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STELLINGEN

behorende bij het proefschrift

BAYESIAN ESTIMATION FOR DECISION-DIRECTED STOCHASTIC CONTROL

1. De bij luchtverkeersgeleiding in gebruik zijnde primaire en secundaire radar-systemen leveren data waarvan de informatie-inhoud op dit moment zeer onvolledig wordt benut. Een vrij volledige benutting is mogelijk door te schatten volgens Bayes op basis van een dynamisch vliegtuigmodel waarvan de toestandsruimte hybride is.
2. De gebruikelijke instelling van radar-plot-extractoren volgens het Neyman-Pearson criterium (vaste vals alarm kans) is niet optimaal voor luchtverkeersgeleiding.
3. Het toekennen van een octrooi op een biologische variëteit is een vorm van kolonisatie in de 21e eeuw.
4. Indien het EEG landbouwbeleid de afgelopen 30 jaar voor Afrika had gegolden, dan had dat werelddeel geen ontwikkelingshulp nodig gehad.
5. Een overheid wiens milieubeleid zo'n 20 jaar achter loopt bij de noodkreten uit de milieubeweging misleidt het publiek door zich milieu-bewust te profileren.
6. Door de regering van Zuid Afrika wordt de beëindiging van 27 jaar ontorechte vrijheidsberoving van één man voorgespiegeld als ware het de opheffing van het 35 jaar durende apartheidregime voor de meerderheid van zijn bevolking.
7. Een verbod op het dragen van een hoofddoek als godsdienstige uiting is in strijd met het recht op vrijheid van godsdienst.
8. Een grondwet die het gebruik van godslasteringen verbiedt en de vrijheid van godsdienst gebiedt, impliceert een verbod op het belasteren van andermans god; met zijn boek "De duivelsverzen" overtrad Rushdie de Nederlandse wet.
9. Niemand zou het in zijn hoofd halen om de plaatsen Enschede, Hengelo (Ov.), Delft, Pijnacker en Naaldwijk samen te voegen tot één gemeente, maar de Nederlandse Antillen worden al tientallen jaren gedwongen een nog vreemdere bestuurlijke samenvoeging in stand te houden.

Corrections of

"Bayesian estimation for decision-directed stochastic control"

p.25, line 16: delete ", N,",

p.31, 2nd line after (19) should refer to "(13) and (14)",

p.32, 6th line: change "H" into "Π",

p.37, first line below (iii): delete "run"

p.38, window in fig.6: change "b" and "c" into "ii" and "iii",

p.39, window in fig.8: change "b" and "c" into "ii" and "iii",

p.53, 7th line: change "(10.a)" into "(10.b)",

p.57, 6th line: change " $p^{-1}$ " into "p",

p.60, (iv), change "i.i.d." into "conditionally independent",

p.61, change eq. (27) into: 
$$\text{Cov}\{\tilde{w}_t | \tilde{\theta}_{t+1} = \theta, \tilde{\theta}_t = \eta, \tilde{x}_{t+1}\} =$$
$$= \text{Cov}\{x_t | \theta_t = \eta\} - \bar{A}(t, \theta, \eta) R(t+1, \theta, \eta) \bar{A}^T(t, \theta, \eta), \quad (27)$$

p.61, 3rd par., 2nd line, insert "not" before "conditionally"

p.61, 3rd par., 4th line, insert "also" at end of line

p.61, 3rd par., 6th line, delete "in law equivalent"

p.61, 3rd par., 7th line, change "proposition 6.2" into "th. 6.1"

p.61, 3rd par., insert at end: ", while this need not hold true

for the in law equivalent time-reversed equations:"

p.61, (iv), change "i.i.d." into "conditionally independent",

p.61, change (iv) into: 
$$\text{Cov}\{\tilde{w}_t | \tilde{\theta}_{t+1} = \theta, \tilde{\theta}_t = \eta, \tilde{x}_{t+1}\} =$$
$$= \text{Cov}\{x_t | \theta_t = \eta\} - \bar{A}(t, \theta, \eta) \text{Cov}\{x_{t+1} | \theta_{t+1} = \theta\} \bar{A}^T(t, \theta, \eta).$$

p.102, (27.c): insert the term " $-\hat{R}_{t, \theta} H_{x, \theta}^T H_{x, \theta} \hat{R}_{t, \theta} dt$ ",

p.138, change "Olsder, 10,19" into "Olsder, 9,19"

p.141, line 4: change "wiskundig aanpak" into "wiskundige aanpak"

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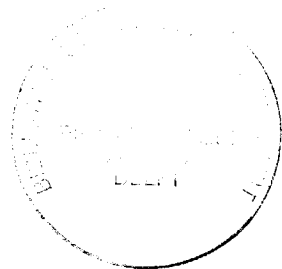
**BAYESIAN ESTIMATION FOR DECISION-DIRECTED  
STOCHASTIC CONTROL**

# BAYESIAN ESTIMATION FOR DECISION-DIRECTED STOCHASTIC CONTROL

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Some of the results described (in chapter V) trace back to my first year (1980) at NLR. At that time my interest aroses in taking a martingale approach to estimation, being stimulated by discussions with prof.dr.ir. H. Kwakernaak (Twente University) and dr.ir. B. Kemp (Academic Hospital Leiden). Subsequently, at NLR, some colleagues supported me in starting this research: ir. T.B. Dalm of the Flight division, ir. G. Moek of the Informatics division and dr.ir. J.P.B. Vreeburg and dr.ir. P.Th.L.M. van Woerkom of the Space division.

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## CHAPTER I

## INTRODUCTION

## 1 Decision-directed stochastic control

For the estimation problem to be studied in this thesis, we need a mathematical characterization of decision-directed stochastic control. This chapter provides that characterization and outlines the present study.

An illustrative example of decision-directed stochastic control is air traffic control. Air traffic control has at its disposal the flightplan of each aircraft under surveillance, listing, amongst others, all altitude changes to be executed at particular way-points. To execute an altitude change the pilot switches the flight regime from altitude hold to climb (or descend) and, on reaching the appropriate altitude, back to altitude hold. If, for example, the reaching of a waypoint is too early/late (due to e.g. unexpected meteorological conditions) or a pilot fails in switching the flight regime appropriately, it may lead to a conflict with another aircraft. Air traffic control monitors the situation and must decide whether and when to interfere with the actions of a pilot. Unfortunately, air traffic control only receives imperfect aircraft state observations. This implies that the latter decision has to be made under some level of uncertainty about the appropriateness of the regime switchings. Air traffic control may decrease the probability of making an erroneous decision by waiting for new observations. However, the risk of such a waiting is that there remains no time left for resolving the conflict.

This type of decision-making occurs for many other stochastic control problems. Other examples are: control of robots in an uncertain environment (Saridis, 1983), control of flexible manufacturing systems (Kimemia and Gershwin, 1983), control of a solar thermal receiver (Sworder and Rogers, 1983), control of an unstable aircraft (Looze et al., 1984), control of large space structures (Williams and Montgomery, 1985), control of electric power distribution networks (Malhame and Chong, 1985). A common property of these decision-directed control systems is that they are event-driven, where the events consist of switchings between operational regimes.

From the controller's point of view, an operational regime is determined by a combination of two types of modes: the process mode and the control mode. Associated with each process mode are

different process properties. Associated with each control mode is a different control law. A switching of a process mode is either intentional (e.g. a switching according to a process plan) or is unintentional (e.g. due to a sudden occurrence). An unintentional switching of the process mode is either according to an internal occurrence (e.g. a sudden failure of a system component) or according to an exogenous occurrence (e.g. a sudden change of the environmental conditions). A general property of processes with decision-directed control is that the state of such processes consists of two components; an  $R^n$ -valued component (the plant state) and a discrete-valued component (the process mode). Hence, from a mathematical point of view processes with a decision-directed control have a hybrid state space.

The ideal reaction of a controller to an unintentional switching of the process mode would be that it is immediately followed by a switching to a control mode that compensates the unintentional switching of the process mode (e.g. a failure of a system component leads to an immediate repair or replacement of that component). There are two reasons why such an ideal reaction is in general not a feasible control policy. Firstly, in many cases, the controller has imperfect information about the present process mode, so that the detection and identification of an unintentional switching of the process mode may be delayed, or may even be false. Secondly, a return to the previous operational regime, through switching the control mode such that the unintentional process mode switching is compensated, is in general simply impossible.

If the ideal reaction is not a feasible control policy, then the best we can do is to develop a control policy such that the controlled system performs optimally w.r.t. some appropriate optimality criterion. In view of the uncertainties involved, the mathematical theory developed for such problems is known as stochastic control theory (Fleming and Rishel, 1975; Dynkin and Yushkevich, 1979; Krylov, 1980; Bensoussan and Lions, 1984; Borkar, 1989). This theory applies to a mathematically well defined class of processes: controlled Markov processes.

A stochastic process is called a Markov process if its "future" is conditionally independent of its "past" given its "present". A stochastic process is called a controlled Markov process if it is conditionally Markov given the control. Hence, to study decision-directed control within the theory of stochastic control, we assume that the process under control is a controlled Markov process, the (Markov) state space of which is hybrid, i.e. a product of a discrete set and a Euclidean space.

To give an idea of the type of results that can be obtained by this abstract approach, we start with some elementary hybrid

state Markov processes that may be controlled through decision-directed control. Although our interest is both in continuous-time and discrete-time models, the examples mentioned are discrete-time. First, in section 2, we present an example of optimal decision-directed control in case of complete mode observations; the so called jump linear quadratic Gaussian (JLQG) control problem. Next, in section 3, we increase the complexity of that problem by assuming that the mode observations are imperfect, and argue that for optimal control this problem may be divided into two independent subproblems:

- Evaluating a so called superstate process while receiving partial process mode observations,
- Controlling that superstate process while receiving perfect superstate observations.

Since the degree of difficulty of each of these subproblems is significant, this thesis is restricted to the study of the first subproblem. To that end, in section 4, we formulate the evaluation of the superstate of a hybrid state Markov process as a Bayesian estimation problem. The approach is general enough to cover the evaluation of the superstate of any of the decision-directed control examples mentioned in the beginning. Finally, in section 5, we present summaries of the studies on Bayesian estimation and hybrid state modelling in this thesis.

The recent monograph of Mariton (1990) gives a comprehensive overview of the material touched upon in this chapter.

## 2 Jump linear quadratic control

Assume that we want to apply an  $R^m$ -valued feedback process  $\{u_t\}$  to control an  $R^n$ -valued plant state process  $\{x_t\}$ , satisfying

$$x_{t+1} = A(\theta_t)x_t + B(\theta_t)w_{t+1} + C(\theta_t)u_{t+1}, \quad (1.a)$$

or

$$x_{t+1} = A(\theta_{t+1})x_t + B(\theta_{t+1})w_{t+1} + C(\theta_{t+1})u_{t+1}, \quad (1.b)$$

where  $\theta_t$  assumes values in MCN ( $N$  is the set of natural numbers),  $\{w_t\}$  is a sequence of i.i.d. Gaussian vectors independent of  $x_0$  and  $\{\theta_t\}$ , while  $\{\theta_t\}$  is a finite state Markov process of which the transition probability matrix is known.

At moment  $t$ , the pair  $(x_t, \theta_t)$  represents the hybrid state of the system;  $x_t$  is the plant state and  $\theta_t$  is the process mode. The pair  $\{x_t, \theta_t\}$  is a conditional Markov process given the control. Hence,  $\{x_t, \theta_t\}$  is a controlled Markov process.

In this section, we consider the optimal control of (1.a) under the assumption of receiving linear Gaussian observations  $\{y_t\}$  of the plant state  $\{x_t\}$  and complete observations of the process

mode  $\{\theta_t\}$ . But before doing that, we consider the situation of receiving complete observations of both the plant state and the process mode. Given the assumption that the observations up to and including  $t$ , say  $y_t = \sigma\{y_s; s \leq t\}$ , are complete in the sense that  $y_t \supset \sigma\{x_s, \theta_s; s \leq t\}$ , then the optimal control policy is a Markov control policy (or Markov strategy), which means that the "future" optimal control is conditionally independent of all "past" if the "present" Markov state is given (Blumenthal and Gettoor, 1968; Krylov, 1980). For system (1.a) this means

$$(y_t, u_t) \xrightarrow{\supset} (x_t, \theta_t) \xrightarrow{\text{Markov CP}} (u_{t+1}).$$

Under the additional assumption that the cost criterium is expectation of a quadratic form in  $(x, u)$ , the optimal control policy for system (1.a) has been characterized as being of jump linear quadratic Gaussian (JLQG) type (Blair and Sworder, 1975; Morozan, 1979; Chizeck et al., 1986; Ji and Chizeck, 1988). The associated optimal control policy is of the following form:

$$u_{t+1} = F(\theta_t, t)x_t, \quad (2)$$

where the mapping  $F$  is defined by the solution of a system of coupled matrix-Riccati equations; for each element of  $M$  one such Riccati equation. Remarkably, the mapping  $F$  is not influenced by the noise coefficient  $B$ , as the system of coupled matrix-Riccati equations does not involve the noise coefficient  $B$ . If  $A$ ,  $B$  and  $C$  are time-invariant, if the time-horizon is infinite and if the solution is stable, then  $F(\cdot, t)$  is invariant w.r.t.  $t$ . A similar result is known for system (1.b) (Birdwell et al., 1978, 1986).

Our next step is to replace the assumption of receiving complete observations of the plant state  $x_t$  by the assumption of receiving linear Gaussian observations of  $x_t$ . Then the control policy of (2) no longer applies, and we need another approach. The linear Gaussian observations of the plant state  $x_t$  and the complete observations of the system mode  $\theta_t$ , allow a transformation of system (1.a) into a Kalman filter form;

$$\hat{x}_{t+1} = A(\theta_t)\hat{x}_t + K_{t+1}v_{t+1} + C(\theta_t)u_{t+1}, \quad (3)$$

with:  $\hat{x}_t \triangleq E\{x_t | y_t, u_t\}$ ,  
 while  $\{v_t\}$  is an innovation process and  $K_t$  forms the solution of an appropriate matrix Riccati equation that is governed by  $\sigma\{\theta_s; s \leq t\}$ . With this, and a replacement of all terms in the expectation of quadratic cost by conditional expectation terms (see Davis, 1977), it can easily be found that the optimal

control of (3), given  $(\hat{x}_t, \theta_t)$ , is the certainty equivalent (CE) version of the optimal control of (1.a) given  $(x_t, \theta_t)$  (Mariton,

1987b; Chizeck and Ji, 1988). Hence, the optimal control policy consists of Kalman filtering followed by a certainty equivalent (CE) Markov control policy:

$$(y_t, u_t) \xrightarrow{\text{Kalman}} (\theta_t, \hat{x}_t) \xrightarrow{\text{CE Markov CP}} (u_{t+1})$$

The latter simply means that the optimal control satisfies

$$u_{t+1} = F(\theta_t, t) \hat{x}_t, \quad (4)$$

where  $F$  is equivalent to the one in (2). To make the connection with a decision-directed control system explicit, we express (4) as a function of the control mode  $\kappa_t$  and the conditional expectation of  $x_t$ :

$$u_{t+1} = F(\kappa_{t+1}, t) \hat{x}_t, \quad (5.a)$$

which is optimal if

$$\kappa_{t+1} = \theta_t. \quad (5.b)$$

Substitution of (5.a) in (1.a) yields the following decision-directed control system:

$$x_{t+1} = A(\theta_t)x_t + B(\theta_t)w_{t+1} + C(\theta_t)F(\kappa_{t+1}, t)\hat{x}_t, \quad (6)$$

where the assumption of having complete mode observation implied that the optimal control mode  $\{\kappa_t\}$  satisfies (5.b).

A similar JLQG solution has been recovered for the continuous time situation (Krasovskii and Lidskii, 1961; Florentin, 1961; Lidskii, 1963; Wonham, 1970; Sworder, 1969, 1972b; Mariton, 1986, 1987a,b, 1988). Despite all these continuous-time results, their formal derivation involves the solution of some serious mathematical problems, due to the fact that the optimal control  $\{u_t\}$  is singular, i.e. has discontinuities (Rishel, 1975a,b, 1977; Boel and Varaiya, 1977; Davis and Elliott, 1977; Kushner and DiMasi, 1978; Akella and Kumar, 1986).

Changing any of the other JLQG assumptions leads to optimal control policies, of which explicit determination is often very difficult (Sworder, 1976). However, many of these policies still involve a switching of the control mode, as is illustrated by studies of the following additional complications:

- Limited renewal capability (Ratner and Luenberger, 1969; Sworder, 1970; Sworder and Kazangey, 1972; Sworder, 1983),
- Unknown transition probabilities (Pierce and Sworder, 1971; Sworder, 1972a; Ji and Chizeck, 1989),
- State dependent transition probabilities (Sworder and Robinson,

- 1973; Kushner and DiMasi, 1978; Chizeck, 1984; Sreenath and Chizeck, 1988),
- Time-optimal control (Olsder and Suri, 1980),
  - Semi-Markovian  $\{\theta_t\}$  (Sworder, 1980; Mariton, 1989a),
  - Non-linear system equations (Sworder and Rogers, 1983; Sworder and Chou, 1985, 1986),
  - Non-quadratic cost (Akella and Kumar, 1986).

### 3 Control under imperfect mode observations

The assumption of having perfect observations of the process mode  $\{\theta_t\}$  is totally unrealistic in many practical cases. A more realistic assumption is that the observations of the process mode are corrupted by noise. It is clear that, under such an assumption, the derivation of the certainty equivalent Markov control policy of section 2 no longer applies, even when the process mode is not switching. From an abstract point of view, however, there still exists a Markov control policy that is optimal. To show that, we introduce the concept of a superstate. With the superstate of a Markov state we mean the conditional distribution of that Markov state, given the past and present observations (and controls). If the Markov state is hybrid, then we call the superstate hybrid. Instead of evolving in the hybrid space  $R^n \times M$ , a hybrid superstate evolves in the space of real-valued functions on  $R^n \times M$ . The interesting point is that, under quite general conditions, the resulting superstate process is a controlled Markov process (Gertner and Rappaport, 1977; Rishel, 1981, 1986; Hijab, 1983, 1987; Caines and Chen, 1985; Caines, 1988; Borkar, 1989). Hence, upon replacing all terms in the expected cost by their conditional expectations, it can readily be seen that an optimal control of  $x_t$  is equivalent to an optimal Markov control of the superstate. Hence, the optimal control policy can be represented by two subsequent steps; the first step is to determine the hybrid superstate while the second step is to determine the optimal Markov control of that hybrid superstate,  $\mu_{x_t, \theta_t | y_t, u_t}$ :

$$(y_t, u_t) \xrightarrow{\text{Superstate evaluation}} \mu_{x_t, \theta_t | y_t, u_t} \xrightarrow{\text{Superstate CP}} (u_{t+1})$$

This splits the study of optimal control under partial process mode observations into two independent studies:

- Evaluation of the hybrid superstate through determination of the conditional distribution of the hybrid Markov state,
  - Evaluation of the optimal control through determining the optimal Markov control policy for the hybrid superstate.
- Solving either one of these two subproblems is very difficult.

The evaluation of the hybrid superstate is the subject of the following chapters, while the determination of good hybrid superstate control policies will not explicitly be considered. There is little motivation to start a thorough study on the evaluation of a hybrid superstate, while it is not clear that good hybrid superstate control policies might be recovered. To make ourselves easy on that score, we notice that there already exist some interesting control policies which may use the eventual results of a study on evaluating a hybrid superstate. To give a quick review of these policies, we first notice that, under imperfect process mode observations, the control input has in general a dual effect; besides controlling the plant state it may also assist in identifying the process mode. Determining an optimal control with such a dual effect is very complicated because the control input that provides optimal assistance in system identification may differ significantly from the control that is optimal if the present process mode was perfectly identified. (Feldbaum, 1965; Aoki, 1967; Åström, 1970; Bar-Shalom and Tse, 1976). As a result, there only are some marginal results in determining an optimal dual control policy (Bar-Shalom and Sivan, 1969; Tse and Bar-Shalom, 1973; Saridis, 1977; Wenk and Bar-Shalom, 1980; Griffiths and Loparo, 1985; Casiello and Loparo, 1985). Presently, all practically useful control policies are suboptimal, and in general passive in their assistance to identifying the process mode. The latter class of control policies are those of the so called open loop feedback type (Aoki, 1967; Bar-Shalom and Tse, 1976), which means that the control policy is determined under the assumptions that no future observations will be available, although all observations made will be used for estimation.

Three well known examples of, in some way optimal, open loop feedback control policies for system (1.a) are the robust control policy of Ladde and Siljak (1983), the multiple model adaptive control (MMAC) policy (Deshpande et al., 1973; Athans et al, 1977; Hijab, 1986) and the mode switching control (or reconfiguration control) policy; Looze et al., 1984; Basseville, 1986):

- The certainty equivalent version of the robust control policy of Ladde and Siljak (1983) for system (1.a) satisfies

$$u_{t+1} = F'(t) \hat{x}_t, \quad (7)$$

with  $F'(t)$  an optimal feedback matrix.

- The MMAC policy for system (1.a) satisfies

$$u_{t+1} = \sum_{i \in M} \hat{P}_{t,i} F(i,t) \hat{x}_{t,i}, \quad (8)$$



$$\text{where: } \hat{P}_{t,i} \triangleq P\{\theta_t=i | y_t, u_t\},$$

$$\hat{x}_{t,i} \triangleq E\{x_t | y_t, u_t, \theta_t=i\},$$

while  $F$  is equivalent to the one in (2) or (4).

- The mode switching control policy for (1.a) reads:

$$u_{t+1} = F(k_{t+1}, t) \hat{x}_{t, k_{t+1}}, \quad (9.a)$$

$$k_{t+1} = D(k_t, \hat{P}_{t,1}, \dots, \hat{P}_{t,N}), \quad (9.b)$$

where  $F$  is equivalent to the one in (2) or (4), while  $D$  is a decision function mapping  $M \times R^N$  into  $M$ . Preferably, the parametrization of the decision function  $D$  is optimized, which can be done with the help of the theory of optimal mode switching (Doshi, 1981; Lenhart and Belbas, 1983). Since the latter approach is a very difficult one, the practical solution to determining a parametrization of the decision function is largely done through evaluation and experimentation (Kerr, 1983; Chow and Willsky, 1984; Adams and Gross, 1984; Looze et al., 1984; Eterno et al., 1985; Williams and Montgomery, 1985; Loparo et al., 1987a,b; Mariton, 1989b).

We need to evaluate the processes  $\hat{x}_t$ ,  $\hat{P}_{t,i}$  and  $\hat{x}_{t,i}$ , for all  $i \in M$ , to implement the open loop feedback control policies above. This evaluation can be done through mapping the superstate, for each  $t$ , appropriately into  $R^n$ ,  $R^N$  and  $R^n \times M$ , respectively. Of the open loop feedback control policies above, the mode switching control policy is the only one which switches the control mode  $\{k_t\}$ . Since this switching is based on making a decision, mode switching control is of decision-directed type. It has been shown that such a decision-directed control policy may yield a far better control than the robust control policy (Mariton and Bertrand, 1986). Another interesting observation is that the open loop feedback approach to system (1.a) leads to controls that are (passively) adaptive: this always is the case for the MMAC and the mode switching control approaches, while if the estimation of  $\{x_t\}$  is adaptive it also is the case for the robust control approach.

#### 4 Bayesian estimation of hybrid state Markov processes

The series of studies in the following chapters is directed to the evolution of the superstate of a decision-directed control process. For this, a hybrid state controlled Markov process is considered as a general model of a process with decision-directed control. The problem of the evolution of the superstate of a Markov process is a problem of Bayesian estimation, i.e. the

determination of the conditional distribution of the state of a Markov process. In a discrete-time setting, the theory of Bayesian estimation can be found in Jazwinski (1970) or Maybeck (1979, 1982). For completeness, we give a short outline of Bayesian estimation.

Given a discrete index set  $T$ , a Markov process  $\{\xi_t\}$  with state space  $Z$ , an observations process  $\{y_t\}$  with state space  $Y$ , and two arbitrary time points  $v$  and  $s$ , the Bayesian estimation problem is to determine the conditional distribution  $\mu_{\xi_v|y_s}$  under the

following assumptions:

- The initial distribution  $\mu_{\xi_0}(\cdot)$  and the transition measure  $\mu_{\xi_{t+1}|\xi_t}(\cdot|\xi)$  are given for all  $t$  and all  $\xi \in Z$ . Moreover, these measures admit density-mass functions.
- The present observation is conditionally independent of all past observations, given the present state of the Markov process. Moreover, the measure  $\mu_{y_t|\xi_t}(\cdot|\xi)$  is given for all  $\xi$  and  $t$ , and admits a density-mass function.

There are three types of problems in Bayesian estimation of  $\xi_v$  given  $y_s$ ; we call it Bayesian filtering if  $s=v$ , Bayesian prediction if  $s < v$  and Bayesian smoothing if  $s > v$ . Of these types, Bayesian filtering plays the key role.

The Bayesian filtering problem may be presented as a sequence of identical subproblems, each of which consists of evolution of the conditional distribution from one moment in time to the next moment in time, i.e. from  $t$  to  $t+1$ . Moreover, each of these subproblems consists of solving the following two subsequent steps:

- The unconditional evolution of  $\mu_{\xi_t|y_t}(\cdot)$  to  $\mu_{\xi_{t+1}|y_t}(\cdot)$ , which is characterized by a Chapman-Kolmogorov equation.
- The conditional update of  $\mu_{\xi_{t+1}|y_t}(\cdot)$  to  $\mu_{\xi_{t+1}|y_{t+1}}(\cdot)$ , which is characterized by Bayes formula.

It is obvious that the Bayesian prediction problem may be reduced to a Bayesian filtering problem. The simplest approach is to exploit the fact that the conditional distribution of  $\xi_v$  given  $y_s$ , with  $s < v$ , be equivalent to the conditional distribution of  $\xi_v$  given  $y'_v$ , generated by a modified observations process  $\{y'_t\}$  being defined through

$$y'_t(\omega) \stackrel{\Delta}{=} y_t(\omega), \quad t \leq s,$$

$$y'_t(\omega) \stackrel{\Delta}{=} y_s(\omega), \quad t > s.$$

The result of this transformation is that the modified observations process  $\{y'_t\}$  satisfies the assumptions necessary for Bayesian filtering.

During the evaluation of a particular stochastic control policy in practice, we often like to reconstruct the true trajectory of the controlled process. Therefore, it is necessary to estimate the trajectory more accurately than is possible through Bayesian filtering. Such a more accurate estimation can be realized through Bayesian smoothing. It is interesting to notice that there are several approaches to reducing the Bayesian smoothing problem to a Bayesian filtering problem. One approach is to exploit the fact that the conditional distribution of  $x_v$  given  $y_s$ , with  $s > v$ , is equivalent to the conditional distribution of  $x'_s$  given  $y_s$ , if  $\{x'_t\}$  is defined by:

$$x'_t(\omega) \stackrel{\Delta}{=} x_t(\omega), \quad t \leq v,$$

$$x'_t(\omega) \stackrel{\Delta}{=} x_v(\omega), \quad t > v.$$

With this, the Bayesian smoothing problem reduces to one of Bayesian filtering of the pair  $\{x_t, x'_t\}$ .

An intuitively more appealing approach is to exploit the fact that the Markov property is invariant with respect to the time direction. With this, the Bayesian smoothing problem reduces to Bayesian filtering both in forward and in reverse-time direction and merging the two filtering results. An additional complication of this approach is to characterize the Markov process  $\{x_t\}$  in time-reversed direction.

In the following chapters, several aspects of these Bayesian estimation problems will be studied, both in a discrete-time setting and a continuous-time setting. The series of studies starts with Bayesian filtering for discrete-time hybrid state Markov processes, in chapter II. Next, chapter III studies the time-reversal of a discrete-time hybrid state Markov process, with application to smoothing. After these discrete-time results, the study continues in a continuous-time setting. First, in chapter IV, with the modelling of continuous-time hybrid state Markov processes. Next, in chapter V, with continuous-time hybrid state Bayesian filtering. The continuous-time material in chapters IV and V requires some background on the theory of discontinuous stochastic processes, which is provided in the appendices following chapter V.

## 5 Summaries of chapters II through V

## II. The Interacting Multiple Model algorithm.

An important problem in filtering for discrete-time linear systems with Markovian switching coefficients (dynamic Multiple Model systems) is the one of management of hypotheses, which is necessary to limit the computational requirements. A novel approach to hypotheses merging is presented for this problem. The novelty lies in the timing of hypotheses merging. When applied to the problem of filtering for a linear system with Markovian coefficients this yields an elegant way to derive the Interacting Multiple Model (IMM) algorithm. Evaluation of the IMM algorithm makes clear that it performs very well at a relatively low computational load. These results imply a significant change in the practical application of approximate Bayesian filtering for systems with Markovian coefficients.

## III. Time-reversal with application to smoothing

The reversal in time of a stochastic difference equation in a hybrid space, with a Markovian solution, is presented. The reversal is obtained by a martingale approach, which previously led to reverse time forms for stochastic equations with Gauss-Markov or diffusion solutions. The reverse time equations follow from a particular non-canonical martingale decomposition, while the reverse time equations for Gauss-Markov and diffusion solutions followed from the canonical martingale decomposition. The need for this non-canonical decomposition stems from the hybrid state space situation. Moreover, the non-Gaussian discrete time situation leads to reverse time equations that incorporate a Bayesian estimation step. The latter step is carried out for linear systems with Markovian switching coefficients, and the result is shown to provide the solution to the problem of fixed-interval smoothing. For an application of this smoothing approach to a trajectory with sudden manoeuvres, simulation results are given to illustrate the practical use of the reverse time equations obtained.

## IV. Continuous-time hybrid state Markov processes.

A remarkably general subclass of hybrid state Markov processes is formed by the so called Piecewise Deterministic (PD) Markov processes. In contrast to most other hybrid state processes, they exclude diffusion, but include a jump reflecting boundary, a random jump rate and hybrid jumps. With the latter we mean jumps of the Euclidean valued process component that anticipate a simultaneous switching of the discrete-valued process component. As such, they cover a wide variety of non-diffusion Markov processes. Because PD processes are defined in a pathwise way, they provide a framework to study the control of non-diffusion processes along the same lines as that of diffusions.

An important generalization is to include diffusion in PD processes. To that end, we present quasi-left continuous PD Markov processes as pathwise unique solutions of an Itô stochastic differential equation (SDE) on a hybrid space, which is driven by a Poisson random measure. Since such an SDE permits the inclusion of diffusion, this approach leads to a large variety of hybrid state Jump diffusion Markov processes, represented by pathwise unique SDE solutions.

#### V. Filtering of continuous-time hybrid state processes.

In this chapter, we consider the problem of evaluating the conditional distribution of a quasi-left continuous hybrid state Markov process. From a mathematical point of view, the theory for such an evolution is well developed. Unfortunately, for non-trivial hybrid state Markov processes, the dimensionality of a Bayesian filter is too large to allow for straightforward practical application. For applications, we are interested in low-dimensional approximations of the conditional distribution. In this chapter, we develop a particular such approximation (the Interacting Multiple Model filter) for linear systems with randomly switching coefficients, and we show its relation with the discrete-time IMM algorithm.

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## CHAPTER II

## THE INTERACTING MULTIPLE MODEL ALGORITHM

## 1 Introduction

In this chapter we present the Interacting Multiple Model (IMM) algorithm as an approximate Bayesian filter for a linear system with Markovian switching coefficients. For this, we mainly consider the for filtering commonly studied equations,

$$x_t = A(\theta_t)x_{t-1} + B(\theta_t)w_t, \quad t \in \{1, 2, \dots\}, \quad (1)$$

$$y_t = H(\theta_t)x_t + G(\theta_t)v_t, \quad (2)$$

where  $\{y_t\}$  is the observed process taking values in  $\mathbb{R}^m$ ,  $\{\theta_t\}$  is a finite state Markov chain taking values in MCN according to a known initial distribution and a transition matrix  $\Pi$ ,  $\{x_t\}$  assumes values in  $\mathbb{R}^n$ ,  $\{w_t\}$  and  $\{v_t\}$  are sequences of i.i.d. standard Gaussian vectors,  $\{\theta_t\}$ ,  $\{w_t\}$ ,  $\{v_t\}$  and  $x_0$  are independent, while, for every  $\theta$ ,  $G(\theta)G(\theta)^T$  is non-degenerate.

With  $N$  denoting the cardinality of  $M$ , after  $t$  steps in time there are  $N^t$  possible path hypotheses for  $\{\theta_t\}$ . Hence, if we assume that  $x_0$  has a Gaussian density, then the exact Bayesian filter is a Multiple Model (MM) filter with a bank of  $N^t$  Kalman filters, one for each hypothesis, and a system to compute the desired output of the filter. If  $\Pi$  is sparse, some of the  $N^t$  hypotheses have zero probability and can be pruned, which reduces the complexity of the algorithm. But, only if  $\Pi$  is a permutation matrix the complexity does not grow with time. The problem is to avoid the growth of the number of hypotheses in an efficient way.

This hypotheses management problem is also known for several other filtering situations (Pattipati and Sandell, 1983; Chong et al., 1982; Makowski et al., 1984; Mori et al., 1986; Blom, 1986; Stirling, 1987). All these problems have stimulated the development of a large variety of approximation methods during the last two decades, a review of which is given in section 2. For our problem the majority of these are techniques that reduce the number of Gaussian hypotheses, by pruning and/or merging of hypotheses. These algorithms, however, lack good performance at modest computational load in too many situations. In view of this unsatisfactory situation and the practical importance of better solutions, the filtering problem for (1), (2) needed further study.

One item that has not received much attention in the past is the timing of hypotheses reduction. It is common practice to reduce

the number of Gaussian hypotheses immediately after a measurement update. Indeed, on first sight there does not seem to be a better moment. However, in two recent publications, (Blom, 1984b; Andersson, 1985), this point has been exploited to develop, respectively, the so called IMM (Interacting Multiple Model) and AFMM (Adaptive Forgetting through Multiple Models) algorithms. The latter exploits pruning to reduce the number of hypotheses, while the IMM exploits merging. The IMM algorithm was the reason for a further evaluation of the timing of hypotheses reduction. In section 3, a novel approach to hypotheses merging is presented for a dynamic MM situation, which leads to an elegant derivation of the IMM algorithm. In section 4, Monte Carlo simulations are presented to judge the state of the art in filtering for systems with Markovian switching coefficients after the introduction of the IMM algorithm. Next, in section 5, the IMM algorithm is applied to tracking a suddenly accelerating object.

Parts of this chapter appeared in Blom and Bar-Shalom (1988). For an extension of the IMM algorithm to the semi-Markov switching coefficient case see Campo et al. (1988). For studies of the situation that the matrices  $\Pi$ ,  $A$  or  $B$  are partly known, see Tugnait (1982b), Hågglund (1984) and Yang et al. (1989).

## 2 Review of approximations in dynamic MM estimation

To escape from the intractability of the exact solution we may proceed in two ways. We may avoid the problem by approximating the linear system with Markovian coefficients by a system for which finite filtering algorithms exist, or we may follow the more basic path and apply hypotheses management techniques. We briefly review these approaches and their resulting algorithms.

### Model approximations

This path has led to algorithms for approximated models of the system with Markovian coefficients. The most relevant ones are the so called modified Multiple Model (modified MM) algorithm, the Modified Gain Extended Kalman (MGEK) filter (Song and Speyer, 1983) and detection-filtering methods (Willsky et al., 1980, Basseville and Benveniste, 1986; White and Speyer, 1987). The latter stands for the running of one single filter, the parameters and estimates of which are adjusted upon the detection of an additive signal in the filter residue. To apply this to a system with Markovian switching coefficients, that system has to be approximated by a system with additive residue changes that are sequentially detectable (Perriot-Mathonna, 1984; Willsky, 1984). For the modified MM approach we have to adopt the approximation  $\Pi=I$ , i.e. no mode switching, and subsequently we have to bound the weights in the associated MM algorithm from

below (Willsky et al., 1980). To apply the MGEK approach, the Markov coefficients have to be approximated by a Markov process in an Euclidian space, after which the filtering can be done by a MGEK filter (Marcus and Westwood, 1984; Westwood, 1984).

### Hypotheses reduction techniques

These can be classified in: heuristic pruning, unlikely hypotheses pruning, fixed depth hypotheses merging and adaptive hypotheses merging. The application of any of these reduction techniques leads to an algorithm, which consists of a bank of Kalman filters, a hypotheses weights evaluation system and a hypotheses reduction system. Heuristic pruning is the most simple approach; apply at some moments in time some heuristic rule to prune hypotheses. For an overview of the main heuristic pruning rules see Pattipati and Sandell (1983) and Chong, Mori, Tse and Wislner (1982). A well known example is the rule to save only a fixed number,  $N$ , of the most likely hypotheses. Application of this rule immediately after a measurement update yields the class of Detection Estimation (DE) algorithms (Tugnait, 1982a).

The unlikely hypothesis pruning means that pruning is only done for those hypotheses that have a zero probability of becoming the most likely hypothesis in future. Methods using this approach are quite scarce. One such method is based on a Viterbi-like approach (Bruneau, 1982). Unfortunately, the number of unpruned hypotheses may still increase rapidly with time.

Hypotheses merging implies that a weighted sum of two or more Gaussian hypotheses is approximated by one single Gaussian hypothesis. Fixed depth hypotheses merging means that only those hypotheses are merged for which the Markov chain paths are equivalent during the recent past of some fixed depth. The complexity of the resulting algorithms does not grow with time, which makes them useful for real time application. Fixed depth hypotheses merging based algorithms have mainly been developed during the seventies. For an overview of these algorithms see Tugnait (1982a). The most important ones are the so called GPB<sup>k</sup> (Generalized Pseudo Bayes) algorithms. A GPB algorithm with  $N^k$  Kalman filters in its bank preserves after each measurement update the different hypotheses over the last  $(k-1)$  steps in time ( $k \geq 1$ ). For short we write GPB( $k$ ) or GPB $k$ . Performance comparisons of GPB $k$  algorithms of low order ( $k=1,2,3$ ) have been made by Tugnait (1982a), by Marcus and Westwood (1984), by Weiss (1983) and by Smith and Makov (1980). In their examples the GPB2 algorithm performs significantly better than the GPB1 algorithm, and almost as good as the GPB3 algorithm. Adaptive hypotheses merging implies that only those hypotheses are merged which do not differ too much. To quantify such a difference several distance measures have been proposed (See Pattipati and Sandell, 1983, Weiss et al., 1983, and Deacon and Atherton, 1985). A still

open problem is which distance measure to choose in a particular situation. For systems with Markovian coefficients it is useful to separate these adaptive merging techniques into two classes (Weiss, 1983). In the first class the merging, at moment  $t$ , is restricted to Gaussian hypotheses that are equivalent about  $\theta_t$ , while in the second class hypotheses that differ about  $\theta_t$  can be merged. For systems with Markovian coefficients the latter implies in general that the Markov property of the pair  $(x_t, \theta_t)$  is not fully exploited. Presently the proposed distance measures are such that the resulting adaptive merging techniques are mainly of the second type. For systems with Markovian coefficients we need distance measures such that the resulting adaptive merging techniques of the first class become truly available.

### Performance comparison

A general comparison of the different types of algorithms is hampered by the analytical complexity of the problem (Weiss et al., 1983, Washburn et al., 1985). Therefore the available comparisons of the different types of algorithms mainly rely on Monte Carlo simulations for specific examples. In this way the GPB algorithms have been compared both with the DE filters and with the modified MM algorithm and MGEK filter. The references for these comparisons are respectively Tugnait (1982a), Westwood (1984) and Marcus and Westwood (1984). Their main criterion of comparison was the RMS error in the estimate of  $x_t$ . Willsky et al. (1980) compared detection-filtering methods with the modified MM algorithm, where their main criterion was fast decision making. These comparisons indicated that if its complexity can be handled, the GPB2 algorithm is a good choice, as it performs almost optimally. Furthermore it appeared that if the implementation of  $N^2$  Kalman filters is too complex, the best thing left is to accept an often significantly lower performance and to choose either a DE filter with less than  $N^2$  Kalman filters, the GPB1 algorithm, the heuristically modified MM algorithm, the MGEK filter or a detection-filtering method of low enough complexity. The available comparisons indicate that the modified MM algorithm is in general a less good choice, while the other four algorithms rank best in performance for different situations.

In practice, for many applications an algorithm with  $N^2$  Kalman filters is too complex, and we have to choose one of the alternatives. According to the preceding discussion, this often leads to a significantly lower performance, which situation is not satisfactory for many practical applications. The expectation is that the above sketched situation will improve by the further development of the adaptive merging techniques (Weiss et al., 1983). Recently (Blom, 1984b) a new fixed depth merging algorithm (IMM) has been introduced that improves the above situation. Namely, it yields almost the performance of the GPB2 algorithm

for the cost of the GPB1 algorithm. In the next section this IMM algorithm is derived using a novel hypothesis merging approach. The novelty lies in an alternative timing of hypothesis merging.

### 3 Timing of hypotheses reduction

To show the possibilities of timing the hypotheses reduction, we start with a filter cycle from one measurement update up to and including the next measurement update, under the assumption that the conditional distribution of our hybrid state Markov process admits a conditional density-mass function. As such, we consider a cycle of recursions for the evolution of the conditional density-mass of  $(x_t, \theta_t)$ . Globally, this cycle reads as follows:

$$\begin{array}{ccc} p_{x_{t-1}, \theta_{t-1} | y_{t-1}} & \xrightarrow{\text{I}} & p_{x_t, \theta_t | y_{t-1}} \\ p_{x_t, \theta_t | y_{t-1}} & \xrightarrow{\text{II}} & p_{x_t, \theta_t | y_t} \end{array}$$

with  $y_{t-1} = \sigma\{y_s; s \leq t-1\}$ .

Since  $\{x_t, \theta_t\}$  is a Markov process, transition I is specified by a Chapman-Kolmogorov equation;

$$p_{x_t, \theta_t | y_{t-1}}(\cdot) = \int_{\mathbb{R}^n} \sum_i p_{x_t, \theta_t | x_{t-1}, \theta_{t-1}}(\cdot | x, i) \cdot p_{x_{t-1}, \theta_{t-1} | y_{t-1}}(x, i) dx.$$

Since  $x_0, \{\theta_t\}, \{v_t\}$  and  $\{w_t\}$  are independent, a characterization of transition II follows from (2) and Bayes formula;

$$p_{x_t, \theta_t | y_t}(x, \theta) = p_{y_t | x_t, \theta_t, y_{t-1}}(x, \theta) \cdot p_{x_t, \theta_t | y_{t-1}}(x, \theta) / p_{y_t | y_{t-1}}.$$

For output purposes we may use the following equations:

$$\begin{aligned} p_{x_t | y_t}(x) &= \sum_{i \in \mathbb{M}} p_{x_t, \theta_t | y_t}(x, i), \\ p_{\theta_t | y_t}(i) &= \int_{\mathbb{R}^n} p_{x_t, \theta_t | y_t}(x, i) dx, \end{aligned}$$

and, if  $p_{\theta_t | y_t}(i) > 0$ ,

$$p_{x_t | \theta_t, y_t}(x | i) = p_{x_t, \theta_t | y_t}(x, i) / p_{\theta_t | y_t}(i).$$



To allow for a more detailed evaluation of timing hypotheses reduction, we next decompose transitions I and II. The result is a more detailed cycle of recursions;

$$p_{\theta_{t-1}|y_{t-1}} \xrightarrow{\text{Ch.-Kolm.}} p_{\theta_t|y_{t-1}}, \quad (3)$$

if  $p_{\theta_t|y_{t-1}}(i) = 0$  prune hypothesis  $\theta_t = i$ ,

$$p_{x_{t-1}|\theta_{t-1}, y_{t-1}} \xrightarrow{\text{Mixing}} p_{x_{t-1}|\theta_t, y_{t-1}}, \quad (4)$$

$$p_{x_{t-1}|\theta_t, y_{t-1}} \xrightarrow{\text{Evolution}} p_{x_t|\theta_t, y_{t-1}}, \quad (5)$$

$$p_{\theta_t|y_{t-1}} \xrightarrow{\text{Bayes}} p_{\theta_t|y_t}, \quad (6)$$

$$p_{x_t|\theta_t, y_{t-1}} \xrightarrow{\text{Bayes}} p_{x_t|\theta_t, y_t}. \quad (7)$$

Let us take a closer look at the characterization of the detailed cycle above. Characterizations of steps (6) and (7) follow immediately from the Chapman-Kolmogorov equation of transition II and the law of total probability. A characterization of step (5) follows from the evolution of system (1). The Chapman-Kolmogorov equation for the Markov chain,  $\{\theta_t\}$ , characterizes (3);

$$p_{\theta_t|y_{t-1}}(i) = \sum_{j \in \mathcal{M}} \Pi_{ij} p_{\theta_{t-1}|y_{t-1}}(j). \quad (8)$$

The remaining problem is to characterize transition (4). For this we start with the law of total probability,

$$p_{x_{t-1}|\theta_t, y_{t-1}}(x|i) = \sum_{j \in \mathcal{M}} p_{x_{t-1}|\theta_{t-1}, \theta_t, y_{t-1}}(x|j, i) \cdot p_{\theta_{t-1}|\theta_t, y_{t-1}}(j|i). \quad (9)$$

As  $\theta_t$  is conditionally independent of  $x_{t-1}$  given  $\theta_{t-1}$ , we obtain

$$p_{x_{t-1}|\theta_{t-1}, \theta_t, y_{t-1}}(x|j, i) = p_{x_{t-1}|\theta_{t-1}}(x|j), \text{ all } i \in \mathcal{M}. \quad (10)$$

Moreover, since  $\theta_t$  is conditionally independent of  $y_{t-1}$  given  $\theta_{t-1}$ , we obtain

$$p_{\theta_{t-1} | \theta_t, y_{t-1}}(j|i) = \pi_{ij} p_{\theta_{t-1} | y_{t-1}}(j) / p_{\theta_t | y_{t-1}}(i).$$

Substitution of this and (10) in (9) yields the desired characterization of transition (4),

$$p_{x_{t-1} | \theta_t, y_{t-1}}(x|i) = \sum_{j \in M} \pi_{ij} p_{\theta_{t-1} | y_{t-1}}(j) \cdot p_{x_{t-1} | \theta_{t-1}, y_{t-1}}(x|j) / p_{\theta_t | y_{t-1}}(i), \quad (11)$$

which can be seen as a "mixing" of conditional densities of type  $p_{x_{t-1} | \theta_{t-1}, y_{t-1}}$ . Notice that this "mixing" in (11) is explicitly related to the underlined properties of the process  $\{x_t, \theta_t\}$ .

According to the detailed filtering cycle there are, at any moment in time,  $N$  densities on  $\mathbb{R}^n$  and  $N$  scalars. Unfortunately, the densities on  $\mathbb{R}^n$  are rarely Gaussian. Even if the initial densities  $p_{x_0 | \theta_0}(\cdot|i)$ ,  $i \in M$ , are Gaussian, a realization of  $p_{x_t | \theta_t, y_t}(\cdot|i)$  is in general a mixture of  $N^t$  weighted Gaussians. Explicit recursions for the resulting  $N^{t+1}$  individual Gaussians and their weights can simply be obtained from the above filter cycle. Obviously the  $N$  times increase of the number of Gaussians during each filter cycle is caused by (4) only.

In the sequence of elementary transitions, (3) through (7), we can apply a hypotheses reduction either after (4), after (5) or after (7). We review these reduction timing possibilities for the fixed depth merging hypotheses reduction. This fixed depth merging approach implies that the Gaussian hypotheses, for which the Markov chain paths are equivalent during the recent past of some fixed depth, are merged in one moment-matched Gaussian hypothesis. The degrees of freedom in applying this fixed depth merging approach are the choice of the depth,  $d$  ( $\geq 1$ ), and the moment of application; after (4), after (5) or after (7).

If the application is immediately after each measurement update pass, (7), depth  $d \geq 1$  yields the GPB( $d+1$ ) algorithm (Tugnait, 1982a; Weiss, 1983). If the application is immediately after the mixing pass (4), the resulting algorithms are the IMM algorithm, for  $d=1$ , and the GPB( $\geq 2$ ) algorithms, for  $d \geq 2$ . If the application is immediately after pass (5), the resulting algorithms are disguised but more complex versions of the IMM and GPB( $\geq 2$ ) algorithms. In the next section we derive the IMM algorithm by applying the fixed depth merging approach with depth,  $d=1$ , after each pass of (4).

4 The IMM algorithm

Each cycle of the IMM algorithm consists of the following four steps, the first three of which are illustrated in figure 1.

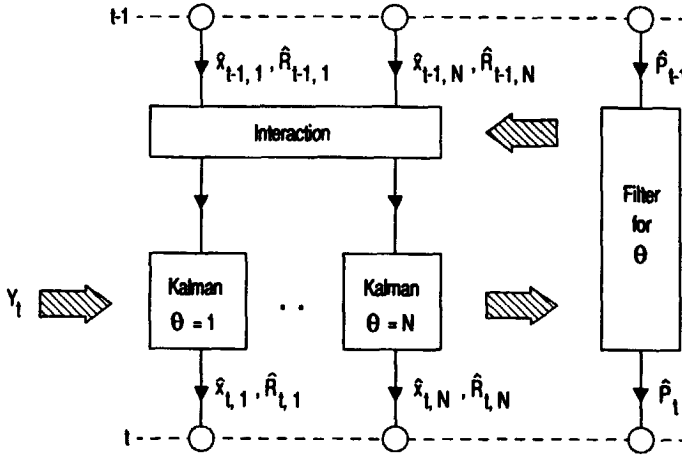


Fig. 1 The IMM algorithm

Step 1

Starting with the N weights  $\hat{P}_{t-1,i} \triangleq p_{\theta_{t-1}=i | y_{t-1}}$  (i), the N means

$\hat{x}_{t-1,i} \triangleq E\{x_{t-1} | \theta_{t-1}=i, y_{t-1}\}$  and the N associated covariances

$\hat{R}_{t-1,i}$ , one computes the mixed initial condition for the filter matched to  $\theta_t=i$ , according to the following equations:

$$\bar{P}_{t,i} = \sum_{j \in M} \pi_{ij} \hat{P}_{t-1,j} \tag{12}$$

if  $\bar{P}_{t,i} = 0$  prune hypothesis  $\theta_t=i$ ,

$$\hat{x}_{t-1,i}^* = \sum_{j \in M} \pi_{ij} \hat{P}_{t-1,j} \hat{x}_{t-1,j} / \bar{P}_{t,i} \tag{13}$$

$$\hat{R}_{t-1,i}^* = \sum_{j \in M} \pi_{ij} \hat{P}_{t-1,j} [\hat{R}_{t-1,j} + [\hat{x}_{t-1,j} - \hat{x}_{t-1,i}^*][\dots]^T] / \bar{P}_{t,i} \tag{14}$$

Step 2

Each of the N pairs  $\hat{x}_{t-1,i}^*, \hat{R}_{t-1,i}^*$  is used as input to a Kalman filter matched to  $\theta_t=i$ . Time-extrapolation yields

$$\bar{x}_{t,i} = A(i) \hat{x}_{t-1,i}^*$$

$$\bar{R}_{t,i} = A(i) \hat{R}_{t-1,i}^* A(i)^T + B(i) B(i)^T,$$

and then, measurement updating yields

$$\hat{x}_{t,i} = \bar{x}_{t,i} + K_{t,i} [y_t - H(i)\bar{x}_{t,i}],$$

$$\hat{R}_{t,i} = \bar{R}_{t,i} - K_{t,i} \bar{R}_{t,i} H(i)^T,$$

with

$$K_{t,i} \triangleq H(i)\bar{R}_{t,i} [H(i)\bar{R}_{t,i}H(i)^T + G(i)G(i)^T]^{-1}.$$

### Step 3

The N weights  $\bar{P}_{t,i}$  are updated from the innovations of the N Kalman filters,

$$\hat{P}_{t,i} = c \cdot \bar{P}_{t,i} \cdot \|Q_{t,i}\|^{-1} \exp\{-\frac{1}{2} \nu_{t,i}^T Q_{t,i}^{-1} \nu_{t,i}\} \quad (15)$$

with c denoting a normalizing constant,

$$\nu_{t,i} = y_t - H(i)\bar{x}_{t,i}, \quad (16)$$

$$Q_{t,i} = H(i)\bar{R}_{t,i}H(i)^T + G(i)G(i)^T. \quad (17)$$

### Step 4

For output purpose,  $\hat{x}_t$  and  $\hat{R}_t$  are computed according to

$$\hat{x}_t = \sum_{i \in M} \hat{P}_{t,i} \hat{x}_{t,i}, \quad (18)$$

$$\hat{R}_t = \sum_{i \in M} \hat{P}_{t,i} [\hat{R}_{t,i} + [\hat{x}_{t,i} \quad -\hat{x}_t][\dots]^T]. \quad (19)$$

Only step 1 is typical for the IMM algorithm. Specifically the mixing represented by equations (11) and (12) and by the Interaction box in figure 1, cannot be found in the GPB algorithms. This is the key novelty of the approach to the timing of fixed depth hypotheses merging that yields the IMM algorithm. We give a derivation of the key Step 1:

Application of fixed depth merging with  $d=1$  implies that,

$$p_{x_{t-1} | \theta_{t-1}, y_{t-1}}(\cdot | i) \sim N(\cdot; \hat{x}_{t-1,i}, \hat{R}_{t-1,i}).$$

Substitution of this in (11) immediately yields (13) and (14),

with  $\hat{x}_{t-1,i} \triangleq E\{x_{t-1} | \theta_{t-1}=i, y_{t-1}\}$  and  $\hat{R}_{t-1,i}$  the associated covariance. Finally we introduce the approximation,

$$p_{x_{t-1} | \theta_t, y_{t-1}}(\cdot | i) \sim N(\cdot; \hat{x}_{t-1,i}, \hat{R}_{t-1,i}),$$

which guarantees that all subsequent IMM steps fit correctly.

Remark 1: In (1),  $\theta_t$  is often replaced by  $\theta_{t-1}$ ; i.e.

$$x_t = A(\theta_{t-1})x_{t-1} + (\theta_{t-1})w_t.$$

This leads to an IMM which executes the Interaction of Step 1 after the Kalman time extrapolations of Step 2 (Blom, 1985).

Remark 2: The GPB1 algorithm can be obtained from the IMM algorithm by introducing the following approximation into the IMM

scheme: Replace  $\hat{x}_{t-1,i}^*$  and  $\hat{R}_{t-1,i}^*$  in step 1 by  $\hat{x}_{t-1}$  and  $\hat{R}_{t-1}$ . Together with (12) this approximates (13) and (14) in step 1 by,

$\hat{x}_{t-1,i}^* \approx \hat{x}_{t-1}$  and  $\hat{R}_{t-1,i}^* \approx \hat{R}_{t-1}$ . These equations are equivalent to (13) and (14) if each component of  $H$  equals  $1/N$ , which implies that  $\{\theta_t\}$  is a sequence of mutually independent stochastic variables. The latter is hardly ever the case and we conclude that the reduction of the IMM to GPB1 leads to a significant performance degradation. Obviously the computational loads of IMM and GPB1 are almost equivalent.

## 5 Performance of the IMM algorithm

At present, a comparison of the different filtering algorithms for systems with Markovian coefficients with respect to their performance is hampered by the analytical complexity of the problem (Weiss, 1983; Washburn et al., 1985). Because of this, such comparisons necessarily rely on Monte Carlo simulations for specific examples. For our simulated examples we used the set of 19 cases that have been developed by Westwood (1984). To make the comparison more precise, we specify these cases and summarize the observed performance results. In all 19 cases both  $\{x_t\}$  and  $\{y_t\}$  are scalar processes, which satisfy,

$$x_t = a(\theta_t)x_{t-1} + b(\theta_t)w_t + u(t),$$

$$y_t = h(\theta_t)x_t + g(\theta_t)v_t,$$

with:  $u(t) = 10 \cdot \cos(2\pi t/100)$ ,  $\theta_t \in \{0,1\}$ ,  $x_0$  a Gaussian variable with expectation 10 and variance 10,  $P\{\theta_0=1\} = P\{\theta_0=0\} = \frac{1}{2}$ , while  $\Pi_{00} = (1-1/\tau_0)$  and  $\Pi_{11} = (1-1/\tau_1)$ . The parameters  $a, b, h, g, \tau_0$  and  $\tau_1$  of these 19 cases are given in Table 1.

The results of Westwood (1984) show that, in all 19 cases the differences in performance of the GPB2 and the GPB3 algorithms are negligible, while in only seven cases (5,6,8,16,17,18,19) the differences in performance of the GPB1 and the GPB2 algorithms are negligible. To our present comparison the other twelve cases (1,2,3,4,7,9,10,11,12,13,14,15) are interesting. For each of these twelve cases we simulated the GPB1, the GPB2 and the IMM algorithms and ran Monte Carlo simulations, consisting of 100 runs from  $t=0$  to  $t=100$ . For simplicity of interpretation of the results we used one fixed path of  $\theta$  during all runs:  $\theta=0$  on the time interval  $[0,30]$ ,  $\theta=1$  on the interval  $[31,60]$  and  $\theta=0$  on the interval  $[61,100]$ .

CASE #	$\tau$ -VALUES		$\theta$ -DEPENDENT VALUES			
	$\tau_0$	$\tau_1$	a(0), a(1)	b(0), b(1)	h(0), h(1)	g(0), g(1)
1	40	20	.995,.990	1.0	1.0	1.0
2	40	20	.995,.990	.5	1.0	.5
3	40	20	.995,.990	.1	1.0	5.0
4	200	100	.995,.990	.1	1.0	5.0
5	40	20	.995,.990	8.0	1.0	1.0
6	40	20	.995,.990	1.0	1.0	.3
7	40	20	.995,.900	.5	1.0	2.0
8	40	20	.995,.750	1.0	1.0	.6
9	40	20	.995	2.0	1.0,.95	.5
10	40	20	.995	1.0	1.0,.80	.2
11	40	20	.995	.5	1.0,.80	.8
12	4	2	.995	.5	1.0,.80	.8
13	200	100	.995	.5	1.0,.80	.8
14	40	20	.995	.1,5.0	1.0	1.0
15	40	20	.995	1.0	1.0	.1,5.0
16	10	2	.95	.5	1.0,0.0	1.0,2.0
17	200	5	.950,0.0	1.0	1.0	1.0
18	50	5	.950,1.2	1.0	1.0	1.0
19	10	2	.95	.5	1.0	1.0,40.0

Table 1 The parameters of the 19 cases of Westwood (1984)

The results of our simulations for the twelve interesting cases are as follows. In six cases (1,2,7,12,14,15) both the IMM and the GPB2 performed slightly better than the GPB1, while the IMM and the GPB2 performed equally well. For typical results see figure 2. In the other six cases both the IMM and the GPB2 performed significantly better than the GPB1. For typical results see figures 3 and 4. Of these six cases the IMM and the GPB2 performed four times equally well (cases 3, 4, 11 and 13) and two times significantly different (cases 9 and 10).

On the basis of these simulations we can conclude that the IMM performs almost as well as the GPB2, while its computational load is about that of GPB1. We can further differentiate this overall conclusion:

- Increasing the parameters  $\tau_0$  and  $\tau_1$  increases the difference in performance between GPB1 and GPB2, but not between IMM and GPB2.
- If a is being switched, then the IMM performs as good as the GPB2, while the GPB1 sometimes stays significantly behind.
- If the white noise gains, b or g, are being switched, then the IMM performs as good as the GPB2, while the GPB1 sometimes stays significantly behind.
- If only h is being switched, then in some cases the IMM and more often the GPB1 tend to diverge while the GPB2 works well.

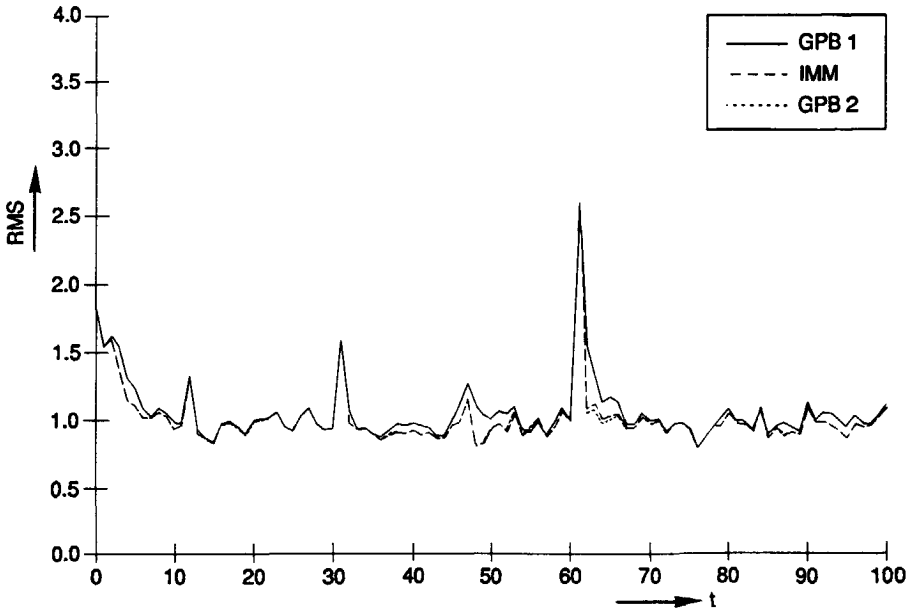


Fig. 2 RMS (Root Mean Square) of the filter errors for case 7, illustrative of the six cases (1,2,7,12,14,15) where both IMM and GPB2 perform slightly better than GPB1

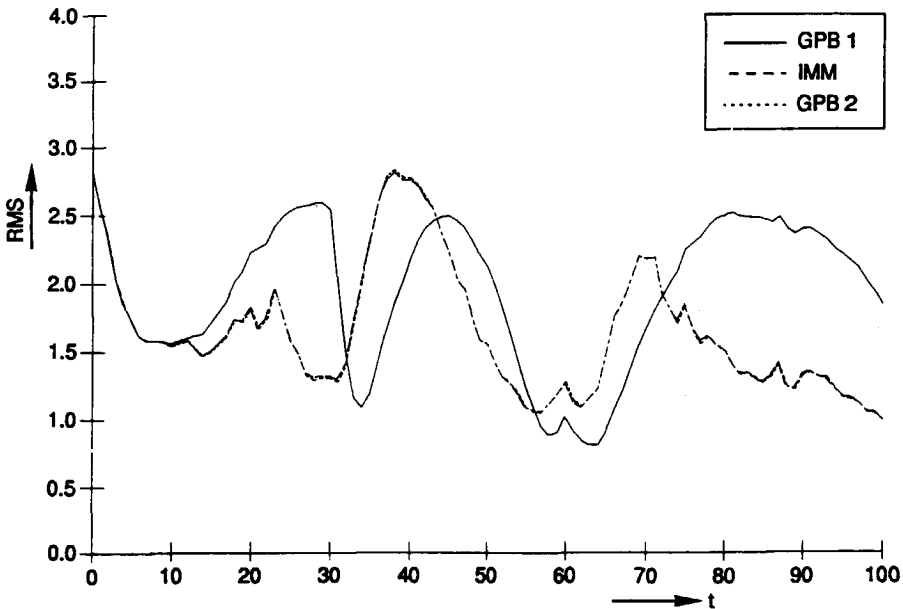


Fig. 3 RMS of the filter errors for case 3, illustrative of the four cases (3,4,11,13) where both IMM and GPB2 perform significantly better than PGB1, while IMM and GPB2 perform equally well

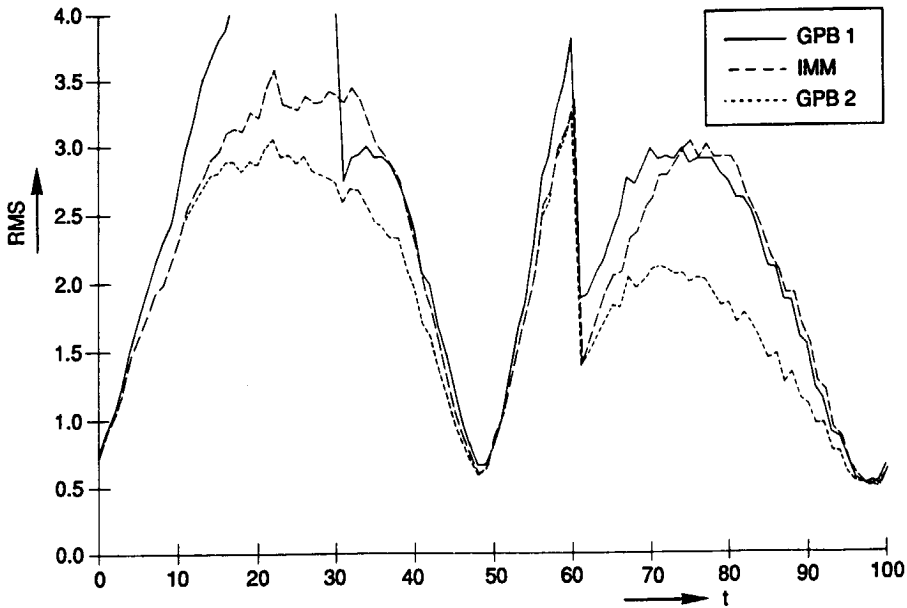


Fig. 4 RMS of the filter errors for case 9, illustrative of the two cases (9 and 10) where IMM performs better than GPB1, but worse than GPB2 (in these two cases only  $h$  switches)

Another interesting question is how the IMM compares with the modified MM algorithm and the MGEK filter. Apart from the GPB algorithms Westwood (1984) also evaluated four more filters, the MM, the modified MM, the MGEK and a MGEK with a "post-processor". For the 19 cases there was only one algorithm that outperformed the GPB1 algorithm in some cases. It was the MGEK filter in the cases 1, 3 and 4. He also found that the MGEK filter performed in these cases marginally or significantly less good than the GPB2 algorithm. As these experiments showed that for cases 1, 3 and 4 the GPB2 and the IMM algorithm performed equally well, one can conclude that the MM, the modified MM, the MGEK, the MGEK with "postprocessor" and the GPB1 are in all 19 cases outperformed by the IMM algorithm.

On the basis of these comparisons one can conclude that for practical filtering applications with  $N=2$ , the IMM algorithm is the best first choice. As the IMM algorithm has been developed on the basis of some general hypotheses reduction principles, one can reasonably well expect that this is also true for larger  $N$ . But it is unlikely that the IMM performs in all applications almost as good as the exact filter. Therefore, if the IMM performs not well enough in a particular application one should consider using a suitable GPB ( $\geq 2$ ) or DE algorithm (Tugnait, 1982), or one might try to design a better algorithm by using adaptive merging techniques (Weiss, 1983). The DE algorithm might



possibly be improved by the novel timing of hypotheses reduction (Andersson, 1985). If for a particular application the performance of the selected algorithm has a too high computational load, than it is best to try to exploit some geometrical structure of the problem considered (Basseville and Benveniste, 1986; Raisch, 1987).

## 6 Filtering sudden manoeuvres

One of the filtering problems for processes with sudden additive jumps, that received considerable attention in the past, is the tracking of a suddenly manoeuvring object. For this problem a large variety of practical filter methods have been developed. If the sudden jumps are approximated by a Gaussian process, the optimal tracker is a Kalman filter. Since a Kalman filter responds too slowly to a sudden acceleration, several other methods have been developed:  $\alpha$ - $\beta$ -tracking, Multiple Model Adaptive Estimation (MMAE) filter (Moose et al., 1979), the Variable Dimension (VD) filter (Bar-Shalom and Birmiwal, 1982) and the Input Estimation (IE) algorithm (Chan et al., 1979; Bogler, 1987). An interesting question is then whether there is an hybrid model and an appropriate IMM algorithm that competes with these methods.

### Manoeuvre tracking with the IMM algorithm

We observe white Gaussian noise corrupted position measurements of an object that undergoes a sudden constant acceleration. For simplicity we restrict ourselves to one dimensional motion.

Our manoeuvring object model is (1) with the following state components: mode  $\theta_t \in \{0,1\}$ , position,  $x_{1t} \in \mathbb{R}$ , speed,  $x_{2t} \in \mathbb{R}$ , potential acceleration,  $x_{3t} \in \mathbb{R}$ , satisfying system (1), with  $\Pi_{00} = (1-1/\tau_0)$ ,  $\Pi_{11} = (1-1/\tau_1)$ ,

$$A(0) = \begin{bmatrix} 1 & \tau & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 1 & \tau & \frac{1}{2}\tau^2 \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{bmatrix},$$

$$B(0) = [0 \ 0 \ 1]^T, \quad B(1) = [0 \ 0 \ 0]^T,$$

and parameters:  $\tau=10$ ,  $\tau_0=1000$  and  $\tau_1=10$ . The true acceleration satisfies:  $\text{acceleration}_t = x_{3t}\theta_t$ .

The observation equation is (2), with  $H(\cdot)=[1 \ 0 \ 0]$  and  $G(\cdot)=100$ . The algorithms considered are respectively the IMM and the GPB2 algorithms, and the Kalman filter for a linear Gaussian system on  $\mathbb{R}^3$  with second order stochastic properties being equivalent to those of position, speed and acceleration in the model above. The latter leads to the following linear Gaussian system:

$$x'_t = A'x'_t + B'w'_t,$$

with:

$$A' = A(1) \text{ Diag}\{1, 1, \Pi_{11}\},$$

$$B' = A(1) \text{ Col}\{0, 0, \sqrt{\beta}\},$$

$$\beta = (1 + \Pi_{11}) / (\tau_0 + \tau_1),$$

and  $\{w'_t\}$  a sequence of i.i.d. standard Gaussian variables.

For the evaluation we implemented a simulator of the switching model above, but with deterministic  $\{\theta_t\}$ ;  $\theta_t=0$  everywhere except on the interval  $(t_B, t_E]$ , where  $\theta_t=1$ . This yields a simulation with uniform motion for  $t \leq t_B$ , constant acceleration of value  $w$  on the interval  $(t_B, t_E]$  and again uniform motion for  $t > t_E$ . On the time interval  $[0, 100]$  we used Monte Carlo simulations of 100 runs with the following three sets of parameters:

- (i)  $w_{t_B} = 1.0$  ,  $t_B = 20$  ,  $t_E = 60$  ,
- (ii)  $w_{t_B} = 0.4$  ,  $t_B = 40$  ,  $t_E = 45$  ,
- (iii)  $w_{t_B} = 0.1$  ,  $t_B = 40$  ,  $t_E = 60$  .

During each Monte Carlo simulation run we measured the RMS of the filter errors in position, speed and acceleration. Some results measured on the interval  $[0, 100]$  are for the IMM algorithm given by Figs. 5 through 8, for Monte Carlo simulations (i), (ii) and (iii). For the Kalman filter and the GPB2 algorithm we present some results for Monte Carlo simulation (i) in Figs. 5 and 7. Because of the Kalman filter's linearity and the similarity between GPB2 and IMM, this covers also the Kalman and GPB2 results measured during Monte Carlo simulations (ii) and (iii).

The differences in performance of the IMM algorithm and the Kalman filter are quite large:

- During uniform motion (filters converged) the RMS errors of the IMM algorithm for position, speed and acceleration are respectively 1.5, 5 and 4 times smaller than those of the Kalman filter.
- During constant acceleration of simulation (i) (filters converged) the RMS errors of the IMM algorithm for position, speed and acceleration are respectively 2.5, 3 and 1.5 times smaller than those of the Kalman filter.
- During simulation (i), both after a transition from uniform motion to constant acceleration or from constant acceleration to uniform motion, the IMM algorithm converges about two times faster than the Kalman filter.
- The IMM algorithm's estimate of the covariance of its errors in position, speed and acceleration is remarkably consistent; a property completely lacking in case of the Kalman filter.

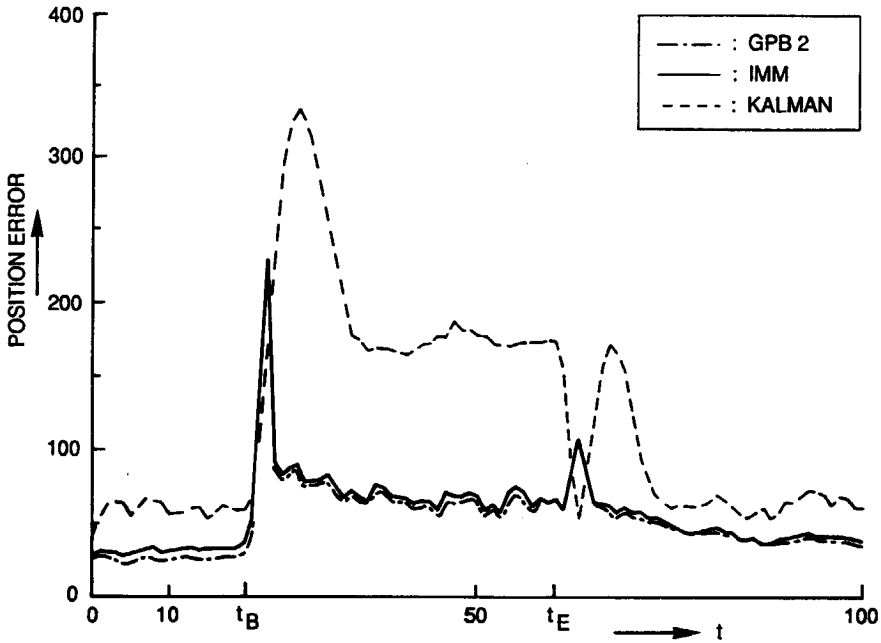


Fig. 5 RMS of the Kalman, IMM and GPB2 filter errors in position, measured during type (i) simulations of a suddenly accelerating object

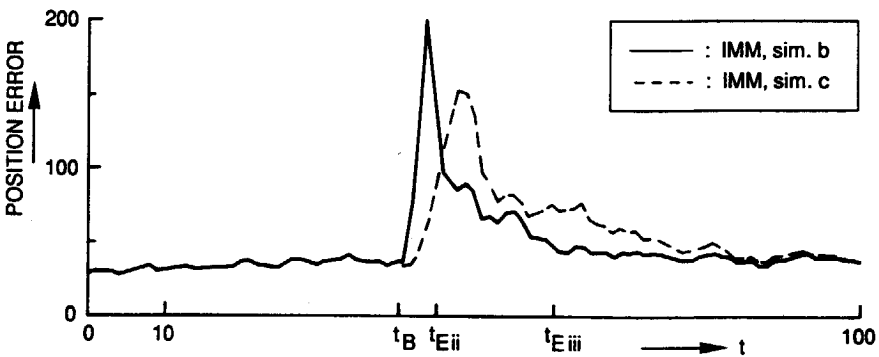


Fig. 6 RMS of the IMM filter errors in position, measured during type (ii) and (iii) simulations of a suddenly accelerating object

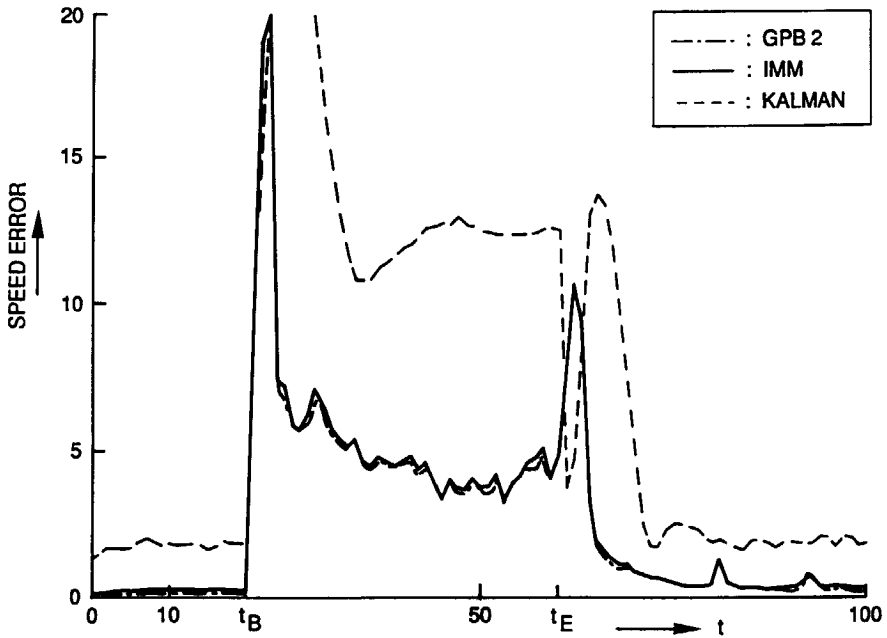


Fig. 7 RMS of the Kalman, IMM and GPB2 filter errors in speed, measured during type (i) simulations of a suddenly accelerating object

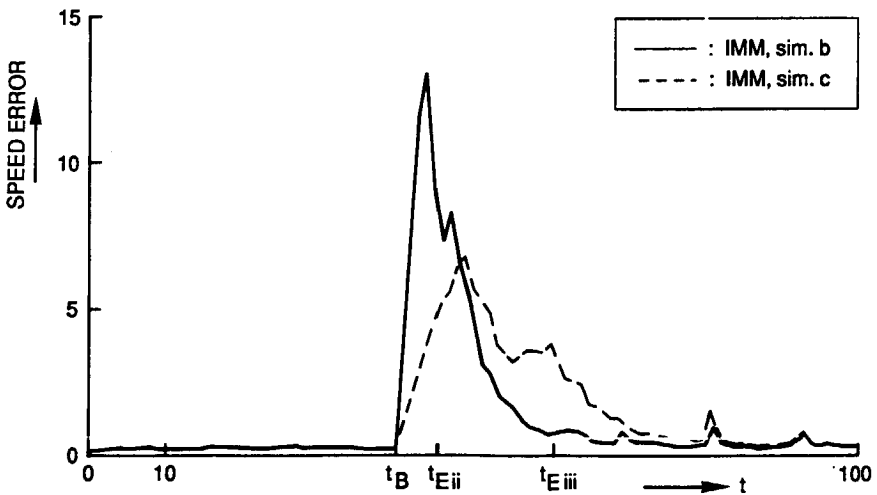


Fig. 8 RMS of the IMM filter errors in speed, measured during type (ii) and (iii) simulations of a suddenly accelerating object

The overall conclusion is that in this example the IMM algorithm performs 1.5 up to 5 times better than the Kalman filter. To complete the picture we also checked the processing load. In this example ( $N=2$ ) the IMM algorithm uses about 2.2 times the CPU- and memory-load of the Kalman filter. On the basis of this evaluation it can be concluded that for this example the IMM algorithm is at least as efficient as the Kalman filter; for the IMM algorithm the average increase in performance is slightly higher than the increase in computational load. For an approximated Bayesian method this is a very good ratio.

The differences in performance we found between the IMM and the GPB2 are very small (See figures 5 and 7). The responses of IMM and GPB2 on a jump are the same. The largest difference occurs during uniform motion, where the GPB2 reaches slightly lower RMS errors than the IMM. These improvements are respectively about 15% in position and about 25% in speed. In view of the IMM performing close to GPB2, we also may expect that IMM performs close to the exact Bayesian filter. Finally we ran some additional Monte Carlo simulations to confirm that the IMM and the GPB2 algorithms are significantly less sensitive to the setting of the jump parameters,  $\tau_0$  and  $\tau_1$ , than the Kalman filter for the linear Gaussian approximated model.

#### Comparison with other manoeuvre tracking methods

As remarked, a large variety of other filter methods have been developed, such as  $\alpha$ - $\beta$  tracking, the Multiple Model Adaptive Estimation (MMAE) filter (Moose et al., 1979), the Variable Dimension (VD) filter (Bar-Shalom and Birmiwal, 1982) and the Input Estimation (IE) algorithm (Chan et al., 1979; Bogler, 1987). Recent overviews of these methods are given by Woolfson (1985) and Bogler (1987). Woolfson compares two Kalman filters, the  $\alpha$ - $\beta$  tracker, the MMAE filter and the VD filter and concludes that the VD filter yields the best overall performance, at the cost of  $\sim 10$  times higher computational requirement than that of the  $\alpha$ - $\beta$  tracker. Bogler (1987) shows that a proper IE algorithm performs better than the VD filter, at the cost of  $\sim 100$  times higher computational load than that of the  $\alpha$ - $\beta$  tracker. Recently Bar-Shalom et al. (1988) have shown that the IMM algorithm performs as good as or better than a proper IE algorithm, while the computational requirements of IMM are far smaller.

Real tracking models are more complicated than the examples above. Apart of the sudden starting and stopping of several kinds of accelerations there are other complications, such as the nonlinear (continuous-time) aircraft dynamics and the ambiguity of measurement origin. As such, for an Air Traffic Control surveillance application we implemented a continuous-discrete IMM

algorithm (see chapter V, section 5) with four extended Kalman-Bucy filters in its bank and cooperating with the so called Probabilistic Data Association (PDA) for MM filters (Kuilder, 1981; Gauvrit, 1984; Houles and Bar-Shalom, 1987). The resulting IMM-PDA based trackers outperform state-of-the-art tracking systems (Blom, 1984a; Bar-Shalom, 1989). The conclusion is that the IMM algorithm can very successfully be combined with approximated Bayesian methods that have been developed to cope with other tracking complications.

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## CHAPTER III

## TIME-REVERSAL WITH APPLICATION TO SMOOTHING

## 1 Introduction

This chapter addresses the problem of time-reversion of a hybrid state Markov process which is given as the solution of a stochastic difference equation. The desired result is a similar equation but running in reverse-time direction while having a solution that is respectively pathwise and in probability law equivalent to the solution of the forward equation.

The motivation to study this problem stems from two different kinds of application. The first is to approach the solution of a Bayesian smoothing problem by a merging of the estimates of two Bayesian filters: one filter matches the original model and is applied in the usual time direction while the other filter matches the time-reversed model and is applied in the reverse-time direction (e.g. Anderson and Rhodes, 1983; Bagchi, 1986). The second application is the determination of a particular error lower bound for discrete-time Bayesian filtering, by the method of Galdos. This method requires a Monte Carlo simulation in reverse-time direction of model matching trajectories, starting from a prespecified end point (Galdos, 1981; Washburn et al., 1985). For both of these applications it is necessary to have a reverse-time difference equation for which the Markovian solution is in probability law equivalent to the original solution.

Our problem falls in the category of how to reverse a Markov process in time. The Markov property implies that the past and the future are independent under the condition that the present state is known (Wentzell, 1981). This invariance with respect to the time direction is the key property used in time-reversion studies. There are two types of studies that deal with this problem; a classical type and a systems-type. The classical type of study assumes that the transition measure or the generator of a Markov process is given and then tries to characterize the transition measure in reverse-time direction (Nagasawa, 1964; Kunita and Watanabe, 1966; Chung and Walsh, 1969; Azéma, 1973; Hasegawa, 1976; Dynkin, 1978; Kelly, 1979; D. Williams, 1979; R.J. Williams, 1988). The systems-type of study assumes that a stochastic equation with a Markovian solution is given for which it tries to characterize the time-reversed equation. The first time-reversed equations were obtained by orthogonality arguments, for the linear Gaussian situation (Ljung and Kailath, 1976; Lainiotis, 1976). For general diffusions, it has already been pointed out by Stratonovich (1960) how to obtain their

time-reversed equations by actually following the classical approach: from a stochastic equation via the generator and the time-reversed generator to the time-reversed equation.

A truly systems-type of study has been started by Verghese and Kailath (1979), by showing how for a linear Gaussian system a more direct martingale approach leads in a simpler way to time-reversed equations. Moreover, by this approach it was possible to obtain a reverse-time equation with a pathwise equivalent solution. Early elaborations of these ideas led, along different routes, to reverse-time equations with pathwise equivalent solutions (Anderson, 1982; Castanon, 1982; Pardoux, 1983). During subsequent studies, quite large classes of stochastic differential equations and their time-reversed equations have been identified (Elliott and Anderson, 1985; Pardoux, 1985; Elliott, 1986a, 1986b; Haussmann and Pardoux, 1986; Pardoux, 1986). Recently, some of these classes have been extended by using a Girsanov transformation of Brownian motion (Picard, 1986; Protter, 1987) and by using a stochastic calculus of variations technique (Millet et al., 1989).

From a constructive point of view, the martingale approach is the most appealing approach to reversing diffusions in time. The martingale approach roughly consists of checking if the time-reversed driving noise sequence can be decomposed in a suitable reverse-time martingale part and its complement and next, if such a decomposition exists (Jacod and Shiryaev, 1987; Jacod and Protter, 1988), selecting such a decomposition. The final step is to characterize both the martingale part and its complement. Unfortunately, there is an additional problem in using this martingale approach to the reversion of an equation with a solution that is not continuous, since a martingale decomposition is then not unique. This makes the selection of a suitable martingale decomposition far from trivial in the hybrid state space situation, because a worse choice yields unnecessarily complicated time-reversed equations. This complication is unsolved at present, both in continuous-time and in discrete-time. It will be solved in the sequel for quite general difference equations in a hybrid space. With that result we subsequently reverse the considered equation in time.

The chapter is organized as follows. In section 2 we define the hybrid state stochastic difference equation that will be considered and briefly compare its time-reversion with the time-reversion of a linear Gaussian equation. In section 3 we specify the time-reversion requirements. Next, in sections 4 and 5 we consider, respectively, the pathwise time-reversion and the in probability law equivalent time-reversion. In section 6 we make the results obtained more specific for a linear system with

Markovian switching coefficients. In section 7 we apply the time-reversion results to fixed-interval smoothing of a jump linear system, and show results for a trajectory with sudden manoeuvres.

This chapter partly appeared in Blom and Bar-Shalom (1989) and will appear in the IEEE Tr. on Information Theory (Vol.36, 1990).

## 2 The stochastic difference equation considered

The stochastic difference equation we consider in the sequel is the following system, on an appropriate stochastic basis

$(\Omega, \mathcal{F}, P, T)$  with  $T = N \cap [0, T]$ ,  $T < \infty$ ,

$$x_{t+1} = a(\theta_{t+1}, \theta_t, x_t, w_t), \quad (1.a)$$

$$\theta_{t+1} = b(\theta_t, v_t), \quad (1.b)$$

$$y_t = c(\theta_t, x_t, w_t, u_t), \quad (1.c)$$

where  $\{w_t\}$ ,  $\{u_t\}$  and  $\{v_t\}$  are i.i.d. standard Gaussian sequences of dimension  $p$ ,  $q$  and  $1$  respectively, the initial distribution of  $(x_0, \theta_0)$  admits the density-mass function  $p_{x_0, \theta_0}$ , and  $\{w_t, v_t, u_t\}$

is independent of  $(x_0, \theta_0)$ . Further  $x_t$ ,  $\theta_t$  and  $y_t$  have respectively  $R^n$ -,  $M$ - and  $R^m$ -valued realizations (with  $M$  a countable set), while  $a$ ,  $b$  and  $c$  are measurable mappings of appropriate dimensions such that system (1) has a unique solution for each initial  $(x_0, \theta_0)$  with  $p_{x_0, \theta_0}(x_0, \theta_0) \neq 0$ . The mappings  $a$ ,  $b$  and  $c$  are time-invariant for notational simplicity only.

In the model above, the pair  $(x_t, \theta_t)$  represents the hybrid system state, while  $y_t$  represents the measurement. For such hybrid system models, the second order dependence of (1.a) on  $\{\theta_t\}$  is quite uncommon (Blom, 1986). Obviously, (1.a) reduces to the more common situation of first order dependence, only if  $a(\theta, \eta, \dots)$  is invariant w.r.t. either  $\theta$  or  $\eta$ . The interpretation of (1.a) as an equation with a second order dependence on  $\{\theta_t\}$

suggests the substitution of  $\theta_{t+1}^* = \Delta(\theta_{t+1}, \theta_t)$  in (1.a). On doing this (1.a) reduces to the more common equation, and it follows immediately that  $\{\theta_t^*\}$  and  $\{\theta_t^*, x_t\}$  are Markov processes. However, as the state space of  $\theta_t^*$  is significantly larger than the state space of  $\theta_t$ , this is a rather brute force transformation of (1.a). A more elegant transformation of (1.a) to the more common equation consists of substituting (1.b) in (1.a), which yields an equation of the following form,

$$x_{t+1} = a'(\theta_t, x_t, w_t, v_t).$$

Instead of a state space expansion, there appears an additional noise term,  $v_t$ . From the latter representation, it follows immediately that the processes  $\{\theta_t, x_t\}$  and  $\{\theta_t\}$  are Markov processes. The latter transformation shows that  $\{x_t\}$  has first order dependence of  $\{\theta_t\}$  only if  $a'(\dots, v)$  is  $v$ -invariant.

Hence, (1.a) is indeed more general than the more commonly studied equation with first order dependence of  $\{\theta_t\}$ . With the study of this more general equation, we also anticipate the time-reversion results obtained; in the sequel it will turn out that a time-reversed equation of (1.a) has, in general, a second order dependence on the time-reversed  $\{\theta_t\}$ , even when  $a(\theta, \eta, \dots)$  is  $\eta$ -invariant. In view of this, it is natural to study the more general form described above.

In the sequel we consider the time-reversion of system (1) under the following assumptions:

A.1

$a(\theta, \eta, \dots, w)$  has an inverse  $a^*: M^2 \times R^n \times R^p \rightarrow R^n$ , such that for any  $(\theta, \eta, w) \in M^2 \times R^p$ ,

$$a^*(\theta, \eta, a(\theta, \eta, x, w), w) = x; \quad \text{all } x \in R^n. \quad (2)$$

A.2

$b(\cdot, v)$  has an inverse  $b^*: M \times R \rightarrow M$ , such that for any  $v \in R$ ,

$$b^*(b(\theta, v), v) = \theta; \quad \text{all } \theta \in M. \quad (3)$$

Assumptions A.1 and A.2 suggest the transformation of (1.a,b,c) to the following time-reversed model,

$$\begin{aligned} x_t &= a^*(\theta_{t+1}, \theta_t, x_{t+1}, w_t), \\ \theta_t &= b^*(\theta_{t+1}, v_t), \\ y_t &= c(\theta_t, x_t, w_t, u_t). \end{aligned}$$

Because  $(w_t, v_t)$  and the future (= reverse-time past),  $\mathcal{F}_{t+1} = \sigma\{(Y_s, X_s, \theta_s); s \in [t+1, T]\}$ , are dependent, this is not the time-reversed system we should look for. Unfortunately, it is not clear how to continue from here. To develop some insight, we take a quick look at the time-reversion of a linear Gaussian system.

Linear Gaussian example

As a special case of (1.a) consider the following linear Gaussian system

$$x_{t+1} = Ax_t + Bw_t.$$

Assumption A.1 implies that  $A$  is invertible, by which

$$x_t = A^{-1} [x_{t+1} - Bw_t].$$

Obviously  $w_t$  and the future  $\mathcal{F}_{t+1}$  are dependent, which requires a martingale decomposition of  $w_t$ . In this linear Gaussian case the canonical martingale decomposition is the appropriate one. It consists of decomposing  $w_t$  in its reverse-time predictable part,  $E\{w_t | \mathcal{F}_{t+1}\}$ , and its complement  $w_t^*$ ;

$$w_t = E\{w_t | \mathcal{F}_{t+1}\} + w_t^*.$$

The problem is now to write the predictable part as a function of  $x_{t+1}$  (if possible) and to characterize the covariance of  $w_t^*$ . As pointed out by Verghese and Kailath (1979) it follows readily from orthogonality arguments that

$$E\{w_t | \mathcal{F}_{t+1}\} = E\{w_t | x_{t+1}\},$$

while the fundamental formula for LLSE estimation yields

$$\begin{aligned} E\{w_t | x_{t+1}\} &= B^T R^{-1}(t+1)x_{t+1}, \\ \text{Cov}\{w_t^* | x_{t+1}\} &= I - B^T R^{-1}(t+1)B, \end{aligned}$$

where  $R(t+1)$  is the covariance of  $x_{t+1}$ , which must be invertible. By a straightforward substitution of these results we obtain

$$x_t = A^{-1} [x_{t+1} - B B^T R^{-1}(t+1)x_{t+1} - B w_t^*],$$

which yields the desired reverse-time system:

$$\tilde{x}_t = A^{-1} [\tilde{x}_{t+1} - B B^T R^{-1}(t+1)\tilde{x}_{t+1} - B\tilde{w}_t].$$

The orthogonality arguments and the LLSE estimation step, used in the above procedure, prevent a straightforward extension of that procedure to equation (1).

In the sequel we replace the orthogonality arguments and the LLSE estimation step respectively by Markov duality arguments and a Bayesian estimation step. Besides this, we have to select an appropriate martingale decomposition. Following the linear Gaussian case, the canonical martingale decomposition seems a good candidate:

$$(w_t, v_t) = (w_t^*, v_t^*) + E\{(w_t, v_t) | \mathcal{F}_{t+1}\}.$$

Unfortunately, this decomposition leads to very complicated elaborations of the Bayesian estimation step. To avoid these complications, we use the following decomposition:

$$(w_t^*, v_t^*) = (w_t, v_t) - (\hat{w}_t, \hat{v}_t),$$

with:  $\hat{v}_t \stackrel{\Delta}{=} E\{v_t | \mathcal{F}_{t+1}\}$  and  $\hat{w}_t \stackrel{\Delta}{=} E\{w_t | \mathcal{F}_{t+1}, v_t\}$ .

The main step, that must be carried out, is to prove that the latter is a martingale decomposition, and to elaborate on the Bayesian estimation step. For the presentation of these results a constructive approach is taken, starting with a precise description of the time-reversion objectives.

### 3 Time-reversion objectives

We want to obtain a time-reversed version of system (1), such

that its solution,  $\{\tilde{y}_t, \tilde{x}_t, \tilde{\theta}_t\}$ , is in some sense equivalent to  $\{y_t, x_t, \theta_t\}$ . To make this objective explicit it needs both a specification of what we mean by a time-reversion of (1), and a specification of the desired sense of process equivalence.

By a reverse-time system we mean a stochastic difference equation which starts at time  $T$  and runs in reverse time direction on the interval  $[0, T]$ . We require from a time-reversion of system (1) that it does not change the state space and that the solution of the resulting reverse-time system represents a process

$\{\tilde{y}_t, \tilde{x}_t, \tilde{\theta}_t\}$ . More specifically,  $\{\tilde{y}_t, \tilde{x}_t, \tilde{\theta}_t\}$  must be the solution of

the following system of stochastic difference equations:

$$\tilde{x}_t = \tilde{a}(t, \tilde{\theta}_{t+1}, \tilde{\theta}_t, \tilde{x}_{t+1}, \tilde{w}_t), \quad (4.a)$$

$$\tilde{\theta}_t = \tilde{b}(t, \tilde{\theta}_{t+1}, \tilde{x}_{t+1}, \tilde{v}_t), \quad (4.b)$$

$$\tilde{y}_t = \tilde{c}(t, \tilde{\theta}_{t+1}, \tilde{\theta}_t, \tilde{x}_{t+1}, \tilde{x}_t, \tilde{w}_t, u_t); \quad \text{all } t \in [0, T-1], \quad (4.c)$$

where  $\tilde{a}$ ,  $\tilde{b}$  and  $\tilde{c}$  are deterministic mappings of appropriate

dimensions and  $(\tilde{w}_t, \tilde{v}_t)$  is a noise sequence to be specified. For a better understanding of (4), notice that the substitutions of (4.a) in (4.c) and of (4.b) in (4.a,c) transform (4) to a reverse-time system of the more common form:

$$\tilde{x}_t = \tilde{a}(t, \tilde{\theta}_{t+1}, \tilde{x}_{t+1}, \tilde{w}_t, \tilde{v}_t),$$

$$\tilde{\theta}_t = \tilde{b}(t, \tilde{\theta}_{t+1}, \tilde{x}_{t+1}, \tilde{v}_t),$$

$$\tilde{y}_t = \tilde{c}(t, \tilde{\theta}_{t+1}, \tilde{x}_{t+1}, \tilde{w}_t, \tilde{v}_t, u_t); \quad \text{all } t \in [0, T-1].$$

To be a useful reverse-time system,  $(\tilde{w}_t, \tilde{v}_t)$  should, as much as possible, be independent of the future (= reverse-time past) information

$$\tilde{\mathcal{F}}_{t+1} = \sigma\{(\tilde{y}_s, \tilde{x}_s, \tilde{\theta}_s, \tilde{w}_s, \tilde{v}_s, u_s); s \in [t+1, T]\}.$$

A minimal requirement is then, that the conditional expectation of  $(\tilde{w}_t, \tilde{v}_t)$ , given  $\tilde{\mathcal{F}}_{t+1}$ , should be zero. Since  $\{\tilde{\mathcal{F}}_t; t \in [0, T]\}$  is a decreasing sequence of  $\sigma$ -algebras, the latter can most easily be put in martingale language :

$(\tilde{w}_t, \tilde{v}_t)$  in (4) should be a reverse-time Martingale Difference (MD) sequence w.r.t.  $\{\tilde{\mathcal{F}}_t\}$  (see Elliott, 1982; Kumar and Varaiya, 1986; and the definitions below).

### 3.1 Definition

Let  $\{\beta_t\} = \{\beta_t; t \in [0, T]\}$  be an increasing sequence of  $\sigma$ -algebras; i.e.  $\beta_{s-1} \subset \beta_s$ , for any  $s \in [1, T]$ .

A random sequence  $\{\xi_t\}$  is said to be a Martingale Difference sequence w.r.t.  $\{\beta_t\}$  if, for all  $t \in [0, T]$ ,

- (i)  $\xi_t$  is  $\beta_t$ -measurable,
- (ii)  $E\{|\xi_t|\} < \infty$ ,
- (iii)  $E\{\xi_t | \beta_s\} = 0$  a.s.; for all  $s \in [0, t-1]$ .

### 3.2 Definition

Let  $\{\mathcal{F}_t\} = \{\mathcal{F}_t; t \in [0, T]\}$  be a decreasing sequence of  $\sigma$ -algebras; i.e.  $\mathcal{F}_s \subset \mathcal{F}_{s-1}$ , for any  $s \in [1, T]$ .

A random sequence  $\{\xi_t\}$  is said to be a reverse-time Martingale Difference sequence w.r.t.  $\{\mathcal{F}_t\}$  if, for all  $t \in [0, T]$ ,

- (i)  $\xi_t$  is  $\mathcal{F}_t$ -measurable,
- (ii)  $E\{|\xi_t|\} < \infty$ ,
- (iii)  $E\{\xi_t | \mathcal{F}_s\} = 0$  a.s. ; for all  $s \in [t+1, T]$ .

Having specified the desired type of reverse-time system, next we specify the types of equivalence of solutions of systems (1) and (4), in which we are interested. For stochastic processes several useful types of equivalence have been defined and named in the past. We restrict ourselves to the two most important types of equivalence and their unambiguous names:

- strict sense pathwise equivalent (indistinguishable),
- same finite-dimensional distributions (equivalent in law).

Two processes  $\{\xi_t\}$  and  $\{\tilde{\xi}_t\}$ ,  $t \in [0, T]$ , are said to be indistinguishable if they are defined on the same probability space  $(\Omega, \mathcal{F}, P)$  and

$$P\{\xi_t = \tilde{\xi}_t, \text{ all } t \in [0, T]\} = 1. \quad (5)$$

For discrete-time processes (5) is satisfied if and only if, for all  $t \in [0, T]$ ,  $\xi_t = \tilde{\xi}_t$  almost surely.

Two processes  $\{\xi_t\}$  and  $\{\tilde{\xi}_t\}$ ,  $t \in [0, T]$ , are said to be equivalent in law, if they have the same state space,  $E$ , and for all  $0 \leq t_1 < t_2 < \dots < t_k \leq T$ ,

$$P\{(\xi_{t_1}, \dots, \xi_{t_k}) \in dX\} = P\{(\tilde{\xi}_{t_1}, \dots, \tilde{\xi}_{t_k}) \in dX\}, \quad (6)$$

for any  $k$  and all measurable  $dX \in E^k$ .

Our objective in the sequel is to obtain time-reversed systems of type (4), with solutions that are respectively indistinguishable and equivalent in law w.r.t. the solution of (1). As two indistinguishable processes are equivalent in law, in practice it is easier to demonstrate equivalence in law than to demonstrate indistinguishability.

#### 4 Indistinguishable time reversion

In this section we derive a type (4) version of system (1), such that their solutions,  $\{\tilde{y}_t, \tilde{x}_t, \tilde{\theta}_t\}$  and  $\{y_t, x_t, \theta_t\}$ , are indistinguishable, and illustrate these results for a jump-linear example. The first step of our derivation consists of using A.1 and A.2, to transform (1) to the in section 2 discussed time-reversed form,

$$x_t = a^*(\theta_{t+1}, \theta_t, x_{t+1}, w_t), \quad (7.a)$$

$$\theta_t = b^*(\theta_{t+1}, v_t), \quad (7.b)$$

$$y_t = c(\theta_t, x_t, w_t, u_t). \quad (7.c)$$

Although (7) and (4) look similar, one requirement is not met:



the driving noise in (7) is not a reverse-time Martingale Difference sequence w.r.t.  $\{\mathcal{F}_t\}$ , with

$$\mathcal{F}_t = \sigma\{(Y_s, x_s, \theta_s, w_s, v_s, u_s); s \in [t, T]\}. \quad (8)$$

Therefore our next step is to introduce a particular reverse-time Martingale Difference sequence,  $\{w_t^*, v_t^*\}$ , as follows,

$$(w_t^*, v_t^*) = (w_t, v_t) - (\hat{w}_t, \hat{v}_t), \quad (w_T^*, v_T^*) = 0, \quad (9)$$

$$\text{with: } \hat{v}_t = E\{v_t | \mathcal{F}_{t+1}\},$$

$$\hat{w}_t = E\{w_t | \mathcal{F}_{t+1}, v_t\}; \quad \text{all } t \in [0, T-1].$$

Notice that the definition of  $\hat{w}_t$  differs significantly from the reverse-time predictable process  $E\{w_t | \mathcal{F}_{t+1}\}$ . As such the decomposition in (9) is not the unique canonical decomposition (see Appendix A5). The introduction of this non-canonical decomposition is a crucial step necessary for obtaining the time-reversion of hybrid state system (1).

In the sequel we verify that  $\{w_t^*, v_t^*\}$  is indeed a reverse-time Martingale Difference sequence w.r.t.  $\{\mathcal{F}_t\}$ , and thus also w.r.t.  $\{\mathcal{F}_t^*\}$ , with  $\mathcal{F}_t^* = \sigma\{\mathcal{F}_t \cup \sigma\{(w_s^*, v_s^*); s \in [t, T]\}\}$ . Moreover we show

that, due to the duality of the Markov property,  $(\hat{w}_t, \hat{v}_t)$  is conditionally independent of  $\mathcal{F}_{t+2}$  given  $(x_{t+1}, \theta_{t+1})$ .

#### 4.1 Theorem

Let  $\{x_t, \theta_t\}$  and  $\{w_t^*, v_t^*\}$  satisfy (1) and (9), and let the assumptions formulated after (1) be satisfied. Then  $\{w_t^*, v_t^*\}$  is a reverse-time Martingale difference sequence w.r.t.  $\{\mathcal{F}_t^*\}$ , while

$\hat{w}_t$  and  $\hat{v}_t$  satisfy:

$$\hat{w}_t = E\{w_t | \theta_{t+1}, \theta_t, x_{t+1}\}, \quad (10.a)$$

$$\hat{v}_t = E\{v_t | \theta_{t+1}, x_{t+1}\}, \quad \text{all } t \in [0, T-1]. \quad (10.b)$$

#### Proof:

Since  $\{w_t, v_t\}$  satisfies (i) and (ii) of definition 3.2, also  $\{w_t^*, v_t^*\}$  satisfies these properties. Furthermore, for any  $s \geq t+1$ ,

$$\begin{aligned} E\{w_t^* | \mathcal{F}_s^*\} &= E\{w_t - \hat{w}_t | \mathcal{F}_s^*\} = E\{w_t - E\{w_t | \mathcal{F}_{t+1}, v_t\} | \mathcal{F}_s^*\} = \\ &= E\{w_t | \mathcal{F}_s^*\} - E\{w_t | \mathcal{F}_s^*\} = 0, \end{aligned}$$

and a similar result for  $v_t^*$ . This corresponds with (iii) of definition 3.2, and verifies the reverse-time MD property of  $\{w_t^*, v_t^*\}$  w.r.t.  $\{\mathcal{F}_t^*\}$ .

To show (10) we notice that for all dB,

$$P\{(w_t, v_t) \in dB | \mathcal{F}_{t+1}\} =$$

$$= P\{(w_t, v_t) \in dB | y_{t+1}, x_{t+1}, \theta_{t+1}, w_{t+1}, v_{t+1}, u_{t+1}, \mathcal{F}_{t+2}\},$$

and that the sextet  $\{y_t, x_t, \theta_t, w_t, v_t, u_t\}$  is a Markov process with respect to the filtration  $\{\mathcal{F}_t\}$ . Then the past-present duality of this Markov property and some manipulations yield, for all

measurable  $dB = dR^P dx dR \subset R^P \times R$ ,

$$\begin{aligned} P\{(w_t, v_t) \in dB | \mathcal{F}_{t+1}\} &= P\{(w_t, v_t) \in dB | y_{t+1}, x_{t+1}, \theta_{t+1}, w_{t+1}, v_{t+1}, u_{t+1}\} = \\ &= P\{(w_t, v_t) \in dB | x_{t+1}, \theta_{t+1}\} = \\ &= P\{w_t \in dR^P | v_t \in dR, x_{t+1}, \theta_{t+1}\} \cdot P\{v_t \in dR | x_{t+1}, \theta_{t+1}\} \end{aligned}$$

where the first equality follows from the Markov property of the sextet, the second equality follows from (1), while the last is a straightforward decomposition. The latter result verifies (10.a), and because  $\theta_t$  is  $(v_t, \theta_{t+1})$ -measurable it also verifies (10.a).  
Q.E.D.

Theorem 4.1 implies that  $\hat{w}_t$  and  $\hat{v}_t$  can be written as

$$\hat{w}_t = f(t, \theta_{t+1}, \theta_t, x_{t+1}), \quad (11.a)$$

$$\hat{v}_t = g(t, \theta_{t+1}, x_{t+1}). \quad (11.b)$$

Substitution of (9.a) and (11.a,b) in (7.a,b,c) yields

$$x_t = \tilde{a}(t, \theta_{t+1}, \theta_t, x_{t+1}, w_t^*), \quad (12.a)$$

$$\theta_t = \tilde{b}(t, \theta_{t+1}, x_{t+1}, v_t^*), \quad (12.b)$$

$$y_t = \tilde{c}(t, \theta_{t+1}, \theta_t, x_{t+1}, x_t, w_t^*, u_t), \quad (12.c)$$

$$\text{with: } \tilde{a}(t, \theta, \eta, x, w^*) = a^*(\theta, \eta, x, w^* + f(t, \theta, \eta, x)), \quad (13.a)$$

$$\tilde{b}(t, \theta, x, v^*) = b^*(\theta, v^* + g(t, \theta, x)), \quad (13.b)$$

$$\tilde{c}(t, \theta, \eta, x, z, w^*, u) = c(\eta, z, w^* + f(t, \theta, \eta, x), u). \quad (13.c)$$

The above result is summarized by the following corollary.

#### 4.2 Corollary

Under assumptions A.1 and A.2, the solution  $\{\tilde{y}_t, \tilde{x}_t, \tilde{\theta}_t\}$  of the reverse-time system (4) is indistinguishable from the solution  $\{y_t, x_t, \theta_t\}$  of system (1) if

$$(I) \quad (\tilde{y}_T, \tilde{x}_T, \tilde{\theta}_T) = (y_T, x_T, \theta_T) \quad \text{a.s.},$$

$$(II) \quad \tilde{a}, \tilde{b} \text{ and } \tilde{c} \text{ satisfy (13.a,b,c),}$$

$$(III) \quad (\tilde{w}_t^*, \tilde{v}_t^*) = (w_t^*, v_t^*) \quad \text{a.s. ; all } t \in [0, T-1],$$

with  $w_t^*$  and  $v_t^*$  satisfying (9) and (10).

#### Jump-linear example

To illustrate the results obtained so far, let us consider the particular situation of a linear system with first order Markovian switching coefficients and observation noise independent of the system driving noise. Both  $a(\theta, \eta, x, w)$  and  $c(\eta, x, w, u)$  are then linear in  $(x, w)$ , while the first is  $\eta$ -invariant and the second is  $w$ -invariant, by which system (1)

$$\begin{aligned} \text{simplifies to,} \quad x_{t+1} &= A(\theta_{t+1})x_t + B(\theta_{t+1})w_t, \\ \theta_{t+1} &= b(\theta_t, v_t), \\ y_t &= G(\theta_t)x_t + H(\theta_t)u_t. \end{aligned}$$

Then from Corollary 4.2 we readily find the indistinguishable time-reversed system,

$$x_t = A^{-1}(\theta_{t+1}) [x_{t+1} - B(\theta_{t+1}) (\hat{w}_t + w_t^*)],$$

$$\theta_t = b^*(\theta_{t+1}, \hat{v}_t + v_t^*),$$

$$y_t = G(\theta_t)x_t + H(\theta_t)u_t,$$

where  $\{w_t^*, v_t^*\}$  is the reverse-time MD-sequence of Theorem 4.1,

$\hat{w}_t = f(t, \theta_{t+1}, \theta_t, x_{t+1})$ ,  $\hat{v}_t = g(t, \theta_{t+1}, x_{t+1})$  and  $f$ ,  $g$  and  $b^*$  are according to (11) and (13.b). The difference equation for  $x_t$  is similar to the one for the linear Gaussian example in section 2.

But due to  $\hat{w}_t$ , it may even be nonlinear in  $x_{t+1}$ . At the end of the next section we will show that there are some further simplifications possible for this example, in case of in probability law equivalence.

## 5 Equivalent in law time-reversion

In this section we derive conditions under which the solutions of (1) and (4) are equivalent in law, and discuss these results for a jump-linear example. So far our line of reasoning is quite similar to the martingale approach of time-reversing a diffusion. However, things are quite different now we require equivalence in law only. The reason is that, while in the diffusion situation

this requires that  $\{\tilde{dw}_t\}$  and  $\{dw_t\}$  are equivalent in law, no similar simple results hold in the discrete-time situation. Instead of this, we identify the relation between conditional

laws of  $\{\tilde{w}_t\}$  and  $\{w_t\}$  by a Bayesian estimation step. Next we characterize  $f$  and the required law of  $\{w_t^*\}$ .

### 5.1 Theorem

Under assumption A.1 the solution  $\{\tilde{y}_t, \tilde{x}_t, \tilde{\theta}_t\}$  of reverse-time system (4) is equivalent in law w.r.t. the solution  $\{y_t, x_t, \theta_t\}$  of system (1) if,

$$(i) \quad \mu_{\tilde{y}_T, \tilde{x}_T, \tilde{\theta}_T} (dX) = \mu_{y_T, x_T, \theta_T} (dX);$$

for any measurable  $dX \subset \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{M}$ ,

$$(ii) \quad \tilde{a} \text{ and } \tilde{c} \text{ satisfy (13.a,c),}$$

$$(iii) \quad p_{\tilde{\theta}_t | \tilde{\theta}_{t+1}, \tilde{x}_{t+1}} (\eta | \theta, x) = p_{\theta_t | \theta_{t+1}, x_{t+1}} (\eta | \theta, x),$$

$$(iv) \quad \mu_{\tilde{w}_t | \tilde{\theta}_{t+1}, \tilde{\theta}_t, \tilde{x}_{t+1}} (\cdot | \theta, \eta, x) = \mu_{w_t^* | \theta_{t+1}, \theta_t, x_{t+1}} (\cdot | \theta, \eta, x),$$

for all  $(x, \theta, \eta, t) \in \mathbb{R}^n \times \mathbb{M}^2 \times [0, T-1]$ , with  $w_t^*$  and  $f$  satisfying (9), (10.a) and (11.a).

Proof:

Under the conditions of Corollary 4.2,  $\{\tilde{y}_t, \tilde{x}_t, \tilde{\theta}_t\}$  and  $\{y_t, x_t, \theta_t\}$  are indistinguishable and thus equivalent in law. This property is preserved if (I) is replaced by (i) and the involved reverse-time Martingale Difference sequence w.r.t.  $\{\mathcal{F}_t^*\}$  is replaced by one that is equivalent in law; i.e. (III) is replaced by (III');

$$\begin{aligned} \mu_{\tilde{w}_t, \tilde{v}_t | \tilde{\theta}_{t+1}, \tilde{x}_{t+1}, (\tilde{\theta}_s, \tilde{x}_s; s \in [t+2, T])} (\dots | \theta, x, (S)) &= \\ &= \mu_{w_t^*, v_t^* | \theta_{t+1}, x_{t+1}, (\theta_s, x_s; s \in [t+2, T])} (\dots | \theta, x, (S)), \end{aligned}$$

for every  $(x, \theta)$  and every realization  $S$  of  $(\theta_s, x_s; s \in [t+2, T])$ .

From the proof of Theorem 4.1 we know that

$$\begin{aligned} P\{(w_t, v_t) \in dB | \mathcal{F}_{t+1}^*\} &= P\{(w_t, v_t) \in dB | \theta_{t+1}, x_{t+1}\} = \\ &= P\{w_t \in dR^D | v_t \in dR, \theta_{t+1}, x_{t+1}\} \cdot P\{v_t \in dR | \theta_{t+1}, x_{t+1}\}, \end{aligned}$$

and thus also

$$P\{(w_t^*, v_t^*) \in dB | \mathcal{F}_{t+1}^*\} = P\{(w_t^*, v_t^*) \in dB | \theta_{t+1}, x_{t+1}\}.$$

Hence, (III') simplifies to (III");

$$\mu_{\tilde{w}_t, \tilde{v}_t | \tilde{\theta}_{t+1}, \tilde{x}_{t+1}} (\dots | \theta, x) = \mu_{w_t^*, v_t^* | \theta_{t+1}, x_{t+1}} (\dots | \theta, x),$$

for every  $(\theta, x)$ . Together with (II), the latter equality transforms straightforwardly to (ii), (iii) and (iv). Q.E.D.

Our remaining problem is the characterization of the conditional law of  $w_t^*$ . As this is actually a discrete-time Bayesian estimation problem, it can be done by applying Bayes formula. We do this under the following additional assumptions:

- A.3. The a priori distribution of  $(x_t, \theta_t)$  permits a density-mass function for all  $t \in [0, T]$ .
- A.4.  $a^*(\theta, \eta, x, w)$  is once differentiable in  $x \in \mathbb{R}^n$  for all  $(\theta, \eta, w) \in \mathbb{M}^2 \times \mathbb{R}^D$ .

5.2 Proposition

Under assumptions A.3 and A.4, the distribution in (iv) of Theorem 5.1 permits a density which is characterized by

$$p_{\tilde{w}_t | \tilde{\theta}_{t+1}, \tilde{\theta}_t, \tilde{x}_{t+1}}(w | \theta, \eta, x) = p_{w_t | \theta_{t+1}, \theta_t, x_{t+1}}(w + f(t, \theta, \eta, x) | \theta, \eta, x), \quad (14.a)$$

$$p_{w_t | \theta_{t+1}, \theta_t, x_{t+1}}(\cdot | \theta, \eta, x) = c'(\theta, \eta, x) |\nabla_x a^{*T}(\theta, \eta, x, \cdot)| p_{w_t}(\cdot) \cdot p_{x_t | \theta_t}(a^*(\theta, \eta, x, \cdot) | \eta), \tag{14.b}$$

for all  $(x, \theta, \eta, t) \in \mathbb{R}^n \times M^2 \times [0, T-1]$ , with  $\nabla_x$  the gradient and  $c'(\theta, \eta, x)$  a normalizing factor, such that  $c'(\theta, \eta, x) \stackrel{\Delta}{=} 0$  if  $p_{x_{t+1} | \theta_{t+1}, \theta_t}(x | \theta, \eta) = 0$ . Moreover,

$$p_{\tilde{\theta}_t | \tilde{\theta}_{t+1}, \tilde{x}_{t+1}}(\eta | \theta, x) = p_{\theta_t | \theta_{t+1}}(\eta | \theta) \cdot p_{x_{t+1} | \theta_{t+1}, \theta_t}(x | \theta, \eta) \cdot p_{x_{t+1} | \theta_{t+1}}^{-1}(x | \theta). \tag{15}$$

**Proof:**

Due to A.3, application of Bayes formula yields,

$$p_{w_t | \theta_{t+1}, \theta_t, x_{t+1}}(\cdot | \theta, \eta, x) = p_{x_{t+1} | w_t, \theta_{t+1}, \theta_t}(x | \cdot, \theta, \eta) \cdot p_{w_t | \theta_{t+1}, \theta_t}(\cdot | \theta, \eta) / p_{x_{t+1} | \theta_{t+1}, \theta_t}(x | \theta, \eta),$$

for any  $(\theta, \eta, x)$  with  $p_{x_{t+1} | \theta_{t+1}, \theta_t}(x | \theta, \eta) > 0$ ; else the right hand side is zero.

In view of (1.b) and the independence of  $\{w_t\}$  and  $\{\theta_t, v_t\}$ ,

$$p_{w_t | \theta_{t+1}, \theta_t}(\cdot | \theta, \eta) = p_{w_t}(\cdot)$$

and from (1.a) and A.4,

$$p_{x_{t+1} | w_t, \theta_{t+1}, \theta_t}(x | \cdot, \theta, \eta) = |\nabla_x a^{*T}(\theta, \eta, x, \cdot)| p_{x_t | \theta_{t+1}, \theta_t}(a^*(\theta, \eta, x, \cdot) | \theta, \eta)$$

Substitution of the last two equations in the first one, and using the conditional independence of  $x_t$  and  $\theta_{t+1}$  given  $\theta_t$ , yields (14.b).

Next, substitution of (9) into the right hand side of (iv) yields:

$$\begin{aligned} \mu_{\tilde{w}_t | \tilde{\theta}_{t+1}, \tilde{\theta}_t, \tilde{x}_{t+1}}(\cdot | \theta, \eta, x) &= \\ &= \mu_{(w_t - \hat{w}_t) | \theta_{t+1}, \theta_t, x_{t+1}}(\cdot | \theta, \eta, x), \end{aligned}$$

for all  $(x, \theta, \eta, t) \in \mathbb{R}^n \times M^2 \times [0, T-1]$ .

Since the latter conditional distribution admits a density, substitution of (11.a) yields (14.a).

Finally, a repeated application of Bayes formula yields:

$$\begin{aligned}
 p_{\theta_t | \theta_{t+1}, x_{t+1}}(\eta | \theta, x) &= \\
 &= p_{\theta_t, \theta_{t+1} | x_{t+1}}(\eta, \theta | x) / p_{\theta_{t+1} | x_{t+1}}(\theta | x) = \\
 &= [p_{\theta_t, \theta_{t+1}}(\eta, \theta) / p_{\theta_{t+1}}(\theta)] \cdot [p_{x_{t+1} | \theta_{t+1}, \theta_t}(x | \theta, \eta) / p_{x_{t+1} | \theta_{t+1}}(x | \theta)] = \\
 &= p_{\theta_t | \theta_{t+1}}(\eta | \theta) \cdot p_{x_{t+1} | \theta_{t+1}, \theta_t}(x | \theta, \eta) / p_{x_{t+1} | \theta_{t+1}}(x | \theta).
 \end{aligned}$$

Hence, (15) follows from (iii). Q.E.D.

Jump-linear example

For a linear system with first order Markovian switching coefficients we arrived, in section 4, at the following time-reversed equation:

$$x_t = A^{-1}(\theta_{t+1}) [x_{t+1} - B(\theta_{t+1}) [\hat{w}_t + w_t^*]],$$

with  $w_t^*$  the reverse-time MD sequence and  $\hat{w}_t = E\{w_t | \theta_{t+1}, \theta_t, x_{t+1}\}$ . Because  $a^*(\dots, x, w)$  is linear in  $(x, w)$ , its gradient w.r.t.  $x$  is  $w$ -invariant, by which (14.b) in proposition 5.2 yields

$$\begin{aligned}
 p_{w_t | \theta_{t+1}, \theta_t, x_{t+1}}(w | \theta, \eta, x) &= \\
 &= c_1(\theta, \eta, x) p_{w_t}(w) p_{x_t | \theta_t}(A^{-1}(\theta) [x - B(\theta)w] | \eta).
 \end{aligned}$$

In spite of the simplification this is a form which is in general quite complex, and  $\hat{w}_t$  still may be a nonlinear function of  $x_{t+1}$ . Obviously, this type of complexity could have been expected, as it is well known that a discrete-time Bayesian estimation step leads to nonlinear equations, unless the prior densities involved are Gaussian. Therefore, we consider in the next section the special situation that for all  $\eta \in M$ ,  $p_{x_t | \theta_t}(\cdot | \eta)$  is Gaussian, to

recover that  $\hat{w}_t$  is then a linear function of  $x_{t+1}$ , and that  $\{w_t^*\}$  is then white Gaussian noise, the covariance of which has a second order dependence on  $\{\theta_t\}$ .

6 Linear systems with Markovian switching coefficients

In this section we consider both the indistinguishable and the in law equivalent time-reversion of the following linear system with Markovian switching coefficients:

$$x_{t+1} = A(\theta_{t+1}, \theta_t)x_t + B(\theta_{t+1}, \theta_t)w_t, \tag{16.a}$$

$$\theta_{t+1} = b(\theta_t, v_t), \tag{16.b}$$

under assumptions A.1, A.2 and A.5:

A.5. The a priori conditional density of  $x_t$ , given  $\theta_t = \eta$ , is a non-singular Gaussian for all  $(t, \eta) \in [0, T] \times M$ .

With the help of Corollary 4.2 and Proposition 5.2, one can show the following.

### 6.1 Theorem

Under assumptions A.1, A.2 and A.5, the solution  $\{\tilde{x}_t, \tilde{\theta}_t\}$  of

$$\tilde{x}_t = \tilde{A}(t, \tilde{\theta}_{t+1}, \tilde{\theta}_t) \tilde{x}_{t+1} + \tilde{B}(\tilde{\theta}_{t+1}, \tilde{\theta}_t) \tilde{w}_t, \quad (17.a)$$

$$\tilde{\theta}_t = \tilde{b}(t, \tilde{\theta}_{t+1}, \tilde{x}_{t+1}, \tilde{v}_t), \quad (17.b)$$

is indistinguishable from the solution  $\{x_t, \theta_t\}$  of system (16) if:

$$(I) \quad (\tilde{x}_T, \tilde{\theta}_T) = (x_T, \theta_T) \quad \text{a.s.},$$

$$(II) \quad \tilde{A}(t, \theta, \eta) = A^{-1}(\theta, \eta) [I - B(\theta, \eta) B^T(\theta, \eta) R^{-1}(t+1 | \theta, \eta)], \quad (18.a)$$

$$\tilde{B}(\theta, \eta) = -A^{-1}(\theta, \eta) B(\theta, \eta), \quad (18.b)$$

$\tilde{b}$  satisfies (13.b),

$$(III) \quad (\tilde{w}_t, \tilde{v}_t) = (w_t^*, v_t^*) \quad \text{a.s. ; all } t \in [0, T-1],$$

with  $w_t^*$  and  $v_t^*$  satisfying (9.a) and (10), and  $R(t+1 | \theta_{t+1}, \theta_t)$  the conditional covariance of  $x_{t+1}$ , given  $\theta_{t+1}$  and  $\theta_t$ :

$$R(t+1 | \theta, \eta) = A(\theta, \eta) \text{Cov}\{x_t | \theta_t = \eta\} A^T(\theta, \eta) + B(\theta, \eta) B^T(\theta, \eta). \quad (19)$$

Proof:

(I) and (III) are as in Corollary 4.2, and it remains to show (II). For short we write  $\underline{A}$  for  $A(\theta_{t+1}, \theta_t)$ ,  $\underline{R}$  for  $R(t+1, \theta_{t+1}, \theta_t)$ , etc.. Assumption A.5 implies

$$p_{x_t | \theta_t}(x) = N\{x; 0, C_t\},$$

where  $C_t$  represents the conditional covariance of  $x_t$ , given  $\theta_t$ . Because A.5 implies A.3 and (16.a) implies A.4, proposition 5.2 holds true. Next, substitution in (15) of

$$a^{*T}(\theta, \eta, x, w) = A^{-1}(\theta, \eta) [x - B(\theta, \eta) w] \quad \text{and} \quad p_{w_t}(w) = N\{w; 0, I\} \quad \text{yields,}$$

$$\begin{aligned} p_{w_t | \theta_{t+1}, \theta_t, x_{t+1}}(w) &= \underline{c}_1(x_{t+1}) N\{w; 0, I\} N\{\underline{A}^{-1}[x_{t+1} - \underline{B}w]; 0, C_t\} = \\ &= \underline{c}_2(x_{t+1}) \exp\{-\frac{1}{2} w^T w - \frac{1}{2} [\underline{A}^{-1}[x_{t+1} - \underline{B}w]]^T C_t^{-1} [\dots]\} = \\ &= \underline{c}_3(x_{t+1}) \exp\{-\frac{1}{2} [w - \underline{F}x_{t+1}]^T \underline{Q}^{-1} [\dots]\}, \end{aligned}$$

with:

$$\underline{Q}^{-1} = I + [\underline{A}^{-1} \underline{B}]^T C_t^{-1} \underline{A}^{-1} \underline{B},$$

$$\underline{Q}^{-1} \underline{F} = [\underline{A}^{-1} \underline{B}]^T C_t^{-1} \underline{A}^{-1}.$$

From the matrix inversion lemma it follows that

$$\underline{Q} = \underline{I} - \underline{B}^T [\underline{B}\underline{B}^T + \underline{A}\underline{C}_t\underline{A}^T]^{-1} \underline{B} = \underline{I} - \underline{B}^T \underline{R}^{-1} \underline{B},$$

with  $\underline{R}$  the covariance of  $x_{t+1}$ , given  $(\theta_{t+1}, \theta_t)$ ;  $\underline{R} = \underline{A}\underline{C}_t\underline{A}^T + \underline{B}\underline{B}^T$ , which corresponds to (19).

Next the evaluation of  $\underline{F}$  yields

$$\begin{aligned} \underline{F} &= \underline{Q} [\underline{A}^{-1} \underline{B}]^T \underline{C}_t^{-1} \underline{A}^{-1} = \\ &= [\underline{A}^{-1} \underline{B}]^T \underline{C}_t^{-1} \underline{A}^{-1} - \underline{B}^T \underline{C}_t^{-1} \underline{B} [\underline{A}^{-1} \underline{B}]^T \underline{C}_t^{-1} \underline{A}^{-1} = \\ &= [\underline{A}^{-1} \underline{B}]^T \underline{C}_t^{-1} \underline{A}^{-1} - \underline{B}^T \underline{R}^{-1} [\underline{R} - \underline{A}\underline{R}_t\underline{A}^T] (\underline{A}^T)^{-1} \underline{C}_t^{-1} \underline{A}^{-1} = \\ &= [\underline{A}^{-1} \underline{B}]^T \underline{C}_t^{-1} \underline{A}^{-1} - \underline{B}^T (\underline{A}^T)^{-1} \underline{C}_t^{-1} \underline{A}^{-1} + \underline{B}^T \underline{R}^{-1} = \\ &= \underline{B}^T \underline{R}^{-1}. \end{aligned}$$

The above results mean that the conditional density of  $w_t$ , given  $(\theta_{t+1}, \theta_t, x_{t+1})$  is Gaussian with mean and covariance,

$$E\{w_t | \theta_{t+1}, \theta_t, x_{t+1}\} = \underline{B}^T \underline{R}^{-1} x_{t+1}, \quad (20.a)$$

$$\text{Cov}\{w_t | \theta_{t+1}, \theta_t, x_{t+1}\} = \underline{I} - \underline{B}^T \underline{R}^{-1} \underline{B}. \quad (20.b)$$

From (20.a) follows that

$$w_t = \tilde{w}_t + \underline{B}^T \underline{R}^{-1} x_{t+1}. \quad (20.c)$$

Substitution of (20.c) in (16) and time-reversion yield (17) and (18). Q.E.D.

The change of the indistinguishable time-reversion to the equivalent in law time-reversion can simply be made by replacing (I) and (III) of Theorem 6.1 by equivalent in law conditions such as in Theorem 5.1. For  $\tilde{w}_t$  this implies a Gaussian density with zero mean and covariance according to (20.b),

$$\text{Cov}\{\tilde{w}_t | \theta_{t+1}, \theta_t, x_{t+1}\} = \underline{I} - \underline{B}^T (\theta_{t+1}, \theta_t) \underline{R}^{-1} (\theta_{t+1}, \theta_t) \underline{B}. \quad (21)$$

Due to (18), this equivalent in law time-reversion requires the quite restrictive assumptions of invertible A's. Fortunately, there exists an equivalent in law time-reversed system if the A's are not invertible. Although, the intermediate step of indistinguishable time-reversion can no longer be used for a full derivation, that route can still be used to construct the form of the less restrictive time-reversed system. That construction runs as follows:



From (ii) and (iv) of Theorem 5.1 we obtain the following alternative in law equivalent time-reversed system,

$$\tilde{\underline{A}} = \underline{A}^{-1}[\underline{I} - \underline{B}\underline{B}^T\underline{R}^{-1}], \quad (22.a)$$

$$\tilde{\underline{B}} = \underline{I}, \quad (22.b)$$

$$\text{Cov}\{\tilde{w}_t | \theta_{t+1}, \theta_t\} = \underline{A}^{-1}\underline{B}[\underline{I} - \underline{B}^T\underline{R}^{-1}\underline{B}]\underline{B}^T(\underline{A}^{-1})^T, \quad (22.c)$$

where the underline notation is as in the proof of theorem 6.1. To avoid the use of  $\underline{A}^{-1}$  we use the following equality,

$$\underline{B}\underline{B}^T = \underline{R} - \underline{A}\underline{C}_t\underline{A}^T. \quad (23)$$

Substitution of (23) in (22.a) and evaluation yields a form that does not involve the inverse of  $\underline{A}$ :

$$\tilde{\underline{A}} = \underline{C}_t\underline{A}^T\underline{R}^{-1}. \quad (24)$$

In a similar way we can transform (22.c):

$$\begin{aligned} \text{Cov}\{\tilde{w}_t | \theta_{t+1}, \theta_t\} &= \underline{A}^{-1}[\underline{B}\underline{B}^T - \underline{B}\underline{B}^T\underline{R}^{-1}\underline{B}\underline{B}^T](\underline{A}^{-1})^T = \\ &= \underline{A}^{-1}[\underline{I} - \underline{B}\underline{B}^T\underline{R}^{-1}]\underline{B}\underline{B}^T(\underline{A}^{-1})^T = \tilde{\underline{A}}\underline{B}\underline{B}^T(\underline{A}^{-1})^T = \\ &= \tilde{\underline{A}}[\underline{R} - \underline{A}\underline{C}_t\underline{A}^T](\underline{A}^{-1})^T = \tilde{\underline{A}}\underline{R}(\underline{A}^{-1})^T - \tilde{\underline{A}}\underline{A}\underline{C}_t = \\ &= \underline{C}_t\underline{A}^T\underline{R}^{-1}\underline{R}(\underline{A}^{-1})^T - \tilde{\underline{A}}\underline{A}\underline{C}_t = \underline{C}_t - \tilde{\underline{A}}\underline{A}\underline{C}_t = \\ &= \underline{C}_t - \underline{R}_t\underline{A}^T\underline{R}^{-1}\underline{A}\underline{C}_t = \underline{C}_t - \tilde{\underline{A}}\underline{R}\tilde{\underline{A}}^T, \end{aligned} \quad (25)$$

which is also free of the inverse of  $\underline{A}$ .

If the  $\underline{A}$ 's are not invertible one can show that (24) and (25) still are the correct in law equivalent time-reversed forms:

## 6.2 Proposition

Under assumption A.5 the solution  $\{\tilde{x}_t, \tilde{\theta}_t\}$  of reverse-time system (17) is equivalent in law w.r.t. the solution  $\{x_t, \theta_t\}$  of system (16) if:

$$(i) \quad P\{(\tilde{x}_T, \tilde{\theta}_T) \in dX\} = P\{(x_T, \theta_T) \in dX\}, \text{ all measurable } dX \subset \mathbb{R}^n \times M,$$

$$(ii) \quad \tilde{\underline{A}}(t, \theta, \eta) = \text{Cov}\{x_t | \theta_t = \eta\} \underline{A}^T(\theta, \eta) \underline{R}^{-1}(t+1 | \theta, \eta), \quad (26.a)$$

$$\tilde{\underline{B}}(\theta, \eta) = \underline{I}, \quad (26.b)$$

$$(iii) \quad P\{\tilde{\theta}_t = \eta | \tilde{\theta}_{t+1} = \theta, \tilde{x}_{t+1} = x\} = P\{\theta_t = \eta | \theta_{t+1} = \theta, x_{t+1} | \theta_{t+1}, \theta_t\} \cdot P(x | \theta, \eta) \cdot P^{-1}(x | \theta).$$

(iv)  $\{\tilde{w}_t\}$  is a sequence of i.i.d. Gaussian variables of conditional covariance:

$$\text{Cov}\{\tilde{w}_t | \theta_{t+1}, \theta_t, x_{t+1}\} =$$

$$= \text{Cov}\{x_t | \theta_t\} - \tilde{A}(t, \theta_{t+1}, \theta_t) R(t+1, \theta_{t+1}, \theta_t) \tilde{A}^T(t, \theta_{t+1}, \theta_t), \quad (27)$$

where  $R(t+1, \theta, \eta)$  satisfies (19).

**Proof:** Omitted. It can be obtained by a straightforward but lengthy evaluation of the transition function of the time-reversed Markov process.

It can easily be verified that even when  $A(\theta, \eta)$  and  $B(\theta, \eta)$  are  $\eta$ -invariant,  $R(t+1 | \theta_{t+1}, \theta_t)$  is in general conditionally independent of  $\theta_t$  given  $\theta_{t+1}$ , and the time-reversed equations have a second order dependence of  $\{\theta_t\}$ . If, however,  $x_{t+1}$  is conditionally independent of  $\theta_t$  given  $\theta_{t+1}$  then it can easily be verified that the in law equivalent time-reversed equations of proposition 6.2 have a first order dependence of  $\{\theta_t\}$ :

### 6.3 Corollary

If A.5 is satisfied and  $x_{t+1}$  is conditionally independent

of  $\theta_t$  given  $\theta_{t+1}$ , then the solution  $\{\tilde{x}_t, \tilde{\theta}_t\}$  of reverse-time system (17) is equivalent in law w.r.t. the solution  $\{x_t, \theta_t\}$  of system (16) if:

(i)  $P\{(\tilde{x}_T, \tilde{\theta}_T) \in dX\} = P\{(x_T, \theta_T) \in dX\}$ , all measurable  $dX \subset \mathbb{R}^n \times M$ ,

(ii)  $\tilde{A}(t, \theta, \eta) = \text{Cov}\{x_t | \theta_t = \eta\} A^T(\theta, \eta) \text{Cov}^{-1}\{x_{t+1} | \theta_{t+1} = \theta\}$ ,

$$\tilde{B}(\theta, \eta) = I,$$

(iii)  $P\{\tilde{\theta}_t = \eta | \tilde{\theta}_{t+1} = \theta, \tilde{x}_{t+1} = x\} = P\{\theta_t = \eta | \theta_{t+1} = \theta\}$

(iv)  $\{\tilde{w}_t\}$  is a sequence of i.i.d. Gaussian variables of conditional covariance:

$$\text{Cov}\{\tilde{w}_t | \theta_{t+1}, \theta_t, x_{t+1}\} =$$

$$= \text{Cov}\{x_t | \theta_t\} - \tilde{A}(t, \theta_{t+1}, \theta_t) \text{Cov}\{x_{t+1} | \theta_{t+1}\} \tilde{A}^T(t, \theta_{t+1}, \theta_t)$$

## 7 Smoothing for a trajectory with manoeuvres

In this section we apply the time reversion results for a linear system with Markovian switching coefficients (jump-linear system) to a particular problem of Bayesian smoothing on a finite time interval. As this smoothing works by way of time reversal, we first introduce the system considered and derive its time-reversed counterpart.

Assume we are provided partial observations  $\{y_t\}$  of  $\{x_t\}$  on a finite time interval,  $[0, T]$ ,

$$y_t = Hx_t + Gu_t,$$

with  $H=[1 \ 0 \ 0]$ ,  $G=100$  and  $\{u_t\}$  a sequence of i.i.d. standard Gaussian variables. The process  $\{x_t\}$  satisfies system (16.a);

$$x_{t+1} = A(\theta_{t+1}, \theta_t)x_t + B(\theta_{t+1}, \theta_t)w_t,$$

where  $\{\theta_t\}$ ,  $\{w_t\}$  and  $\{u_t\}$  are independent,  $\{w_t\}$  is a sequence of i.i.d. standard Gaussian variables and  $\{\theta_t\}$  switches between 0 and 1 with probabilities depending on the parameters,  $\tau_0$  and  $\tau_1$ , as follows:

$$P\{\theta_0=1\} = 1 - P\{\theta_0=0\} = \tau_1 / (\tau_0 + \tau_1),$$

$$P\{\theta_{t+1}=1 | \theta_t=0\} = 1/\tau_0,$$

$$P\{\theta_{t+1}=0 | \theta_t=1\} = 1/\tau_1.$$

The process  $\{x_t\}$  has three scalar components: position, velocity and potential acceleration, while its initial distribution is, for all  $\theta_0$ , Gaussian of zero mean and covariance

$$\text{Cov}\{x_0 | \theta_0\} = \text{Diag}\{1/\epsilon^2, 1/\epsilon, 1\},$$

with  $\epsilon$  a small positive scalar, approaching 0.

The parameters of the above model are:

$$A(1,1) = A(1,0) = \begin{bmatrix} 1 & \tau & \frac{1}{2}\tau^2 \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{bmatrix}, \quad A(0,0) = A(0,1) = \begin{bmatrix} 1 & \tau & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix},$$

$$B(1,1) = B(1,0) = [0 \ 0 \ 0]^T, \quad B(0,0) = B(0,1) = [0 \ 0 \ \sqrt{1-\alpha^2}]^T,$$

with  $\alpha = \sqrt{\epsilon}$ . The acceleration  $a_t$  is defined by:  $a_t = \theta_t x_{3,t}$ , where  $x_{3,t}$  represents the potential acceleration component of  $x_t$ .

With  $\alpha=0$ , the model above corresponds with the manoeuvre model of section II.6. With  $\alpha \downarrow 0+$ , the model above is such that both the prior distribution and the parameters of the time-reversed equations can be determined explicitly by analysis.

In particular,

$$\lim_{\epsilon \downarrow 0+} p_{x_{t+1} | \theta_{t+1}, \theta_t} = p_{x_{t+1} | \theta_{t+1}},$$

while the latter is Gaussian, with covariance

$$\text{Cov}\{x_t | \theta_t\} = \begin{bmatrix} 0(1/\epsilon^2) & 0(1/\epsilon) & 0(\alpha) \\ 0(1/\epsilon) & 0(1/\epsilon) & 0(\alpha) \\ 0(\alpha) & 0(\alpha) & 1 \end{bmatrix}, \quad \text{all } t \in [0, T],$$

where  $\lim_{\alpha \downarrow 0} 0(\alpha) = 0$ . Hence,

$$\text{Cov}^{-1}\{x_t | \theta_t\} = \begin{bmatrix} 0(\epsilon^2) & 0(\epsilon^2) & 0(\alpha\epsilon^2) \\ 0(\epsilon^2) & 0(\epsilon) & 0(\alpha\epsilon) \\ 0(\alpha\epsilon^2) & 0(\alpha\epsilon) & 1+0(\epsilon) \end{bmatrix}.$$

From theorem 6.1 and some evaluation, the indistinguishable time-reversed system becomes:

$$\tilde{Y}_t = H\tilde{X}_t + Gu_t,$$

$$\tilde{X}_t = \tilde{A}(t, \tilde{\theta}_{t+1}, \tilde{\theta}_t)\tilde{X}_{t+1} + \tilde{B}(\tilde{\theta}_{t+1}, \tilde{\theta}_t)\tilde{w}_t,$$

with  $\tilde{w}_t$  white Gaussian noise and:

$$\tilde{A}(t, 1, 1) = \tilde{A}(t, 1, 0) = \begin{bmatrix} 1 & -\tau & \frac{1}{2}\tau^2 \\ 0 & 1 & -\tau \\ 0 & 0 & 1 \end{bmatrix},$$

$$\tilde{A}(t, 0, 1) = \tilde{A}(t, 0, 0) = \begin{bmatrix} 1 & & \\ 0 & -\tau & 0 \\ 0 & 0 & \alpha + 0(\epsilon\alpha) \end{bmatrix},$$

$$\tilde{B}(1, 1) = \tilde{B}(1, 0) = [0 \ 0 \ 0]^T,$$

$$\tilde{B}(0, 1) = \tilde{B}(0, 0) = [0 \ 0 \ -\sqrt{1-\alpha^2}/\alpha]^T.$$

Consequently, it follows from (21) that

$$\text{Cov}\{\tilde{w}_t | \theta_{t+1}=0, \theta_t\} = 1 - (1-\alpha^2)[1+0(\epsilon)] = \alpha^2 + 0(\epsilon),$$

by which we arrive at the following in law equivalent time-reversed system:

$$\tilde{x}_t = \tilde{A}(t, \tilde{\theta}_{t+1}, \tilde{\theta}_t)\tilde{x}_{t+1} + B(\tilde{\theta}_{t+1}, \tilde{\theta}_t)\tilde{w}_t, \quad (28)$$

with  $\tilde{w}_t$  white Gaussian noise and

$$\text{Cov}\{\tilde{w}_t\} = [\alpha^2 + 0(\epsilon)] (1-\alpha^2)/\alpha^2 = 1 + 0(\alpha^2 + \epsilon), \quad (29)$$

while the density of  $\tilde{x}_T$  is Gaussian of zero mean and covariance

$$\text{Cov}\{\tilde{x}_T | \tilde{\theta}_T\} = \begin{bmatrix} 0(1/\epsilon^2) & 0(1/\epsilon) & 0(\alpha) \\ 0(1/\epsilon) & 0(1/\epsilon) & 0(\alpha) \\ 0(\alpha) & 0(\alpha) & 1 \end{bmatrix}. \quad (30)$$

With this, we substitute  $\alpha = \sqrt{\epsilon}$  and take limit for  $\epsilon \downarrow 0$ , by which several parameters of (28), (29) and (30) simplify:

$$\text{Cov}\{\tilde{w}_t\} = 1,$$

$$\text{Cov}\{\tilde{x}_T | \tilde{\theta}_T\} = \begin{bmatrix} U & 0 \\ 0 & \sigma_w^2 \end{bmatrix}, \quad \text{with } U = \lim_{\epsilon \downarrow 0} \begin{bmatrix} 1/\epsilon^2 & 0(1/\epsilon) \\ 0(1/\epsilon) & 1/\epsilon \end{bmatrix},$$

$$\tilde{A}(t, 0, 1) = \tilde{A}(t, 0, 0) = \begin{bmatrix} 1 & -\tau & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Finally, since  $\lim_{\epsilon \downarrow 0} p_{x_{t+1} | \theta_{t+1}, \theta_t} = p_{x_{t+1} | \theta_{t+1}}$ , from (iii) in corollary 6.3 it follows that

$$P\{\tilde{\theta}_t=1|\tilde{\theta}_{t+1}=0\} = 1/\tau_0 ,$$

$$P\{\tilde{\theta}_t=0|\tilde{\theta}_{t+1}=1\} = 1/\tau_1 ,$$

$$P\{\tilde{\theta}_T=1\} = \tau_1/(\tau_1+\tau_0) .$$

Now that we have both a forward model and an equivalent in law time-reversed model, we consider the problem of estimating  $\{x_t\}$  from the information field  $Y_T$ . One approach of this so called fixed-interval smoothing problem is to apply Bayesian filtering of  $Y_T$  both in forward and in reverse-time direction, independently from each other. The forward filtering assumes the forward model, the reverse-time filter assumes the time-reversed model. After that, the smoothed estimates can be obtained by a merging of the forward and the reverse-time estimates according to the following version of Bayes formula (e.g., Anderson and Rhodes, 1983):

If  $p_{x_t, \theta_t}(x, i) \neq 0$

$$p_{x_t, \theta_t|Y_T}(x, i) = \frac{p_{x_t, \theta_t|Y_t}(x, i) p_{\tilde{x}_t, \tilde{\theta}_t|\tilde{Y}_{t+1}}(x, i)}{p_{x_t, \theta_t}(x, i) \text{Constant}_{t|Y_T}} , \quad (31)$$

else  $p_{x_t, \theta_t|Y_T}(x, i) = 0$ ,

where  $\tilde{Y}_t \triangleq \{y_s; s \in [t, T]\}$ , and of course under the assumption that the density-mass functions involved are non-singular.

With this and the above time reversion results we have a closed form representation of the exact solution of the fixed interval smoothing problem. Obviously, for practical application of this closed form representation, we need algorithms to compute the density-mass functions that appear at the right hand side of (31). If we have an algorithm to compute  $p_{x_t, \theta_t|Y_t}$  then that algorithm can certainly be used to compute  $p_{x_t, \theta_t}$ . Moreover, in view of the above time reversion results, that algorithm can also be used to compute  $p_{\tilde{x}_t, \tilde{\theta}_t|\tilde{Y}_{t+1}}$ .

As is well known (Tugnait, 1982), the exact computation of  $p_{x_t, \theta_t|Y_t}$ , from  $p_{x_{t-1}, \theta_{t-1}|Y_{t-1}}$  and  $Y_t$ , involves the running of  $2^t$  Kalman filters, which leads to a prohibitive computational load. The only way out of this combinatorial explosion is to accept approximations. Fortunately, for this type of filtering there exist algorithms that compute very close approximations of  $p_{x_t, \theta_t|Y_t}$ , while using only a fixed number of Kalman filters. The

most efficient algorithm of these is the Interacting Multiple Model (IMM) algorithm of chapter II. The IMM algorithm runs one Kalman filter for each possible value of  $\theta_t$  (mode), evaluates the conditional mode probabilities and mixes the estimates of all these filters in a particular way. Depending of some particular properties of the coefficients A and B, the latter mixing occurs either at the beginning or at the end of each IMM filter cycle. The mixing occurs at the beginning of each IMM filter cycle if  $A(\theta, \eta)$  and  $B(\theta, \eta)$  are  $\eta$ -invariant. The mixing occurs at the end of each IMM filter cycle, if  $A(\theta, \eta)$  and  $B(\theta, \eta)$  are  $\theta$ -invariant (see remark 2 in section II.4). In view of the above, the application of the IMM approach to smoothing our particular trajectory with manoeuvres implies the following procedure:

- Application, in forward time direction, of a common IMM algorithm (section II.6) with two Kalman filters and matched to the parameters  $\tau_0, \tau_1, A, B, H$  and  $G$ ,
- Application, in reverse-time direction, of a similar IMM algorithm with mixing at the end of each cycle and matched to the parameters  $\tau_0, \tau_1, \tilde{A}, \tilde{B}, H$  and  $G$ ,
- Determination of the parameters of the prior distribution,
- Appropriate merging of the prior distribution and the filtered estimates computed by the forward and the reverse-time applied IMM algorithms.

For short, we refer to this procedure as the IMM smoothing algorithm or the IMM smoother.

The performance of this IMM smoother has been evaluated and compared to the performance of the optimal linear smoother for the process  $\{x_t\}$ , i.e. a Kalman smoother (Sidhu and Desai, 1976). The details of the linear Gaussian model underlying the Kalman smoother are given in section II.6. In view of the lack of analytical tools for these performance evaluations, we had to rely on Monte Carlo simulations. During these Monte Carlo simulations, the parameters of the underlying models are assumed to be:  $\tau=10$ ,  $\tau_0=10$ ,  $\tau_1=1000$ ,  $\sigma_w=1$ ,  $\sigma_u=100$ . Both the parameters of the IMM smoother and the optimal linear (Kalman) smoother were set according to these parameters. To show the transient behaviour of the smoothers, a fixed trajectory has been simulated, with  $T=100$  and  $x_0=[0 \ 1000 \ 0]^T$ , while the acceleration was piecewise constant: zero on the intervals  $[0,29]$ , 1 on the interval  $[30,70]$  and again zero on  $[71,100]$ . During the Monte Carlo simulations we measured the RMS of the smoothing errors in position, speed and acceleration, the results of which are given by solid lines in figures 1 through 6. Moreover, we measured the average value (RMS) of the covariances that were estimated by the smoothers. The dashed lines in figures 1 through 6 represent the diagonal elements of these RMS values.

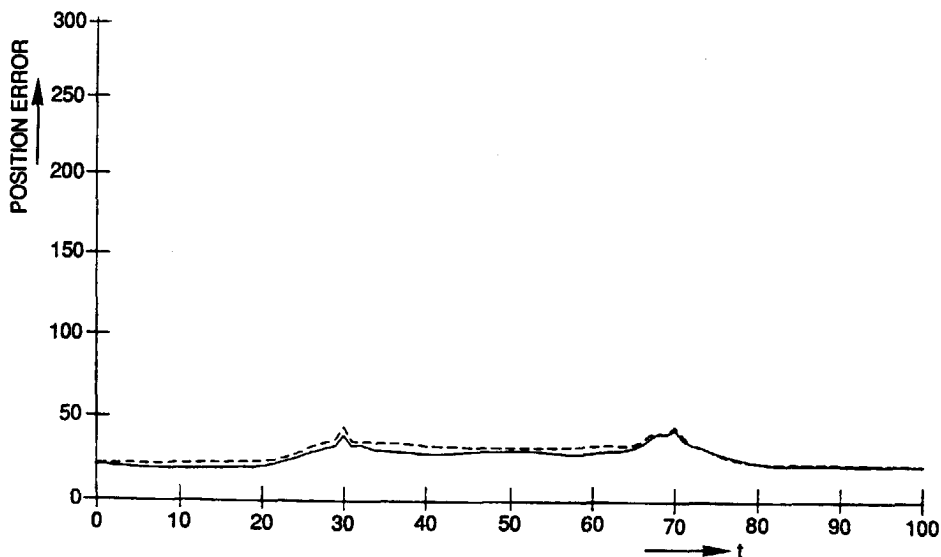


Fig. 1 Measured and expected RMS of the position errors of the IMM smoother;  
 — = measured RMS, - - - = expected RMS (RMS)

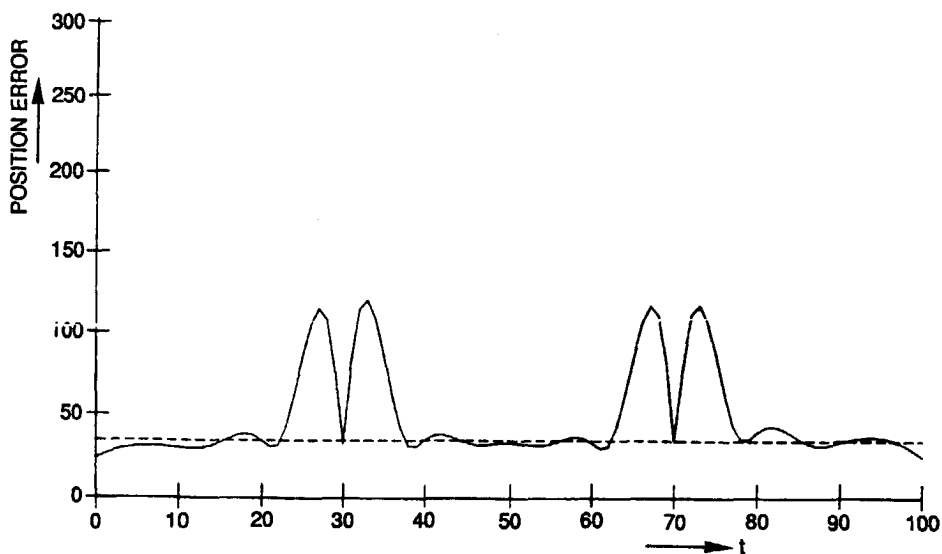


Fig. 2 Measured and expected RMS of the position errors of the Kalman smoother;  
 — = measured RMS, - - - = expected RMS (RMS)

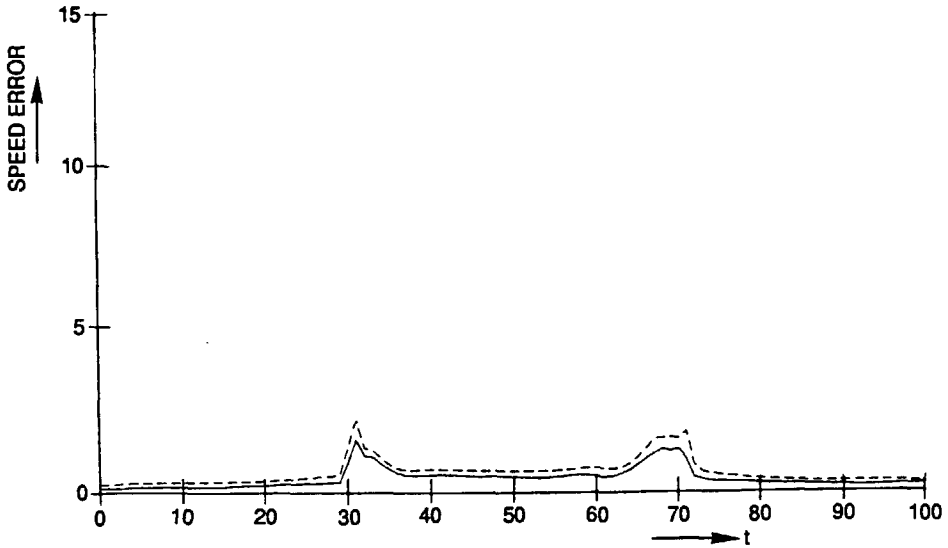


Fig. 3 Measured and expected RMS of the speed errors of the IMM smoother;  
 — = measured RMS, - - - = expected RMS (RMS)

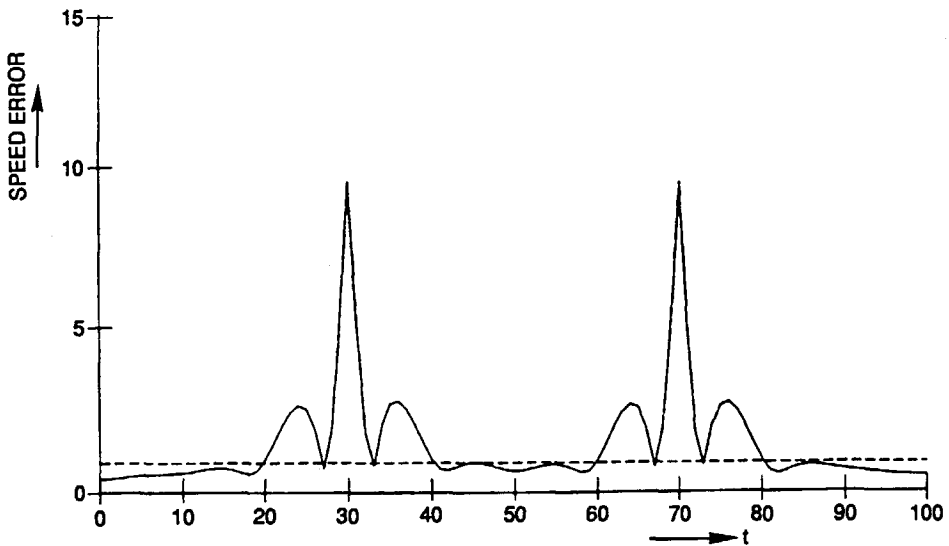


Fig. 4 Measured and expected RMS of the speed errors of the Kalman smoother;  
 — = measured RMS, - - - = expected RMS (RMS)



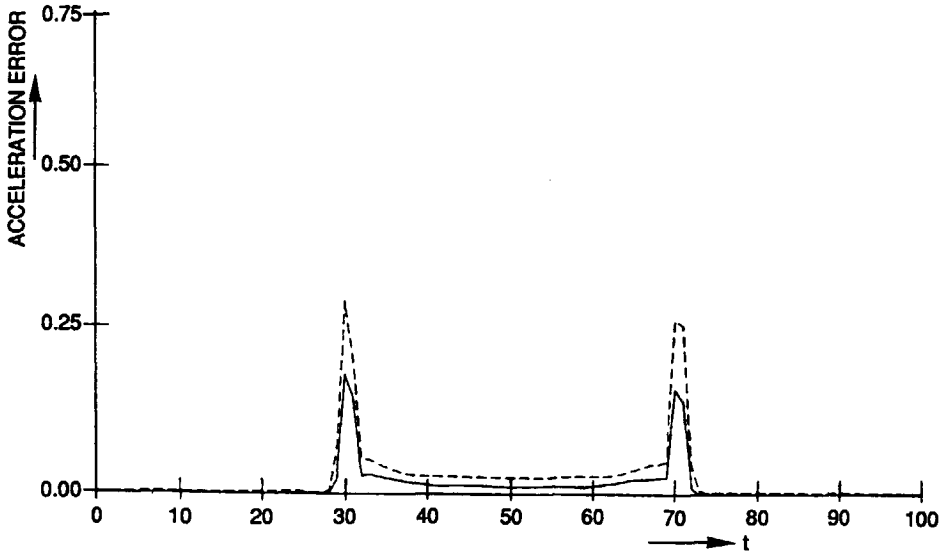


Fig. 5 Measured and expected RMS of the acceleration errors of the IMM smoother;  
 — = measured RMS, - - - = expected RMS (RMS)

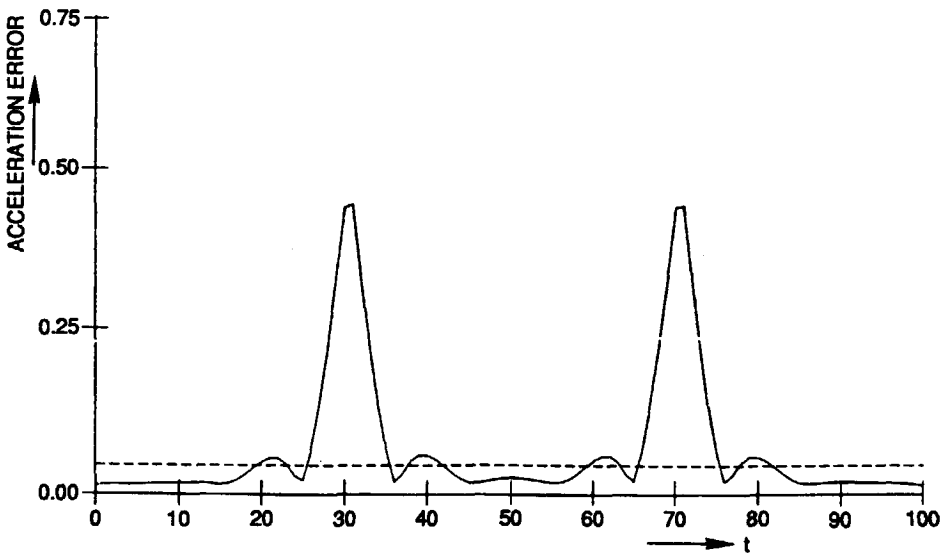


Fig. 6 Measured and expected RMS of the acceleration errors of the Kalman smoother; — = measured RMS, - - - = expected RMS (RMS)

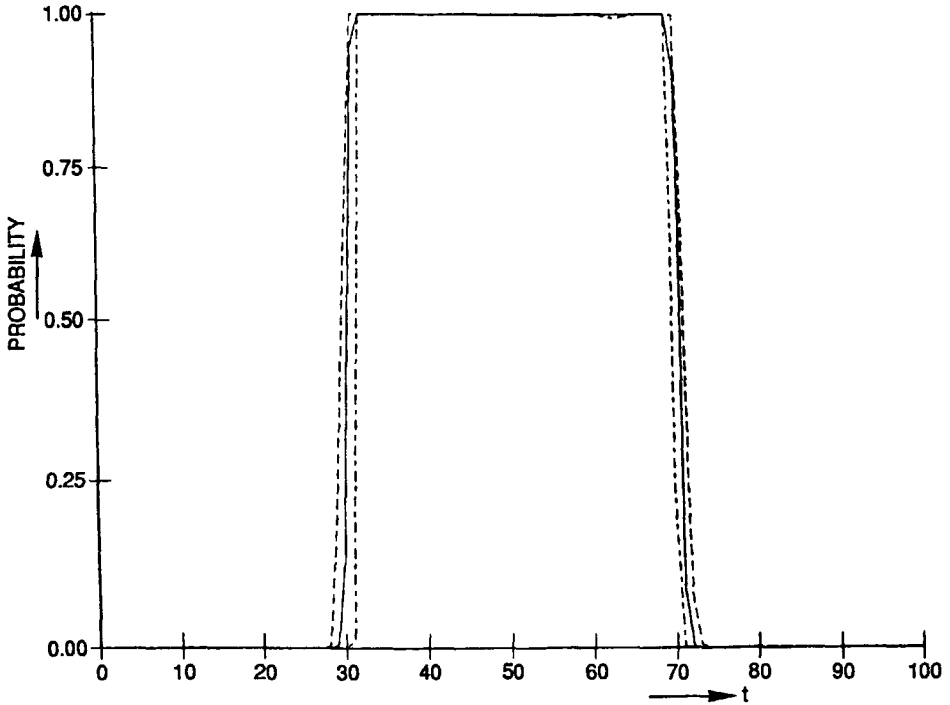


Fig. 7 Mean (—), maximal (----) and minimal (-·-·-) values of the IMM smoother estimated probability of acceleration.

The figures clearly indicate that the IMM smoother outperforms the Kalman smoother in all aspects: significantly more accurate during constant acceleration, significantly smaller errors around acceleration discontinuities, and significantly more reliable estimates of its own RMS errors. These performance improvements come at the cost of about a doubling of the computational load, which is a remarkably small increase for the size of performance improvements obtained. Moreover, the IMM smoother provides very reliable conditional probabilities of either being in the non-accelerating or the accelerating mode (see fig. 7).

From the above, it may be expected that IMM smoothing is a serious competitor of other fixed-interval smoothing algorithms. This leads immediately to the question if and how the smoothing approach presented can be extended to fixed-lag smoothing (Mathews and Tugnait, 1983). Obviously, the study of these interesting questions falls beyond the scope of the present thesis. For now we conclude that the smoothing results obtained form a good illustration of the practical use of the time-reversion results developed.

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## Chapter IV

## CONTINUOUS-TIME HYBRID STATE MARKOV PROCESSES

## 1 Introduction

Because many of the stochastic processes that we meet in nature have a state space that is the product of a Euclidean space and a discrete set, we often need pathwise models on such a hybrid state space. As a result, several classes of hybrid state space models have been developed, such as diffusions with Markovian switching parameters, doubly stochastic counting processes and Markov decision drift processes. These models are used in quite different fields of applications, because of which their studies have often evolved separately. An important reason to study hybrid state space processes within a common framework is that their martingale parts are in general discontinuous. This property has attracted a lot of attention, and is by now very well documented (Jacod, 1979; Cinlar et al., 1980; Bremaud, 1981; Elliott, 1982; Bensoussan and Lions, 1984; Ethier and Kurtz, 1986; Jacod and Shiryaev, 1987). It is quite clear from these results that, to study hybrid state Markov processes along the same lines as diffusions, we need pathwise representations of those processes. Unfortunately, for hybrid state Markov processes there is presently a lacuna of pathwise representations. To show this lacuna we depict the main classes of hybrid state Markov processes in a Venn-diagram.

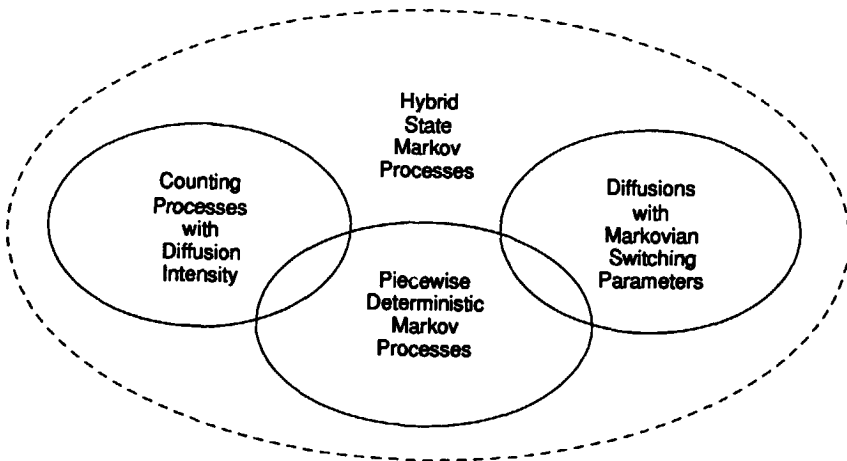


Fig. 1 Main classes of hybrid state Markov processes

There exist pathwise representations of counting processes with diffusion intensity (Snyder, 1975; Marcus, 1978), of diffusions with Markovian switching parameters (Wonham, 1970; Brockett and Blankenship, 1977) and of Piecewise Deterministic (PD) Markov processes (Davis, 1984). For many other Markov processes in figure 1, no such pathwise representations exist (Kingman, 1975; Anulova, 1979, 1982; Çinlar and Jacod, 1981; Bensoussan and Lions, 1984; Belbas and Lenhart, 1986).

Actually, PD Markov processes seem the most interesting of all processes in figure 1, as they cover all major non-diffusion Markov processes; PD Markov processes exclude diffusion, but include a random switching intensity, a jump reflecting boundary and hybrid jumps. With the latter we mean jumps of the Euclidean-valued process component that anticipate a simultaneous switching of the discrete valued process component. As such, PD Markov processes provide a framework to study Markov decision drift processes (Hordijk and Van der Duyn Schouten, 1983; Yushkevich, 1983; Van der Duyn Schouten, 1986) along the same line as diffusions (Vermes, 1985). With this, an interesting generalization is to extend the spectrum of hybrid state Markov processes with Jump Diffusion (JD) Markov processes, by including diffusion into PD Markov processes. As the present definition of PD processes does not seem to have an opening left for that inclusion (Davis, 1984), we need a different approach.

The approach that overcomes this difficulty, presented in the sequel, is to assume a stochastic differential equation (SDE) in a hybrid space and to construct rather large classes of PD and JD Markov processes from it. With respect to the state space we restrict our attention to a hybrid subset of a Euclidean space. Then the most general SDE is of Itô-Skorohod type (Métivier, 1982, pp. 240-272),

$$d\xi_t = \alpha(\xi_t)dt + \beta(\xi_t)dw_t + \int_U \psi(\xi_{t-}, u) p_p(dt, du),$$

where  $\{w_t\}$  is a Brownian motion, and  $p_p$  is a Poisson random measure (defined in section 2) on  $(0, \infty) \times U$ . The path of a solution of this SDE is right continuous and has left hand limits:

$$\xi_{t-} = \lim_{\Delta \downarrow 0} \xi_{t-\Delta}.$$

Further, if  $p_p$  generates a multivariate point  $(t, u_t)$ , then the path of  $\xi$  has a discontinuity:

$$\xi_t = \xi_{t-} + \psi(\xi_{t-}, u_t).$$

In the sequel we extend the Itô-Skorohod equations to cover more general Markov semimartingales. Therefore we shall focus on pathwise unique solutions. The classical result for the existence of such solutions requires that  $\psi$  is sufficiently continuous

(Gihman and Skorohod, 1972), which restricts the SDE essentially to systems with Markovian switching coefficients. However, there are some non-classical pathwise uniqueness results that allow a discontinuous  $\psi$  (Lepeltier and Marchal, 1976; Jacod and Protter, 1982; Situ, 1985; Veretennikov, 1988). Taking these results as a starting point, we introduce and evaluate a particular form for  $\psi$  in section 3, which models a random jump rate.

Next, in section 4, we consider the hybrid state space situation. The most interesting effect of the hybrid state space assumption is that it leads to a particular type of jumps: jumps in the continuous state component of  $\{\xi_t\}$  that anticipate a simultaneous switching of the discrete component of  $\{\xi_t\}$ . This type of jumps has been introduced by Gnedenko and Kovalenko (1968) for piecewise linear processes and by Sworder (1972) for systems with Markovian switching coefficients. For short we refer to these anticipating simultaneous jumps as hybrid jumps. The SDE framework of this paper provides an elegant way of representing the hybrid jumps of PD Markov processes and their Jump Diffusion generalizations. Section 5 concludes the results with SDE representations for different types of linear systems with switching coefficients.

Finally, we might put the question how to model a process with a jump reflecting boundary as a solution of an SDE. Given a boundary  $\partial O$  of an open subset  $O$  of  $R^n$ , the problem is to construct an SDE such that  $\{\xi_t\}$  undergoes an instantaneous jump into the interior of  $O$  if  $\{\xi_t\}$  tries to cross or to travel through  $\partial O$ . The simplest solution would be that the Poisson random measure  $p_p$  instantaneously generates a point each time  $\{\xi_t\}$  hits  $\partial O$ . However, this is not possible as a Poisson random measure generates almost surely no point at such an arbitrary time. To overcome this problem, we might think of adding to  $p_p$  another random measure, which generates one point each time  $\{\xi_t\}$  would otherwise cross or travel through  $\partial O$ . Although this extension seems possible, it will not be treated in the sequel since it falls outside the scope of our present study on Bayesian estimation.

Some other interesting generalizations of PD Markov processes, not considered in the sequel, are the inclusion of continuously reflecting or sticky boundaries. The inclusion of a continuously reflecting boundary, while preserving pathwise uniqueness, seems possible if that boundary is smooth enough (Chaleyat-Maurel et al., 1980; Menaldi and Robin, 1985; Frankowska, 1985; Saisho, 1987). The inclusion of a sticky boundary without losing pathwise uniqueness seems difficult if not impossible (Kingman, 1975; Anulova, 1979, 1982).



## 2 Poisson random measure

Following Jacod and Shiryaev (1987), we introduce in this section Poisson random measures and their integration. Throughout this chapter we always work within a complete stochastic basis  $(\Omega, \mathcal{F}, P, T)$ , with  $T = \mathbb{R}_+$  and  $\mathcal{F}$  right continuous (see Appendix A2).

### 2.1 Definition

Let  $U$  denote a finite dimensional Euclidean space.

A random measure  $p(\cdot; dt, du)$ , or  $p(dt, du)(\cdot)$ , on  $\mathbb{R}_+ \times U$  is a family  $(p(\omega; dt, du); \omega \in \Omega)$  of nonnegative measures on  $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(U)$ , such that  $p(\omega; \{0\} \times U) = 0$  for all  $\omega$ .

Let  $\mathcal{E}$  denote the  $\sigma$ -algebra on  $T \times \Omega$  that is generated by all cadlag adapted processes and let  $\mathcal{P}$  denote the  $\sigma$ -algebra on  $T \times \Omega$  that is generated by all adapted left-continuous processes.

Let  $p$  be a random measure and let  $W(\cdot)$  denote an  $\mathcal{E} \times \mathcal{B}(U)$ -measurable mapping of  $\Omega \times \mathbb{R}_+ \times U$  into  $\mathbb{R}$ , then we define the  $\mathbb{R}U\{\omega\}$ -valued integral process  $\{[W^*p]_t\}$  as follows:

$$\begin{aligned}
 [W^*p]_t(\omega) &= \int_{[0, t] \times U} W(\omega, s, u) p(\omega; ds, du), \\
 & \quad \text{if } \int_{[0, t] \times U} |W(\omega, s, u)| p(\omega; ds, du) < \infty, \\
 &= +\infty, \quad \text{otherwise.}
 \end{aligned}$$

A random measure is called optional if the process  $\{[W^*p]_t\}$  is optional for every  $\mathcal{E} \times \mathcal{B}(U)$ -measurable mapping  $W$ . A random measure is called predictable if the process  $\{[W^*p]_t\}$  is predictable for every  $\mathcal{P} \times \mathcal{B}(U)$ -measurable mapping  $W$ .

A random measure is said to be  $\mathcal{P} \times \mathcal{B}(U)$ - $\sigma$ -finite if there exists a  $\mathcal{P} \times \mathcal{B}(U)$ -measurable partition  $(A_i)$  of  $\Omega \times \mathbb{R}_+ \times U$ , such that each  $[1_{A_i}^*p]_\infty$  is integrable.

Jacod and Shiryaev (1987; J&S) prove a slightly more general version of the following (J&S, Th. 1.8, p.66): Let  $p(\omega; dt, du)$  be an optional  $\mathcal{P} \times \mathcal{B}(U)$ -measurable random measure. There exists a predictable random measure, called the compensator of  $p$  (or (dual) predictable compensator of  $p$ ) and denoted by  $p^D$ , which is unique up to a  $P$ -null set, and which satisfies:

$$E([W^*p^D]_\infty) = E([W^*p]_\infty)$$

for every nonnegative  $\mathcal{P} \times \mathcal{B}(U)$ -measurable function  $W$ .

### 2.2 Definition

An integer-valued random measure is an optional  $\mathcal{P} \times \mathcal{B}(U)$ - $\sigma$ -finite random measure  $p(\omega; dt, du)$  satisfying:

- $p(\omega; \{t\} \times U) \leq 1$ , for every  $\omega$ ,
- for each  $A \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(U)$ ,  $p(\cdot; A)$  assumes values in  $\mathbb{N}U\{\omega\}$ .

### 2.3 Proposition

Let  $p(\omega; dt, du)$  be an integer-valued random measure.

The set  $D$ , defined by

$$D = \{(\omega, t); p(\omega; \{t\} \times U) = 1\},$$

is a thin set (see appendix A2), and there exists an  $U$ -valued optional process  $\{u_t\}$  such that, for every  $\omega$ ,

$$p(\omega; \cdot) = \sum_s 1_D(\omega, s) \delta_{(s, u_s(\omega))}(\cdot),$$

where  $\delta_\alpha$  denotes the Dirac measure at point  $\alpha$  (i.e.,  $\delta_\alpha(A) = 1$  for all measurable  $A$  containing point  $\alpha$ , while  $\delta_\alpha(A) = 0$  otherwise).

Proof: See J&S, pp.68-69.

Hence, if  $(\tau_i)$  is an exhausting sequence for the thin set  $D$ , the integer-valued random measure  $p(\omega; \cdot)$  admits the following representation:

$$p(\omega; \cdot) = \sum_{(i; \tau_i < \infty)} \delta_{(\tau_i(\omega), u_{\tau_i}(\omega))}(\cdot), \text{ for all } \omega,$$

and if  $W(\cdot)$  is  $\mathcal{E} \times \mathcal{B}(U)$ -measurable and nonnegative, the integral expression for the integrated process  $\{[W^*p]_t\}$  reduces to:

$$[W^*p]_t(\omega) = \sum_{(i)} W(\omega, \tau_i(\omega), u_{\tau_i}(\omega)) 1_{\{\tau_i \leq t\}}(\omega).$$

Jacod and Shiryaev (1987) prove the following (Prop. 1.17, p.70): If  $p(\omega; dt, du)$  is an integer-valued random measure and  $p^p$  its compensator,  $p^p$  admits a predictable version  $p'$ , such that  $p'(\omega; \{t\} \times U) \leq 1$  for all  $(\omega, t)$ , while the set  $\{(\omega, t); p'(\omega; \{t\} \times U) > 0\}$  is exhausted by a sequence of predictable times.

The intensity measure  $\nu(dt, du)$  of an integer-valued random measure is defined by  $\nu(A) = E\{p(\cdot; A)\}$ ;  $\nu$  is said to be  $\sigma$ -finite if there exists a sequence of sets  $A_i \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)$ , such that  $A_i \uparrow \mathbb{R}_+ \times U$  for increasing  $i$ , while  $\nu(A_i) < \infty$  for every  $i$ .

Obviously, we are specially interested in integer-valued random measures that are not predictable. This leads us to Poisson and extended Poisson random measures.

### 2.4 Definition (J&S, p.70, Def.1.20a)

An extended Poisson random measure on  $\mathbb{R}_+ \times U$ , relative to the filtration  $\mathcal{F}$ , is an integer-valued random measure  $p(\omega; dt, du)$  which satisfies:

- its intensity measure  $\nu$  is  $\sigma$ -finite,
- for every  $t \in \mathbb{R}_+$  and every  $A \in \mathcal{B}(t, \infty) \times \mathcal{B}(U)$  such that  $\nu(A) < \infty$ , the variable  $p(\cdot; A)$  is independent of the  $\sigma$ -algebra  $\mathcal{F}_t$ .

The best known type of extended Poisson random measure is the Poisson random measure.

### 2.5 Definition

A Poisson random measure is an extended Poisson random measure, the intensity measure  $\nu$  of which satisfies  $\nu(\{t\}, U) = 0$ , for all  $t$ . A Poisson random measure is said to be homogeneous if its intensity measure is of the form  $\nu(dt, du) = dt.m(du)$ .

A Poisson random measure is the counting measure of a Poisson point process (see Jacod and Shiryaev, p.71).

Next, from Jacod and Shiryaev (pp.104-106) we have some characterizations of an extended Poisson random measure:

### 2.6 Proposition

An integer-valued random measure  $p(\omega; \cdot)$ , with intensity measure  $\nu(\cdot)$  and compensator  $p^P(\omega; \cdot)$ , is an extended Poisson random measure if and only if  $\nu(\cdot)$  is  $\sigma$ -finite,  $\nu(\{t\} \times U) \leq 1$  for every  $t$ , and  $p^P(\omega; \cdot) = \nu(\cdot)$  for every  $\omega$ .

Moreover, if we associate to an extended Poisson random measure  $p(\omega; \cdot)$  two new measures  $p_F(\omega; \cdot)$  and  $p_p(\omega; \cdot)$  as follows,

$$\begin{aligned} p_F(\omega; dt, dx) &= p(\omega; dt, dx) \mathbf{1}_{\{s; \nu(\{s\} \times U) > 0\}}(t), \\ p_p(\omega; \cdot) &= p(\omega; \cdot) - p_F(\omega; \cdot), \end{aligned}$$

then  $p_p$  and  $p_F$  are independent and  $p_p$  is a Poisson random measure.

Proof: See J&S, pp. 105-106.

## 3 The SDE of Lepeltier and Marchal

We assume a complete stochastic basis  $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{P}_+, \mathbb{R}_+)$ , endowed with an  $m$ -dimensional standard Wiener process,  $\{w_t\}$ , and a Poisson random measure,  $p_p(dt, du)(\cdot) = p_p(\cdot; dt, du)$  on  $\mathbb{R}_+ \times U$ , with intensity measure  $\nu(dt, du) = dt.m(du)$ , and consider the following stochastic differential equation (SDE) in  $\mathbb{R}_+ \times \mathbb{R}^n$ ,

$$\begin{aligned} d\xi_t &= \alpha(\xi_t)dt + \beta(\xi_t)dw_t + \int_{U_1} \psi(\xi_{t-}, u) q(dt, du) + \\ &\quad + \int_{U_2} \psi(\xi_{t-}, u) p_p(dt, du), \end{aligned} \quad (1)$$

where  $q$  is the martingale measure of  $p_p$ ; i.e.

$$q(dt, du)(\cdot) = p_p(dt, du)(\cdot) - \nu(dt, du),$$

$\xi_0$  is an  $\mathcal{F}_0$ -measurable  $\mathbb{R}^n$ -valued random variable, while  $\alpha$ ,  $\beta$  and  $\psi$  are measurable mappings of appropriate dimensions (with domains  $\mathbb{R}^n$ ,  $\mathbb{R}^n \times \mathbb{R}^m$  and  $\mathbb{R}^n$ , respectively).

Although, in the sequel, we are not really using the third right hand term, we start from (1) to notice the main difference between the roles played by the first and fourth right hand terms. The set-up commonly used is to partition  $U$  into  $U_1$  and  $U_2$  and to assume conditions such that:

$\{ \int_0^t \int_{U_1} \psi(\xi_{s-}, u) q(ds, du) \}$  is a local martingale, while the process

$\{ \int_0^t \int_{U_2} \psi(\xi_{s-}, u) p_p(ds, du) \}$  has finite variation over each finite

interval. The classical reference for an SDE of type (1) is Gihman and Skorohod (1972), who considered the situation  $U_2 = \emptyset$ . Extensions of their results have been obtained by Lepeltier and Marchal (1976) in their study of the relation between an integro-differential operator and an SDE of type (1), with  $U_1 = \{u; |u| \leq 1\}$  and  $U_2 = \{u; 1 < |u| < \infty\}$ .

The partition we use in the sequel is:

$$U_1 = R^- \times R^d, \quad \text{with } R^- = (-\infty, 0),$$

$$U_2 = R_+ \times R^d, \quad \text{with } R_+ = [0, \infty).$$

Hence, the results of Lepeltier and Marchal can easily be translated to our case by introducing measurable mappings, of  $\{u \in R^{d+1}; |u| \leq 1\}$  into  $R^- \times R^d$  and of  $\{u \in R^{d+1}; 1 < |u| < \infty\}$  into  $R_+ \times R^d$ , and subsequently transforming  $m$  and  $\psi$  correspondingly. Consequently, the results of Lepeltier and Marchal can immediately be used during our present study.

To that end we make the following assumptions:

A.1 There is a constant  $K$  such that, for all  $\xi \in R^n$ ,

$$|\alpha(\xi)|^2 + \|\beta(\xi)\|^2 + \int_{R^- \times R^d} |\psi(\xi, u)|^2 m(du) \leq K(1 + |\xi|^2).$$

A.2 For all  $k \in \mathbb{N}$  there exists a constant  $L_k$  such that, for all  $\xi$  and  $y$  in the ball  $B_k = \{x \in R^n; |x| \leq k+1\}$ ,

$$|\alpha(\xi) - \alpha(y)|^2 + \|\beta(\xi) - \beta(y)\|^2 + \int_{R^- \times R^d} |\psi(\xi, u) - \psi(y, u)|^2 m(du) \leq L_k |\xi - y|^2.$$

A'.3  $m(R_+ \times R^d) < \infty$ .

A'.4 For every  $k \in \mathbb{N}$  there exists a constant  $M_k$ , such that

$$\sup_{|\xi| \leq k} \int_{R_+ \times R^d} |\psi(\xi, u)| m(du) \leq M_k.$$

### 3.1 Proposition

Let A.1, A.2, A'.3 and A'.4 be satisfied and let  $U_1 = \mathbb{R}^- \times \mathbb{R}^d$  and  $U_2 = \mathbb{R}_+ \times \mathbb{R}^d$ . Then equation (1) has for every initial condition  $\xi_0(\omega) = \xi \in \mathbb{R}^n$  a pathwise unique solution,  $\{\xi_t\}$ , which is cadlag and adapted. Moreover, there exists a measurable random function  $f(t, \xi, \omega)$  such that  $\xi_t(\cdot) = f(t, \xi, \cdot)$  almost surely for every  $t$ .

Proof: See Lepeltier and Marchal (1976, Theorem III<sub>4</sub>, pp.84-85).

Remarks:

- With a pathwise unique solution we mean that if there is more than one solution, those solutions are modifications of each other.
- If the fourth right hand term of equation (1) vanishes, then it is well known that A.1 and A.2 are sufficient conditions (Gihman and Skorohod, 1972, pp. 273-275). Lepeltier and Marchal show that adding the fourth right hand term does not change the existence of a pathwise unique solution, under A'.3 and A'.4.
- Recently the result in proposition 3.1 has been extended to the situation that  $\beta$  is not Lipschitzian (Kleptsyna, 1984; Situ, 1986; Veretennikov, 1988).
- The existence of an in law unique solution has been shown to hold true under more general non-Lipschitz conditions (Komatsu, 1984a,b; Negoro and Tsuchiya, 1989).

The interesting aspect of proposition 3.1 is that the coefficients of the fourth right hand term of (1) may be discontinuous in  $\xi$ . This is exactly what we need to construct a class of hybrid state Markov processes that is larger than the class of solutions of systems with Markovian switching coefficients. The first step towards this construction is using an idea of Jacod and Protter (1982; Protter, 1983), by assuming

$$m(du) = \begin{cases} du_1 \times \mu(du), & \text{on } [-c, C] \times \mathbb{R}^n, \\ 0, & \text{else,} \end{cases}$$

for some pair  $(c, C)$  of values in  $\mathbb{R}_+$ , and replacing  $\psi(\xi, u)$  by  $\psi'(\xi, u)$ :

$$\psi'(\xi, u) = \psi(\xi, \underline{u}) \mathbf{1}_{[-c, \Lambda(\xi)]}(u_1),$$

where  $\Lambda$  is a measurable mapping of  $\mathbb{R}^n$  into  $\mathbb{R}_+$ , while  $\psi$  is a measurable mapping of  $\mathbb{R}^n \times \mathbb{R}^d$  into  $\mathbb{R}^n$  and  $\mu$  is a probability measure (underlining of the vector  $u$  refers to all, except the first components of  $u$ ).

From now on, we restrict our attention to the situation that  $c=0$ . With this, the replacement of  $\psi$  by  $\psi'$  in (1) yields the following SDE:

$$d\xi_t = \alpha(\xi_t)dt + \beta(\xi_t)dw_t + \int_{R_+ \times R^d} \psi(\xi_{t-}, \underline{u}) 1_{[0, \Lambda(\xi_{t-})]}(u_1) p_P(dt, du). \quad (2)$$

Next, we introduce the following assumptions:

A.3  $\Lambda(\xi)$  is twice continuously differentiable in  $\xi$ .

A.4 For all  $k \in \mathbb{N}$  there exists a constant  $M_k$ , such that  $\sup_{|\xi| \leq k} \int_{R^d} |\psi(\xi, \underline{u})| \mu(d\underline{u}) \leq M_k$ .

A.5 There is a constant  $C$  such that  $\Lambda(\xi) \leq C$ , for every  $\xi$ .

3.2 Corollary

Let  $\alpha$  and  $\beta$  satisfy A.1 and A.2, while  $\psi$  and  $\Lambda$  satisfy A.3, A.4 and A.5. Then for every initial condition  $\xi_0(\omega) = \xi$ , equation (2) has a pathwise unique solution,  $\{\xi_t\}$ , which is cadlag and adapted. Moreover, there exists a measurable random function  $f(t, \xi, \omega)$  such that  $\xi_t(\cdot) = f(t, \xi, \cdot)$  almost surely for every  $t$ .

Proof:

On defining the mapping  $x(\xi, u_1)$  by

$x(\xi, u_1) = 1_{[0, \Lambda(\xi)]}(u_1)$ , for every  $\xi$  and every  $u_1$ , the third right hand term of (2) can be replaced by

$$\int_{R_+ \times R^d} \psi(\xi_{t-}, \underline{u}) x(\xi_{t-}, u_1) p_P(dt, du),$$

which implies that (2) is an equation of type (1). Due to A.3,  $x(\xi, u_1)$  is measurable in  $(\xi, u_1)$ . Hence A.3, A.4 and A.5 imply that the conditions of proposition 3.1 are satisfied. Q.E.D.

Next, we give a semimartingale characterization of the solution  $\{\xi_t\}$  of equation (2).

3.3 Theorem

Let the assumptions of corollary 3.2 hold true. Then for all  $\xi_0 \in R^n$ ,  $\{\xi_t\}$  is a semimartingale Markov process, and its extended generator,  $\mathcal{A}$ , is given by:

$$\mathcal{A}f = \mathcal{A}f + \mathcal{G}f, \quad \text{for all } f \in D(\mathcal{A}) \supset C^2(R^n), \quad (3)$$

where

$$\mathcal{A}f(\xi) = \sum_{i=1}^n \alpha_i(\xi) f_{\xi_i}(\xi) + \frac{1}{2} \sum_{i,j=1}^n [\beta(\xi)\beta(\xi)^T]_{ij} f_{\xi_i \xi_j}(\xi), \quad (4)$$

$$\mathcal{G}f(\xi) = \Lambda(\xi) \int_{R^n - \{0\}} [f(\zeta) - f(\xi)] S(\xi; d\zeta), \quad (5)$$

and for all Borel  $A \subset R^n - \{0\}$ ,

$$S(\xi; A) = \int_{R^d} 1_A(\xi + \psi(\xi, \underline{u})) \mu(d\underline{u}). \quad (6)$$

Proof:

The existence of a measurable  $f$  in Corollary 3.2 implies the Markov property.

Due to A.3 and A.4, the predictable part  $\{a_t\}$  of  $\{\xi_t\}$  satisfies

$$a_t = \int_0^t \alpha(\xi_s) ds + \int_0^t \Lambda(\xi_{s-}) \int_{\mathbb{R}^d} \psi(\xi_{s-}, \underline{u}) \mu(d\underline{u}) ds,$$

up to indistinguishability. Obviously,  $\{a_t\}$  is of finite variation on any finite time interval, while  $\{\xi_t - a_t - \xi_0\}$  is a local martingale (see Appendix A.4). Hence,  $\{\xi_t\}$  is a semimartingale (see Appendix A.5), by which the extended generator  $\mathcal{A}$  follows from Itô's differentiation rule for discontinuous semimartingales. Q.E.D.

Now we are prepared to consider the hybrid state space situation.

#### 4 The hybrid state space situation

In this section we explicitly consider the hybrid state space situation for a system of the form (2). The main objective of this section is to show that the last term of (2) generates a particular type of jump: a jump in  $\{\xi_t\}$  that anticipates a simultaneous switching of  $\{\xi_t^1\}$ . For short we refer to this type of jump as hybrid jump. Notice that such a hybrid jump is in some sense unexpected, as all coefficients of (2) are non-anticipating. Before showing this, we rewrite (2) as follows:

$$d\xi_t = \alpha(\xi_t) dt + \beta(\xi_t) dw_t + \int_{\mathbb{R}^d} \psi(\xi_{t-}, \underline{u}) p_I(dt, \mathbb{R}_+ x d\underline{u}), \quad (7.a)$$

with  $p_I$  the integer-valued random measure

$$p_I(dt, A) = \int_A \mathbf{1}_{[0, \Lambda(\xi_{t-})]}(u_1) p_p(dt, du), \quad (7.b)$$

for every  $A \in \beta(U)$ .

To consider the hybrid state space situation, we assume that the first component of  $\xi_t$  is  $M$ -valued, with  $M = \{1, 2, \dots, N\}$ , by which we can write the first scalar equation of (7.a) as follows:

$$d\xi_t^1 = \int_{\mathbb{R}^d} \psi_1(\xi_{t-}, \underline{u}) p_I(dt, du), \quad (8)$$

with  $\psi_1$  a mapping of  $\mathbb{R}^n \times \mathbb{R}^d$  into  $Z = \{., -2, -1, 0, 1, 2, .\}$ .

Further, we assume that  $\psi$  satisfies

$$\psi(\xi, \underline{u}) = \sum_{\eta \in M} \varphi(\eta, \xi, \underline{u}) \mathbf{1}_{(\Sigma_{\eta-1}(\xi), \Sigma_{\eta}(\xi))}(u_1), \quad (9)$$

with:  $\Sigma_{\eta}(\xi) = \sum_{i=0}^{\eta} \rho(i, \xi)$ , for  $\eta \in Z$ ,

$$\mu(d\underline{u}) \stackrel{\Delta}{=} \mu_1(d\underline{u}_1) \underline{\mu}(d\underline{u}),$$

where  $\mu_1(\cdot)$  is a uniform distribution on  $[0,1]$ ,  $\varphi$  is a measurable mapping of  $M \times \mathbb{R}^n \times \mathbb{R}^{d-1}$  into  $Z \times \mathbb{R}^{n-1}$ , and  $\rho$  is a measurable mapping of  $N \times \mathbb{R}^n$  into  $\mathbb{R}_+$ , such that

$$\begin{aligned} \rho(i, \cdot) &= 0, & i \in N/M, \\ \sum_{i \in N} \rho(i, \xi) &= 1. \end{aligned}$$

Moreover, we assume that for all  $\eta \in M$ ,  $\xi \in Z \times \mathbb{R}^{n-1}$  and  $\underline{u} \in \mathbb{R}^{d-1}$ ,

$$\varphi_1(\eta, \xi, \underline{u}) = \eta - \xi_1, \quad (10)$$

which, together with (9) and  $\rho(i, \cdot) = 0$  for all  $i \in N/M$ , implies that if  $\xi_0^1(\omega) \in M$ , then  $\xi_t^1(\omega) \in M$ , for all  $t$ . Substitution of (9) and (10) into (7.a) yields:

$$d\xi_t^1 = \sum_{\eta \in M} (\eta - \xi_{t-}^1) p_{I, \eta}(dt, \mathbb{R}_+ \times \mathbb{R}^d), \quad (11.a)$$

$$d\underline{x}_t = \underline{\alpha}(\xi_t) dt + \underline{\beta}(\xi_t) d\underline{w}_t + \sum_{\eta \in M} \int_{\mathbb{R}^{d-1}} \varphi(\eta, \xi_{t-}, \underline{u}) p_{I, \eta}(dt, \mathbb{R}_+ \times \mathbb{R}^d \times d\underline{u}), \quad (11.b)$$

with, for all  $A \in \mathcal{B}(U)$ ,

$$p_{I, \eta}(dt, A) = \int_A 1_{(\Sigma_{\eta-1}(\xi_{t-}), \Sigma_{\eta}(\xi_{t-}))}(\underline{u}_1) p_I(dt, d\underline{u}), \quad (11.c)$$

$$p_I(dt, A) = \int_A 1_{[0, \Lambda(\xi_{t-})]}(\underline{u}_1) p_P(dt, d\underline{u}). \quad (11.d)$$

Next, we introduce the following assumptions:

**B.1** There is a constant  $K$  such that, for all  $\xi \in M \times \mathbb{R}^{n-1}$ ,

$$|\alpha(\xi)|^2 + \|\beta(\xi)\|^2 \leq K(1 + |\xi|^2).$$

**B.2** For all  $k \in \mathbb{N}$  there exists a constant  $L_k$  such that, for all  $\xi$  and  $\gamma$  in the set  $\{x \in M \times \mathbb{R}^{n-1}; |x| \leq k+1\}$ ,

$$|\alpha(\xi) - \alpha(\gamma)|^2 + \|\beta(\xi) - \beta(\gamma)\|^2 \leq L_k |\xi - \gamma|^2.$$

**B.3**  $\Lambda(\xi)$  is twice continuously differentiable in  $\xi$ .

**B.4** For all  $k \in \mathbb{N}$  there exists a constant  $M_k$ , such that

$$\sup_{|\xi| \leq k} \sum_{\eta \in M} \rho(\eta, \xi) [|\eta - \xi_1| + \int_{\mathbb{R}^{d-1}} |\varphi(\eta, \xi, \underline{u})| \underline{\mu}(d\underline{u})] \leq M_k.$$

**B.5** There is a constant  $C$  such that  $\Lambda(\xi) \leq C$ , for every  $\xi$ .

**B.6**  $\rho(\cdot, \xi)$  is twice continuously differentiable in  $\xi$ .



#### 4.1 Corollary

Given the hybrid space  $O = M \times R^{n-1}$ . Under assumptions B.1 through B.6, the system of equations (11.a,b,c,d) has for any  $\xi_0(\omega) = \xi \in O$  a pathwise unique solution  $\{\xi_t\}$ , which is cadlag adapted. Moreover  $\{\xi_t\}$  is then a semimartingale Markov process in  $R_+ \times O$ .

##### Proof:

Due to B.6, (9) defines  $\varphi$  as a measurable mapping (see proof of corollary 3.2). Hence, (11) is a special case of (7.a,b), and it remains to be shown that A.1-A.5 are satisfied. For A.1, A.2, A.3 and A.5 this is obvious, and we only have to show that B.4 implies A.4. Therefore, we subsequently substitute (10) in B.4, interchange order of integration and substitute (9):

$$\begin{aligned} & \sum_{\eta \in M} \rho(\eta, \xi) \left[ |\eta - \xi| + \int_{R^{d-1}} |\varphi(\eta, \xi, \underline{u})| \mu(d\underline{u}) \right] = \\ & = \sum_{\eta \in M} \rho(\eta, \xi) \int_{R^{d-1}} |\varphi(\eta, \xi, \underline{u})| \mu(d\underline{u}) = \\ & = \sum_{\eta \in M} \int_{R^{d-1}} \mathbb{1}_{(\Sigma_{\eta-1}(\xi), \Sigma_{\eta}(\xi))}(\omega_1) d\omega_1 \int_{R^{d-1}} |\varphi(\eta, \xi, \underline{u})| \mu(d\underline{u}) = \\ & = \int_{R^{d-1}} \int_{R^{d-1}} \sum_{\eta \in M} \mathbb{1}_{(\Sigma_{\eta-1}(\xi), \Sigma_{\eta}(\xi))}(\omega_1) |\varphi(\eta, \xi, \underline{u})| d\omega_1 \mu(d\underline{u}) = \\ & = \int_{R^d} |\psi(\xi, \omega)| \mu(d\omega), \end{aligned}$$

which implies that B.4  $\Rightarrow$  A.4.

Q.E.D.

To make the hybrid jumps more explicit, we notice that due to the definition of  $p_{I, \eta}$ ,

$$p_{I, \eta}(\{t\}, R_+ \times R^d) \in \{0, 1\}, \text{ any } \eta \in M,$$

$$\sum_{\eta \in M} p_{I, \eta}(\{t\}, R_+ \times R^d) = 1.$$

This means that (11.b) can be simplified to:

$$d\xi_t = \underline{\alpha}(\xi_t) dt + \underline{\beta}(\xi_t) d\omega_t + \int_{R^{d-1}} \varphi(\xi_t^1, \xi_{t-}, \underline{u}) p_I(dt, R_+ \times R^d d\underline{u}), \quad (12)$$

By taking a closer look at (12), we notice the interesting aspect that  $\xi_t^1$  appears in the coefficient of (12)'s third right hand term. This means that this coefficient anticipates a switching from  $\xi_{t-}^1$  to  $\xi_t^1$ , and thus a jump of  $\{\xi_t\}$  anticipates a simultaneous transition of  $\{\xi_t^1\}$ ; i.e.  $\{\xi_t\}$  has hybrid jumps. Since there is no anticipating coefficient in equation (11.b), and the solutions of equations (11.b) and (12) are indistinguishable, we conclude that (11.b) forms a kind of canonical representation of a system with hybrid jumps. However, (12) is the representation that is more useful when it comes to the practical realization of Markov models with hybrid jumps.

### 5 Linear systems with switching coefficients

For  $\varphi=0$ , system (12) has had a lot of attention in the past, mainly in the case that  $\{\xi_t^1\}$  is a Markov process and that  $\xi_t$  is conditionally Gaussian given the  $\sigma$ -algebra generated by  $\{\xi_s^1; s \leq t\}$ . It would be interesting to study (12) under similar conditions when  $\varphi \neq 0$ . For this, the differentials in (12) should be linear in  $(\xi_t, dw_t, \underline{u})$ . In addition, for conditionally Gaussian  $\xi_t$  the mapping  $\varphi(\theta, (\theta, \cdot), \underline{u})$  should vanish for all  $\theta$ . For Markovian  $\{\xi_t^1\}$ , the intensity  $\lambda(\xi) \cdot \rho(\theta, \xi)$  should be  $\xi$ -invariant for all  $(\theta, \xi^1)$ .

To introduce similar conditions when  $\varphi \neq 0$ , we rewrite (12) as follows:

$$\begin{aligned} d\xi_t &= \underline{\alpha}(\xi_t)dt + \underline{\beta}(\xi_t)dw_t + \\ &+ \int_{\mathbb{R}^d} \varphi(\xi_t^1, \xi_{t-}, \underline{u}) \mathbf{1}_{\{0\}}(\Delta\xi^1 t) p_I(dt, R_+ \times \mathbb{R}^d \times d\underline{u}) + \\ &+ \int_{\mathbb{R}^{d-1}} \varphi(\xi_t^1, \xi_{t-}, \underline{u}) \mathbf{1}_{Z/\{0\}}(\Delta\xi^1 t) p_I(dt, R_+ \times \mathbb{R}^d \times d\underline{u}). \end{aligned} \quad (13)$$

Next, we introduce the following assumptions:

$$\begin{aligned} \underline{L1} \quad \underline{\alpha}(\xi) &= A_C(\xi^1) + A_X(\xi^1)\xi, \quad \underline{\beta}(\xi) = B(\xi^1), \\ \varphi(\theta, \xi, \underline{u}) &= C_C(\theta, \xi^1) + C_X(\theta, \xi^1)\xi + C_u(\theta, \xi^1)\underline{u}. \end{aligned}$$

L2  $\lambda(\xi) \cdot \rho(\theta, \xi)$  is  $\xi$ -invariant and bounded.

L3  $\rho(\theta, (\theta, \xi)) = 0$  for all  $(\theta, \xi)$ .

L2'  $\lambda(\theta, \xi) \cdot \rho(\theta, (\theta, \xi))$  is  $\xi$ -invariant and bounded.

L3'  $\rho(\theta, (\theta, \xi)) = 0$  for all  $(\theta, \xi) \in \partial\mathcal{O}$ .

#### 5.1 Definition

System (13) is called a linear system with randomly switching coefficients if L1, L2' and L3' hold true.

#### 5.2 Definition

System (13) is called a system with Markovian switching coefficients if L2 holds true.

#### 5.3 Corollary

System (13) is called a linear system with Markovian switching coefficients if L1 and L2 hold true.

#### 5.4 Definition

System (13) is called a linear Gaussian system with randomly switching coefficients if L1 and L3 hold true and  $\underline{\mu}$  is Gaussian.

#### 5.5 Corollary

System (13) is called a linear Gaussian system with Markovian switching coefficients if L1, L2 and L3 hold true and  $\underline{\mu}$  is Gaussian.

The classical sense definitions of linear systems with randomly switching coefficients assume, in addition to L1, that  $C_i(\theta, \eta) = 0$  for all  $\theta \neq \eta$ ,  $i=0,1,2$  (see Brockett and Blankenship, 1977). Hence the classical sense definitions exclude hybrid jumps. Although scarce, there are some applications reported with hybrid jumps: Sworder (1972), Blom (1984), Mariton (1987).

A very interesting subclass of systems with randomly switching coefficients are those with semi-Markovian switching coefficients. To represent them by SDE's, we assume that  $\underline{\xi}_t = \text{Col}\{\underline{\xi}_H, t, \underline{\xi}_L, t\}$  and that  $\Lambda$  and  $\rho$  are invariant w.r.t.  $\underline{\xi}_L, t$ . Moreover, we assume that during each jump,  $\{\underline{\xi}_H, t\}$  starts independent of the past; i.e.  $[\varphi_H(\eta, \underline{\xi}, \underline{\mu}) + \underline{\xi}_H]$  is  $\underline{\xi}$ -invariant, say  $f(\eta, \underline{\mu})$ . The above implies that  $\{\underline{\xi}_H, t\}$  and  $\{\underline{\xi}_L, t\}$  are pathwise unique solutions of:

$$d\underline{\xi}_H, t = \underline{\alpha}_H(\underline{\xi}_t)dt + \underline{\beta}_H(\underline{\xi}_t)d\underline{w}_t + \int_{\mathbb{R}^d} d-1 [f(\underline{\xi}_t^1, \underline{\mu}) - \underline{\xi}_H, t-] p_I(dt, \mathbb{R}_+ \times \mathbb{R}^d \times d\underline{\mu}), \quad (14.a)$$

$$d\underline{\xi}_L, t = \underline{\alpha}_L(\underline{\xi}_t)dt + \underline{\beta}_L(\underline{\xi}_t)d\underline{w}_t + \int_{\mathbb{R}^d} d-1 \varphi_L(\underline{\xi}_t^1, \underline{\xi}_t-, \underline{\mu}) p_I(dt, \mathbb{R}_+ \times \mathbb{R}^d \times d\underline{\mu}). \quad (14.b)$$

Now we introduce the following assumptions:

S1  $\underline{\alpha}_H(\underline{\xi})$ ,  $\underline{\beta}_H(\underline{\xi})$  and  $\Lambda(\underline{\xi})\rho(\theta, \underline{\xi})$  are  $\underline{\xi}_L$ -invariant,

S2  $\underline{\alpha}_L(\underline{\xi})$ ,  $\underline{\beta}_L(\underline{\xi})$  and  $\varphi_L(\theta, \underline{\xi}, \underline{\mu})$  are  $\underline{\xi}_H$ -invariant,

#### 5.6 Definition

Let  $\{\underline{\xi}_H, t\}$  satisfy (14.a). System (14.b) is called a system with semi-Markovian switching coefficients if S1 and S2 hold true.

#### 5.7 Corollary

Let  $\{\underline{\xi}_H, t\}$  satisfy (14.a). System (14.b) is a linear system with semi-Markovian switching coefficients iff  $\underline{\alpha}_L, \underline{\beta}_L$  and  $\varphi_L$  satisfy L1 and S1 and S2 hold true.

### 5.8 Corollary

Let  $\{\xi_H, \tau\}$  satisfy (14.a). System (14.b) is a linear Gaussian system with semi-Markovian switching coefficients iff  $\alpha_L, \beta_L$  and  $\varphi_L$  satisfy S1,  $\rho(\theta, \theta, \xi) = 0$  for all  $(\theta, \xi)$ ,  $\underline{u}$  is Gaussian and conditions S1 and S2 hold true.

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## Chapter V

## FILTERING CONTINUOUS-TIME HYBRID STATE PROCESSES

## 1 Introduction

This chapter addresses the problem of evaluating the conditional distribution of a hybrid state Markov process given a filtration  $\{\mathcal{Y}_t\}$  generated by an observations process  $\{y_t\}$ . The hybrid Markov process has the components  $\{x_t\}$  and  $\{\theta_t\}$ , assuming values in  $\mathbb{R}^n$  and  $M=\{1, \dots, N\}$ , respectively. Since a realization of such a conditional distribution is a nonnegative measure on  $\mathbb{R}^n \times M$ , its numerical evaluation is in general not a simple problem. For an exact numerical evaluation we need a finite-dimensional sufficient statistic which characterizes the conditional distribution uniquely. For practical filter applications, where the dimension may not be "too large", the problem even is to introduce a not "too large" statistic which characterizes the conditional distribution either exactly or approximately. In the sequel, we develop a low dimensional approximate Bayesian filter for linear systems with Markovian switching coefficients. Before starting with this development, we review the existing, exact and approximate finite dimensional results.

For the evaluation of the conditional distribution of a hybrid state Markov process two approaches exist: a classical approach and a Bayesian approach. The classical approach evaluates all possible paths of  $\{\theta_s; s \leq t\} \sim \theta_t$ , under the condition that there are countably many such paths. For each path  $\theta_t$ , the evaluation characterizes the conditional distribution of  $x_t$  given  $(\theta_t, \mathcal{Y}_t)$  and the conditional mass of  $\theta_t$  given  $\mathcal{Y}_t$ . Next, the conditional distribution  $\Pi_t = \mu_{x_t, \theta_t | \mathcal{Y}_t}$  of  $(x_t, \theta_t)$  given  $\mathcal{Y}_t$  becomes:

$$\Pi_t(\cdot, \theta) = \sum_{\theta_{t-1}} \mu_{x_t | \theta_t, \mathcal{Y}_t}(\cdot | (\theta_{t-1}, \theta)) p_{\theta_t | \mathcal{Y}_t}(\theta_{t-1}, \theta). \quad (1)$$

This leads to a finite dimensional solution iff, for every  $t$ , there are finitely many possible paths  $\theta_t$ , and each conditional distribution of  $x_t$  given  $\theta_t$  is finite dimensional. Both conditions are satisfied in the multiple model (MM) situation:  $N$  is finite,  $\{\theta_t\}$  is continuous (no switching) and  $\{x_t, y_t\}$  is the solution of a linear Gaussian system, the coefficients of which are governed by  $\{\theta_t\}$ . If  $\{\theta_t\}$  is switching at random moments in time, the possible paths of  $\{\theta_t\}$  are not countable (Fujishige and Sawaragi, 1974), and the classical approach does not lead to a finite dimensional solution.



The Bayesian approach consists of evaluating the conditional distribution of the Markov state as an Hilbert space valued solution of a closed system of stochastic differential equations (SDE's). For the evaluation of an  $R^n$ -valued diffusion, the Bayesian approach has led to the identification of a large class of problems admitting a finite dimensional sufficient statistic. These are the "almostly linearizable" diffusions (Zeitouni, 1984; Shukhman, 1985; Kolodziej and Mohler, 1986; Haussmann and Pardoux, 1988) and processes that are immersible in conditionally "almostly linearizable" diffusions, while no other examples are known. "Almostly linearizable" diffusions consist of combinations of the following situations:

- Conditionally Gaussian systems (Liptser and Shiryaev, 1978),
- Almost linear Gaussian systems (Benes, 1981, 1985, 1987; Wong, 1983; Zeitouni and Bobrovsky, 1986),
- Linear Gaussian systems with non-Gaussian initial conditions (Ocone, 1980; Makowski, 1982, 1986; Benes and Karatzas, 1983),
- Systems that are immersible in linear Gaussian systems (Wong, 1983; Lévine and Marino, 1986).

In the hybrid state situation, the Bayesian approach yields  $\{\Pi_t\}$  as the solution of a Hilbert space valued stochastic differential equation. Obviously,  $\Pi_t$  admits the following decomposition:

$$\Pi_t(\cdot, \theta) = \mu_{x_t | \theta_t, \mathcal{Y}_t}(\cdot | \theta) p_{\theta_t | \mathcal{Y}_t}(\theta), \quad \text{for every } \theta \in M, \quad (2)$$

which means that a realization of  $\Pi_t$  is the product of a counting measure on  $M$  and Lebesgue measures on  $R^n$  (one for each  $\theta \in M$ ). Hence, the Bayesian approach leads to the evaluation of a counting measure on  $M$  and  $N$  Lebesgue measures on  $R^n$ , which means that the complexity of evaluating  $\{\Pi_t\}$  is time-invariant. The solution of this evaluation is finite dimensional iff  $N$  is finite and each of the  $N$  Lebesgue measures admits a finite dimensional parametrization. Examples of the latter are the "almostly linearizable" MM case and the situation that  $\{y_t\}$  consists of two components,  $\{y_t^x\}$  and  $\{y_t^\theta\}$ , while  $\{x_t, y_t^x\}$  and  $\{\theta_t, y_t^\theta\}$  are independent and  $\{x_t, y_t^x\}$  is immersible in a conditionally "almostly linearizable" diffusion. Unfortunately, these are rather degenerate hybrid state examples, while no other finite dimensional examples are known. Furthermore, there even is significant evidence (Roth and Loparo, 1986) that no other finite dimensional hybrid state example exists.

In view of the lack of finite dimensional sufficient statistics for the conditional distributions of non-trivial hybrid state Markov processes, we have to accept finite dimensional approximations of  $\{\Pi_t\}$ . Inherent to the problem, any approximation is a compromise between approximation "accuracy"

and statistic "dimension", while the best compromise depends of the particular application. Since hybrid state Markov processes appear so often in practice, several approximate Bayesian filters have been developed in the past; mainly for linear Gaussian systems with Markovian switching coefficients. The most common approach is to approximate  $\{\theta_t\}$  by a process  $\{\theta'_t\}$  the random switching of which only may happen at a finite number of discrete moments on any finite time interval. This results in countably many approximated paths and allows application of the classical approach. As the number of possible paths grows in general exponentially with time (Fujishige and Sawaragi, 1974), it is common practice to truncate less likely paths heuristically (Au et al., 1978, 1982; Loparo et al., 1984, 1986). For a reasonable approximation of the conditional distribution, many ( $\gg N$ ) paths often need be evaluated. Even if the switching rate of  $\{\theta_t\}$  approaches arbitrarily close to zero, the number of paths to be evaluated does not approach to  $N$  (Marcus and Westwood, 1984).

With the help of the general MMSE (minimum mean square error) filter theory for semimartingales, some low dimensional MMSE filters have been developed (Davis, 1975; Björk, 1980, 1982; Krishnan, 1981). For linear systems with Markovian switching coefficients, Björk (1980) identifies some problems admitting a low dimensional sufficient statistic w.r.t. MMSE filtering, while the conditional distribution admits not a finite dimensional sufficient statistic. In a subsequent study, these results are extended to some nonlinear system situations (Björk, 1982). The filters of Björk all have in common that the observation process  $\{y_t\}$  is independent of  $\{\theta_t\}$ , which is a rather degenerate hybrid state space situation. In an early paper, Davis (1975) developed a heuristic approximation to the optimal detection of a failuring drift coefficient in a linear Gaussian system. The result is a modified type of MM filter. Since this filter can not be applied if the failure switches the diffusion coefficient only, Krishnan (1981) developed another approximate filter (of "significantly higher dimension"). Of all these filters it is unclear whether they might be represented as approximate Bayesian filters.

Presently, the spectrum of exact and approximate Bayesian filters lacks one of "low dimensionality". Therefore, we develop such a filter in the sequel. In order of keeping the presentation self contained, we first give an overview of the main body of filter theory. First, in section 2, we present the fundamental filtering theorem. Next, in section 3, we develop SDE's for the conditional distribution of a Markov process. To support practical implementation, in addition to the commonly used Itô characterizations, we present a Wiener characterization. Subsequently, in section 4, we develop our low dimensional approximate Bayesian filter for linear systems with Markovian switching coefficients, and mention the

relation with the filters of Björk and Davis. Finally, in section 5, we develop the associated continuous-discrete filter equations.

## 2 Fundamental filtering theorem

In this section, we develop the fundamental filtering theorem under quite general conditions. Therefore we assume to work on a complete stochastic basis  $(\Omega, \mathcal{F}, P, R_+)$  with a right continuous filtration  $F$ . We consider a pair of adapted processes  $\{x_t\}$  and  $\{y_t\}$ , with  $\{x_t\}$  real valued and  $\{y_t\}$   $R^m$ -valued, which are pathwise unique solutions of the following equations:

$$x_t = x_0 + \int_0^t a_s ds + m_t, \quad (3)$$

$$y_t = y_0 + \int_0^t b_s ds + w_t, \quad (4)$$

both up to indistinguishability, with  $\{a_t\}$  and  $\{b_t\}$  progressive finite-valued processes,  $\{m_t\}$  and  $\{w_t\}$  local martingales with a predictable quadratic covariation

$$\langle M, W^i \rangle_t = \int_0^t \lambda_s^i ds, \quad i \in [1, m], \quad (5)$$

up to indistinguishability, while  $\{\lambda_t\}$  is progressive. The problem we consider is to develop a differential equation,

the pathwise unique solution of which is  $\{\hat{x}_t\}$ ;

$$\hat{x}_t \stackrel{\Delta}{=} E\{x_t | \mathcal{F}_t^V\},$$

up to indistinguishability. In the sequel, this will be done under the following assumptions:

**A.1**  $\{b_t\}$  is  $P$ -integrable.

**A.2**  $\{w_t\}$  is an  $m$ -dimensional standard Wiener process.

**A.3**  $\{x_t\}$  and  $\{a_t\}$  are  $P$ -integrable.

**A.4**  $\{\lambda_t\}$  is  $P$ -integrable.

**A.5**  $\{x_t b_t\}$  is  $P$ -integrable.

### 2.1 Theorem (fundamental filtering)

Let assumptions **A.1** through **A.5** hold true and let  $F^V = F^Y$ . Then

$$\hat{x}_t = \hat{x}_0 + \int_0^t \hat{a}_s ds + \int_0^t [\hat{\lambda}_s + E\{x_s b_s | \mathcal{F}_s^V\} - \hat{x}_s \hat{b}_s]^T dv_s, \quad (6)$$

$$v_t = y_t - \int_0^t \hat{b}_s ds, \quad (7)$$

both up to indistinguishability, with  $^T$  denoting transpose.

Remark: There are three approaches in deriving the fundamental filtering theorem: an abstract Bayesian approach, an innovations approach and a change of measure approach (Liptser and Shiryaev, 1977; Kallianpur, 1980; Elliott, 1982; Davis, 1984; Wong and Hajek, 1985; Rogers and Williams, 1987). Although the most general results have been obtained by the change of measure approach (Davis and Marcus, 1981; Kunita, 1982; Mitter, 1982; Pardoux, 1982), we prefer to follow the intuitively more appealing innovations approach. Unfortunately, the innovations approach has largely been based on Kailath's conjecture that there are mild conditions under which  $F^V = F^Y$ , while the validity of this conjecture has been proven under rather restrictive conditions only; see Krylov (1979), Allinger and Mitter (1981), Toronjadze (1986) and Situ (1987).

Proof:

First we show that  $\{v_t\}$  is a Brownian motion. For every  $t \geq s$

$$\begin{aligned} E\{v_t - v_s | \mathcal{F}_s^Y\} &= E\{w_t - w_s - \int_s^t [b_u - E\{b_u | \mathcal{F}_u^Y\}] du | \mathcal{F}_s^Y\} = \\ &= E\{w_t - w_s | \mathcal{F}_s^Y\} = 0, \end{aligned}$$

which implies that  $\{v_t - v_0\}$  is a local wide sense  $F^Y$ -martingale.

Due to A.2,  $\{w_t\}$  is continuous. Due to A.1,  $\{\int_0^t [b_s - E\{b_s | \mathcal{F}_s^Y\}] ds\}$

is continuous. Hence,  $\{v_t\}$  is continuous and thus a local  $F^Y$ -martingale. Next, application of Itô's differentiation rule to  $\{v_t^i v_t^j\}$  and (3) yields:

$$dv_t^i v_t^j = [v_t^i dv_t^j + v_t^j dv_t^i] dt + d\langle W^i, W^j \rangle_t.$$

Hence,

$$d\langle v^i, v^j \rangle_t = d\langle W^i, W^j \rangle_t = 1_{(i=j)} dt,$$

thus  $\{v_t\}$  is an  $m$ -dimensional standard Brownian motion.

Next, we define a process  $\{n_t\}$  as follows:

$$n_t = \hat{x}_t - \int_0^t \hat{a}_s ds, \quad (\text{with } n_0 = \hat{x}_0) \quad (8)$$

up to indistinguishability. Hence,  $\{n_t\}$  is cadlag,  $F^V$ -adapted and

$$\begin{aligned} E\{n_t - n_s | \mathcal{F}_s^V\} &= E\{\hat{x}_t - \hat{x}_s - \int_s^t \hat{a}_u du | \mathcal{F}_s^V\} = \\ &= E\{x_t - x_s - \int_s^t a_u du | \mathcal{F}_s^V\} = \\ &= E\{m_t - m_s | \mathcal{F}_s^V\} = 0, \end{aligned}$$

which implies that  $\{n_t - n_0\}$  is a local  $F^V$ -martingale. But then, the martingale representation theorem (See appendix A.3) yields: There exists an  $F^V$ -predictable process  $\{h_t\}$ , such that

$$n_t = \hat{x}_0 + \int_0^t h_s^T dv_s,$$

up to indistinguishability, while  $\{\int_0^t h_s^T h_s ds\}$  is locally integrable. Hence,

$$\hat{x}_t = \hat{x}_0 + \int_0^t \hat{a}_s ds + \int_0^t h_s^T dv_s. \quad (9)$$

The remaining task is to characterize  $\{h_t\}$ .

Application of Itô's differentiation rule to  $\{x_t y_t\}$  yields

$$x_t y_t = x_0 y_0 + \int_0^t x_s (b_s ds + dw_s) + y_s (a_s ds + dm_s) + \lambda_s ds,$$

up to indistinguishability. In view of this, we define  $\{f_t\}$  as

$$f_t = \hat{x}_t y_t - \int_0^t [E\{x_s b_s | \mathcal{F}_s^V\} + y_s \hat{a}_s + \hat{\lambda}_s] ds,$$

up to indistinguishability.

Hence,  $\{f_t\}$  is cadlag,  $F^V$ -adapted and

$$\begin{aligned} E\{f_t - f_s | \mathcal{F}_s^V\} &= \\ &= E\{\hat{x}_t y_t - \hat{x}_s y_s - \int_s^t [E\{x_u b_u | \mathcal{F}_u^V\} + y_u \hat{a}_u + \hat{\lambda}_u] du | \mathcal{F}_s^V\} = \\ &= E\{x_t y_t - x_s y_s - \int_s^t [x_u b_u + y_u a_u + \lambda_u] du | \mathcal{F}_s^V\} = \\ &= E\{\int_s^t x_u (b_u du + dw_u) + y_u (a_u du + dm_u) + \lambda_u du + \\ &\quad - [x_u b_u + y_u a_u + \lambda_u] du | \mathcal{F}_s^V\} = \\ &= E\{\int_s^t x_u dw_u + y_u dm_u | \mathcal{F}_s^V\} = 0. \end{aligned}$$

But this implies:  $\{f_t - f_0\}$  is a local  $F^V$ -martingale and  $\{\hat{x}_t y_t\}$  is an  $F^V$ -semimartingale, with decomposition

$$\hat{x}_t y_t = \bar{e}_t + f_t, \quad (f_0 = \hat{x}_0 y_0) \quad (10)$$

$$\bar{e}_t = \int_0^t [E\{x_s b_s | \mathcal{F}_s^V\} + y_s \hat{a}_s + \hat{\lambda}_s] ds, \quad (11)$$

both up to indistinguishability. Because  $\{\bar{e}_t\}$  is continuous,  $\{\bar{e}_t\}$  is  $F^V$ -predictable, and  $\{\hat{x}_t y_t\}$  is special, with (10) its canonical martingale decomposition. To obtain another characterization of

$\{\bar{e}_t\}$ , we next apply Itô's differentiation rule to  $\{\hat{x}_t y_t\}$ :

$$\begin{aligned} \hat{x}_t y_t &= \hat{x}_0 y_0 + \int_0^t \hat{x}_s dy_s + y_s d\hat{x}_s + h_s ds = \\ &= \hat{x}_0 y_0 + \int_0^t \hat{x}_s (\hat{b}_s ds + dv_s) + y_s (\hat{a}_s ds + h_s^T dv_s) + h_s ds, \end{aligned}$$

up to indistinguishability. Hence, comparison with (10) yields

$$\bar{e}_t = \int_0^t [\hat{x}_s \hat{b}_s + y_s \hat{a}_s + h_s] ds, \quad (12)$$

up to indistinguishability. Comparison of (11) with (12) yields

$$\int_0^t [\hat{x}_s \hat{b}_s + h_s] ds = \int_0^t [E\{x_s b_s | \mathcal{F}_s^V\} + \hat{\lambda}_s] ds,$$

and since  $\{h_t\}$  is  $F^V$ -predictable,

$$h_t = \mathcal{L}\hat{\lambda}_t + \mathcal{L}E\{x_t b_t | \mathcal{F}_t^V\} - \mathcal{L}[\hat{x}_t \hat{b}_t],$$

up to indistinguishability. This and (9) yields (6).

Q.E.D.

If the P-integrability conditions in A.1 through A.5 are replaced by local integrability conditions, then it can be verified that the fundamental filtering theorem holds true on a finite interval  $[0, T]$ . For extensions of the fundamental filtering theorem to point process observations see Van Schuppen (1979), Wong and Hajek (1985) and Cohen (1988).

### 3 Evolution of the conditional distribution

In general, the fundamental filtering formula does not provide a closed form solution of the filtering problem. As such, we next develop filter equations for the conditional distribution of a semimartingale  $X$ -valued Markov process  $\{\xi_t\}$ , with extended generator  $\mathcal{A}$  acting on mappings in its domain  $D(\mathcal{A})$ .  $\mathcal{A}$  is assumed to be time-invariant for notational simplicity only.

Hence, for every  $f \in D(\mathcal{A})$ , the process  $\{f(\xi_t)\}$  satisfies,

$$f(\xi_t) = f(\xi_0) + \int_0^t \mathcal{A}f(\xi_{s-}) ds + m_t, \quad (13)$$

$$y_t = y_0 + \int_0^t h(\xi_s) ds + w_t, \quad (14)$$

both up to indistinguishability, with  $\{m_t\}$  a local martingale and  $\{w_t\}$  an  $m$ -dimensional standard Wiener process, such that

$$\langle M^C, W^i \rangle_t = \int_0^t \mathcal{B}^i f(\xi_s) ds, \text{ for all } i \in [1, m], \quad (15)$$

up to indistinguishability.

Now we introduce the following conditions:

B.1  $h \in C^2$ .

B.2  $h(\xi_0)$  is integrable.

B.3  $f \in C^2$ .

B.4  $f(\xi_0)$  and  $f(\xi_0)h(\xi_0)$  are integrable.

B.5  $\{\mathcal{A}f(\xi_t)\}$  and  $\{\mathcal{B}f(\xi_t)\}$  are locally integrable.

Next, application of the fundamental filtering theorem yields a recursive equation for the process  $\{\hat{f}(\xi_t)\}$ ;

$$\hat{f}(\xi_t) \stackrel{\Delta}{=} E\{f(\xi_t) | \mathcal{F}_t^V\}. \quad (16)$$

3.1 Corollary (Fujisaki-Kallianpur-Kunita (FKK) equation)

Let  $\{y_t\}$  be observations of a quasi-left continuous semimartingale Markov process  $\{\xi_t\}$  with extended generator  $[\mathcal{A}, D(\mathcal{A})]$  satisfying (13), (14) and (15), let conditions B.1 and B.2 hold true and let  $F^V = F^Y$ . If  $S$  contains all  $f \in D(\mathcal{A})$  for which B.3, B.4 and B.5 are satisfied, then, for every  $f \in S$ , and  $t \leq T < \infty$ ,

$$\hat{f}(\xi_t) = \hat{f}(\xi_0) + \int_0^t E[\mathcal{A}f(\xi_s) | \mathcal{F}_s^V] ds + \int_0^t [E[\mathcal{B}f(\xi_s) + h(\xi_s)f(\xi_s) | \mathcal{F}_s^V] - \hat{h}(\xi_s)\hat{f}(\xi_s)]^T dv_s, \quad (17.a)$$

$$v_t = y_t - \int_0^t \hat{h}(\xi_s) ds, \quad (17.b)$$

both up to indistinguishability.

What we actually like to have, is a differential equation for the conditional "density" of a Markov process  $\{\xi_t\}$ . If the state space is hybrid, such "density" is a mixed density-probability function:

3.2 Proposition (Kushner-Stratonovich-Wonham (KSW) equation)

Let  $\{y_t\}$  be observations of a quasi-left continuous semimartingale Markov process  $\{\xi_t\}$  with extended generator  $[\mathcal{A}, D(\mathcal{A})]$  and satisfying (13), (14) and (15), let  $P(\xi_0 \in dx | \mathcal{F}_0^Y)$

admit the "density"  $\hat{\pi}_0(x)$ , let conditions B.1, and B.2 hold true, let  $F^V = F^Y$  and let  $\mathcal{A}^*$  and  $\mathcal{B}^*$  be the adjoints of  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. If there is, on  $[0, T]$ , a Hilbert space valued

pathwise unique solution  $\{\hat{\pi}_t(\cdot)\}$  of

$$\hat{\pi}_t(\cdot) = \hat{\pi}_0(\cdot) + \int_0^t \mathcal{A}^* \hat{\pi}_s(\cdot) ds + \int_0^t [\mathcal{B}^* \hat{\pi}_s(\cdot) + \hat{\pi}_s(\cdot) [h(\cdot) - \hat{h}(\xi_s)]^T] dv_s, \quad (18)$$

then  $\{P(\xi_t \in dx | \mathcal{F}_t^V)\}$  admits  $\{\hat{\pi}_t(x)\}$  as its "density" on  $[0, T]$ .

Proof: If (18) has a pathwise unique solution, then

$$\int_X f(x) \hat{\pi}_t(x) dx = \int_X f(x) \hat{\pi}_0(x) dx + \int_0^t \int_X f(x) \mathcal{A}^* \hat{\pi}_s(x) dx ds + \int_0^t \int_X [f(x) \mathcal{B}^* \hat{\pi}_s(x) + f(x) \hat{\pi}_s(x) [h(x) - \hat{h}(\xi_s)]^T] dv_s dx,$$

up to indistinguishability, for every  $f \in S$ .

The forward generators,  $\mathcal{A}^*$  and  $\mathcal{B}^*$ , are such that, for all  $f \in S$ ,

$$\int_X \hat{\pi}_t(x) \mathcal{A}f(x) dx = \int_X f(x) \mathcal{A}^* \hat{\pi}_t(x) dx,$$

$$\int_X \hat{\pi}_t(x) \mathcal{B}f(x) dx = \int_X f(x) \mathcal{B}^* \hat{\pi}_t(x) dx,$$

both up to indistinguishability. Substituting this yields

$$\int_{\mathbf{X}} f(x) \hat{\pi}_t(x) dx = \int_{\mathbf{X}} f(x) \hat{\pi}_0(x) dx + \int_0^t \int_{\mathbf{X}} \hat{\pi}_s(x) \mathcal{A}f(x) ds dx + \\ + \int_0^t \int_{\mathbf{X}} \hat{\pi}_s(x) [\mathfrak{B}f(x) + h(x)f(x) - \hat{h}(\xi_s)f(x)]^T dx dv_s,$$

for all  $f \in \mathcal{S}$ , up to indistinguishability. To match (17.a),  $\{\hat{\pi}_t(\cdot)\}$  must be the conditional "density" on  $[0, T]$ . Q.E.D.

For sufficient conditions under which (18) has a pathwise unique solution see Liptser and Shirayev (1977, 1978), Pardoux (1979), Bismut & Michel (1982), Baras et al., (1983), Fujisaki (1988), Kurtz & Ocone (1988), Haussmann (1988).

Notice that the last term of the KSW equation is nonlinear. To simplify the situation, we introduce a transformation that removes this nonlinearity.

### 3.3 Theorem (Duncan-Mortensen-Zakai (DMZ) equation)

Let the conditions of proposition 3.2 be satisfied. Then,

$$\hat{\pi}_t(\cdot) = \hat{\rho}_t(\cdot) / \int_{\mathbf{X}} \hat{\rho}_t(\xi) d\xi, \quad t \in [0, T], \quad (19.a)$$

where  $\{\hat{\rho}_t\}$  is an "unnormalized conditional density", satisfying the DMZ equation

$$\hat{\rho}_t(\cdot) = \hat{\rho}_0(\cdot) + \int_0^t \mathcal{A}^* \hat{\rho}_s(\cdot) ds + \int_0^t [\mathfrak{B}^* \hat{\rho}_s(\cdot) + \hat{\rho}_s(\cdot) h^T(\cdot)] dy_s, \quad (19.b)$$

up to indistinguishability on  $[0, T]$ .

Proof: Define a scalar valued process  $\{\Lambda_t\}$ ,

$$d\Lambda_t = \Lambda_t \hat{h}(\xi_t)^T dy_t, \text{ up to indistinguishability,}$$

with  $\Lambda_0 = 1$ . Next, define an Hilbert space valued  $\{\hat{\rho}_t\}$  by

$$\hat{\rho}_t(\cdot) = \Lambda_t \hat{\pi}_t(\cdot), \text{ up to indistinguishability.}$$

Hence, application of Itô's differentiation rule yields

$$d\hat{\rho}_t(x) = \mathcal{A}^* \hat{\rho}_t(x) dt + [\mathfrak{B}^* \hat{\rho}_t(x) + \hat{\rho}_t(x) h(x)^T] dy_t, \text{ on } [0, T].$$

Furthermore,

$$d[\int_{\mathbf{X}} \hat{\rho}_t(x) dx] = \int_{\mathbf{X}} d\hat{\rho}_t(x) dx = \int_{\mathbf{X}} \hat{\rho}_t(x) h(x)^T dy_t dx = \\ = \Lambda_t \int_{\mathbf{X}} \hat{\pi}_t(x) h(x)^T dy_t = \Lambda_t \hat{h}(\xi_t)^T dy_t = d\Lambda_t,$$

which implies (19.a,b).

Q.E.D.

Obviously, the KSW and the DMZ equations involve stochastic integrals which are not of Wiener type. Unfortunately, it is not always clear how to interpret such a stochastic integral equation physically, while numerical integration is complicated



(Pardoux and Talay, 1985). If, however,  $\{\xi_t\}$  and  $\{w_t\}$  are independent, the DMZ equation admits a Wiener representation:

### 3.4 Corollary (log-Zakai equation)

Let the conditions of proposition 3.2 be satisfied, let  $\{\xi_t\}$  and  $\{w_t\}$  be independent and let  $\pi_t(\cdot) > 0$  for all  $t$ . The unnormalized density  $\{\hat{\rho}_t\}$  satisfies, up to indistinguishability on  $[0, T]$ ,

$$\begin{aligned} \mathcal{L}\hat{\rho}_t(\cdot) = \mathcal{L}\pi_0(\cdot) + \int_0^t [\hat{\rho}_s(\cdot)]^{-1} \mathcal{A}^* \hat{\rho}_s(\cdot) ds + \\ + \int_0^t h(\cdot)^T dy_s - \frac{1}{2} \int_0^t h(\cdot)^T h(\cdot) ds. \end{aligned} \quad (20)$$

Notice that the log-Zakai equation has a nonlinear drift but a Wiener type of diffusion. This implies that the physical interpretation of the log-Zakai equation is unambiguous, and thus the physical interpretations of the KSW and DMZ equations are unambiguous too. The log-Zakai equation can also be obtained when  $\{\xi_t\}$  and  $\{w_t\}$  are dependent, but the diffusion term is than not of Wiener type. Remarkably, the log-Zakai equation has gone rather unnoticed, although it already appears in Zakai (1965) for a finite state Markov process. For diffusions, it appears in Fleming and Mitter (1982; Fleming, 1982; Blom, 1981). Better known is a disguised log-Zakai version: the "Multiplicative Functional Filter" (MFF) equation (Clark, 1978; Davis, 1980, 1984; Elliott and Kohlmann, 1981; Wong and Hajek, 1985; Davis and Spathopoulos, 1987). To apply the MFF approach,  $\pi_t$  need not be strictly positive.

Unfortunately, the total measure,  $\Lambda_t$ , of  $\hat{\rho}_t$  increases rapidly with time. To see this, substitute

$$dy_t = \hat{h}(\xi_t)dt + dv_t,$$

in the SDE for  $\Lambda_t$ . Hence,

$$d\Lambda_t = \Lambda_t \hat{h}(\xi_t)^T \hat{h}(\xi_t) dt + \Lambda_t \hat{h}(\xi_t)^T dv_t,$$

which implies that  $\{\Lambda_t\}$  is an  $F^V$ -supermartingale, which increases rapidly with time. Hence, the solutions of the Zakai, the log-Zakai and the MFF equations are not stable. To prevent this, we might better use a quotient as an unnormalized "density":

$$\hat{q}_t(\cdot) \stackrel{\Delta}{=} \hat{\pi}_t(\cdot) / \hat{\pi}_t(x) = \hat{\rho}_t(\cdot) / \hat{\rho}_t(x), \quad (21)$$

with  $x \in X$ , such that  $h(x)=0$  (if there is no such  $x$ , we can create one by modifying the observations process  $\{y_t\}$  to  $\{y'_t\}$ , satisfying  $dy'_t = dy_t - h(x)dt$ , for some  $x \in X$ ).

Subsequent application of Itô's differentiation rule to (20),

(21) and  $\{\mathcal{L}\hat{q}_t\}$  yields a log-Quotient equation on  $[0, T]$ :

$$\begin{aligned} \ln \hat{q}_t(\cdot) = & \int_0^t [\hat{q}_s(\cdot)^{-1} \mathcal{A}^* \hat{q}_s(\cdot) - \mathcal{A}^* \hat{q}_s(x)] ds + \\ & + \int_0^t h(\cdot)^T dy_s - \frac{1}{2} h(\cdot)^T h(\cdot) ds. \end{aligned} \quad (22)$$

Notice that the only difference between the log-Quotient equation and the log-Zakai equation is the extra drift term  $\mathcal{A}^* \hat{q}_t(x)$ . If the Markov state space is  $M = \{1, \dots, N\}$ , then the log-Quotient equation consists of  $N$  scalar Wiener differential equations. As demonstrated in (Kemp and Blom, 1981; Kemp, 1987) this allows for straightforward practical implementation.

#### 4 The Interacting Multiple Model filter

If the Markov state space is hybrid ( $\mathbb{R}^n \times M$ ), finite dimensional sufficient statistics for the conditional "density" are only known in rather degenerate cases (Roth and Loparo, 1986). Even for linear Gaussian systems with Markovian switching coefficients we have to accept approximate numerical evaluations of the conditional distribution. In this section we develop a "low dimensional" such approximation. This approximation originally appears in Blom (1982, 1984), for less general situations.

Let  $\{x_t\}$  be the evolving state of a linear system with Markovian switching coefficients  $\{\theta_t\}$ , that is observed through the  $\mathbb{R}^m$  valued process  $\{y_t\}$ :

$$dy_t = h(\theta_t, x_t) dt + dw_t. \quad (23)$$

Following chapter IV (see eqs. 11 and 12), we write  $\{\theta_t, x_t\}$  as a pathwise unique solution of a system of stochastic equations;

$$d\theta_t = \sum_{\eta \in M} (\eta - \xi^1_{t-}) p_{I, \eta}(dt, R_+ \times \mathbb{R}^d), \quad (24.a)$$

$$\begin{aligned} dx_t = & \alpha(\theta_t, x_t) dt + \beta_V(\theta_t, x_t) dv_t + \beta_W(\theta_t, x_t) dw_t + \\ & + \int_{\mathbb{R}^{d-1}} \varphi(\theta_t, \theta_{t-}, x_{t-}, \underline{u}) p_I(dt, R_+ \times \mathbb{R}^d \times \underline{u}), \end{aligned} \quad (24.b)$$

with, for all  $A \in \mathcal{B}(U)$ ,

$$p_{I, \eta}(dt, A) = \int_A \mathbf{1}_{(\Sigma_{\eta-1}(\xi_{t-}), \Sigma_{\eta}(\xi_{t-}))} (u_1) p_I(dt, du),$$

$$p_I(dt, A) = \int_A \mathbf{1}_{[0, \Lambda(\xi_{t-})]} (u_1) p_P(dt, du),$$

$$\Sigma_{\eta}(\xi) = \sum_{i=0}^{\Delta} \sum_{\eta} \rho(i, \xi), \quad \eta \in \mathbb{Z},$$

where  $\rho$  is a measurable mapping of  $N \times \mathbb{R}^n$  into  $\mathbb{R}_+$ , such that

$$\rho(i, \cdot) = 0, \quad i \in N/M, \quad \text{and} \quad \sum_{i \in N} \rho(i, \xi) = 1,$$

while  $\theta_t(\omega) \in M = \{1, \dots, N\}$ ,  $x_t(\omega) \in \mathbb{R}^n$  and  $\theta_0, x_0, \{w_t\}$  and  $\{v_t\}$

independent. For simplicity, we assume that (23,24.b) is a linear Gaussian system with Markovian switching coefficients, i.e. (see chapter IV, corollary 5.5):  $\underline{\mu}$  is Gaussian and

$$\underline{L0} \quad h(\theta, x) = H_C(\theta) + H_X(\theta)x.$$

$$\underline{L1} \quad \alpha(\theta, x) = A_C(\theta) + A_X(\theta)x, \quad \beta_-(\theta, x) = B_-(\theta), \\ \varphi(\eta, \theta, x, \underline{u}) = C_C(\eta, \theta) + C_X(\eta, \theta)x + C_U(\eta, \theta)\underline{u}.$$

$$\underline{L2} \quad \Lambda(\theta, x) \cdot \rho(\eta, \theta, x) \text{ is } x\text{-invariant and bounded.}$$

$$\underline{L3} \quad \rho(\theta, \theta, x) = 0, \text{ for all } (\theta, x).$$

Since a density-mass function  $\hat{\pi}_t$  on  $R^{n \times M}$  satisfies

$$\hat{\pi}_t(x, \theta) = \hat{p}_t(x|\theta) \cdot \hat{P}_{t, \theta},$$

with  $\hat{p}_t(x|\theta)$  denoting the conditional density  $p_{x_t|\theta_t, \mathcal{Y}_t}(x|\theta)$  and

$$\hat{P}_{t, \theta} = P(\theta_t = \theta | \mathcal{Y}_t), \quad (25.a)$$

our problem really is to evaluate  $\hat{p}_t(\cdot|\theta)$  numerically for all  $\theta$ . To find a low dimensional approximation, we consider the first

and second central moments of  $\hat{p}_t(\cdot|\theta)$  on  $[0, T]$ ;

$$\hat{X}_{t, \theta} \stackrel{\Delta}{=} E\{x_t | \theta_t = \theta, \mathcal{Y}_t\}, \quad (25.b)$$

$$\hat{R}_{t, \theta} \stackrel{\Delta}{=} E\{(x_t - \hat{X}_{t, \theta})(\dots)^T | \theta_t = \theta, \mathcal{Y}_t\}. \quad (25.c)$$

#### 4.1 Theorem

Let the processes  $\{y_t\}$  and  $\{\theta_t, x_t\}$  satisfy (23), (24), L0, L1, L2 and L3, let  $\theta_0, x_0, \{w_t\}$  and  $\{v_t\}$  be independent, and let:

$$P\{\theta_t = \theta\} > 0, \text{ for all } \theta \in M,$$

$$\underline{\mu}(d\underline{u}) = N(d\underline{u}; \underline{\epsilon}_u, \underline{\sigma}_u \underline{\sigma}_u^T), \text{ with } \|\underline{\sigma}_u \underline{\sigma}_u^T\| < \infty.$$

If  $\{x_t\}$ ,  $\{x_t x_t^T\}$  and  $\{x_t x_t^T \otimes x_t\}$  are locally integrable and the

innovations conjecture holds true, then the processes  $\{\hat{P}_{t, \theta}\}$ ,

$\{\hat{X}_{t, \theta}\}$  and  $\{\hat{R}_{t, \theta}\}$ , defined by (25.a,b,c) on  $[0, T]$ , are pathwise unique solutions of the following equations:

$$\hat{P}_{t, \theta} = \hat{I}_{t, \theta} / \sum_{\eta \in M} \hat{I}_{t, \eta}, \quad (26)$$

$$d \ln \hat{I}_{t, \theta} = [\hat{I}_{t, \theta}]^{-1} \sum_{\eta \in M} [\lambda_{\theta\eta} \hat{I}_{t, \eta} - \lambda_{\eta\theta} \hat{I}_{t, \theta}] dt + \\ + (H_{C, \theta} + H_{X, \theta} \hat{X}_{t, \theta})^T dy_t - \frac{1}{2} (H_{C, \theta} + H_{X, \theta} \hat{X}_{t, \theta})^T (\dots) dt, \quad (27.a)$$

$$d \hat{X}_{t, \theta} = \hat{I}_{t, \theta}^{-1} \sum_{\eta \in M} \lambda_{\theta\eta} \hat{I}_{t, \eta} [\hat{x}_{t, \eta} - \hat{X}_{t, \theta}] dt + \\ + [A_{C, \theta} + A_{X, \theta} \hat{X}_{t, \theta}] dt + [B_{W, \theta} + \hat{R}_{t, \theta} H_{X, \theta}^T] dv_{t, \theta}, \quad (27.b)$$

$$d \hat{R}_{t, \theta} = \hat{I}_{t, \theta}^{-1} \sum_{\eta \in M} \lambda_{\theta\eta} \hat{I}_{t, \eta} [\hat{x}_{t, \eta} \hat{x}_{t, \eta}^T - \hat{R}_{t, \theta} + (\hat{x}_{t, \eta} - \hat{X}_{t, \theta})(\dots)^T] dt + \\ + [A_{X, \theta} \hat{R}_{t, \theta} + \hat{R}_{t, \theta} A_{X, \theta}^T + B_{V, \theta} B_{V, \theta}^T] dt + Z_{t, \theta} dv_{t, \theta}; \quad (27.c)$$

$$\text{with: } \hat{x}_{t,\theta\eta} = C_{c,\theta\eta} + [I+C_{x,\theta\eta}]\hat{x}_{t,\eta}, \quad (28.a)$$

$$\hat{x}_{t,\theta\eta} = [I+C_{x,\theta\eta}]\hat{R}_{t,\eta}[I+C_{x,\theta\eta}]^T + C_{u,\theta\eta}\sigma_u\sigma_u^T C_{u,\theta\eta}^T, \quad (28.b)$$

$$dv_{t,\theta} = dy_t - (H_{c,\theta} + H_{x,\theta}\hat{x}_{t,\theta})dt, \quad (29)$$

$$\lambda_{\theta\eta} = \Lambda(\eta, x)\rho(\theta, \eta, x), \text{ any } x, \quad (30)$$

$$Z_{t,\theta} = \sum_{i=1}^n E\{(x_t - \hat{x}_{t,\theta})(\dots)^T(\dots)_i | \theta_t = \theta, y_t\} (H_{x,\theta}^T)_i, \quad (31)$$

and  $(H_{x,\theta}^T)_i$  the  $i$ -th row of  $(H_{x,\theta}^T)$ .

Proof: The result can be obtained both from the KSW equation and from the fundamental filtering theorem. We follow the latter approach: Define the processes  $\{x_{t,\theta}\}$ ,  $\{s_{t,\theta}\}$  and  $\{Q_{t,\theta}\}$  by

$$x_{t,\theta} = 1 \text{ if } \theta_t = \theta, \\ = 0 \text{ if } \theta_t \neq \theta,$$

$$s_{t,\theta} = x_{t,\theta} x_t, \\ Q_{t,\theta} = x_{t,\theta} x_t x_t^T,$$

up to indistinguishability, for each  $\theta \in M$ . Then a repeated use of Itô's differentiation rule and some lengthy evaluations, yield:

$$dx_{t,\theta} = \sum_{\eta \in M} [\lambda_{\theta\eta} x_{t,\eta} - \lambda_{\eta\theta} x_{t,\theta}] dt + dm_{t,\theta}^x,$$

$$ds_{t,\theta} = [A_{c,\theta} x_{t,\theta} + A_{x,\theta} s_{t,\theta}] dt + B_{v,\theta} x_{t,\theta} dv_t + B_{w,\theta} x_{t,\theta} dw_t + \\ + \sum_{\eta \in M} [\lambda_{\theta\eta} y_{t,\eta} - \lambda_{\eta\theta} s_{t,\theta}] dt + dm_{t,\theta}^s,$$

$$dQ_{t,\theta} = [A_{c,\theta} s_{t,\theta}^T + s_{t,\theta} A_{c,\theta}^T + A_{x,\theta} Q_{t,\theta} + Q_{t,\theta} A_{x,\theta}^T] dt + \\ + [B_{v,\theta} B_{v,\theta}^T + B_{w,\theta} B_{w,\theta}^T] x_{t,\theta} dt + \\ + [B_{v,\theta} s_{t,\theta}^T + s_{t,\theta} B_{v,\theta}^T] dv_t + [B_{w,\theta} s_{t,\theta}^T + s_{t,\theta} B_{w,\theta}^T] dw_t + \\ + \sum_{\eta \in M} [\lambda_{\theta\eta} Q_{t,\eta} - \lambda_{\eta\theta} Q_{t,\theta}] dt + dm_{t,\theta}^Q,$$

$$\text{with: } y_{t,\theta\eta} = C_{c,\theta\eta} x_{t,\eta} + [I+C_{x,\theta\eta}] s_{t,\eta},$$

$$Q_{t,\theta\eta} = C_{c,\theta\eta} y_{t,\theta\eta}^T + y_{t,\theta\eta} C_{c,\theta\eta}^T + \\ + [I+C_{x,\theta\eta}] Q_{t,\eta} [I+C_{x,\theta\eta}]^T + C_{u,\theta\eta} \sigma_u \sigma_u^T C_{u,\theta\eta}^T x_{t,\eta},$$

while the components of  $\{m_{t,\theta}^x\}$ ,  $\{m_{t,\theta}^s\}$  and  $\{m_{t,\theta}^Q\}$  are purely discontinuous martingales. Subsequent application of the fundamental filtering theorem (2.1) yields, on  $[0, T]$ :

$$d\hat{x}_{t,\theta} = \sum_{\eta \in M} [\lambda_{\theta\eta} \hat{x}_{t,\eta} - \lambda_{\eta\theta} \hat{x}_{t,\theta}] dt + \\ + [\hat{x}_{t,\theta} (H_{c,\theta} - \hat{h}_t)^T + \hat{s}_{t,\theta}^T H_{x,\theta}^T] dv_t, \quad (32.a)$$

$$d\hat{s}_{t,\theta} = [A_{c,\theta} \hat{x}_{t,\theta} + A_{x,\theta} \hat{s}_{t,\theta}] dt + B_{w,\theta} \hat{x}_{t,\theta} dw_t +$$

$$\begin{aligned}
& + \sum_{\eta \in \mathbf{M}} [\lambda_{\theta\eta} \hat{y}_{t,\theta\eta} - \lambda_{\eta\theta} \hat{s}_{t,\theta}] dt + \\
& + [\hat{S}_{t,\theta} (H_{C,\theta} - \hat{h}_t)^T + \hat{Q}_{t,\theta} H_{X,\theta}^T] dv_t, \quad (32.b)
\end{aligned}$$

$$\begin{aligned}
d\hat{Q}_{t,\theta} = & [A_{C,\theta} \hat{S}_{t,\theta}^T + \hat{S}_{t,\theta} A_{C,\theta}^T + A_{X,\theta} \hat{Q}_{t,\theta} + \hat{Q}_{t,\theta} A_{X,\theta}^T] dt + \\
& + [B_{W,\theta} B_{W,\theta}^T + B_{V,\theta} B_{V,\theta}^T] \hat{x}_{t,\theta} dt + [B_{W,\theta} \hat{S}_{t,\theta}^T + \hat{S}_{t,\theta} B_{W,\theta}^T] dv_t + \\
& + \sum_{\eta \in \mathbf{M}} [\lambda_{\theta\eta} \hat{z}_{t,\theta\eta} - \lambda_{\eta\theta} \hat{Q}_{t,\theta}] dt + \\
& + [\hat{Q}_{t,\theta} (H_{C,\theta} - \hat{h}_t)^T + \sum_{i=1}^n E\{Q_{t,\theta} x_t^i | \mathcal{Y}_t\} (H_{X,\theta}^T)_i] dv_t, \quad (32.c)
\end{aligned}$$

with:  $dv_t = dy_t - \hat{h}_t dt$ ,

$$\hat{h}_t = \sum_{\eta \in \mathbf{M}} (H_{C,\eta} \hat{x}_{t,\eta} + H_{X,\eta} \hat{s}_{t,\eta}),$$

$$\hat{y}_{t,\theta\eta} = C_{C,\theta\eta} \hat{x}_{t,\eta} + [I + C_{X,\theta\eta}] \hat{s}_{t,\eta},$$

$$\begin{aligned}
\hat{z}_{t,\theta\eta} = & C_{C,\theta\eta} \hat{y}_{t,\theta\eta}^T + \hat{y}_{t,\theta\eta} C_{C,\theta\eta}^T + \\
& + [I + C_{X,\theta\eta}] \hat{Q}_{t,\eta} [I + C_{X,\theta\eta}]^T + C_{U,\theta\eta} \sigma_u \sigma_u^T C_{U,\theta\eta}^T \hat{x}_{t,\eta}.
\end{aligned}$$

Now we identify the processes  $\{\hat{x}_{t,\theta}\}$ ,  $\{\hat{s}_{t,\theta}\}$  and  $\{\hat{Q}_{t,\theta}\}$  through an evaluation of their defining equations:

$$\hat{x}_{t,\theta} = E\{x_{t,\theta} | \mathcal{Y}_t\} = P\{x_{t,\theta} = 1 | \mathcal{Y}_t\} = P\{\theta_t = \theta | \mathcal{Y}_t\},$$

$$\hat{s}_{t,\theta} = E\{x_{t,\theta} x_t | \mathcal{Y}_t\} = \sum_{\eta \in \mathbf{M}} \hat{x}_{t,\eta} E\{x_{t,\theta} x_t | \mathcal{Y}_t, \theta_t = \eta\} =$$

$$= \hat{x}_{t,\theta} E\{x_{t,\theta} x_t | \mathcal{Y}_t, \theta_t = \theta\} = \hat{x}_{t,\theta} E\{x_t | \mathcal{Y}_t, \theta_t = \theta\} =$$

$$= \hat{x}_{t,\theta} \hat{x}_{t,\theta},$$

$$\hat{Q}_{t,\theta} = E\{x_{t,\theta} x_t x_t^T | \mathcal{Y}_t\} = \sum_{\eta \in \mathbf{M}} \hat{x}_{t,\eta} E\{x_{t,\theta} x_t x_t^T | \mathcal{Y}_t, \theta_t = \eta\} =$$

$$= \hat{x}_{t,\theta} E\{x_{t,\theta} x_t x_t^T | \mathcal{Y}_t, \theta_t = \theta\} = \hat{x}_{t,\theta} E\{x_t x_t^T | \mathcal{Y}_t, \theta_t = \theta\} =$$

$$= \hat{x}_{t,\theta} [\hat{R}_{t,\theta} + \hat{x}_{t,\theta} \hat{x}_{t,\theta}^T],$$

up to indistinguishability. These equalities imply that

$$\hat{P}_{t,\theta} = \hat{x}_{t,\theta},$$

$$\hat{x}_{t,\theta} = \hat{x}_{t,\theta}^{-1} \hat{s}_{t,\theta},$$

$$\hat{R}_{t,\theta} = \hat{x}_{t,\theta}^{-1} \hat{Q}_{t,\theta} - \hat{x}_{t,\theta} \hat{x}_{t,\theta}^T,$$

up to indistinguishability. Application of Itô's differentiation rule to the latter equalities and evaluation yield (27.b,c) and

$$d\hat{P}_{t,\theta} = \sum_{\eta \in \mathbf{M}} [\lambda_{\theta\eta} \hat{P}_{t,\eta} - \lambda_{\eta\theta} \hat{P}_{t,\theta}] dt + \hat{P}_{t,\theta} [H_{C,\theta} + H_{X,\theta} \hat{x}_{t,\theta} - \hat{h}_t]^T dv_t.$$

Finally, (27.a) follows from the latter SDE, in a similar way as the log-Zakai equation follows from the KSW equation. Q.E.D.

Due to the term  $Z_{t,\theta} dv_{t,\theta}$ , (27.a,b,c) does not provide a closed system of SDE's, unless  $Z_{t,\theta} = 0$  for all  $\theta$ . This observation suggests the following approximate filter:

#### 4.2 Definition

The Interacting Multiple Model (IMM) filter, for linear Gaussian systems with Markovian switching coefficients, satisfies the system of equations (26) through (30), with  $Z_{t,\theta} = 0$  for all  $\theta$ .

From this definition follows that the IMM filter consists of a number of coupled subfilters: a filter (27.a) for the unnormalized conditional probabilities of  $\{\theta_t\}$ , and a bank of  $N$  interacting Kalman-Bucy-like filters (27.b,c), one for each  $\theta \in M$ . If  $\lambda_{\theta\eta} = 0$  for all  $(\theta, \eta)$  then the latter filters are Kalman-Bucy filters and the IMM filter is equivalent to the MM filter, which is in that case exact if  $x_0$  is conditionally Gaussian distributed given  $\theta_0$ . Notice that all IMM equations except (27.a) are of Wiener type.

Remark 1: If we assume that  $\Lambda(\xi) = \epsilon$ , by which  $\lambda_{\theta\eta} = \epsilon \rho(\theta, \eta, x)$ , and we let  $\epsilon$  approach zero, then the IMM filter converges in general not to the MM filter. Similarly, the exact filter converges then in general not to the MM filter (Marcus and Westwood, 1984). An interesting open question is whether the IMM filter then converges to the exact filter.

Remark 2: Equations (32.a,b) for  $\{\hat{X}_{t,\theta}\}$  and  $\{\hat{S}_{t,\theta}\}$ , in the proof of theorem 4.1, originally appear in Davis (1975; Björk, 1980; Krishnan, 1981) under some additional assumptions, such as  $\lambda_{\theta\eta} = 0$  for all  $(\theta, \eta)$ . Björk (1980) notices that the pair (32.a,b) forms a closed system of equations if  $H_{x,\theta} = 0$ , for all  $\theta$ . In that case, however, the resulting statistics need not be sufficient w.r.t. the conditional distribution.

Remark 3: In their search for low dimensional approximations, Davis (1975) and Krishnan (1981) propose some approximations in (32.a,b), which become superfluous with the IMM approach.

Remark 4: Actually, theorem 4.1 also holds true without assumption L3. If  $\lambda_{\theta\eta} = 0$  for all  $\theta \neq \eta$  (i.e. no switching, and no interaction between the filters in the bank) and  $\lambda_{\theta\theta} \neq 0$ , then the filters in the bank are equivalent to second order filters for linear systems driven by Brownian motion and compound Poisson processes (Kwakernaak, 1975; Rogers, 1983).

Remark 5: To apply the IMM filter to linear systems with randomly switching coefficients one might think of approximating the  $N$ -state process  $\{\theta_t\}$  by a conditionally finite state Markov process. The simplest way to accomplish this is to replace the

switching parameters  $\lambda_{\theta\eta}$  by  $\lambda_{\theta\eta,t}$ , with

$$\lambda_{\theta\eta,t} \triangleq \Lambda(\eta, \hat{X}_t, \eta) \rho(\theta, \eta, \hat{X}_t, \eta). \quad (33)$$

A better, but more costly, approach is to approximate  $\{\theta_t\}$  by a "larger than N" state Markov process (Rudemo, 1973).

## 5 Continuous-discrete IMM filtering

Often, the observations process is of discrete-parameter type, while the to be estimated process is of continuous-parameter type. To develop an IMM filter for such situations, we consider the following continuous-discrete filtering problem. The hybrid state Markov process  $\{\theta_t, x_t\}$  of section 4 is observed, at discrete moments in time,  $\tau \in \mathbb{N} \cap [0, T]$ , by the process  $\{y_\tau\}$ ,

$$y_\tau = \sum_{i \in \mathbb{M}} \chi_{\theta}(\theta_\tau) [H_{\theta} x_\tau + G_{\theta} v_\tau], \quad (34)$$

where  $\{v_\tau\}$  is a sequence of i.i.d. standard Gaussian variables,  $G$  and  $H$  are mappings of  $\mathbb{M}$  in appropriate Euclidean spaces,  $G_{\theta} G_{\theta}^T$  is non-degenerate, while  $\{\theta_t, x_t\}$  and  $\{v_\tau\}$  are independent.

Our first step is to characterize the evolution of the conditional distribution of  $(x_t, \theta_t)$  given  $\mathcal{Y}_t$  under the

assumption that it admits a density-mass function  $\hat{p}_t: \mathbb{R}^n \times \mathbb{M} \rightarrow \mathbb{R}$ . The evolution of this density-mass between two successive measurement moments, is governed by the following equation:

$$\frac{\partial}{\partial t} \hat{p}_t = \mathcal{A}^* \hat{p}_t, \quad t \in [\tau-1, \tau), \quad (35.a)$$

where  $\mathcal{A}^*$  is the adjoint of the generator of the Markov process  $\{x_t, \theta_t\}$ . The conditional evolution of this density mass is completed by applying Bayes formula at each moment  $\tau$  that a measurement according to (34) is received:

$$\hat{p}_\tau(\cdot) = c \cdot \hat{p}_{\tau^-}(\cdot) p_{y_\tau | x_\tau, \theta_\tau}(y_\tau | \cdot), \quad (35.b)$$

with  $c$  a normalizing constant. Similar as in the discrete-time and continuous-time situations, the above filter equations admit in general no low-dimensional representation. Between two successive observation instants, however, there exists a low-dimensional characterization of the zero-th, the first and the second conditional central moments, if  $\{x_t\}$  is the solution of a linear Gaussian system with Markovian switching coefficients. This characterization follows from Theorem 4.1:

### 5.1 Corollary

Let the processes  $\{y_\tau\}$  and  $\{\theta_t, x_t\}$  satisfy (34), (24), L1, L2 and L3, let  $\theta_0, x_0, \{w_t\}$  and  $\{v_\tau\}$  be independent, and let

$$B_w(\theta) = 0, \text{ for all } \theta \in M,$$

$$P\{\theta_t = \theta\} > 0, \text{ for all } \theta \in M,$$

$$\mu(d\underline{u}) = N(d\underline{u}; \underline{\epsilon}_u, \underline{\sigma}_u \underline{\sigma}_u^T), \text{ with } \|\underline{\sigma}_u \underline{\sigma}_u^T\| < \infty.$$

If  $\{x_t\}$ ,  $\{x_t x_t^T\}$  and  $\{x_t x_t^T \otimes x_t\}$  are locally integrable, then the processes  $\{\hat{P}_{t,\theta}\}$ ,  $\{\hat{X}_{t,\theta}\}$  and  $\{\hat{R}_{t,\theta}\}$  (defined by (25.a,b,c)) are, between two successive measurement moments, the pathwise unique solutions of the following ordinary differential equations:

$$\frac{\partial}{\partial t} \hat{P}_{t,\theta} = \sum_{\eta \in M} [\lambda_{\theta\eta} \hat{P}_{t,\eta} - \lambda_{\eta\theta} \hat{P}_{t,\theta}], \quad (36.a)$$

$$\frac{\partial}{\partial t} \hat{X}_{t,\theta} = \hat{P}_{t,\theta}^{-1} \sum_{\eta \in M} \lambda_{\theta\eta} \hat{P}_{t,\eta} [\hat{x}_{t,\theta\eta} - \hat{X}_{t,\theta}] + A_{C,\theta} + A_{X,\theta} \hat{X}_{t,\theta}, \quad (36.b)$$

$$\begin{aligned} \frac{\partial}{\partial t} \hat{R}_{t,\theta} = & \hat{P}_{t,\theta}^{-1} \sum_{\eta \in M} \lambda_{\theta\eta} \hat{P}_{t,\eta} [\hat{x}_{t,\theta\eta} - \hat{R}_{t,\theta} + (\hat{x}_{t,\theta\eta} - \hat{X}_{t,\theta}) (\cdot)^T] + \\ & + A_{X,\theta} \hat{R}_{t,\theta} + \hat{R}_{t,\theta} A_{X,\theta}^T + B_{V,\theta} B_{V,\theta}^T, \end{aligned} \quad (36.c)$$

with:

$$\hat{x}_{t,\theta\eta} = C_{C,\theta\eta} + [I + C_{X,\theta\eta}] \hat{X}_{t,\eta},$$

$$\hat{x}_{t,\theta\eta} = [I + C_{X,\theta\eta}] \hat{R}_{t,\eta} [I + C_{X,\theta\eta}]^T + C_{u,\theta\eta} \underline{\sigma}_u \underline{\sigma}_u^T C_{u,\theta\eta}^T,$$

$$\lambda_{\theta\eta} = \Lambda(\eta, x) \rho(\theta, \eta, x), \text{ any } x.$$

The finite-dimensionality problem turns up as soon as we want to evaluate Bayes formula (35). This is caused by the fact that the first and second (central) moments say nothing about the shape of  $p_t$ . Hence, for a low-dimensional approximation we propose to apply the measurement update equations of the IMM algorithm in chapter II:

$$\hat{P}_{\tau,\theta} \approx c_{\tau} \hat{P}_{\tau-,\theta} \|U_{\tau,\theta}\|^{-\frac{1}{2}} \exp\{-\frac{1}{2} \nu_{\tau,\theta}^T U_{\tau,\theta}^{-1} \nu_{\tau,\theta}\}, \quad (37.a)$$

$$\hat{X}_{\tau,\theta} \approx \hat{X}_{\tau-,\theta} + K_{\tau,\theta} [y_{\tau} - H_{\theta} \hat{X}_{\tau-,\theta}], \quad (37.b)$$

$$\hat{R}_{\tau,\theta} \approx \hat{R}_{\tau-,\theta} - K_{\tau,\theta} H_{\theta} \hat{R}_{\tau-,\theta}, \quad (37.c)$$

with:  $\nu_{\tau,\theta} = y_{\tau} - H_{\theta} \hat{X}_{\tau-,\theta},$

$$U_{\tau,\theta} = H_{\theta} \hat{R}_{\tau-,\theta} H_{\theta}^T + G_{\theta} G_{\theta}^T,$$

$$K_{\tau,\theta} = \hat{R}_{\tau-,\theta} H_{\theta}^T U_{\tau,\theta}^{-1},$$

and  $c_{\tau}$  such that  $\sum_{\theta \in M} \hat{P}_{\tau,\theta} = 1.$

One cycle of the continuous-discrete IMM filter than consists of Corollary 5.1 and the IMM measurement update equations above.



### A scheme to integrate (36) numerically

For practical application of the continuous-discrete IMM filter, we have to integrate (36.a,b,c) numerically. The  $N$  equations of (36.a) are linear, but the  $N(n+n^2)$  equations of (36.b,c) are nonlinear. Due to these nonlinearities a direct numerical integration may be complicated. To simplify this, we introduce

the processes  $\{X_{t,\theta}\}$  and  $\{R_{t,\theta}\}$ , as follows:

$$X_{t,\theta} \stackrel{\Delta}{=} \hat{X}_{t,\theta} \hat{P}_{t,\theta} \quad \text{and} \quad R_{t,\theta} \stackrel{\Delta}{=} \hat{R}_{t,\theta} \hat{P}_{t,\theta}.$$

Application of the ordinary differentiation rule to the latter and (36) yields the following differential equations,  $t \in [\tau, \tau-1]$ :

$$\frac{\partial}{\partial t} X_{t,\theta} = A_{C,\theta} + A_{X,\theta} X_{t,\theta} + \sum_{\eta \in M} [\lambda_{\theta\eta} \hat{P}_{t,\eta} \hat{x}_{t,\theta\eta} - \lambda_{\eta\theta} X_{t,\theta}], \quad (39.a)$$

$$\begin{aligned} \frac{\partial}{\partial t} R_{t,\theta} = & A_{X,\theta} R_{t,\theta} + R_{t,\theta} A_{X,\theta}^T + B_{\theta} B_{\theta}^T \hat{P}_{t,\theta} + \\ & + \sum_{\eta \in M} [\lambda_{\theta\eta} \hat{P}_{t,\eta} (\hat{x}_{t,\theta\eta} + (\hat{x}_{t,\theta\eta} - \hat{X}_{t,\theta}) (\dots)^T) - \lambda_{\eta\theta} R_{t,\theta}]. \quad (39.b) \end{aligned}$$

Now, we propose to integrate alternately the jump terms of (36) according to (36.a) and (39.a,b), and the diffusion terms of (36) according to (36.b,c). This yields the following fractional step integration method: partition each interval  $[\tau, \tau+1]$  in equidistant intervals of length  $\Delta$  and integrate (36), on each  $\Delta$ -interval, by two fractional steps.

During each first fractional step we integrate (36.a) and the jump terms in (39.a,b). During each second fractional step we

first transform  $(\hat{P}_{t,\theta}, X_{t,\theta}, R_{t,\theta})$  to  $(\hat{P}_{t,\theta}, \hat{X}_{t,\theta}, \hat{R}_{t,\theta})$  and then integrate the diffusion terms in (36.b,c). Euler integration of (36.a) and the jump terms of (39.a,b), during each first fractional step, yields the scheme:

$$\hat{P}_{s+\Delta,\theta} = \sum_{\eta \in M} [\lambda_{\theta\eta} \Delta \hat{P}_{s,\eta} + (1 - \lambda_{\theta\eta} \Delta) \hat{P}_{s,\theta}], \quad (40.a)$$

$$X_{s+\Delta,\theta} = \sum_{\eta \in M} [\lambda_{\theta\eta} \Delta \hat{P}_{s,\eta} \hat{x}_{s,\theta\eta} + (1 - \lambda_{\theta\eta} \Delta) \hat{P}_{s,\theta} \hat{X}_{s,\theta}], \quad (40.b)$$

$$\begin{aligned} R_{s+\Delta,\theta} = & \sum_{\eta \in M} [\lambda_{\theta\eta} \Delta \hat{P}_{s,\eta} (\hat{x}_{s,\theta\eta} + (\hat{x}_{s,\theta\eta} - \hat{X}_{s+\Delta,\theta}) (\dots)^T) + \\ & + (1 - \lambda_{\theta\eta} \Delta) \hat{P}_{s,\theta} \hat{R}_{s,\theta}], \quad (40.c) \end{aligned}$$

with the  $\hat{x}$ 's and  $\hat{X}$ 's defined in Corollary 5.1. During each second fractional step we first apply the transformation:

$$\hat{X}_{s+\Delta,\theta} = X_{s+\Delta,\theta} / \hat{P}_{s+\Delta,\theta}, \quad (41.a)$$

$$\hat{R}_{s+\Delta,\theta} = R_{s+\Delta,\theta} / \hat{P}_{s+\Delta,\theta}, \quad (41.b)$$

and then numerically integrate the diffusion terms of (36.b,c) over a  $\Delta$ -interval through numerical integration of

$$\frac{\partial}{\partial t} \hat{X}_{t,\theta} \approx A_{C,\theta} + A_{X,\theta} \hat{X}_{t,\theta}, \quad (42.a)$$

$$\frac{\partial}{\partial t} \hat{R}_{t,\theta} \approx A_{X,\theta} \hat{R}_{t,\theta} + \hat{R}_{t,\theta} A_{X,\theta}^T + B_\theta B_\theta^T. \quad (42.b)$$

The latter may be done with the help of appropriate Kalman time-extrapolation equations.

Remark: For  $C_{C,\theta} = C_{X,\theta} = C_{U,\theta} = 0$ , equation (39.a) and the non-central version of (39.b) appear in Morrison (1972), while (40.a,b,c) and (41.a,b) are equivalent to the equations characterizing the interaction between the Kalman filters of the IMM algorithm in chapter II.

## 6 References

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## APPENDICES A1 - A5

These appendices provide the mathematical background for the material in chapters IV and V, as far as it goes beyond standard texts on analysis, like Rudin (1964, 1966), on probability, like Chung (1974) or Shiryaev (1984), and on Itô equations with diffusion solutions, like Wong (1971), Friedman (1975), Schuss (1981) or Gard (1988). Stimulated by the complexity and the generality of the subject, a comprehensive theory of discontinuous processes has been developed. This is illustrated by an impressive and still growing number of mathematically oriented texts on discontinuous processes. The background presented in these appendices only covers a small part of this theory. The main sources used for this presentation are Dellacherie and Meyer (1978), Elliott (1982) and Jacod and Shiryaev (1987).

The appendices are organized as follows. In appendix A1, we present general background. In appendices A2 and A3, we present measurability and predictability concepts, respectively. In appendices A4 and A5, we present local martingales and semimartingales, respectively.



## APPENDIX A1

## GENERAL BACKGROUND

A probability space  $(\Omega, \mathcal{F}, P)$  consists of a measurable space  $(\Omega, \mathcal{F})$  and a probability measure  $P$  defined on  $\mathcal{F}$ . An element of  $\mathcal{F}$  is called an event. An event of  $P$ -measure zero is called a P-null set of  $\mathcal{F}$ . In general a state space (or phase space)  $X$  is a metric space with Borel  $\sigma$ -algebra  $\beta(X)$ . An index set (or time set)  $T$  is either  $\mathbb{R}_+$  (called continuous) or  $\mathbb{N}$  (called discrete). Whenever  $\omega$ ,  $t$  (or  $s$ ) and  $x$  (or  $y$ ) appear in the sequel they denote, without qualification, elements of  $\Omega$ ,  $T$  and  $X$ , respectively. A state space  $X$  is said to be hybrid if  $X = \mathbb{R}^n \times M$ , with MCN.

Given an index set  $T$ , a state space  $X$  and a probability space  $(\Omega, \mathcal{F}, P)$ , a stochastic process (or simply a process)  $\{x_t\}$  is a family of measurable mappings  $x_t$  of  $\Omega$  into  $X$ , indexed by  $t$ . It is often convenient to represent a stochastic process  $\{x_t\}$  as a mapping of  $T \times \Omega$  into  $X$ , i.e.  $X: (t, \omega) \rightarrow x_t(\omega)$ . In that case we write  $\{x_t\} \sim X$  and say  $\{x_t\}$  is measurable if  $X$  is a measurable mapping. Additional measurability concepts, which are really useful for stochastic processes, are discussed in appendix A2. A trajectory (or sample path or realization) of a process  $\{x_t\}$  is a family of outcomes  $\{x_t(\omega)\} = X(t, \omega)$ , for a given  $\omega \in \Omega$ . Given the finite ordered sequence,  $0 \leq t_1 < t_2 < \dots < t_k \leq \infty$  of elements in  $T$  and a Borel measurable set  $\Gamma \subset X^k$ , then we call  $P(\{\omega; [x_{t_1}(\omega), x_{t_2}(\omega), \dots, x_{t_k}(\omega)] \in \Gamma\})$  a finite-dimensional distribution of  $\{x_t\}$ .

Now we fix  $t$  and proceed with some terminology for  $x_t$  as a random vector. By its definition,  $x_t$  is a measurable mapping of  $\Omega$  into  $X$ . The  $\sigma$ -algebra  $\sigma\{\omega; x_t(\omega) \in A\}; A \in \beta(X)\}$  is called the  $\sigma$ -algebra generated by  $x_t$ , and is often denoted as  $\sigma x_t$ . We say  $x_t$  is  $\mathcal{F}'$ -measurable if  $\mathcal{F}' \supset \sigma x_t$ . An  $\mathbb{R}$ -valued random variable  $x_t$  is said to be P-integrable (or integrable) if  $\int_{\Omega} |x_t(\omega)| dP(\omega)$  exists and is finite. A vector of random variables is said to be integrable if each of its components is integrable. Let  $x_t$  and  $y_t$  be  $\mathbb{R}_+$ -valued random variables, of which  $y_t$  is measurable w.r.t. some  $\sigma$ -algebra  $\mathcal{F}' \subset \mathcal{F}$ . We say  $y_t$  is a version of the conditional expectation of  $x_t$  given  $\mathcal{F}'$  if  $\int_A x_t(\omega) dP(\omega) = \int_A y_t(\omega) dP(\omega)$ , for all  $A \in \mathcal{F}'$ . In that case we write  $y_t = E\{x_t | \mathcal{F}'\}$  a.s.. More generally,  $E\{x_t | \mathcal{F}'\}$  denotes the generalized conditional expectation of  $x_t$  given  $\mathcal{F}'$ , which is defined as follows;

$$E\{x_t | \mathcal{F}'\} = \begin{cases} E\{x_t^+ | \mathcal{F}'\} - E\{x_t^- | \mathcal{F}'\}, & \text{if } E\{x_t^+ | \mathcal{F}'\} < \infty, \\ \infty, & \text{otherwise.} \end{cases}$$

A particular form of conditional expectation is conditional probability. The conditional probability of a set  $A \in \mathcal{F}$ , given a

$\sigma$ -algebra  $\mathcal{F}' \subset \mathcal{F}$ , is defined by  $P(A|\mathcal{F}') = E\{1_A|\mathcal{F}'\}$ , where  $1_A(\omega)$  is the indicator of  $\omega \in A$ . Two random vectors  $x_t$  and  $y_t$  are said to be conditionally independent given the  $\sigma$ -algebra  $\mathcal{F}' \subset \mathcal{F}$ , if for all Borel  $(A, B)$ :

$P(\{\omega; x_t(\omega) \in A, y_t(\omega) \in B\}|\mathcal{F}') = P(\{\omega; x_t(\omega) \in A\}|\mathcal{F}') \cdot P(\{\omega; y_t(\omega) \in B\}|\mathcal{F}')$ .  
 A regular conditional distribution (or conditional distribution)  $\mu_{x_t|\mathcal{F}'}$  of  $x_t$  given  $\mathcal{F}'$  is for each  $\omega$  a probability measure on  $\beta(X)$ , such that for every  $B \in \beta(X)$ ,

$$\mu_{x_t|\mathcal{F}'}(B; \omega) = P(\{\omega' \in \Omega; x_t(\omega') \in B\}|\mathcal{F}')(\omega), \text{ a.s.}$$

If the state space of  $x_t$  is a complete separable metric space (e.g.  $R^n$ ) such a regular conditional distribution always exists (Shiryayev, 1984, p. 228).

The variation process  $\{\text{Var}^{x_t}\}$  is defined for a measurable real valued process  $\{x_t\}$  by:

$\text{Var}^{x_t}(\omega) \equiv \text{Sup}_{\Pi(t)} \sum_i |x(t_{i+1}, \omega) - x(t_i, \omega)| = \int_{(0, t]} |dx_s(\omega)|$ , where the supremum is taken with respect to all finite partitions  $\Pi(t)$ , with parameters  $0 = t_0 < t_1 < \dots < t_k = t$ . A real valued process  $\{x_t\}$  is said to have a finite variation if  $\text{Sup}_{t \in T} \text{Var}^{x_t}(\omega) < \infty$  for every  $\omega$ ; to have a finite variation over  $[0, T]$  if  $\text{Sup}_{t \in [0, T]} \text{Var}^{x_t}(\omega) < \infty$  for every  $\omega$ ; to have an integrable variation if  $E\{\text{Var}^{x_t}\} < \infty$  for every  $t$ ; to be P-integrable if  $x_t$  is P-integrable for every  $t$ . The  $\sigma$ -algebra generated by  $(x_s; s \in [0, t])$  is denoted by the script capital version of  $x_t$ ; i.e.

$$\mathcal{X}_t = \sigma[\{\omega; x_s(\omega) \in A\}; \text{ all } s \in [0, t] \cap T, A \in \beta(X)].$$

It is a common practice to use the following short notations:

$\{x_t \in A\}$	for	$\{\omega; x_t(\omega) \in A\}$
$P\{x_t \in A \mathcal{F}'\}$	for	$P(\{\omega; x_t(\omega) \in A\} \mathcal{F}')$
$P\{x_t \in A x_s\}$	for	$P(\{\omega; x_t(\omega) \in A\} \sigma x_s)$
$\sigma\{x_s; s \in [0, t]\}$	for	$\sigma[\{\omega; x_s(\omega) \in A\}; \text{ all } s \in [0, t], \text{ all } A \in \beta(X)]$ ,
$\mu_{x_t \mathcal{F}'}(B)$	for	$\mu_{x_t \mathcal{F}'}(B; \cdot)$ .

However, to improve the understanding of the material in the following appendices, there we often use the full notations.

## APPENDIX A2

## PROGRESSIVELY MEASURABLE PROCESSES

Throughout this and the following appendices, we assume that the index set  $T$  is either  $R_+$  or  $N$ .

Equivalence of processes

If  $\{x_t\}$  and  $\{y_t\}$  are processes with the same index set and state space, while their finite-dimensional distributions are equal, then  $\{x_t\}$  and  $\{y_t\}$  are said to be equivalent in law. Two stronger notions of equivalence between two processes are modification and indistinguishability. If  $\{x_t\}$  and  $\{y_t\}$  are two measurable processes with the same index set, state space and probability space while  $P\{\omega; x_t(\omega) \neq y_t(\omega)\} = 0$  for every  $t$ , then  $\{x_t\}$  is called a modification of  $\{y_t\}$  and then we write: " $x_t = y_t$  a.s. for every  $t$ ". If the latter condition is replaced by  $P\{\omega; \exists t$  with  $x_t(\omega) \neq y_t(\omega)\} = 0$  then  $\{x_t\}$  is said to be indistinguishable from  $\{y_t\}$ , and we write: " $x_t = y_t$ , up to indistinguishability". Obviously, indistinguishable is the strongest: it implies modification, which in turn implies equivalent in law. There are some elegant conditions under which modification implies indistinguishable. For this we need some additional terminology.

If  $T$  is continuous and the path  $\{x_t(\omega)\}$  is continuous (respectively right/left continuous, has right-/left- hand limits) for every  $\omega \in \Omega$ , then we say  $\{x_t\}$  is continuous (respectively right/left continuous, has right-/left-hand limits). We add almost surely when the above holds true for  $P$ -almost all paths.

2.1 Proposition

The processes  $\{x_t\}$  and  $\{y_t\}$  are indistinguishable if  $\{x_t\}$  is a modification of  $\{y_t\}$  and one of the following conditions is satisfied:

- $\{x_t\}$  and  $\{y_t\}$  are of discrete-parameter type,
- $\{x_t\}$  and  $\{y_t\}$  are right continuous, almost surely,
- $\{x_t\}$  and  $\{y_t\}$  are left continuous, almost surely.

Proof: For each rational number  $r \in Q \cap T$   $P\{\omega; x_r(\omega) \neq y_r(\omega)\} = 0$ . Hence,  $P\{\cup_{r \in Q \cap T} \{\omega; x_r(\omega) \neq y_r(\omega)\}\} = 0$ . Due to the assumptions above,  $\{\omega; \exists t$  with  $x_t(\omega) \neq y_t(\omega)\} \subset \cup_{r \in Q \cap T} \{\omega; x_r(\omega) \neq y_r(\omega)\}$ ; by which  $P\{\omega; \exists t$  with  $x_t(\omega) \neq y_t(\omega)\} = 0$ . Q.E.D.

Stochastic basis

Our next step is introducing some additional structure of a given probability space  $(\Omega, \mathcal{F}, P)$  by defining a filtration  $F$  as an increasing family of sub- $\sigma$ -algebras of  $\mathcal{F}$ , i.e.

$F \triangleq \{\mathcal{F}, (\mathcal{F}_t; t \in T), \mathcal{F}\}$ , with  $\mathcal{F} \subset \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$  for every  $s \leq t$ . The sub- $\sigma$ -algebra  $\mathcal{F}$  is called the initial  $\sigma$ -algebra. Notice that we do not follow the usual convention that  $\mathcal{F} = \mathcal{F}_0$ . This is to distinguish between the different roles played in Bayesian estimation by the initial  $\sigma$ -algebra  $\mathcal{F}$  and by the first moment  $\sigma$ -algebra  $\mathcal{F}_0$ .

By convention, we set  $\mathcal{F}_\infty \triangleq \mathcal{F}$ ,  $\mathcal{F}_{\infty-} \triangleq \sigma[\cup_{s \in T} \mathcal{F}_s]$ ,  $\mathcal{F}_{t-} \triangleq \sigma[\cup_{s < t} \mathcal{F}_s]$  for  $t > 0$ ,  $\mathcal{F}_{0-} \triangleq \mathcal{F}_0$  and  $\mathcal{F}_{t+} \triangleq \cap_{s > t} \mathcal{F}_s$ . A filtration  $\{\mathcal{F}_t\}$  is called right continuous if the index set  $T$  is continuous and  $\mathcal{F}_t = \mathcal{F}_{t+}$ , for every  $t$ .

## 2.2 Definition

A stochastic basis (or filtered probability space) is a quintet  $(\Omega, \mathcal{F}, F, P, T)$ , consisting of a probability space  $(\Omega, \mathcal{F}, P)$ , an index set  $T$  and a filtration  $F$ .

A stochastic basis  $(\Omega, \mathcal{F}, F, P, T)$  or filtration  $F$  is called complete if  $\mathcal{F}$  is  $P$ -complete and the initial  $\sigma$ -algebra  $\mathcal{F}$  contains all  $P$ -null sets of  $\mathcal{F}$ . Obviously, any stochastic basis can be completed, and such a completion is unique.

Given an  $X$ -valued process  $\{x_t\}$  on the stochastic basis  $(\Omega, \mathcal{F}, F, P, T)$ . The presence of the filtration  $F$  makes it possible to consider, for every  $t$ , the measurability of  $\{x_s; s \leq t\}$  with respect to sub- $\sigma$ -algebra  $\mathcal{F}_t$ . We say  $\{x_t\}$  is adapted if  $x_t$  is  $\mathcal{F}_t$ -measurable for every  $t$ . For a joint  $(t, \omega)$ -measurability concept we consider the restriction of the mapping  $X$  to  $[0, t] \times \Omega$ :

## 2.3 Definition

We say that  $\{x_t\}$  is progressively measurable (or simply progressive) if, for every  $t$ , the restriction of  $X$  to  $[0, t] \times \Omega$  is a measurable mapping of  $\mathcal{B}[0, t] \times \mathcal{F}_t$  into  $\mathcal{B}(X)$ .

Obviously, progressive is stronger than adapted, hence for the converse we need additional conditions.

## 2.4 Proposition

An  $X$ -valued process  $\{x_t\}$  on a stochastic basis  $(\Omega, \mathcal{F}, F, P, T)$  is progressive if  $\{x_t\}$  is adapted and one of the following conditions is satisfied:

- $\{x_t\}$  is of discrete-parameter type,
- $\{x_t\}$  is right continuous,
- $\{x_t\}$  is left continuous.

**Proof:** Fix an arbitrary  $t \in T$  and consider a partition of  $[0, t]$  into  $2^k$  equal intervals. If  $\{x_t\}$  is right continuous, we define a process  $\{y_{k,t}\}$  as follows:

For all  $\omega \in \Omega$  set  $y_{k,t}(\omega) = x_t(\omega)$  and  
 $y_{k,s}(\omega) = x_{t_i/2^k}(\omega)$ , for all  $s \in [t(i-1)/2^k, t_i/2^k)$ ,  $1 \leq i \leq 2^k$ .

As  $\{y_{k,t}\} \sim Y_k$  is a process with simple sample paths, the restriction of the mapping  $Y_k$  to  $[0, t] \times \Omega$  is a measurable mapping of  $\mathcal{B}[0, t] \times \mathcal{F}_{t+\epsilon}$  into  $\mathcal{B}(X)$ , for every  $t$  and every  $\epsilon > 1/2^k$ . With  $k$  to infinity  $\{y_{k,t}\}$  becomes indistinguishable from  $\{x_t\}$ . Hence, for every  $t$ , the restriction of  $X$  to  $[0, t] \times \Omega$  is a measurable mapping of  $\mathcal{B}[0, t] \times \mathcal{F}_{t+\epsilon}$  into  $\mathcal{B}(X)$ , for every every  $\epsilon > 0$ . The final step is to show that the latter also holds true for  $\epsilon = 0$ . For  $s \leq t$  we can write

$$x_s = \lim_{\epsilon \rightarrow 0} x_s 1_{[0, t-\epsilon)}(s) + x_t 1_{\{t\}}(s),$$

the right hand side of which is measurable, since  $x_t$  is  $\mathcal{F}_t$ -measurable and the restriction of  $X$  to  $[0, t] \times \Omega$  is a measurable mapping of  $\mathcal{B}[0, t] \times \mathcal{F}_{t+\epsilon}$  into  $\mathcal{B}(X)$ , for every every  $\epsilon > 0$ . Hence, for every  $t$ , the restriction of  $X$  to  $[0, t] \times \Omega$  is a measurable mapping of  $\mathcal{B}[0, t] \times \mathcal{F}_t$  into  $\mathcal{B}(X)$ .

Similar for the left continuous situation.

Q.E.D.

### 2.5 Definition

A stopping (or optional) time  $\tau$  (relative to  $\mathcal{F}$ ) is a mapping of  $\Omega$  into  $\text{TU}\{\infty\}$  such that  $\{\omega; \tau(\omega) \leq t\} \in \mathcal{F}_t$ , for every  $t$ .

### 2.6 Definition

A wide sense stopping (or Markov) time  $\tau$  (relative to  $\mathcal{F}$ ) is a mapping of  $\Omega$  into  $\text{TU}\{\infty\}$  such that  $\{\omega; \tau(\omega) < t\} \in \mathcal{F}_t$ , for every  $t$ .

### 2.7 Proposition

A stopping time is a Markov time. Moreover, a Markov time is a stopping time, if  $\mathcal{F}$  is right continuous or  $T$  is discrete.

**Proof:** Only in case  $T$  is continuous;

First, let  $\tau$  be an  $\mathcal{F}$ -stopping time; i.e.  $\{\omega; \tau(\omega) \leq t\} \in \mathcal{F}_t$ , for every  $t$ . Because  $\{\omega; \tau(\omega) < t\} = \bigcup_{n \in \mathbb{N}} \{\omega; \tau(\omega) \leq t - 1/n\}$ , we also have  $\{\omega; \tau(\omega) < t\} \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_{t-1/n} \subset \mathcal{F}_t$ . Hence,  $\{\omega; \tau(\omega) < t\} \in \mathcal{F}_t$ , for every  $t$ , which implies  $\tau$  is a Markov time.

Next, let  $\mathcal{F}$  be right continuous and let  $\tau$  be an  $\mathcal{F}$ -Markov time. Then, for every  $t$ ,  $\{\omega; \tau(\omega) < t\} \in \mathcal{F}_t$ , which implies  $\{\omega; \tau(\omega) < t + 1/n\} \in \mathcal{F}_{t+1/n}$ , if  $n \geq m$ . But then, as  $\{\omega; \tau(\omega) \leq t\} = \bigcap_{n \in \mathbb{N}} \{\omega; \tau(\omega) < t + 1/n\}$ ,  $\{\omega; \tau(\omega) \leq t\} \in \bigcap_{n \in \mathbb{N}} \mathcal{F}_{t+1/n} = \mathcal{F}_t$ , for every  $t$ . Hence, for every  $t$ ,  $\{\omega; \tau(\omega) \leq t\} \in \mathcal{F}_t$ , which implies  $\tau$  is a stopping time. Q.E.D.

Propositions 2.4 and 2.7 indicate that, if  $T$  is continuous, we would best simplify measurability issues by restricting our attention to right continuous filtrations and to processes that are either right continuous or left continuous. Fortunately, the material to be presented in the sequel does not suffer from such restrictions.

The  $\sigma$ -algebra  $\mathcal{F}_\tau$  of events prior to or at stopping time  $\tau$  is:

$$\mathcal{F}_\tau \triangleq \{A \in \mathcal{F}; A \cap \{\omega; \tau(\omega) \leq t\} \in \mathcal{F}_t \text{ for every } t\},$$

and the  $\sigma$ -algebra  $\mathcal{F}_{\tau-}$  of events strictly prior to  $\tau$  is:

$$\mathcal{F}_{\tau-} \triangleq \sigma[\mathcal{F}_0 \cup \{A \cap \{\omega; \tau(\omega) > t\}; A \in \mathcal{F}_t, t \in T\}].$$

It can readily be verified that this agrees with the definition of  $\mathcal{F}_{t-}$ : when  $\tau(\cdot) \stackrel{\Delta}{=} t$ , then  $\mathcal{F}_\tau = \mathcal{F}_t$  and  $\mathcal{F}_{\tau-} = \mathcal{F}_{t-}$ .

With  $\tau$  a stopping time, the process  $\{x_\tau^T\} \stackrel{\Delta}{=} \{x_{t \wedge \tau}\}$  is called a process stopped at  $\tau$ . It can readily be verified that if  $\{x_t\}$  is progressive and  $\tau$  is a stopping time, then  $x_\tau 1_{\{\omega; \tau(\omega) < \infty\}}$  is  $\mathcal{F}_\tau$ -measurable and  $\{x_\tau^T\}$  is progressive.

### 2.8 Definition

A subset  $A$  of  $\Omega \times \mathbb{R}_+$  is called thin if there is a sequence  $(\tau_i)$  of stopping times, such that  $A = \bigcup_i \{(\omega, t); t \in \mathbb{R}_+, t = \tau_i(\omega)\}$ .

An immediate consequence of this is (J&S, p.8, Lemma 1.3):

Any thin random set  $A$  admits a sequence  $(\alpha_i)$  of stopping times such that:

$$A = \bigcup_i \{(\omega, t); t \in \mathbb{R}_+, t = \alpha_i(\omega)\}, \text{ and for all } i \neq j,$$

$$\{(\omega, t); t \in \mathbb{R}_+, t = \alpha_i(\omega)\} \cap \{(\omega, t); t \in \mathbb{R}_+, t = \alpha_j(\omega)\} = \emptyset.$$

Such a sequence  $(\alpha_i)$  is called an exhausting sequence for  $A$ .

## APPENDIX A3

## CADLAG PROCESSES AND PREDICTABILITY

Troughout this and the following appendices, we assume a stochastic basis  $(\Omega, \mathcal{F}, F, P, T)$  where either  $T$  is discrete or  $F$  is right continuous.

Cadlag processes

If an  $X$ -valued process is both right continuous and has left-hand limits, then the process is called cadlag ("continu à droite avec des limites à gauche"; or corlo1). When  $\{x_t\}$  is cadlag we always assume the following convention:  $\lim_{s \uparrow t-} x_s \stackrel{\Delta}{=} x_0$ .

3.1 Proposition

If  $\{x_t\}$  is a cadlag adapted process,  $\{(\omega, t); x_t(\omega) \neq x_{t-}(\omega)\}$  is a thin set.

Proof: See J&S (p.8, Prop. 1.32).

If  $\{x_t\}$  is cadlag we define two other processes  $\{\ell x_t\}$  and  $\{\Delta x_t\}$ , as follows: for every  $\omega$ ,

$$\ell x_t(\omega) \stackrel{\Delta}{=} x_{t-}(\omega) \stackrel{\Delta}{=} \lim_{s \uparrow t-} x_s(\omega), \text{ for every } t,$$

$$\Delta x_t(\omega) \stackrel{\Delta}{=} x_t(\omega) - \ell x_t(\omega) \quad (\text{hence } \Delta x_0(\omega) = 0).$$

Notice that the above may not be possible when  $\{x_t\}$  is simply right continuous.

To recover the discrete-parameter versions of the above introduced processes  $\{\ell x_t\}$  and  $\{\Delta x_t\}$ , we introduce a particular one-to-one mapping of the class of discrete-parameter processes into the class of cadlag processes.

Given an  $X$ -valued process  $\{x_i\}$  on the discrete stochastic basis  $(\Omega, \mathcal{F}, F, P, N)$  we associate to it a right continuous  $X$ -valued process  $\{\underline{x}_t\}$  on a right continuous stochastic basis

$(\Omega, \mathcal{F}, \underline{F}, P, R_+)$ , with  $\underline{F} \stackrel{\Delta}{=} \{\mathcal{F}, (\mathcal{F}_t; t \in R_+), \mathcal{F}\}$  and, for every  $t \in [i, i+1)$ ,

$$\underline{x}_t \stackrel{\Delta}{=} x_i \text{ and } \underline{\mathcal{F}}_t \stackrel{\Delta}{=} \mathcal{F}_i, \quad i \in N.$$

With this mapping, the discrete-parameter processes are:

$$\begin{aligned} \bullet \quad \ell \underline{x}_t &= x_{t-1}, \quad \text{for } t \geq 1, \text{ and } \ell x_0 = x_0, \\ \Delta \underline{x}_t &= x_t - \ell x_t \quad (\text{hence } \Delta x_0 = 0). \end{aligned}$$

For a detailed analysis of this set-up see J&S (pp. 13-15).

Predictable processes

The simplest example of a predictable process is the following:  
If  $\{x_t\}$  is cadlag adapted, the process  $\{\mathcal{L}x_t\}$  is predictable.

3.2 Definition

The predictable  $\sigma$ -algebra is the  $\sigma$ -algebra  $\mathcal{P}$  on  $T \times \Omega$ , that is generated by all adapted left-continuous processes (considered as mappings on  $T \times \Omega$ ).

3.3 Proposition

The predictable  $\sigma$ -algebra is related to the filtration as follows:

$$\mathcal{P} = \sigma \left[ \left\{ \{0\} \times A; A \in \mathcal{F}_0 \right\} \cup \left\{ (s, t] \times A; s < t, A \in \mathcal{F}_s \right\} \right],$$

and to stopping times  $\tau$  as follows:

$$\mathcal{P} = \sigma \left[ \left\{ \{0\} \times A; A \in \mathcal{F}_0 \right\} \cup \left\{ (t, \omega); 0 \leq t \leq \tau(\omega), \text{ all } \tau \right\} \right].$$

Proof: See J&S (p. 16, Th. 2.2).

3.4 Definition

An  $X$ -valued process  $\{x_t\} \sim X$  is said to be predictable if  $X$  is  $\mathcal{P}$ -measurable.

The following are immediate consequences of definition 3.4:

- If  $\{x_t\}$  is cadlag and adapted then  $\{\mathcal{L}x_t\}$  is predictable.
- If  $\{x_t\}$  is cadlag and predictable then  $\{\Delta x_t\}$  is predictable.
- If  $\{x_t\}$  is a predictable process and  $\tau$  is a stopping time, then the stopped process  $\{x_t^{\tau}\}$  is predictable.

3.5 Definition

A predictable time  $\tau$  is a mapping of  $\Omega$  into  $T \cup \{\infty\}$ , such that

$$\left\{ (t, \omega); 0 \leq t < \tau(\omega) \right\} \in \mathcal{P}.$$

Some consequences of definition 3.5 are:

- A predictable time is a stopping time.
- If  $(\tau_i)$  is a sequence of stopping times increasing to a  $\tau > 0$  while  $\tau_i < \tau$  for all  $i$ , then  $\tau$  is a predictable time, and we call  $(\tau_i)$  an announcing sequence for  $\tau$ .
- If  $\tau$  is a predictable time  $> 0$ , there exists an increasing sequence  $(\tau_i)$  of predictable times, such that  $\tau_i < \tau$  a.s.,  $\tau_i(\omega) \leq \tau$  for all  $\omega$ , and  $\lim_i \tau_i = \tau$  a.s. If the stochastic basis is complete, then the additions "a.s." may be deleted (Dellacherie and Meyer, 1978, p.132).

3.6 Proposition

Let the stochastic basis be complete, and let  $\{x_t\}$  and  $\{y_t\}$  be two cadlag adapted processes. Then  $\{x_t\}$  and  $\{y_t\}$  are indistinguishable if and only if  $x_{\tau} 1_{(\tau < \infty)} = y_{\tau} 1_{(\tau < \infty)}$  a.s., for every stopping time  $\tau$ .



Proof: See Elliott (1982, 6.25)

Next we give a characterization of predictable processes.

### 3.7 Definition

A stopping time  $\alpha$  is called totally inaccessible if

$P\{\omega; \alpha(\omega) = \tau(\omega) < \infty\} = 0$  for all predictable times  $\tau$ .

An adapted cadlag process  $\{x_t\}$  is said to charge a stopping time  $\tau$  if  $P\{\omega; [x_\tau(\omega) \neq x_{\tau-}(\omega)] \text{ and } [\tau(\omega) < \infty]\} > 0$ .

### 3.8 Theorem

Let the stochastic basis be complete. Let  $1_{(\tau < \infty)}$  denote a  $\{0,1\}$ -valued random variable, such that  $1_{(\tau < \infty)}(\omega) = 1$  iff  $\tau(\omega) < \infty$ . If  $\{x_t\}$  is an adapted cadlag process which does not charge any totally inaccessible stopping time, and the random variable  $x_\tau 1_{(\tau < \infty)}$  is  $\mathcal{F}_{\tau-}$ -measurable for every predictable time  $\tau$ , then  $\{x_t\}$  is predictable. Conversely, if  $\{x_t\}$  is predictable, then  $x_\tau 1_{(\tau < \infty)}$  is  $\mathcal{F}_{\tau-}$ -measurable for every stopping time  $\tau$ .

Proof: See Elliott (1982; 6.30 and 6.47).

### 3.9 Definition

A process  $\{x_t\}$  is called quasi-left continuous if  $\{x_t\}$  is cadlag and  $\Delta x_\tau = 0$ , almost surely on the set  $\{\omega; \tau(\omega) < \infty\}$ , for every predictable time  $\tau$ .

An immediate consequence of definition 3.9 is that a discrete-parameter process is quasi-left continuous if and only if  $x_t = x_0$  for all  $t$ , up to indistinguishability.

### 3.10 Definition

A complete right-continuous filtration  $F$  is said to be quasi-left continuous (or predictable) if  $\mathcal{F}_{\tau-} = \mathcal{F}_\tau$ , for every predictable time  $\tau$ .

## APPENDIX A4

## LOCAL MARTINGALES

In this section we only give a brief outline of the main local martingale framework, and refer to other texts for proofs. Throughout this and the following appendix we assume a complete stochastic basis  $(\Omega, \mathcal{F}, F, P, T)$  with either  $T$  discrete or F right continuous. Our first step is to define a martingale with respect to the filtration  $F$ .

4.1 Definition

A real valued process  $\{x_t\}$  on the basis  $(\Omega, \mathcal{F}, F, P, T)$  is said to be a wide sense martingale (w.r.t.  $F$ ) if  $\{x_t\}$  is adapted and  $F$ -integrable, while:

$$x_s = E\{x_t | \mathcal{F}_s\} \text{ for all } s \leq t.$$

A wide sense martingale  $\{x_t\}$  is said to be a martingale if  $\{x_t\}$  is either cadlag or of discrete parameter type.

It can be verified that each wide sense martingale admits a modification that is a martingale (due to the right continuity of  $F$  when  $T$  is continuous). We denote the class of real-valued martingales by  $\mathcal{M}^1$ . A vector valued process is said to be a martingale if each of its components is in  $\mathcal{M}^1$ .

4.2 Theorem (Optional stopping)

Let  $\{m_t\}$  be a real-valued  $F$ -martingale and let  $(\alpha, \tau)$  be a pair of bounded  $F$ -stopping times such that  $\alpha(\omega) \leq \tau(\omega)$ , for every  $\omega$ .

Then the random variables  $m_\alpha$  and  $m_\tau$  are integrable and

$$E\{m_\tau | \mathcal{F}_\alpha\} = m_\alpha \quad \text{a.s.}$$

Proof: See Elliott (1982, 4.12 and 4.13).

4.3 Definition

A Wiener process on  $(\Omega, \mathcal{F}, F, P, R_+)$  (or relative to  $F$ ) is a continuous adapted real-valued process  $\{w_t\}$ , such that  $w_0=0$  and, for every  $t$ ,

$$(i) \quad E\{w_t\} = 0,$$

$$(ii) \quad E\{w_t^2\} < \infty,$$

$$(iii) \quad (w_t - w_s) \text{ is independent of the } \sigma\text{-algebra } \mathcal{F}_s \text{ for all } s \leq t.$$

A Wiener process  $\{w_t\}$  is said to be standard if  $E\{w_t^2\} = t$ . An n-dimensional Wiener process is an  $n$ -vector of independent Wiener processes. A process  $\{b_t\}$  is called a standard Brownian motion if there exists a stochastic basis, say  $(\Omega, \mathcal{F}, F, P, R_+)$ , such

that  $b_0$  is  $\mathcal{F}_0$ -measurable and  $\{b_t - b_0\}$  is a standard Wiener process relative to  $F$ .

4.4 Definition

A square-integrable martingale is a real valued  $F$ -martingale  $\{x_t\}$ , such that  $\sup_t E\{|x_t|^2\} < \infty$ . The class of all square-integrable martingales is denoted as  $\mathcal{M}^2$ .

Localization

Let  $\mathcal{E}$  denote a particular class of processes. We denote by  $\mathcal{E}_{loc}$  the localized class of  $\mathcal{E}$ , defined as follows: a process  $\{x_t\}$  belongs to  $\mathcal{E}_{loc}$  if there exists an increasing sequence  $(\tau_i)$  of stopping times (depending on  $\{x_t\}$ ), such that  $\lim_i \tau_i = \infty$  almost surely and that each stopped process  $\{x_{t \wedge \tau_i}\}$  belongs to  $\mathcal{E}$ .  $(\tau_i)$  is called a localizing sequence for  $\{x_t\}$  relative to  $\mathcal{E}$ .

Hence, a process is said to be a local martingale if it belongs to the class  $\mathcal{M}_{loc}^1$ . A process is said to be a locally-square integrable martingale if it belongs to the class  $\mathcal{M}_{loc}^2$ . A process  $\{x_t\}$  is said to be locally bounded if  $\{x_t\}$  is in the localized class of bounded processes. A real valued process  $\{x_t\}$  is said to be locally integrable if  $\{x_t\}$  is in the localized class of integrable processes; to be locally of integrable variation if  $\{x_t\}$  is in the localized class of processes of integrable variation; to be locally of finite variation if  $\{x_t\}$  is in the localized class of finite variation processes.

4.5 Definition

If  $\{x_t\} \sim X$  and  $\{y_t\} \sim Y$  are locally square-integrable martingales, then the predictable quadratic covariation (or angle bracket, or quadratic characteristic) of the pair  $(\{x_t\}, \{y_t\})$  is the process  $\langle X, Y \rangle_t$  such that  $\{x_t y_t - \langle X, Y \rangle_t\}$  is a local martingale, and that  $\{\langle X, Y \rangle_t\}$  is a cadlag predictable process which is locally of finite variation, while  $\langle X, Y \rangle_0 = 0$ .

4.6 Proposition

The predictable quadratic covariation, of definition 4.5, exists and is unique up to indistinguishability. Moreover,

$$\langle X, Y \rangle_t = [ \langle X+Y, X+Y \rangle_t - \langle X-Y, X-Y \rangle_t ] / 4$$

and the predictable quadratic variance  $\{\langle X, X \rangle_t\}$  admits a continuous version if and only if  $\{x_t\}$  is quasi-left continuous.

Proof: See J&S, p.38-39.

4.7 Proposition

A Wiener process  $\{w_t\}$  is a continuous martingale, and its angle bracket satisfies  $\langle W, W \rangle_t(\omega) = \sigma^2(t) \stackrel{\Delta}{=} E\{w_t^2\}$ .

Proof: That  $\{w_t\}$  is a continuous martingale follows readily from (i) and (iii) in definition 4.3. Due to (ii),  $\sigma^2(t) < \infty$ , hence we can define a process  $\{x_t\}$  as follows:

$$x_t = w_t^2 - \sigma^2(t), \text{ up to indistinguishability.}$$

Evaluation yields:

$$x_t - x_s = (w_t - w_s)^2 - [\sigma^2(t) - \sigma^2(s)] + 2w_s(w_t - w_s).$$

Hence, for  $t \geq s$ ,  $E[x_t - x_s | \mathcal{F}_s] = E\{(w_t - w_s)^2 | \mathcal{F}_s\} - [\sigma^2(t) - \sigma^2(s)] = 0$ , which implies that  $\langle W, W \rangle_t = \sigma^2(t)$ . Q.E.D.

#### Orthogonality and characterization

Two local martingales  $\{m_t\}$  and  $\{m'_t\}$  are called orthogonal if their product  $\{m_t m'_t\}$  is a local martingale. A local martingale  $\{m_t\}$  is called a purely discontinuous local martingale if  $m_0 = 0$  and  $\{m_t\}$  is orthogonal to every continuous local martingale.

#### 4.8 Theorem

Any local martingale  $\{m_t\}$  admits a unique decomposition

$$m_t = m_0 + m_t^c + m_t^d, \text{ up to indistinguishability,}$$

such that  $m_0^c = m_0^d = 0$ ,  $\{m_t^c\}$  is a continuous local martingale and  $\{m_t^d\}$  is a purely discontinuous local martingale. Moreover, if  $\{m_t\}$  is locally of finite variation, then  $m_t^c = 0$ , for all  $t$ .

Proof: Uniqueness is trivial. For existence, see J&S, pp.42-43.

Further, it can easily be verified that if  $\{m_t\}$  is a local martingale such that  $m_0$  is square-integrable and  $\{\Delta m_t\}$  is locally bounded, then  $\{m_t\}$  is a locally square-integrable martingale.

#### 4.9 Theorem (Levy's characterization of a local martingale)

A continuous local martingale  $\{w_t\}$  with  $w_0 = 0$  is a Wiener process if and only if its angle bracket  $\langle W, W \rangle$  is deterministic, say  $\langle W, W \rangle_t = \sigma^2(t)$  for some increasing continuous function  $\sigma^2(\cdot)$ . Then  $\sigma^2(\cdot)$  is the variance function of  $\{w_t\}$ , and for all  $s \leq t$  the variable  $w_t - w_s$  is Gaussian, centered, with variance  $\sigma^2(t) - \sigma^2(s)$ .

Proof: See J&S, p.102 (with the help of theorem 5.5 of the next section).

#### 4.10 Definition

We define the filtration  $F^X$  generated by a process  $\{x_t\}$  as:

$$F^X \stackrel{\Delta}{=} \{\mathcal{F}, (\mathcal{F}_t^X; t \in T), \mathcal{F}^X\},$$

with  $\mathcal{F}$  the initial  $\sigma$ -algebra (see Appendix A2) and:

$$\mathcal{F}_t^X \stackrel{\Delta}{=} \sigma[\mathcal{F} \cup \{x_s; s \leq t\}] = \sigma[\mathcal{F} \cup \mathcal{X}_t], \text{ for every } t,$$

$$F^X \stackrel{\Delta}{=} \sigma[\cup_{s \in T} \mathcal{F}_s^X].$$

If  $F$  is complete,  $\mathcal{F}$  contains all  $P$ -null sets of  $\mathcal{G}$ . Hence,  $\mathcal{F}$  contains all  $P$ -null sets of  $\mathcal{G}^X$  and  $(\Omega, \mathcal{G}^X, F^X, P, T)$  is complete. However, even if  $\{x_t\}$  is cadlag and adapted and  $F$  is right continuous,  $F^X$  is in general not right continuous. An interesting exception is the following.

4.11 Proposition (Conditional Zero-One Law)

Let  $\{w_t\}$  be a Wiener process relative to a complete  $F$ . Then,  $F^W$  is right continuous.

Proof: See Wong and Hajek, 1985, p. 245.

4.12 Theorem (Martingale representation)

Let  $\{w_t\}$  be a Wiener process relative to a complete  $F$ , and let  $F^W$  be the filtration generated by  $\{w_t\}$ . If  $\{m_t\}$  is a local  $F^W$ -martingale, there is an  $F^W$ -predictable process  $\{h_t\}$ , such that:

$$m_t = m_0 + \int_0^t h_s dw_s, \quad \text{up to indistinguishability,}$$

$$\left\{ \int_0^t (h_s)^2 ds \right\} \text{ is locally integrable.}$$

Proof: See Wong and Hajek, 1985, pp. 246-248.

The martingale representation theorem above can be extended in many other interesting directions (J&S, pp. 166-178).

## APPENDIX A5

## SEMIMARTINGALES

Although it will not become apparent from the text below, the class of semimartingales is a very desirable one, as it is closed under a large class of transformations:

- stopping,
- localization,
- change of time,
- absolutely continuous change of probability measure,
- changes of filtration.

Moreover, the class of semimartingales is the largest possible class of processes with respect to which one may "reasonably" integrate all bounded predictable processes (see Dellacherie and Meyer, 1982; Jacod, 1979; Dellacherie, 1980).

5.1 Definition

A real valued process  $\{x_t\}$  is said to be a semimartingale if  $x_0$  is an  $\mathcal{F}_0$ -measurable random variable and  $\{x_t\}$  admits the representation

$$x_t = x_0 + m_t + a_t, \quad \text{up to indistinguishability,}$$

with  $\{m_t\}$  a local martingale and  $\{a_t\}$  a cadlag adapted process which has a finite variation over each finite interval, while  $m_0 = a_0 = 0$ .

A decomposition of the latter type is called a martingale decomposition, and is in general not unique. Further, it can easily be verified that if  $\{x_t\}$  is a semimartingale then it admits a martingale decomposition

$$x_t = x_0 + m_t + a_t, \quad \text{up to indistinguishability,}$$

such that  $\{m_t\}$  is a locally square integrable martingale and  $\{a_t\}$  is a cadlag adapted process which is locally of finite variation (see remark in Wong and Hajek, 1985, p.234). Further, it can easily be verified that a real valued process  $\{x_t\}$  is a deterministic semimartingale if and only if we can write  $x_t(\omega) = f(t)$  up to indistinguishability, where  $f(t)$  is a real-valued function which is cadlag and locally of finite variation. When  $T$  is discrete, then a real valued process  $\{x_t\}$  is a semimartingale if and only if  $\{x_t\}$  is adapted. An  $\mathbb{R}^n$ -valued process  $\{x_t\}$  is said to be a semimartingale if each of its  $n$  components is a semimartingale.

5.2 Proposition

Let  $\{x_t\}$  be a real valued semimartingale. Then every martingale decomposition of the type in definition 5.1;

$$x_t = x_0 + m_t + a_t$$

yields the same continuous local martingale  $\{m_t^c\}$ , with  $m_0^c = 0$ .

Proof:  $\{x_t\}$  admits a martingale decomposition;

$$x_t = x_0 + m_t + a_t, \quad \text{up to indistinguishability,}$$

of which  $\{m_t\}$  admits a unique decomposition (theorem 4.8):

$$m_t = m_t^c + m_t^d.$$

Assume:  $a_t = m'_t + a'_t$ , up to indistinguishability, where  $\{m'_t\}$  is a local martingale and  $\{a'_t\}$  is locally of finite variation.

As  $\{a_t\}$  and  $\{a'_t\}$  are locally of finite variation,  $\{m'_t\}$  is locally of finite variation. Hence from theorem 4.8, it follows that  $\{m'_t\}$  is purely discontinuous, which implies that every martingale decomposition yields the same  $\{m_t^c\}$ . Q.E.D.

### 5.3 Definition

The martingale decomposition of (5.1) is called canonical if  $\{a_t\}$  is predictable.

A semimartingale  $\{x_t\}$  is said to be special if it admits a canonical martingale decomposition.

### 5.4 Theorem

Let  $\{x_t\}$  be a semimartingale. There is equivalence between:

- (i)  $\{x_t\}$  is a special semimartingale,
- (ii) There exists a decomposition of type 5.1 such that  $\{a_t\}$  has locally integrable variation,
- (iii) Every decomposition of type 5.1 yields an  $\{a_t\}$  the variation of which is locally integrable,
- (iv) The process  $\{\text{Sup}_{s \leq t} |x_s - x_0|; t \in T\}$  is locally integrable.

Proof: See J&S, p.44.

### Itô's differentiation rule

Several generalizations of Itô's differentiation rule for stochastic integrals with respect to Brownian motion eventually led to the differentiation rule of Doleans-Dade and Meyer (1970) for  $\mathbb{R}^n$ -valued semimartingales. Many of the most important results of probability can easily be obtained by using this rule; such as Lévy's Brownian motion characterization of local martingales (due to Kunita and Watanabe) and the Girsanov transformation of local martingales (due to Van Schuppen and Wong).

### 5.5 Theorem (Semimartingale differentiation rule)

If  $\{x_t\}$  is an  $\mathbb{R}^n$ -valued semimartingale and  $f$  is a twice continuously differentiable mapping of  $\mathbb{R}^n$  into  $\mathbb{R}$ , then  $f(x_t)$  is a semimartingale satisfying:

$$\begin{aligned} f(x_t) = & f(x_0) + \sum_{i=1}^n \int_0^t \frac{\partial}{\partial x^i} f(x_{s-}) dx_s^i + \\ & + \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2}{\partial x^i \partial x^j} f(x_{s-}) d\langle m^{ic}, m^{jc} \rangle_s + \\ & + \sum_{0 < s \leq t} \left[ f(x_s) - f(x_{s-}) - \sum_{i=1}^n \frac{\partial}{\partial x^i} f(x_{s-}) \Delta x_s^i \right] \end{aligned}$$

up to indistinguishability, where  $m^{ic}$  is the  $i$ -th component of the continuous martingale part of  $\{x_t\}$ . Moreover, if  $\{x_t\}$  is a special semimartingale, then  $\{f(x_t)\}$  is a special semimartingale.

Proof: See Elliott (1982; pp. 128-140), Wong and Hajek (1985, pp.238-240) and, for special semimartingales, Métivier (1982, pp. 231-232).

The concept of a semimartingale can be generalized to Hilbert space valued processes, and it has been shown that Itô's rule still holds true (Kusssmaul, 1977; Métivier and Pellaumail, 1980; Métivier, 1982; Gyöngy and Krylov, 1982; Kopp, 1983).

As is well known, there exists an ordinary differentiation rule for continuous semimartingales, the result of which implies a Stratonovich type of integration. Recently, this ordinary differentiation rule has been extended to discontinuous semimartingales (Ferreira, 1987).





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## LIST OF SYMBOLS

$N$	$= \{0, 1, 2, \dots\}$ ,
$Z$	$= \{\dots, -2, -1, 0, 1, 2, \dots\}$ ,
$Q$	$=$ Set of rational numbers,
$R$	$= (-\infty, \infty)$ ,
$R_+$	$= [0, \infty)$ ,
$R_-$	$= (-\infty, 0]$ ,
$R^+$	$= (0, \infty)$ ,
$R^-$	$= (-\infty, 0)$ ,
$R^{\cdot\cdot}$	$=$ Closure of $R^{\cdot\cdot}$ .
$a \vee b$	$= \sup(a, b)$ ,
$a \wedge b$	$= \inf(a, b)$ .
$x^+$	$= x \vee 0$ , for $x \in R$ ,
$x^-$	$= (-x) \vee 0$ , for $x \in R$ .
$1_A(a)$	$= 1$ , $a \in A$ ,
$1_A(a)$	$= 0$ , else.
$\underline{u}$	$= \text{Col}\{u_2, \dots, u_n\}$ , if $u = \text{Col}\{u_1, \dots, u_n\}$ .
$\ a\ ^2$	$= \sum_{i,j} a_{ij}^2$ , if $a$ is a matrix.
$ a ^2$	$= \sum_i a_i^2$ , if $a$ is a vector.
$\sigma[\mathcal{Y}]$	$:$ smallest $\sigma$ -algebra containing the collection $\mathcal{Y}$ .
$\mathfrak{I}_t$	$:$ smallest $\sigma$ -algebra generated by $\{x_s; s \leq t\}$ .
$\xi_t^i$	$:$ $i$ -th component of $\xi_t$ .
$\partial 0$	$:$ boundary of the closure of set $0$ .
$C^b(A)$	$:$ the set of all real-valued functions that are bounded and continuous, everywhere on $A$ .
$C^k(A)$	$:$ the set of all real-valued functions that are, on $A$ , $k$ times continuously differentiable. The superscript is deleted if $k=0$ . If $k$ is followed by $b$ , then $f$ and the $k$ derivatives are bounded on $A$ .
$\mathcal{D}(A)$	$:$ domain of operator $A$ .



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## SUMMARY (In Dutch)

SCHATTEN VOLGENS BAYES TEN BEHOEVE VAN  
OP BESLISSINGEN GERICHTE STOCHASTISCHE BESTURING

## Samenvatting

Stochastische processen met een op beslissingen gerichte besturing worden beschouwd als bestuurd Markov processen waarvan de toestandsruimte "hybride" is; dat wil zeggen een produkt van een discrete set en een Euclidische ruimte. Deze wiskundig aanpak is algemeen genoeg om als model te dienen voor velerlei op beslissingen gerichte stochastische besturingsproblemen.

In het algemeen leiden de in het "verleden" en "heden" gedane waarnemingen van het bestuurd Markov proces niet tot volledige zekerheid over het "heden" van de discreet-waardige toestandscomponent. De optimale besturing kan dan worden verkregen door het uitvoeren van twee opeenvolgende stappen:

- Het schatten volgens Bayes (het evalueren van de conditionele distributie) van het Markov proces,
- Het besturen van de conditionele distributie op basis van perfecte kennis van zijn evolutie.

Helaas levert de uitvoering van elk van deze stappen aanzienlijke problemen in het geval dat de Markov toestand hybride is.

Het proefschrift richt zich op de modellering van hybride toestand Markov processen en op de oplossing van problemen die zich voordoen bij het schatten volgens Bayes voor dergelijke processen.

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