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Abstract: A novel generating mechanism for non-strict bivariate Archimedean copulas via the Lorenz curve of a non-negative random variable is proposed. Lorenz curves have been extensively studied in economics and statistics to characterize wealth inequality and tail risk. In this paper, these curves are seen as integral transforms generating increasing convex functions in the unit square. Many of the properties of these "Lorenz copulas", from tail dependence and stochastic ordering, to their Kendall distribution function and the size of the singular part, depend on simple features of the random variable associated to the generating Lorenz curve. For instance, by selecting random variables with a lower bound at zero it is possible to create copulas with asymptotic upper tail dependence. An "alchemy" of Lorenz curves that can be used as general framework to build multiparametric families of copulas is also discussed.

Keywords: Lorenz curves, Archimedean copulas, stochastic ordering, tail dependence, Gini index

MSC: 62H05, 62H10

1 Introduction

The paper introduces and studies Lorenz copulas, a novel class of non-strict bivariate Archimedean copulas [35], whose generators are defined in terms of Lorenz curves [28]. A well-known tool in the study of socio-economic inequality and statistical size distributions [4, 12, 25], the Lorenz curve is a very interesting mathematical object, characterized by several useful properties, which prove extremely powerful when imported in the world of copulas and dependence modeling.

For instance, the use of Lorenz generators not only allows for the definition of several brand new copulas, given the richness of Lorenz curves available in the literature [25], but it also gives the possibility of immediately knowing fundamental characteristics of the resulting non-strict copulas, from their upper-tail dependence structure, to the presence or not of a singular part, from the features of the zero set, to the behavior of concordance measures like the Kendall's τ. Moreover, the use of Lorenz generators suggests an easy way to develop multiparametric families of non-strict bivariate Archimedean copulas.

The paper is organized as follows: the next three subsections deal with the basic notation and the necessary tools; Section 2 introduces the novel Lorenz copulas; Section 3 contains some explicit examples, like the lognormal and the uniform Lorenz copulas; Section 4 discusses how to obtain new generators exploting the so-called alchemy of Lorenz curves, and how to create new multiparametric families of copulas; Section 5 closes the paper, suggesting possible extensions and research directions.

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1.1 Bivariate Archimedean copulas: a quick review

First introduced by Sklar [48], copulas represent a convenient way of modeling multivariate phenomena, by disentangling the joint dependence structure from the marginal behavior. This is particularly true for applications, where the flexibility of copulas appears preferable to the direct fitting of multivariate distributions, which may be difficult to define and deal with [35]. For the sake of completeness, not all statisticians agree with this view: for example Mikosch [33] argues that the static separation of the dependence function from the marginal distributions gives a biased view of stochastic dependence.

In the bivariate framework, consider the two-variable function $C : [0,1]^2 \to [0,1]$. $C(u,v)$ is a two-dimensional copula if:

1. $C(u,0) = C(0,v) = 0$ for every $u, v \in [0,1]$;
2. $C(u,1) = u$ and $C(1,v) = v$ for every $u, v \in [0,1]$;
3. $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$ for every $u_1, v_1, u_2, v_2 \in [0,1]$, such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

Sklar's theorem [48] shows that, if (X_1, X_2) is a random vector with joint distribution F and margins F_1 and F_2, then

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)) = C(u,v), \qquad u, v \in [0,1].$$

In other words, it is possible to represent the bivariate distribution F of the random vector (X_1, X_2) in terms of the copula function C, and of two uniform margins obtained via the probability integral transform. A bivariate copula is thus nothing more than a bivariate distribution with uniform margins. If the marginals F_1 and F_2 are continuous then C is unique, otherwise it is uniquely determined on the Cartesian product of the support of the two marginals distributions.

It can be easily verified that, for every copula $C(u,v)$, one has

$$W(u,v) = \max(u + v - 1, 0) \leq C(u,v) \leq \min(u,v) = M(u,v),$$

where W and M are known as the Fréchet-Hoeffding lower and upper bound respectively [35]. In the bivariate framework here considered, both W and M are proper copula functions.

A copula $C(u,v)$ is Archimedean if it is associative, that is $C(u, C(z,w)) = C(C(u,z), w)$ for every $u, z, w \in [0,1]$, and if its diagonal $\delta_C(x) = C(x,x) < x$ for every $x \in (0,1)$ [2, 35].

The associative nature of Archimedean copulas allows for a very convenient representation in terms of a one-place function called the generator.

Theorem 1. *Let $C(u,v) \in [0,1]^2$ be an Archimedean copula, then there exists a strictly decreasing convex function $\varphi : [0,1] \to \mathbb{R}^+$, with $\varphi(0) \leq \infty$ and $\varphi(1) = 0$, such that*

$$C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)) \quad u, v \in [0,1], \tag{1}$$

where $\varphi^{[-1]}$ is the pseudo-inverse [35] of φ.

For details and a proof we refer to [35]. If one is familiar with non-Newtonian calculus [22], Equation (1) can be recognized as a φ-arithmetic operation, a φ-sum specifically [31]. This suggests that a bivariate Archimedean copula endows the interval [0, 1] with a semi-group structure.

To any Archimedean copula $C(u,v)$ it is always possible to associate a co-copula

$$\hat{C}(u,v) := u + v - C(u,v),$$

where $\hat{C}(u,v)$ is an S-norm [2], whose generator $\hat{\varphi}$ is a strictly increasing convex function with swapped boundary conditions with respect to those in Theorem 1.

In the literature, several functions φ have been proposed over the years, from the well-known logarithmic and exponential generators behind the famous Clayton, Gumbel, Joe and Independence copulas [35], to those based on the inverse Laplace and the Williamson transforms [30]. In particular, this last class of generators provides a solution to the problem of finding d-monotonic functions, thus extending the Archimedean construction to an arbitrary number of dimensions.

Naturally, a large number of generators has given birth to a large number of copulas, and this richness (and flexibility) is one of the reasons of the popularity of the Archimedean family, in particular in applications [49].

An appealing feature of the Archimedean family is that the properties of the generator determine the properties of the corresponding copula. For example, looking at the value $\varphi(0)$, it is possible to distinguish between strict, for $\varphi(0) = \infty$, and non-strict, for $\varphi(0) < \infty$, copulas [35].

Non-strict copulas are the object of interest of this paper. A non-strict copula $C(u, v)$ is such that a subset of its domain has zero probability mass but possibly positive Lebesgue measure [35]. Such a subset takes the name of zero set, or $Z(C)$.

While is not difficult to verify that a non-strict Archimedean copula $C(u, v)$ is not able to model independence directly *i.e.* $C(u, v) \neq uv \neq \Pi(u, v)$, it can be extremely useful when dealing with phenomena that exhibit upper tail dependence, or when one is interested in the dependence structure of random quantities that do not take on low quantiles at the same time [5, 11, 26]. In economics, for instance, a situation in which a non-strict copula could be a viable tool for data modeling is given by the presence of minimum production cost (including minimum wages), or the existence of some sort of technological frontier [43].

Strictly related to $Z(C)$ is the zero curve $v = \kappa(u)$, that is the level curve separating the zero set from the part of the copula domain with positive probability mass. For a non-strict Archimedean copula the zero curve can be easily derived in terms of generator φ, by setting $C(u, v) = 0$ in Equation (1), so that

$$\kappa(u) = \varphi^{[-1]}(\varphi(0) - \varphi(u)), \qquad \forall u \in [0, 1].$$

When dealing with bivariate copulas, and in particular with Archimedean copulas, a very important object of study is the Kendall distribution function $K(t)$ [20]. Such a function represents the bivariate equivalent of the univariate probability integral transform, and it is formally defined as

$$K(t) = P(C(U, V) \leq t), \qquad t \in [0, 1],$$

where U and V are standard uniforms on $[0, 1]$. For a fixed $t \in [0, 1]$, the Kendall distribution function can be seen as the measure–also known as the C–measure [2]–of the set $\{(u, v) \in [0, 1]^2 : C(u, v) \leq t\}$.

In an Archimedean copula with differentiable generator φ, $K(t)$ can be obtained as

$$K(t) = t - \frac{\varphi(t)}{\varphi'(t^+)}, \quad t \in [0, 1], \tag{2}$$

with $\varphi'(t^+)$ denoting the right derivative of φ at t. It is important to recall that for bivariate Archimedean copulas with differentiable generator the function K can be used to determine the corresponding copula via its generator φ leading to possible estimation strategies, see [20] for more details.

The Kendall distribution function has many applications in copula theory [36, 37]. In what follows we will mainly exploit the following ones:

- $K(\cdot)$ induces a dependence ordering in the set of copulas, known as the Kendall stochastic ordering [7].
- $K(\cdot)$ can be used to obtain association and dependence measures between random variables. For instance, a well-known concordance measure like the Kendall's τ can be computed as

$$\tau = 3 - 4 \int_0^1 K(t)\mathrm{d}t. \tag{3}$$

As we will further investigate in Sections 2 and 3, Equations (2) and (3) suggest that it is possible to build a direct connection between measures like τ and the generator φ, via the Kendall distribution function K.

1.2 The Lorenz curve

The Lorenz curve L of a non-negative random variable X, with finite expectation and cumulative distribution function F, is defined as

$$L(p) = \frac{\int_0^p F^{-1}(y)\mathrm{d}y}{\int_0^1 F^{-1}(y)\mathrm{d}y}, \quad p \in [0, 1], \tag{4}$$

where F^{-1} is the quantile function of X [17], that is the quasi-inverse of F. A Lorenz curve completely characterizes the corresponding distribution function F up to a scale transformation [25].

Introduced by Max Lorenz in 1905 [28], the Lorenz curve is a well-known tool in the study of wealth and income inequality [12]. When the random variable X represents wealth in a given society, the curve $L(p)$ represents the percentage of wealth owned by the lower $p \times 100\%$ of the population. This makes the curve L essential to study economic size distributions, and to verify the so-called Pareto principle [4, 13].

Given a Lorenz curve it is then possible to construct a large number of inequality indices [12]. Among them, a famous one is the Gini G index [21], defined as (5).

$$G = 2\int_0^1 \left(p - L(p)\right) \mathrm{d}p. \tag{5}$$

Clearly $G \in [0, 1]$. A Gini equal to 0 indicates a society in which everyone possesses the same amount of wealth. A Gini equal to 1 describes the opposite situation: one individual owns everything and all the others nothing. All other values represent intermediate situations: the higher the Gini, the more unequal the society.

As observed in [51], all inequality indices are nothing more than generalizations and improvements of some common measures of variability like the variance or the standard deviation. This justifies the rising interest for their application outside inequality studies, in fields like biostatistics and finance, see for example [15] and references therein.

The following proposition collects some useful properties of the Lorenz curve which will be exploited later. For further properties and all proofs, please refer to [3].

Proposition 1. *Let L be the Lorenz curve associated with the random variable $X \geq 0$, with $E[X] = \mu < \infty$ and distribution function F. The following holds:*

1. *$L(p)$ is a non-decreasing and convex function in $p \in [0, 1]$, with $L(0) = 0$ and $L(1) = 1$. If F is strictly increasing, then L is strictly convex.*
2. *$L(p)$ is strictly increasing if and only if $F(0) = 0$.*
3. *$L(p)$ is non-differentiable in $p \in (0, 1)$ if and only if the quantile function of X, $F^{-1}(p)$, has jump discontinuities in $(0, 1)$.*
4. *If $L(p)$ admits a first derivative then $L'(p) = \frac{F^{-1}(p)}{\mu}$;*
5. *For all $p \in [0, 1]$, the curve $L(p)$ is always bounded from above by $L_{PE}(p) = p$ and from below by $L_{PI}(p) = 0$, for all $p \in [0, 1)$ and $L_{PI}(1) = 1$. The curves $L_{PE}(p)$ and $L_{PI}(p)$ are respectively called perfect equality and perfect inequality lines.*
6. *L can be seen as a distribution function. In particular, it represents the distribution function of the random variable $Y_L = L^{-1}(U)$, with U following a standard uniform, and where $L^{-1}(p)$ is the quasi-inverse of $L(p)$.*

Note that here and in the rest of the paper $L^{-1}(p)$ is defined as the quasi-inverse of the Lorenz curve being the $L(p)$ not necessarily strictly increasing, as the case when $F(0) = m$ with $m \in (0, 1)$.

From Proposition 1 one can derive two important facts. First, every non-decreasing convex function $g : [0, 1] \to [0, 1]$, such that $g(0) = 0$ and $g(1) = 1$ is a Lorenz curve [51] corresponding to some random variable $X \geq 0$ with finite expectation. Second, every Lorenz curve can be seen and used as a distortion function [6, 29], *i.e.* an integral transform generating increasing convex functions in the unit square. This second fact will prove useful later in the paper.

1.3 Orders

Dependence orderings are multivariate stochastic orders defining posets among copulas [44]. The following definition introduces three cases relevant for the paper.

Definition 1. Let $C_1(u, v)$ and $C_2(u, v)$ be two copulas, with Kendall distribution functions $K_1(t)$ and $K_2(t)$ respectively. Denote by ${}_xC(v)$, the conditional probability $P(V \leq v \,|\, U \leq x)$, and by ${}_xC^{-1}$ its inverse. We have

1. Left tail decreasing order (LTD): $C_1 \succ^{\mathrm{LTD}} C_2$, if ${}_{x'}C_1({}_xC_1^{-1}(u)) \le {}_{x'}C_2({}_xC_2^{-1}(u))$, for $0 \le x < x' \le 1$.
2. Positive Kendall order (PK): $C_1 \succ^{\mathrm{PK}} C_2$, if $K_1(t) \le K_2(t)$ for every $t \in [0,1]$.
3. Positive quadrant order (PQD): $C_1 \succ^{\mathrm{PQD}} C_2$, if $C_1(u,v) \ge C_2(u,v)$, for all $(u,v) \in [0,1]^2$.

If $C_1(u,v)$ and $C_2(u,v)$ are Archimedean, one has the following relevant implications

$$LTD \Rightarrow PK \Rightarrow PQD.$$

As proven in [5], for Archimedean copulas the conditions given in Definition 1 can be restated in terms of their generators, as in the following theorem.

Theorem 2. *Let $C_1(u,v)$ and $C_2(u,v)$ be two Archimedean copulas with corresponding generators φ_1 and φ_2. The following holds:*
1. *$C_1 \succ^{LTD} C_2$, if and only if $\varphi_1(\varphi_2^{[-1]}(y))$ is convex;*
2. *$C_1 \succ^{PK} C_2$, if and only if $\varphi_1(\varphi_2^{[-1]}(y))$ is star-shaped[1];*
3. *$C_1 \succ^{PQD} C_2$, if and only if $\varphi_1(\varphi_2^{[-1]}(y))$ is super-additive.*

As in the case of copulas, it is possible to define notions of stochastic ordering which make the set of non-negative random variables with finite expectations a poset [3]. The following definition introduces two useful stochastic orders related to the Lorenz curve.

Definition 2. Let X_1 and X_2 be two non-negative random variables with finite expectation, and let L_1, L_2 be their Lorenz curves. The Lorenz order and the star order are defined as follows:
1. Lorenz order: $X_1 \succ^{\mathrm{L}} X_2$, if $L_1(p) \le L_2(p)$ for all $p \in [0,1]$;
2. Star order: $X_1 \succ^{*} X_2$, when $L_1(L_2^{-1}(p))$ is convex in $p \in [0,1]$.

Observe that, if $X_1 \succ^{\mathrm{L}} X_2$, then $G_1 \ge G_2$, where G_1 and G_2 are the Gini indices of X_1 and X_2 respectively. Furthermore, notice that the star order condition in Definition 2 is not the usual one relying on quantile functions [44]. However, as shown in the appendix, the two definitions are equivalent. Finally, it can be easily shown that the star order implies the Lorenz order [3].

2 Lorenz generators and Lorenz copulas

We now have all the ingredients to finally introduce the new class of Lorenz copulas, that is bivariate non-strict Archimidean copulas, whose generators are defined in terms of Lorenz curves.

Let $\mathcal{L}_\varphi$ be the set of all strictly increasing Lorenz curves. In terms of the random variable X, with distribution function F, such a set is characterized as

$$\mathcal{L}_\varphi := \{L : X \ge 0,\ E[X] < \infty,\ F(0) = 0\}.$$

All Lorenz curves in $\mathcal{L}_\varphi$ are strictly increasing homeomorphisms of $[0,1]$.

Let $L \in \mathcal{L}_\varphi$, for $p \in [0,1]$ define the mirrored Lorenz curve as

$$\bar{L}(p) := L(1-p).$$

This new function is strictly decreasing and convex, with $\bar{L}(0) = 1$ and $\bar{L}(1) = 0$.

Recalling Theorem 1, it is evident that $\bar{L}(p)$ is a valid generator for a bivariate Archimedean copula $C(u,v)$. Since $\bar{L}(0) < \infty$, the copula will be non-strict. Conversely, the curve $L(p)$ is a proper generator for the co-copula $\hat{C}(u,v)$.

1 A function $h : [0,\infty) \to [0,\infty)$ with $h(0) = 0$ is star-shaped if, for every $\alpha \in [0,1]$ and every x, one has $h(\alpha x) \le \alpha h(x)$.

The Gini index corresponding to $\bar{L}$ is given by $\bar{G} = 2\int_0^1 (1-p-\bar{L}(p))\mathrm{d}p$, and it is easy to verify that its value will coincide with that of the standard Gini G associated to L. For this reason, the notation G will be used to indicate both indices.

Definition 3 (Lorenz copula)**.** Let $L \in \mathcal{L}_\varphi$ be the Lorenz curve of the random variable X, and let $\bar{L}$ be its mirrored version. The corresponding bivariate non-strict Archimedean copula $C(u,v)$, in short Lorenz copula, is given by

$$C(u,v) = \bar{L}^{[-1]}(\bar{L}(u)+\bar{L}(v)) = 1 - L^{[-1]}(L(1-u)+L(1-v)), \qquad u, v \in [0,1]^2. \tag{6}$$

$C(u,v)$ is thus the Lorenz copula generated by L or $\bar{L}$, or the copula associated with X. In what follows, $C(u,v)$ will always indicate a Lorenz copula.

Every non-strict copula is characterized by the presence of a zero set and the relative zero curve. For a Lorenz copula, the zero curve $\kappa(p)$ is easily derived to be

$$\kappa(p) = 1 - L^{[-1]}(1 - L(1-p)), \quad p \in [0,1]. \tag{7}$$

Interestingly, the function $\kappa(p)$ is itself a mirrored Lorenz curve, given that it follows the composition rules described in [3]. This means that a particular Gini index, the zero-Gini G_κ, can be computed from it. Such an index has an appealing interpretation in terms of the corresponding Lorenz copula: G_κ measures indeed how far $C(u,v)$ is from its Fréchet-Hoeffding bounds, *i.e.* from co- and countermonotonicity. In particular $G_\kappa \to 0$ indicates that the copula tends, in the limit, to the Fréchet-Hoeffding lower bound W, while $G_\kappa = 1$ indicates that $C(u,v)$ coincides with the Fréchet-Hoeffding upper bound M.

The following proposition clarifies the relation between G_κ and the Gini G associated with the Lorenz generator $\bar{L}$.

Proposition 2. *Let G_κ be the zero-Gini of the Lorenz copula $C(u,v)$, while G is the Gini associated to the Lorenz generator $\bar{L}$. Then*

$$G_\kappa \geq G.$$

Moreover, if X_1 and X_2 are two non-negative random variables and $X_1 \succ^L X_2$, one has

$$G_\kappa^1 \geq G_\kappa^2,$$

where G_κ^i, $i = 1,2$, is the zero-Gini of the copula associated with X_i.

Proof. To prove the first statement, notice that

$$G_\kappa - G \propto \int_0^1 \left(L^{[-1]}(1-L(1-p)) - 1 + L(1-p)\right)\mathrm{d}p. \tag{8}$$

In order for $G_\kappa \geq G$ to hold true it is sufficient to show that the right-hand side of Equation (8) is non-negative.

Setting $y = 1 - L(1-p)$, one gets that (8) is proportional to

$$\int_0^1 \frac{1}{F^{-1}(L^{[-1]}(1-y))}(L^{[-1]}(y) - y)dy \geq 0, \tag{9}$$

since, by construction, $L^{[-1]}(y) \geq y$ for all $y \in [0,1]$.

Hence $G_\kappa \geq G$.

To prove the second statement it is sufficient to show that

$$\int_0^1 L_1^{[-1]}(1-L_1(1-p))\mathrm{d}p \geq \int_0^1 L_2^{[-1]}(1-L_2(1-p))\mathrm{d}p, \tag{10}$$

where $L_i(p)$ is the Lorenz curve of X_i, $i = 1,2$, such that $L_1(p) \leq L_2(p)$ for every $p \in [0,1]$.

Observing that $L^{[-1]}(p)$ is an increasing concave function for $p \in [0,1]$, and that $L_1^{[-1]}(p) \geq L_2^{[-1]}(p)$ for all $p \in [0,1]$ when $X_1 \succ^L X_2$, by monotonicity we can conclude that Equation (10) holds true. □

Notice that the opposite assertion does not hold, namely $G_\kappa^1 \geq G_\kappa^2$ does not imply $X_1 \succ^L X_2$. This is due to the fact that the order induced by the zero-Gini is a total one, while the Lorenz order is only partial. Therefore it is always possible to define two intersecting Lorenz curves resulting in either $G_\kappa^1 \leq G_\kappa^2$ or $G_\kappa^1 \geq G_\kappa^2$.

Thanks to Proposition 2, a necessary condition for $G_\kappa^1 \geq G_\kappa^2$ is that $G_1 \geq G_2$. Such a result proves useful when generating Lorenz copulas that need to satisfy some specific conditions in terms of their zero sets and zero curves (more in Section 3).

It is no surprise that many analytic properties of the copula $C(u, v)$ trace back to the random variable X, its Lorenz curve L and the associated mirrored version $\bar{L}$. These properties are collected per topic in the following.

2.1 Bounds and singularities

The next proposition clarifies under which conditions a Lorenz copula replicates the Fréchet-Hoeffding lower bound.

Proposition 3. *Let $C(u, v)$ be a non-strict Archimedean copula, while $\bar{L}$ is its Lorenz generator obtained from the curve L of the non-negative finite-mean random variable X. The following statements are then equivalent:*
1. *$C(u, v)$ coincides with the Fréchet-Hoeffding lower bound $W(u, v) = \max(u + v - 1, 0)$;*
2. *X is characterized by perfect equality,* i.e. *$L(p) = L_{PE}(p)$;*
3. *$G = 0$.*

Proof. The goal is to show that $1 \Leftrightarrow 2$ and $2 \Leftrightarrow 3$.

Assume that $L(p) = L_{PE}(p)$. The mirrored Lorenz is $\bar{L}_{PE}(p) = 1 - p$. By applying Definition 3, one gets $C(u, v) = u + v - 1$ if $u + v > 1$, and $C(u, v) = 0$ otherwise. Hence $C(u, v) = W(u, v) = \max(u + v - 1, 0)$. The opposite implication is then straightforward.

To prove that 2 implies 3, it is sufficient to compute the Gini index in Equation (5) for $L(p) = p$, obtaining $G = 0$.

Finally $3 \Rightarrow 2$ holds since $L(p) \leq p$, therefore the only solution for the functional equation $\int_0^1 p - L(p)\,\mathrm{d}p = 0$ is $L(p) = p$, which concludes the proof. □

Regarding the Fréchet-Hoeffding upper bound, no Archimedean structure is able to replicate it exactly [35]. Therefore, an if-and-only-if characterization can neither be given for a Lorenz copula. However, it is possible to show that, if a Lorenz curve converges point-wise to the perfect inequality case, and thus its Gini tends to 1, then the corresponding $C(u, v)$ will have the upper bound M as its limit.

Proposition 4. *Let $\{C_n(u, v)\}_{n \geq 0}$ be a sequence of Lorenz copulas generated by a sequence of curves $L_n \in \mathcal{L}_\varphi$. We have that $C_n(u, v) \to M(u, v) = \min(u, v)$, as $n \to \infty$, if and only if $L_n \to L_{PI}$ (or $\bar{L}_n \to \bar{L}_{PI}$), as $n \to \infty$.*

Proof. A necessary and sufficient condition for a sequence of Archimedean copulas with generators φ_n to attain the Fréchet-Hoeffding upper bound is given in [35] as

$$\lim_{n \to \infty} \frac{\varphi_n(p)}{\varphi_n'(p)} = 0, \quad p \in (0, 1), \tag{11}$$

where φ' is the right-side derivative.

By trivial substitution, in terms of Lorenz generators $\{L_n\}$, such a condition is restated as

$$\lim_{n \to \infty} \frac{\bar{L}_n(p)}{\bar{L}_n'(p)} = 0, \qquad p \in (0, 1). \tag{12}$$

The only Lorenz satisfying such a condition is the perfect inequality line, or $L_{PI} \notin \mathcal{L}_\varphi$, the lower bound for every Lorenz curve (recall Proposition 1). All curves $L \in \mathcal{L}_\varphi$ are in fact strictly increasing and bounded, hence they cannot satisfy $\lim_{n \to \infty} \bar{L}_n'(p) = \infty$ for every $p \in (0, 1)$, which would be the only condition for Equation (12) to hold for $L \neq L_{PI}$.

The opposite direction is a trivial consequence of Proposition 5 in [8], noticing that for us $\lambda(p) = \frac{\bar{L}_n(p)}{\bar{L}_n'(p)}$. □

Proposition 4 has an immediate corollary in terms of Gini indices, whose proof is a direct application of Equation (5).

Corollary 1. $L_n \to L_{PI}$ (or $\bar{L}_n \to \bar{L}_{PI}$) and $C_n(u,v) \to M(u,v)$, as $n \to \infty$, if and only if $G_n \to 1$.

Let $L_\theta \in \mathcal{L}_\varphi$ be a Lorenz curve with parameter θ. Thanks to Propositions 3 and 4, it is clear that the associated Lorenz copula $C_\theta(u,v)$ can effectively interpolate between the Fréchet-Hoeffding bounds W and M, only if L_θ is able to reach the limiting cases L_{PE} and L_{PI}. As observed in [25], this not always the case.

An important feature of non-strict Archimedean copulas is the presence or the absence of a singular part. For a Lorenz copula this completely depends on the continuity of the quantile function of the underlying random variable X.

Proposition 5. *The Lorenz Copula $C(u,v)$ associated to the random variable X, with Lorenz curve $L \in \mathcal{L}_\varphi$, exhibits a singular part if and only if the quantile function of X has a jump, or $X \le b < \infty$.*

Proof. Recall that an Archimedean copula exhibits a singular part when there exists a level curve with positive mass [35]. The result then immediately follows by adapting Equation 4.3.2 in [35] to the Lorenz framework.

We know in fact that the mass $m(t)$ on a level curve at $t \in [0,1)$ is representable as

$$m(t) = \mu \bar{L}(t)\left(\frac{1}{F^{-1}(1-t^+)} - \frac{1}{F^{-1}(1-t^-)}\right), \tag{13}$$

with $\mu = E(X)$, and where $F^{-1}(1-t^+)$ and $F^{-1}(1-t^-)$ are the left and the right limits of the quantile function $F^{-1}(1-t)$ of X, respectively. Clearly $m(t) \neq 0$ if and only if the two limits differ, implying a jump in the quantile function.

The mass of a zero set is given by the mass of the associated zero curve, which is obtained via Equation (13), setting $t = 0$, *i.e.*

$$m(0) = \mu \frac{\bar{L}(0)\mu}{F^{-1}(1)} = \frac{\mu}{b}. \tag{14}$$

Therefore the result follow by noting that $b < \infty$. □

Proposition 5 describes the behaviour of the singular component of the Lorenz copula in terms of the underlying random variable X. It follows immediately that if the density of X is positive almost everywhere on the real line, then the associated Lorenz copula has no singular part.

By looking at Equation (13), it is worth observing that the levels at which the Lorenz copula exhibits singular components actually correspond to the jump points in the quantile function. Consider for example a random variable X, whose density consists of two triangles of height 1 placed over $[0,1]$ and $[3,4]$. The associated quantile function exhibits a jump of size 2 at $p = 0.5$. Note also that $X \le 4$. Proposition 5 and Equation (13) tell us that the associated Lorenz copula exhibits singular components: one on the zero curve of mass $\frac{1}{2}$, and another one on the level $C(u,v) = 0.5$, with mass $\frac{1}{6}$. Therefore the Lorenz copula has both singular and continuous components. The first one of mass $\frac{2}{3}$ and the latter of mass $\frac{1}{3}$.

The following proposition, under some additional integrability conditions, presents an interesting convergence result.

Proposition 6. *Let $X_1, \ldots, X_n$ be a uniformly integrable sequence of random variables with distribution functions $F_1, \ldots, F_n$ and Lorenz curves $L_1, \ldots, L_n \in \mathcal{L}_\varphi$. If $X_n \to X$ weakly, then $L_n \to L$ point-wise in $p \in [0,1]$, and the associated Lorenz copulas $C_n(x,y) \to C(x,y)$ point-wise in $x, y \in [0,1]$.*

Proof. For X_n, it is well-known that an equivalent definition of Lorenz curve is given by $L_n(y) = \frac{\int_0^y x dF_n(x)}{\mu_{X_n}}$ with $y = F_n^{-1}(p) \in [0,\infty]$ and $\mu_{X_n} = E[X_n]$. By the Portmanteau Lemma [14], we know that weak converge implies the convergence of the integrals of bounded functions. For fixed $y \in [0,\infty)$ we can re-write the Lorenz curve as an expectation $L_n(y) = \frac{E_n(1_{[0,y]}x)}{\mu_X n}$ where $1_{[0,y]}x \leq y$ for every $y \in [0, \infty)$ and $E_n(1_{[0,y]}x) := \int_0^y x dF_n(x)$. Therefore we have point-wise convergence of the Lorenz curve $\forall y \in [0, \infty]$, where convergence for $\{y = \infty\}$ is granted being just a countable set. Uniform integrability allows to state the convergence of $\mu_{Xn} \to \mu_X$ completing the step.

Finally, the convergence of the associated Lorenz copula is obtained by applying Proposition 2 in [8] to the ratio $\frac{L_n(y)\mu_{Xn}}{F^{-1}(y)_n}$ which converges to the corresponding limit $\frac{L(y)\mu_X}{F^{-1}(y)}$. □

Remark 1. The requirement of uniform integrability in Proposition 6 is not necessary for the convergence of the Lorenz copula. In fact, to have the point-wise convergence of $C_n(x, y) \to C(x, y)$, one needs the ratio between the generator and its derivative, which for the Lorenz generator simplifies, cancelling out the mean.

2.2 Dependence and inequality orders

In the study of copulas, a fundamental topic is the analysis of the dependence structure induced by a given copula function. For Lorenz copulas the dependence structure is directly connected to the underlying variable X and to the stochastic orders discussed in Subsection 1.3.

Proposition 7. *Let C_1 and C_2 be two Lorenz copulas associated to X_1 and X_2. One has that $C_1 \succ^{LTD} C_2$ if and only if $X_1 \succ^{*} X_2$.*

Proof. The proof is a straightforward application of Theorem 2 and Proposition 11 in the appendix. □

If two Lorenz copulas are left-tail ordered, then one of the associated random variables is more unequal than the other. In terms of Gini indices, one can easily verify that a necessary condition for $C_1 \succ^{\mathrm{LTD}} C_2$ is that $G_1 \geq G_2$.

Proposition 8. *Let C_1 and C_2 be two Lorenz copulas associated with the non-negative finite-mean random variables X_1 and X_2, with Gini indices G_1 and G_2. Then $C_1 \succ^{PK} C_2$ only if $G_1 \geq G_2$.*

Proof. By Theorem 2 a necessary and sufficient condition for $C_1 \succ^{\mathrm{PK}} C_2$ is $L_1(L_2^{[-1]}(u))$ being star-shaped.

Looking at derivatives, the star shape condition implies that $\frac{L_1(u)}{L_2(u)}$ is increasing. This is equivalent to $Y_{L_2} \succ^{*} Y_{L_1}$, where $Y_L \sim L$ (recall the last point in Proposition 1).

Therefore, remembering that the star order implies the Lorenz one [3], and setting $L^{(1)}(p) = \int_0^p L(t)\mathrm{d}t$, one gets that a necessary condition for the Kendall order is that

$$L_1^{(1)}(p) \geq L_2^{(1)}(p), \quad \forall p \in [0, 1]. \tag{15}$$

Equation (15) can be rewritten in terms of the original quantile functions, *i.e.*

$$\int_0^p \int_0^u F_1^{-1}(t)\mathrm{d}t\mathrm{d}s \geq \int_0^p \int_0^s F_2^{-1}(t)\mathrm{d}t\mathrm{d}s, \qquad \forall p \in [0, 1]. \tag{16}$$

This condition represents an ordering on non-negative random variables called inverse stochastic dominance of third degree (ISD(3)) between X_1 and X_2, and it can be shown [34] that the condition $G_1 \geq G_2$ is a necessary one for ISD(3) to hold. □

Here below a graphical summary of the relations among the orders cited in this section is provided.

LTD		$\Rightarrow$		PK	$\Rightarrow$	PQD	$\Rightarrow$	τ
$\Updownarrow$				$\Downarrow$				
$\star$	$\Rightarrow$	L	$\Rightarrow$	ISD(3)	$\Rightarrow$	G		

As stated, the LTD and * orders imply each other. LTD then implies PK and PQD, and from PQD one can derive that if $C_1(u,v) \succ^{PQD} C_2(u,v)$, then $\tau_1 > \tau_2$. Similarly, * implies L, which implies ISD(3). From ISD(3) one can finally obtain an order on the Gini indices.

2.3 Upper tail dependence

Following [9], we analyze some of the asymptotic dependence properties (of the survival function) of Lorenz copulas. In particular, we re-state the conditions given in in [9], about the generator of the copula, in terms of the distribution of the random variable X generating L. Since we are dealing with non-strict copulas, we just focus our attention on the upper tail, being the behaviour of the lower one trivial.

For example, we show that the Lorenz copula associated to a lognormally distributed random variable exhibits asymptotic tail dependence, providing a brand new example of Archimdean copula with such a rare behavior.

To simplify treatment, in what follows we assume that a given Lorenz curve is sufficiently smooth to allow derivatives to exist when needed.

Provided that the limit $\lim_{t\to 0} -\frac{t\varphi'(1-t)}{\varphi(1-t)}$ exists, the upper tail behavior of the survival function of an Archimedean copula, when one of the components vanishes to zero, can be summarized as follows [9]:

1. Tail independence, if and only if
$$\varphi'(1) < 0. \tag{17}$$
2. Asymptotic upper tail dependence, if and only if $\varphi'(1) = 0$ and $\lim_{t\to 0} -\frac{t\varphi'(1-t)}{\varphi(1-t)} = 1$.
3. Upper tail dependence, if and only if $\varphi'(1) = 0$ and $\lim_{t\to 0} -\frac{t\varphi'(1-t)}{\varphi(1-t)} > 1$.

For Lorenz copulas, a necessary and sufficient condition for tail independence is that the support of the underlying random variable X has a lower bound strictly larger than zero. In fact, it is sufficient to observe that, thanks to Proposition 1 and the definition of $\bar{L}$, Equation (17) becomes

$$\bar{L}'(1) = -\frac{F^{-1}(0)}{\mu}, \tag{18}$$

with $\bar{L}'$ indicating the appropriate derivative of $\bar{L}$.

Upper tail dependence, be it asymptotic or not, requires $\bar{L}'(1) = 0$, therefore the lower bound of X needs to be 0, or we would fall back in the case of independence we have just considered.

In order to decide whether upper tail dependence is asymptotic or not, one then needs to study the behavior of $\lim_{p\to 0} -\frac{p\bar{L}'(1-p)}{\bar{L}(1-p)}$. For sufficiently smooth Lorenz curves the condition can be re-written in terms of the cumulative function F of X, and of its derivatives. By setting $y = F^{-1}(p)$, and by applying L'Hôpital's rule twice, we get

$$\lim_{p\to 0} -\frac{p\bar{L}'(1-p)}{\bar{L}(1-p)} = \lim_{y\to 0} \frac{F''(y)F(y)}{(F'(y))^2} \le 1. \tag{19}$$

If the inequality in Equation 19 is strict, one has tail dependence, while the asymptotic case appears for the equality.

Observe that Equation (19) can be read as a limiting log-concavity condition for the cumulative distribution function of X. Using this observation we can state the following proposition, whose proof is immediate, about a sufficient condition for the upper tail dependence of a Lorenz copula.

Proposition 9. *Let $L \in \mathcal{L}_\varphi$. The associated Lorenz copula exhibits asymptotic tail dependence if $\lim_{t\to 0} -\frac{t\varphi'(1-t)}{\varphi(1-t)}$ exists and the random variable X generating L has a strict log-concave cumulative distribution function F.*

Unfortunately, Proposition 9 has no strong applicability. Finding distributions satisfying it is not immediate. An interesting case, however, is the one we obtain for X lognormally distributed.

Lemma 3. Let X be a lognormally distributed random variable, then Equation (19) holds in the equality, *i.e.* the associated Lorenz copula shows asymptotic upper tail dependence.

Proof. Let $\Phi(\cdot)$ be the cumulative distribution function of a standard normal, and $\phi(\cdot)$ the density. For a lognormally distributed random variable, Equation (19) becomes

$$\lim_{y\to 0} \frac{\Phi(\log(y))\left(\frac{1}{y^2}(\phi'(\log(y)) - \phi(\log(y))\right)}{\left(\frac{1}{y^2}\phi(\log(y))\right)^2} = \lim_{z\to-\infty} \frac{\Phi(z)(-1-z)}{\phi(z)}, \tag{20}$$

with $z = \log(y)$ and recalling that $\phi'(z) = -z\phi(z)$ [40].

Using L'Hôpital's rule twice, Equation (20) turns into

$$\lim_{z\to-\infty} \frac{\phi'(z)(-1-z) - 2\phi(z)}{\phi''(z)} = \lim_{z\to-\infty} \frac{z^2+z-2}{z^2-1} = 1, \tag{21}$$

since $\phi''(z) = (z^2-1)\phi(z)$, and the proof is complete. □

Even if we are not able to prove it, we conjecture that all Lorenz curves connected to lognormal-like random variables [25] can be suitable generators for non-strict Archimedean copulas characterized by asymptotic upper tail dependence.

3 Examples of Lorenz copulas

The present section is devoted to the illustration of some explicit Lorenz copulas. Playing with Lorenz generators, it is not only possible to recover very well-known models, but–more interestingly–one can obtain new non-strict Archimedean copulas, with useful tail properties.

3.1 The lognormal Lorenz copula

The lognormal Lorenz copula is obtained by taking X to be lognormally distributed, so that the generator is

$$\bar{L}_{LN}(p) = \Phi\left(\Phi^{-1}(1-p) - \sigma\right), \qquad p \in [0,1], \tag{22}$$

where Φ and Φ^{-1} are respectively the cumulative distribution and the quantile function of a standard normal, and $\sigma > 0$ is a scale parameter (equal to the standard deviation of $\log(X)$). Equation (22) can be obtained by substituting $y = F^{-1}(x)$ in Equation (4), and then by using the properties of the Gaussian integral in Φ, when multiplied by an exponential function [40]; finally, reverting the substitution $y = F^{-1}(x)$ leads to the desired result.

Figure 1 shows some examples of the lognormal generator for different values of σ.

Notice that the quantity $\Phi(\Phi^{-1}(y) - \sigma)$, with $\sigma > 0$, is a well-known distortion function in actuarial mathematics, usually called Wang transform, and it has powerful applications in the fields of asset pricing, risk theory and utility theory [29, 50]. Furthermore, in terms of non-Newtonian calculus [22], it represents the pseudo-difference of a variable y and a constant σ: just notice that, given the continuity of Φ^{-1}, one has $\sigma = \Phi^{-1}(\Phi(\sigma))$, so that $\Phi(\Phi^{-1}(y) - \sigma) = \Phi\left(\Phi^{-1}(y) - \Phi^{-1}(\Phi(\sigma))\right)$ [31, 39].

Using Equation (6), the functional form of the lognormal Lorenz copula is:

$$C_{LN}(u,v) = \max\left(1 - \Phi(\Phi^{-1}(\Phi(\Phi^{-1}(1-u) - \sigma) + \Phi(\Phi^{-1}(1-v) - \sigma)) + \sigma, 0\right). \tag{23}$$

Figure 2 shows the surface of $C_{LN}(u,v)$ for $\sigma = 0.5$ and $\sigma = 2$.

Since the lognormal variable X has an unbounded support (its right-end point is $x_F = +\infty$), Proposition 5 guarantees that the lognormal Lorenz copula has no singular part. Moreover, Lemma 3 tells us that such a

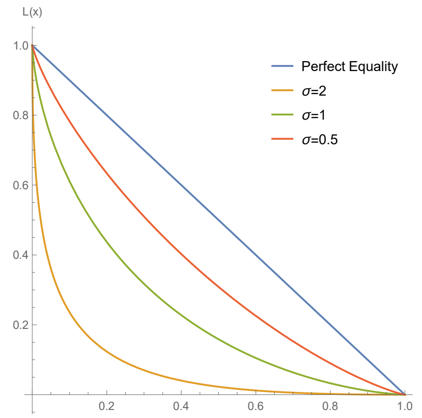
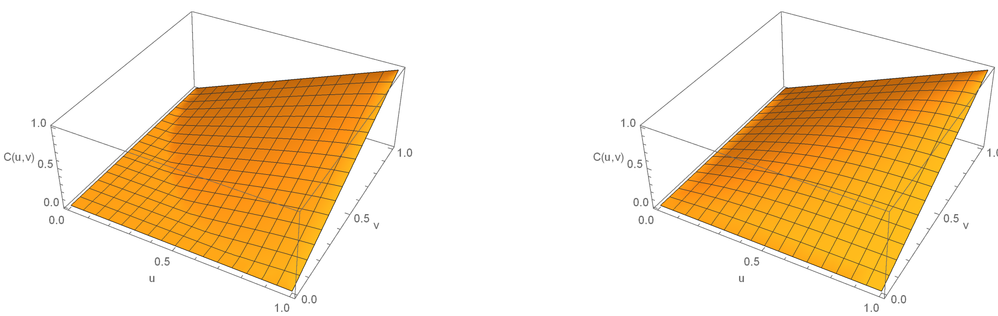


Figure 1: Examples of lognormal generators with different σ parameters.



(a) $\sigma = 0.5$

(b) $\sigma = 2$

Figure 2: Surfaces of a lognormal Lorenz copula for different values of σ.

copula is characterized by asymptotic tail dependence, whose strength grows with σ. In Figure 3 two simulations are given, and in both of them it is possible to notice the expected tail behavior: observations in the top right corner are definitely more dependent.

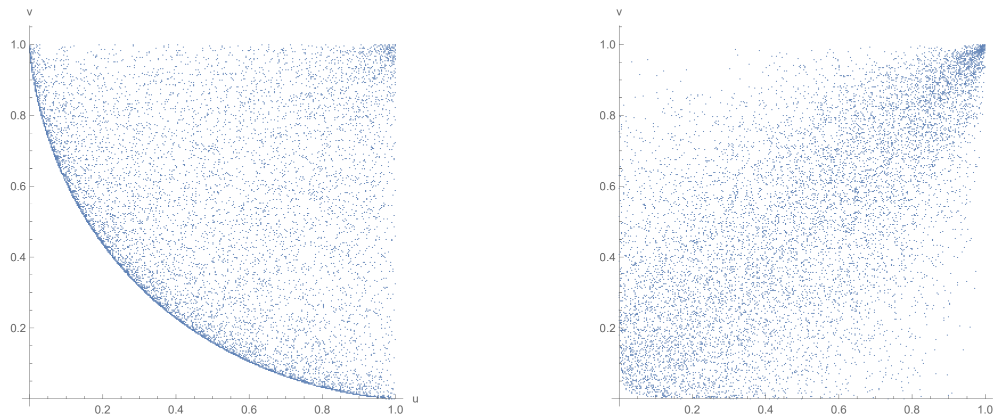
The Kendall distribution function associated to $C_{LN}(u, v)$ is given by:

$$K_{LN}(t) = t + \Phi(\Phi^{-1}(1-t) - \sigma)e^{\frac{\sigma^2}{2} - \Phi^{-1}(1-t)}, \quad t \in (0, \infty), \tag{24}$$

which is obtained by noting that the quantile function of a lognormal distribution rescaled by the mean is given by $e^{-(\frac{\sigma^2}{2} - \Phi^{-1}(1-t))}$.

From Equation (24), one can then obtain the value of the Kendall's τ, but this is only possible numerically, the analytical derivation being unfeasible. In Figure 4 the relation between τ and the parameter σ is presented. It is clear that monotonic dependence grows with σ. This is somehow expected by looking back at Figure 3, where not only tail dependence gets stronger as σ becomes larger, but the size of the zero set decreases. In the limit, for $\sigma \to \infty$, the lognormal Lorenz copula tends to the Fréchet-Hoeffding upper bound M (and to W for $\sigma \to 0$).

Finally, some considerations in terms of stochastic orders. It is known that the lognormal distribution is star ordered in σ [25]. Namely, if $\sigma_1 > \sigma_2$ then $X_1 \succ^* X_2$, with $X_i \sim LN(\mu, \sigma_i)$, $i = 1, 2$. Thanks to Proposition 7 this means that the lognormal Lorenz copula is LTD ordered, and this also implies the positive quadrant dependence and the Kendall order.



(a) $\sigma = 0.5$ **(b)** $\sigma = 2$

Figure 3: Two simulations from a lognormal Lorenz copula for different σ.

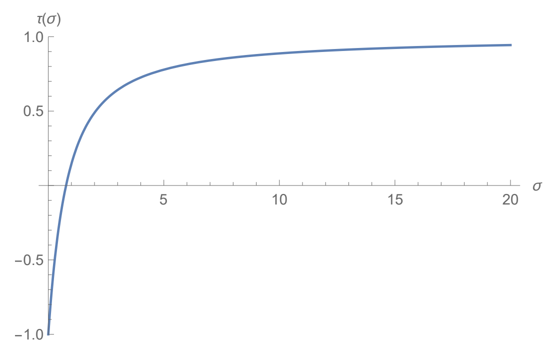


Figure 4: Lognormal Lorenz Copula Kendall's τ_{LN} as a function of σ.

3.2 The shifted exponential Lorenz copula

The shifted exponential Lorenz copula is obtained via the generator

$$\bar{L}_{SE}(p) = (1-p) + 2gp\log(p), \tag{25}$$

with $g \in (0, \frac{1}{2}]$. When $g = \frac{1}{2}$ the mirrored Lorenz curve in Equation (25) corresponds to that of a standard exponential random variable $X \sim \text{Exp}(\lambda)$. Notice that, for all exponentials, the (mirrored) Lorenz curve does not depend on λ, *i.e.* all exponentials share the same Lorenz curve, as observed in [15]. For $g < \frac{1}{2}$ the random variable X is shifted away from zero by a factor equal to $(1-2g)\lambda$. Figure 5 shows some examples of the generator $\bar{L}_{SE}(p)$ for different values of g.

The shifted exponential Lorenz copula obtained from Equation (25) is

$$C_{SE}(u,v) = \max\left(\exp\left(\frac{1}{2g} + W_{-1}\left(\frac{2g(u\log(u) + v\log(v)) - (u+v) + 1}{2g\exp\left(\frac{1}{2g}\right)}\right)\right), 0\right), \tag{26}$$

where W_{-1} is the lower-branch of the Lambert W function [1]. Figure 6 presents two examples of $C_{SE}(u,v)$, for $g = 0.5$ and $g = 0.2$. In the appendix, the details of the derivation of Equation (26) are presented.

The copula $C_{SE}(u,v)$ is characterized by some relevant properties and facts, which we can list as follows:

1. The shifted exponential Lorenz copula has no singular part. This is a consequence of the unbounded support of the exponential distribution.
2. The shifted Exponential random variables are star ordered with g [23]. Therefore, by Proposition 7, shifted exponential Lorenz copulas are ordered according to the LTD dependence order.

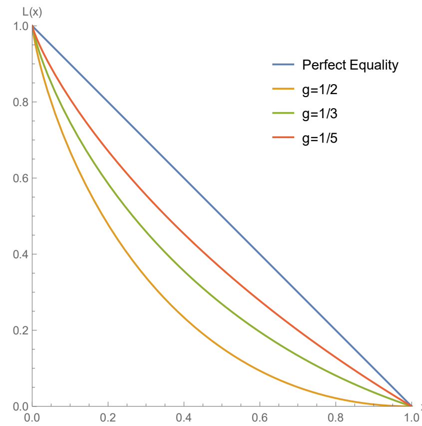
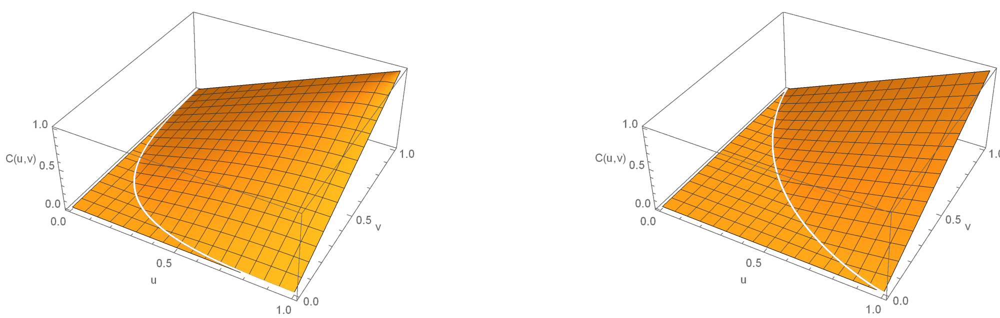


Figure 5: Examples of shifted exponential generators for different values of g.



(a) $g = \frac{1}{2}$ (b) $g = \frac{1}{5}$

Figure 6: Surfaces of a shifted exponential Lorenz copula for different values of g.

3. Since the Lorenz curve of perfect inequality is never attained by a shifted exponential random variable [25], Proposition 4 suggests that $C_{SE}(u, v)$ is always bounded away from the Fréchet-Hoeffding upper bound M.
4. The Kendall's τ_{SE} of the shifted exponential Lorenz copula cannot be written in closed form, but only in terms of Gamma and Exponential Integral functions [1]. However, it can be easily evaluated numerically. Figure 7 shows its behaviour as a function of the parameter g. Interestingly, its range of variation is [-1,0.227], in line with the previous point.
5. The copula exhibits tail dependence only when $g = \frac{1}{2}$, *i.e.* when the support of the underlying random variable starts in 0. For all the other values of g, $C_{SE}(u, v)$ is tail independent. Figure 8 shows two simulations from $C_{SE}(u, v)$, with $g = 0.5$ and $g = 0.2$. As expected, the former case shows tail dependence, while the latter manifests tail independence.

3.3 The Pareto Lorenz copula

The Pareto Lorenz copula emerges when X is Pareto distributed with shape/tail parameter α and scale $x_m > 0$, with mirrored Lorenz curve equal to

$$\bar{L}_P(p) = 1 - p^{1-\frac{1}{\alpha}}, \qquad \alpha > 1. \tag{27}$$

For a Pareto random variable, $\alpha > 1$ is required in order to guarantee that $E[X] < \infty$, so that the Lorenz curve is defined. In Figure 9 some examples of L_P for varying α are given.

The Pareto Lorenz copula is

$$C_P(u,v) = \max\left(\left(u^{1-\frac{1}{\alpha}} + v^{1-\frac{1}{\alpha}} - 1\right)^{\frac{1}{1-\frac{1}{\alpha}}}, 0 \right), \tag{28}$$

again with $\alpha > 1$.

It is worth noticing that, by setting $\theta = \frac{1}{\alpha} - 1$, the Pareto Lorenz copula coincides with the non-strict Clayton family, obtained for $\theta \in (-1, 0)$ [35].

Since the support of a Pareto random variable starts in $x_m > 0$, the Pareto Lorenz copula is upper tail independent for every choice of α. Moreover, since $X \sim Pareto(\alpha, x_m)$ is unbounded from above, $C_P(u,v)$ has no singular component.

As shown in the appendix, the Pareto Lorenz copula is LTD ordered, as expected being a subset of the Clayton family. Recall that LTD then implies PK and PQD.

Finally, it is interesting to look at the role of the tail parameter α in the Kendall's τ of $C_P(u,v)$. One can easily verify that

$$\tau_P = \frac{1-\alpha}{1+\alpha} < 0, \qquad \alpha > 1. \tag{29}$$

Equation (29) can be re-written in terms of the Paretian Gini index $G_P = \frac{1}{2\alpha-1}$ [25], getting

$$\tau_P = \frac{G_P - 1}{3G_P + 1}. \tag{30}$$

Equation (30) shows that τ_P is an increasing function of $G_P \in [0,1]$, moving from -1 towards 0. This implies that the intensity of the association between two random variables coupled with a Pareto Lorenz copula decreases–in absolute value–with an increase in the inequality (in socio-economic terms) of the underlying Pareto random variable X.

Besides the Paretian case, it is worth stressing that, in general, the connection between τ and G is always rather interesting in Lorenz copulas.

3.4 The uniform Lorenz copula

The uniform Lorenz Copula represents another interesting case. The underlying non-negative finite-mean variable X is taken to be uniformly distributed on $[a,b]$, with $0 \le a < b < \infty$. One has

$$\bar{L}_U(p) = \frac{2a(1-p) + (b-a)(1-p)^2}{a+b}. \tag{31}$$

When $a = 0$ and $b = 1$, *i.e.* $X \sim U[0,1]$, Equation (31) simplifies to $\bar{L}_U(p) = (1-p)^2$. As usual, Figure 10 presents some examples of uniform Lorenz generators.

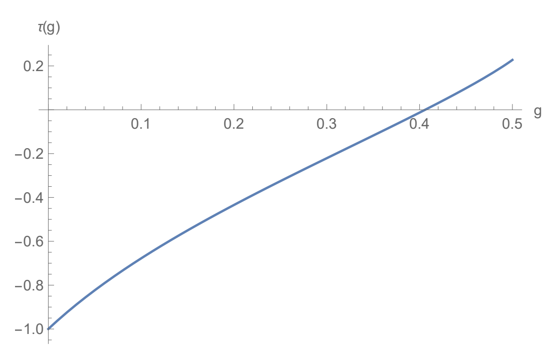
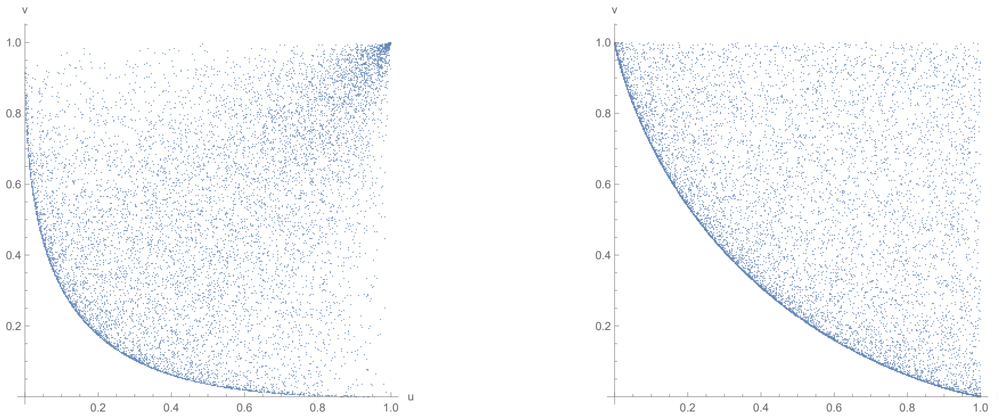


Figure 7: Shifted exponential Lorenz Copula Kendall's τ_{SE} as a function of g.



(a) $g = 0.5$ **(b)** $g = 0.2$

Figure 8: Two simulations from a shifted exponential Lorenz copula for different g.

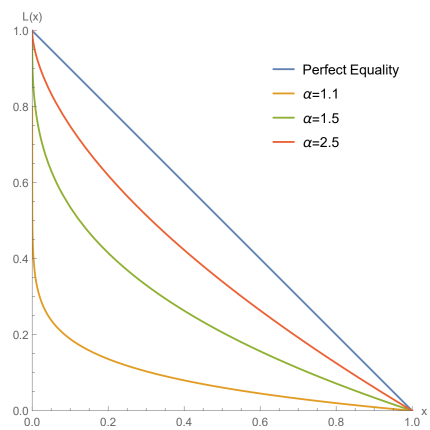


Figure 9: Examples of Pareto generators for different values of α.

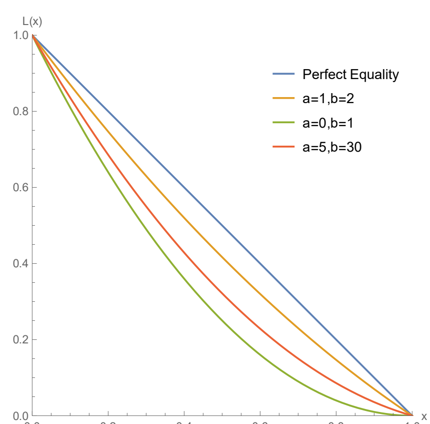
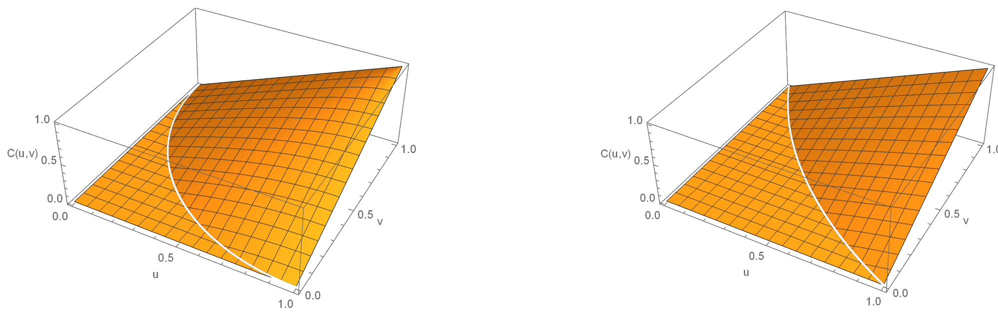


Figure 10: Examples of uniform Lorenz generators for some combinations of a and b.

The uniform Lorenz copula is explicitly given as

$$C_U(u,v) = \max\left(1 - \frac{a - \sqrt{b^2(2+(-2+u)u+(-2+v)v) + 2ab(u-u^2+v-v^2) + a^2(-1+u^2+v^2)}}{(a-b)}, 0\right), \tag{32}$$

and two examples of surfaces are given in Figure 11.



(a) $a = 0, b = 1$

(b) $a = 1, b = 2$

Figure 11: Copula surface for Uniform Lorenz copula.

As far as the properties of $C_U(u, v)$ are concerned, one can observe the following:

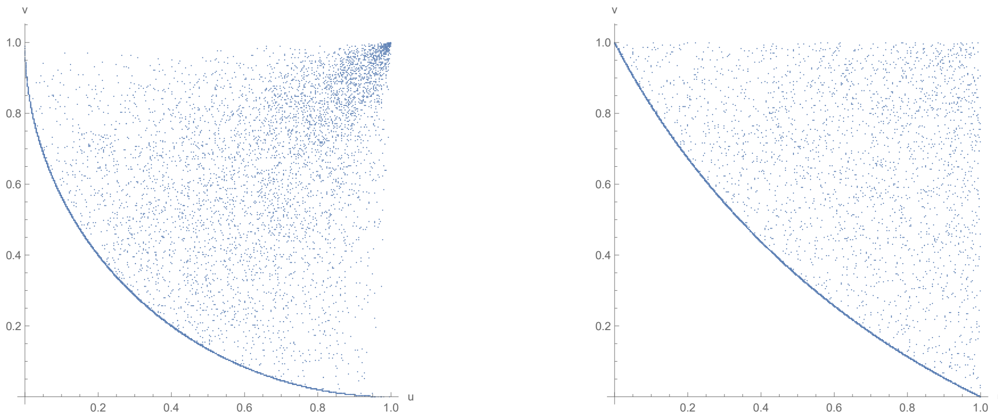
1. The uniform Lorenz copula always possesses a singular part. From Equation (14), the C–measure is $\frac{a+b}{2b} > \frac{1}{2}$.
2. $C_U(u, v)$ exhibits tail dependence for $a = 0$, and tail independence for all $a > 0$.
3. The uniform family is star ordered with respect to a and b, therefore one can conclude that uniform Lorenz copulas are ordered according to the LTD (PK and PQD) dependence order.
4. The Gini index of the uniform family is $G_U = \frac{b-a}{3(a+b)}$, which can never be equal to one. Therefore, by Proposition 4, the uniform Lorenz copula will never attain its upper Fréchet-Hoeffding bound M.
5. Using Equation (3) it is possible to obtain a closed form formula for the Kendall's τ associated to the uniform Lorenz copula.
$$\tau_U = \frac{2a(a - b - a\log(ab))}{(a - b)^2}.$$
Observe that, for $a = 0$, one has $\tau_U = 0$ for every choice of b. But for $a = 0$ the uniform Lorenz copula necessarily exhibits tail dependence. This simple pathological case suggests, should it be necessary, that single measures of association should not be trusted uncritically, when dealing with multivariate data [33, 35]. Imagine in fact that only the Kendall's τ were to be computed for data coming from a uniform Lorenz copula with $a = 0$. One could easily draw the conclusion of no association nor dependence, while Figures 11 and 12 show that those data would be actually characterized by a non trivial form of dependence. In a sense, it is like when people insist in only using Pearson's correlation to measure dependence, even when there are evident signals of non-linearity in the data [29]. The moral of the story is therefore to always use more than just one measure (of dependence, but also of variability and so on), and to look at the data and their background carefully.

Figure 12 shows two samples drawn from two different uniform Lorenz copulas. The presence of tail dependence/independence given the values of a is evident.

4 Alchemies and multiparametric extensions

The class of Lorenz copulas is extremely rich and flexible. In the previous section a few examples were considered, starting from some well-known size distributions, but they only represent a small set of all the possible copulas one can actually generate. Just think about all the Lorenz curves currently available in the literature [3, 10, 25, 41, 51].

Besides flexibility, an appealing characteristic of the Lorenz approach to copulas is the possibility of importing into the Archimedean family many results developed in the study of inequality. A particularly interesting example is represented by the so-called "alchemy of Lorenz curves" discussed in [3, 41]. The evocative



(a) $a = 0$, $b = 1$ **(b)** $a = 1$, $b = 2$

Figure 12: Two simulations from a uniform Lorenz copula with $X \sim U[0, 1]$ and $X \sim U[1, 2]$.

term alchemy is used by Sarabia and Arnold to indicate a set of techniques for generating new Lorenz curves starting from given ones. Some relevant cases are presented in the following proposition, for the proof of which see [3].

Proposition 10. *Let $L_1(p)$ and $L_2(p)$ be two Lorenz curves. Then, a new Lorenz curve $L(p)$ can be obtained, for instance, via*
1. *Exponentiation: $L(p) = L_1(p)^\alpha$, with $\alpha > 0$;*
2. *Composition: $L(p) = L_1(L_2(p))$;*
3. *(Generalized) Multiplication: $L(p) = L_1(p)^\alpha L_2(p)^\beta$, with $\alpha, \beta > 0$.*
4. *Convex combination: $L(p) = wL_1(p) + (1 - w)L_2(p)$, with $w \in [0, 1]$.*
5. *Maximization: $L(p) = \max(L_1(p), L_2(p))$.*

It is straightforward to verify that the statements in Proposition 10 are easily extended to any finite collection of Lorenz curves, so that, for example, $L(x) = \prod_{i=1}^n L_i(p)^{\alpha_i}$ and $\max(L_1(p), \dots, L_n(p))$ are proper Lorenz curves.

Some of the transformations in Proposition 10 are already known for Archimedean generators. For instance, Nelsen's α and β families [35]–when restricted to non-strict generators–can be obtained by combining Statements 1. and 2. of Proposition 10. The same holds for most of the transformations presented in Proposition 1 of [19].

Furthermore, the alchemy of Lorenz curves can be used to create new multiparametric Lorenz copulas, providing a possible solution to the growing demand for more flexible copulas [35]. Consider, for instance, the Paretian generator in Equation (27). By applying exponentiation and multiplication one can easily obtain the following family of three-parameter Lorenz generators

$$\bar{L}_{3P}(p) = (1 - p)^\eta (1 - p^\theta)^\gamma, \tag{33}$$

with $\eta \geq 0$, $\theta \in (0, 1)$ and $\gamma \geq 1$. The related Lorenz curves have been studied extensively in [42].

From Equation (33) it is possible to obtain a three-parameter Lorenz copula, whose properties can be studied using the results of Section 2. For example, one can quickly find out that the copula obtained from L_{3P} is almost never tail independent. This comes from the fact that $\bar{L}'_{3P}(1) = 0$ for every choice of the parameters except for $\eta = 0$ and $\gamma = 1$.

By taking $\eta = 0$ in Equation (33), one obtains the mirrored Lorenz curve of a Sigh-Maddala random variable [47], which represents a pseudo-translation of a standard Pareto [25, 39], so that the new variable has its lower bound shifted to zero. Because of this new lower bound the original Paretian tail independence is lost.

By setting $\gamma = \frac{1}{\theta}$, Equation (33) becomes the famous Genest and Ghoudi's generator [18] behind copula 4.2.15 in [35]. Thanks to the Lorenz approach, it is immediate to study the properties of the associated copula.

First notice that the new generator corresponds to the mirrored Lorenz curve of a Lomax random variable [25]. The Lomax distribution has a lower bound at zero and no upper bound, hence the associated copula is absolutely continuous and never tail independent. Moreover, looking at the behavior of the density we conjecture that the copula will be tail dependent for every choice of the parameter (the Lomax is indeed a lognormal-like distribution [25]). Finally, by noting that the Lomax family is star ordered with respect to the parameter θ, the associated family of copulas is stochastically ordered, as noted by Genest and Ghoudi [18]. In particular, Proposition 7 guarantees that the family is LTD (PK and PQD) ordered.

Table 1 summarizes and extends the results presented so far, listing some Lorenz generators and the properties of the related Lorenz copulas.

5 Conclusions

We have proposed an alternative approach to the generation of non-strict bivariate Archimedean copulas using the Lorenz curve, a powerful tool in the study of socio-economic inequality[12, 13, 25, 51] and risk management [15, 16, 45, 46]. The main advantages of the Lorenz-generation of copulas are threefold. First, the great number of Lorenz curves available in the literature allows for the generation of a large amount of copulas, which include existing cases, but also novelties. Second, every Lorenz copula can be easily characterized looking at some basic features of the non-negative finite-mean variable X underlying the Lorenz generator. In particular, we have shown that quantities like the Kendall's τ, or properties like upper tail dependence and stochastic dominance, can be inferred from X. Third, the possibility of importing into the world of copulas many of the results developed in the studies of inequality allows for many interesting considerations: from a novel perspective on the Gini index, as a measure of the distance of a Lorenz copula from its Fréchet-Hoeffding bounds, to the possibility of generating multiparametric copulas using some useful compositions rules for Lorenz curves.

Regarding the last point, it is interesting to notice that the opposite direction works as well. It is in fact possible to borrow tools from the theory of copulas and to apply them to the study of socio-economic inequality.

Consider for example the Kendall's τ. In terms of Lorenz curve, one has

$$\tau = 1 - 4 \int_0^1 \frac{L(p)\mu}{F^{-1}(p)} \mathrm{d}p. \tag{34}$$

Since $\tau \in [-1, 1]$, Equation (34) is hard to interpret as an inequality index, according to the usual framework [25, 51]. But we can easily apply a MinMax transformation, to get $\bar{\tau} = 1 - 2\int_0^1 \frac{L(p)\mu}{F^{-1}(p)} \mathrm{d}p$, which now lies in the unit interval [0, 1]. It is easy to observe that $\bar{\tau} = 0$ when the $L(p) = L_{PE}(p)$, while $\bar{\tau} = 1$ for $L(p) = L_{PI}(p)$.

Observe that $U(p) = \int_0^p \frac{\mu}{F^{-1}(t)} \mathrm{d}t$ is an increasing function. Therefore one can rewrite Equation (34) as

$$\bar{\tau} = 1 - 2 \int_0^1 L(p) \mathrm{d}U(p) = 2 \int_0^1 \frac{pF^{-1}(p) - L(p)\mu}{F^{-1}(p)} \mathrm{d}p. \tag{35}$$

Following [32], $\bar{\tau}$ is therefore a valid inequality index with weighting function U. In particular, $\bar{\tau}$ measures the distance between the Lorenz curve L and the line of perfect equality L_{PE}, and in this it is similar to the Gini index. However, differently from the standard Gini, $\bar{\tau}$ weights both L and L_{PE} for the actual value of wealth, as represented by the quantile function $F^{-1}(p)$. As a consequence, $\bar{\tau}$ could be used for direct comparisons among countries, something not immediately possible using the Gini index, given its scale free nature [51].

As far as future work is concerned, it would be interesting to investigate the possibility of extending the Lorenz approach to d-dimensional copulas, with $d \geq 3$. In the case of Archimedean copulas this boils down to the property of d-monotonicity of the generator. For the Lorenz copulas, however, it does not seem straightforward to derive conditions to test the sign of the d-derivative of the inverse of the generator. Assuming that the

Lorenz generator is smooth enough, using the the Faà di Bruno's formula [24], we can formulate the condition for 3-monotonicity as

$$F^{-1''}(1-x)F^{-1}(1-x) - 3(F^{-1'}(1-x))^2 \leq 0, \tag{36}$$

where $F^{-1''}(1-x)$ is the second derivative of the quantile function of X.

One can verify that Equation (36) is satisfied by the uniform Lorenz copula, just after observing that the second derivative of the quantile function of the uniform distribution is zero. If we consider instead a Pareto Lorenz copula, we can observe the following: for $\alpha < 2$, the left-hand side of Equation (36) is positive, while the same quantity becomes negative for $\alpha > 2$. This shows that there is no clear condition for the d-monotonicity of the Lorenz copula, but also that the moments of X could play a role. More research is therefore needed.

Relying on the d-monotonicity of the generator is not, however, the only viable approach. Other possibilities could be the the exploitation of nested constructions [35], or the use of multivariate Lorenz curves, as for example the Lorenz zonoid of [27]. For this second direction, anyway, one needs to remember that there exist more definitions of multivariate Lorenz curves [3, 25], and there is no guarantee that they may all serve for the purpose.

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Table 1: Some examples of Lorenz copulas with their properties. Legend: TI = tail independence, AD = asymptotic tail dependence, TD = tail dependence, * = star order (which then implies the other orders as discussed in Section 1.3), – not available.

Generator	Underlying X	Also known as	Tail Dependence	Singular	Kendall's τ Range	Order
1. $1-x$	Perfect equality	M	TI	No	1	–
2. $\Phi(\Phi^{-1}(1-x)-\sigma)$	Lognormal	–	AD	No	$[-1,1]$	*
3. $(1-x)+2gx\log(x)$	Shifted Exponential	–	$\begin{cases}\text{TD} & \text{if } g=1/2\\ \text{TI} & \text{if } g<1/2\end{cases}$	No	$[-1,0.227]$	*
4. $1-x^{1-\frac{1}{\alpha}}$	Pareto	Non-strict Clayton	TI	No	$[-1,0]$	*
5. $\frac{2a(1-x)+(b-a)(1-x)^2}{a+b}$	$U[a,b]$	–	$\begin{cases}\text{TD} & \text{if } a=0, b>0\\ \text{TI} & \text{if } a>0, b>0\end{cases}$	$\frac{a+b}{2b}$	$\begin{cases}0 & \text{if } a=0, b>0\\ [-1,0) & \text{if } 0<a<b<\infty\end{cases}$	*
6. $(1-x)^{\eta}(1-x^{\theta})^{\gamma}$	$\begin{cases}\text{Pareto} & \text{if } \eta=0,\gamma=1\\ \text{Singh-Maddala} & \text{if } \eta=0\\ \text{See [42]} & \text{otherwise}\end{cases}$	$\begin{cases}4.2.15 & \text{if } \eta=0,\gamma=\frac{1}{\theta}\\ - & \text{otherwise}\end{cases}$	$\begin{cases}\text{TI} & \text{if } \eta=0,\gamma=1\\ \text{TD} & \text{otherwise}\end{cases}$	No	$[-1,1]$	See [42]

[!h]

Appendix

The appendix collects comments and results that, for the sake of space, are not part of the main narrative of the paper. To ease the readability, we refer to the corresponding sections of the work.

Subsection 1.3 - Star order equivalence

When dealing with the star order, one can show the equivalence between the standard definition in terms of quantile functions [3, 44] and Definition 2 via the Lorenz curve.

Proposition 11. *Let X_1, X_2 be positive random variables with finite mean and quantile function $F_i^{-1}(p)$, $p \in [0, 1]$, $i \in \{1, 2\}$. Let $L_i(p)$ be the associated Lorenz curves. The condition $L_1(L_2^{-1}(x))$ being convex, where $L^{-1}(x)$ denotes the quasi-inverse of the Lorenz, is equivalent to $X_1 \succ^\star X_2$.*

Proof. Recall that two non-negative random variables X_1 and X_2 are star ordered, *i.e.* $X_1 \succ^\star X_2$, when the ratio of their quantile functions

$$\frac{F_2^{-1}(p)}{F_1^{-1}(p)}, \quad p \in [0, 1], \tag{37}$$

is non-increasing in p. Or, equivalently when the ratio $\frac{F_1^{-1}(p)}{F_2^{-1}(p)}$ is non-decreasing.

Given a Lorenz curve, its inverse is clearly an non-decreasing function. We now evaluate Equation (37) in $L_1^{-1}(p)$ and multiply it by the ratio $\frac{\mu_1}{\mu_2}$ obtaining

$$\frac{F_2^{-1}(L_1^{-1}(p))}{\mu_2} \frac{\mu_1}{F_1^{-1}(L_1^{-1}(p))} \quad p \in [0, 1]. \tag{38}$$

Observe that Equation (38) is still an non-decreasing function, with some possibly bounded jump discontinuities.

We know integrate Equation (38), and by the chain rule we get

$$L_2(L_1^{-1}(p)) = \int_0^p \frac{F_2^{-1}(L_1^{-1}(u))}{\mu_2} \frac{\mu_1}{F_1^{-1}(L_1^{-1}(u))} du \quad p \in [0, 1]. \tag{39}$$

The convexity of $L_2(L_1^{-1}(p))$ is granted by the boundedness of the jumps of the quantile function [38].

The opposite direction can be obtained by sub-differentiation of Equation (39). □

Subsection 3.2 - Shifted exponential Lorenz copula

Consider the Lorenz curve of the shifted exponential distribution, *i.e.*

$$L(p) = p + 2g(1 - p)\log(1 - p). \tag{40}$$

The mirrored Lorenz is easily obtained as $\bar{L}(p) = L(1 - p)$, while its inverse is $\bar{L}^{-1}(y) = \left(1 - L^{-1}(y)\right) 1_{y \in [0,1]}$.

In order to get $L^{-1}(y)$ some manipulations are needed, starting from $L(p) = y$. In particular,

$$-e^{\log(1-p)} + 2ge^{\log(1-p)} \log(1 - p) = y - 1$$
$$e^{\log(1-p)} \left(\log(1 - p) - \frac{1}{2g}\right) = \frac{y - 1}{2g}$$
$$\frac{e^{\frac{1}{2g}}}{e^{\frac{1}{2g}}} e^{\log(1-p)} \left(\log(1 - p) - \frac{1}{2g}\right) = \frac{y - 1}{2g}$$

$$e^{\log(1-p)-\frac{1}{2g}}\left(\log(1-p)-\frac{1}{2g}\right)=\frac{y-1}{2ge^{\frac{1}{2g}}}$$

By setting $z=\log(1-p)-\frac{1}{2g}$, and noting that $z<-1$ for every choice of $p\in[0,1]$ and $g\in[0,\frac{1}{2}]$, one gets

$$e^z z=\frac{y-1}{2ge^{\frac{1}{2g}}}. \tag{41}$$

Using the Lambert function W, defined as $f^{-1}(xe^x)=W(xe^x)$, Equation (41) becomes

$$z=W_{-1}\left(\frac{y-1}{2ge^{\frac{1}{2g}}}\right), \tag{42}$$

where the lower branch W_{-1} is chosen being $z<-1$ [1]. Recalling that $z=\log(1-p)-\frac{1}{2g}$, the inverse of the shifted exponential Lorenz is

$$L^{-1}(y)=\exp\left(W_{-1}\left(\frac{y-1}{2ge^{\frac{1}{2g}}}\right)+\frac{1}{2g}\right)-1. \tag{43}$$

The rest of the derivation is straightforward: one needs to combine Equations (40) and (43) according to Definition 3.

Notice that the maximum appearing in Equation (26) takes care of the fact that $\bar{L}(u)+\bar{L}(v)$ may be larger than 1.

Subsection 3.3 - Star order and Pareto random variables

Assume that $X_1\sim\text{Par}(x_m,\alpha_1)$ and $X_2\sim\text{Par}(x_m,\alpha_2)$, with $1<\alpha_1<\alpha_2$. One can then show that $X_1\succ^{*}X_2$.

Consider the quantile function of a Pareto random variable, $F^{-1}(p)=x_m(1-p)^{-\frac{1}{\alpha}}$. Equation (37) becomes

$$\frac{F_2^{-1}(p)}{F_1^{-1}(p)}=\frac{x_m(1-p)^{-\frac{1}{\alpha_2}}}{x_m(1-p)^{-\frac{1}{\alpha_1}}}=(1-p)^{-\frac{1}{\alpha_2}+\frac{1}{\alpha_1}}, \tag{44}$$

which is decreasing for every $\alpha_1<\alpha_2$. Hence we can conclude that the shape parameter α orders Pareto random variables in the star sense.

Proposition 7 then guarantees that the Pareto Lorenz copulas C_1 and C_2 associated to X_1 and X_2 are such that $C_1\succ^{\text{LTD}}C_2$.

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