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ORIGINAL RESEARCH

PGP for portfolio optimization: application to ESG index family

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Abstract
The conventional portfolio design approach assumes Gaussian return distributions, but this is not accurate in practice. Asymmetric and heavy-tailed return distributions necessitate consideration of higher-order moments such as skewness and kurtosis, in addition to mean and variance. This study proposes a multi-objective approach using a mean-variance-skewness-kurtosis model to construct a diversified portfolio. A parametrized polynomial goal programming (PGP) method is used to optimize the portfolio by maximizing returns and skewness while minimizing variance and kurtosis. Empirical data from the S&P ESG index family is used, and PGP generates multiple portfolios reflecting investors' preferences for the four moments. To compare between the obtained portfolios, we represent the empirical cumulative distribution of the portfolio returns for all studied values of weights and show how this can be used to assist the inverstor in selecting the best set of weights.

Keywords High-order portfolios · Skewness · Kurtosis · PGP method

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1 Introduction

The early works of Harry Markowitz's groundbreaking article in 1952 laid the principles of the mean-variance approach under the fundamental assumption of Gaussian distributed asset returns. However, implementation of the mean-variance based portfolios has rapidly shown their limitations, and were as a result generally discarded in practice. The reason lies in that they are designed for Gaussian distributed asset returns, an assumption that is not satisfied by real asset returns, which contains outliers and thus are better modeled by asymmetric and heavy-tailed distributions [for more information, see (Dcock et al., 2015; Resnick, 2007; Harvey & Siddique, 2000; Jobst & Zenios, 2001; Ang et al., 2006)].
To compensate for the shortcomings of the mean-variance approach, several works proposed techniques that consider the higher-order moments of the portfolio returns. In this line, the authors (Keung et al., 2006; Jean-Luc & Marwa, 2010) advocate the use of the mean-variance-skewness-kurtosis (MVSK) approach, where the skewness and kurtosis standing for the third and fourth central moments of portfolio returns describe asymmetry and heavy-tailedness of the portfolio returns. This technique, consisting of maximizing the mean and skewness while reducing the variance and kurtosis (Boudt et al., 2015; Kshatriya & Prasanna, 2018; Dinh & Niu, 2011) allows for generating positively skewed portfolios presenting smaller tails. The robustness of MVSK based portfolios towards heavy tailed distributed asset returns has motivated Boudt and Cornilly (2020) to propose portfolio titlting technique which aims to move an original portfolio to a MVSK efficient portfolio. They show that the benefits of tilting come at the cost of deviating from the initial optimality criterion. For the sake of illustration, they apply the tilting technique to equally-weighted, equal-risk-contribution, and maximum-diversification portfolios in a UCITS-compliant asset allocation setting.
Despite its numerous benefits, the design of MVSK portfolios is challenging. As a matter of fact, when including higher-order moments, the objective functions become non-convex, which makes the problems in general NP-hard (Murty & Kabadi, 1985). In addition, the computational complexity in calculating the gradients of high-order moments significantly increases as more portfolios are considered. To handle this issue, several metaheuristic optimization approaches, such as differential evolution (Maringer & Parpas, 2009) and genetic algorithms (Kshatriya & Prasanna, 2018), have been traditionally proposed. Recently, Zhou and Palomar (2021) propose a very efficient and convergence-provable algorithm framework based on the sequential convex approximation (SCA) method to solve high-order portfolios. They also propose a numerical experiment to show the algorithm framework's efficiency.

In this paper, we reexamine the multi-objective portfolio selection model, where investors aim to identify portfolios with optimal trade-offs between maximizing return and skewness, and minimizing variance and kurtosis. We focus on Polynomial Goal Programming (PGP), which uses a set of four parameters to adjust the contribution of the fourth moments in the objective function and seeks portfolios that achieve a specific trade-off. The main challenge lies in determining the appropriate parameters for the Polynomial Goal Programming (PGP) model. This is a complex issue because it is difficult to convert the higher-order moments into a meaningful metric that aligns with the investor's strategy.
To address this issue, we introduce a practical tool based on the visualization of empirical distribution functions for the asset returns of all generated portfolios. This allows investors to efficiently evaluate their portfolio performance and identify the most suitable parameters for their investment strategy. Our goal is to bridge the gap between the theoretical foundations of portfolio optimization and the practical requirements of investors in the Environmental, Social, and Governance (ESG) space.

Using empirical data sets from the S&P ESG index family, we demonstrate the benefits of our approach, which summarizes portfolio performance in a single scalar value. This value, while dependent on all high-order moments, is more relevant and can easily guide investors in selecting the best parameters for their strategy.
This paper is organized as follows: We first propose the problem formulation for portfolio optimization using polynomial goal programming in Section 2. In Section 3, we present our empirical analysis. Finally, the conclusion of this paper is summarized in Section 4.

2 The portfolio optimization within the high-order moments

In this section, we discuss the polynomial goal programming (PGP) approach for resolving the portfolio selection problem in the mean-variance-skewness-kurtosis framework. This method consists of changing the PGP's mean-variance-skewness-kurtosis dimension of portfolio selection to measure the deviation of returns from normality. So, we look at the issue of an investor choosing the best portfolio from a set of N risky assets in the presence of skewness and kurtosis in the return distribution. This problem involves balancing two conflicting goals: maximizing the expected return and the skewness while simultaneously minimizing the variance and the kurtosis. Formally, we consider a portfolio return computed from the returns of N assets $\boldsymbol{R} = (R_1, \ldots, R_N)^T$ and the asset weights $\mathbf{w} = (w_1, \ldots, w_N)^T$ where w_i is the weight invested in the i^{th} asset with $\sum_{i=1}^{N} w_i = \mathbf{w}^T \mathbb{1} = 1$, where $\mathbb{1}$ is the vector of all ones. The expected return of this portfolio is $\psi_1(\mathbf{w}) = \mathbb{E}[\mathbf{w}^T \boldsymbol{R}] = \mathbf{w}^T \boldsymbol{\mu}$, where $\boldsymbol{\mu}$ is the mean vector of the asset's returns $\boldsymbol{\mu} = \mathbb{E}[\boldsymbol{R}]$.

Given that the q^{th} central moment of the portfolio retun is $\mathbb{E}[\mathbf{w}^T \tilde{\boldsymbol{R}}]^q$ with $\tilde{\boldsymbol{R}} = \boldsymbol{R} - \boldsymbol{\mu}$, the second, third and fourth moments of the returns are given by:

$$\psi_2(\mathbf{w}) = \mathbb{E}[(\mathbf{w}^T \tilde{\boldsymbol{R}})^2] \tag{2.1}$$

$$= \mathbb{E}[\mathbf{w}^T \tilde{\boldsymbol{R}} \tilde{\boldsymbol{R}}^T \mathbf{w}] \tag{2.2}$$

$$= \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} \tag{2.3}$$

$$\psi_3(\mathbf{w}) = \mathbb{E}[(\mathbf{w}^T \tilde{\boldsymbol{R}})^3] \tag{2.4}$$

$$= \mathbb{E}[\mathbf{w}^T \tilde{\boldsymbol{R}} \tilde{\boldsymbol{R}}^T \mathbf{w} \tilde{\boldsymbol{R}}^{\mathbf{T}} \mathbf{w}] \tag{2.5}$$

$$= \mathbf{w}^T \boldsymbol{\Phi} (\mathbf{w} \otimes \mathbf{w}) \tag{2.6}$$

$$\psi_4(\mathbf{w}) = \mathbb{E}[(\mathbf{w}^T \tilde{\boldsymbol{R}})^4] \tag{2.7}$$

$$= \mathbb{E}[\mathbf{w}^T \tilde{\boldsymbol{R}} \tilde{\boldsymbol{R}}^T \mathbf{w} \tilde{\boldsymbol{R}}^{\mathbf{T}} \mathbf{w} \tilde{\boldsymbol{R}}^{\mathbf{T}} \mathbf{w}] \tag{2.8}$$

$$= \mathbf{w}^T \boldsymbol{\Psi} (\mathbf{w} \otimes \mathbf{w} * \otimes \mathbf{w}) \tag{2.9}$$

where $\otimes$ denotes the Kronecker product, and $\boldsymbol{\Phi} \in \mathbb{R}^{N \text{x} N^2}$ the co-skewness and $\boldsymbol{\Psi} \in \mathbb{R}^{N \text{x} N^3}$ the co-kurtosis matrices are given, respectively, by:

$$\boldsymbol{\Phi} = \mathbb{E}[\tilde{\boldsymbol{R}}(\tilde{\boldsymbol{R}}^T \otimes \tilde{\boldsymbol{R}}^T)], \tag{2.10}$$

$$\boldsymbol{\Psi} = \mathbb{E}[\tilde{\boldsymbol{R}}(\tilde{\boldsymbol{R}}^T \otimes \tilde{\boldsymbol{R}}^T \otimes \tilde{\boldsymbol{R}}^T)] \tag{2.11}$$

In practice, these moments are estimated from computing for the logarithmic return $\mathbf{r}_i$, their associated empirical means, variance, skewness and kurtosis denoted by $\hat{\mu}_i, \hat{\sigma}_i, \hat{s}_i$ and $\hat{k}_i$, respectively as well as the cross-product statistics of the joint distribution of risky asset returns

R_i and R_j, namely, the variance-covariance $\hat{\sigma}_{ij}$, the skewness-coskewness referred to as by s_{iij} and s_{ijj}, and kurtosis-cokurtosis denoted by k_{iiij}, k_{ijjj}, and k_{iijj}. More specifically, these sample estimates are given by:

$$\hat{\sigma}_{ij} = \frac{1}{T}\sum_{i=1}^{N}\sum_{j=1}^{N}(\tilde{R}_i\tilde{R}_j) \tag{2.12}$$

$$\hat{s}_{iij} = \frac{1}{T}\sum_{i=1}^{N}\sum_{j=1}^{N}(\tilde{R}_i^2\tilde{R}_j) \tag{2.13}$$

$$\hat{s}_{ijj} = \frac{1}{T}\sum_{i=1}^{N}\sum_{j=1}^{N}(\tilde{R}_i\tilde{R}_j^2) \tag{2.14}$$

$$\hat{k}_{iiij} = \frac{1}{T}\sum_{i=1}^{N}\sum_{j=1}^{N}(\tilde{R}_i^3\tilde{R}_j) \tag{2.15}$$

$$\hat{k}_{ijjj} = \frac{1}{T}\sum_{i=1}^{N}\sum_{j=1}^{N}(\tilde{R}_i\tilde{R}_j^3) \tag{2.16}$$

$$\hat{k}_{iijj} = \frac{1}{T}\sum_{i=1}^{N}\sum_{j=1}^{N}(\tilde{R}_i^2\tilde{R}_j^2) \tag{2.17}$$

With these quantities at hand, the estimates for the second, third and fourth moments can be expressed as"

$$\hat{\psi}_2(\mathbf{w}) = \sum_{i=1}^{N} w_i^2\hat{\sigma}_i^2 + \sum_{i=1}^{N}\sum_{j=1, j<i}^{N} w_i w_j \hat{\sigma}_{ij} \tag{2.18}$$

$$\hat{\psi}_3(\mathbf{w}) = \sum_{i=1}^{N} w_i^3\hat{s}_i^3 + 3\sum_{i=1}^{N}(\sum_{j=1, j\neq i}^{N} w_i^2 w_j \hat{s}_{iij} + \sum_{j=1, j\neq i}^{N} w_i w_i^2 \hat{s}_{ijj}) \tag{2.19}$$

$$\hat{\psi}_4(\mathbf{w}) = \sum_{i=1}^{N} w_i^4 k_i^4 + 4\sum_{i=1}^{N}(\sum_{j=1, j\neq i}^{N} w_i^3 w_j \hat{k}_{iiij} + \sum_{j=1, j\neq i}^{N} w_i w_j^3 \hat{k}_{ijjj}) \tag{2.20}$$

$$+ 6\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} w_i^2 w_j^2 \hat{k}_{iijj} \tag{2.21}$$

The polynomial goal programming involves a multi-objective optimization technique in which the goal is to maximize the expected and skewness of portfolio return while minimizing variance and kurtosis. The optimal portfolio in the multiobjective optimization process can be represented by:

$$\begin{aligned} &\max_{\mathbf{w}} \;\; \psi_1(\mathbf{w}) \\ &\max_{\mathbf{w}} \;\; \psi_3(\mathbf{w}) \\ &\min_{\mathbf{w}} \;\; \psi_2(\mathbf{w}) \\ &\min_{\mathbf{w}} \;\; \psi_4(\mathbf{w}) \end{aligned} \tag{2.22}$$

$$
\begin{aligned}
\text{s.t. } & \mathbf{w}^T \mathbb{1} = 1 \\
& w_i \geq 0, i = 1, \ldots, N
\end{aligned}
$$

This model can be solved by a single utility function with the aforementioned objectives. By adding skewness and kurtosis to the Markowitz portfolio, it becomes the mean-variance-skewness-kurtosis (MVSK) portfolio as follows:

$$
\begin{aligned}
\min_{\mathbf{w}} \ & f(\mathbf{w}) = -\lambda_1 \psi_1(\mathbf{w}) + \lambda_2 \psi_2(\mathbf{w}) - \breve{\ }_{\mathbf{3}}\,{}_{\mathbf{3}}(\mathbf{w}) + \breve{\ }_{\mathbf{4}}\,{}_{\mathbf{4}}(\mathbf{w}) \\
\text{s.t. } & \mathbf{w}^T \mathbb{1} = 1 \\
& w_i \geq 0, i = 1, \ldots, N
\end{aligned}
\tag{2.23}
$$

where the parameters λ_1, λ_2, λ_3 and λ_4 are positive and are used to combine the four moments of the portfolio return. Solving Problem (2.23) leads to determining the MVSK efficient frontier, for which any moment could not get improved without making the other moments worse. In practice, investors may be interested in changing their current portfolio denoted by $\mathbf{w}_0$ to make it similar to the ones lying in the MVSK efficient portfolio. This can be done by tilting their portfolios in a way that raises their first and third central moments and lowers their second and fourth central moments (Zhou & Palomar, 2021).

To solve the optimization problem in (2.22), we invoked the PGP technique, which involves two main steps. The first step aims to find the optimal costs of all problems by solving each problem equation (2.22) one at a time:

$$
\begin{aligned}
\max_{\mathbf{w}} \ & \psi_1^* = \mathbf{w}^T \boldsymbol{\mu} \\
\text{s.t. } & \mathbf{w}^T \mathbb{1} = 1 \\
& w_i \geq 0, i = 1, \ldots, N
\end{aligned}
\tag{2.24}
$$

$$
\begin{aligned}
\min_{\mathbf{w}} \ & \psi_2^* = \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} \\
\text{s.t. } & \mathbf{w}^T \mathbb{1} = 1 \\
& w_i \geq 0, i = 1, \ldots, N
\end{aligned}
\tag{2.25}
$$

$$
\begin{aligned}
\max_{\mathbf{w}} \ & \psi_3^* = \mathbf{w}^T \boldsymbol{\Phi} (\mathbf{w} \otimes \mathbf{w}) \\
\text{s.t. } & \mathbf{w}^T \mathbb{1} = 1 \\
& w_i \geq 0, i = 1, \ldots, N
\end{aligned}
\tag{2.26}
$$

$$
\begin{aligned}
\min_{\mathbf{w}} \ & \psi_4^* = \mathbf{w}^T \boldsymbol{\Psi} (\mathbf{w} \otimes \mathbf{w} \otimes \mathbf{w}) \\
\text{s.t. } & \mathbf{w}^T \mathbb{1} = 1 \\
& w_i \geq 0, i = 1, \ldots, N
\end{aligned}
\tag{2.27}
$$

Solving the problems (2.24)-(2.27) can be then performed by using linear and nonlinear programming techniques. However, this step is challenging, not only because of the non-convexity of functions $\psi_3(\mathbf{w})$ and $\psi_4(\mathbf{w})$ but also because $\boldsymbol{\Psi}$ has a dimension of NxN^3 which requires a memory complexity of order $\mathcal{O}(N^4)$ and a computational complexity for each single evaluation of the fourth moment of order $\mathcal{O}(N^4)$. In that case, neither the general gradient descent method nor the backtracking line search can be used to solve the high-order portfolio problem.

This is the reason why we did not used traditional convex optimization techniques, more suitable to convex problems or general gradient approach because of the high cost of gradient calculation. Instead, we invoked the Successive Convex Approximation (SCA) algorithm,

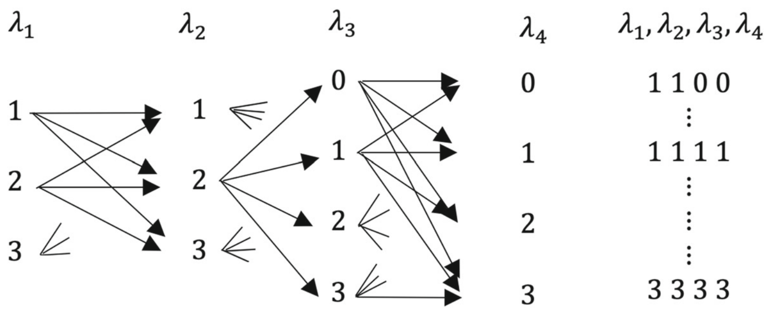


Fig. 1 Different levels of λ's preferences

which was proposed by Rui Zhou and Daniel Palomar in 2020. Having solved the problems (2.24)-(2.27), in the second step, we feed their optimal costs into the PGP model, which consists of minimizing the following Minkowski distance (Keung et al., 2006)

$$Z = \left(\sum_{k=1}^{m} \left| \frac{\psi_k(\mathbf{w}_k)}{\psi^\star} \right|^p \right)^{1/p}$$

where the k^{th} central moment is normalized by ψ_k. We include four parameters, λ_1,λ_2, λ_3, and λ_4, to reflect the various investors' preferences for the mean, variance, skewness, and kurtosis of the portfolio return. Given that the λ_i parameters indicate the investor's subjective level of preferences, the higher the λ_i, the more important the corresponding moment of portfolio return is to the investor. By introducing the auxiliary variables d_k, $k = 1, \ldots, 4$ the PGP model can be rewritten as:

$$\begin{aligned} \min_{\mathbf{w},\mathbf{d}} Z(d) = & \left(\left| \frac{d_1}{\psi_1^*} \right| \right)^{\lambda_1} + \left(\left| \frac{d_2}{\psi_2^*} \right| \right)^{\lambda_2} + \left(\left| \frac{d_3}{\psi_3^*} \right| \right)^{\lambda_3} + \left(\left| \frac{d_4}{\psi_4^*} \right| \right)^{\lambda_4} \\ \text{s.t. } & \psi_1(\mathbf{w}) + \mathbf{d_1} = \overset{*}{\mathbf{1}} \\ & \psi_2(\mathbf{w}) - \mathbf{d_2} = \overset{*}{\mathbf{2}} \\ & \psi_3(\mathbf{w}) + \mathbf{d_3} = \overset{*}{\mathbf{3}} \\ & \psi_4(\mathbf{w}) - \mathbf{d_4} = \overset{*}{\mathbf{4}} \\ & \mathbf{w}^T \mathbb{1} = 1 \\ & d_1, d_2, d_3, d_4 \geq 0, w_i \geq 0, i = 1, \ldots, N \end{aligned} \tag{2.28}$$

where the variables d_1, d_2, d_3 and d_4 are used to minimize the deviations from the desired values. Solutions to problem (2.28) for various combinations of λ_i in Fig. 1 form the set of efficient portfolios. By varying these values, we can get can get the outcomes of different combinations of portfolio items, similarly to MV and MVSK.

For any possible choice of $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, the resulting portfolio achieves an average portfolio return that is always less than ψ_1^*. However, these values of λ can be useful in practice, as they may lead to lower risks and, as a result, a higher probability for the portfolio to achieve greater returns than the average portfolio return ψ_1^* obtained from $\lambda = (1, 0, 0, 0)$. In other words, they may result in better trade-offs between the maximization of this probability and the minimization of risk. However, it remains unclear how the objective function in (2.24) could help achieve this goal for some values of λ. Clearly, a "good" value of λ should lead to a portfolio that protects the investor from experiencing losses while achieving better returns

Table 1 Descriptive statistics and normality test results for international dataset

Variable	Mean (%)	StDev	Skew	Kur	JB test	p-value
SP500.ESG	0.2738	0.0205	−0.7074	6.2417	896.00	0.0000
Europe.ESG	0.1466	0.0246	−1.2238	10.6614	2617.49	0.0000
Japan.ESG	0.1481	0.0236	−0.1577	8.2267	1482.63	0.0000
Canada.ESG	0.1246	0.0235	−1.3988	8.9752	1933.33	0.0000
Korea.ESG	0.1588	0.0329	0.04827	9.0827	1804.78	0.0000
Asia.ESG	0.1394	0.0218	−0.3436	8.3120	1521.65	0.0000
Emerging.ESG	0.0687	0.0241	−0.5248	2.9368	212.76	0.0000

The mean, stabdard deviation (StDev), skewness (Skew), and kurtosis (Kur) values of the returns are shown as descriptive statistics; JB is the value of the Jarque-Bera test for normalcy. The chi-square distribution of the JB test statistic has two degrees of freedom. At the 5% level of significance, the critical value for the JB test is 5.99

than ψ_1^* with a higher probability than with the portfolio obtained for $\lambda = (1, 0, 0, 0)$. As will be seen next, such a behavior can be obtained for some values of λ_i. Indeed, the values of λ_i affect how the portfolio's return is spread out, with some of them resulting in the desired distribution.

Our focus in this research is on the empirical distribution of portfolio returns for different λ_i values. The distribution function is instrumental in determining the probability that the portfolio is above or below a threshold multiplied by ψ_1^*.

3 Empiriacl analysis

3.1 Data description

The sample data for this study consists of the S&P ESG Index Family, which offers investors exposure to companies based on their ESG profiles through country-specific and regional indexes. This index family relies on S&P DJI ESG Scores, derived from the annual S&P Global Corporate Sustainability Assessment (CSA). The scores encompass a total company-level ESG score for a fiscal year, composed of individual scores for environmental (E), social (S), and economic and governance (G) components, as well as 21 industry-specific criterion scores that serve as average ESG signals. Our choice of using the S&P ESG index family is driven by the growing significance of Environmental, Social, and Governance (ESG) factors in investment decisions. The S&P ESG index family is a widely recognized and popular set of indices that incorporate ESG criteria, making it a pertinent and valuable source for analyzing portfolio performance within the context of our study.

The index family includes a selection of the following broad equity indices S&P 500 ESG, Europe ESG, Japan ESG, Canada ESG, Korea.ESG, Asia.ESG, and Emerging.ESG. The data used in this study represent weekly samples and are taken from the following web site: spglobal.com/esg/p performance/indices/esg-index-family. The entire data set covers the period from 30 Mars, 2002 to 26 April, 2022, with a total of 3675 observations. We use the weekly returns of these indices, which are defined for each index (i) as follows: $R_{i,t} = p_{i,t}/p_{i,t-1} - 1$ where $p_{i,t}$ is the price of the stock index traded at time t.

The statistics in Table 1 provide some information on the characteristics of the return data. According to the first column, all indexes have a positive mean return. S&P 500.ESG has

Table 2 The desired optimal levels of four portfolio return moments

$\psi_1(\mathbf{w})$ 10^{-2}	$\psi_2(\mathbf{w})$ (10^{-3})	$\psi_3(\mathbf{w})$ (10^{-6})	$\psi_4(\mathbf{w})$ (10^{-6})
0.2665	0.4238	- 6.5781	1.6736
0.2161	**0.3692**	- 4.6399	1.3129
0.1553	0.7502	**4.2266**	7.2071
0.1899	0.3694	- 4.3988	**1.1552**
$\psi_1^*(\mathbf{w})$ 10^{-2}	$\psi_2^*(\mathbf{w})$ (10^{-3})	$\psi_3^*(\mathbf{w})$ (10^{-6})	$\psi_4^*(\mathbf{w})$ (10^{-6})
0.2665	0.3692	4.2266	1.1552

The bolded values indicate the four optimal portfolio return moments

the greatest average rate of return, (0.27%), followed by Korea.ESG (0.15%), Japan.ESG (0.148%), and Europe.ESG (0.146%). Table 1 also shows the skewness and kurtosis values for each of the index returns. As can be seen, the skewness is negative except for Korea.ESG while the kurtosis is high. The Jarque-Bera test rejects the null hypothesis of normality at the 5% significance level for the majority of the return distributions of empirical datasets, which clearly shows that none of the returns can be reasonably assumed to be normal. It is thus of interest to include the higher-order moments in the design and analysis of portfolios. For this purpose, we invoke the Polynomial Goal Programming (PGP). which allows us to account for the skewness and kurtosis when designing the portfolio.

3.2 Results of the empirical study

In this section, we present the results of the optimization problems presented in the previous section. As previously suggested, we begin by determining the best value for each of the four portfolio return moments (ψ_1^*, ψ_2^*, ψ_3^*, andψ_4^*) as expressed in the equations system (2.24) through (2.27). We get the following set of results:

In accordance with Table 2, Fig. 2 shows that the optimal portfolio chosen maximizing the first moment of portfolio returns, which is generally chosen by most investors, corresponds to investing approximately 90% in SP500.ESG asset and a maximum of 10% in other assets. Such a result can already be anticipated from Table 1, which shows that the SP500.ESG asset achieves the highest average rate of return. Overall, the optimal portfolio in terms of first moment maximization has the following characteristics: a mean return of 0.26%, a variance of 0.04%, a kurtosis of −2.99 and a skewness of −0.72. As for the portfolio minimizing the second moment, it is obtained with about 56% in SP500.ESG asset, 20% in Asia.ESG asset, 11% in Japan asset, and no more than 10% in other assets.

Now, focusing on the portfolio that maximizes the third moment, Fig. 2 shows that, in accordance with Table 1, it is obtained by investing 67% in Korea.ESG asset and 33% in Europe.ESG asset, which together represent the highest skewness. Similarly, according to Table 1, the SP500.ESG, Emerging.ESG and Japan.ESG assets have the lowest kurtosis, which explains why most investors only invest in them when they look to minimize the fourth moment of the portfolio return.

As an additional experiment, we search to maximize (resp. minimize) the 3rd moment (resp. the 4th moment), assuming that the other moments are fixed. This amounts to maximizing the skewness (or minimizing the kurtosis) for a given variance and return, which are chosen in the Markowitz (1952) efficient frontier. We then display the skewness shape (resp. kurtosis shape) for all efficient portfolios in Fig. 3 (resp. Fig. 4). As can be seen, all these skewnesses are negative because they are caused by positive coskewness. Moreover, we can

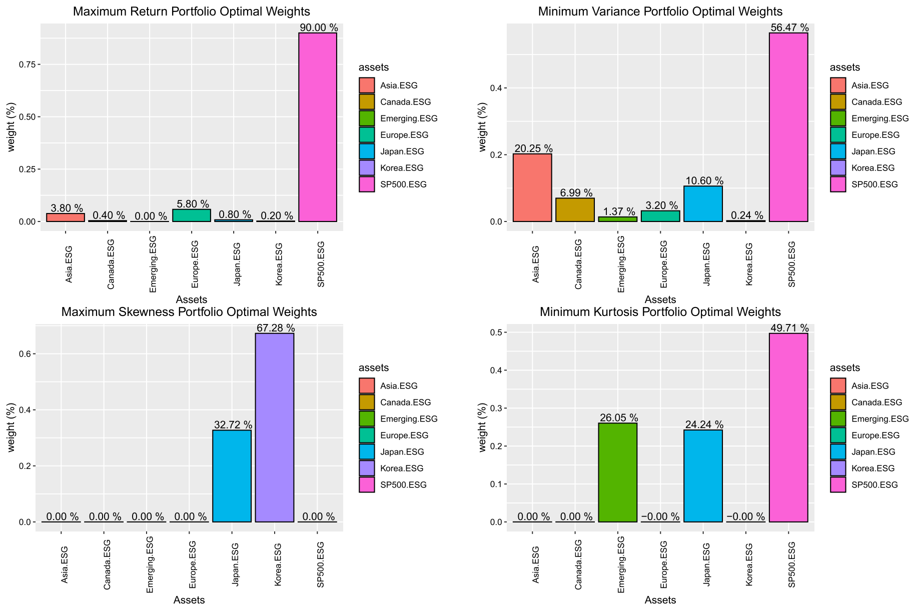
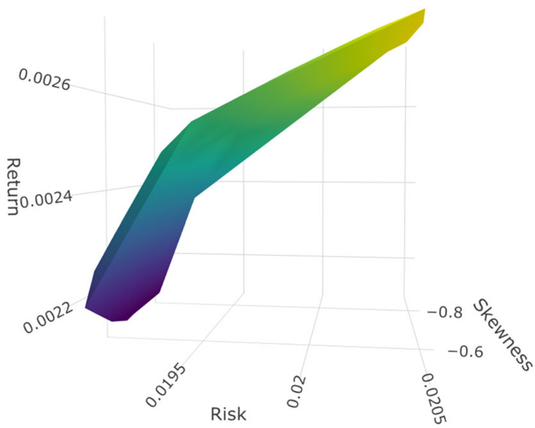


Fig. 2 Asset allocation for different portfolio moments

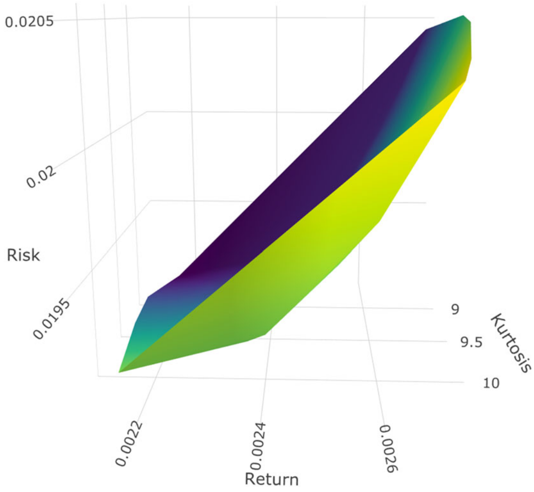
Fig. 3 Shape of the mean-variance-skewness efficient frontier



see from Fig. 4 that due to negative cokurtosis, diversification is likely to promote kurtosis minimization.

With the four subproblems ((2.24)–(2.27)) being solved, we are now ready to study the PGP approach and solve the PGP model in (2.28). Prior research into the PGP approach examined how investors' preferences influenced portfolio selection for restricted $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ combinations. Particularly, the studied investors' preferences were for the following values of lambda: (3, 1, 1, 0), (3, 1, 2, 1), (3, 1, 3, 1), (1, 3, 1, 1), (1, 1, 1, 3), (1, 3, 1, 3), (1, 2, 3, 2), (3, 1, 2, 3), (2, 3, 3, 1), (1, 1, 1, 1), and (1, 1, 0, 0), which allow investors to place more or less importance onto each of the fourth first moments. For instance, the (1, 1, 1, 1) preference structure considers that the mean, variance, skewness, and kurtosis of the excess return are all important to investors. On the other hand, the results based on the preference

Fig. 4 Shape of the mean-variance-kurtosis efficient frontier



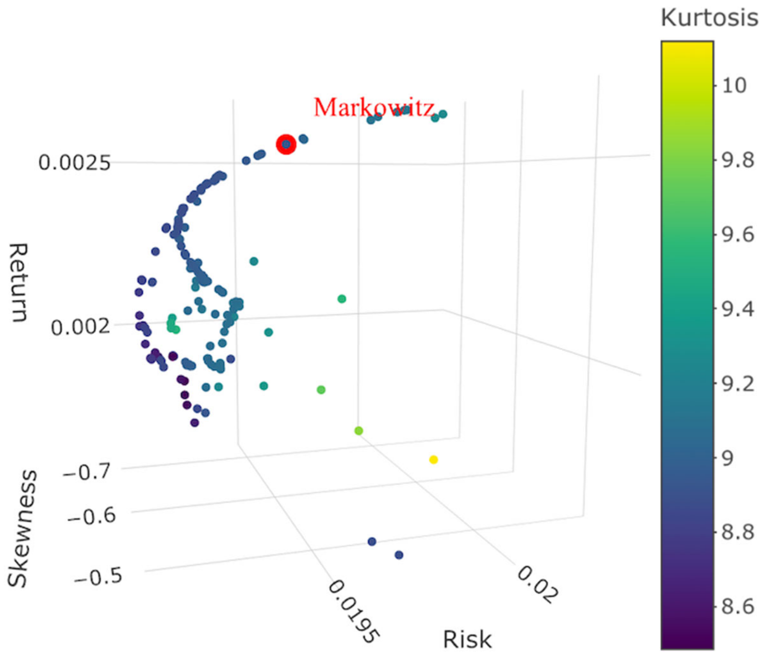


Fig. 5 The shape of the MVSK efficient frontier from PGP

structures of (1, 3, 1, 1), (1, 1, 1, 3), and (1, 3, 1, 1) place more emphasis on risk control, while those based on (3, 1, 1, 0), (3, 1, 2, 1), and (3, 1, 3, 1) imply that investors are willing to pursue more excess returns regardless of risk level. However, the investors who select lambda among the following sets (1, 2, 3, 2), (3, 1, 2, 3), and (2, 3, 3, 1) aim to account for multiple objectives. The benchmark case, which represents the Markowitz mean-variance portfolio, is represented by the (1, 1, 0, 0).

Table 3 Asset allocations and moment statistics for optimal portfolios with a high return compared to Markowitz

Portfolio	A	B	C	D
λ_1	1	1	2	1
λ_2	3	3	3	1
λ_3	0	0	1	0
λ_4	0	2	3	0
SP500.ESG	0.922	0.864	0.936	0.8500
Europe.ESG	0.000	0.000	0.000	0.000
Japan.ESG	0.130	0.063	0.147	0.149
Canada.ESG	0.000	0.000	0.000	0.000
Korea.ESG	0.000	0.001	0.000	0.000
Asia.ESG	0.000	0.001	0.000	0.000
Emerging.ESG	0.000	0.002	0.000	0.000
Mean (%)	0.264	0.256	0.266	0.2549
Var (10^{-4})	3.998	3.877	4.030	3.847
Skew	−0.687	−0.670	−0.691	−0.661
Kur	9.104	9.124	9.030	9.133

In our study, in order to further investigate the effect of investors' preferences on portfolio selection, different levels of preference have been investigated by allowing each value of lambda to take values in the set 0, 1, 2, 3. This gives in total 144 possible values of λ's preferences illustrated in Fig. 1. Based on the results for all these values of investor's preferences, we build the efficient frontier for MVSK. Figure 5 provides a three-dimensional representation of the efficient frontier for MVSK. The x-axis represents the expected return, the y-axis represents the volatility (i.e., standard deviation), and the z-axis represents the skewness. The kurtosis is shown as a color map, with yellow indicating high kurtosis and blue indicating low kurtosis. By constructing the efficient frontier for MVSK, investors can identify the optimal portfolios that balance risk and return based on their preferences for skewness and kurtosis. As the figure shows, higher expected returns are generally associated with higher volatility, which is consistent with the standard mean-variance analysis. However, the efficient frontier for MVSK reveals that portfolios can have different skewness and kurtosis levels while still experiencing the same expected return and volatility. This means that investors can optimize their portfolios not only for risk and return but also for skewness and kurtosis based on their preferences. Furthermore, the efficient frontier for MVSK shows that the PGP of some portfolios dominates that of others. This dominance is due to the tradeoff between the four moments of the distribution, which causes at least one of the statistics for the other three moments to worsen. For example, a portfolio with high expected returns and low volatility may have a negative skewness or high kurtosis. To avoid such extreme outcomes, investors can use the efficient frontier for MVSK to identify the optimal portfolios that meet their preferences for skewness and kurtosis while maximizing their risk-adjusted returns.

Table 3 presents some portfolios that have a higher average return compared to Markowitz. It clearly shows that investor preferences affect the four portfolio moments. In general, the more importance investors attach to a certain moment, i.e., the higher the preference parameter, the more favorable this moment statistic is in the optimal portfolio. For instance, the lowest kurtosis is achieved in Portfolio C. But, interestingly, this behavior is not observed for skewness. In fact, the highest skewness is achieved in portfolio D, where the investor

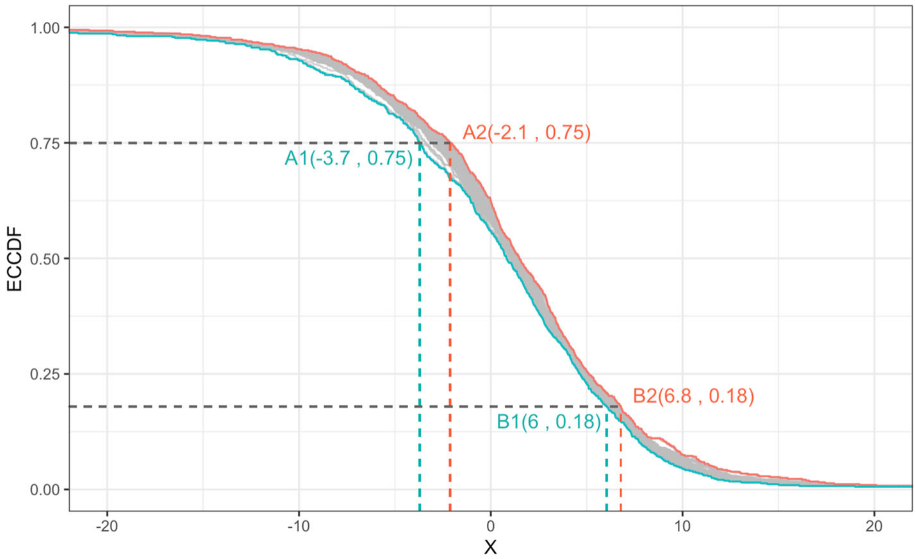


Fig. 6 ECCDF of the portfolio's returns from different λ_i values

preference is (1, 1, 0, 0), which clearly illustrates that this rule could not be taken for granted. For this reason, we shift our interest in the next step of our research to the empirical distribution of portfolio returns obtained for all studied values of lambda.

Figure 6 shows the Empirical Complementary Cumulative Distribution Functions (ECCDF) graphs of several portfolio returns normalized by $\psi_1^\star$ obtained by selecting $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ from a set of 144 possible values. For a given value of λ, the corresponding curve represents the probability that the corresponding portfolio return divided by $\psi_1^\star$ is greater than a threshold shown on the abscissa axis in Fig. 6. The red (resp. green) curve in this figure depicts the highest (resp. lowest) probability for any given threshold value from the available set of ECCDF graphs. These curves can alternatively assist in indicating the greater (resp. lesser) value of the threshold for a given probability value.

The represented curves may be used to guide the investor in choosing the suitable value of λ. For instance, fixing a target probability of 18% in the ECCDF, the investor will select point B2 (6.8, 0.18) rather than point B2 (6, 0.18) because it leads to achieving with a nearly one-fourth chance a return higher than 6.8 rather than 6. In other words, given that $\psi_1^* = 0.266510^{-2}$, the investor will take a higher risk by selecting Lambda =(2, 1, 2, 0), hoping to earn up to 0.018122 ($= 6.8 * 0.266510^{-2}$) per week. The same reasoning holds for the point A1 and A2, for which we can easily see that the investor is better off moving towards point A1.

4 Conclusion

This study examines the Polynomial Goal Programming (PGP) approach, which seeks to create a class of portfolios that optimize a parameterized function based on the first four moments of their achieved returns. The PGP parameters reflect an investor's preferences concerning expected return, variance, skewness, and kurtosis of the portfolio returns. Our primary research question explores how these parameters enable investors to accomplish multiple, conflicting objectives, such as maximizing expected return and skewness while minimizing variance and kurtosis. Intriguingly, we demonstrate that, in some instances, considering combinations of investor preferences may result in higher returns for certain realizations compared to solely maximizing the expected return, although the average return remains lower in this case. To assist investors in identifying the most suitable set of parameters, we introduce a practical tool based on the representation of the empirical distribution function of portfolio returns. We demonstrate that this tool enables investors to easily select the optimal parameters that maximize the probability of their portfolio return (normalized by the average return obtained through expected return maximization) exceeding a specific threshold. This innovative approach not only aids investors in achieving their desired investment outcomes but also contributes significantly to the risk management field by offering a practical method for balancing various risk factors and performance objectives in portfolio optimization.

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