RANDOMIZED UMD BANACH SPACES AND DECOUPLING INEQUALITIES FOR STOCHASTIC INTEGRALS

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Abstract. In this paper we prove the equivalence of decoupling inequalities for stochastic integrals and one-sided randomized versions of the UMD property of a Banach space as introduced by Garling.

1. Introduction

In recent years, decoupling inequalities have been used to construct theories of stochastic integration in UMD Banach spaces [4, 13, 15]. The basic idea underlying this approach is to use abstract decoupling inequalities to estimate stochastic integrals

$$\int_0^T \phi(t)\, dW(t),$$

where ϕ is a process with values in a UMD space E and W is a standard Brownian motion, with its decoupled analogue

$$\int_0^T \phi(t)\, d\tilde{W}(t),$$

where $\tilde{W}$ is a standard Brownian motion independent of ϕ and W. This decoupled integral is easier to handle, as it is defined in a pathwise sense. Indeed, using a general two-sided decoupling inequality for E-valued tangent sequences, McConnell [13] was able to show that a strongly measurable E-valued process is stochastically integrable with respect to W if and only if its trajectories, viewed as E-valued functions, are stochastically integrable with respect to $\tilde{W}$. His techniques depend heavily on the equivalence of the UMD property and geometric notions related to ζ-convexity. Decoupling inequalities for tangent sequences may be found in [7, 9, 13, 14, 17].

Earlier, Garling [4] had derived a two-sided decoupling inequality for stochastic integrals of elementary E-valued processes directly from the definition of the UMD property. More precisely, he proved that a Banach space E is a UMD space if and only if for some (for all) $1 < p < \infty$ there exist constants $0 < c \leq C < \infty$ such that for all elementary E-valued processes ϕ, we have

$$c\,\mathbb{E}\Big\|\int_0^T \phi\, d\tilde{W}(t)\Big\|^p \leq \mathbb{E}\Big\|\int_0^T \phi\, dW(t)\Big\|^p \leq C\,\mathbb{E}\Big\|\int_0^T \phi\, d\tilde{W}(t)\Big\|^p. \tag{1.1}$$

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These inequalities, combined with the operator-theoretic approach to stochastic integration of Banach space-valued functions developed in [16], was used in [15] to construct a systematic theory of stochastic integration for E-valued processes. In particular, necessary and sufficient conditions for L^p-stochastic integrability were obtained, analogues of the Itô isometry and the Burkholder-Davis-Gundy inequalities were proved, and McConnell's result was recovered as a corollary via standard stopping time arguments.

Various applications of the decoupling inequalities in (1.1) require only one of the two a priori estimates. An analysis of the proof of (1.1) in [4] shows moreover that one-sided decoupling inequalities can be derived from one-sided versions of the UMD property which were introduced subsequently by Garling in [5]. These properties are called UMD^- and UMD^+ below. These properties can be used as in [15] to obtain generalized theories of stochastic integration in which the necessary and sufficient conditions and two-sided estimates for stochastic integrals are replaced by necessary conditions or sufficient conditions, respectively, with one-sided estimates.

The stochastic integration theory in [15] has many consequences and applications. For instance, many results in the theory of stochastic evolution equations in Hilbert spaces (cf. [3] and the references therein), have analogues in UMD^-_{PW} Banach spaces. Therefore, we believe it is important to know the largest class of spaces for which one can construct a stochastic integration theory as in [15]. The aim of the present paper is to show that this is the class of UMD^-_{PW} Banach spaces. It is shown that the validity of the second one-sided a priori estimate in (1.1) for all elementary processes implies the UMD^-_{PW} property. With the same ideas one can prove that E has property UMD^+_{PW} if for some $1 < p < \infty$ the left estimate in (1.1) holds for all elementary E-valued processes, so we include this too. The proofs are based on Skorohod embedding techniques from [4], the Maurey-Pisier characterization of finite cotype and estimates for randomized sums in spaces of finite cotype.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 1}, \mathbb{P})$ be a filtered probability space, and let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be a probability space. Both probability spaces are assumed to be rich enough for constructions as below. We shall consider random variables and processes on $(\Omega\times\tilde{\Omega}, \mathcal{F}\times\tilde{\mathcal{F}}, \mathbb{P}\times\tilde{\mathbb{P}})$. On this probability space we use the filtration $(\mathcal{F}_n\otimes\tilde{\mathcal{F}})_{n\geq 1}$. In most cases our random variables and processes are extensions to $\Omega\times\tilde{\Omega}$ of variables and processes on Ω or $\tilde{\Omega}$. Integration over Ω and $\tilde{\Omega}$ will be denoted by $\mathbb{E}$ and $\tilde{\mathbb{E}}$.

Let $(r_n)_{n\geq 1}$ be a Rademacher sequence on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}_0 = \{\emptyset, \Omega\}$ and $\mathcal{G}_n = \sigma(r_k, k = 1, \ldots, n)$. Recall that a martingale difference sequence $(d_n)_{n=1}^N$ is a *Paley-Walsh martingale difference sequence* if it is a martingale difference sequence with respect to the filtration $(\mathcal{G}_n)_{n=0}^N$.

Recall that a Banach space E is a *UMD(p) space* for $p \in (1,\infty)$ if there exists a constant $C_p > 0$ such that for every $N \geq 1$, every martingale difference sequence $(d_n)_{n=1}^N$ in $L^p(\Omega, E)$ and every $\{-1, 1\}$-valued sequence $(\varepsilon_n)_{n=1}^N$, we have

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N \varepsilon_n d_n\Big\|^p\Big)^{\frac{1}{p}} \leq C_p \Big(\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p\Big)^{\frac{1}{p}}.$$

Similarly, we say E is a $\text{UMD}_{\text{PW}}(p)$ space if one only considers Paley-Walsh martingales in the definition of UMD(p). In [11], Maurey has shown that $\text{UMD}_{\text{PW}}(p)$ already implies UMD(p). It was shown by Burkholder in [1] that if E is UMD(p)

space for some $p \in (1,\infty)$, then E is a UMD(p) space for all $p \in (1,\infty)$. Spaces with this property will be referred to as *UMD spaces*. For the theory of UMD spaces we refer the reader to [1, 2] and references given therein.

Let $(\tilde{r}_n)_{n\geq 1}$ be a Rademacher sequence on $\tilde{\Omega}$.

Definition 1.1. *Let E be a Banach space and let $p \in (1,\infty)$.*

(1) *The space E is* a $\mathrm{UMD}^-_{\mathrm{PW}}(p)$ space *if there exists a constant $C_p^- > 0$ such that for every $N \geq 1$, every Paley-Walsh martingale difference sequence $(d_n)_{n=1}^N$ in $L^p(\Omega, E)$, we have*

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p\Big)^{\frac{1}{p}} \leq C_p^- \Big(\mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n d_n\Big\|^p\Big)^{\frac{1}{p}}. \tag{1.2}$$

(2) *The space E is* a $\mathrm{UMD}^+_{\mathrm{PW}}(p)$ space *if there exists a constant $C_p^+ > 0$ such that for every $N \geq 1$, every Paley-Walsh martingale difference sequence $(d_n)_{n=1}^N$ in $L^p(\Omega, E)$, we have*

$$\Big(\mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n d_n\Big\|^p\Big)^{\frac{1}{p}} \leq C_p^+ \Big(\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p\Big)^{\frac{1}{p}}. \tag{1.3}$$

The corresponding notion of UMD^- and UMD^+ spaces, where arbitrary martingale difference sequences are allowed, has been studied by Garling in [5]. It was shown there that if E is a $\mathrm{UMD}^\pm(p)$ space for some $p \in (1,\infty)$, then E is a $\mathrm{UMD}^\pm(p)$ space for all $p \in (1,\infty)$. Thus, both definitions are independent of $p \in (1,\infty)$ and spaces with this property will be referred to as *UMD^- and UMD^+ spaces*. In [5] these properties are called LERMT (Lower Estimates for Random Martingale Transforms) and UERMT (Upper Estimates for Random Martingale Transforms) respectively. We preferred the notation UMD^- and UMD^+, since it emphasizes the relation with UMD. Here the superscript $^-$ stands for Lower and the superscript $^+$ stands for Upper. Similarly, one can show that $\mathrm{UMD}^\pm_{\mathrm{PW}}(p)$ are p-independent and these will denoted by *$\mathit{UMD}^\pm_{\mathrm{PW}}$*. It seems to be an open problem if $\mathrm{UMD}^-_{\mathrm{PW}}$ implies UMD^- and if $\mathrm{UMD}^+_{\mathrm{PW}}$ implies UMD^+.

We list some results on UMD^- and UMD^+ spaces, the proofs of which can be found in [5]:

- If E is a UMD^+ space, then its dual E^* is a UMD^- space. If E^* is a UMD^+ space, then its predual E is a UMD^- space.
- Every UMD^- space has finite cotype. Every UMD^+ space is super-reflexive.
- E is a UMD space if and only if it is both UMD^- and UMD^+.

Similar results hold for $\mathrm{UMD}^-_{\mathrm{PW}}$ and $\mathrm{UMD}^+_{\mathrm{PW}}$ spaces.

It was shown in [5] that l^1 is a UMD^- space. It can be shown that if E is a UMD^- space and if (S,Σ,μ) is a σ-finite measure space, then $L^p(\Omega;E)$ is a UMD^- space for all $p \in [1,\infty)$. A similar result holds for UMD^+ for $p \in (1,\infty)$.

Apart from trivial cases, the space $L^1(S,\mu)$ is an example of a UMD^- space that is not UMD. It appears to be unknown if there exist UMD^+ or $\mathrm{UMD}^+_{\mathrm{PW}}$ spaces that are not UMD (cf. [6, Problem 4.2]).

2. Main result

Let W be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(\mathcal{F}_t)_{t\geq 0}$ be the augmented filtration induced by W. Similarly, let $\tilde{W}$ be a Brownian motion on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and let $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ be the augmented filtration induced by $\tilde{W}$.

Let E be a real Banach space. A process $\phi : [0,\infty) \times \Omega \to E$ will be called an *elementary process* if it is of the form

$$\phi(t,\omega) = \mathbf{1}_{[0]}(t)\xi_0(\omega) + \sum_{n=1}^{N} \mathbf{1}_{(t_{n-1},t_n]}(t)\xi_n(\omega),$$

where $0 \leq t_0 < \cdots < t_N < \infty$, ξ_n is an elementary $\mathcal{F}_{t_{n-1}}$-measurable random variable, $n = 1, \ldots, N$ and ξ_0 is $\mathcal{F}_0$-measurable. The stochastic integral $\int_0^\infty \phi(t)\, dW(t)$ is defined in the usual way and is an element of $L^p(\Omega; E)$ for all $p \in [1,\infty)$.

Theorem 2.1 (Garling)**.** *For a UMD space E and $p \in (1,\infty)$ the following statements hold:*

(1) *There exists a constant $c_p > 0$ such that for all elementary processes ϕ,*

$$\mathbb{E}\Big\|\int_0^\infty \phi(t)\, dW(t)\Big\|^p \leq c_p^p \mathbb{E}\,\tilde{\mathbb{E}}\Big\|\int_0^\infty \phi(t)\, d\tilde{W}(t)\Big\|^p. \tag{2.1}$$

(2) *There exists a constant $c_p > 0$ such that for all elementary processes ϕ,*

$$\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\int_0^\infty \phi(t)\, d\tilde{W}(t)\Big\|^p \leq c_p^p \mathbb{E}\Big\|\int_0^\infty \phi(t)\, dW(t)\Big\|^p. \tag{2.2}$$

Conversely, if (2.1) *and* (2.2) *hold for all elementary processes ϕ, then E is a UMD space.*

Inspection of the proof in [4, Theorem 2] shows that (2.1) only requires UMD$^-$ and (2.2) only requires UMD$^+$. The main result of this paper reads as follows.

Theorem 2.2. *Let E be a Banach space E and let $p \in (1,\infty)$. The following statements hold:*

(1) *If there exists a constant $c_p > 0$ such that* (2.1) *holds for all elementary processes, then E is a UMD$^-_{\rm PW}$ space.*
(2) *If there exists a constant $c_p > 0$ such that* (2.2) *holds for all elementary processes, then E is a UMD$^+_{\rm PW}$ space.*

Although these results are in some sense not surprising, they appear to be new and nontrivial to prove.

For the proof we need some lemmas. The first lemma is well-known and follows from the strong Markov property.

Lemma 2.3. *Let $\tau_0 = 0$ and define inductively*

$$\tau_n = \inf\{t \geq \tau_{n-1} : |W_t - W_{\tau_{n-1}}| = 1\}, \quad 1 \leq n \leq N.$$

Then $(\tau_n)_{n=1}^N$ is an increasing sequence of stopping times and $(\Delta\tau_n, \Delta W_n)_{n=1}^N$ is an i.i.d. sequence of random vectors, where

$$\Delta\tau_n = \tau_n - \tau_{n-1}, \quad \Delta W_n = W_{\tau_n} - W_{\tau_{n-1}}, \quad 1 \leq n \leq N.$$

Moreover $(\Delta W_n)_{n=1}^N$ is a Rademacher sequence adapted to $(\mathcal{F}_{\tau_n})_{n=1}^N$.

The next lemma gives some important properties of the independent Brownian motion $\tilde{W}$ at random times. Such stopped Brownian motions $\tilde{W}$ are not Gaussian random variables in general, but in this case they inherit some important properties.

Lemma 2.4. *For $1 \leq n \leq N$, let $\Delta\tilde{W}_n = \tilde{W}_{\tau_n} - \tilde{W}_{\tau_{n-1}}$. Then $(\Delta\tilde{W}_n)_{n=1}^N$ is an i.i.d. sequence of symmetric random variables, which is independent of $(\Delta W_n)_{n=1}^N$. Furthermore, each ΔW_n has finite moments of all orders.*

Proof. For all $1 \leq n \leq N$, $\Delta\tilde{W}_n$ is symmetric, because $\Delta\tilde{W}_n(\omega,\cdot)$ is symmetric for each $\omega \in \Omega$. It follows from the strong Markov property of $(W, \tilde{W})$ that $(\Delta W_n, \Delta\tilde{W}_n)_{n=1}^N$ is an i.i.d. sequence. So in order to prove the independence of $(\Delta\tilde{W}_n)_{n=1}^N$ and $(\Delta W_n)_{n=1}^N$, it is enough to show that $\Delta W_1 = W_{\tau_1}$ and $\Delta\tilde{W}_1 = \tilde{W}_{\tau_1}$ are independent. The following argument is shown to us by Tuomas Hytönen. For every Brownian motion B on Ω we introduce the following two stopping times:

$$\tau_{\pm}^B = \inf\{t \geq 0 : B_t = \pm 1\}.$$

Note that $\tau_1 = \tau_-^W \wedge \tau_+^W$ and for the Brownian motion $-W$, we have $\tau_+^{-W} = \tau_-^W$ and $\tau_-^{-W} = \tau_+^W$. Let $B \in \mathbb{R}$ be some Borel measurable set. Since $(W, \tilde{W})$ is identically distributed with $(-W, \tilde{W})$ it follows that

$$\begin{aligned}\mathbb{P}(W_{\tau_1} = 1, \tilde{W}_{\tau_1} \in B) &= \mathbb{P}(\tau_+^W < \tau_-^W, \tilde{W}_{\tau_1} \in B)\\ &= \mathbb{P}(\tau_-^{-W} < \tau_+^{-W}, \tilde{W}_{\tau_1} \in B) = \mathbb{P}(W_{\tau_1} = -1, \tilde{W}_{\tau_1} \in B).\end{aligned}$$

Clearly,

$$\mathbb{P}(W_{\tau_1} = 1, \tilde{W}_{\tau_1} \in B) + \mathbb{P}(W_{\tau_1} = -1, \tilde{W}_{\tau_1} \in B) = \mathbb{P}(\tilde{W}_{\tau_1} \in B).$$

Hence

$$\mathbb{P}(W_{\tau_1} = 1, \tilde{W}_{\tau_1} \in B) = \frac{1}{2}\mathbb{P}(\tilde{W}_{\tau_1} \in B) = \mathbb{P}(W_{\tau_1} = 1)\mathbb{P}(\tilde{W}_{\tau_1} \in B).$$

The same holds for -1. This proves the independence.

For $0 < p < \infty$ we have

$$\mathbb{E}\,\tilde{\mathbb{E}}\,|\Delta\tilde{W}_n|^p = \mathbb{E}\,\tilde{\mathbb{E}}\,|\tilde{W}_{\tau_1}|^p = g_p^p \mathbb{E}\,\tau_1^{p/2},$$

where g_p^p is the p-th moment of a standard Gaussian random variable and the statement follows from the elementary fact that τ_1 has finite moments of all orders. $\square$

Below we will consider adapted and measurable processes $\phi : [0,\infty)\times\Omega \to E$ that take values in a finite-dimensional subspace of E. Since n-dimensional subspaces of E are isomorphic to $\mathbb{R}^n$, one may construct the stochastic integral for such processes ϕ that satisfy $t \mapsto \phi(t,\omega) \in L^2(0,\infty,E)$ for almost all $\omega \in \Omega$. By the Burkholder-Davis-Gundy inequalities we have for all $p \in (1,\infty)$ and for ϕ as above, $\int_0^\infty \phi(t)\,dW(t) \in L^p(\Omega;E)$ if $\phi \in L^p(\Omega;L^2(0,\infty;E))$. In this case the decoupled stochastic integral $\int_0^\infty \phi(t)\,d\tilde{W}(t)$ is defined pathwise as an element of $L^p(\Omega;L^p(\tilde{\Omega};E))$. Moreover, if (2.1) or (2.2) holds for all elementary processes one may extend this to all processes as above. In fact, Garling proved (2.1) and (2.2) for this class of processes.

The next lemma is a variation of an example in [5]. We include a proof for convenience.

Lemma 2.5. *Let $E = c_0$ and $p \in [1,\infty)$. There does not exist a constant $c_p > 0$ such that for all elementary processes ϕ, (2.1) holds.*

Proof. Assume there exists a constant $c_p > 0$ such that for all elementary processes ϕ, (2.1) holds. Then we may extend (2.1) to all measurable and adapted processes $\phi \in L^p(\Omega; L^2(0,\infty;E))$ that take values in a finite-dimensional subspace of E. For each $N \geq 1$, we will construct a process ϕ as above and such that

$$\Big(\mathbb{E}\Big\|\int_0^\infty \phi(t)\,dW(t)\Big\|^p\Big)^{1/p} = N \text{ and } \Big(\mathbb{E}\tilde{\mathbb{E}}\Big\|\int_0^\infty \phi(t)\,d\tilde{W}(t)\Big\|^p\Big)^{1/p} \leq K_p\sqrt{N}.$$

Here $K_p > 0$ is some universal constant. This gives a contradiction.

We modify an example in [5] in such a way that the martingale differences arise as stochastic integrals. We use the notation of Lemmas 2.3 and 2.4. Fix an integer $N \geq 1$. Let $D = \{-1,1\}^N$, and for each $e = (e_n)_{n=1}^N \in D$ define the process $\phi_e : [0,\infty)\times\Omega \to \mathbb{R}$ by

$$\phi_e(t) = \begin{cases} e_n \mathbf{1}_{A_{e,n}} & \text{for } t \in (\tau_{n-1},\tau_n],\ n = 1,\dots,N, \\ 0 & \text{for } t = 0 \text{ or } t > \tau_N, \end{cases}$$

where $A_{e,1} = \Omega$ and for $2 \leq n \leq N$,

$$A_{e,n} = \{\Delta W_1 = e_1, \dots, \Delta W_{n-1} = e_{n-1}\}.$$

Then each ϕ_e is stochastically integrable with

$$\int_0^\infty \phi_e(t)\,dW(t) = \sum_{n=1}^N \Delta W_n e_n \mathbf{1}_{A_{e,n}}.$$

Define $\phi : [0,\infty)\times\Omega \to l^\infty(D)$ by $\phi = (\phi_e)_{e\in D}$. Then ϕ is stochastically integrable and for almost all $\omega \in \Omega$ and $e \in D$ we have $\left|\Big(\int_0^\infty \phi(t)\,dW(t)\Big)(\omega)(e)\right| \leq N$. For almost all $\omega \in \Omega$ and $e = (\Delta W_n(\omega))_{n=1}^N$ we have $\left|\Big(\int_0^\infty \phi(t)\,dW(t)\Big)(\omega)(e)\right| = N$. This shows that

$$\Big(\mathbb{E}\Big\|\int_0^\infty \phi(t)\,dW(t)\Big\|^p_{l^\infty(D)}\Big)^{1/p} = N, \quad \text{for all } p \in [1,\infty).$$

On the other hand, we have

$$\int_0^\infty \phi(t)\,d\tilde{W}(t) = \sum_{n=1}^N \Delta\tilde{W}_n v_n,$$

where for $1 \leq n \leq N$, $v_n = (e_n \mathbf{1}_{A_{e,n}})_{e\in D}$.

For $\omega \in \Omega$ and $e \in D$ let $k(\omega,e)$ be 0 if $\Delta W_1(\omega) \neq e_1$ and let $k(\omega,e)$ be the maximum of all integers $n \leq N$ such that $\Delta W_i(\omega) = e_i$ for all $i \leq n$ if $\Delta W_1(\omega) = e_1$. For almost all $\omega \in \Omega$ and for all $e \in D$, $\Big(\sum_{n=1}^N \Delta\tilde{W}_n v_n\Big)(\omega)(e)$ is equal to

$$\begin{array}{ll} -\Delta\tilde{W}_{k(\omega,e)+1}(\omega,\cdot)\Delta W_{k(\omega,e)+1}(\omega) + \sum_{n=1}^{k(\omega,e)} \Delta\tilde{W}_n(\omega,\cdot)\Delta W_n(\omega), & \text{if } k(\omega,e) < N, \\ \sum_{n=1}^N \Delta\tilde{W}_n(\omega,\cdot)\Delta W_n(\omega), & \text{if } k(\omega,e) = N. \end{array}$$

Of course we have for all $k \leq N$,

$$-\tilde{W}_k\Delta W_k + \sum_{n=1}^{k-1}\Delta\tilde{W}_n\Delta W_n = 2\sum_{n=1}^{k-1}\Delta\tilde{W}_n\Delta W_n - \sum_{n=1}^{k}\Delta\tilde{W}_n\Delta W_n.$$

We obtain that for almost all $\omega \in \Omega$,

$$\Big\|\int_0^\infty \phi(t,\omega)\,d\tilde{W}(t)\Big\|_{l^\infty(D)} \leq 3\sup_{k\leq N}\Big|\sum_{n=1}^{k} \Delta\tilde{W}_n(\omega,\cdot)\Delta W_n(\omega)\Big|.$$

Since for almost all $\omega \in \Omega$, $(\Delta\tilde{W}_n(\omega,\cdot))_{n=1}^N$ is a sequence of independent centered Gaussian random variables on $\tilde{\Omega}$, we have by the Lévy-Octaviani inequalities for independent symmetric random variables (see [9, Section 1.1]) for almost all $\omega \in \Omega$,

$$\begin{aligned}\tilde{\mathbb{E}}\sup_{k\leq N}\Big|\sum_{n=1}^{k} \Delta\tilde{W}_n(\omega,\cdot)\Delta W_n(\omega)\Big|^p &\leq 2^p\tilde{\mathbb{E}}\,\Big|\sum_{n=1}^{N} \Delta\tilde{W}_n(\omega,\cdot)\Delta W_n(\omega)\Big|^p \\ = 2^p\tilde{\mathbb{E}}\Big|\sum_{n=1}^{N} \Delta\tilde{W}_n(\omega,\cdot)\Big|^p &= 2^p\tilde{\mathbb{E}}\,|\tilde{W}_{\tau_N}(\omega,\cdot)|^p = 2^p g_p^p \tau_N(\omega)^{p/2}.\end{aligned}$$

Here g_p^p is the p-th moment of a standard Gaussian random variable. We may conclude that

$$\Big(\mathbb{E}\,\tilde{\mathbb{E}}\,\Big\|\int_0^\infty \phi(t)\,d\tilde{W}(t)\Big\|_{l^\infty(D)}^p\Big)^{1/p} \leq 6g_p(\mathbb{E}\,\tau_N^{p/2})^{1/p}.$$

Recall that the sequence $(\tau_n - \tau_{n-1})_{n=1}^N$ is identically distributed. For $p = 2$ we obtain

$$\begin{aligned}(\mathbb{E}\,\tau_N^{p/2})^{1/p} = (\mathbb{E}\,\tau_N)^{1/2} &= \Big(\mathbb{E}\sum_{n=1}^{N} \tau_n - \tau_{n-1}\Big)^{1/2} \\ &= \Big(\sum_{n=1}^{N} \mathbb{E}\,(\tau_n - \tau_{n-1})\Big)^{1/2} = \Big(\sum_{n=1}^{N} \mathbb{E}\,\tau_1\Big)^{1/2} = \sqrt{N}\sqrt{\mathbb{E}\,\tau_1}.\end{aligned}$$

For $1 \leq p < 2$ we have by Hölder's inequality,

$$(\mathbb{E}\,\tau_N^{p/2})^{1/p} \leq (\mathbb{E}\,\tau_N)^{1/2} = \sqrt{N}\sqrt{\mathbb{E}\,\tau_1}.$$

Finally for $p > 2$, by the triangle inequality in $L^{p/2}(\Omega)$,

$$\begin{aligned}(\mathbb{E}\,\tau_N^{p/2})^{1/p} &= \Big(\mathbb{E}\Big(\sum_{n=1}^{N} \tau_n - \tau_{n-1}\Big)^{p/2}\Big)^{1/p} \leq \Big(\sum_{n=1}^{N}(\mathbb{E}\,(\tau_n - \tau_{n-1})^{p/2})^{2/p}\Big)^{1/2} \\ &= \Big(\sum_{n=1}^{N}(\mathbb{E}\,\tau_1^{p/2})^{2/p}\Big)^{1/2} = \sqrt{N}(\mathbb{E}\,\tau_1^{p/2})^{1/p}.\end{aligned}$$

By Lemma 2.4 this proves that for all $p \in [1,\infty)$ and some universal constant K_p,

$$\Big(\mathbb{E}\,\tilde{\mathbb{E}}\,\Big\|\int_0^\infty \phi(t)\,d\tilde{W}(t)\Big\|^p\Big)^{1/p} \leq K_p\sqrt{N}.$$

Since $l^\infty(D)$ can be identified isometrically with a finite-dimensional subspace of c_0, this completes the proof. □

Corollary 2.6. *Let E be a Banach space. If there exists a constant $c_p > 0$ such that for all elementary processes* (2.1) *holds, then E has finite cotype.*

Proof. It follows from the above example that c_0 is not finitely representable in E. Hence the Maurey-Pisier Theorem (see [12]) implies that E has finite cotype. □

Proof of Theorem 2.2. We may assume that the martingale starts at zero (see [1, Remark 1.1]). Let $(r_n)_{n=1}^N$ be a Rademacher sequence on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(d_n)_{n=1}^N$ be an E-valued martingale difference sequence with respect to the filtration $(\sigma(r_1, r_2, \ldots, r_n))_{n=0}^N$. We may write $d_n = r_n f_n(r_1, \ldots, r_{n-1})$ for $n = 1, \ldots, N$, for some $f_n : \{-1, 1\}^{n-1} \to E$. Let $(\tilde{r}_n)_{n=1}^N$ be a Rademacher sequence on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.

(1): We will show that there exists a constant $C_p^- > 0$ only depending on E such that

$$\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p \leq (C_p^-)^p \mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n d_n\Big\|^p. \tag{2.3}$$

We use the notation of Lemmas 2.3 and 2.4. Define a process $\phi : [0, \infty) \times \Omega \to E$ by

$$\phi(t) = \begin{cases} f_n(\Delta W_1, \ldots, \Delta W_{n-1}) & \text{for } t \in (\tau_{n-1}, \tau_n],\ n = 1, \ldots, N \\ 0 & \text{for } t = 0 \text{ or } t > \tau_N. \end{cases}$$

The process ϕ is stochastically integrable and we have

$$\begin{aligned} \mathbb{E}\Big\|\int_0^\infty \phi(t)\,dW(t)\Big\|^p &= \mathbb{E}\Big\|\sum_{n=1}^N \Delta W_n f_n(\Delta W_1, \ldots, \Delta W_{n-1})\Big\|^p \\ &= \mathbb{E}\Big\|\sum_{n=1}^N r_n f_n(r_1, \ldots, r_{n-1})\Big\|^p = \mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p. \end{aligned}$$

Also, we have

$$\tilde{\mathbb{E}}\,\mathbb{E}\Big\|\int_0^\infty \phi(t)\,d\tilde{W}(t)\Big\|^p = \mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \Delta\tilde{W}_n f_n(\Delta W_1, \ldots, \Delta W_{n-1})\Big\|^p.$$

By Lemma 2.4, Corollary 2.6 and [10, Proposition 9.14], we have

$$\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \Delta\tilde{W}_n x_n\Big\|^p \leq K_p \mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n x_n\Big\|^p,$$

where $(x_n)_{n=1}^N$ is a sequence in E and $K_p > 0$ is some constant depending only on E and p. By conditioning (cf. [8, Lemma 3.11]) this result extends to

$$\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \Delta\tilde{W}_n X_n\Big\|^p \leq K_p \mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n X_n\Big\|^p, \tag{2.4}$$

where $(X_n)_{n=1}^N$ is a sequence of E-valued random variables independent of $(\Delta\tilde{W}_n)_{n=1}^N$ and independent of $(\tilde{r}_n)_{n=1}^N$. By Lemmas 2.3 and 2.4, we may apply (2.4) to the random variables $X_n = f_n(\Delta W_1, \ldots, \Delta W_{n-1})$ for $1 \leq n \leq N$ to obtain:

$$\begin{aligned} \mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \Delta\tilde{W}_n f_n(\Delta W_1, \ldots, \Delta W_{n-1})\Big\|^p &\leq K_p^p \mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n f_n(\Delta W_1, \ldots, \Delta W_{n-1})\Big\|^p \\ &= K_p^p \mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n f_n(r_1, \ldots, r_{n-1})\Big\|^p \overset{\text{(i)}}{=} K_p^p \mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n r_n f_n(r_1, \ldots, r_{n-1})\Big\|^p \\ &= K_p^p \mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n d_n\Big\|^p. \end{aligned}$$

In (i), we used that $(r_1,\ldots,r_N,\tilde r_1,\ldots,\tilde r_N)$ and $(r_1,\ldots,r_N,r_1\tilde r_1,\ldots,r_N\tilde r_N)$ are identically distributed. By assumption we have

$$\mathbb{E}\Big\|\int_0^\infty \phi(t)\,dW(t)\Big\|^p \le c_p^p\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\int_0^\infty \phi(t)\,d\tilde W(t)\Big\|^p.$$

We may conclude that (2.3) holds with constant c_pK_p.

(2): We will show that there exists a constant $C_p^+>0$ only depending on E such that

$$\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde r_n d_n\Big\|^p \le (C_p^+)^p\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p. \tag{2.5}$$

Let ϕ be as before. By Lemmas 2.3, 2.4 and [10, Lemma 4.5] and the same arguments as before we have

$$\begin{aligned}\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde r_n d_n\Big\|^p &= \mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde r_n f_n(r_1,\ldots,r_{n-1})\Big\|^p\\ &= \mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde r_n f_n(\Delta W_1,\ldots,\Delta W_{n-1})\Big\|^p\\ &\le \frac{1}{(\mathbb{E}\,\tilde{\mathbb{E}}\,|\tilde W_1|)^p}\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \Delta\tilde W_n f_n(\Delta W_1,\ldots,\Delta W_{n-1})\Big\|^p.\end{aligned}$$

By assumption we have

$$\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\int_0^\infty \phi(t)\,d\tilde W(t)\Big\|^p \le c_p^p\mathbb{E}\Big\|\int_0^\infty \phi(t)\,dW(t)\Big\|^p.$$

We may conclude that (2.5) holds with constant $\dfrac{c_p}{\mathbb{E}\,\tilde{\mathbb{E}}\,|\tilde W_1|}$. $\square$

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