 2 \int_{R+h}^{\infty} \frac{dr}{r\sqrt{\frac{r^2}{J^2}\left(1 + \eta_0 \mathrm{e}^{-\frac{r-R}{H}}\right)^2 - 1}} \tag{3.20}
$$

This seems quite a daunting expression, but the derivation of this is quite simple. Normally, when going in a straight line, θ starts at π when coming from the left side into infinity. Then, when it travels through the universe and goes into infinity on the right side, θ goes to zero. So the difference in angle between the begin and the end is π. Now, in the case where a light ray travels through the atmosphere, we have first that r is at minus infinity. Then the light ray

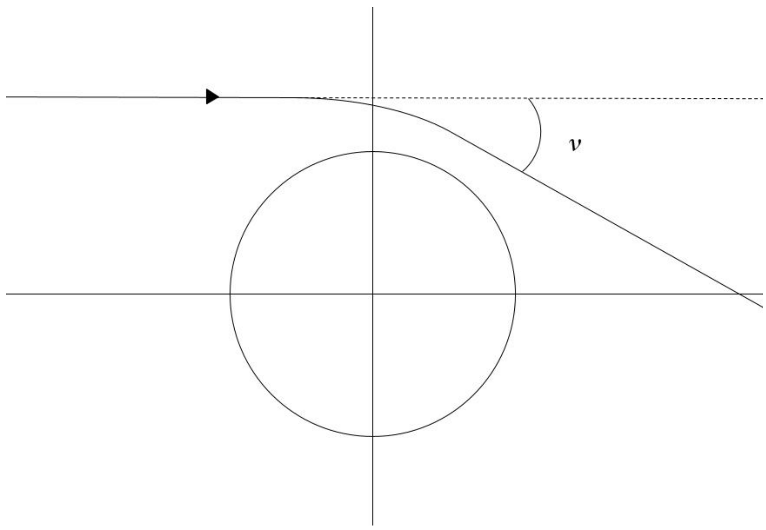

Figure 8: The definition of the deflection angle ν

travels until r reaches its minimum, that is $R + h$, and then it will increase until infinity again. The difference in angle can then be divided into two integrals, one going from minus infinity until $R + h$ and one going from $R + h$ until infinity, which is what we see in equation 3.20.

Now we want to find a solution for this equation. For that we will use that we know that $J = (R + h)n(R + h)$ by equation 3.16. So that means that we can write J as follows:

$$J = (R + h)\left(1 + \eta_0 \mathrm{e}^{-h/H}\right) \tag{3.21}$$

Now we take one more intermediate step before we fill in this definition for J into our equation 3.20; we will do a Taylor expansion in η_0 in r^2n^2. This goes as follows:

$$\begin{aligned} r^2n^2 &= r^2\left(1 + \eta_0 \mathrm{e}^{-\frac{r-R}{H}}\right)^2 \\ &= r^2 + 2r^2\mathrm{e}^{-\frac{r-R}{H}}\eta_0 + \cdots \\ &= r^2\left(1 + 2\eta_0 \mathrm{e}^{-\frac{r-R}{H}}\right) + \cdots \end{aligned} \tag{3.22}$$

Here we neglect the terms with higher order η_0, which we can do because η_0 is very small. Now combining this with the formulation we found for J, we see the following:

$$\begin{aligned} \frac{r^2n^2}{J^2} &= \frac{r^2n(r)^2}{(R+h)^2n(R+h)^2} \\ &= \frac{r^2}{(R+h)^2}\frac{1 + 2\eta_0\mathrm{e}^{(r-R)/H)} + \cdots}{1 + 2\eta_0\mathrm{e}^{-h/H} + \cdots} \end{aligned} \tag{3.23}$$

Multiplying this fraction with $\frac{1-2\eta_0\mathrm{e}^{-h/H}}{1-2\eta_0\mathrm{e}^{-h/H}}$ and neglecting the higher order terms we get the following:

$$\frac{r^2n^2}{J^2} \approx \frac{r^2}{(R+h)^2}\left[1 + 2\eta_0\left(\mathrm{e}^{-(r-R)/H} - \mathrm{e}^{-h/H}\right)\right] \tag{3.24}$$

However, in equation 3.20 we see that we have this in the denominator, inside a square. So that is what we'll do next.

$$\left(\frac{r^2n^2}{J^2}-1\right)^{-\frac{1}{2}} = \left(\frac{r^2}{(R+h)^2}\left[1+2\eta_0\mathrm{e}^{-(r-R)/H}+2\eta_0\mathrm{e}^{-h/H}\right]-1\right)^{-\frac{1}{2}} \tag{3.25}$$

To simplify this, we'll yet again use a Taylor expansion of this equation as a function of η_0 around 0. Neglecting higher order terms we see this gives us the following equation.

$$\left(\frac{r^2n^2}{J^2}-1\right)^{-\frac{1}{2}} = \left(\frac{r^2}{(R+h)^2}-1\right)^{-\frac{1}{2}} - \frac{1}{2}\left(\frac{r^2}{(R+h)^2}-1\right)^{-\frac{3}{2}}\frac{2\eta_0 r^2}{(R+h)^2}\left[\mathrm{e}^{-(r-R)/H}-\mathrm{e}^{-h/H}\right] \tag{3.26}$$

Now we will fill this quite monstrous equation into our formula for ν (equation (3.20)).

$$\nu = \frac{2\eta_0}{(R+h)^2}\int_{R+h}^{\infty}\frac{r\mathrm{d}r\left(-\mathrm{e}^{-(r-R)/H}+\mathrm{e}^{-h/H}\right)}{\left(\frac{r^2}{(R+h)^2}-1\right)^{\frac{3}{2}}} \tag{3.27}$$

Here we will introduce a change of variables to make this integral a little easier to grasp. We say:

$$\begin{aligned} x &= \frac{r}{R+h} \\ \mathrm{d}r &= (R+h)\mathrm{d}x \\ r-R &= x(R+h)-R = R(x-1)+hx \end{aligned} \tag{3.28}$$

Using this we see we get:

$$\nu = \frac{2\eta_0}{(R+h)^2}\int_{1}^{\infty}\frac{(R+h)^2\mathrm{d}x\left(-\mathrm{e}^{-R(x-1)-hx}+\mathrm{e}^{-h/H}\right)}{(x^2-1)^{\frac{3}{2}}} \tag{3.29}$$

Simplifying this gives us our final integral from for the deflection angle.

$$\nu = 2\eta_0\int_{1}^{\infty}\frac{\mathrm{e}^{-h/H}\left(1-\mathrm{e}^{-(x-1)\frac{(R+h)}{H}}\right)x}{(x^2-1)^{\frac{3}{2}}}\mathrm{d}x \tag{3.30}$$

To get a solution for this integral we used Wolfram Mathematica to give us our deflection angle in terms of the minimal height the light ray reaches.

$$\nu = 2\eta_0\mathrm{e}^{\frac{R}{H}}\frac{R+h}{H}\mathrm{K}_0\left(\frac{R+h}{H}\right) \tag{3.31}$$

Here K_0 is the modified Bessel function of the second type, which is given by:

$$\mathrm{K}_0(x) = \int_0^{\infty}\frac{\cos(xt)\mathrm{d}t}{\sqrt{t^2+1}} \tag{3.32}$$

We will however, not use this expression for the Bessel function, but the following approximation [13]:

$$K_0(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x} \tag{3.33}$$

This gives us the following equation for the deflection angle:

$$\nu \approx 2\eta_0 e^{R/H} \frac{R+h}{H} \sqrt{\frac{\pi}{2\left(\frac{R+h}{H}\right)}} e^{-\frac{R+h}{H}} = 2\eta_0 e^{-h/H} \sqrt{\frac{\pi(R+h)}{2H}} \tag{3.34}$$

This is a simple enough function that Python is very easily able to plot the deflection angle ν as a function of h, the minimal distance of a light ray from the origin. This is shown in figure 9. The values we used for the constants in equation 3.34 are shown in table 1.

R	6.371×10^6 m
H	8.5×10^3 m
η_0	2.73×10^{-4}

Table 1: The values for the radius of the Earth , the height of the atmosphere and the refractive index of air at sea level minus 1 [14]

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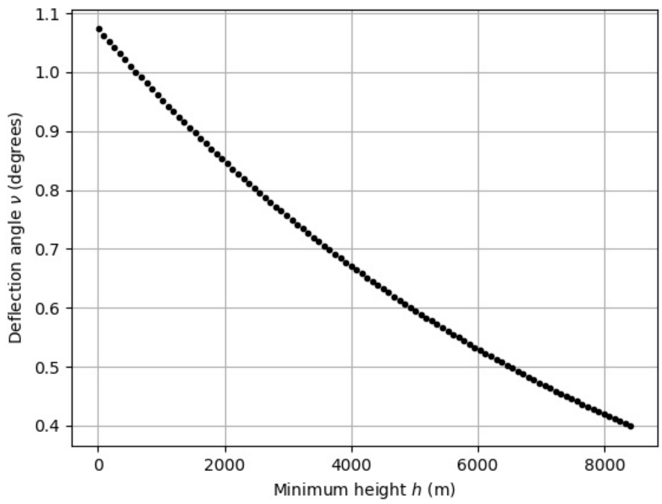


Figure 9: The deflection angle ν as a function of the minimal height h reached by a light ray travelling through the earth's atmosphere.

Now that we have formula 3.34 for the deflection angle, we can start to find equations for the incoming and outgoing asymptote of the lightray. We had already seen in section 3.1.1.1 that in the vacuum in space the path of a lightray is a straigth line described by equation 3.19. The way we choose our incoming lightray, we will say that y is a constant and therefore does not depend on x, so $\tan(C) = 0 \Rightarrow C = 0$. This means that our incoming asymptote can be given by:

$$y = \pm J \tag{3.35}$$

Where $J = (R+h)n(R+h)$, with h the minimal height reached.

Then for the outgoing asymptote, we know that the deflection angle is equal to ν, so the outgoing asymptote can be written as follows.

$$y = \tan(\nu)x - \frac{J}{\cos(\nu)} \tag{3.36}$$

3.1.2. Numerical calculation of the path

Unfortunately, we haven't been able to find a complete analytical solution for the path of a light ray through the atmosphere. So we will look for a numerical solution instead. In order to do this we won't start with minimizing the optical path length as we did in section 3.1.1. Instead we're going to look at the Euler-Lagrange equations of motion.

We will use the form of these equations of motion as described below [12].

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\partial \mathcal{L}}{\partial r'} \tag{3.37}$$

Then we can use the formulation of the Lagrangian we found in section 3.1.1, namely equation 3.5. Filling this into equation gives us the equation of motion in terms of r, r' and θ,

$$\frac{\partial n(r)\sqrt{\dot{r}^2 + r^2}}{\partial r} = \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\partial n(r)\sqrt{\dot{r}^2 + r^2}}{\partial \dot{r}} \tag{3.38}$$

Using the product rule on the left side, and noticing that $n(r)$ does not depend on $theta$ or $\dot{r}$ we see that this is equal to the following.

$$\sqrt{\dot{r}^2 + r^2}\frac{\partial n(r)}{\partial r} + n(r)\frac{r}{\sqrt{\dot{r}^2 + r^2}} = n(r)\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\dot{r}}{\sqrt{\dot{r}^2 + r^2}} \tag{3.39}$$

Then dividing both sides by $n(r)$ and using first the product rule, and afterwards the chain rule on the right side we see that this then becomes

$$\sqrt{\dot{r}^2 + r^2}\frac{\partial \log(n(r))}{\partial r} + \frac{r}{\sqrt{\dot{r}^2 + r^2}} = \frac{\ddot{r}}{\sqrt{\dot{r}^2 + r^2}} - \frac{\dot{r}^2(\ddot{r} + r)}{(\dot{r}^2 + r^2)^{3/2}} \tag{3.40}$$

Then we multiply both sides of the equation by $(\dot{r}^2 + r^2)^{3/2}$, and take all the loose terms to the right and we find a second order differential equation:

$$\left(\dot{r}^2 + r^2\right)^2 \frac{\partial \log(n(r))}{\partial r} = r^2\ddot{r} - 2\dot{r}r - r^3 \tag{3.41}$$

Now we try to find an expression for the derivative of $\log(n(r))$. This is quite trivial, with just using the chain rule. For $n(r)$ we use equation 3.1 We find:

$$\frac{\partial \log(n(r))}{\partial r} = -\frac{\eta_0}{H}\frac{\mathrm{e}^{(R-r)/H}}{1 + \eta_0\mathrm{e}^{(R-r)/H}} \tag{3.42}$$

Plugging this into equation 3.41 we find our second order differential equation.

3.1.2.1 Initial conditions

The last thing we need to find a numeric path of our lightray is our initial conditions. We need an initial distance r_0 and an inital 'velocity' $\dot{r}$ where $\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\theta}$. We want our ray to come from a star, which is very far away. For the sake of simplicity in the code, we are going to ray start somewhere the positive x direction. We want the ray to make a horizontal line towards the earth until it hits the atmosphere. So we say that $y' = 0$. Filling in the definition of y in polar coordinates gives us:

$$y' = r' \sin\theta + r\cos\theta = 0 \tag{3.43}$$

We want to find r_0', so we rewrite this, and we see:

$$r'(\theta_0) = \frac{-r(\theta_0)}{\tan\theta_0} \tag{3.44}$$

Then our θ_0 is defined as follows from the figure 10.

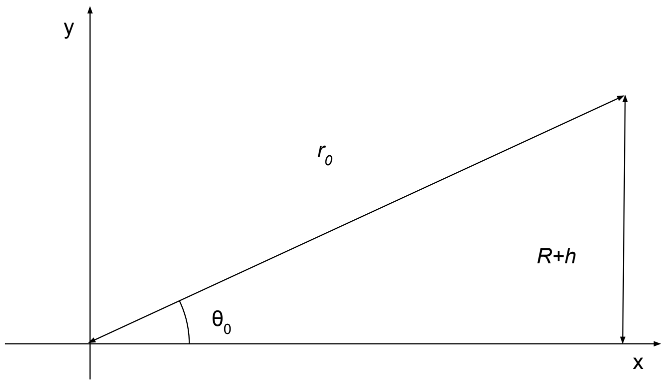


Figure 10: Diagram of θ with r_0 the initial distance, and $R + h$ the initial height

Clearly now we can see that $\theta_0 = \arcsin\left(\frac{R+h}{r_0}\right)$. Filling this in gives us our initial conditions:

$$\begin{cases} r(\theta_0) = r_0 \\ \dot{r}(\theta_0) = \frac{r_0}{\tan(\theta_0))} \\ \theta_0 = \arcsin(\frac{R+h}{r_0}) \end{cases} \tag{3.45}$$

Now we have all the information to solve the ODE for the path of the light ray.

3.1.3. Comparing numeric and analytic methods

The results of our code can be seen in the following few figures. We made a figure of the deflection angle, both calculated analytically and by solving equation 3.41 in figure 11.

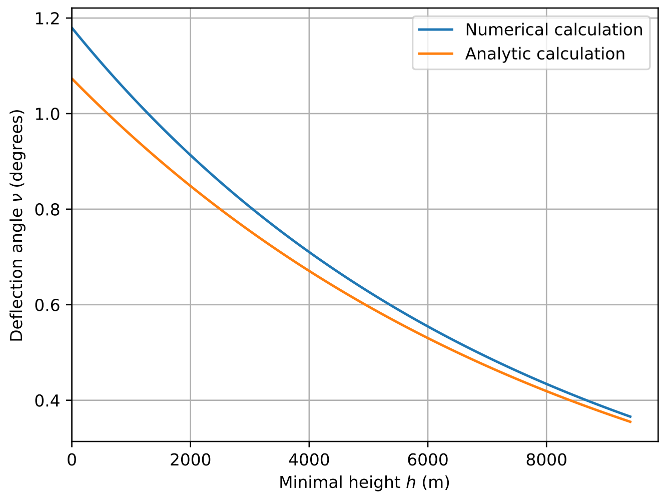


Figure 11: The deflection angle calculated both analytically (the same as figure 9) and numerically.

Clearly there is a difference between the two methods. This is probably due to the approximations used in the analytical way, such as the approximation for the Bessel function, and the Taylor expansion where the higher order terms were neglected. Unfortunately this difference is significant, and therefore the analytical way is not usable for the model of the Atmoscope.

3.2. Modeling many light rays

In this section we will be looking at many light rays, instead of just one like the previous section. We will model these light rays in the way described in section 2.

3.2.1. Equation of motion

To adapt the method described in section 2, we need to plug in our refractive index, equation (3.1), into our equations of motion. equations (2.28a)-(2.28c). We see we need the derivatives of the refractive index with respect to all coordinates. First we will look at taking the derivative with respect to x. This is simply using the chain rule twice.

$$\begin{aligned}\frac{\partial n(r)}{\partial x} &= \eta_0 \frac{d \exp\left(-\frac{r-R}{H}\right)}{dr}\frac{\partial r}{\partial x} \\ &= -\frac{\eta_0}{H}\exp\left(-\frac{r-R}{H}\right)\frac{d\sqrt{x^2+y^2+z^2}}{dx} \\ &= -\frac{\eta_0 x}{H\sqrt{x^2+y^2+z^2}}\exp -\frac{(r-R)}{H} \\ &= -\frac{\eta_0 x}{Hr}\exp -\frac{(r-R)}{H}\end{aligned} \tag{3.46}$$

And similarly we can get results for the partial derivatives to y and z. Filling these into our general equations of motion we obtain our final equations of motion for a spherical earth.

$$u_3 = \sqrt{1 - u_1^2 - u_2^2} \tag{3.47}$$

$$\begin{pmatrix} \frac{dx}{dz} \\ \frac{dy}{dz} \end{pmatrix} = \begin{pmatrix} \frac{u_1}{u_3} \\ \frac{u_2}{u_3} \end{pmatrix} \tag{3.48}$$

$$\begin{pmatrix} \frac{du_1}{dz} \\ \frac{du_2}{dz} \end{pmatrix} = \frac{-\eta_0}{n(x,y,z)Hr} \begin{pmatrix} u_3 x - u_1 z + \frac{u_2^2 x}{v_3} - \frac{u_1 u_2 y}{u_3} \\ u_3 y - u_2 z + \frac{u_1^2 y}{v_3} - \frac{u_1 u_2 x}{u_3} \end{pmatrix} \exp \frac{-(r-R)}{H} \tag{3.49}$$

With $r = \sqrt{x^2 + y^2 + z^2}$.

These equations we have implemented into a function in our code, which in listing 1, is called `dvdz`, but in the code below is called `TotalFunction`. which is shown below. The function depends on `v` and `z`, where `v` is an array in the shape of $[x, y, u_1, u_2]$. The result is an array in the shape $\left[\frac{dx}{dz}, \frac{dy}{dz}, \frac{du_1}{dz}, \frac{du_2}{dz}\right]$

```
def TotalFunction(v,z):     #this is the differential equation for the propagation in z-direction of photons
    r1=sqrt(v[0]**2+v[1]**2+z**2)
    n1= 1+eta*exp(-(r1-R)/H)

    x=v[0]
    y=v[1]
    v1=v[2]
    v2=v[3]
    v3= sqrt(1-v1**2-v2**2)

    dv11= -eta / H / r1 /n1 * exp(-(r1-R)/H)*(v3*x - v1*z + v2**2*x/v3 - v1*v2*y/v3)
    dv22 = -eta / H / r1 /n1 * exp(-(r1-R)/H)*(v3*y - v2*z + v1**2*y/v3 - v1*v2*x/v3)
    result = [v[2],v[3], dv11,dv22]
    return result
```

Listing 2: The equations of motion written in such a way that they work with the propagating code from section 2.2.1

3.2.2. Initial conditions

In this section we will shortly describe specifically which initial conditions we will use for our spherical planet model. In section 2.2.2, we said we wanted to model the propagation of a thin band. The light rays on this thin band we describe by the following equation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \\ -(R+10H) \end{pmatrix} \tag{3.50}$$

Where $\theta \in [0, 2\pi]$ and $r \in [R_{band} - 0.5k, R_{band} + 0.5k]$ Where the band is k m thick, and has an average radius of R_{band} m above the surface of the planet if there were no atmosphere.

3.2.3. Results

Now that we have all the necessary ingredients to run our model, lets do exactly that. The variables we used are shown in table 2, and the flash map is shown in figure 12.

R	$6.371 \times 10^6\,\mathrm{m}$
H	$8.5 \times 10^4\,\mathrm{m}$
η_0	2.73×10^{-4}
z_{span}	$[-(R + 10H), R + 10H]$
R_{band}	$1.5 \times 10^4\,\mathrm{m}$
k	$2\,\mathrm{m}$
r-steps	2780
θ-steps	720
total #light rays	2001600

Table 2: All the variables we used in our model for the spherical planet [14]

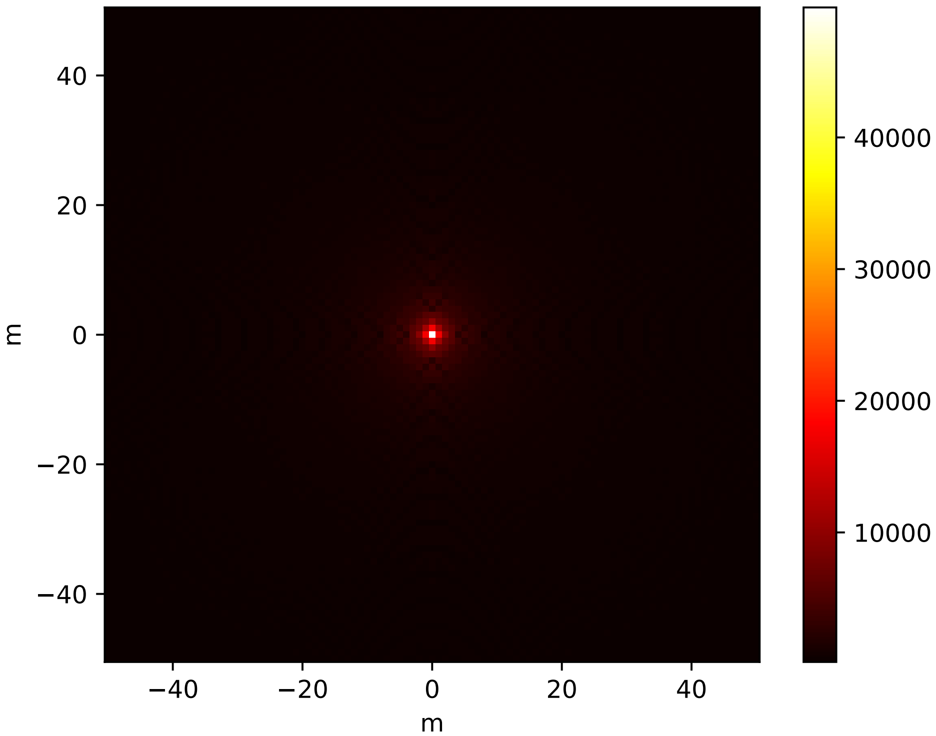


Figure 12: The amplification map of light rays passing through the atmosphere of a spherical earth. In this particular picture we used a planet with a radius $R = 6.371 \times 10^6\,\mathrm{m}$, and an atmosphere with scale height $H = 8.5 \times 10^4\,\mathrm{m}$. The initial conditions of the light rays was a band with a width of 2 meter and a radius of $1.5 \times 10^4\,\mathrm{m}$. To make this figure we used 2780 steps in the r-direction, and 720 steps in the θ-direction. Each pixel in this figure is 1 meter squared, which gives us an amplification factor $C_{amp} = 40.09$.

We see in this picture that all light rays converge to the center, which is exactly as we expected. A check for our code is that for pixels of 1x1 m, our amplification factor is about 40, which means that in the 4 brightest center pixels we have a little over 1400 light rays that end up in each pixel, which seems reasonable, since each concentric band of light rays in our thin band has about 720 rays. Furthermore we see the maximum amplification is about 50000 for pixels of $1\,\mathrm{m}^2$, which is also about the same as Kipping found in his paper.

It is interesting to see the effect of pixelsize on our maximum amplification, which is what we have done graphically in the figure below.

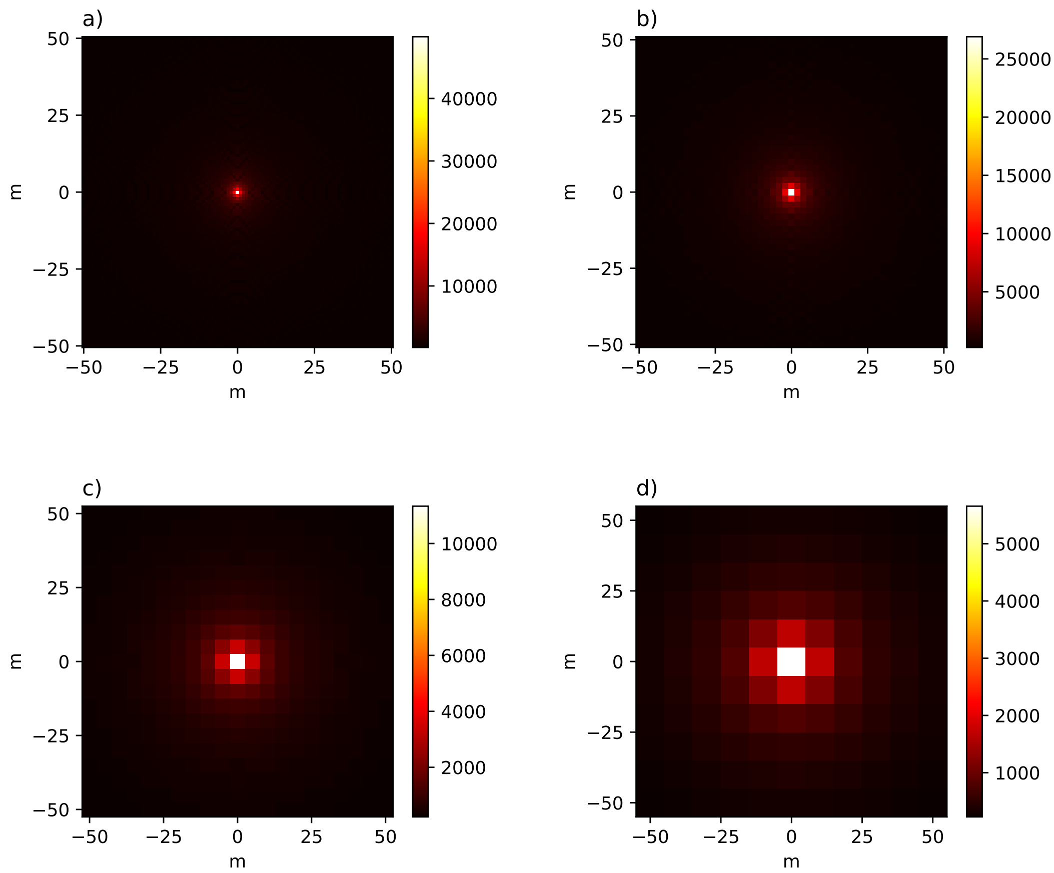


Figure 13: The difference in amplification from the same model with varying pixelsizes. a), b), c) and d) have pixel-width 1, 2, 5 and 10 meter respectively.

We investigate the relation between maximum amplification and pixelsize further in figure 14. We fit a curve through the data, which is a $A = \frac{a}{D} + b$ fit, where D is the pixel width. The fit resulted in the parameters $a = 52099$ and $b = -272$.

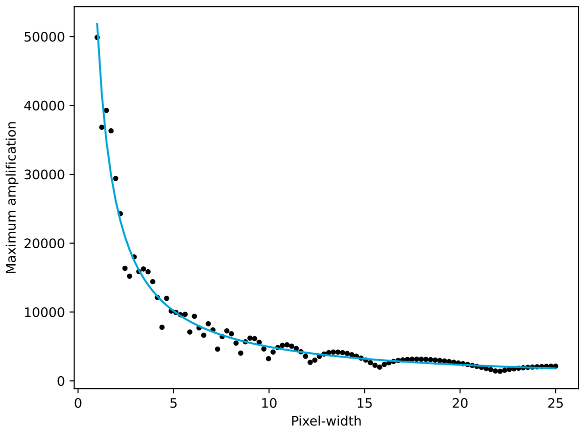


Figure 14: Maximum amplification as a function over pixelsize, the dots are our datapoints, and the blue line is a $A = \frac{a}{D} + b$ *curve fit, where* D *is the pixel width, plotted over the points, and parameters* $a = 52099$ *and* $b = -272$.

4. Amplification by the atmosphere of an ellipsoidal planet

The next step in making the model more realistic is realizing that nearly no planet is a perfect spherical object. Therefore, it might be of great value to look at the effect of the oblateness of the planet. Important to note is that in the equations of motion, the only part of the shape of the planet that we use is in the refractive index function. We see that the refractive index is a function of distance from the planet's surface. For a spherical planet, this is simple; it's just the distance from the origin minus the planet's radius. For an ellipsoidal earth however, distance from the surface is not as easily available. Thus, we need to find a way to express distance from the surface of an ellipsoid, which we can then insert into the already known equations of motion. In this chapter we start by doing this, and then we take a intermezzo by looking at a geometric explanation of what will happen when light bends towards the surface of an ellipsoid, after which we will look at the results and compare this to what we found in the previous section. In figure 15 we show graphically what we mean with an ellipsoid.

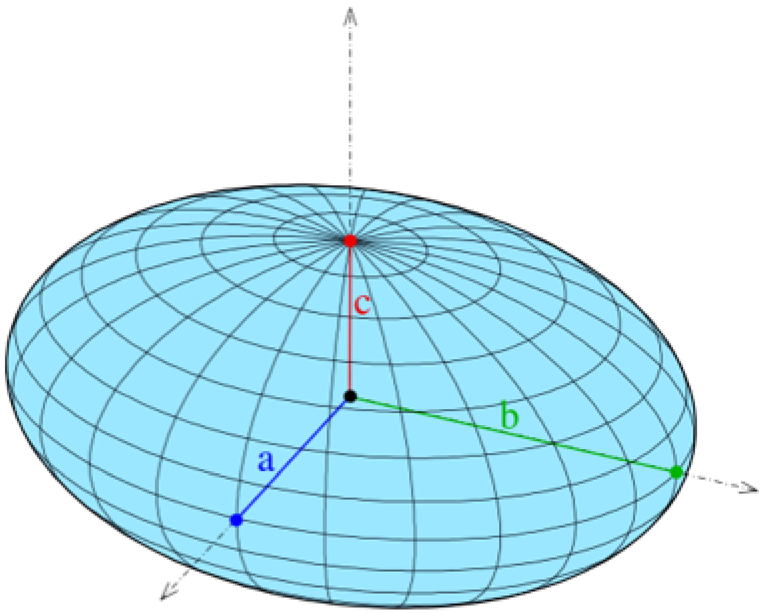


Figure 15: An ellipsoid with principal semi-axes a, b, and c [15]

4.1. Distance from an ellipsoid

4.1.1. Method 1

In order to apply the equations of motion found in section 2.1 to an ellipsoidal planet, we need to find a way to express the height above an ellipsoid, since the refractive index is a function of height. We look for the length of a line normal to the surface of the ellipse through the point we want to know the height of. In this section we're looking at two different ways to find this distance, and showing how they are similar to the first order.

We start our search by looking at how we write an ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{4.1}$$

Lets take a point $A = (x_0, y_0, z_0)$ somewhere in the atmosphere above our ellipsoid. We want to see how far above the ellipsoid it is. We get the normal vector **N** to the surface as follows:

$$\mathbf{N} = \begin{pmatrix} \frac{2x_0}{a^2} \\ \frac{2y_0}{b^2} \\ \frac{2z_0}{c^2} \end{pmatrix} \tag{4.2}$$

Using this we can find the parametrization of a line that goes through our point A and is perpendicular to the surface of the planet:

$$\vec{r}(t) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} \frac{2x_0}{a^2} \\ \frac{2y_0}{b^2} \\ \frac{2z_0}{c^2} \end{pmatrix} \tag{4.3}$$

Now we want to find the point on this line that is on the ellipsoid, so we substitute this parametrization into our equation for the ellipsoid.

$$\frac{\left(x_0 + 2t\frac{x_0}{a^2}\right)^2}{a^2} + \frac{\left(y_0 + 2t\frac{y_0}{b^2}\right)^2}{b^2} + \frac{\left(z_0 + 2t\frac{z_0}{c^2}\right)^2}{c^2} = 1 \tag{4.4}$$

We want to solve this for t, so first we take out the x_0, y_0 and z_0:

$$\frac{x_0^2}{a^2}\left(1 + \frac{4t}{a^2} + \frac{4t^2}{a^4}\right) + \frac{y_0^2}{b^2}\left(1 + \frac{4t}{b^2} + \frac{4t^2}{b^4}\right) + \frac{z_0^2}{a^2}\left(1 + \frac{4t}{c^2} + \frac{4t^2}{c^4}\right) = 1 \tag{4.5}$$

Since a, b, c are very big compared to the height above the surface, we will neglect the highest order terms in this equation and we get:

$$\frac{x_0^2}{a^2}\left(1 + \frac{4t}{a^2}\right) + \frac{y_0^2}{b^2}\left(1 + \frac{4t}{b^2}\right) + \frac{z_0^2}{c^2}\left(1 + \frac{4t}{c^2}\right) = 1 \tag{4.6}$$

Now we factor out t, and we get

$$4t\left(\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} + \frac{z_0^2}{c^4}\right) = 1 - \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2} \tag{4.7}$$

Which we then solve for t and we get:

$$t = \frac{1}{\frac{4x_0^2}{a^4} + \frac{4y_0^2}{b^4} + \frac{z_0^2}{c^4}}\left(1 - \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2}\right) < 0 \tag{4.8}$$

The distance between A and the surface of the planet is given by:

$$h(x_0, y_0, z_0) = (-t)\sqrt{\left(\frac{2x_0^2}{a^2}\right)^2 + \left(\frac{2y_0^2}{b^2}\right)^2 + \left(\frac{2z_0^2}{c^2}\right)^2} \tag{4.9}$$

Then filling in the value for t we found we get

$$h(x_0, y_0, z_0) = \frac{\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} - 1}{2\sqrt{\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} + \frac{z_0^2}{c^4}}} \tag{4.10}$$

So finally we have a new formula for the refractive index:

$$n(x, y, z) = 1 + \eta_0 \exp \frac{-h(x, y, z)}{H}, \tag{4.11}$$

where $h(x, y, z)$ is as defined in equation (4.10).

Now we still need to fill this in into our equations of motion, which means we need the partial derivatives for the refractive index.

$$\frac{\partial n}{\partial x} = \eta_0 \frac{-x}{H} \exp \frac{-h}{H} \frac{\partial h(x, y, z)}{\partial x} \tag{4.12}$$

These partial derivatives of h are quite gruesome to get, and therefore it might be worth looking into a simpler way to get the height, which is what we'll do in the next paragraph. We'll also show that this different, simpler way, is a first order approximation of the method described in this paragraph.

4.1.2. Method 2

At a place close to the surface of the planet, you can suppose that in a point (x, y, z) at a height h you are in another ellipsoid:

$$\frac{x^2}{a^2(1 + h/a)^2} + \frac{y^2}{b^2(1 + h/a)^2} + \frac{z^2}{c^2(1 + h/a)^2} = 1 \tag{4.13}$$

Solving this for h gives:

$$h(x, y, z) = \left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} - 1 \right) \cdot a \tag{4.14}$$

This only works if a, b and c are close in size, which for Earth they are. At Earth a and c are equal, which is what we will assume for the rest of the equations as well. Hence

$$h(x, y, z) = \left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2}} - 1 \right) \cdot a \tag{4.15}$$

4.1.3. Comparing the methods

In this paragraph we will compare the two different methods found to calculate the distance h from an ellipsoid. Let us first look at our first equation, here we have only assumed a, b and c are big. In the second method however, we have also assumed $a \approx b \approx c$, and that our chosen x, y and z are on an ellipsoid as well. Lets also add these assumptions to our first model. That means we can make the following approximations:

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \approx \frac{1}{a^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) \approx \frac{1}{a^2} \tag{4.16}$$

This gives us the following approximation for the height.

$$h_1(x,y,z) \approx \frac{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1}{2\sqrt{\frac{1}{a^2}}} = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)\frac{a}{2} \tag{4.17}$$

Now we look at our second method. We work from the approximation we found for our first method, and show it is also an approximation for the second method.

$$\begin{aligned}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)\frac{a}{2} &= \left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} + 1\right)\left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} - 1\right)\frac{a}{2} \\ &\approx \left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} - 1\right)a = h_2(x,y,z)\end{aligned} \tag{4.18}$$

In the last step here we have used that the first term of the right hand side is approximately 2. So now we've shown that the two methods are the same up to the first order. So, we will use method 1 to describe the height in our model.

4.2. Equations of motion

Just like in section 3.2.1, we need to fill in our refractive index into the general equations of motion: equation (2.28a)-(2.28c). We have for our refractive index the following formula:

$$n(x,y,z) = 1 + \eta_0 \exp\frac{-h(x,y,z)}{H} \tag{4.19}$$

Where for $h(x,y,z)$ we use the expression we found in the second method: equation (4.15). The partial derivatives of this refractive index are quite simple to obtain with the equation for h from the second model:

$$\frac{\partial n}{\partial x} = -\frac{x}{aH}\frac{n(x,y,z)-1}{\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2}}} \tag{4.20}$$

$$\frac{\partial n}{\partial y} = -\frac{ya}{b^2H}\frac{n(x,y,z)-1}{\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2}}} \tag{4.21}$$

$$\frac{\partial n}{\partial z} = -\frac{z}{aH}\frac{n(x,y,z)-1}{\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2}}} \tag{4.22}$$

Now that we have these derivatives, we fill this into equation (2.28c), which, together with equations (2.28a) and (2.28b) gives us the equations of motion for an ellipsoidal planet.

$$u_3 = \sqrt{1 - u_1^2 - u_2^2} \tag{4.23}$$

$$\begin{pmatrix} \frac{dx}{dz} \\ \frac{dy}{dz} \end{pmatrix} = \begin{pmatrix} \frac{u_1}{u_3} \\ \frac{u_2}{u_3} \end{pmatrix} \tag{4.24}$$

$$\begin{pmatrix} \frac{du_1}{dz} \\ \frac{du_2}{dz} \end{pmatrix} = \frac{-\eta_0}{n(x,y,z)h(x,y,z)H} \begin{pmatrix} \frac{u_3 x}{a} - \frac{u_1 z}{a} + \frac{u_2^2 x}{v_3 a} - \frac{u_1 u_2 y}{u_3 b^2} \\ \frac{u_3 y}{b^2} - \frac{u_2 z}{a} + \frac{u_1^2 y}{u_3 b^2} - \frac{u_1 u_2 x}{v_3 a} \end{pmatrix} \exp - \frac{h(x,y,z)}{H} \tag{4.25}$$

We can now use these equations to slightly adjust the code from section 3.2.1, which is shown in the listing below.

```
def TotalFunction(v,z):
    r1 = sqrt( v[0]**2 / a**2 + v[1]**2 / b**2 + z**2 /c**2)
    x = v[0]
    y = v[1]
    h1 = (sqrt( x**2 /a**2 + y**2 /b**2 + z**2 / c**2) - 1) * a
    n1 = 1+eta*exp(-(h1)/H)
    v1 = v[2]
    v2 = v[3]
    v3 = sqrt( 1 - v1**2 - v2**2)
    dv11 = -eta / H / r1 /n1 * exp(-(h1)/H)*(v3*x/a - v1*z/a + v2**2*x/v3/a - v1*v2*y*a/v3/b**2)
    dv22 = -eta / H / r1 /n1 * exp(-(h1)/H)*(v3*y*a/b**2 - v2*z/a + v1**2*y*a/v3/b**2 - v1*v2*x/v3/a)
    result = [v[2],v[3], dv11,dv22]
    return result
```

Listing 3: The equations of motion for an ellipsoidal planet written in such a way that they work with the propagating code from section 2.2.1

4.3. Initial conditions

Now that we have the full equations of motion for our ellipsoidal system, we only need to describe the initial conditions for our system. As described in section 2.2.2, we want to divide the light rays homogeneously onto a thin band just outside of the atmosphere. The way we've described this band is as follows:

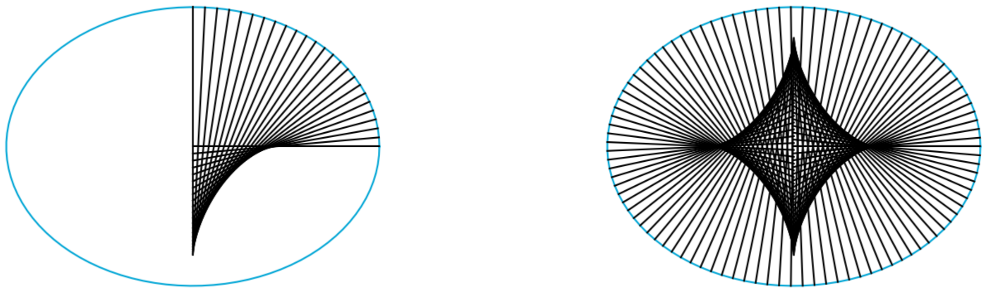
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a\cos(\theta)\left(1+\frac{r}{a}\right) \\ b\sin(\theta)\left(1+\frac{r}{a}\right) \\ -R+10H \end{pmatrix} \tag{4.26}$$

We take $\theta \in [0, 2\pi]$, and $r \in [R_{band} - 0.5k, R_{band} + 0.5k]$, where R_{band} is the average radius of the thin band, and k is the width of the band. We choose how many light rays we want to model, and take the appropriate amount of steps in the r - and θ-direction.

4.4. Caustics in the flash map: Evolute of an ellipse

For both the spherical case and the ellipsoidal case, it is helpful to think geometrically about what is actually happening when we run our model. This is why we take this quick detour about the evolute of an ellipse. An evolute can be described as the envelope of the normals to a curve. So why is this useful? When we are looking at our model, the light rays bend in the direction perpendicular to the surface of the planet. We are also mainly interested in the points to where these light rays converge, since that is where the intensity and amplification is the highest. This is translated to geometry with the evolute. The evolute of a circle is a point: all normals cross through the center of the circle. This is what we see when we look at the flash map of the results of a spherical earth as well; we only see a small point with high intensity.

For an ellipse, finding the evolute is a bit more work. Let's look at what we're expecting to find. The figure below shows what we're trying to find.



(a) Quarter ellipse *(b) Full ellipse*

Figure 16: Showing the evolute of the ellipse, by drawing lines that are perpendicular to the ellipse and seeing where they intersect the line next to it. In figure (a) we see only a quarter of the ellipse, to show for clarity what actually happens when you do this. In figure (b) this has been done throughout the whole ellipse, which results in a diamond shape appearing, this is the evolute.

We want to find an equation for this diamond shape, and see how it depends on the shape of the ellipse, so we know what to expect for different shaped planets. First, we look at the equation we have for an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4.27}$$

The gradient for this function is given by

$$\begin{pmatrix} \frac{2x}{a^2} \\ \frac{2y}{b^2} \end{pmatrix} \tag{4.28}$$

So we can take the normal to be

$$\begin{pmatrix} \frac{x}{a^2} \\ \frac{y}{b^2} \end{pmatrix} \tag{4.29}$$

Now we will use a parametrization of the ellipse:

$$\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} a\cos\theta \\ b\sin\theta \end{pmatrix} \tag{4.30}$$

We want to find lines that cross the ellipse, and are parallel to the normal, so we fill in the parametrization into the normal. These lines we'll denote by:

$$\bar{r}_\theta(t) = \begin{pmatrix} a\cos\theta \\ b\sin\theta \end{pmatrix} + t\begin{pmatrix} \frac{\cos\theta}{a} \\ \frac{\sin\theta}{b} \end{pmatrix} \tag{4.31}$$

Since the evolute is the envelope of these lines we just created, we want to find an expression of where these lines cross each other. Let's therefore find the intersection between two of these lines $\bar{r}_{\theta_1}(t)$ and $\bar{r}_{\theta_2}(s)$.

$$\begin{pmatrix} a\cos\theta_1 + t\frac{\cos\theta_1}{a} \\ b\sin\theta_1 + t\frac{\sin\theta_1}{b} \end{pmatrix} = \begin{pmatrix} a\cos\theta_2 + s\frac{\cos\theta_2}{a} \\ b\sin\theta_2 + s\frac{\sin\theta_2}{b} \end{pmatrix} \tag{4.32}$$

We have two unknowns, t and s, so lets rewrite this a little bit to pull those out, while also moving the a's and b's to one side:

$$\begin{pmatrix} \cos\theta_1 & -\cos\theta_2 \\ \sin\theta_1 & -\sin\theta_2 \end{pmatrix}\begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} a^2(\cos\theta_2 - \cos\theta_1) \\ b^2(\sin\theta_2 - \sin\theta_1) \end{pmatrix} \tag{4.33}$$

To find expressions for t and δ, we find the inverse of the leftmost matrix. To do that we use that for a 2 by 2 matrix the inverse can be given by:

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{4.34}$$

Using this, we get:

$$\begin{pmatrix} t \\ s \end{pmatrix} = \frac{1}{\cos\theta_1\sin\theta_2 - \cos\theta_2\sin\theta_1}\begin{pmatrix} -\sin\theta_2 & \cos\theta_2 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix}\begin{pmatrix} a^2(\cos\theta_2 - \cos\theta_1) \\ b^2(\sin\theta_2 - \sin\theta_1) \end{pmatrix} \tag{4.35}$$

Using goniometric formula's and multiplying the matrix with the vector we get:

$$\begin{pmatrix} t \\ s \end{pmatrix} = -\frac{1}{\sin(\theta_2 - \theta_1)}\begin{pmatrix} -a^2\sin\theta_2(\cos\theta_2 - \cos\theta_1) + b^2\cos\theta_2(\sin\theta_2 - \sin\theta_1) \\ -a^2\sin\theta_1(\cos\theta_2 - \cos\theta_1) + b^2\cos\theta_1(\sin\theta_2 - \sin\theta_1) \end{pmatrix} \tag{4.36}$$

Now we say $\theta_2 = \theta_1 + d\theta$, where $d\theta$ is small. We do this, since we want to know where two lines that start closest to each other intersect. This gives us:

$$\begin{pmatrix} t \\ s \end{pmatrix} = -\frac{1}{d\theta}\begin{pmatrix} -a^2\sin\theta_2(\cos\theta_2 - \cos\theta_1) + b^2\cos\theta_2(\sin\theta_2 - \sin\theta_1) \\ -a^2\sin\theta_1(\cos\theta_2 - \cos\theta_1) + b^2\cos\theta_1(\sin\theta_2 - \sin\theta_1) \end{pmatrix} \tag{4.37}$$

Now using sum to product goniometric identities we can rewrite everything between brackets.

$$\begin{pmatrix} t \\ s \end{pmatrix} = -\frac{1}{d\theta}\begin{pmatrix} a^2\sin\theta_2(\sin\theta_1 d\theta) + b^2\cos\theta_2(\cos\theta_1 d\theta) \\ a^2\sin\theta_1(\sin\theta_1 d\theta) + b^2\cos\theta_1(\cos\theta_1 d\theta) \end{pmatrix} \tag{4.38}$$

Since θ_d is small, we can say that $\sin\theta_2 \approx \sin\theta_1$ and similarly for the cosines. This gives us our final equation for t and s (which have become equal, like we would have hoped).

$$\begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} -a^2\sin^2\theta_1 - b^2\cos^2\theta_1 \\ -a^2\sin^2\theta_1 - b^2\cos^2\theta_1 \end{pmatrix} \tag{4.39}$$

Now we can fill this value for t we found into the $\bar{r}_\theta(t)$, which now becomes only dependent on θ.

$$\bar{r}_\theta(t) = \begin{pmatrix} a\cos\theta - (a^2\sin^2\theta + b^2\cos^2\theta)\frac{\cos\theta}{a} \\ b\sin\theta - (a^2\sin^2\theta + b^2\cos^2\theta)\frac{\sin\theta}{b} \end{pmatrix} \tag{4.40}$$

Now we want to simplify this, so we expand the brackets, and we get:

$$\bar{r}_\theta(t) = \begin{pmatrix} a\cos\theta - a\sin^2\theta\cos\theta - \frac{b^2}{a}\cos^3\theta \\ b\sin\theta - b\cos^2\theta\sin\theta - \frac{a^2}{b}\cos^3\theta \end{pmatrix} \tag{4.41}$$

Factoring out $a\cos\theta$ and $b\sin\theta$ in the top and bottom equation respectively, we get:

$$\bar{r}_\theta(t) = \begin{pmatrix} a\cos\theta(1-\sin^2\theta) - \frac{b^2}{a}\cos^3\theta \\ b\sin\theta(1-\cos^2\theta) - \frac{a^2}{b}\cos^3\theta \end{pmatrix} \tag{4.42}$$

Now very clearly we know that $\cos^2\theta + \sin^2\theta = 1$ so we finally get:

$$\bar{r}(\theta) = \begin{pmatrix} \frac{a^2-b^2}{a}\cos^3\theta \\ \frac{b^2-a^2}{b}\sin^3\theta \end{pmatrix} \tag{4.43}$$

We can compare this with our 'idea' of the evolute in figure 16, which we've shown in the figure below.

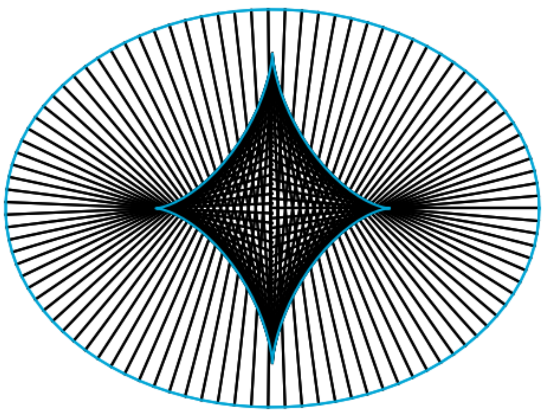


Figure 17: The evolute of an ellipse, which is drawn in blue over the drawing from the previous figure. Here the evolute is described by equation 4.43

.

As you can see, this fits perfectly. Now, it might be nice to know more about the specifics of this curve, like how wide and high is it? This we can find out very simply by plugging in the right θ in equation 4.43. When we plug in $\theta = 0$, we get the right point of the curve, and for

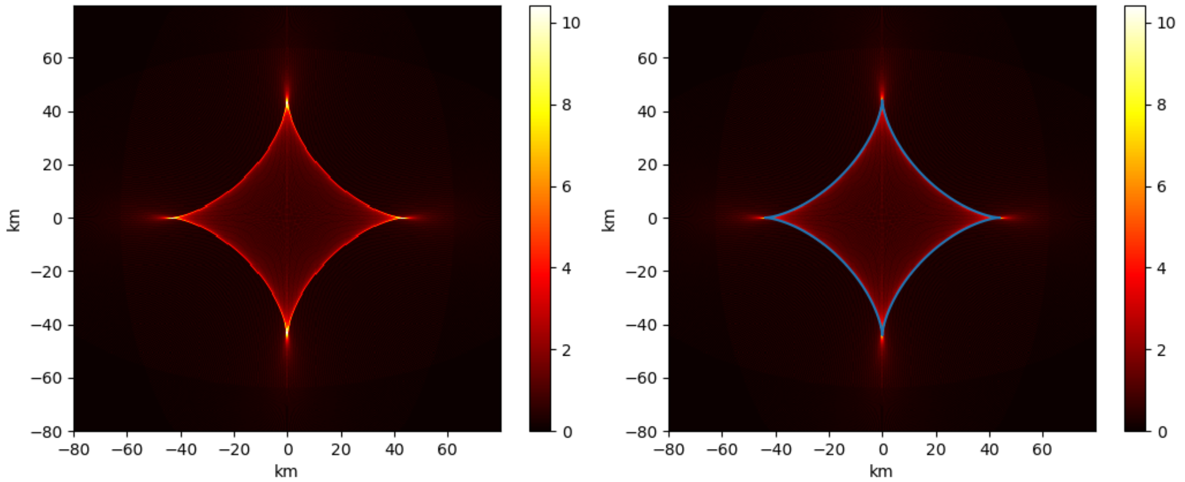
$\theta = \pi/2$, we get the bottom point of the curve.

$$\bar{r}(0) = \begin{pmatrix} \frac{a^2-b^2}{a} \\ 0 \end{pmatrix}$$
$$\bar{r}\left(\frac{\pi}{2}\right) = \begin{pmatrix} 0 \\ \frac{b^2-a^2}{b} \end{pmatrix} \tag{4.44}$$

So the diamond will be $2(a^2 - b^2)/a$ wide, and $2(a^2 - b^2)/b$ high (assuming $a > b$). For our actual model we will assume that a and b are very similar, so this then will simplify to that the diameter of the diamond will be $2(a - b)$.

4.5. Results

Firstly we will show you the flash map of a planet that we modeled as closely to the earth as possible, which you can see in the graph below. Next to it is the exact same graph, but with an overlay of the shape and size we would expect this diamond to be according to section 4.4. As you can see these overlap as good as perfectly, so these results are right as expected.



(a) Without overlay *(b) With overlay*

Figure 18: The amplification map of light rays passing through the atmosphere of an ellipsoidal earth. In these particular pictures we used a planet with 2 semi-major exes of 6.278×10^6 m *and a semi-minor axis of* 6.356×10^6 m*. The atmosphere has a scale height H of* 8.5×10^3 m*. The initial conditions of the light rays was a band with a width of* 110 m *and a height of* 1.5×10^4 m*. To make this figure, we divided 2001600 photons homogeneously onto this band: 2780 steps in the r-direction, and 720 steps in the* θ*-direction. The pixels in this figure are* 450 m *wide and* 450 m *tall. The amplification factor is 0.0217. In figure (b) we drew the evolute: the curve follows the caustics of the flash map.*

Just as in section 3.2.3, we can look at the effect of the pixel size on the maximum amplification, which we have shown in the figure below.

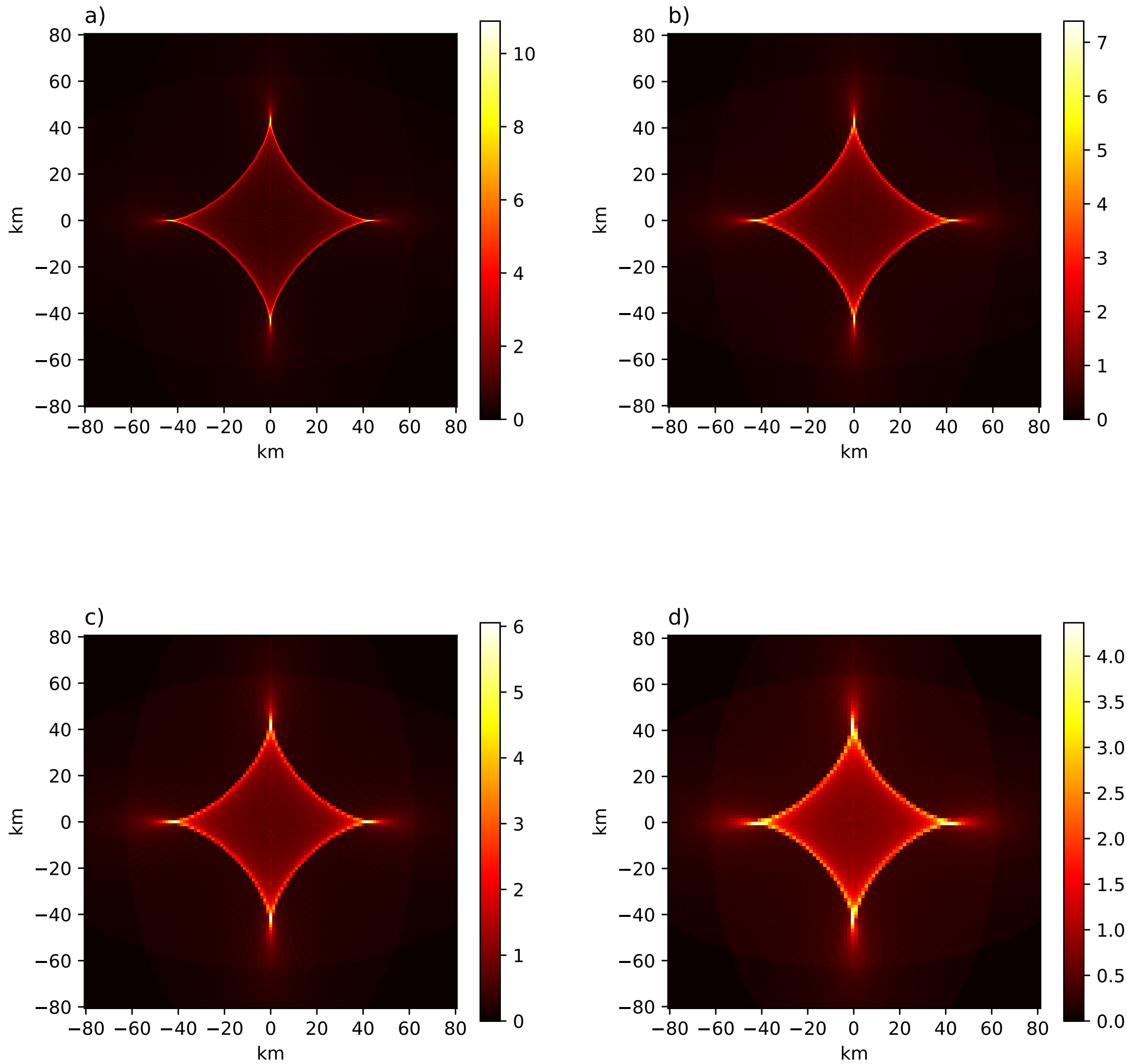


Figure 19: The difference in amplification from the same model with varying pixelsizes. a), b), c) and d) have pixel-width 450, 900, 1250 and 1500 respectively.

Just as in the spherical case we see that the maximum amplification goes down when the size of the pixels increases. However, contrary to the spherical case, in these figures you can see that the arches connecting the four corner points of the diamond shape, have roughly the same amplification: around 3.

4.6. Comparing spherical and ellipsoidal planets

An interesting question is to compare the behavior of spherical and ellipsoidal planets. To do that, we simply look at planets with different eccentricities. Eccentricity is a measure of oblateness, defined as

$$\epsilon = \sqrt{1 - \frac{b^2}{a^2}}, \tag{4.45}$$

where $b < a$.

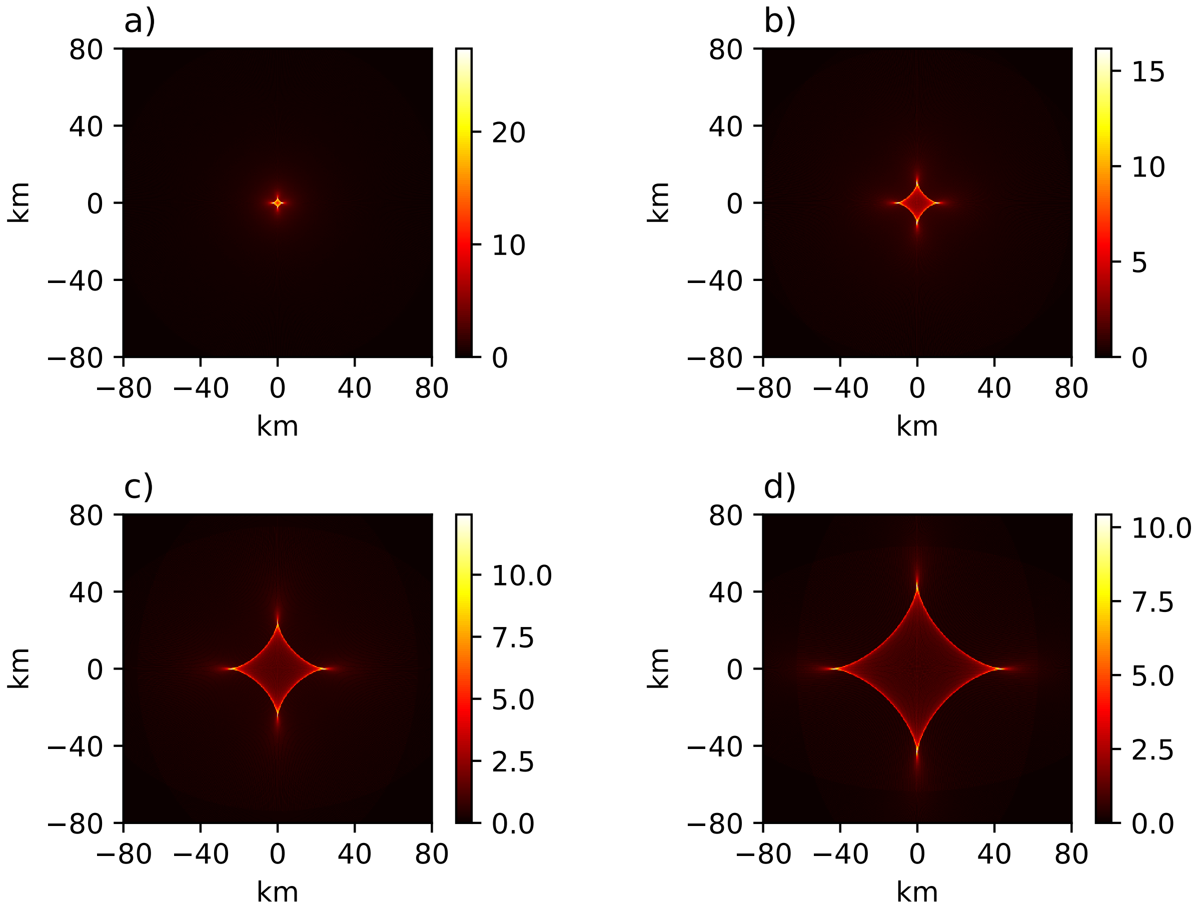


Figure 20: 4 different flash maps, for planets with different eccentricities. For a), d) the eccentricities are $0.021, 0.041, 0.062, 0.083$ *respectively. Note different colorbars on each plot; the maximum amplification is different for each. This plot was made with running the model with 20 million light rays per each figure.*

As expected, the diamond becomes smaller when the eccentricities of the planet becomes smaller. A more in-depth look is shown in the plot below.

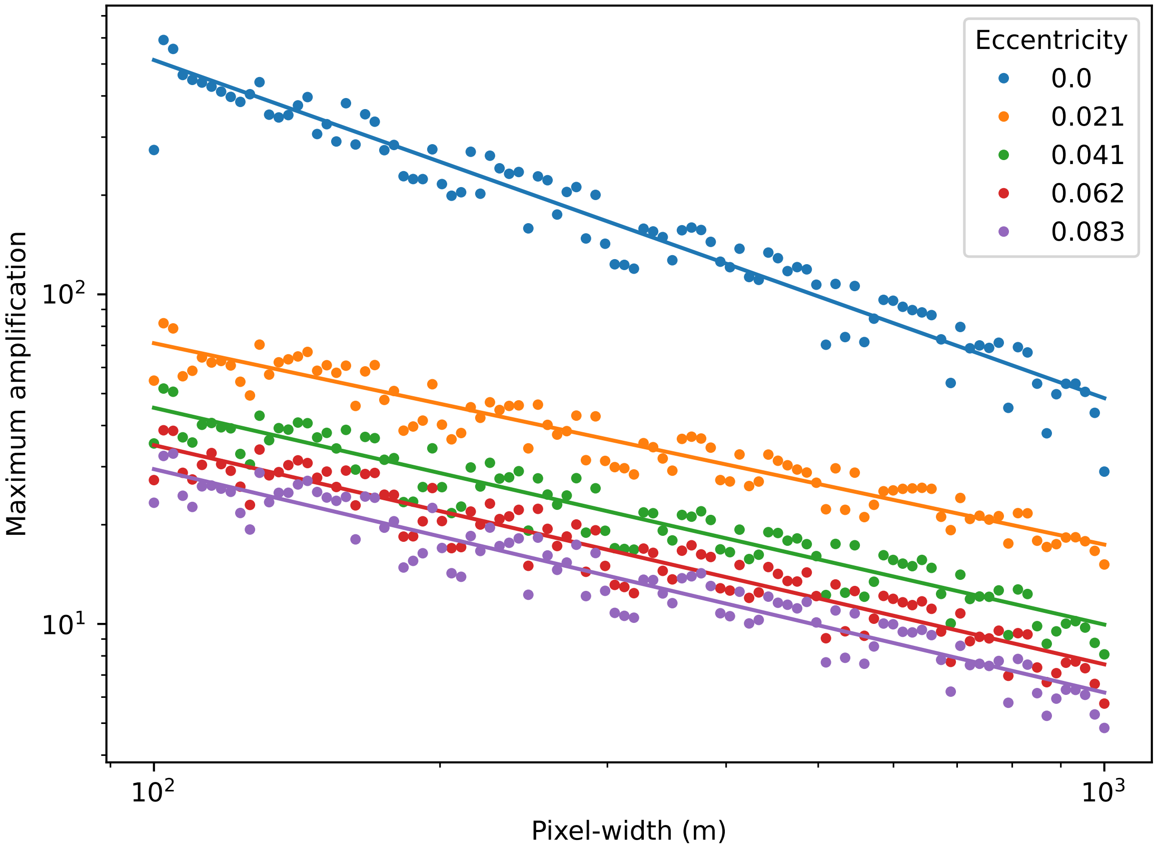


Figure 21: The maximum amplification as a function of the pixelsize, for various values of the eccentricity of the planet. For each eccentricity we have modeled a band of 2001600 light rays, with a width $110\,\mathrm{m}$. *The lines plotted in this double log plot through the points are fitted, from lowest eccentricity to highest, and have slopes* -1.025, -0.6109, -0.6577, -0.6650 *and* -0.6776

5. Modeling turbulence in the atmosphere: Diffusion

So far we have only looked at a planet where the atmosphere is perfectly described by equation (3.1) or equation (4.19). This is clearly just an approximation, as there are many things that can change the refraction index in the atmosphere, such as turbulence, temperature, etc. To account for this, we will add a diffusion term to he propagation in the atmosphere. We're using Ito diffusion.

We're using velocity diffusion inside the atmosphere to account for turbulence. To recap, without diffusion we have seen that we could get the following expression.

$$\frac{d}{dt}\hat{\mathbf{v}} = \frac{1}{n(\mathbf{r})^2}(\nabla n(\mathbf{r}) - \hat{\mathbf{v}}\hat{\mathbf{v}} \cdot \nabla n(\mathbf{r})) \tag{5.1}$$

Now we take out the factor $\nabla n(\mathbf{r})$ and see that $\frac{\nabla n(\mathbf{r})}{n(\mathbf{r})}$ can be written as $\nabla \log n$, where for ease's sake, we leave out the dependency on $\mathbf{r}$ when writing (it is of course still implied). We also multiply both by dt to obtain the following expression for the change in direction of the velocity:

$$d\hat{\mathbf{v}} = (1 - \hat{\mathbf{v}}\hat{\mathbf{v}}\cdot)\frac{\nabla n}{n^2}dt \tag{5.2}$$

This is the point where we will be adding in the diffusion term, since the diffusion creates a difference in the direction of the velocity. We will use the following equation for our ito diffusion:

$$d\hat{\mathbf{v}} = (1 - \hat{\mathbf{v}}\hat{\mathbf{v}}\cdot)\frac{\nabla n}{n^2}dt + (1 - \hat{\mathbf{v}}\hat{\mathbf{v}}\cdot)\frac{n-1}{n^2}\frac{\nabla n}{|\nabla n|}\sqrt{Ddt}\xi \tag{5.3}$$

Let me take you through everything we have in this added diffusion term. We have a term $(1 - \hat{\mathbf{v}}\hat{\mathbf{v}}\cdot)$. This term is added so that the diffusion is always perpendicular to the original velocity. The term $\frac{n-1}{n^2}$ we have added, because we expect there to be more diffusion the lower we are in the atmosphere, since we both travel through the atmosphere more, and the lower you are in the atmosphere, the higher the refraction index, and we expect there to be greater differences at lower heights. The $\frac{\nabla n}{|\nabla n|}$ term is there to make sure we take the vertical diffusion. ξ is the random factor in this diffusion model. Important to note is that the mean of it, $\langle \xi \rangle = 0$: we expect diffusion to happen in both up or down with the same probability. Another prerequisite of this random variable is that $\langle \xi^2 \rangle = 1$ [16]. The term D is called the diffusion coefficient, and this is the one that determines how much diffusion actually happens. The calculation of D will follow later. For now, we want to rewrite this equation in such a way that we can implement it into our model. To do that, first note that since we have taken the scale of $c = 1$ where c is the speed of light, and the correlation between velocity and the refractive index is given by

$$n = \frac{c}{v} = \frac{1}{v} \tag{5.4}$$

In our model, we are not integrating with respect to time, but w.r.t z. So we need to replace the dt's in equation 5.3. To do so, we see that

$$\frac{dz}{dt} = vu_3 = \frac{u_3}{n} \tag{5.5}$$

So multiplying both sides by dt and taking all none-dt terms to one side, we see:

$$dt = \frac{ndz}{u_3} \tag{5.6}$$

We can fill this into our equation 5.3, and we get

$$d\hat{\mathbf{v}} = (1 - \hat{\mathbf{v}}\hat{\mathbf{v}}\cdot)\frac{\nabla n}{n^2}\frac{ndz}{u_3} + (1 - \hat{\mathbf{v}}\hat{\mathbf{v}}\cdot)\frac{n-1}{n^2}\frac{\nabla n}{|\nabla n|}\sqrt{D\frac{ndz}{u_3}}\xi \tag{5.7}$$

Here we can simplify the first tirm, since we have by factoring out and if we also factor out the $(1 - \hat{\mathbf{v}}\hat{\mathbf{v}}\cdot)\frac{\nabla n}{n}$

$$d\hat{\mathbf{v}} = (1 - \hat{\mathbf{v}}\hat{\mathbf{v}}\cdot)\frac{\nabla n}{n}\left(\frac{dz}{v_3} + \frac{n-1}{n}\frac{1}{|\nabla n|}\sqrt{D\frac{ndz}{u_3}}\xi\right) \tag{5.8}$$

Now we start by looking at the x-component of $d\hat{\mathbf{v}}$:

$$du_1 = \left(\frac{\partial n}{\partial x} - u_1\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}\cdot\begin{pmatrix}\frac{\partial n}{\partial x}\\\frac{\partial n}{\partial y}\\\frac{\partial n}{\partial z}\end{pmatrix}\right)\left(\frac{dz}{nu_3} + \frac{n-1}{n^2}\frac{\sqrt{D\frac{ndz}{u_3}}\xi}{\sqrt{\left(\frac{\partial n}{\partial x}\right)^2 + \left(\frac{\partial n}{\partial y}\right)^2 + \left(\frac{\partial n}{\partial z}\right)^2}}\right) \tag{5.9}$$

Taking the inner product of the two vectors we get

$$du_1 = \left(\frac{\partial n}{\partial x} - u_1\left(u_1\frac{\partial n}{\partial x} + u_2\frac{\partial n}{\partial y} + u_3\frac{\partial n}{\partial z}\right)\right)\left(\frac{dz}{nu_3} + \frac{n-1}{n^2}\frac{\sqrt{D\frac{ndz}{u_3}}\xi}{\sqrt{\left(\frac{\partial n}{\partial x}\right)^2 + \left(\frac{\partial n}{\partial y}\right)^2 + \left(\frac{\partial n}{\partial z}\right)^2}}\right) \tag{5.10}$$

To obtain du_2, we follow the exact same steps, so that will not explicitly be written down here. The next step is to fill in all the partial derivatives. We start by looking at the spherical case.

5.1. Spherical planet

Recall that for the spherical planet we have

$$n(r) = 1 + \eta_0 \exp(-\frac{r-R}{H}), \tag{5.11}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. The partial derivatives are also shown in section 3.2.1, but for comprehension's sake we write them down here again.

$$\frac{\partial n}{\partial x} = \frac{-\eta_0 \exp\frac{-(r-R)}{H}x}{Hr} = \frac{(1-n)x}{Hr} \tag{5.12}$$

With diffusion present, we see we also need the partial derivatives squared:

$$\left(\frac{\partial n}{\partial x}\right)^2 = \frac{\eta_0^2 \exp(\frac{-2(r-R)}{H})x^2}{H^2 r^2} \tag{5.13}$$

Adding up all the squared partial derivatives and taking the square root of these, while realizing the definition of r gives us:

$$\sqrt{\left(\frac{\partial n}{\partial x}\right)^2+\left(\frac{\partial n}{\partial y}\right)^2+\left(\frac{\partial n}{\partial z}\right)^2}=\sqrt{\frac{\eta_0^2\exp(\frac{-2(r-R)}{H})}{H^2(x^2+y^2+z^2)}(x^2+y^2+z^2)} \\ =\frac{\eta_0\exp(-\frac{r-R}{H})}{H}=\frac{n-1}{H} \tag{5.14}$$

This we can fill in what we found into equation 5.10. This gives us:

$$du_1=\frac{1-n}{nHr}\left(x-u_1\left(u_1x+v_2y+u_3z\right)\right)\left(\frac{dz}{u_3}+\frac{n-1}{n}\frac{\sqrt{D\frac{ndz}{u_3}}\xi H}{n-1}\right) \tag{5.15}$$

Crossing out the $n-1$ term gives us finally:

$$du_1=\frac{1-n}{nHr}\left(x-u_1\left(u_1x+u_2y+v_3z\right)\right)\left(\frac{dz}{u_3}+\frac{\sqrt{D\frac{ndz}{u_3}}\xi H}{n}\right) \tag{5.16}$$

And similarly we for du_2 we get:

$$du_2=\frac{1-n}{nHr}\left(y-u_2\left(u_1x+u_2y+u_3z\right)\right)\left(\frac{dz}{u_3}+\frac{\sqrt{D\frac{ndz}{u_3}}\xi H}{n}\right) \tag{5.17}$$

5.2. Ellipsoidal planet

For an ellipsoidal planet, the partial derivatives of n are different, as we have seen in section 4.2. This means that we also get different resulting equations of motion for the case with diffusion incorporated. Again, as a recap, below we show the partial derivatives (these are directly copies of equations 4.20 - 4.22), where for ease's sake we did not write out the dependency on x, y and z, but it still is implied.

$$\frac{\partial n}{\partial x}=-\frac{x}{aH}\frac{n-1}{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{a^2}}} \tag{5.18}$$

$$\frac{\partial n}{\partial y}=-\frac{ya}{b^2H}\frac{n-1}{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{a^2}}} \tag{5.19}$$

$$\frac{\partial n}{\partial z}=-\frac{z}{aH}\frac{n-1}{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{a^2}}} \tag{5.20}$$

We also need the square root of the sum of the squares of these partial derivatives.

$$\sqrt{\left(\frac{\partial n}{x}\right)^2+\left(\frac{\partial n}{y}\right)^2+\left(\frac{\partial n}{z}\right)^2}=\frac{n-1}{H\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{a^2}}}\sqrt{\frac{x^2}{a^2}+\frac{y^2a^2}{b^4}+\frac{z^2}{a^2}} \tag{5.21}$$

Filling all this into equation 5.10, we get the following equation for du_1 in the ellipsoidal case.

$$du_1 = \frac{1-n}{H\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{a^2}}}\left(\frac{x}{a}-u_1\left(\frac{u_1x}{a}+\frac{u_2ay}{b^2}+\frac{u_3z}{a}\right)\right)\left(\frac{dz}{u_3}+\frac{H\sqrt{D\frac{ndz}{u_3}}\xi\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{a^2}}}{n\sqrt{\frac{x^2}{a^2}+\frac{y^2a^2}{b^4}+\frac{z^2}{a^2}}}\right) \tag{5.22}$$

We take into account the assumption that $a \approx b$, which means that the two square root terms in the last term cancel out. This gives us our expression for du_1

$$du_1 = \frac{1-n}{u_3nH\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{a^2}}}\left(\frac{x}{a}-u_1\left(\frac{u_1x}{a}+\frac{u_2ay}{b^2}+\frac{u_3z}{a}\right)\right)\left(dz+\frac{H\sqrt{Dndzu_3}\xi}{n}\right) \tag{5.23}$$

Similarly, we find for du_2

$$du_2 = \frac{1-n}{u_3nH\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{a^2}}}\left(\frac{ya}{b^2}-u_2\left(\frac{u_1x}{a}+\frac{v_2ay}{b^2}+\frac{u_3z}{a}\right)\right)\left(dz+\frac{H\sqrt{Dndzu_3}\xi}{n}\right) \tag{5.24}$$

We see the change in direction of the instantaneous velocity is now a function of dz and of $\sqrt{dz}$.

5.3. Diffusion coefficient

Before the equations of motion found in the previous section can be implemented into the model, first the diffusion coefficient D needs to be determined. The definition used for this constant is given in formula (5.25) [16].

$$D = \lim_{dt\to 0}\frac{\left\langle (v(t+dt)-v(t))^2\right\rangle}{dt} \tag{5.25}$$

This results for the spherical system in the following diffusion constant:

$$D = \frac{9}{20}\frac{C_0}{H^2}L_0^{\frac{5}{3}} \tag{5.26}$$

Here C_0 is an approximation for the refractive index structure parameter C_n^2 at height 0, which is a parameter that describes the effect of turbulence on electromagnetic waves [17], where the value $C_0 = 9.0\times 10^{-17}\,\mathrm{m}^{-\frac{2}{3}}$ is used in the model. The constant L_0 is called the integral length scale. It is the size of the longest eddy current in the atmosphere. We will show the results for a few different values of L_0.

5.4. Implementation in the code

In the previous sections of the report we used *Leapfrog* as a numerical integrator, but *Leapfrog* uses acceleration at a certain time to compute the velocity. This was implemented before, when the equations of motion expressed the acceleration $\frac{d\hat{\mathbf{v}}}{dz}$. However, now there is no expression for $\frac{d\hat{\mathbf{v}}}{dz}$: in equations (5.23) and (5.24) there is also a dependency on $\sqrt{dz}$, so dz cannot simply be divided out of the equation. So there is no z-independent expression for acceleration. Therefore an adaptation of *Leapfrog* is required. This adaptation is shown in equation (5.27). Knowing the position x_i, velocity v_i, z-independent part of acceleration a_i and z-dependent part of acceleration b_i at a certain position z, this method gives the new position and new velocity at a later time $z + \Delta z$.

$$
\begin{aligned}
x_{i+1} &= x_i + v_{1,i}\Delta z + \frac{1}{2}a_{x,i}\Delta z^2 \\
y_{i+1} &= y_i + v_{2,i}\Delta z + \frac{1}{2}a_{x,i}\Delta z^2 \\
v_{1,i+1} &= v_{1,i} + \frac{1}{2}\left(a_{x,i} + a_{x,i+1}\right)\Delta z + \frac{1}{2}(b_{x,i} + b_{x,i+1})\sqrt{\Delta z} \\
v_{2,i+1} &= v_{2,i} + \frac{1}{2}\left(a_{y,i} + a_{y,i+1}\right)\Delta z + \frac{1}{2}(b_{x,i} + b_{x,i+1})\sqrt{\Delta z}
\end{aligned}
\tag{5.27}
$$

This is implemented in the code in the following way:

```
def LeapFrogSolve(dvdz, zspan, v0, n):  # adapted leapfrog integration method
  z0 = zspan[0]
  zstop = zspan[1]
  dz = ( zstop - z0 ) / n

  z = np.zeros( n + 1 )
  v = np.zeros( [ n + 1, 4 ] )

  for i in range ( 0, n + 1 ):
    if ( i == 0 ): #setting initial values
      z[0]    = z0
      v[0,0] = v0[0]
      v[0,1] = v0[1]
      v[0,2] = v0[2]
      v[0,3] = v0[3]
      anew    = dvdz( v[i,:], z[i] )[0]
      v3new = dvdz( v[i,:], z[i] )[1]
      nnew = dvdz ( v[i,:], z[i] )[2]
    else:   #updating all values using the adapted leapfrog
      z[i]    = z[i-1] + dz
      aold    = anew
      v3old   = v3new
      nold = nnew

      #defining b_i
      scalar = *(1/nold *sqrt(D*nold*v3old) * np.random.normal()*H)
      bold = [aold[2]*scalar,  aold[3]*scalar]
      #updating position
      v[i,0] = v[i-1,0] + dz * ( v[i-1,2] + 0.5 * dz * aold[2] )
      v[i,1] = v[i-1,1] + dz * ( v[i-1,3] + 0.5 * dz * aold[3] )

      anew    = dvdz ( v[i,:], z[i] )[0]
      v3new   = dvdz ( v[i,:], z[i] )[1]
      nnew = dvdz ( v[i,:], z[i] )[2]

      #defining b_i+1
      scalar = (1/nnew *sqrt(Diff*nnew*v3new)*np.random.normal()*H)
      bnew = [anew[2]*scalar, anew[3]*scalar]
```

```

    #updating velocities
    v[i,2] = v[i-1,2] + 0.5  * ( aold[2] + anew[2] )* dz \
            + 0.5 * (bold[0] + bnew[0]) *sqrt(dz)
    v[i,3] = v[i-1,3] + 0.5  * ( aold[3] + anew[3] )* dz \
            + 0.5 * (bold[1] + bnew[1]) *sqrt(dz)
  return v
```

Listing 4: The code used to propagate light rays using the adapted Leapfrog numerical method, to include diffusion.

The result of this code is a $(n+1) \times 4$ matrix `v`, where row `i` is an array of the shape $[x_i, y_i, u_{1_i}, u_{2_i}]$ at that z-location. This code is very similar to listing 1, where all the variables used are defined. One may note that here `dvdz` is now callable: this function has changed to also return u_3 and the refractive index at the given location, since these are needed to adapt the diffusion.

5.5. Results

We show in this section the results for implementing diffusion into the models for both a spherical and an ellipsoidal planet. In figure 22 the flash map of implementing diffusion for the spherical case is shown, and in figure 23 the flash maps of diffusion is shown for the ellipsoidal case.

In figure 22 we looked at 2 different values of L_0: $1 \times 10^8\,\mathrm{m}$ and $1 \times 10^9\,\mathrm{m}$. We see for L_0 the maximum amplification is about 5.8×10^4. This is higher than without diffusion. A possible explanation for this would be that fluctuations get compensated: say a ray that without diffusion would end up in the middle gets deflected slightly due to diffusion and no longer is in the center, but rays that would not end up in the middle get deflected just so that it does end up in the middle. For $L_0 = 1 \times 10^9\,\mathrm{m}$ this apparently does not hold up. Here the diffusion is so great that everything spreads away from the center, and in this thin band not enough light rays get deflected back to the center. This points us to conclude that for higher diffusion constants we need to look at a thicker ring of light rays: when the diffusion is greater, light rays from further out from the band might be deflected towards the center.

In figure 23 we looked at the same values for L_0 for ellipsoidal planets. On this scale, the effect of the diffusion is not obvious. One would need to zoom into one of the corner points of the diamond to see on a small scale how the diffusion influences the maximum amplification. However, even without doing that, we do see a small effect: the maximum amplification varies slightly.

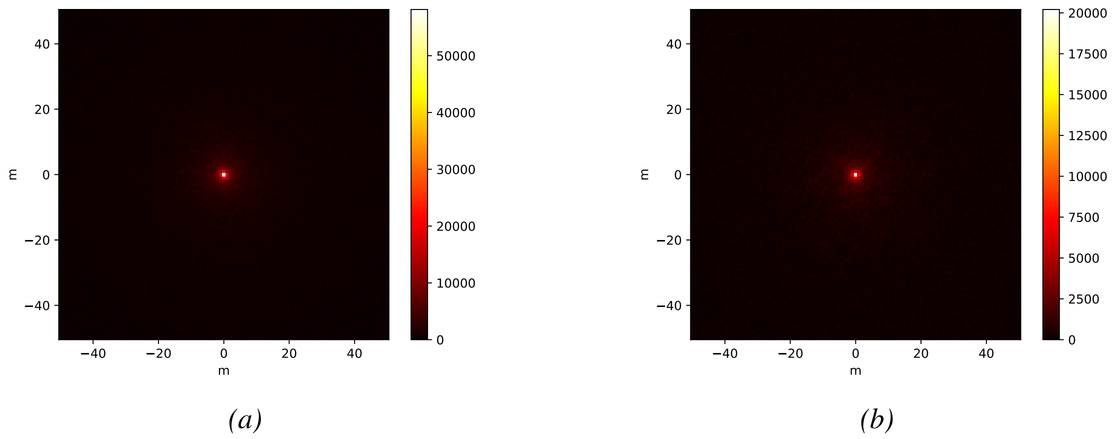


Figure 22: Flash maps a spherical planet with diffusion in the atmosphere implemented. Figure (a) *and* (b) *are the flash maps of a spherical planet with radius* $R = 6.371 \times 10^6\,\mathrm{m}$, *with* $L_0 = 1 \times 10^8\,\mathrm{m}$, *and* $1 \times 10^9\,\mathrm{m}$ *respectively.*

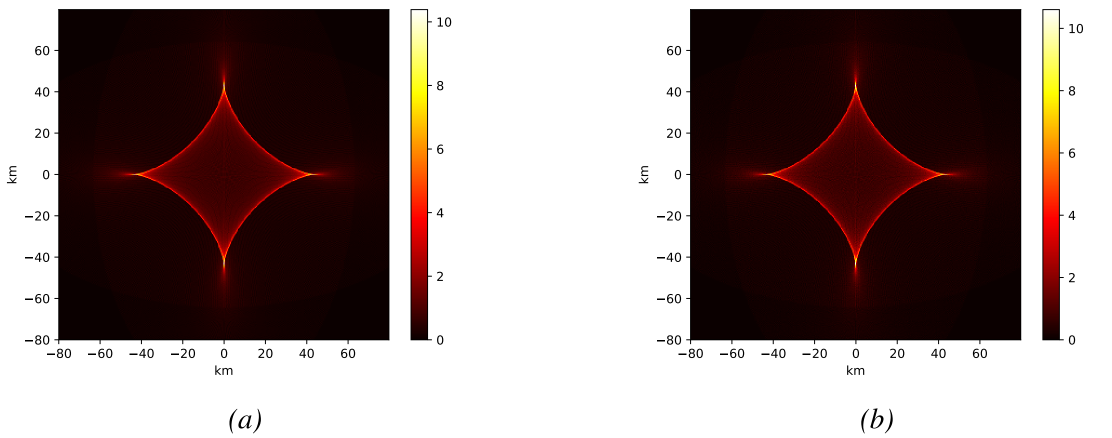


Figure 23: Figure (a) *and* (b) *are flash maps with diffusion incorporated of an ellipsoidal earth with semi-major axis* $a = 6.378 \times 10^6\,\mathrm{m}$ *and semi-minor axis* $b = 6.356 \times 10^6\,\mathrm{m}$. *Figure* (a) *has* $L_0 = 1 \times 10^5\,\mathrm{m}$ *and figure* (b) *has* $L_0 = 1 \times 10^7\,\mathrm{m}$.

6. Discussion

In the model for a spherical earth, without diffusion implemented, we found that the Atmoscope yields a maximum amplification of about 50,000: celestial objects would look 50,000 times brighter when using a detector placed 1.8×10^9 m away from Earth, than if they would be observed without the Atmoscope. This lines up with Kippings conclusions: he found an amplification of 55,000. We also found that the amplification for this system is inversely proportional to the pixelsize: $A \propto \frac{1}{D}$.

When implementing the oblateness of a planet into the model, the biggest difference found was the spreading of the focus: in the spherical case all the rays converge to a single point, yet for an ellipsoid caustics are formed. These caustics now have 4 main points where the amplification is the highest. The relation between amplification and pixel size is no longer $1/D$ but rather approximately $A \propto \frac{1}{D^{0.65}}$. For a pixel size of 100 m by 100 m, we see that a planet with Earths eccentricity has a maximum amplification of order 10^1, while a spherical planet has a maximum amplification of order 10^3. This means the loss of amplification due to the oblateness of planets is significant. We made a few approximations to obtain the expression for the height, and these might cause errors in the results for the ellipsoidal planets. However, since the shape for the caustics we calculated analytically, and these match perfectly with the caustics we found, we may assume these errors are small.

The final implementation to the model is adding diffusion to model fluctuations in air. For the spherical case we looked at four possible values for the diffusion constant. For the smallest of these, we saw very little difference with the model without diffusion. We expected to see a decrease in maximum amplification when diffusion was implemented, since we imagine that the rays do not get focused as neatly, but instead for one of the values for L_0 we found the opposite. This could be explained by realizing that for every ray that gets deflected out of the center, there might be rays that used to be further out from the center that now deflect inwards towards the center. For the other value of L_0 there was a decrease in amplification: instead of 5×10^4 , it now had a maximum amplification of about 2×10^4. This would be what we expected initially, but after the different result for lower L_0 we must also consider that now perhaps using the thin band is not an adequate way to observe the effect of the Atmoscope. When the deflections in the direction of the light rays are large enough, there will be light rays that we did not program that would get deflected enough to end up in the center. To test this out we recommend running more tests, and varying the band width to see whether with a thicker band the amplification returns to around 5×10^4.

For the ellipsoidal planet diffusion has also been implemented, however the effect of this is even for the largest chosen L_0 relatively small. Where without diffusion the maximum amplification was around 10.5 for a pixel-size of 450 m, with the two chosen values of L_0 the maximum amplification goes down to 10.38 for the smaller L_0, but then up to 10.60 for the larger L_0. A possible explanation for this small difference is the same concept of compensation as we posed for the spherical case. We recommend researching this effect further, and zooming in on the corner points of the diamond shapes, and see how those are effected by the diffusion.

All the results obtained were if the detector was placed at 1.9×10^9 m, which is outside of the Hill Radius of Earth. This could have been prevented by making the radius of the thin band smaller, which due to time has unfortunately not been done. However, when not accounting for physical effects inside the atmosphere, this does not truly affect the results. Light that enters the atmosphere higher, refracts less, but it still refracts.

7. Conclusion

7.1. Is the Atmoscope feasible?

The aim of this thesis was to further research whether the Atmoscope can be a useful telescope. This meant for this report specifically looking at the effect of the geometry of the planet used for the Atmoscope, and implementing diffusion to account for any physical effects inside the atmosphere such as turbulence. In chapter 2 of this report we developed the ray tracing model for gradient-index media. This means the path of light rays was determined using the way each ray interacts with the atmosphere via the refractive index. We found a set of ordinary differential equations that describe the path of a light ray, which we call the equations of motion. *Leapfrog* was used as the numerical integrator for these equations, and for the results we used a thin band model. Most code used in the report used the method described in this chapter, except for the code to describe diffusion due to turbulence, which is explained in chapter 5.

In chapter 3 we applied the established code to a spherical planet. The data found in this chapter was in agreement with previous results by Kipping. This was reason to assume the model used was a good option to continue with.

In chapter 4 we applied the model to an ellipsoidal planet. Here we had to describe the height above the surface of an ellipsoid in order to use the equations of motion, and we found an expression for the caustics that appear due to the ellipsoidal shape of the planet. We saw that the 'radius' of the spread of light rays is $2(a-b)$ where $a-b$ is the difference between the semi-major and semi-minor axes of the ellipsoid. The difference between ellipsoidal and spherical planets is also discussed in this chapter, where we concluded that the relation between maximum amplification and pixel size was $A \propto 1/D$ for spherical planets and $A \propto D^{-0.66}$ for ellipsoidal planets. We postulate that the theoretical exponent is $-\frac{2}{3}$.

In chapter 5 we looked at the effect of physical effects within the atmosphere such as turbulence, and incorporated this with a diffusion model. This model was then applied to both spherical planets and ellipsoidal planets. Unlike our expectation for the diffusion, the amplification did not decrease for all cases: in 1 case for the spherical planet and for both the cases of the ellipsoidal planet the amplification stayed roughly the same or even increased slightly. This result we might be able to explain by imagining that even when a light ray that lands in the center without diffusion gets deflected out of the center, other light rays might get deflected into the center. For the remaining case of the spherical planet where the maximum amplification did decrease significantly, we should investigate further that this is not only due to that the set of initial light rays might be in a band that is too thin: if the diffusion coefficient gets very big, light rays that would be just outside of the thin band could get deflected into the band, and increase the maximum amplification that way. From the results we obtained we cannot conclude that fluctuations in the refractive index have a significant effect on the maximum amplification.

Thus, only the oblateness of Earth is an effect that hinders the usefulness of the Atmoscope.

7.2. Future work

There are still many aspects of the Atmoscope that require more research before any true missions can start. In this report, light was treated as a purely geometric effect. We did not look at specific wavelengths and what their effect on the results are, which Kipping has done. Combining this with the research in this report might lead to valuable insights.

Effects of the atmosphere have been touched upon in this report, but there is room to investigate this further. One could make a map of the refractive index throughout the atmosphere, and find the fluctuation of the light rays by using this non-homogeneous refractive index. Furthermore, more research on the diffusion model used in this report is in order, especially the idea of using a thicker band in which the initial light rays are placed should be looked at.

In this report the main focus for all the results we model that come as close to Earth as possible. For future work it is interesting to look at different celestial objects and different atmospheres, where we recommend a planet or moon that is as close to a sphere as possible, since the effect of the oblateness of a planet is so significant.

Another interesting question is to research what happens when the propagation direction of the light rays is not along the z-axis, or what happens if the detector is not placed on the same axis as on which the star and the planet are aligned. This latter question has been discussed in Kippings paper, but only for a spherically symmetric Earth.

Lastly, we would recommend researching the Atmoscope not per se as a way to observe and research star light, but to use it to observe atmospheres of celestial objects: from the central flash of a planet or moon we can learn a lot about its atmosphere, and by understanding the Atmoscope, we can interpret the signal in central flash maps to extract information about turbulent motion in upper atmospheres of Solar System objects.

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