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Discrete-time Linear Time-invariant Distributed Minimum Energy Estimator

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Abstract: Proper monitoring of large complex spatially critical infrastructures often requires a sensor network capable of inferring the state of the system. Such networks enable the design of distributed estimators considering only local (partial) measurements, local communication capabilities with nearby sensors, as well as the system model. Solutions often assume perfect knowledge of the system model, and white process and measurement noise, which are limiting in engineering settings. In this paper, we consider the minimum energy setting where the model uncertainty and process and measurement noises are bounded but unknown. We provide the first distributed minimum energy estimator for discrete-time linear time-invariant systems, and we show that the error dynamics is input-to-state stable. Lastly, we illustrate the performance in some pedagogical examples, and compare the performance with respect to the centralized implementation of the minimum energy estimator.

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Keywords: distributed estimation, minimum energy, consensus

1. INTRODUCTION

Large-scale dynamical systems, such as power networks (Xie et al., 2012), increasingly have many distributed sensors and actuators that need to communicate to estimate the global state of the system and regulate it, respectively. In the presence of a central coordination unit that receives information from all the sensors, the global state of the system can be estimated (provided that certain conditions are met). This is called centralized state estimation, or simply state estimation (Rego et al., 2019).

The problem with centralized state estimation in large complex spatially critical infrastructures is that they often require the aid of a wireless sensor network where sensors are located at relatively large distances from each other, and wireless communication is costly, privacy sensitive, and possibly prone to errors (Kar and Moura, 2013). To address these challenges, we need a different approach, namely distributed estimation. This method aims to asymptotically reconstruct the centralized state estimate at each sensor using solely its own measurements and limited information from neighbouring sensors (Mitra and Sundaram, 2018).

Typically, distributed approaches are based on centralized estimation approaches such as Kalman filters and Luenberger observers for systems under stochastic noise or noiseless processes, respectively (Rego et al., 2019). These approaches are often adapted by integrating a consensus protocol that reaches agreement on measurements or estimates while simultaneously performing a filter update on the state estimate. In particular, we are interested in single time-scale filters that minimize the communication load.

Under certain assumptions on the noise, the Kalman filter offers an optimal (i.e., minimum variance) estimate of the state according to the conditional maximum likelihood. Distributed Kalman filtering schemes have been studied extensively, see for instance Khan et al. (2010), Kar and Moura (2013), Olfati-Saber (2007), and Mosquera and Jayaweera (2008). Nevertheless, the use of Kalman filter requires the use of exact descriptions of the system model and the knowledge of the statistical characterization of the white noise describing the process disturbances and measurement noises. As such, hereafter we adhere with a setting where the model uncertainty (as long it is bounded) can be captured as part of the process and measurement errors, and refrains from any statistical assumptions.

A possible approach is to consider a *minimum energy estimator*. The earliest references to the minimum energy type estimators are made by Mortensen (1968) and Swerling (1971), where they propose a recursive solution to the state estimation problem without the use of any statistical concepts. Later, Willems (2004) argues for the use of the deterministic interpretation over the probabilistic one, suggesting that it is more pragmatic as it avoids any claims of knowledge on the statistical properties of the uncertainty in a model. Simply said, there are only few applications for which one could justify any sort of precise knowledge on the distribution of the noise.

Up until recently, studies on the minimum energy estimator have upheld the assumption that the discrete-time version of the estimator follows from the continuous-time version straightforwardly, without any additional derivation. However, recently, Buchstaller et al. (2021) has argued that this is not the case. In their work they provide an alternative comprehensive proof that establishes a

minimum energy formulation for discrete-time linear time-invariant (LTI) systems.

That said, whereas a distributed minimum energy estimator was proposed by [Zamani and Ugrinovskii \(2014\)](#) for continuous-time case LTI systems, to the best of the author's knowledge, there are currently no existing results and studies on the derivation and application of a distributed minimum energy estimator for discrete-time LTI systems.

The contribution of this paper is as follows: we propose a distributed minimum energy estimator. Additionally, we show that it converges to the centralized state estimate with bounded error under mild conditions, i.e., the error dynamics are input-to-state stable (ISS).

The rest of the paper is structured as follows. In Section 2, we introduce the problem statement and the concept of minimum energy estimation and its recursive formulation for a centralized problem. Subsequently, in Section 3, we firstly implement a consensus protocol in combination with the minimum energy recursion to obtain a distributed algorithm. Secondly, we show that the error dynamics are stable and the estimates converge to within the neighbourhood of the true state. Finally, in Section 4, we provide a simulated example to show the convergence of the estimator.

2. PROBLEM STATEMENT

Consider a large-scale dynamical system with spatially distributed sensors where each sensor is able to monitor a subset of the states of the system at all times (see Fig. 1).

Specifically, consider a discrete-time LTI system

$$x_{k+1} = Ax_k + B_d w_k, \quad (1a)$$

$$y_k = Cx_k + v_k, \quad (1b)$$

where $A \in \mathbb{R}^{m \times m}$, $B_d \in \mathbb{R}^{m \times n_w}$, and $C \in \mathbb{R}^{n_y \times m}$. Moreover, in contrast to the approach taken in much of the literature on distributed estimation, we do not state any assumptions on the stochastic nature of the disturbances on the system. Instead, we simply assume that the input and output disturbances, $w_k \in \mathbb{R}^m$ and $v_k \in \mathbb{R}^n$ respectively, are unknown, bounded, and deterministic.

2.1 Minimum energy estimation

Minimum energy estimation involves finding the most probable state trajectory based on measurements by minimizing the unknown disturbances through a weighted least-squares. Specifically, let us define a weighted least-squares objective for $k \in [0, \tau]$,

$$J(\tau, x_0, w_0^\tau, v_0^\tau) = \|x_0 - \hat{x}_0\|_{\Sigma_0^{-1}}^2 + \sum_{k=0}^{\tau} \left(\|w_k\|_{Q^{-1}}^2 + \|v_k\|_{R^{-1}}^2 \right), \quad (2)$$

for which we seek a minimal solution $(\tilde{x}_0, \tilde{w}_0^\tau, \tilde{v}_0^\tau)$ that defines the trajectory \tilde{x}_0^τ (i.e., the evolution of the state x_k over the interval $[0, \tau]$). This trajectory is the minimum energy trajectory compatible with data y_k ([Zamani and Ugrinovskii \(2014\)](#)).

The problem of minimizing (2) is a quadratic program that grows in complexity as time progresses. Therefore,

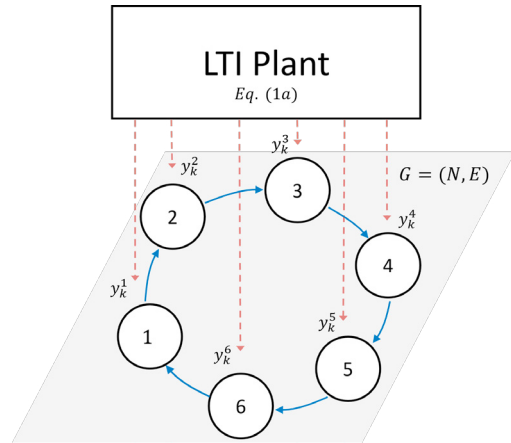


Fig. 1. A schematic illustration of a communication graph with six nodes (numbered circles) and six edges (blue arrows) and its interaction with the dynamical system.

minimum energy estimation seeks a recursive algorithm to reduce the computational complexity. In continuous-time, the recursion for the minimum energy estimator is obtained using Hamilton-Jacobi theory, see [Mortensen \(1968\)](#). As previously argued, the discrete-time minimum energy does not follow straightforwardly from the continuous-time version. Yet, there exists also a recursive algorithm to compute \tilde{x}_0^τ which we present in the next result.

Theorem 1. ([Buchstaller et al. \(2021\)](#)). Consider the system described in (1), and let the current state estimate of the system, corresponding to the solution of (2), be denoted by $x_{k|k-1}$. Then, the discrete-time recursion can be described as

$$x_{k+1|k} = Ax_{k|k-1} + K_k(y_k - Cx_{k|k-1}), \quad (3)$$

with

$$K_k = \Sigma_{k|k-1} C^\top [C \Sigma_{k|k-1} C^\top + R]^{-1}, \quad (4)$$

and recursively updates the intensity gain matrix, Σ , by

$$\begin{aligned} \Sigma_{k|k} &= (I - K_k C) \Sigma_{k|k-1}, \\ \Sigma_{k+1|k} &= A \Sigma_{k|k} A^\top + B_d Q B_d^\top, \end{aligned} \quad (5)$$

where the remaining matrices have appropriate dimensions.

2.2 Distributed setting: wireless sensor network and communication graph

The result from Theorem 1 applies only to single entity systems that collect measurements centrally as in (1b). We assume that gathering all sensor information by a central coordinator is not possible, or undesirable. Therefore, in the design of our estimator we want to employ a wireless sensor network to allow communication between neighbouring sensors. We would like each sensor to be able to infer the global state (i.e., all states) of the system in order to provide any local controllers with sufficient information about the system.

Therefore, we assume a wireless sensor network where each sensor (or, agent) collects the following set of measurements:

$$y_k^i = C_i x_k + v_k^i, \quad i = 1, \dots, n. \quad (6)$$

Additionally, each sensor also has communication capabilities and can communicate with a set of neighbouring sensors. We consider the wireless sensor network represented by a connected directed graph $G = (N, E)$ with nodes $N = \{1, \dots, n\}$ and edges $E \subseteq N \times N$. Fig. 1 depicts the interaction of the communication graph with the dynamical system.

Therefore, the problem we seek to address in this paper is as follows:

Problem statement: Determine a distributed solution to the minimum energy estimator problem for discrete-time linear time-invariant system described in (1a), where measurements are collected by different agents as in (6) who are capable of exchanging data with their neighbors defined by the communication graph G .

Notice that the original measurement equation from (1b) can be recovered when we consider $y_k = [y_k^1, \dots, y_k^n]^\top$, $C = [C_1^\top, \dots, C_n^\top]^\top$, and $R = \text{diag}(R_1, \dots, R_n)$. Denote the estimate of $x_k \in \mathbb{R}^m$ at node i using $x_{k|k}^i$. Then, $x_{k|k}$ now denotes the stacked vector of $x_{k|k}^i$ for $i = 1, \dots, n$. Furthermore, we do not require that (A, C_i) is observable (or even detectable), only that the centralized system (i.e., if all measurements are collected by a central coordinator) is detectable. Hence, we explicitly state the assumption that is required to ensure asymptotic stability of the estimator error dynamics and boundedness of Σ (Anderson and Moore, 2012).

Assumption 1. The system (A, C) is detectable. Specifically, this means that

$$\text{rank} \begin{pmatrix} \lambda_i I - A \\ C \end{pmatrix} = m, \quad \text{if } |\lambda_i| \geq 1. \quad (7)$$

Furthermore, if G is directed, we must also have that the subsystem associated with each *source component* is detectable. For a given $G = (N, E)$, a source component (N_s, E_s) is a strongly connected component for which there exist no edges from the nodes in $N \setminus N_s$ to those in N_s (Mitra and Sundaram, 2018).

The proposed solution is presented in the next section.

3. DISCRETE-TIME LINEAR TIME-INVARIANT DISTRIBUTED MINIMUM ENERGY ESTIMATOR

Firstly, we apply the theory of discrete-time minimum energy estimation and distributed estimation to obtain a distributed version of the minimum energy estimator, given in Algorithm 1. Secondly, we show that the proposed implementation works by showing that the error dynamics satisfy input-to-state stability assumptions. Specifically, in Theorem 2, we provide bounds on the error between centralized and distributed implementations.

3.1 Distributed minimum energy recursion

The discrete-time static consensus iteration is formulated by

$$x_{k+1} = Px_k, \quad (8)$$

where $P = I - \epsilon L$ is the Perron matrix. Here, $\epsilon \in (0, 1/\Delta)$ is the discretization step-size, and Δ represents

the maximum degree of G (Olfati-Saber et al., 2007). The matrix P is irreducible and stochastic. The condition $\epsilon < 1/\Delta$ is necessary to ensure that the largest eigenvalue of P has modulus 1.

In particular, we consider a distributed recursive estimation algorithm that follows a single time-scale communication scheme that requires only a single consensus update step at each iteration of the estimator update, drawing inspiration from works such as Khan et al. (2010), Kar and Moura (2013), and Xie et al. (2012) that focus on distributed Kalman filter solutions. Specifically, we propose the following *discrete-time linear time-invariant distributed minimum energy estimator*.

First, the estimator update as a weighted combination of a consensus step and an innovation update is formulated by

$$x_{k+1|k}^i = a_k \sum_{j \in \mathcal{N}_i} w_{ij} \times \left[x_{k|k-1}^j + b_k \sum_{l \in \mathcal{N}_j} C_l^\top R_l^{-1} (y_k^l - C_l x_{k|k-1}^j) \right], \quad (9)$$

where $a_k, b_k > 0$ are potentially time-varying weights that relate to the system dynamics and the confidence in local estimate, respectively. The weights w_{ij} are part of consensus (Perron) matrix $P = \{w_{ij}\}$ and $\mathcal{N}_i := \{j \in N : (j, i) \in E\}$.

To this end, we introduce a distributed recursive scheme to update locally the state estimate based on exchange of measurements and estimates. The estimation scheme that we adopt to solve the problem is described in Algorithm 1. Within the algorithm, we briefly describe the steps taken to compute the estimate.

Algorithm 1 Estimation scheme

1: **for** each agent $i \in N$ **do**:

2: *Initialize* $x_{0|0}^i = \hat{x}_0$ and $\Sigma_{0|0}^i = \Sigma_0$.

3: *Predict*

$$\begin{aligned} \Sigma_{k|k-1}^i &= A \Sigma_{k-1|k-1}^i A^\top + B_d Q B_d^\top, \\ x_{k|k-1}^i &= A x_{k-1|k-1}^i. \end{aligned} \quad (10)$$

4: *Exchange* $C_i R_i^{-1} y_k^i$ and $C_i R_i^{-1} C_i$ with all $j \in \mathcal{N}_i$.

5: *Estimate*

$$\psi_{k|k}^i = x_{k|k-1}^i + \Sigma_{k|k}^i (s_k^i - S_k^i x_{k|k-1}^i), \quad (11)$$

where ψ is a local estimate and

$$S_k^i = \sum_{j \in \mathcal{N}_i} C_j^\top R_j^{-1} C_j, \quad s_k^i = \sum_{j \in \mathcal{N}_i} C_j^\top R_j^{-1} y_k^j,$$

$$\text{and } \Sigma_{k|k}^i = \left[(\Sigma_{k|k-1}^i)^{-1} + S_k^i \right]^{-1}.$$

6: *Exchange* local estimates $\psi_{k|k}^i$ with all $j \in \mathcal{N}_i$.

7: *Consensus*

$$x_{k|k}^i = \sum_{j \in \mathcal{N}_i} w_{ij} \psi_{k|k}^j. \quad (12)$$

8: **end for**

Remark 1. We have opted to go for a separation between the estimator update in (11) and the consensus update in (12), rather than a combined consensus and innovation

approach used in Khan et al. (2010), Kar and Moura (2013), and Xie et al. (2012), which can be described by

$$x_{k+1|k}^i = a_k \sum_{j \in \mathcal{N}_i} w_{ij} x_{k|k-1}^j + b_k \sum_{j \in \mathcal{N}_i} C_j^\top R_j^{-1} (y_k^j - C_j x_{k|k-1}^i). \quad (13)$$

While this approach may be more convenient for communication and simplicity, it has negative impact on the stability of the error dynamics compared to the adopted method. In other words, the error dynamics of the method from (13) are more likely to have eigenvalues larger than 1 in absolute value compared to the separate update resulting in (9). This is likely due to the extended propagation of information that occurs in (9) by having a separate instance of communication. Consequently, when consensus is performed, the agent has access to a more accurate representation of the state, resulting in a better estimate. Nonetheless, this observation requires further investigation that lies outside the scope of the present paper. \circ

Next, we discuss notions of stability which ensure that the minimum energy estimator converges within the neighbourhood of the optimal solution.

3.2 Error analysis

First, we rewrite (11) using the matrix inversion lemma.

Let us consider a vector $\mathbf{C}_i = \text{col}(C_j)_{j \in \mathcal{N}_i} \in \mathbb{R}^{n_y n_i \times m}$, diagonal matrix $\mathbf{R}_i = \text{diag}(R_j)_{j \in \mathcal{N}_i}$, and combined measurement vector $\mathbf{y}_k^i = \text{col}(y_k^j)_{j \in \mathcal{N}_i}$. Then, the measurement and consensus update equations for a single agent are

$$\begin{aligned} K_k^i &= \Sigma_{k|k-1}^i \mathbf{C}_i^\top \left[\mathbf{C}_i \Sigma_{k|k-1}^i \mathbf{C}_i^\top + \mathbf{R}_i \right]^{-1}, \\ \Sigma_{k|k}^i &= \Sigma_{k|k-1}^i - K_k^i \mathbf{C}_i \Sigma_{k|k-1}^i, \\ x_{k|k}^i &= \sum_{j \in \mathcal{N}_i} w_{ij} \left[x_{k|k-1}^j + K_k^j (y_k^j - \mathbf{C}_j x_{k|k-1}^i) \right]. \end{aligned} \quad (14)$$

Note that we study the expression in this form solely for analysis of the error, hence, we are able to use information here (i.e., \mathbf{C}_i , \mathbf{R}_i , and \mathbf{y}_k^i) that is not available to a single agent at each time instance.

Subsequently, let us define $x_{k|k} = \text{col}(x_{k|k}^i)_{i \in \mathcal{N}} \in \mathbb{R}^{mn}$, $\mathbf{y}_k = \text{col}(\mathbf{y}_k^i)_{i \in \mathcal{N}} \in \mathbb{R}^{n_y n}$, $K_k = \text{blkdiag}(K_k^i)_{i \in \mathcal{N}}$, $\mathbf{C} = \text{blkdiag}(\mathbf{C}_i)_{i \in \mathcal{N}} \in \mathbb{R}^{n_y n \times mn}$, and $\mathbf{R} = \text{blkdiag}(\mathbf{R}_i)_{i \in \mathcal{N}} \in \mathbb{R}^{n_y n \times n_y n}$. Furthermore, we define the consensus matrix $P = I - \epsilon L$. This results in the combined update step

$$x_{k|k} = (P \otimes I_m) [x_{k|k-1} + K_k (\mathbf{y}_k - \mathbf{C} x_{k|k-1})]. \quad (15)$$

Then, we can describe the error dynamics at all agents as

$$\begin{aligned} e_{k+1} &= (\mathbf{1}_n \otimes x_{k+1}) - x_{k+1|k}, \\ e_{k+1} &= (I_n \otimes A)(\mathbf{1}_n \otimes x_k) + (I_n \otimes B_d)(\mathbf{1}_n \otimes w_k) \\ &\quad - (P \otimes A) [x_{k|k-1} + K_k (\mathbf{y}_k - \mathbf{C} x_{k|k-1})], \\ e_{k+1} &= (P \otimes A)e_k + (I_{mn} - P \otimes I_m)(\mathbf{1}_n \otimes x_k) \\ &\quad + (I_n \otimes B_d)(\mathbf{1}_n \otimes w_k) \\ &\quad - (P \otimes A)K_k (\mathbf{C} x_k \mathbf{1}_n + \mathbf{v}_k - \mathbf{C} x_{k|k-1}), \end{aligned} \quad (16)$$

where the term $(I_{mn} - P \otimes I_m)(\mathbf{1}_n \otimes x_k) = 0$. Finally, we express the dynamics of the error as a linear time-varying system as follows:

$$e_{k+1} = F_k e_k + G_k d_k, \quad (17)$$

with

$$\begin{aligned} F_k &= (P \otimes A)(I - K_k \mathbf{C}), \\ G_k &= [I_n \otimes B_d, -(P \otimes A)K_k], \text{ and} \\ d_k &= \begin{bmatrix} \mathbf{1}_n \otimes w_k \\ \mathbf{v}_k \end{bmatrix}. \end{aligned}$$

3.3 Time-invariance of the estimator

Due to the time-varying component of the system matrices F_k and G_k , we cannot use straightforward stability arguments applicable to LTI systems. Results on stability for time-varying systems exist but are generally complex (Anderson and Moore, 2012). Therefore, we formulate a time-invariant notion of stability for the error dynamics of the minimum energy estimator.

First, we proceed with two results before giving a formal definition of stability.

Proposition 1. For any A, B_d , and C satisfying Assumption 1, and for any $\Sigma_0 \geq 0$, we have that

$$\lim_{k \rightarrow \infty} \Sigma_{k+1|k} = \bar{\Sigma}, \quad (18)$$

where $\bar{\Sigma}$ is constant and independent of Σ_0 , and $\Sigma_{k+1|k}$ follows the recursion from (10).

Proof. Following the analysis in Anderson and Moore (2012), page 78, we can readily obtain the result in Proposition 1. In short, the result is derived by showing that for an arbitrary Σ_0 , any subsequent $\Sigma_{k|k-1}$ is bounded for all k . Under global detectability of (A, C) , there exists a suboptimal estimator with gain K^s for which $\rho(F^s) < 1$. The stability of the suboptimal estimator implies boundedness of the solution on $\Sigma_{k|k-1}^s$. If we compare the suboptimal estimator with the optimal estimator with the same initial values, we must have that the optimal estimator is bounded by $\Sigma_{k|k-1}^s \geq \Sigma_{k|k-1} \geq 0$. Secondly, one can prove by induction that $\Sigma_{k|k-1}$ is monotonically increasing by keeping fixed initial conditions but shifting backwards the initial time. Together, these results imply that the Riccati equation has a steady state solution. \circ

The second result requires that the matrix F_k is stable in the limit. Using Proposition 1, we define time-invariant matrices F and G for which we substitute in a constant intensity gain matrix K which is the limiting solution of the recursive update K_k from (14).

Additionally, building upon the results in Khan et al. (2010), we can readily obtain the following proposition.

Proposition 2. For any estimator that satisfies Proposition 1, such that $F = (P \otimes A)(I - K \mathbf{C})$ with $K = \bar{\Sigma} \mathbf{C}^\top [\mathbf{C} \bar{\Sigma} \mathbf{C}^\top + R]^{-1}$, the nominal error dynamics from (17) are asymptotically stable if matrices A and P are contained in the set

$$\mathcal{S} := \{A, P : \rho(F) < 1\}. \quad (19)$$

Proof. For a constant $\bar{\Sigma}$, we have a constant K gain. Hence, the internal dynamics of the error can be described by a LTI system with $F = (P \otimes A)(I - K \mathbf{C})$. Thus, any A

and P that ensure that the spectral radius of F is smaller than one, implies that the system driven by F is globally asymptotically stable at zero (0-GAS). \circ

3.4 Input-to-state stability

In the previous section, we show that the error dynamics are asymptotically stable at zero. However, due to the deterministic assumptions on the disturbance, we cannot guarantee convergence to zero in general. Hence, the notion of input-to-state stability (ISS) can be used to define the bound on the error.

To this end, we can say that the error dynamics are asymptotically ISS if and only if Proposition 1 and Proposition 2 (0-GAS) hold, and the asymptotic gain property is satisfied, i.e., there is some $\gamma \in \mathcal{K}_\infty$ such that:

$$\lim_{k \rightarrow +\infty} |e_k| \leq \gamma(\|d_k\|_\infty), \quad \forall e_0, d_k, \quad k \in \mathbb{N}. \quad (20)$$

Due to nominal stability of the dynamics, the error will converge to a neighborhood around the origin, with the bound depending on the disturbance. Therefore, we can find γ as a function of the disturbance.

Now that we have formulated the notion of ISS, we can use it to specify the bounds on the error between distributed estimates, as well as between any distributed estimate and the centralized estimate.

Theorem 2. Considering the distributed implementation of the minimum energy estimator provided in Algorithm 1, the following ISS expression for the error between any local state estimate holds:

$$\lim_{k \rightarrow +\infty} |\epsilon_k^{ij}| \leq \gamma_i(\|d_k^i\|_\infty) + \gamma_j(\|d_k^j\|_\infty), \quad (21)$$

$$\forall e_0, d_k^i, d_k^j, \quad k \in \mathbb{N}, \quad i, j \in N, \quad \gamma_i, \gamma_j \in \mathcal{K}_\infty.$$

Similarly, we can use an identical argument to show that the error between the centralized estimate and any local distributed estimate is bounded in the limit by

$$\lim_{k \rightarrow +\infty} |\epsilon_k^{cj}| \leq \gamma_c(\|d_k^c\|_\infty) + \gamma_j(\|d_k^j\|_\infty), \quad (22)$$

$$\forall e_0, d_k^c, d_k^j, \quad k \in \mathbb{N}, \quad j \in N, \quad \gamma_c, \gamma_j \in \mathcal{K}_\infty.$$

Proof. We define the error between two estimates to be

$$\epsilon_k^{ij} = x_{k|k-1}^i - x_{k|k-1}^j, \quad i, j \in N, j \neq i. \quad (23)$$

On the other hand, we know that the error with respect to the true state is

$$e_k^i = x_k - x_{k|k-1}^i, \quad i \in N, \quad (24)$$

as defined in (16). Therefore, we can substitute (24) into (23) to get the following expression:

$$\epsilon_k^{ij} = e_k^j - e_k^i. \quad (25)$$

From this result, it follows straightforwardly that

$$|\epsilon_k^{ij}| \leq \max(|e_k^j| + |e_k^i|). \quad (26)$$

Replacing the local estimate i with the estimate of the centralized solution and applying the exact same reasoning results in

$$|\epsilon_k^{cj}| \leq \max(|e_k^j| + |e_k^c|). \quad (27)$$

Therefore, equations (21) and (22) from Theorem 2 follow readily from (26) and (27), respectively, by substituting in (20). This concludes the proof. \circ

Ultimately, we have defined a notion of input-to-state stability for the error between the distributed estimate and the true state, the error between the consensus error (i.e., the error between estimates), and the error between the centralized and distributed estimates. As expected, the bounds on the error dynamics depend on the size of the unknown disturbances acting on the system.

4. EXAMPLE

Consider a system with dynamics according to (1a) with $m = 11$, and measurements according to (6) with $n = 6$. Firstly, we generate an A matrix with a specific diagonal structure with three sparsely connected subsystems,

$$A = \begin{bmatrix} A_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_{22} & A_{23} \\ A_{31} & \mathbf{0} & A_{33} \end{bmatrix}. \quad (28)$$

Entries of A_{ii} are generated by placing a set of randomly generating poles using the MATLAB *place* command. We set the eigenvalues to be $\gamma_1(A_{11}) = 1.05$, $\gamma_{1,2}(A_{22}) = 0.9 \pm 0.5i$, and all others $|\gamma_j(A_{ii})| < 1$. For reference, the full A matrix is given here:

$$\underbrace{\begin{bmatrix} 1.57 & -0.36 & 0.28 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2.18 & -0.675 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.74 & -0.78 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.998 & -1.6 & -0.16 & 3.0 & -0.46 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.76 & -0.3 & 1.05 & 3.03 & -2.02 & 0 & 0 & -0.03 \\ 0 & 0 & 0 & 1.29 & -0.815 & 1.38 & 1.41 & -1.38 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.33 & -0.53 & 0.07 & 1.32 & -0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.68 & -1.37 & 0.83 & 1.58 & -1.15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.68 & 0.05 & 0.005 \\ 0.115 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.21 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.265 \end{bmatrix}}_A.$$

Remark 2. This specific structure is chosen because it allows us to create a dynamical system for which (A, C) is detectable but *not* observable, and such that (A, C_i) is *not* locally detectable for any i . Specifically, by keeping $A_{12} = A_{21} = 0$, any node that measures a state related to a subsystem A_{11} or A_{22} will not be able to observe any states from the other subsystems. Hence, those nodes cannot have local detectability as A_{11} and A_{22} both contain at least one unstable mode. Furthermore, we ensure that the entire system is not globally observable because there is no node that measures any states represented in A_{33} . That said, since A_{33} is stable, all the unobservable modes will vanish, rendering the system (A, C) detectable. As a result, under specific conditions on matrices A_{31} and A_{23} we have the desired properties for our matrix. \circ

4.1 Network

We consider a circularly shaped network represented by a strongly connected directed graph $G = (N, E)$ with nodes $N = \{1, \dots, n\}$ and edges $E \subseteq N \times N$. Each node is connected with a neighbour in an ascending order (i.e., node 1 connects to node 2, node 2 connects to node 3). The circle is closed by connecting node 6 to node 1, see Fig. 1 for a schematic depiction. Each nodes measurement model (C_i) is represented by a row of the global C matrix

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Furthermore, we set disturbances w_k and v_k to vary randomly over time according to a uniform distribution on $[-0.3, 0.7]$ and $[-0.5, 0.5]$, respectively.

Finally, we choose the following parameters for the algorithm:

$$\begin{aligned} \epsilon &= 0.1, \\ \hat{x}_0 &= \mathbf{0}, \quad x_0 = \text{rand}(m, 1), \quad x_0 \in [-10, 10], \\ \Sigma_0^i &= 10I_m, \quad Q = I_m, \quad \text{and} \quad R = I_n. \end{aligned}$$

4.2 Error comparison

Due to the instability of the dynamical system, we consider a performance metric that evaluates the error of the estimator in a relative fashion. Specifically, we consider the average absolute error between the centralized estimate and any local estimate of the nodes in G . The metric is defined by

$$\pi_k^j = \frac{1}{m} \sum_{p=1}^m \frac{|x_{k|k}^{p,j} - x_{k|k}^{p,c}|}{1 + |x_{k|k}^{p,c}|}. \quad (29)$$

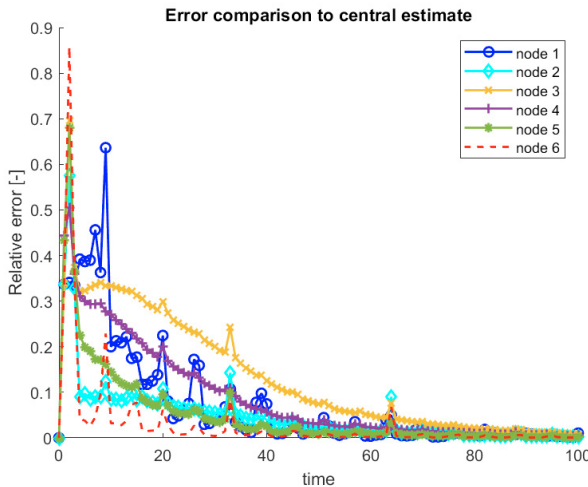


Fig. 2. The average state estimation error for all states over time according to relative error metric given in (29).

The results are plotted in Fig. 2, where it is clear that π_k^j converges to (the neighbourhood of) zero for each of the nodes. Notably, the error trajectories show oscillatory behaviour which is caused by the complex eigenvalue pairs present in the system.

5. CONCLUSION

We propose for the first time a distributed implementation of a minimum energy estimator for discrete-time LTI

systems. We provide the equations needed to conduct a single time-scale recursion that is of bounded complexity. Furthermore, we offer a comprehensive analysis on the error convergence of the proposed estimator, showing that it is input-to-state stable with a bound depending on unknown input and output disturbances. Finally, we present a simple pedagogical example to demonstrate the performance of the estimator.

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