Master Thesis Applied Mathematics

Binomial formulas for Macdonald polynomials

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Abstract

Symmetric and nonsymmetric Macdonald polynomials associated to root systems are very general families of orthogonal polynomials in multiple variables. Their definition is quite complex, but in certain cases one can define so-called interpolation polynomials that have a surprisingly simple definition and are related to the Macdonald polynomials by a binomial formula. In this thesis we will discuss such formulas for two kinds of root systems: type A and type $(\mathrm{C}^\vee, \mathrm{C})$. For the latter case, there are still some open questions that remain unanswered.

Contents

Voorwoord

In september 2015 ben ik begonnen met de master Applied Mathematics aan de TU Delft. Dit na een dubbele bachelor wis- en natuurkunde aan de UvA te hebben afgerond in 2014. Altijd ben ik zoekende geweest naar mijn vervolgpad. Wat is de volgende stap? In eerste instantie heb ik na mijn bachelor gekozen voor een master Mathematical Physics aan de UvA, maar ik merkte dat ik hier heel weinig motivatie voor had en dat dit me veel stress opleverde. Dit koppelde ik aan het feit dat ik deze master zo abstract vond en dat de link met de 'echte wereld' voor mij heel ver te zoeken was. Daarom ging ik op zoek naar iets meer toegepasts. Zo kwam ik terecht op een open dag van de TU Delft bij een voorlichting van de master Applied Mathematics. In alle eerlijkheid was ik niet laaiend enthousiast, maar samen met het vooruitzicht dat ik in een andere omgeving kwam en voor het eerst op mezelf ging wonen (en niet meer bij mijn ouders), vond ik het toch aantrekkelijk.

Van mijn voornemen om wiskunde te doen die meer 'toegepast' is, is weinig terechtgekomen. Het dichtste bij kwam ik bij de vakken over optimalizatie en het vak Computational Fluid Dynamics. Dit laatste vak heeft weinig te maken met de rest van mijn studieprogramma, maar juist daarom was het interessant om te volgen. Los van deze meer toegepaste vakken, merkte ik dat ik toch weer werd aangetrokken tot abstractere wiskunde, en in het bijzonder tot analyse, waar ik nog steeds een soort haat-liefdeverhouding mee heb. Sommige vakken kan ik heel inspirerend vinden, maar zodra ik echt zelf iets moet doen, zoals een scriptie schrijven, dan heb ik het idee dat ik het alleen moet doen. Dit was overigens ook al het geval bij mijn profielwerkstuk op de middelbare school en bij mijn bachelorscriptie aan de UvA. In deze momenten komt aan het licht dat (abstracte) wiskunde misschien niet mijn toekomst is, omdat ik merk dat er weinig intrinsieke motivatie is, terwijl ik andere dingen in het leven heb ontdekt waar ik die motivatie wel heel sterk voor heb. Daarom heb ik serieus getwijfeld over het afmaken van mijn studie. Iedereen raadde me af om te stoppen. Dit bracht veel spanningen met zich mee, want wat als ik wel zou stoppen? Ik had het gevoel dat ik dan niet meer geaccepteerd zou worden en minder kans meer had op een mooie toekomst.

In zekere zin klopt dit natuurlijk en daarom ben ik blij dat ik het toch heb afgemaakt.

Wat ik na mijn studie ga doen, weet ik nog niet. Er zijn veel ideen, veel mogelijkheden, om de wereld een mooiere plek te maken. 'Verbeter de wereld, begin bij jezelf', dat is een van mijn favoriete uitspraken. Ik ben dankbaar voor de vele lessen die ik heb geleerd tijdens mijn tijd in Delft, hoe moeilijk ze soms ook waren.

Introduction

In this thesis, we will give an overview of the theory of binomial formulas related to Macdonald polynomials. Classically, Macdonald polynomials have been defined by orthogonality conditions, but we will define them as eigenfunctions of certain difference operators[1]. These operators are the so-called Y-operators of the Cherednik representation of the (affine) Hecke algebra related to a root system. This procedure, for general root systems, is explained in chapter 2. We will treat the polynomials related to root systems of type A and $(\mathrm{C}^\vee, \mathrm{C})$ explicitly. The former are studied in chapter 1, and the latter (called Koornwinder polynomials) in chapter 3.

The number of parameters in the polynomials is equal to the number of orbits in the root system under the action of the Weyl group. It is the highest for the root system of type $(\mathrm{C}^\vee, \mathrm{C})$ (namely 5), so in a sense this case is the most complex. Taking limits of parameters or specializing to the one-dimensional case, the polynomials become other well-known families of orthogonal polynomials. For example, the one-dimensional Koornwinder polynomials are the well-known Askey-Wilson polynomials.

It turns out to be possible to define certain polynomials that are related to the Macdonald polynomials and are defined by certain interpolation properties. The *main results* of this thesis will be binomial type formulas relating the Macdonald polynomials (whose definition is complex) to the interpolation Macdonald polynomials (whose definition is very simple).

We first discuss the article of Knop [5], in which the non-symmetric and symmetric interpolation polynomials are defined for type A. In this article, the interpolation polynomials are defined using the interpolation points and it is shown that they are eigenfunctions of certain non-symmetric analogues of Cherednik operators. After that, we define a Fourier pairing and check that the non-symmetric interpolation polynomials are orthogonal with respect to this pairing. From this we derive a binomial formula. In the second chapter, we discuss the general theory of root systems, Weyl groups, affine

[1] It can be shown that these two definitions result in the same polynomials, see for example [8]

Hecke algebras and Macdonald polynomials, following the approach in [8]. In the third and last chapter, we discuss the same theory (symmetric and non-symmetric) for type $(C^\vee, C)$. In the symmetric case, we prove a binomial type formula using the article [11]. We end with a discussion of the non-symmetric case. A binomial formula for this case is not known in the literature. We derive such a formula for the one-dimensional case, and discuss shortly the case of general dimension, which was treated recently in [2].

Applications

Macdonald polynomials and their specializations appear in a lot of different areas. To name one example, in [3] they appear in the study of the representation theory of quantum groups. More recently, they appear in mathematical physics in the study of gauge theories (see [4]). And this is just the tip of the iceberg when it comes to possible reasons for studying this abstract theory.

Chapter 1

Type A Macdonald polynomials

In this chapter, we will discuss most of the results of the first four sections of [5], and our sections (except for the last one) correspond to the sections in this article. The main result of [5] is that the nonsymmetric Macdonald interpolation polynomials associated to the root system of type A (that are defined by certain interpolation properties) form a simultaneous eigenbasis of certain inhomogeneous versions of the Cherednik operators.

First, for the sake of completeness, we introduce the notation that is used in the article. We fix n and write $\Lambda = (\mathbb{N} \cup \{0\})^n$ and $\Lambda^+ \subset \Lambda$ the subset of partitions (a *partition* is an element $\lambda \in \Lambda$ with $\lambda_i \geq \lambda_j$ when $i < j$). Let $\lambda \in \Lambda$. Then we define the weight of λ to be the sum of its elements and denote it by $|\lambda|$. Moreover, we define the length of λ to be the number of nonzero elements and denote it by $l(\lambda)$.

We fix a field k of characteristic zero and denote the ring of polynomials over k in n variables by $\mathcal{P}$ and the ring of Laurent polynomials over k in n variables by $\mathcal{P}'$. We will need two parameters, so we will fix $q, t \in k$. The group $W = S_n$ has a natural action on Λ (by permuting the elements) and on $\mathcal{P}$ and $\mathcal{P}'$ (by permuting the variables). For $\lambda \in \Lambda$, we denote by $w_\lambda \in W$ the shortest permutation σ that satisfies $\sigma(\lambda^+) = \lambda$, where λ^+ is the unique partition in the W-orbit of λ. Furthermore, we define $\rho = (1, t^{-1}, t^{-2}, \ldots, t^{-n+1})$.

For $\lambda \in \Lambda$, we define $\bar{\lambda} = w_\lambda(q^{\lambda^+}\rho)$. More explicitly, we have

$$\bar{\lambda}_i = q^{\lambda_i} t^{-k_i},$$

where k_i is given by

$$k_i = k_i(\lambda) = \#\{j = 1, \ldots, i-1 \mid \lambda_j \geq \lambda_i\} + \#\{j = i+1, \ldots, n \mid \lambda_j < \lambda_i\}.$$

These will be our interpolation points. The reason for this choice is that these are the eigenvalues of the Cherednik operators (we will prove this later).

Example 1.1. Let $n = 5$ and $\lambda = (3, 0, 2, 5, 2)$. Then λ is not a partition. We have $|\lambda| = 12$, $l(\lambda) = 4$ and $w_\lambda = (1452) \in S_n$. Finally, we have $\bar{\lambda} = (q^3t^{-1}, t^{-4}, q^2t^{-2}, q^5, q^2t^{-3})$.

1.1 The vanishing condition

With all the notations in place, we can prove the first lemma of the paper.

Lemma 1.2 (2.1). *Let $\lambda \in \Lambda$ and define $\lambda^* = (\lambda_n - 1, \lambda_1, \ldots, \lambda_{n-1})$. Then $\overline{\lambda^*} = (\overline{\lambda}_n/q, \overline{\lambda}_1, \ldots \overline{\lambda}_{n-1})$.*

Proof. In the equality that we want to prove, the powers of q clearly coincide. To see that the powers of t also coincide, define $\mu = (\lambda_1, \ldots, \lambda_{n-1})$. Then if $\lambda_i \geq \lambda_n$ and $i \neq n$, we have $k_i(\lambda) = k_i(\mu) = k_{i+1}(\lambda^*)$ and if $\lambda_i < \lambda_n$, we have $k_i(\lambda) = k_i(\mu) + 1 = k_{i+1}(\lambda^*)$. Finally $k_1(\lambda^*) = k_n(\lambda)$. □

The following theorem tells us that we can construct interpolation polynomials for arbitrary values at the interpolation points.

Theorem 1.3 (2.2). *For $d \in \mathbb{N}$ let $S(n, d)$ be the set of all $\overline{\lambda}$ where $\lambda \in \Lambda$ and $|\lambda| \leq d$. Let $\overline{f} : S(n, d) \to \mathbb{F}$ be a mapping. Then there exists a unique polynomial $f \in \mathcal{P}$ of degree at most d such that $f(z) = \overline{f}(z)$ for all $z \in S(n, d)$.*

Proof. See [5]. In this proof it is used that the cardinality of $S(n, d)$ is equal to the cardinality of the space of polynomials of degree at most d. We remark that in order for this to be true, we need conditions on q and t, otherwise points in $S(n, d)$ can coincide. In [5], it is assumed that $q^at^b \neq 1$ for all $a, b \in \mathbb{N}$. However, only this condition is not sufficient, as can be seen from the example $n = 2$, $d = 4$, $q = 1$, $t = 0.5$, $\lambda = (3, 1)$ and $\mu = (2, 1)$. It turns out that in order to ensure the stated property, we need the additional condition that q is not a root of unity (it is not immediately clear that these conditions together are sufficient, but we leave the proof to the reader, since the conditions will not play a role in the rest of the thesis). □

The above theorem also has a symmetric version. Since the proof in [5] is quite condensed, we will give a more elaborate proof below. In the proof, we will denote by m_λ the mononial symmetric polynomial associated to λ, defined by the orbit sum under W of $\prod_i x^{\lambda_i}$. It is immediate from the definition that the polynomials m_λ form a basis for the space of symmetric polynomials.

Theorem 1.4 (2.3). *For $d \in \mathbb{N}$ let $S^+(n, d)$ be the set of all $\overline{\lambda} = q^\lambda \rho$ where $\lambda \in \Lambda^+$ and $|\lambda| \leq d$. Let $\overline{f} : S^+(n, d) \to \mathbb{F}$ be a mapping. Then there exists a unique symmetric polynomial $f \in \mathcal{P}$ of degree at most d such that $f(z) = \overline{f}(z)$ for all $z \in S^+(n, d)$.*

Proof. If we write $f = \sum_{|\lambda|\le d} c_\lambda m_\lambda$, then we see that $f(z) = \overline{f}(z)$ for $z \in S^+(n,d)$ is a square system of equations for the coefficients c_λ. So existence for every $\overline{f}$ will imply uniqueness. It remains to prove existence. Because $m_\lambda(z_1 - t^{-n+1}, \dots, z_{n-1} - t^{-n+1})$ form a basis for the symmetric polynomials on $n-1$ variables (where λ runs over the partitions of $n-1$), we can define a map from symmetric polynomials in $n-1$ variables to symmetric polynomials in n variables that sends $m_\lambda(z_1 - t^{-n+1}, \dots, z_{n-1} - t^{-n+1})$ to $m_{\lambda,0}(z_1 - t^{-n+1}, \dots, z_n - t^{-n+1})$. We denote the image of a function g under this map by g^+. The identity $g^+(z_1, \dots, z_{n-1}, t^{-n+1}) = g(z_1, \dots, z_{n-1})$ holds, because it holds for all elements in the basis. Let f be an arbitrary symmetric polynomial in n variables. Then by defining $g(z_1, \dots, z_{n-1}) = f(z_1, \dots, z_{n-1}, t^{-n+1})$ and

$$h(z_1/q, \dots, z_n/q) = \frac{f(z_1, \dots, z_n) - g^+(z_1, \dots, z_n)}{\prod_{i=1}^n (z_i - t^{-n+1})}, \tag{1.1}$$

we can write f in the form

$$f(z_1, \dots, z_n) = g^+(z_1, \dots, z_n) + \prod_{i=1}^n (z_i - t^{-n+1}) h(z_1/q, \dots, z_n/q). \tag{1.2}$$

So finding a function f is equivalent to finding both g and h. Now the theorem is proved by induction on $n+d$. The equation $f(\overline{\lambda}) = \overline{f}(\overline{\lambda})$ for $\overline{\lambda} \in S^+(n,d)$ with $\lambda_n = 0$ uniquely determines g, by the induction hypothesis. For the remaining $\overline{\lambda} \in S^+(n,d)$, we get equation (1.1) with $\overline{f}$ instead of f for the function h. Denoting the right hand side of this equation by $\overline{h}(z)$, we see that it is equivalent to

$$h(z_1, \dots, z_n) = \overline{h}(qz_1, \dots, qz_n)$$

for $z \in S^+(n, d-n)$. By the induction hypothesis, we can find a symmetric function h of degree at most $d-n$ such that these equations are satisfied. Hence the function f defined by (1.2) satisfies the desired interpolation properties. □

Now we specify the value at the interpolation points to be zero, except at one point. This gives us the Macdonald interpolation polynomials.

Theorem 1.5 (2.4). *(i) For every $\lambda \in \Lambda$ there is a unique polynomial E_λ with $E_\lambda(\overline{\mu}) = 0$ for all $\mu \in \Lambda$ with $|\mu| \le |\lambda|$, $\mu \ne \lambda$ and which has an expansion $E_\lambda = \sum_\mu e_{\lambda\mu} z^\mu$ with $e_{\lambda\lambda} = 1$.*

(ii) For every $\lambda \in \Lambda^+$ there is a unique symmetric polynomial P_λ with $P_\lambda(\overline{\mu}) = 0$ for all $\mu \in \Lambda^+$ with $|\mu| \le |\lambda|$, $\mu \ne \lambda$ and which has an expansion $P_\lambda = \sum_\mu p_{\lambda\mu} m_\mu$ with $p_{\lambda\lambda} = 1$.

Proof. The proof of (i) is given in [5]. Here we prove (ii). Theorem 1.4 implies that there exists a symmetric polynomial P_λ that satisfies the vanishing conditions with $P_\lambda(\overline{\lambda}) \neq 0$. We only need to show that it contains m_λ with non-zero coefficient, or equivalently, that it contains z^λ with non-zero coefficient. As in the proof of theorem 1.4, we write

$$P_\lambda(z_1, \ldots, z_n) = g^+(z_1, \ldots, z_n) + \prod_{i=1}^{n}(z_i - t^{-n+1})h(z_1/q, \ldots, z_n/q).$$

Again, we prove this by induction on $n + d$. If $\lambda_n = 0$, then g is a multiple of $P_{\lambda'}$, where $\lambda' = (\lambda_1, \ldots, \lambda_{n-1})$. By the induction hypothesis, g contains m_λ, hence it contains z^λ, hence g^+ contains z^λ, hence P_λ contains m_λ. If $\lambda_n \neq 0$, then $g = 0$ and h is a multiple of $P_{\lambda^\star}$, where $\lambda^\star = (\lambda_1 - 1, \ldots, \lambda_n - 1)$. By the induction hypothesis, h contains $z^{\lambda^\star}$, hence P_λ contains z^λ, hence also m_λ. □

Now we define the operators Δ and Φ by

$$\Delta f(z_1, \ldots, z_n) = f(z_n/q, z_1, \ldots, z_{n-1})$$

and $\Phi = (z_n - t^{-n+1})\Delta$. The following is the symmetric analogue of corollary 2.5 in [5].

Corollary 1.6 (2.5b). *For $\lambda \in \Lambda^+$ with $\lambda_n \neq 0$, let $\lambda^\star = (\lambda_1 - 1, \ldots, \lambda_n - 1)$. Then $P_\lambda = (\prod_{i=1}^n q^{\lambda_i - 1})\Phi^n(P_{\lambda^\star}) = q^{|\lambda|-n} \cdot \Phi^n(P_{\lambda^\star})$.*

Proof. From the above proof, it follows that this equation holds up to a constant. The normalization is correct, because the coefficient of z^λ in $\Phi^n(z^{\lambda^\star})$ is $q^{-(|\lambda|-n)}$. □

For specific values of t and q the Macdonald interpolation polynomials specialize to other known orthogonal polynomials. In [5], two examples are given without proof. Here we will provide the proofs. We use the same notation as in [5]. That is, we denote by $[z;k]_q$ the q-factorial polynomial $(z-1)(z-q)\ldots(z-q^{k-1})$. Also, we define the q-factorial Schur function by $\mathfrak{s}_\lambda(z;q) := a^{-1}\det[z_i; \lambda_j + n - j]_q$, where $a = \prod_{i<j}(z_i - z_j)$ is the Vandermonde determinant.

Proposition 1.7 (2.7). *Let $t = 1$. Then $E_\lambda(z;q,1) = [z_1;\lambda_1]_q \ldots [z_n;\lambda_n]_q$ and $P_\lambda(z;q,1)$ is the symmetrization of it.*

Proof. Let $\mu \in \Lambda$, $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$. Then there exists a value of i such that $\mu_i < \lambda_i$. It follows that $[\overline{\mu}_i; \lambda_i]_q = 0$, so the vanishing conditions are satisfied. Moreover, the coefficient of z^λ is clearly 1. The symmetrization of this polynomial gives a symmetric polynomial of the right degree, that satisfies the right conditions. □

Proposition 1.8 (2.8)**.** *Let $t = q$. Then $P_\lambda(z; q, q) = q^{-(n-1)|\lambda|}\mathfrak{s}_\lambda(q^{n-1}z; q)$.*

Proof. First we note that the function is indeed a symmetric polynomial and that it is of the right degree. Using the well known formula for expansion of a determinant, we get

$$\mathfrak{s}_\lambda(q^{n-1}\overline{\mu}; q) = \sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=1}^{n} [q^{\mu_i+n-i}; \lambda_{\sigma(i)} + n - \sigma(i)]_q.$$

If $\mu \neq \lambda$, then for every $\sigma \in S_n$, there exists i such that $\mu_i - i < \lambda_{\sigma(i)} - \sigma(i)$. For this i, the q-factorial polynomial vanishes. It follows that the given polynomial satisfies the right vanishing conditions. Now we check the normalisation. We use induction on $n + d$. First suppose that $\lambda_n \neq 0$. Using corollary 2.5b, we get

$$\begin{aligned} P_\lambda(z; q, q) &= q^{|\lambda|-n} q^{-(n-1)(|\lambda|-n)} \Phi^n(\mathfrak{s}_{\lambda^\star}(q^{n-1}z; q)) \\ &= q^{|\lambda|-n} \left(\prod_{i=1}^{n}(z_i - t^{-n+1})\right) q^{-(n-1)(|\lambda|-n)} \mathfrak{s}_{\lambda^\star}(q^{n-1}(z/q); q). \end{aligned}$$

By the induction hypothesis, the coefficient of $z^{\lambda^\star}$ in the last part of this expression is $q^{-|\lambda^\star|} = q^{-(|\lambda|-n)}$ (because the polynomial is evaluated in z/q instead of z). It follows that the coefficient of z^λ in P_λ is indeed equal to 1. The case $\lambda_n = 0$ is left, and in this case the coefficient of z^λ is equal to the coefficient of z^λ in $P_{\lambda'}$, where $\lambda' = (\lambda_1, \ldots, \lambda_{n-1})$ (this follows from the proof of theorem 2.4). By the induction hypothesis, this coefficient is equal to one. Hence the normalization is correct. □

1.2 Hecke operators

A main result in the theory of Macdonald polynomials is that these polynomials can be written as simultaneous eigenfunctions of a certain family of commuting difference operators (the so-called Y-operators or Cherednik operators). These operators are defined using the basic representation of the affine Hecke algebra. Here we will describe the affine Hecke algebra for type A.

Let $V = \mathbb{R}^n$ and $\hat{V} = V \oplus \mathbb{R}\delta$. The affine root system S of type A_{n-1} is the subset of $\hat{V}$ given by

$$S = \{\epsilon_i - \epsilon_j + m\delta \mid i \neq j,\ m \in \mathbb{Z}\}.$$

The associated Weyl group W_0 has a cycle as its Coxeter graph. Writing $s_n \coloneqq s_0$, this means that the Braid relations are

$$\begin{cases} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & i = 0, \dots, n-1 \\ s_i s_j = s_j s_i & \text{for the remaining pairs } (i,j). \end{cases} \tag{1.3}$$

Contrary to the general case (as discussed in [8] and chapter 2), we define a slightly different extended Weyl group by

$$W = W_0 \ltimes t(L), \tag{1.4}$$

where L is the $\mathbb{Z}$-lattice spanned by the basis vectors $\epsilon_1, \dots, \epsilon_n$ and $t(L)$ is the corresponding group of translations. From this extended affine Weyl group, we can define an affine Hecke algebra in the same way as in the general case that is explained in chapter 2. We will call this algebra the affine Hecke algebra for GL_n, like in [13].

Proposition 1.9. *The affine Hecke algebra for GL_n, which we will denote by $\mathcal{H}_n$, is the k-algebra with generators $T_1, \dots, T_{n-1}$ and $Y_1^{\pm 1}, \dots, Y_n^{\pm 1}$ and relations*

- $(T_i - \tau)(T_i + \tau^{-1}) = 0$ *for* $i = 1, \dots, n-1$
- $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ *for* $i = 1, \dots, n-2$ *and* $T_i T_j = T_j T_i$ *if* $|i-j| > 1$
- $Y_{i+1} = T_i Y_i T_i$ *for* $i = 1, \dots, n-1$ *and* $T_i Y_j = Y_j T_i$ *if* $j \neq i, i+1$
- $Y_i Y_j = Y_j Y_i$ *and* $Y_i Y_i^{-1} = 1$ *for* $i, j = 1, \dots, n$.

These relations are called, respectively, the quadratic relations, the braid relations, the action relations and the Laurent relations. This presentation of the algebra is derived from the decomposition (1.4). A different presentation of the same algebra is derived from the decomposition

$$W = W_S \ltimes \Omega$$

of the Weyl group and is called the Coxeter presentation of the algebra. We give it in the following proposition.

Proposition 1.10. *The algebra $\mathcal{H}_n$ is isomorphic to the algebra with generators $T_1, \dots, T_{n-1}$ and $\pi^{\pm 1}$ and relations*

- *The quadratic and braid relations involving the elements T_i.*
- $\pi T_i = T_{i+1} \pi$ *for* $i = 1, \dots, n-2$.

- *π^n is central in $\mathcal{H}_n$, i.e. it commutes with T_i for $i = 1, \dots, n-1$.*

The relation between the two presentations is given by

$$\pi = T_1^{-1} \cdots T_{i-1}^{-1} Y_i^{-1} T_i \cdots T_{n-1}, \tag{1.5}$$

which is independent of i because of the action relations. To prove proposition 1.10, one first checks that the relations are satisfied by the algebra given in proposition 1.9. Then one defines Y_i in the algebra of proposition 1.10 by using (1.5) and checks that the action relations and Laurent relations hold.

Now we define a representation of $\mathcal{H}_n$ on the algebra $k[L] \cong k[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ in the same way as the Cherednik representation in the general case by

$$\begin{cases} T_i \mapsto \tau s_i - (\tau - \tau^{-1}) z_{i+1} N_i & i = 1, \dots, n-1, \\ \pi(z_1^{\lambda_1} \dots z_n^{\lambda_n}) = q^{-\lambda_n} z_1^{\lambda_n} z_2^{\lambda_1} \dots z_n^{\lambda_{n-1}}, & \end{cases}$$

where $N_i = (z_i - z_{i+1})^{-1}(1 - s_i)$, as in [5] (the relations of proposition 1.10 can be checked directly). From (1.5), we see that the inverses of the Y-operators can be written in terms of the generators of the affine Hecke algebra given in proposition 1.10 as

$$Y_i^{-1} = (T_{i-1} \cdots T_1)\pi(T_{n-1}^{-1} \cdots T_i^{-1}).$$

In what follows, we will adhere to Knop's notation, i.e. we will write $\overline{H}_i := \tau^{-1} T_i$, $H_i := \tau^{-1} T_i^{-1}$, $\Delta = \pi^{-1}$ and $\xi_i = \tau^{n-1} Y_i^{-1}$ (here $\tau = t^{-1/2}$). The main result in Knop is that a small change in this operator results in an inhomogenous operator that has the interpolation polynomials as its eigenfunctions. This new operator is given by

$$\Xi_i = z_i^{-1} + z_i^{-1} H_i \cdots H_{n-1} \Phi H_1 \cdots H_{i-1}.$$

After proving that these operators act on $\mathcal{P}$, Knop states the main result of his paper [5]. We will not replicate the proof here.

Theorem 1.11 (3.6)**.** *The interpolation polynomials E_λ form a simultaneous eigenbasis for the operators $\Xi_1, \dots, \Xi_n$. More specifically, we have $\Xi_i E_\lambda = \bar{\lambda}_i^{-1} E_\lambda$.*

Corollary 1.12 (3.7)**.** *The operators $\Xi_1, \dots, \Xi_n$ commute pairwise.*

Corollary 1.13 (3.8)**.** *Let $p \in \mathcal{P}^W$. Then $\Xi_p := p(\Xi_1, \dots, \Xi_n)$ commutes with all H_i. Moreover, $\Xi_p(P_\lambda) = p(\bar{\lambda}^{-1}) P_\lambda$.*

Proof. We will prove the claim that Ξ_p acts on $\mathcal{P}_\lambda$ as scalar multiplication by $p(\bar{\lambda}^{-1})$. Let $w \in S_n$ and choose $w' \in S_n$ such that $\overline{w\lambda} = w'\bar{\lambda}$. Then

$$\Xi_p E_{w\lambda} = p(\overline{w\lambda}^{-1}) E_{w\lambda} = p((w'\bar{\lambda})^{-1}) E_{w\lambda} = p(w'(\bar{\lambda}^{-1})) E_{w\lambda} = p(\bar{\lambda}^{-1}) E_{w\lambda}.$$

The claim follows, since the $E_{w\lambda}$ form a basis for $\mathcal{P}_\lambda$. The claim that all H_i commute with Ξ_p follows from corollary 3.2 in [5]. □

Now, as an exercise, we will show that the Ξ_i's have a simultaneous eigenbasis without using the existence of the interpolation polynomials. In fact this will give a different proof of the existence of these polynomials. It will not add anything to the results of Knop, but it is a different way of obtaining the same results. For the proof, we will need two properties of the operators: that they are triangular, and that they commute pairwise. We begin with proving the triangularity.

Definition 1.14. Let $\lambda, \mu \in \Lambda^+$. Then we write $\lambda \leq \mu$ if

$$\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$$

for all $i = 1, \ldots, n$. This is called the dominance ordering on partitions.

We can extend this partial ordering to Λ as follows. For $\lambda \in \Lambda$, we write λ^+ for the unique partition (element of Λ^+) in the orbit $W\lambda$ and we write w_λ for the unique shortest element in W such that $\lambda = w_\lambda \lambda^+$.

Definition 1.15. Let $\lambda, \mu \in \Lambda$. Then we write $\lambda \leq \mu$ if either $\lambda^+ \leq \mu^+$ or $\lambda^+ = \mu^+$ and $w_\mu \leq w_\lambda$.

Now we can prove lemma 3.10 in [5].

Lemma 1.16. *The operators Ξ_i are triangular with diagonal elements $\bar{\lambda}_i^{-1}$.*

Proof. We can write $\Xi_i = \xi_i^{-1} + R_i$, where

$$R_i = z_i^{-1} + t^{-n+1} \overline{H}_i \ldots \overline{H}_{n-1} z_n^{-1} \Delta H_1 \ldots H_{i-1}.$$

It is known that the operators ξ_i^{-1} are triangular with the given diagonal elements (these are the Cherednik operators that we will discuss in chapter 2). So it remains to show that R_i is strictly upper triangular. Since R_i decreases the degree, definition 1.15 implies that we have to show that $\mu^+ < \lambda^+$ for every monomial z^μ occurring in $R_i(z^\lambda)$. This follows from the fact that H_i and $\overline{H}_i$ are triangular. This can be easily checked by computing $T_i(z^\lambda)$, using the explicit formula for the operator T_i. □

Lemma 1.17. *The operators Ξ_i commute pairwise. In other words, $\Xi_i \Xi_j = \Xi_j \Xi_i$ for all i, j.*

Proof. Without loss of generality, let's assume that $j > i$. Using that $H_i z_i^{-1} = z_{i+1}^{-1} \overline{H}_i$, it is an easy calculation to show that

$$R_{i,j} \coloneqq [\Xi_i, z_j^{-1}] = -(t-1) z_i^{-1} z_j^{-1} H_i \cdots \hat{H}_{j-1} \cdots H_{n-1} \Phi H_1 \cdots H_{i-1},$$

where the hat means that this operator is omitted. It follows that

$$\begin{aligned} [\Xi_i, \Xi_j] &= R_{i,j} + (t-1) z_j^{-1} H_j \cdots H_{n-1} \Phi H_1 \cdots H_{i-1} \overline{H}_{i+1} \cdots \overline{H}_{j-1} \Xi_j \\ &\quad + R_{i,j} H_j \cdots H_{n-1} \Phi H_1 \cdots H_{j-1} \\ &= (R_{i,j} z_j + (t-1) z_j^{-1} H_j \cdots H_{n-1} \Phi H_1 \cdots H_{i-1} \overline{H}_{i+1} \cdots \overline{H}_{j-1}) \Xi_j. \end{aligned}$$

To prove that this commutator equals zero, it remains to be shown that

$$\begin{aligned} &z_i^{-1} z_j^{-1} H_i \cdots \hat{H}_{j-1} \cdots H_{n-1} \Phi H_1 \cdots H_{i-1} z_j = \\ &z_j^{-1} H_j \cdots H_{n-1} \Phi H_1 \cdots H_{i-1} \overline{H}_{i+1} \cdots \overline{H}_{j-1}, \end{aligned}$$

where we used the explicit expression of $R_{i,j}$. Moving the z_j on the left hand side of the equality to the left side of the expression, and using the relations $\Phi z_j = z_{j-1} \Phi$ and $H_k z_{k+1} = z_k \overline{H}_k$, we see that this equality is equivalent to

$$\begin{aligned} &\overline{H}_i \cdots \overline{H}_{j-2} (H_j \cdots H_{n-1} \Phi H_1 \cdots H_{i-1}) = \\ &(H_j \cdots H_{n-1} \Phi H_1 \cdots H_{i-1}) \overline{H}_{i+1} \cdots \overline{H}_{j-1}. \end{aligned}$$

For $j = i+1$, this is immediately clear (in this case, we don't have any operators of the form $\overline{H}_k$ and we are left with the operators between parantheses, which are the same). If $j > i+1$, then Φ is the only operator that the operators $\overline{H}_k$ $(k = i, \dots, j-2)$ have a non-trivial commutation relation with, namely $\overline{H}_k \Phi = \Phi \overline{H}_{k+1}$. This implies the equality and it follows that $[\Xi_i, \Xi_j] = 0$. □

Now we are ready to prove the following theorem.

Theorem 1.18. *There exists a simultaneous eigenbasis $\{G_\lambda\}_{|\lambda| \leq d}$ of the operators Ξ_i, considered as operators acting on the space of polynomials of degree at most d.*

Proof. Because $\lambda \neq \mu$ implies that $\overline{\lambda} \neq \overline{\mu}$, we can find a polynomial f such that $f(\overline{\lambda}^{-1})$ is different for each $\lambda \in S(n,d)$. Because the operators commute, the operator $f(\Xi_1, \dots, \Xi_n)$ is well defined. Using the triangularity property, we see that

$$f(\Xi_1, \dots, \Xi_n) z^\lambda = f(\overline{\lambda}^{-1}) z^\lambda + \text{l.o.t.}$$

Hence this operator is triangular with different numbers on the diagonal. This implies that it is diagonalizable. In other words, there exist polynomials G_λ ($|\lambda| \leq d$) with leading term z^λ such that

$$f(\Xi_1, \dots, \Xi_n)G_\lambda = f(\bar{\lambda}^{-1})G_\lambda.$$

Applying the operator Ξ_i to both sides and using again that the operators commute, we see that $\Xi_i G_\lambda$ is also an eigenfunction of the operator $f(\Xi_1, \dots, \Xi_n)$ with the same eigenvalue. Hence it must be a multiple of G_λ. Now lemma 1.16 implies that $\Xi_i G_\lambda = \bar{\lambda}_i^{-1} G_\lambda$. □

The existence of the interpolation polynomials E_λ was the content of the first part of theorem 1.5. Here we give an alternative proof of this theorem using the polynomials G_λ (these are the same polynomials, but with a different definition).

Corollary 1.19. *There exist polynomials E_λ with the interpolation properties described in theorem 1.5.*

Proof. We will give an outline of the proof and leave the details to the reader. Let $\lambda \in \Lambda$ and $\mu \in \Lambda$ with $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$. We will prove that $G_\lambda(\bar{\mu}) = 0$ by induction on $|\mu|$ using that $\Xi_i G_\lambda = \bar{\lambda}_i^{-1} G_\lambda$. We use the same decomposition of Ξ_i as in theorem 1.11, namely $\Xi_i = z_i^{-1} + X_i$. Now we need two facts to finish the proof. Firstly, we need that $X_i G_\lambda(\bar{\mu})$ is a linear combination of $E_\lambda(\bar{\nu})$ for elements ν with $|\nu| < |\mu|$. Secondly, we need that if $\mu \neq \lambda$, then there exists a value of i such that $\bar{\mu}_i^{-1} \neq \bar{\lambda}_i^{-1}$. □

1.3 Extra vanishing theorem

Here we discuss section 4 of [5], where it is shown that the polynomials E_λ satisfy the so-called extra vanishing property, which is an unexpected and non-trivial result. This section is extra: we don't use the results anywhere else in this thesis. Before proving the main theorem, we need some definitions and results. For more details, we refer the reader to [5].

Definition 1.20. Let $\lambda, \mu \in \Lambda$. Then we say that $\lambda \preceq \mu$ if there is a permutation $\pi \in W$ such that $\lambda_i < \mu_{\pi(i)}$ if $i < \pi(i)$ and $\lambda_i \leq \mu_{\pi(i)}$ if $i \geq \pi(i)$. Such a permutation π is called a *defining permutation* for $\lambda \preceq \mu$.

The following lemma makes this definition less abstract and allows us to easily check if the relation $\lambda \preceq \mu$ holds for specific $\lambda, \mu \in \Lambda$.

Lemma 1.21. *If $\lambda \preceq \mu$, then the $\pi = w_\mu w_\lambda^{-1}$ is a defining permutation.*

We need one additional definition and result in order to prove the extra vanishing theorem.

Definition 1.22. A set $S \subset \Lambda$ is called closed if $\lambda \in S$ and $\lambda \preceq \mu$ implies $\mu \in S$. For a closed set S, we denote by I_S the ideal of functions that vanish on all points $\bar{\lambda}$ with $\lambda \in \Lambda \setminus S$.

Using the explicit form of the operators Ξ_i, Knop proves that if S is a closed set, then $\Xi_i(I_S) \subseteq I_S$ for all $i = 1, \ldots, n$.

Theorem 1.23 (4.5)**.** *Let $\lambda, \mu \in \Lambda$ with $\lambda \not\preceq \mu$. Then $E_\lambda(\bar{\mu}) = 0$.*

Remark 1.24. Notice that $\lambda \prec \mu$ implies $|\lambda| < |\mu|$, but the reverse is not true. Hence the set of points $\bar{\mu}$ for which the following theorem holds is strictly bigger than the set of points that E_λ is required to vanish on by its definition.

Proof. Fix λ and let $S = \{\nu \in \Lambda \mid \lambda \preceq \nu\}$. Then S is clearly closed and we have to prove that $E_\lambda \in I_S$. First, we claim that there exists a function $f \in I_S$ with the property that $f(\bar{\lambda}) \neq 0$. Indeed, define

$$f(z) = \prod_{\pi \in W} \left[\prod_{i < \pi(i)} \varphi_{\lambda_i+1}(\bar{\lambda}_i^{-1} z_{\pi(i)}) \prod_{i \geq \pi(i)} \varphi_{\lambda_i}(q\bar{\lambda}_i^{-1} z_{\pi(i)}) \right], \tag{1.6}$$

where $\varphi_k(z) := (z-1)(z-q^{-1})\ldots(z-q^{-k+1})$. Then it is clear that $f(\bar{\lambda}) \neq 0$. Moreover, to see that $f \in I_S$, let $\mu \in \Lambda \setminus S$. Then choosing $\pi = w_\mu w_\lambda^{-1}$ in (1.6) and using the definition of the partial ordering, we see that indeed $f(\bar{\mu}) = 0$. The existence of the function f with the stated properties together with the Ξ_i-stability of I_S implies that there exists $E_{\lambda'} \in I_S$ with the same property: $E_{\lambda'}(\bar{\lambda}) \neq 0$. The vanishing properties of $E_{\lambda'}$ imply that either $|\lambda| > |\lambda'|$ or $\lambda = \lambda'$. Also, since $E_{\lambda'} \in I_S$ and $E_{\lambda'}(\lambda') \neq 0$, we have $\lambda' \in S$, so $\lambda \preceq \lambda'$. This implies $|\lambda| \leq |\lambda'|$. Hence $\lambda' = \lambda$, so $E_\lambda \in I_S$. □

1.4 Binomial formula

We have seen that the non-symmetric type A interpolation polynomials can be defined as eigenfunctions of the inhomogeneous Cherednik operators Ξ_i. Likewise, the nonsymmetric type A Macdonald polynomials (which are classically defined by an orthogonality property), can also be defined as eigenfunctions of the homogeneous Cherednik operators ξ_i (see for example [7]). To show that the two definitions result in the same polynomials, one needs an adjointness property of the operators ξ_i with respect to the inner product defined by Macdonald in [7].

In this section, we follow the same steps as in [10], but now for the non-symmetric Macdonald polynomials, which we denote by $\bar{E}_\lambda,\ \lambda \in \Lambda$. We want to express this polynomial as a linear combination of non-symmetric interpolation polynomials. For this, we will use the Fourier pairing, defined as

$$\langle g, f \rangle = [\xi(g) \cdot f](\bar{0}),$$

where $\xi(g) = g(\xi_1, \ldots, \xi_n)$, which is well defined, because the operators ξ_i commute pairwise.

Theorem 1.25. *The non-symmetric interpolation polynomials E_λ are orthogonal with respect to this pairing.*

Proof. With λ fixed, first we assume that $|\mu| \le |\lambda|$ and $\mu \neq \lambda$. We denote the set of all ν for which this holds by $S(n,d)$, where $d \coloneqq |\lambda|$. The vanishing properties of the interpolation polynomials imply that $\langle E_\lambda, \bar{E}_\nu \rangle = 0$ for all $\nu \in S(n,d)$. Since the highest homogeneous term of E_μ is $\bar{E}_\mu$ (theorem 3.9 in [5]), we get

$$\langle E_\lambda, E_\mu \rangle = \langle E_\lambda, \bar{E}_\mu \rangle + \sum_{|\nu|<|\mu|} a_\nu \langle E_\lambda, \bar{E}_\nu \rangle = 0.$$

Note that only triangularity of the polynomials E_μ would not be enough here, because a priori it might happen, if $\mu > \lambda$, that $\bar{E}_\lambda$ appears with non-zero coefficient in the expansion of E_μ.

Now let μ be such that $|\mu| > |\lambda|$. Then

$$\langle E_\lambda, E_\mu \rangle = [\xi(E_\lambda) \cdot E_\mu](\bar{0}) = \sum_{|\nu| \le |\lambda|} c_{\nu\lambda} E_\mu(\bar{\nu}) = 0. \tag{1.7}$$

Here we used that

$$\xi_i f(\bar{\mu}) = \sum_{|\nu|=|\mu|+1} \ldots f(\bar{\nu})$$

for any i, f and μ. This follows from the explicit formula

$$\xi_i = t^{-n+1} \bar{H}_{i-1} \cdots \bar{H}_1 \Delta^{-1} H_{n-1} \cdots H_i.$$

for the operator ξ_i. □

This orthogonality property of the interpolation polynomials with respect to the Fourier pairing allows us to express the nonsymmetric Macdonald polynomials in terms of the nonsymmetric interpolation Macdonald polynomials, which is the main result of this chapter.

Theorem 1.26 (Binomial formula)**.** *For all* $\lambda \in \Lambda$*, we have*

$$\bar{E}_\lambda(x) = \sum_{\mu \leq \lambda} \frac{E_\mu(\bar{\lambda}) E_\mu(x)}{\langle E_\mu, E_\mu \rangle}.$$

Proof. Let $\lambda \in \Lambda$. Because the set $\{E_\mu : \mu \leq \lambda\}$ forms a basis for the space of polynomials of degree at most $|\lambda|$ and $\bar{E}_\lambda$ is an element of this space, there exists an expansion

$$\bar{E}_\lambda(x) = \sum_{\mu \leq \lambda} c_{\mu\lambda} E_\mu(x)$$

The coefficients are determined by using theorem 1.25. □

Corollary 1.27 (Label-argument symmetry for $\bar{E}_\lambda$)**.** *For all* $\lambda, \mu \in \Lambda$*, we have* $\bar{E}_\lambda(\bar{\mu}) = \bar{E}_\mu(\bar{\lambda})$*.*

Now we calculate $\langle E_\mu, E_\mu \rangle$ in two ways. First, we have

$$\langle E_\mu, E_\mu \rangle = \langle E_\mu, \bar{E}_\mu \rangle = E_\mu(\bar{\mu}) \bar{E}_\mu(\bar{0}).$$

We also have

$$\langle E_\mu, E_\mu \rangle = [\xi(E_\mu) \cdot E_\mu](\bar{0}) = c_{\mu\mu} E_\mu(\bar{\mu}),$$

where the number $c_{\mu\mu}$ is from equation (1.7). Combining these two formulas gives the identity $\bar{E}_\mu(\bar{0}) = c_{\mu\mu}$. So when we can make the number $c_{\mu\mu}$ explicit (using equation (1.7)) this gives us a formula for the value of the nonsymmetric Macdonald polynomial in the point $\bar{0}$.

Chapter 2

Intermezzo: general theory

As mentioned earlier, every affine root system has two associated families of orthogonal polynomials (symmetric and non-symmetric). In this chapter, we will define what an affine root system is and we will state how to define the associated families of polynomials. This chapter is quite technical and compact. For more details and proofs, we refer the reader to [8].

2.1 Finite root systems

Definition 2.1. Let $V = \mathbb{R}^n$ equipped with the standard Euclidian inner product $(\cdot, \cdot)$. Then a finite subset $R \subset V$ is called a (finite) root system if the following properties are satisfied:

1. The vectors in R (called roots) span V.
2. For every $\alpha, \beta \in R$, the vector $s_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$ is also in R.
3. For every $\alpha, \beta \in R$, the number $\langle \beta, \alpha \rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.

One can show from these properties that if $\alpha \in R$ and $c \cdot \alpha \in R$, then $c \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$. If R is a root system where additionally $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for every $\alpha \in R$, then it is called a reduced root system. The group W_0 generated by all reflections s_α ($\alpha \in R$) is called the Weyl group of R. A root system R is called reducible if it can be split in two sets of mutually orthogonal vectors that are both root systems and irreducible otherwise. We can divide R in a set of positive roots R^+ and a set of negative roots R^- by choosing a dividing hyperplane that doesn't contain a root. When a set of positive roots is chosen, we denote by Δ the set of positive roots that cannot be written as a linear combination of other positive roots and we call this a base for R. It can be shown that Δ is a basis for V and that every root $\alpha \in R^+$ is a linear combination of the roots in Δ with non-negative coefficients.

Up to a naturally defined isomorphism, all root systems are known and are classified by Dynkin diagrams. There are four families of root systems: type A, B, C, D, which exist for arbitrary dimension, and then there are five exceptional root systems, each of which only exists in a specific dimension.

2.2 Affine root systems

An affine root system is a subset of the affine linear functionals on V satisfying certain axioms similar to the axioms of a finite root system (see for example [8]). With each irreducible root system R we can associate a reduced affine root system $S(R)$ as follows. For $\alpha \in R$ and $r \in \mathbb{Z}$, let $a_{\alpha,r}$ be the affine linear functional defined by

$$a_{\alpha,r}(x) = (\alpha, x) + r.$$

Moreover, denote by $R_1 \subset R$ the subset of indivisible roots and let $R_2 = R \backslash R_1$. Then the set

$$\{a_{\alpha,r} : \alpha \in R_1, r \in \mathbb{Z}\} \cup \{a_{\alpha,r} : \alpha \in R_2, r \in 2\mathbb{Z} + 1\}$$

is a reduced affine root system and every reduced affine root system is of this form. Every non-reduced irreducible affine root system S is of the form $S_1 \cup S_2$, where S_1 and S_2 are respectively the set of indivisible roots and the set of inmultiplyable roots in S.

Just as in the finite non-affine case, we can define a basis Δ of S and we can define positive and negative roots with respect to this basis. We denote the set of positive resp. negative roots by S^+ resp. S^-.

In section 1.4 and in the rest of his book [8], Macdonald distinguishes three cases for the affine root system S and associates to each of these choices a second affine root system S', a pair of finite root systems (R, R') and a pair of lattices (L, L'). These are given by

$$\begin{array}{ll} 1. & S = S(R),\ S' = S(R^\vee),\ R' = R^\vee,\ L = P,\ L' = P^\vee \\ 2. & S = S' = S(R)^\vee,\ R' = R,\ L = L' = P^\vee \\ 3. & S = S' \text{ is of type } (C_n^\vee, C_n),\ R = R' \text{ is of type } C_n,\ L = L' = Q^\vee \end{array} \tag{2.1}$$

Here $Q, Q^\vee$ are the lattices spanned by the roots resp. the coroots and $P, P^\vee$ are the lattices of vectors that have an integral inner product with all elements in respectively $R^\vee$ and R. The lattices P and $P^\vee$ are called the weight and coweight lattice respectively. The three choices in (2.1) don't include all irreducible affine root systems, but all Macdonald polynomials can be described by these three choices, because specializing parameters in the third case gives the Macdonald polynomials for the remaing root systems (see section 5.1 in [8]).

2.3 Weyl group, Braid group, Hecke algebra

Let S be an irreducible affine root system corresponding to one of the three choices above. The Weyl group of S can be decomposed as $W_S = W_0 \ltimes t(Q^\vee)$, where W_0 is the Weyl group of R and $t(Q^\vee)$ is the group of translations of the root lattice of the dual root system $R^\vee$. We can slightly extend this Weyl group by defining $W = W_0 \ltimes t(L')$, where L' is the lattice corresponding to S defined in (2.1).

Remark 2.2. When studying the extended affine Weyl group W of an affine root system S, we may assume without loss of generality that $S = S(R)$, where R is a finite reduced irreducible root system.

Since L' and $Q^\vee$ are both abelian groups, and L' contains $Q^\vee$, we can take the quotient and we write $\Omega = L'/Q^\vee$.

We can define a length function on W by defining the length of an element $w \in W$, which we will denote by $l(w)$ to be the cardinality of the set $S^+ \cap w^{-1}S^-$. It can be shown that Ω is exactly the set with elements of length 0, i.e. Ω permutes the affine basis. The length function is used to define the braid group B of the extended affine Weyl group W: the group B is defined as the group with generators $T(w),\ w \in W$ and relations

$$T(v)T(w) = T(vw) \text{ if } l(v) + l(w) = l(vw).$$

We will write $T_i = T(s_i)$ and $\omega = T(\omega)$. It can be shown that the Braid group is generated by the elements T_i and ω for $i = 0, \ldots, n$ and $\omega \in \Omega$ subject to

- the braid relations coming from the group W between the elements T_i,
- the relations $T(\omega_1)T(\omega_2) = T(\omega_1\omega_2)$ for $\omega_1, \omega_2 \in \Omega$ and
- the relations $\omega T_i = T_j\omega$, where j is determined by $\omega(a_i) = a_j$.

This is the presentation of the group B corresponding to the decomposition $W = W_S \ltimes \Omega$ of the extended affine Weyl group. There is also a presentation corresponding to the decomposition $W = W_0 \ltimes t(L)$. For the sake of simplicity, we take $L = P^\vee$, the dual weight lattice. Defining $Y^\lambda = T(t(\lambda))$ when $\lambda \in L$ is a dominant weight (i.e. it has not only an integral, but also a positive inner product with all roots in R). When $\lambda \in L$ is not dominant, then we can take μ, ν dominant such that $\lambda = \mu - \nu$ and in this case we define $Y^\lambda = Y^\mu (Y^\nu)^{-1}$. It is straightforward to verify that this is well-defined. Now the other presentation of B is given as follows.

Proposition 2.3. *The algebra B defined above is generated by $T_i (i = 1, \ldots, n)$ and Y^λ $(\lambda \in L)$ subject to the braid relations for T_i and the relations*

$$\begin{cases} T_i Y^\lambda = Y^\lambda T_i & \text{if } (\lambda, \alpha_i) = 0 \\ T_i Y^{\lambda - \alpha_i^\vee} T_i = Y^\lambda & \text{if } (\lambda, \alpha_i) = 1 \end{cases}$$

between the operators T_i and Y^λ.

Now we are in a position to define the Hecke algebra of an extended affine Weyl group W. First, let q be a real number between 0 and 1, and let τ_i be positive real numbers such that $\tau_i = \tau_j$ if s_i and s_j are conjugate in W. Defining the number $e \in \mathbb{Z}$ by $(L, L') = e^{-1}\mathbb{Z}$, where L and L' are the lattices defined in (2.1), we let K be a subfield of $\mathbb{R}$ that contains $q^{1/e}$ and all numbers τ_i.

Definition 2.4. The Hecke algebra H of W is the quotient of the group algebra KB of the Braid group B by the ideal generated by the elements

$$(T_i - \tau_i)(T_i + \tau_i^{-1}).$$

2.4 Cherednik representation and Macdonald polynomials

Proposition 2.3 can be used to prove a fundamental relation between the T-operators and the Y-operators that is called Lusztig's relation. This relation implies that the elements

$$T(w) Y^{\lambda'} \ (w \in W_0, \ \lambda' \in L')$$

form an K-basis for H. This implies that

$$H \cong A' \otimes_K H_0,$$

as K-vector spaces, where $A' = K[L']$, the group algebra of the lattice L' and H_0 is the K-subalgebra of H spanned by elements $T(w),\ w \in W_0$. This, together with Lustzig's relation, can be used to prove the following theorem. For details, see [8].

Theorem 2.5 (Cherednik's representation). *Let τ_i and τ_i' be parameters as in [8]. The map $\beta : H \to \mathrm{End}_K(A)$ given by*

$$\beta(T_i) = \tau_i s_i + \frac{(\tau_i - \tau_i^{-1}) + (\tau_i' - (\tau_i')^{-1})X^{a_i}}{1 - (X^{a_i})^2}(1 - s_i),$$
$$\beta(\omega) = \omega$$

is a representation of H on A, where X^{a_i} is the operator of multiplication by e^{a_i}.

The non-symmetric Macdonald polynomials corresponding to the affine root system S can be defined as eigenfunctions of the Y-operators in this representation. An orthogonality property can then be derived by using the self-adjointness of these operators with respect to a suitably chosen inner product. The symmetric Macdonald polynomials can be obtained by symmetrizing the non-symmetric ones and the corresponding operators of which they are eigenfunctions can be obtained by symmetrizing the Y-operators. See section 5 of Macdonald's book [8].

Chapter 3

Koornwinder polynomials

In this chapter, we try to imitate the theory of chapter 1, but now for root systems of type BC. In the symmetric case a binomial formula is known, which we will discuss in section 3.4, after introducing the symmetric Koornwinder polynomials and the symmetric interpolation polynomials. In the non-symmetric case a binomial formula is not yet known in the literature. We will end the chapter with a discussion of this case (first one-dimensional in section 3.5 and then for higher dimensions in section 3.6).

3.1 Symmetric interpolation polynomials

In [9], Okounkov defines natural BC_n-type analogues of the symmetric type A interpolation polynomials studied in [5]. Among other things, he proves that they are related to Koornwinder polynomials (which are the symmetric Macdonald polynomials associated to the root system of type BC) via a binomial formula. Moreover, he proves that they do not satisfy any q-difference equation like in the type A case. However, Rains constructs difference operators that act not only on the variables $x_1, \dots x_n$, but also on the parameter s and that have the Okounkov interpolation polynomials as eigenfunctions [11]. We will describe his approach here.

Definition 3.1. The operator $D^{(n)}(u_1, u_2; q, t)$ acts on Laurent polynomials by

$$(D^{(n)}(u_1,u_2;q,t)f)(x_1,\dots,x_n) = \sum_{\sigma\in\{\pm1\}^n} \prod_{1\le i\le n} \frac{(1-u_1x_i^{\sigma_i})(1-u_2x_i^{\sigma_i})}{(1-x_i^{2\sigma_i})} \prod_{1\le i<j\le n} \frac{(1-tx_i^{\sigma_i}x_j^{\sigma_j})}{(1-x_i^{\sigma_i}x_j^{\sigma_j})} f(x_1q^{\sigma_1/2},\dots,x_nq^{\sigma_n/2}).$$

Lemma 3.2. *The operator $D^{(n)}(u_1, u_2; q, t)$ acts on the space $k[x_i^{\pm1}]^{\mathrm{BC}_n}$.*

Proof. Let $f \in k[x_i^{\pm 1}]^{\mathrm{BC}_n}$. Then the polynomial

$$g(x) = \prod_{1 \le i \le n} \frac{(1 - u_1 x_i)(1 - u_2 x_i)}{(1 - x_i^2)} \prod_{1 \le i < j \le n} \frac{(1 - t x_i x_j)}{(1 - x_i x_j)} f(x_1 \sqrt{q}, \ldots, x_n \sqrt{q})$$

is in $k[x_i^{\pm 1}]^{\mathrm{S}_n}$. Hence the polynomial

$$\sum_{\sigma \in \{\pm 1\}^n} g(x_1^{\sigma_1}, \ldots, x_n^{\sigma_n})$$

is in $k[x_i^{\pm 1}]^{\mathrm{BC}_n}$ and this polynomial equals $D^{(n)}(u_1, u_2; q, t) f$. □

Now we introduce a parameter s and define $K = k[s, 1/s]$. The ring $K[x_i^{\pm 1}]^{\mathrm{BC}_n}$ has k-basis $\{s^k m_\lambda : k \in \mathbb{Z},\ \lambda \in \Lambda^+\}$, where m_λ is the orbit sum of $\prod_i x_i^{\lambda_i}$. We extend the dominance ordering on partitions by defining $(k, \lambda) \ge (l, \mu)$ when $\lambda \ge \mu$ and $|l - k| \le |\lambda| - |\mu|$. Now we define a difference operator that acts on the parameter s in addition to the x variables by

$$(D_s^{(n)}(u; q, t) f)(x_1, \ldots, x_n; s) = (D^{(n)}(s, u/s; q, t) f)(x_1, \ldots, x_n; s\sqrt{q}). \quad (3.1)$$

Lemma 3.3. *The operator $D_s^{(n)}(u; q, t)$ preserves the space $K[x_i^{\pm 1}]^{\mathrm{BC}_n}$ and acts on monomials triangularly with respect to the dominance ordering. In particular,*

$$D_s^{(n)}(u; q, t) s^k m_\lambda = q^{k/2} E_\lambda^{(n)}(u; q, t) s^k m_\lambda + \textit{dominated terms},$$

where

$$E_\lambda^{(n)}(u; q, t) = q^{-|\lambda|/2} \prod_{1 \le i \le n} (1 - q^{\lambda_i} t^{n-i} u).$$

Proof. The factor $q^{k/2}$ comes from the action of $D_s^{(n)}$ on s^k. Hence it suffices to consider the case $k = 0$. We multiply $D_s^{(n)}(u; q, t) m_\lambda$ by the product

$$\prod_{1 \le i \le n} (x_i - x_i^{-1}) \prod_{1 \le i < j \le n} (x_i + x_i^{-1} - x_j - x_j^{-1}). \quad (3.2)$$

This product is BC_n antisymmetric, because the first product produces a minus sign when sending x_i to $1/x_i$ and the second product produces a minus sign when interchanging x_i and x_j. The resulting function is

$$\sum_{\sigma \in \{\pm 1\}^n} \prod_{1 \le i \le n} -\sigma_i x_i^{-\sigma_i} (1 - s x_i^{\sigma_i}) \left(1 - \frac{u}{s} x_i^{\sigma_i}\right) \prod_{1 \le i < j \le n} (1 - t x_i^{\sigma_i} x_j^{\sigma_j})(x_i^{-\sigma_i} - x_j^{-\sigma_j}) \cdot$$
$$m_\lambda(x_1 q^{\sigma_1/2}, \ldots, x_n q^{\sigma_n/2}).$$

By the symmetry of m_λ, we can write the last part of this expression as

$$m_\lambda(\sqrt{q}x_1^{\sigma_1}, \ldots, \sqrt{q}x_n^{\sigma_n}).$$

Hence we can write the expression as

$$(-1)^n \sum_{\sigma\in\{\pm 1\}^n} \left(\prod_{1\le i\le n} \sigma_i R_{x_i}(\sigma_i) \right) F(u; x_1, \ldots, x_n; s) m_\lambda(\sqrt{q}x_1, \ldots, \sqrt{q}x_n), \tag{3.3}$$

where

$$F(u; x_1, \ldots, x_n; s) = \prod_{1\le i\le n} x_i^{-1}(1-sx_i)\left(1-\frac{u}{s}x_i\right) \prod_{1\le i<j\le n}(1-tx_ix_j)(x_i^{-1}-x_j^{-1})$$

and $R_{x_i}(\sigma_i)$ is the homomorphism that sends x_i to $x_i^{\sigma_i}$ and doesn't change the other variables. We see that the leading monomial of F is $x_1^n x_2^{n-1} \ldots x_n$ and hence the leading monomial of Fm_λ is $\prod_i x_i^{\lambda_i+n-i+1}$. Since the function

$$F(u; x_1, \ldots, x_n; s) m_\lambda(\sqrt{q}x_1, \ldots, \sqrt{q}x_n)$$

is S_n-antisymmetric, it follows that (3.3) is BC_n-antisymmetric. So we can divide out again the factor (3.2) to obtain a BC_n-symmetric polynomial dominated by m_λ. Now a straightforward calculation (see [11]) shows that the leading coefficient is indeed equal to $E_\lambda^{(n)}(u; q, t)$. □

Since for generic u all diagonal elements of the operator $D_s^{(n)}(u; q, t)$ with respect to the basis $\{s^k m_\lambda\}$ are distinct, it follows that for each (k, λ), there exists a unique eigenfunction dominated by $s^k m_\lambda$. It turns out that these eigenfunctions are independent of u and satisfy Okounkov's interpolation conditions, which we will prove in the following theorem. Similarly to the notation in the type A case, we write $\bar{\mu}(s) = (q^{\mu_1}t^{n-1}s, q^{\mu_2}t^{n-2}s, \ldots, q^{\mu_n}s)$ for $\mu \in \Lambda^+$ and we will also write $p(\bar{\mu}, s) = p(\bar{\mu}(s), s)$.

Theorem 3.4. *For any partition λ the following three properties hold.*

(a) The operators $D_s^{(n)}(u; q, t)$ for different u commute on the monomials dominated by m_λ, and thus (since they are triangular by lemma 3.3) have a common eigenfunction $s^k \bar{P}_\mu^{(n)}(; s)$ for each leading monomial $s^k m_\mu$ dominated by m_λ;

(b) For any partition μ, we have $\bar{P}_\lambda^{(n)}(\bar{\mu}; s) = 0$ unless $\lambda \subset \mu$;

(c) $\bar{P}_\lambda^{(n)}(\bar{\lambda}; s) \neq 0$.

Proof. We will first prove that if (a) holds for a fixed partition λ, then (b) and (c) also hold for λ. Then we will prove by induction that (a) holds for all partitions.

So let λ be a partition and suppose that (a) holds for λ. Let μ be a partition different from λ and choose l such that $\lambda_l \neq \mu_l$. We have

$$0 = (D_s^{(n)}(q^{-\lambda_l}t^{l-n}; q, t)\bar{P}_\lambda^{*(n)})(\bar{\mu}; s/\sqrt{q}) = \sum_\nu C_{\mu,\nu}\bar{P}_\lambda^{*(n)}(\bar{\nu}; s). \tag{3.4}$$

The first equality holds because the eigenvalue $E_\lambda^{(n)}(u; q, t)$ equals zero for this value of u. In the second equality, the sum runs over all ν such that either $\nu_i = \mu_i$ or $\nu_i = \mu_i - 1$ for all i, because of (3.1) (this depends on σ_i in definition 3.1). Hence a term in this sum where ν is not a partition appears either if $\mu_n = 0$ and $\sigma_n = -1$ or if $\mu_i = \mu_{i+1}, \sigma_i = -1$ and $\sigma_{i+1} = 1$. In the first case, the coefficient $C_{\mu,\nu}$ vanishes, because $(1 - s'\bar{\mu}_n^{-1}(s')) = 0$, where $s' = s/\sqrt{q}$. In the second case, the coefficient also vanishes, because $(1 - t\bar{\mu}(s')_i/\bar{\mu}(s')_{i+1}) = 0$. Hence the sum runs only over partitions $\nu \subseteq \mu$. The coefficient $C_{\mu,\mu}$ is obtained by setting $\sigma_i = 1$ for every i, and a computation of this coefficient shows that it is not zero. Hence equation (3.4) implies that

$$\bar{P}_\lambda^{*(n)}(\bar{\mu}; s) = \sum_{\nu \subsetneq \mu} \tilde{C}_{\mu,\nu}\bar{P}_\lambda^{*(n)}(\bar{\nu}; s).$$

Assuming that we started with a partition μ that doesn't contain λ, induction gives us that $\bar{P}_\lambda^{*(n)}(\bar{\mu}; s)$ is a multiple of $\bar{P}_\lambda^{*(n)}(\bar{0}; s)$. Now equation (3.4) implies that $0 = \bar{P}_\lambda^{*(n)}(\bar{0}; s)$, so it follows that property (b) holds. Moreover, the same induction implies that if $\bar{P}_\lambda^{*(n)}(\bar{\lambda}; s) = 0$, then $\bar{P}_\lambda^{*(n)}(\bar{\mu}; s) = 0$ for every μ, which gives a contradiction. Hence (c) holds.

Now suppose that (a) (and thus (b) and (c)) holds for all partitions $\mu < \lambda$. We want to show that (a) holds for λ and since, by the induction hypothesis, the operators already commute on polynomials that are strictly dominated by m_λ, it suffices to show that they commute on m_λ. We claim that there exists a unique BC_n-symmetric polynomial

$$f(x; s) = m_\lambda(x) + \sum_{\mu<\lambda} c_{\lambda\mu}(s)\bar{P}_\mu^{*(n)}(x; s) \tag{3.5}$$

such that $f(\bar{\mu}; s) = 0$ for all $\mu < \lambda$. Indeed, using the induction hypothesis for $\mu < \lambda$, property (b) implies that we get a triangular system of equations for the coefficients and property (c) implies that it has a nonzero diagonal. The action of the $D_s^{(n)}$ can be naturally extended to polynomials with coefficients in $k(s)$. By the same argument as above, we see that

$$(D_s^{(n)}(u;q,t)f)(\bar{\mu};s) = \sum_{\nu \subseteq \mu} C_{\mu\nu}(s) f(\bar{\nu}; s\sqrt{q}) = 0$$

for every $\mu < \lambda$. Hence $D_s^{(n)} f$ satisfies the same vanishing conditions as f and since f is the unique BC_n-symmetric polynomial (up to a constant) that vanishes at these points, we conclude that $D_s^{(n)} f$ is a multiple of f. It follows that the operators $D_s^n(u;q,t)$ commute on f for different values of u and we also know, by the induction hypothesis, that they commute on the sum in (3.5). Since m_λ is the difference of these two functions, the operators also commute on this polynomial. □

3.2 Nonsymmetric Koornwinder polynomials

In this section we discuss some results in the article [14] by Stokman on nonsymmetric Koornwinder polynomials.

Let $V = \mathbb{R}^n$ with the standard inner product $(\cdot, \cdot)$ and orthonormal basis $\epsilon_1, \ldots, \epsilon_n$. Let δ be the constant function on V equal to one. We define $\hat{V} = V \oplus \mathbb{R}\delta$, where we identify V with its dual using the inner product. So we have

$$(v + m\delta)(w) = (v, w) + m.$$

We extend the inner product from V to $\hat{V}$ by setting $(v + m\delta, w + n\delta) = (v, w)$. Now consider the set

$$S = \left\{ \epsilon_i + \frac{m}{2}\delta,\ 2\epsilon_i + m\delta,\ \pm\epsilon_i \pm \epsilon_j + m\delta \mid i < j,\ m \in \mathbb{Z} \right\},$$

which is a subset of $\hat{V}$. It is known from the theory of root systems that this set satisfies the properties of an irreducible affine root system (these systems are completely classified, see [8]). The Weyl group W of this root system is the subgroup of $GL_{\mathbb{R}}(\hat{V})$ that is generated by the reflections s_β, where

$$s_\beta(\alpha) = \alpha - \langle \alpha, \beta^\vee \rangle \beta = \alpha - \left\langle \alpha, \frac{2\beta}{\langle \beta, \beta \rangle} \right\rangle \beta.$$

This group is uniquely determined by the Coxeter graph corresponding to the root system (see [8]). It is isomorphic to the Coxeter group corresponding to this graph (generated by s_i, subject to braid relations). The root system itself is not uniquely determined by the Coxeter graph: there are several non-isomorphic irreducible root systems with the same Coxeter graph and hence also the same Weyl group. However, the root system is uniquely determined by its Dynkin diagram, which contains more information than the Coxeter graph.

The subset R of S of inmultiplyable roots is a reduced affine root system of type C_n and $S = R \cup R^\vee$. The roots

$$a_0 = \delta - 2\epsilon_1,\ a_i = \epsilon_i - \epsilon_{i+1}\ (i = 1, \ldots, n-1),\ a_n = 2\epsilon_n$$

form a basis for R and the corresponding roots $a_i^\vee$ form a basis for $R^\vee$. These roots without the first one form a basis for the associated non-affine root system Σ of type C_n. We denote by $\Lambda = \mathbb{Z}^n$ the weight lattice of Σ and by Λ^+ the cone of dominant weights. We write $Q^{\vee,+}$ for the positive $\mathbb{Z}$-span of the simple co-roots $a_i^\vee$ $(i = 1, \ldots, n)$.

It is straightforward to check that there are five W-oribits in S, namely

$$Wa_0,\ Wa_0^\vee,\ Wa_i\ (i = 1, \ldots, n-1),\ Wa_n,\ Wa_n^\vee. \tag{3.6}$$

A multiplicity function is a function $t : S \to \mathbb{C},\ \beta \mapsto t_\beta$ that is constant on each orbit. So in this case, the function is determined by five values, which we will denote respectively by $t_0, t_0^\vee, t, t_n, t_n^\vee$. We will also write $t_i = t_{a_i}$ for $i = 1, \ldots, n-1$.

We need the affine Hecke algebra H to define the nonsymmetric Koornwinder polynomials. This is the algebra generated by elements $T_i,\ i = 0, \ldots, n$ subject to the same relations as the s_i in the Weyl group, with the exception that $s_i^2 = 1$ is replaced by the quadratic relation $(T_i - t_i)(T_i + t_i^{-1}) = 0$. Noumi showed that H has a representation on the field $\mathcal{R} = \mathbb{F}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ given by

$$T_i \mapsto t_i + t_i^{-1} v_{a_i}(x)(s_i - 1), \tag{3.7}$$

where

$$v_\beta(x) = \frac{(1 - t_\beta t_{\beta/2} x^{\beta/2})(1 + t_\beta t_{\beta/2}^{-1} x^{\beta/2})}{(1 - x^\beta)},$$

$t_i = t$ for $i \neq 0, n$ and the operators s_i act on $\mathcal{R}$ by

$$\begin{aligned}(s_0 f)(x) &= f(qx_1^{-1}, x_2, \ldots, x_n)\\ (s_i f)(x) &= f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n), \quad i \neq 0, n\\ (s_n f)(x) &= f(x_1, \ldots, x_{n-1}, x_n^{-1}).\end{aligned}$$

This is called the Noumi representation in the literature and is a special case of theorem 2.5. Notice that when all parameters t_i are equal to one, the affine Hecke algebra reduces to the group algebra of the affine Weyl group.

The Y-operators of the affine Hecke algebra are given by

$$Y_i = (T_i \cdots T_{n-1})(T_n \cdots T_0)(T_1^{-1} \cdots T_{i-1}^{-1}). \tag{3.8}$$

In order to define the non-symmetric Koornwinder polynomials as eigenfunctions of these operators, we first prove their triangularity with respect to the partial ordering $\preceq$ on Λ, which we will define now.

Definition 3.5. Let $\lambda, \mu \in \Lambda$.

(i) We write $\lambda \leq \mu$ if $\mu - \lambda \in Q^{\vee,+}$.

(ii) We write $\lambda \preceq \mu$ if $\lambda^+ < \mu^+$ or if $\lambda^+ = \mu^+$ and $\lambda \leq \mu$.

Lemma 3.6. *Let $\mu \in \Lambda$ and $\alpha \in \Sigma^+$ and write $m = \langle \mu, \alpha \rangle$.*

If $m \geq 1$, then $\mu - r\alpha^\vee \prec \mu$ for $r \in [1, m]$ such that $\mu - r\alpha^\vee \in \Lambda$.

If $m < -1$, then $\mu + r\alpha^\vee \prec \mu$ for $r \in [1, -m)$ such that $\mu + r\alpha^\vee \in \Lambda$.

Proof. Suppose that $m \geq 1$ and let r be as in the lemma. We write $\mu_r = \mu - r\alpha^\vee$. Let $w \in W$ be such that $\mu_r^+ = w\mu_r$. If $w\alpha^\vee \in Q^{\vee,+}$, then $\mu_r^+ = w\mu - rw\alpha^\vee < w\mu \leq \mu^+$, since μ^+ is the unique maximum in the orbit $W\mu$. If $w\alpha^\vee \in -Q^{\vee,+}$, then $\mu_r^+ = w\mu - rw\alpha^\vee \leq w\mu - mw\alpha^\vee = (ws_\alpha)\mu \leq \mu^+$. This is either a strict inequality, which immediately implies $\mu_r \prec \mu$, or it is an equality. In this case $\mu_r < \mu$, since $\alpha \in \Sigma^+ \subset Q^{\vee,+}$. The second statement can be proved in the same way. Notice that it is slightly weaker, because the last part of the previous proof (the case where $\mu_r^+ = \mu^+$) does not hold here. $\square$

For $\beta \in R$, define

$$\mathcal{R}(\beta) = t_\beta s_\beta + t_\beta^{-1} v_\beta(x)(1 - s_\beta)$$

and let ϵ be the function that maps positive integers to 1 and strictly negative integers to -1. Then the above lemma implies that

$$\mathcal{R}(\beta)(x^\lambda) = t_\beta^{\epsilon(\langle \lambda, \beta \rangle)} x^\lambda + \text{l.o.t.}, \tag{3.9}$$

where the lower order terms are with respect to the partial ordering $\preceq$. We will denote the diagonal term of a triangular operator $\mathcal{R}$ corresponding to λ by $[\mathcal{R}]_\lambda$, so that we can write this equation as

$$[\mathcal{R}(\beta)]_\lambda = t_\beta^{\epsilon(\langle \lambda, \beta \rangle)}.$$

Moreover, we can write Y_i as a product of operators that are of this form together with $\tau(\epsilon_i)$. To prove this, note that $\mathcal{R}(a_i) = T_i s_i$ for $i = 1, \ldots, n$ and $\mathcal{R}(w(\beta)) = w\mathcal{R}(\beta)w^{-1}$ for all $w \in W$ and all $\beta \in R$. Using this, together with (3.8), we see that

$$\begin{aligned} Y_i =& \mathcal{R}(\epsilon_i - \epsilon_{i+1})\mathcal{R}(\epsilon_i - \epsilon_{i+2}) \cdots \mathcal{R}(\epsilon_i - \epsilon_n)\mathcal{R}(2\epsilon_i) \cdot \\ & \mathcal{R}(\epsilon_i + \epsilon_n) \cdots \mathcal{R}(\epsilon_i + \epsilon_{i+1})\mathcal{R}(\epsilon_i + \epsilon_{i-1}) \cdots \mathcal{R}(\epsilon_i + \epsilon_1) \cdot \\ & \mathcal{R}(\delta + 2\epsilon_i)\tau(\epsilon_i)\mathcal{R}(\epsilon_1 - \epsilon_i)^{-1} \cdots \mathcal{R}(\epsilon_{i-1} - \epsilon_i)^{-1}. \end{aligned} \tag{3.10}$$

This proves the triangularity of the operators Y_i. Note that in formula (3.10), for every positive root $a_i \in \Sigma^+$ in the W-orbit of a_i $(i = 1, \ldots, n-1)$ with $\langle a_i, \epsilon_i \rangle \neq 0$ either $\mathcal{R}(a_i)$ or $\mathcal{R}(a_i)^{-1}$ appears (once) depending on whether $\langle a_i, \epsilon_i \rangle$ is positive or negative. Moreover, together with $\tau(\epsilon_i), \mathcal{R}(2\epsilon_i)$ and $\mathcal{R}(\delta + 2\epsilon_i)$ these are all the operators that appear in (3.10). Together with formula (3.9) and the fact that there are five W-orbits (3.6) in S, this implies that

$$[Y_i]_\lambda = \prod_{\substack{\beta \in W a_1 \\ \langle \beta, \epsilon_i \rangle > 0}} t^{\epsilon(\langle \lambda, \beta \rangle)} \cdot \prod_{\substack{\beta \in W a_1 \\ \langle \beta, \epsilon_i \rangle < 0}} t^{-\epsilon(\langle \lambda, \beta \rangle)} \cdot (t_0 t_n)^{\epsilon(\langle \lambda, 2\epsilon_i \rangle)} \cdot q^{\lambda_i} =: (\gamma_\lambda)_i$$

is the diagonal term of Y_i corresponding to λ. Here $(\gamma_\lambda)_i$ is the i'th component of the vector $\gamma_\lambda \in \mathbb{C}^n$. It is not difficult to verify (see [14]) that

$$\lambda \neq \mu \implies \gamma_\lambda \neq \gamma_\mu.$$

This property allows us to define the non-symmetric Koornwinder polynomials as follows.

Definition 3.7. The non-symmetric Koornwinder polynomials E_λ are defined as the unique polynomials that satisfy the properties

(i) $E_\lambda = x^\lambda + \text{l.o.t.}$

(ii) $Y_i E_\lambda = (\gamma_\lambda)_i E_\lambda$.

Here the lower order terms are of course again with respect to the partial ordering $\preceq$. Equivalently, instead of property (ii), we can require that $f(Y)E_\lambda = f(\gamma_\lambda)E_\lambda$.

3.3 Symmetric Koornwinder polynomials

In [6], Koornwinder defined multiple variable generalizations of the famous Askey-Wilson polynomials in one variable. He proved that these polynomials are eigenfunctions of the difference operator

$$D = \sum_{i=1}^{n} \Phi_i(x)(T_{q,x_i} - 1) + \sum_{i=1}^{n} \Phi_i(x^{-1})(T_{q,x_1}^{-1} - 1), \tag{3.11}$$

where T_{q,x_i} is the ring endomorphism such that $T_{q,x_i}(x_j) = q^{\delta_{ij}} x_j$ and $x^{-1} := (x_1^{-1}, \dots, x_n^{-1})$ and

$$\Phi_i(x) = \frac{(1-ax_i)(1-bx_i)(1-cx_i)(1-dx_i)}{(1-x_i^2)(1-qx_i^2)} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{(1-tx_ix_j^{-1})(1-tx_ix_j)}{(1-x_ix_j^{-1})(1-x_ix_j)}. \tag{3.12}$$

We will show now that the operator D is equal, up to a constant, to the operator

$$m_{\epsilon_1}(Y) = Y_1 + \cdots + Y_n + Y_1^{-1} + \cdots + Y_n^{-1}.$$

First of all, it is well known that if $f \in \mathcal{S} = \mathcal{R}^W$, then $f(Y)$ lies in the center of H. So $f(Y)$ commutes with T_i for every i. It follows that $[v_{a_i}(x)(s_i - 1), f(Y)] = 0$, so

$$v_{a_i}(x)[s_i, f(Y)] + [v_{a_i}(x), f(Y)](s_i - 1) = 0.$$

On $\mathcal{S}$, the second term vanishes and it follows that $f(Y)$ maps $\mathcal{S}$ to $\mathcal{S}$. Secondly, from the formula for the Y-operators, we see that $m_{\epsilon_1}(Y)$ is of the form

$$\sum_{i=1}^{n} \phi_i(x)(T_{q,x_i} - 1) + \sum_{i=1}^{n} \psi_i(x)(T_{q,x_i}^{-1} - 1) + c,$$

where c is independent of x, because $c = m_{\epsilon_1}(Y) \cdot 1$. Thirdly, the operator T_{q,x_1} only appears in Y_1 and it is immediately clear from the formula that $\phi_1(x) = \Phi_1(x)$. Now since $s_1 m_{\epsilon_1}(Y) = m_{\epsilon_1}(Y)$ it follows that $\phi_2(x) = \Phi_2(x)$. Repeating this process, we see that indeed $m_{\epsilon_1}(Y) = D + c$ (to arrive at the the functions ψ_i, one applies s_n). This explains and justifies the name nonsymmetric and symmetric Koornwinder polynomials.

3.4 Binomial formula for Koornwinder polynomials

In what follows, we reparametrize the parameters t_i in order to be consistent with Okounkov [9], Koornwinder [6] and Sahi [12]. Instead of t_i, we now write $t_i^{1/2}$. We define the parameters a_1, a_2, a_3, a_4 and the corresponding dual parameters a_i' as in [9] and we define

$$q^\rho = (t^{n-1}a_1', \ldots, ta_1', a_1')$$
$$q^{\rho'} = (t^{n-1}a_1, \ldots, ta_1, a_1).$$

In our previous notation, we have that $q^\rho = \gamma_0$, where $0 = (0, \ldots, 0)$ is the zero partition. We define a Fourier pairing on the space of BC_n-symmetric polynomials by

$$\langle f, g\rangle = [f(Y) \cdot g](q^{\rho'}), \tag{3.13}$$

where $f(Y) = f(Y_1, \ldots, Y_n)$ is well defined, because the Y-operators commute. To prove the orthogonality of Okounkov's interpolation polynomials with respect to this pairing, we first show that the top homogeneous terms of both $P_\lambda(x; q, t, s)$ and $E_\lambda(x; q, t, s)$ is the type A symmetric Macdonald polynomial $P_\lambda(x; q, t)$. We will do this by comparing the difference operators that have these polynomials as their eigenfunctions. We already gave the operators corresponding to $P_\lambda(x; q, t, s)$ and $E_\lambda(x; q, t, s)$, and Macdonald constructed the symmetric type A polynomials as eigenfunctions of the operators

$$D_n^r = D_n^r(q, t) = \sum_I A_I(x; t) \prod_{i \in I} T_{q, x_i},$$

where the sum is over all r-element subsets I of $\{1, \ldots, n\}$ and

$$A_I(x; t) = t^{r(r-1)/2} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j}.$$

If $r = 1$, we will write $A_{\{i\}} = A_i$.

Lemma 3.8. *The top homogeneous term of the Koornwinder polynomial $P_\lambda(x; q, t, a_1, a_2, a_3, a_4)$ is the type A Macdonald polynomial $P_\lambda(x; q, t)$.*

Proof. The operator D in (3.11) can be rewritten as $D = D_1 + D_2$, where

$$D_1 = s^2 t^{n-1} \sum_{i=1}^n A_i(x; t)(T_{q, x_i} - 1) + t^{n-1} \sum_{i=1}^n A_i(x; t^{-1})(T_{q, x_i}^{-1} - 1)$$

and D_2 is an operator that decreases the degree of a polynomial. Expanding the brackets, we get four sums (together with constants in front of the sums). The two sums involving the difference operators are equal to $D_n^1(q, t)$ and $D_n^1(q^{-1}, t^{-1})$. Both these operators have $P_\lambda(x; q, t)$ as their eigenfunction, since

$$P_\lambda(x; q^{-1}, t^{-1}) = P_\lambda(x; q, t).$$

The other two sums are $\sum A_i(x;t)$ and the same sum with t replaced by t^{-1}. These sums are constants (independent of x). We can see this using the determinant representation of A_i, which is given by

$$A_i(x;t) = \frac{T_{t,x_i}(a_\delta(x))}{a_\delta(x)},$$

where $a_\delta(x)$ is the Vandermonde determinant. It follows that $\sum A_i(x;t)$ is a polynomial (since the numerator is divisible by each factor of the denominator). Since the numerator and the denominator have the same degree, we see that $\sum_i A_i(x;t)$ is a polynomial of degree zero, hence a constant. This proves that $P_\lambda(x;q,t)$ is an eigenfunction of D_1. □

Lemma 3.9. *The top homogeneous term of Okounkov's interpolation polynomial $\bar{P}^{*(n)}_\lambda(x;q,t,s)$ is the type A Macdonald polynomial $P_\lambda(x;q,t)$.*

Proof. For $\sigma \in \{\pm 1\}^n$, we define $|\sigma|$ to be the number of positive ones appearing in σ. Taking the highest homogeneous term in the eigenvalue equation

$$(D^{(n)}_s(u;q,t)\bar{P}^{*(n)}_\lambda)(x;q,t,s) = E^{(n)}_\lambda(u;q,t)\bar{P}^{*(n)}_\lambda(x;q,t,s),$$

we get

$$\sum_{k=0}^{n} \sum_{\substack{\sigma\in\{\pm1\}^n \\ |\sigma|=k}} (-1)^k u^k t^{k(k-1)/2} \prod_{\substack{i,j \\ \sigma_i=1 \\ \sigma_j=-1}} \frac{tx_i - x_j}{x_i - x_j} \hat{P}_\lambda(x_1 q^{\sigma_1/2}, \ldots, x_n q^{\sigma_n/2}; q,t)$$
$$= c_\lambda \hat{P}_\lambda(x;q,t),$$

where $\hat{P}_\lambda$ is the highest homogeneous term of $\bar{P}^{*(n)}_\lambda$ and c_λ is shorthand for $E^{(n)}_\lambda(u;q,t)$. Taking the coefficient of u^1 in this equation, we find that

$$\left(\sum_{i=1}^{n} A_i(x;t) T_{q,x_i} \hat{P}_\lambda\right)(x;q,t) = \left(\sum_{i=1}^{n} q^{\lambda_i} t^{n-i}\right) \hat{P}_\lambda(x;q,t),$$

which is precisely the eigenvalue equation for the type A Macdonald polynomials. Hence $\hat{P}_\lambda(x;q,t) = P_\lambda(x;q,t)$. □

Corollary 3.10. *Okounkov's interpolation polynomials can be written as*

$$P^*_\lambda(x;q,t,a_1) = P_\lambda(x;q,t,a_1,a_2,a_3,a_4) + \sum_{|\mu|<|\lambda|} c_{\mu\lambda} P_\mu(x;q,t,a_1,a_2,a_3,a_4).$$

Theorem 3.11. *Okounkov's interpolation polynomials satisfy the bi-orthogonality relation*

$$\langle P^*_\lambda(\cdot\,; q,t,s), P^*_\mu(\cdot\,; q,t,a_1)\rangle = 0$$

for all $\lambda \neq \mu$ *with respect to the pairing* (3.13)

Proof. We will write $P^*_\lambda = P^*_\lambda(\cdot\,; q,t,s)$, $\tilde{P}^*_\lambda = P_\lambda(\cdot\,; q,t,a_1)$ and $P_\lambda = P_\lambda(\cdot\,; q,t,a_1,a_2,a_3,a_4)$. As in the type A case, we will split the proof in two cases. First suppose that $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$. Then, using corollary 3.10, we have

$$\langle P^*_\lambda, \tilde{P}^*_\mu\rangle = \left\langle P^*_\lambda, P_\mu + \sum_{|\nu|<|\mu|} c_{\nu\mu} P_\nu \right\rangle = 0,$$

because of the property of the pairing that $\langle f, P_\nu\rangle = f(q^{\rho+\nu})P_\nu(q^{\rho'})$ and the vanishing properties of the interpolation polynomials. For $|\mu| > |\lambda|$, we need a property of the operators $D_1,\ldots,D_n$ defined by Van Diejen in [15]. The operator $P^*_\lambda(Y)$ lies in the algebra generated by the operators $D_1,\ldots,D_n$. The property we need is that if ν is a partition, then

$$(D_k\tilde{P}^*_\mu)(q^{\rho'+\nu}) = \sum_\kappa c_{\kappa\mu}\tilde{P}^*_\mu(q^{\rho'+\kappa}),$$

where the sum is only over partitions κ. In other words, the terms in the sum where κ is not a partition vanish. See [16]. This implies that

$$\langle P^*_\lambda, \tilde{P}^*_\mu\rangle = [P^*_\lambda(Y)\cdot\tilde{P}^*_\mu](q^{\rho'}) = \sum_{|\nu|\leq|\lambda|} c_{\nu\lambda}\tilde{P}^*_\mu(q^{\rho'+\nu}) = 0. \qquad \square$$

Corollary 3.12. *Binomial formula for Koornwinder polynomials (see also Okounkov, Rains). We have*

$$P_\lambda(x;q,t,a_1,\ldots,a_4) = \sum_{\mu\subseteq\lambda} \frac{P^*_\mu(q^{\rho+\lambda};q,t,s)}{P^*_\mu(q^{\rho+\mu};q,t,s)}\cdot\frac{P_\lambda(q^{\rho'};q,t,a_1,\ldots,a_4)}{P_\mu(q^{\rho'};q,t,a_1,\ldots,a_4)}P^*_\mu(x;q,t,a_1).$$

Proof. Since the set $\{P^*_\mu\}_{\mu\subseteq\lambda}$ forms a basis for the space spanned by the monomials $\{m_\mu\}_{\mu\subseteq\lambda}$ and since P_λ is an element of this space, there is an expansion

$$P_\lambda(x;q,t,a_1,\ldots,a_4) = \sum_{\mu\subseteq\lambda} c_{\mu\lambda}P^*_\mu(x;q,t,a_1).$$

The coefficients $c_{\mu\lambda}$ can be obtained by taking the Fourier pairing on both sides with the polynomial $P^*_\nu(x;q,t,s)$. $\square$

We will now make this formula explicit in the one-dimensional case. In particular, we will give an expression for the interpolation polynomials P^*_m and we will calculate the value $P_m(a)$, so that the binomial formula gives us an explicit expression for the one-dimensional Koornwinder polynomials. We will see that they are the well-known Askey-Wilson polynomials, which are defined by a certain basic hypergeometric series.

Definition 3.13. The q-shifted factorial $(a;q)_n$ is defined by

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

Lemma 3.14. *In the one-dimensional case, the interpolation polynomials P^*_m are given by*

$$P^*_m(x;q,s) = (-1)^m s^{-m} q^{-m(m-1)/2} \cdot (sx, sx^{-1};q)_m$$

Proof. By theorem 3.4, the polynomial P^*_m has to vanish on the points $q^l s$ for $l < m$, hence it has to contain the factor $(sx^{-1};q)_m$. Because the polynomials must be BC_1-symmetric, it follows that it also contains the factor $(sx;q)_m$. Now the statement follows from the fact that the leading monomial of $P^*_m(x)$ is $x^m + x^{-m}$. □

Lemma 3.15. *The value of the one dimensional Koornwinder polynomial P_m at the point a is given by*

$$P_m(a) = \frac{\tilde{P}^*_m(q^m a)}{P^*_m(q^m s)} \cdot s^{-m} \cdot \Phi(a)\Phi(qa)\cdots\Phi(q^{m-1}a),$$

where $\Phi(x)$ is given by equation (3.12) *with $n=1$, i.e.*

$$\Phi(x) = \frac{(1-ax)(1-bx)(1-cx)(1-dx)}{(1-x^2)(1-qx^2)}.$$

Proof. We will use the method described at the end of section 1.4: we will calculate the value $\langle P^*_m, \tilde{P}^*_m\rangle$ in two different ways. On the one hand, arguing as in the first part of the proof of theorem 3.11, we see that

$$\langle P^*_m, \tilde{P}^*_m\rangle = \langle P^*_m, P_m\rangle = P^*_m(q^m s)\cdot P_m(a). \tag{3.14}$$

On the other hand, arguing as in the second part of the proof of theorem 3.11, we see that

$$\langle P^*_m, \tilde{P}^*_m\rangle = [P^*_m(Y)\cdot \tilde{P}^*_m](a) = [Y^m \cdot \tilde{P}^*_m](a), \tag{3.15}$$

since Y^m is the only term in $P_m^*(Y)$ that contains the operator $T_{q,x}^m$. The coefficient of this operator in Y^m can be calculated by writing down the explicit formula for Y in terms of the operators s_0 and s_1. This can be done using equations (3.8) and (3.7). Since $T_{q,x}^m = (s_1 s_0)^m$, the only term in Y^m that contains this operator is

$$\left(s^{-1}\frac{(1-ax)(1-bx)}{1-x^2}\circ s_1 \circ \frac{(1-cx^{-1})(1-dx^{-1})}{(1-qx^{-2})}\circ s_0\right)^m,$$

which is equal to

$$s^{-m}\Phi(a)\Phi(qa)\cdots\Phi(q^{m-1}a)T_{q,x}^m.$$

Together with (3.15), this implies that

$$\langle P_m^*, \tilde{P}_m^*\rangle = s^{-m}\cdot\Phi(a)\Phi(qa)\cdots\Phi(q^{m-1}a)\tilde{P}_m^*(q^m a).$$

Combining this with equation (3.14) gives us the desired formula. □

Lastly, in order to state the following proposition succinctly, the following definition is useful (see equation (1.4) in [1]).

Definition 3.16. The basic hypergeometric series is defined by

$${}_{r+1}\phi_{r+j}\left(\begin{matrix} a_0,\ldots,a_r \\ b_1,\ldots,b_{r+j}\end{matrix};q,x\right) = \sum_{n=0}^{\infty}\frac{(a_0;q)_n\cdots(a_r;q)_n(-1)^{jn}q^{j\binom{n}{2}}x^n}{(b_1;q)_n\cdots(b_{r+j};q)_n(q;q)_n},$$

Proposition 3.17. *The one-variable Koornwinder polynomials P_n are given by*

$$P_n(x;q,t,a_1,\ldots,a_4) = \frac{(ab,ac,ad;q)_n}{a^n(abcdq^{n-1};q)_n}\,{}_4\phi_3\left(\begin{matrix} q^{-n},abcdq^{n-1},ax,ax^{-1} \\ ab,ac,ad\end{matrix};q,q\right).$$

Proof. Using corollary 3.12 and lemmas 3.14 and 3.15, we get

$$\begin{aligned}P_n(x) = \sum_{m\le n} &\frac{(q^n s^2,q^{-n};q)_m s^{-m}}{(q^n s^2,q^{-n};q)_n s^{-n}}\cdot\frac{(q^n a^2,q^{-n};q)_n a^{-n}}{(q^m a^2,q^{-m};q)_m a^{-m}}\cdot s^{n-m}\cdot\\ &\frac{(q^{n-1}a^2,q^{n-1}ab,q^{n-1}ac,q^{n-1}ad;q^{-1})_{n-m}}{(q^{2n-1}a^2;q^{-1})_{2n-2m}}\cdot(ax,ax^{-1};q)_m\cdot\\ &(-1)^m a^{-m}q^{-m(m-1)/2}.\end{aligned}$$

Now using that

$$(q^{n-1}\alpha;q^{-1})_{n-m} = \frac{(\alpha;q)_n}{(\alpha;q)_m}\text{ for }\alpha = a^2,ab,ac,ad$$

and $(q^{-m};q)_m = (-1)^m q^{-m} q^{-m(m-1)/2}(q;q)_m$, we obtain

$$P_n(x) = \frac{(ab,ac,ad;q)_n}{a^n(abcdq^{n-1};q)_n} \cdot \sum_{m\leq n} \frac{(q^{-n},s^2q^n,ax,ax^{-1};q)_m \cdot q^m}{(ab,ac,ad,q;q)_m} \cdot \frac{(q^na^2,a^2;q)_n(a^2;q)_{2m}}{(q^ma^2,a^2;q)_m(a^2;q)_{2n}}.$$

The second fraction in the sum is equal to 1, and the remaining sum is equal to the hypergeometric function in the proposition (by definition). □

Comparing this with (5.8) in [1], we see that for $n = 1$, the Koornwinder polynomials coincide with the monic Askey-Wilson polynomials.

3.5 Binomial formula for 1D case

In the one-dimensional case, the partial ordering of definition 3.5 becomes

$$0 \prec -1 \prec 1 \prec -2 \prec 2 \prec \dots$$

and the eigenvalues of the Y-operators are $\gamma_n(t) = s^{\epsilon(n)}q^n$, where t is shorthand for the parameters $(t_0, t_0^\vee, t_1, t_1^\vee)$. We can construct interpolation polynomials explicitly by defining

$$E^*_m(x;t) = \begin{cases} (-1)^m q^{-\frac{1}{2}m(m-1)} s^{-m}(sx, qsx^{-1};q)_m, & m \in \mathbb{Z}_{\geq 0} \\ (-1)^{-m-1}s^{1+m}q^{-\frac{1}{2}m(m+1)}x^{-1}(sx;q)_{-m}(qsx^{-1};q)_{-1-m}, & m \in \mathbb{Z}_{<0}, \end{cases} \tag{3.16}$$

where $(x;q)_m = (1-x)(1-qx)\cdots(1-q^{m-1}x)$ is the q-Pochhammer symbol and $(x_1,\dots,x_n;q)_m = (x_1;q)_m \cdots (x_n;q)_m$. Here E^*_m is the unique polynomial (up to normalization) of the form

$$\sum_{k\preceq m} a_k x^k, \quad a_k \in \mathbb{F}(q,s)$$

that vanishes at all the points γ_k^{-1} for $k \prec m$. The normalization constants are chosen in such a way that the coefficient of x^m in E^*_m is equal to 1.

In the same way as in in the symmetric case, we can check that the one-dimensional non-symmetric interpolation polynomials satisfy a bi-orthogonality relation and this allows us to also write down a very similar binomial formula for the non-symmetric Askey-Wilson polynomials:

$$E_n = \sum_{m\preceq n} \frac{E^*_m(\gamma_n)E_n(a)}{E^*_m(\gamma_m)E_m(a)}\tilde{E}^*_m.$$

We can build a similar differential operator as in the type A case that has the polynomials E_m^* as its eigenfunctions. Instead of the operator Δ that is used in the type A case, we will use the operator

$$\Delta : x \to x^{-1}\sqrt{q},\ \ s \to s\sqrt{q}.$$

To see the parallel with the type A case, we will call this operator Δ and we define $\Phi = (sx - 1)\Delta$. Also, we renormalize the polynomials $E_m^*(x;k)$ so that the constant factor in (3.16) doesn't depend on s. Then we get

$$[x^{-1} + x^{-1}\Phi]E_m^* = q^m s E_m^*$$

for $m \geq 0$ and

$$[x + q^{\frac{1}{2}} x^{-1}\Phi]E_m^* = q^{-m} s E_m^*$$

for $m < 0$. Because the operators for $m \geq 0$ and $m < 0$ are different, it is not clear how to generalize these operators to higher dimensions, because there we don't have the division between positive and negative there.

3.6 Higher dimensions

Here we will say a little bit about the possible existence of non-symmetric interpolation Okounkov polynomials for the higher dimensional case. In [2], Disveld, Koornwinder and Stokman proved that, like in the type A case, there exist unique polynomials G_α of degree $|\alpha|$ that vanish on the set

$$\{\gamma_\mu^{-1} : |\mu| \leq |\lambda|,\ \mu \neq \lambda\}. \tag{3.17}$$

Note that in the one-dimensional case, for $m < 0$ the number of interpolation points for G_m is strictly bigger than for E_m. In particular, it follows that $G_m \neq E_m$ for $m < 0$ and the polynomials G_m are not triangular with respect to the ordering $\preceq$. So we cannot have both triangularity and vanishing on the set (3.17). Also, it is not clear how to construct a binomial formula for non-symmetric Koornwinder polynomials using the polynomials G_α, while such a formula is crucial for the usefulness of the interpolation polynomials. It is still an open question if there exist interpolation polynomials for this case that can be used in a binomial type formula.

Bibliography

[1] Richard Askey and James A. Wilson. *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, volume 319. American Mathematical Soc., 1985.

[2] Niels Disveld, Tom H. Koornwinder, and Jasper V. Stokman. A nonsymmetric version of Okounkov's BC-type interpolation Macdonald polynomials. *arXiv preprint arXiv:1808.01221*, 2018.

[3] Pavel Etingof and Alexander Kirillov Jr. Macdonald's polynomials and representations of quantum groups. *arXiv preprint hep-th/9312103*, 1993.

[4] Abhijit Gadde, Leonardo Rastelli, Shlomo S Razamat, and Wenbin Yan. Gauge theories and Macdonald polynomials. *Communications in Mathematical Physics*, 319(1):147–193, 2013.

[5] Friedrich Knop. Symmetric and non-symmetric quantum Capelli polynomials. *Commentarii Mathematici Helvetici*, 72(1):84–100, 1997.

[6] Tom H. Koornwinder. Askey-Wilson polynomials for root systems of type BC. *Contemp. Math*, 138:189–204, 1992.

[7] Ian G. Macdonald. Affine Hecke algebras and orthogonal polynomials. *Séminaire Bourbaki*, 37:189–207, 1996.

[8] Ian G. Macdonald. *Affine Hecke algebras and orthogonal polynomials*, volume 157. Cambridge University Press, 2003.

[9] Andrei Okounkov. BC-type interpolation Macdonald polynomials and binomial formula for Koornwinder polynomials. *Transformation groups*, 3(2):181–207, 1998.

[10] Andrei Okounkov. A remark on the Fourier pairing and the binomial formula for the Macdonald polynomials. *Functional Analysis and Its Applications*, 36(2):134–139, 2002.

[11] Eric M. Rains. BC_n-symmetric polynomials. *Transformation groups*, 10(1):63–132, 2005.

[12] Siddhartha Sahi. Nonsymmetric Koornwinder polynomials and duality. *Annals of mathematics*, 150:267–282, 1999.

[13] Seth Shelley-Abrahamson, José Simental, et al. Lecture notes on Double Affine Hecke algebras and Elliptic Hall algebras, February 2017.

[14] Jasper V. Stokman. Koornwinder polynomials and affine Hecke algebras. *International Mathematics Research Notices*, 2000(19):1005–1042, 2000.

[15] Jan F. Van Diejen. Commuting difference operators with polynomial eigenfunctions. *Compositio Mathematica*, 95(2):183–233, 1995.

[16] Jan F. van Diejen. Self-dual Koornwinder-Macdonald polynomials. *Inventiones mathematicae*, 126(2):319–339, 1996.