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Swaption pricing under the Hull-White One Factor Model

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"Swaption pricing under the Hull-White One Factor Model"

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Abstract

In this thesis we price a swaption, an interest rate derivative, under the Hull-White one factor model. First we introduce the model and the way we can price a swaption under this specific short-rate model. Second we discuss the algorithm used to calibrate the parameters of the model to best fit the observed market prices of a set of swaptions. We discuss the goodness of fit, and evaluate the differences between the observed price and the theoretical price in order to find mispriced swaptions. After finding the mispriced swaptions we develop a trading strategy that aims to exploit this mispricing.

Keywords: Swaptions, Term structure, Interest rates, Hull-White one factor, Black Formula

Contents

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Wouter Tilgenkamp

1 Preliminaries

Although interest rate derivatives are traded in large volumes on a daily basis, these products are not so well-known to the outsider. So to prevent any possible confusion throughout the following thesis this chapter will introduce the reader into the concepts and terminology involved. In the first chapter all the ingredients are introduced that are later needed to price a swaption (financial interest rate derivative). In the second chapter we introduce Black's Model, a pricing model under which a swaption can be priced. Chapter three gives a rough oversight of the different models describing the term structure of interest rates. The implementation details can be found in the fourth chapter after which we discuss the results in chapter five. Chapter six and seven are the conclusion and discussion.

1.1 The Bank Account, Zero-Coupon bonds and Interest Rates

Let us consider a bank account with value $B(t)$ at time $t \geq 0$. We assume that we started our bank account with some initial deposit $B(0) = B_0$, $B_0 \in \mathbb{R}_{>0}$. This bank account is a savings account in the sense that interest is paid according to the short-term interest rate (*short-rate*) $r(t)$. The value of the bank account can be described by the following differential equation,

$$dB(t) = B(t)r(t)dt \tag{1.1}$$

Integration over time and using the initial condition $B(0) = B_0$ gives us an explicit solution for the value of our bankaccount $B(t)$,

$$B(t) = B_0 \cdot exp\left(\int_0^t r(s)ds\right) \tag{1.2}$$

Equation (1.2) tells us that investing an amount of B_0 at time 0 yields at time t the value $B(t)$.

1.2 Compounding Interest Rates

If we invest an amount $A \in \mathbb{R}_{>0}$ over a period of n years with a constant interest rate of R that is compounded m times per annum, the terminal value A_{term} of our initial investment A is,

$$A_{term} = A\left(1 + \frac{R}{m}\right)^{mn} \tag{1.3}$$

For the *continuous compounding* interest rate, we take the limit of the compounding frequency m to infinity,

$$A_{term} = \lim_{m\to\infty} A\left(1 + \frac{R}{m}\right)^{mn} = Ae^{Rn} \tag{1.4}$$

We find that given the continuous compounding interest rate R_c we find the equivalent interest rate R_m compounding m times per annum, through the following equation,

$$A\left(1+\frac{R_m}{m}\right)^{mn} = Ae^{R_c n} \tag{1.5}$$

solving for R_c or R_m yields,

$$R_c = m\ln\left(1+\frac{R_m}{m}\right), \quad R_m = m\left(e^{R_c/m}-1\right) \tag{1.6}$$

We will use this conversion between interest rates later on, when determining the forward swap rates.

1.3 Bonds

A bond is an financial instrument of indebtedness of the bond issuer to the bond holders. Thus a bond can be viewed as a loan. Bonds have specific characteristics like a periodic interest payment (*coupon*), which states the fixed percentage that the bond issuer pays to the bond holder after fixed intervals ((semi)annual, sometimes monthly). The fixed percentage is paid over the *par* value of the bond, this par value is almost always repaid at the end of the term (*maturity date*). The ownership of most bonds is negotiable [1], which means that it can change from owner multiple times during its term. While the par value of the bond is determined at the moment the bond is issued, the *market price* at which the bond trades in the market can fluctuate over time. The most well-known bonds are the government bonds, given out by countries to finance their current expenditure, but they are not the only kind of bond. Multinationals like Shell and Coca-Cola also issue bonds to finance long-term investments. For example in September 2013, the largest corporate bond issuance in history took place by Verizon Communications Inc. that sold 49 billion of dollars worth of bonds.

To summarize:

- *Principal*: The nominal, par or face value is the amount on which the issuer pays interest, this amount is most commonly repaid to the bond holder at the end of the term.
- *Maturity*: The length of the term, this is the date at which the bond expires and the bond holder is repaid the principal value. The obligation between bond issuer and bond holder ends and the contract is terminated.
- *Coupon*: The coupon is the interest rate that the bond issuer pays to the bond holder at fixed intervals. Usually this percentage is fixed over the entire term but there are exceptions.
- *Market price*: The price at which you can buy/sell the bond in the market.

[1] an example of non-tradable bonds are U.S. savings bonds

1.4 Zero-Coupon Bond

A zero-coupon bond is a contract that guarantees its holder the payment of the principal L at the time of maturity T, there are no intermediate payments. From here on out $P(t,T)$ denotes the price, at time t, of a unit face value zero-coupon bond maturing at time T. Let $R(t,T)$ denote the continuous compounded interest rate at time t for a given maturity T, with $T \geq t$. Then $P(t,T)$ and $R(t,T)$ are related in the following way,

$$P(t,T)e^{R(t,T)(T-t)} = 1 \Longleftrightarrow R(t,T) = -\frac{ln(P(t,T))}{T-t} \tag{1.7}$$

Observe that $P(T,T) = 1$, from the definition of $R(t,T)$ we obtain another expression for the short-rate,

$$r(t) = \lim_{T \to t^+} R(t,T) \tag{1.8}$$

Please note that at any point t in time we approximately know the value $P(t,T)$ at which the contract is trading in the market. To elaborate further, we do not know $P(t,T)$ exactly most of the time but we can approximate its value through linear interpolation in the following way. If we want to know $P(t,T)$, we find the zero coupon bonds trading in the market with maturity T^- and T^+ closest to T so that $T^- < T < T^+$. At this point we know $P(t,T^-)$ and $P(t,T^+)$ and linearly interpolate $P(t,T)$,

$$P(t,T) \approx P(t,T^-) + \left(P(t,T^+) - P(t,T^-)\right)\frac{T-T^-}{T^+-T^-} \tag{1.9}$$

From a theoretical point of view this means we now know the present value of the principal at time T.

1.5 Bond Yield and Yield Curve

A bond's yield is the discount rate that, when applied to all the cash flows, gives a bond price equal to the market price observed today. Consider a 2-year bond with a principal of 100 that provides coupons at the rate of 5% annually, let us assume that the current market price is 98. To determine the bond's yield r we solve the following equation,

$$100 \times \left(0,05e^{-r\times 1} + 1,05e^{-r\times 2}\right) = 5e^{-r\times 1} + 105e^{-r\times 2} = 98$$

This can be done iteratively for coupon bearing bonds, and gives in this case $r = 5,91\%$. For a zero-coupon bond we can calculate the bond's yield directly through (1.7). Consider another 2 year bond with principal 100 that does not pay coupons, the current market price is 98. We determine the bond's yield r,

$$100 \times e^{-r\times 2} = 98 \Longleftrightarrow r = -\frac{1}{2}ln\left(\frac{98}{100}\right) = 1,01\%$$

The bond's yield for the 2 year zero coupon bond would also be called the 2 year *zero-rate.* If we have several zero coupon bonds with different maturities we could plot the different zero-rates as a function of the maturity, this is called the *yield curve.* The yield curve shows the relation between the interest rate and the time to maturity, known as the *term.* This relation is called the term structure of interest rates, models and explanations for the term structure will be further discussed in Chapter 3. The yield curve for AAA-rated government bonds in the Euro area on the 2nd of December 2013 can be seen in Figure 1.1.

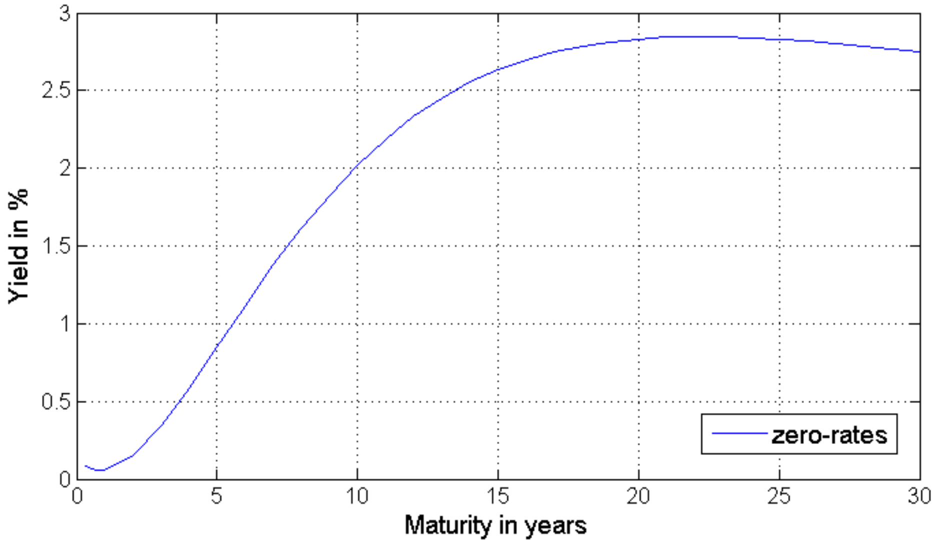


Figure 1.1: AAA-rated Euro area central government bonds yield curve on 2 December 2013

1.6 Forward rates

Forward rates are the rates of interest implied by the current zero rates for periods of time in the future. To determine the continuous compounded forward interest rate $R(t, T)$, we use a no-arbitrage argument. Consider the periods $[0, t]$, $[t, T]$ and $[0, T]$ with $0 < t < T$, for our forward rate $R(t, T)$ to be consistent with the zero rates $R(0, t)$ and $R(0, T)$ we require that an investment of B_0 over the period $[0, T]$ should yield the same result as reinvesting B_0 over period $[0, t]$ followed by period $[t, T]$. This leads to,

$$B_0 \times e^{R(0,t)(t-0)} \cdot e^{R(t,T)(T-t)} = B_0 \times e^{R(0,T)(T-0)} \tag{1.10}$$

solving for $R(t, T)$ yields,

$$R(t, T) = \frac{R(0, T)T - R(0, t)t}{T - t} \tag{1.11}$$

$F(0, t)$ is the instantaneous forward rate that applies to time t as observed at time zero. It can be computed from the initial price of a zero-coupon bond as,

$$F(0, t) = -\frac{\delta \log P(0, t)}{\delta t}$$

1.7 Swap

An *interest rate swap* is an agreement between two parties to exchange cash flows in the future. The first party agrees to pay the other party a cash flow equal to a fixed rate on a notional principal, while receiving from the other party a cash flow equal to the interest at a floating rate on the same notional. The floating rate, that is often used, is the London Interbank Offer Rate (LIBOR). Usually 1-month, 3-month, 6-month and 12-month LIBOR are quoted. The exchange of cash flows will be done at predetermined times over the length of the swap. Figure 1.2 displays a typical 2 year swap agreement that pays annually in which the fixed rate is 5%. The floating rate that is to be paid at the end of the period is recorded at the beginning of the period (reset date). Principal payments are not exchanged in an interest rate swap, however if we assume we did exchange the principal at the end of the swap the value of the contract does not change. The reason is that if we would both pay the principal and receive the principal at the same time net result is zero. From this point of view, the owner of payer swap can be regarded as having bought a floating rate bond and sold a fixed rate bond. So in order to determine the value of our payer swap V_{payer} we have to find the value of the floating rate bond B_{float} and subtract the value of the fixed rate bond B_{fix}.

$$V_{payer} = B_{float} - B_{fix} \tag{1.12}$$

The value of a newly issued floating rate bond that pays 6-month LIBOR every 6 months is always equal to its principal value when we use the same 6-month LIBOR for discounting the payment. The reason being that the bond pays a rate of interest equal to 6-month LIBOR every 6 months, and we use the same LIBOR curve to discount the payments to the present. In other words if the rate of LIBOR used for the floating rate coincides with the interval between payments the floating rate bond of the swap will be worth par. The fixed rate offered on a swap so that the swap is valueless at the start is known as the at-the-money *swap rate*. For our swap to be valueless we look at (1.12) and for now conclude that $B_{float} = B_{fix}$. In section 1.11 we derive the at-the-money swap rate in more detail.

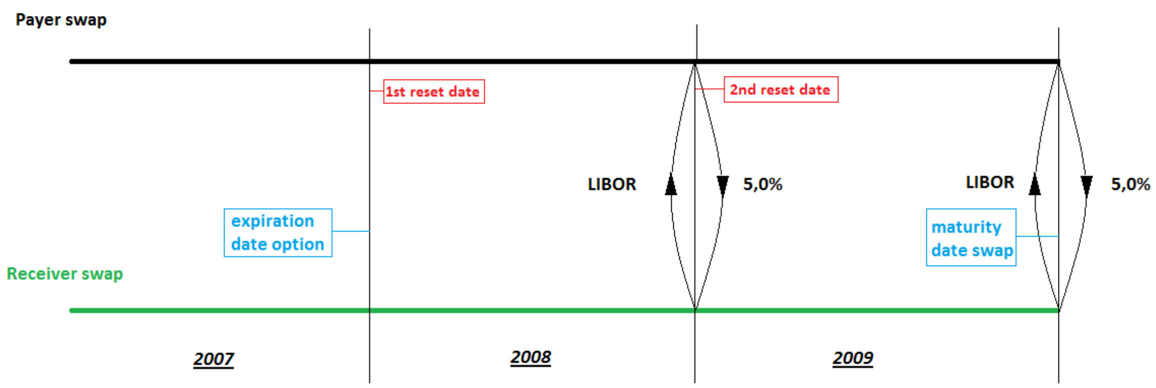


Figure 1.2: The payout structure of a 2 year swap that pays annually, the swap starts at the beginning of 2008 and matures at the end of 2009.

1.8 Swaption

A swap option contract, often referred to as *swaption*, gives the owner the right (but not the obligation) to enter into a certain interest rate swap at a predetermined future point of time. A *payer swaption* gives the owner the right to enter into a swap where they pay the fixed interest rate while receiving the floating interest rate. A *receiver swaption* gives the owner the right to enter into a swap in which they will receive the fixed interest rate and pay the floating interest rate.

There are multiple reasons to enter into a swaption contract, one prominent reason is to transform the nature of an asset. A swap allows a company to transform an asset that earns a fixed rate of interest into an asset earning a floating rate of interest. Likewise a party could cover future interest rate risks, consider a company which has a wide array of loans on which they pay variable interest. For the continuity of the company it could be favorable to eliminate the risk of the variable interest on these loans. They enter a payers swaption with the same maturity and principal of the loans and now they pay a fixed interest rate while receiving the floating rate, completely eliminating the interest rate risk. Swaptions are also used for speculation on the future interest rate, if an institution were to expect future interest rates to decline they could buy a receiver swaption, so that the underlying swap pays the expected lower future interest rate while receiving the higher fixed interest rate. It is reasonable to assume that we would only enter the swap when our swaption expires if the value of the swap is positive at the expiration of the swaption.

As earlier mentioned a floating rate bond that pays 6 month LIBOR and is discounted with 6 month LIBOR is worth its face value. Because of this a payer swaption is therefore a put option on a fixed rate bond with the strike price equal to the face value of the bond. In the same way a receiver swaption is a call option on a fixed rate bond with the strike price equal to the face value of the bond.

1.9 Case Study: Vestia

In January 2012 the journal *''Het Financieel Dagblad''* reported on the financial problems of Vestia, a dutch housing corporation. The company had suffered very significant losses on their derivatives portfolio, due to low interest rates. Vestia had tried to take advantage of the low interest rates and wanted to lock these low rates in for the upcoming years through a portfolio of payer swaps. So if interest rates rose in the next period they make a nice profit. However they didn't quite consider the possibility that the interest rates could fall even lower. As it happened, interest rates dropped even further and with a payer swap portfolio on approximately €9,9 billion in notional principals while their total assets were only worth €5 billion this lead to Vestia getting into big financial trouble. The thing to note here is that had Vestia bought payer swaptions instead of directly entering the payer swaps these problems would not have occurred. With the big fall in interest rates the payer swaptions would have expired valueless and would not be executed, and as a consequence no swap would be entered. The only money lost would have been the purchasing price of the swaptions themselves, instead by entering the swaps they became exposed to a possible almost unlimited loss. This case further tainted the public view on financial derivatives, and gave rise to the discussion if housing corporations should be able to speculate on the derivatives market altogether.

1.10 Derivative pricing

Consider a time horizon $T > 0$, a probability space (Ω, Q_0) and a right-continuous filtration $\mathscr{F} = \{\mathscr{F} : 0 \leq t \leq T\}$. In a given economy $K+1$ non dividend paying securities are traded continuously from time 0 untill time T. Their prices are given by $S_k(t)$, with $S_0(t) = B(t)$ and $k = 0..K$. A trading strategy ϕ is a $K+1$ dimensional stochastic process $\phi = \{\phi(t) : 0 \leq t \leq T\}$ whose components $\phi_0, \phi_1.., \phi_K$ are locally bounded and predictable. $\phi_k(t)$ denotes the amount of shares we have of stock k at time t. The value of this strategy ϕ at time t is defined by,

$$V_\phi(t) = \sum_{k=0}^{K} \phi_k(t) S_k(t) \tag{1.13}$$

This portfolio is *self-financing* if it obeys the following condition,

$$dV_\phi(t) = \sum_{k=0}^{n} \phi_k(t) dS_k(t) \tag{1.14}$$

Intuitively, a trading strategy is self-financing if its value changes only due to changes in the asset prices. No additional cash inflows or outflows occur after the initial time. An *arbitrage opportunity* is a self-financing portfolio with zero initial investment, that has zero probability of losing money, and a positive probability of having a positive return. In mathematical terms,

$$\mathbb{P}\left(V_\phi(t) \geq 0\right) = 1, \;\; \mathbb{P}\left(V_\phi(t) > 0\right) > 0, \;\; V_\phi(0) = 0 \tag{1.15}$$

We also speak of an arbitrage possibility if there is a initial investment $V_\phi(0) > 0$ but the portfolio at time T is at least the initial investment $V_\phi(0)$ plus interest at the risk-free rate and with positive probability of having a value higher than that. In other words, there is a strategy that expects to pay more than the money market account, without the possibility to pay less. From the no-arbitrage assumption we derive the law of one price which states that two portfolios with the same cash flows must have the same price today. Otherwise we create a strategy that consists of selling the expensive portfolio while buying the cheap portfolio. The cash flows between the two portfolios would offset each other and we would make a risk-free profit equal to the difference in price of the initial portfolios.

Due to the fundamental theorem of asset pricing, which implies that in a complete market a derivative's price is the discounted expected value of the future payoff under the unique risk-neutral measure. A complete market is one in which every contingent claim can be replicated with existing assets without transaction costs.

In the theory of derivative pricing, the no-arbitrage assumption, which states that the market is free of arbitrage possibilities is of great importance. How can we determine if a market satisfies the arbitrage-free assumption? The answer is that a market model satisfies the no-arbitrage assumption if and only if there is a unique risk-neutral measure also called an equivalent martingale measure. For a rigorous proof we refer to [16].

1.11 Annuity as Numeraire

A numéraire for a derivative in financial mathematics is the standard to which we relate the value of the derivative. Usually the numéraire chosen is the bank account defined in section 1.1.

If we consider a swap that starts at time T with pay dates $T_1, T_2, ..., T_N$. Assume that the principal $L = 1$. The forward swap rate is defined as being the fixed rate for which the swap has value zero at initiation. At time $t < T$ the forward swap rate is defined as,

$$s(t) \tag{1.16}$$

The value of the fixed side of the swap is,

$$s(t) \cdot A(t) \tag{1.17}$$

with

$$A(t) = \sum_{i=1}^{N} P(t, T_i)(T_i - T_{i-1}) \tag{1.18}$$

Previously we found that if we add the principal to the last payment date (T_N) that the value of the floating side of the swap at time $t = T$ is equal to the principal. So if we add 1 at time T_N the value of the floating side will be equal to 1 at time T.

The value of 1 at T_N discounted to time t is equal to $P(t, T_N)$, obviously the value of 1 received at time T is $P(t, T)$. We conclude that the value of the floating side of the swap at time t is,

$$P(t, T) - P(t, T_N) \tag{1.19}$$

Given that the swap has value zero we equate both the fixed and floating side and find,

$$s(t)A(t) = P(t,T) - P(t,T_N) \tag{1.20}$$

Solving for the at-the-money swap rate at $t = 0$ yields,

$$s(0)^{ATM} = \frac{P(0,T) - P(0,T_N)}{\sum_{i=1}^{N} P(0,T_i)(T_i - T_{i-1})} \tag{1.21}$$

Now it is important to note that if our world is forward risk neutral with respect to $A(t)$, meaning that rather than using the bank account as a numéraire we use $A(t)$. An implication of doing so is that the expected future swap rate equals the forward swap rate observed at time t. Mathematically,

$$s(t) = \mathbb{E}[s(T)] \tag{1.22}$$

1.12 The Greeks

In mathematical finance, the Greeks represent the sensitivities of the price of derivatives to a change in the underlying parameters on which the price depends. In this thesis we are concerned with delta's and vega's. Delta is the first derivative of the value V of the option with respect to the price S of the underlying instrument, in our case the forward swap rate.

$$\triangle = \frac{\delta V}{\delta S} \tag{1.23}$$

The vega[2] is the first order derivative of the option value with respect to the σ of the underlying forward swap rate.

$$\nu = \frac{\delta V}{\delta \sigma} \tag{1.24}$$

Theta Θ denotes the sensitivy of the option value with respect to the time T.

$$\Theta = \frac{\delta V}{\delta T} \tag{1.25}$$

A portfolio consisting of swap(tion)s that has no exposure to the price of the underlying forward swaprate is said to be delta-neutral. The most practical way to obtain the Greeks is by numerical differentiation, therefore the following procedure is used. Once we move the entire yield curve up by one basis point and once we move the yield curve down. For both shifted yield curves we calculate the implied swaption price. The delta is the symmetrical difference of the prices. This means we price the swaption after shifting the entire interest rate curve one basis point up minus the swaption price after shifting the entire interest rate curve one basis point down divided by two. It is important to note that the theoretical delta is uniquely defined based on the model that is being used to price the derivative.

1.13 Brownian Motion

N. Wiener (1931) did the mathematical groundwork for Brownian Motion, therefore it is commonly referred to as *Wiener Process*. Brownian motion refers to the mathematical model that was developed to describe the random movement of particles immersed in a fluid. It is also used to

[2] although vega is not denoted by a greek letter it is part of the "Greeks"

describe random movements that behave in a similar fashion.

Definition A Wiener Process $W(t)$ with $t \geq 0$ is a real-valued, continuous stochastic process with the following properties:

- $W(0) = 0$
- The function $t \to W(t)$ is almost surely everywhere continuous
- $W(t) - W(s) \sim N(0, t-s)$ for $(0 \leq s < t)$. Where $N(0, t-s)$ denotes the Gaussian distribution with mean 0 and variance $t - s$.
- Let $0 \leq s_1 < t_1 \leq s_2 < t_2$ then $W(t_1) - W(s_1)$ and $W(t_2) - W(s_2)$ are independent random variables, the same holds for n increments.

The variance is equal to,

$$var(W(t)) = \mathbb{E}[W(t)^2] - \mathbb{E}^2[W(t)] = \mathbb{E}[W(t)^2] - 0 = \mathbb{E}[W(t)^2] = t \tag{1.26}$$

As you can see from Figure 1.3, a Wiener Process can take on negative values with positive

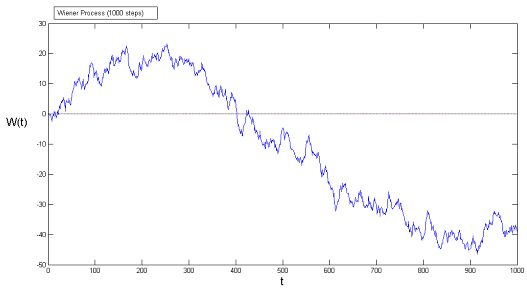


Figure 1.3: sample path Wiener process

probability. It we want to restrict the process from taking on negative values we can consider Geometric Brownian Motion (GBM). Which is the exponential of a standard Wiener Process. GBM is a log normally distributed stochastic process with independent increments. A log normal distribution is a continuous probability distribution of a random variable whose logarithm is normally distributed.

2 Black model

The simplest way to value swaptions is through the Black formula (Black-76) which has been introduced by Fisher Black in 1976. It is widely accepted as the standard market model to price swaptions. It is an extension of the Black-Scholes formula that is used for pricing stock options. The Black model depends on two parameters, the forward swap rate and its volatility. The reason for the wide acceptance of the Black model is twofold. Firstly, under the assumption of log normal distribution of the underlying short-rate, the price is theoretically correct. Second the volatility input for the Black model can be computed from market data. The forward swap rates are observed from the market, while the volatility is available on financial websites like Bloomberg or Thomson Reuters.

2.1 Black Formula

The Black formula is used to value a European swaption assuming that the underlying swap rate at the maturity of the option is log normally distributed. Consider a swaption where the holder has the right to pay a fixed rate s_k and receive LIBOR on a swap that will last n years starting in T years. We suppose that there are m payments per year under the swap and that the notional principal is L. We ignore day count issues and assume that every fixed payment on the swap is equal to $s_k \frac{L}{m}$. A swaption can be regarded as a single option on the forward swap rate with repeated payoffs. Each payment is the payoff from a call option on s_T with strike price s_k. Black formula gives the value of a swaption where the holder has the right to pay s_k as,

$$S_{payer} = \frac{L}{m} \sum_{i=1}^{mn} P(0, T_i)[s_0 N(d_1) - s_k N(d_2)] \tag{2.1}$$

with,

$$d_1 = \frac{ln(\frac{s_0}{s_k}) + \sigma^2 \frac{T}{2}}{\sigma \sqrt{T}} \tag{2.2}$$

$$d_2 = \frac{ln(\frac{s_0}{s_k}) - \sigma^2 \frac{T}{2}}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T} \tag{2.3}$$

The swaptions we consider in this thesis are at-the-money, meaning $s_0 = s_k$, substituting this into the formula yields,

$$d_1 = \frac{\sigma^2 T/2}{\sigma \sqrt{T}} = \frac{\sigma \sqrt{T}}{2} \tag{2.4}$$

$$d_2 = -\frac{\sigma \sqrt{T}}{2} = -d_1 \tag{2.5}$$

$\sum_{i+1}^{mn} P(0,T_i)$ can be regarded as the discount factor for the $m \times n$ payoffs. If we now define,

$$A = \frac{1}{m} \sum_{i=1}^{mn} P(0,T_i) \tag{2.6}$$

we get,

$$S_{payer} = LA[s_0 N(d_1) - s_k N(d_2)] \tag{2.7}$$

Likewise we can value a receiver swaption in which we receive fixed and pay LIBOR, the value of the receiver swaption is,

$$S_{receive} = LA[s_k N(-d_2) - s_0 N(-d_1)] \tag{2.8}$$

In appendix B you can find a worked-out example. For a complete proof of the Black formula we refer to [17]. In Appendix A we show how the Black-Scholes Formula leads to the Black Formula.

2.1.1 Delta Swaption

The delta for a ATM swaption priced through the Black model (see Section 2.1) is defined as,

$$\triangle_{payer} = N(d_1) \tag{2.9}$$
$$\triangle_{receiver} = N(-d_1) \tag{2.10}$$

This can be seen by taking the derivative in the Black Formula.

3 Models for the Term Structure of Interest Rates

The zero-curve, often referred to as the yield curve, is a plot at time t of compounded interest rates for all maturities. This curve is also called *the term structure of interest rates*. It measures the relationship between the yields on (default-free) securities that differ only in their term to maturity. Theories about the behavior of the term structure of interest rates date back to Fisher (1896). By offering interest rates for future periods, the term structure embodies the market anticipations of future events. A proper explanation of this term structure provides us a way to extract this information and to predict how changes in the underlying variables affect the yield curve.

3.1 Expectations Hypothesis

In its simplest form, this theory postulates that bonds are priced so that the implied forward rates are equal to the expected spot rates. This is generally characterized by one of the following propositions:

- The return on holding a long term bond to maturity is equal to the expected return on repeated investment in a series of the short-term bonds.
- The expected rate of return over the next holding period is the same for bonds of all maturities.

3.2 Liquidity Preference Hypothesis

Advanced by Hicks [9] , it asserts that risk aversion causes forward rates to be systematically greater than expected spot rates, usually proportional to increasing maturity. This term-premium is required to induce investors to hold longer-term securities. The underlying assumption for this is that investors prefer liquidity and invest funds for short periods of time. It is consistent with the empirical observation that yield curves tend to be upward sloping most of the time. Intuitively we can see that longer maturity investments are more risky than shorter ones. An investor lending money for a five-year term will demand a higher rate of interest than if he were to lend the same customer money for a five-week term. This is because the borrower may not be able to repay the loan over the longer time period as he could have gone bankrupt in the meanwhile. For this reason longer-dated yields should be higher than short-dated yields. We can consider this theory in terms of inflation expectations as well. Where inflation is expected to remain roughly stable over time, the market would anticipate a positive yield curve. However the expectations hypothesis cannot by itself explain this phenomenon, as under stable inflationary conditions one would expect a flat yield curve. The risk inherent in longer-dated investments, or the liquidity preference theory, seeks to explain a positive shaped curve. Generally borrowers prefer to borrow over as long a term as possible, while lenders will wish to lend over as short a term as possible. Therefore, as we first stated, lenders have to be compensated for lending over the longer term; this compensation is considered a premium for a loss in liquidity for the lender. The premium is increased the further the investor lends across the term structure, so that the longest dated investments will, all else being equal, have the highest yield.

3.3 Market Segmentation Hypothesis

Proposed by Culbertson [11] and others, it gives a different explanation for term premiums. It stresses that individuals have strong personal maturity preferences and that therefore bonds with different maturities trade in separate markets. It implies that there does not need to be a relationship between short-, medium- and long-term interest rates. From this perspective there is no reason for the term premiums to be either positive or to be increasing functions of time. Common sense dictates that this hypothesis has some limitations in the sense that you could view bonds of close maturities as close substitutes.

3.4 Preferred Habitat Hypothesis

Modigliani and Sutch [12] altered the market segmentation hypothesis by recognizing that if nearby yields differ sufficiently, participants will change maturities.

3.5 Ex-Ante, Ex-post

Roll [13] built and tested a mean-variance model which treated bonds symmetrically with other assets under a condition of market efficiency to relate ex-ante and ex-post concepts. If rationality requires that ex post realizations not differ significantly from ex ante views, then we can use statistical tests on ex ante propositions by using ex post data.

3.6 Equilibrium models

Merton (1974) introduced a general equilibrium term structure model by assuming that zero-coupon bonds follow a stochastic process. Many scientists have extended this idea and today an entire framework of equilibrium term structures exists. Vasicek (1977) assumed that the spot interest rate follows a diffusion process. Dothan went one step further and assumed the spot interest rate follows GBM. Cox, Ingersoll, and Ross (1981) found theoretical justification for the local expectations hypothesis. They stressed that returns over the same holding period for similar bonds with different maturities should be the same.

In general, equilibrium models start with assumptions regarding the behavior of basic economic variables. For example Cox, Ingersoll and Ross assumed a single good economy with linear production opportunities and stochastic development of technology. From this they derived the stochastic process of interest rates.

3.7 Vasicek Model

One of the earliest models for the term structure of interest rates was developed by Oldrich Vasicek, the Czech mathematician published a paper on the evolution of the yield curve in 1977. His model is based on the evolution of the short-rate. He proposed that $r(t)$ satisfies the following stochastic differential equation (SDE),

$$dr(t) = a(\beta - r(t))dt + \sigma dW_t, \quad r(0) = r_0 \tag{3.1}$$

where β, a and σ are positive constants and $r(t)$ is the current level of the interest rate. The parameter β is the long term mean. One of the features of the model is that it exhibits mean reversion. For example, if the current interest rate r is lower than the long term value β ($\beta > r_t$), the drift will be positive and the rate will go up on average. Likewise, if $\beta < r_t$ the drift becomes negative so that the interest rate will be pushed down to β. The coefficient a is thus the speed with which the current interest rate is pushed toward the long term mean. This feature is attractive because without it, interest rates could drift permanently up, in the same way stock prices do and this does not relate with what we observe in the market. This particular type of stochastic process is referred to as an Ornstein-Uhlenbeck process [18]. This process has a bounded variance and a stationary probability distribution, in contrast with the earlier mentioned Wiener Process. The variance is given by,

$$var(r(t)) = \frac{\sigma^2}{2a}(1 - e^{-2a(t-s)}) \tag{3.2}$$

As we can see as $t \to \infty$ the long term variance of $r(t)$ is given by,

$$\lim_{t\to\infty} var(r(t)) = \frac{\sigma^2}{2a} \tag{3.3}$$

Note that the variance depends on a en σ in opposing directions. If we increase σ the volatility increases, while if we increase a the variance decreases.

The model is simple and elegant, but in its simplicity lies also the following weaknesses,

- It is impossible to fit the initial yield curve exactly
- There is one volatility (diffusion) parameter σ for calibration, as a result fitting the volatility of the term structure becomes impossible
- With non-zero probability, rates may become negative.

To address the first weakness we consider a slight extension of the model.

3.8 Hull and White One Factor Model

Hull and White in 1990 proposed the following model for the short rate (notation similar to Hull [6]),

$$dr = a\left(\frac{\theta(t)}{a} - r\right)dt + \sigma dW_t \tag{3.4}$$

With a and σ constants. At time t the short-rate reverts to $\frac{\theta(t)}{a}$ with speed a. $\theta(t)$ is calculated from today's term structure,

$$\theta(t) = \frac{\delta F(0,t)}{\delta t} + aF(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) \tag{3.5}$$

The Hull-White model is a no arbitrage model that is able to fit today's term structure. The model assumes the short-rate is normally distributed, as a consequence forward interest rates can be negative with positive probability, just like the Vasicek model. Despite the simple structure of the Hull-White model, most financial derivatives cannot be priced by means of explicit formula, fortunately for swaptions we can price them directly by replicating the payoffs of the swaption through a portfolio of zero coupon bonds. By means of the law of one price we conclude that the price of this portfolio must be equal to the price of the swaption under the assumption of no-arbitrage. This makes it convenient to calibrate the parameters of the model using swaption data from the market.

3.8.1 Bond prices

Hull and White [4] showed that bond prices at time t in the Hull-White model are given by,

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{3.6}$$

$$A(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left(B(t,T)F(0,t) - \frac{\sigma^2}{4a}B(t,T)^2(1-e^{-2at})\right) \tag{3.7}$$

$$B(t,T) = \frac{1-e^{-a(T-t)}}{a} \tag{3.8}$$

3.8.2 Options on Bonds

The price at time $t = 0$ of a European call option that matures at time T on a zero-coupon bond with principal L and strike price K maturing at time s is under the Hull-White model given by,

$$LP(0,s)N(h) - KP(0,T)N(h-\sigma_p) \tag{3.9}$$

The price at time $t = 0$ of a European put option that matures at time T on a zero-coupon bond maturing at time s is.

$$KP(0,T)N(-h+\sigma_p) - LP(0,s)N(-h) \tag{3.10}$$

With σ_p and h defined as,

$$h = \frac{1}{\sigma_p} ln\left(\frac{LP(0,s)}{P(0,T)K}\right) + \frac{\sigma_p}{2} \tag{3.11}$$

$$\sigma_p = \frac{\sigma}{a}\left(1-e^{-a(s-T)}\right)\sqrt{\frac{1-e^{-2aT}}{2a}} \tag{3.12}$$

3.8.3 Options on Coupon Bearing Bonds

In a one-factor model of r, all bond prices are decreasing functions of r. This means that when r decreases all zero-coupon bonds move up in price and vice versa. As a result, a one factor model allows a European option on a coupon bearing bond to be expressed as the sum of European options on zero coupon bonds by Jamshidian's decomposition [15]. The procedure is as follows:

- Determine the critical value r^* of the yield r for which the price of the coupon-bearing bond equals the strike price of the option on the bond at option maturity.

- Calculate the prices of options on the zero-coupon bonds that replicate the coupon-bearing bond. Set the strike price of each option equal to the value the corresponding zero-coupon bond will have at time T when $r = r^*$.

- Sum all the option prices in the above step to find the price of the swaption.

Consider a new coupon bearing bond that pays 5% annually over a notional principal L equal to 100, maturing two years from now. There will be two payments namely 5 after year one and 105 after year two (principal + interest). These payments can be regarded as two zero coupon bonds, namely a one year zero coupon bond with a principal of 5, and in the same way a zero coupon bond with principal 105 maturing two years from now. On these two zero coupon bonds we can get a put/call option price through equations (3.9) and (3.10). The strike prices of these zero coupon bonds are determined by the method described above. The price of our swaption is then equal to the sum of all the individual options on the different zero coupon bonds, that together make up the coupon bearing bond.

4 Calibration

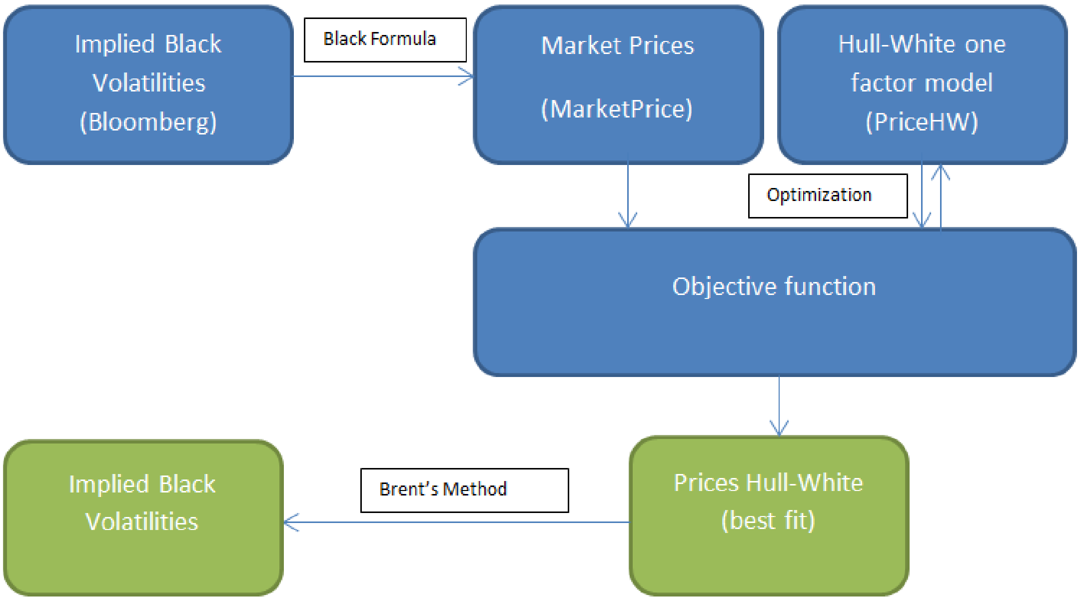


Figure 4.1: Flowchart of the calibration process

4.1 Implied Black Volatility to Market Price

Now that we have developed the Hull-White model we move on to the process of calibration in which we determine the parameters of the Hull and White one factor model. We focus on the implementation details and further practical aspects of applying the theory in practice. In the Hull-White model, the parameter $\theta(t)$ is chosen to make the model consistent with the initial term structure. Determining the remaining model parameters $\hat{a}$ and $\hat{\sigma}$ is to compare our n calibration instruments, in our case the market prices of the swaptions, to the theoretical prices implied by the parameters $\hat{a}$ and $\hat{\sigma}$. The data(format) we were provided with can be found in Figure 4.2. As you can see it is a matrix that contains the implied black volatilities for a set of swaptions given the market price. The implied black volatility of a swaption contract for the market price, is the volatility which, when used in Black's Formula will return the market price. In the first column of the matrix you find the different expiries of the swaptions and on the first row the different maturities of the underlying swap. At the bottom we find details of the swaption on a swap that starts in 1 year and matures 5 years later. The black volatility implied by the market price of this swaption is 50.32 basis points (bps), and the ATM swap rate is 1.1878%. From this we can compute the market prices ($MarketPrice$) through the formulas

(2.7) or (2.8). We determine the ATM swap rates ourselves from today's yield curve through the methodology developed in Section 1.11.

Figure 4.2: Swaption volatilities from Bloomberg terminal on 23 may 2013

4.2 Market Price to calibrated Hull-White model

Now that we have determined the market prices (*MarketPrice*), we want to find the a and σ in our Hull-White model that best fit the market prices of the swaptions. This is an optimization problem. Let *PriceHW* denote the price matrix derived from the calibrated Hull-White model. To assess the goodness of fit of a given a and σ we use the following objective function,

$$obj = \sum_{k=1}^{n} (MarketPrice_k - PriceHW_k)^2 \tag{4.1}$$

We minimize this objective function using optimization algorithms till our objective function no longer improves significantly. The result is a $\hat{a}$ and $\hat{\sigma}$ that produce the best fit for our swaptions.

A possible shortcoming of the objective function is that it minimizes the quadratic difference in price. For us to find mispriced swaptions it is more natural to look at the deviation in price in terms of percentage. Therefore we could argue it makes more sense to minimize for the sum of the relative quadratic deviation in price compared to the market price of the swaptions,

$$obj_{alt} = \sum_{k=1}^{n} \left(\frac{MarketPrice_k - PriceHW_k}{MarketPrice_k} \right)^2 \tag{4.2}$$

We will optimize for both criteria and discuss the results. The chosen optimization algorithm is the Broyden-Fletcher-Goldfarb-Shanno (BFGS) Quasi-Newton algorithm. In Quasi-Newton methods, the Hessian matrix of second derivatives doesn't need to be evaluated directly. Instead, the Hessian matrix is approximated using gradient evaluations. For more insight into optimization algorithms and an illustrative example, we refer to Appendix D.

4.3 Implied Black volatilities from Hull-White swaption price

Once we have the calibrated Hull-White parameters $\hat{a}$ and $\hat{\sigma}$ we compute the prices of the swaptions under this calibrated model. We can compare the market prices to the Hull-White prices but we can also can compare the implied black volatilities of the market prices and the black volatility implied by the Hull-White prices. We will evaluate both. The process of finding the implied Black volatility of the Hull-White prices can be done with root-finding algorithms like bisection, interpolation, Newton-Rhapson or Brent's method. For robustness and efficiency we choose Brent's method, which is a combination of bisection, secant and inverse quadratic interpolation methods.

5 Results

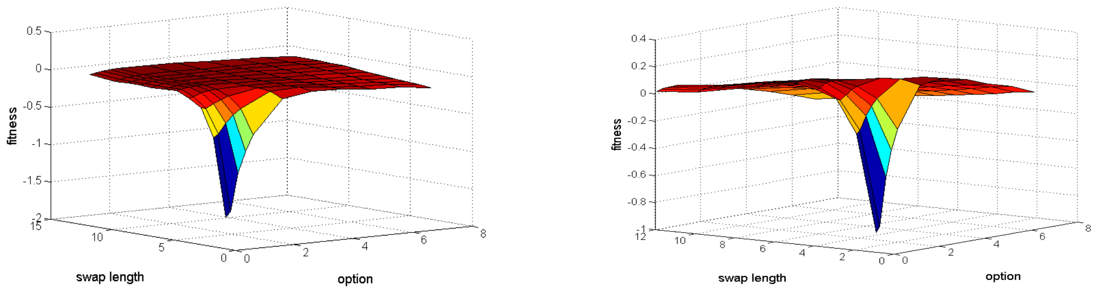
This chapter is dedicated to the results. Table E.1 shows the prices derived from the Black Formula for payer swaptions (2.7) on page 17. The calibrated Hull and White one factor model parameters for objective functions (4.1) and (4.2) can be found in table 5.1.

$\hat{a}$	0.023	$\hat{a}_{alt}$	0.042
$\hat{\sigma}$	0.044	$\hat{\sigma}_{alt}$	0.030

Table 5.1: Hull-White parameters under both objective functions

The market prices derived from the Black Formula can be found in the Appendix Table E.1. The prices derived from the Hull-White model for both objective functions can be found in Appendix Table E.2 and E.3. Figure 5.1a shows the 3D-surface of the relative price differences calculated through the following formula,

$$\frac{MarketPrice - PriceHW}{MarketPrice} \tag{5.1}$$



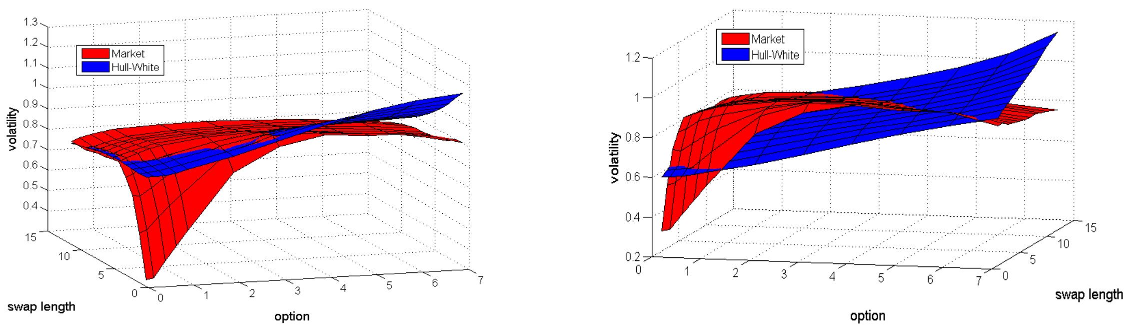
(a) Deviation under (4.1)

(b) Deviation under (4.2)

Figure 5.1: Deviation in price between market price and Hull-White price under the two objective functions

As you can see the Hull-White price for the swaption expiring in one month with a maturity of one year for the underlying forward swap differs over 150% from the market price for optimization under the first objective function. Given that the price for relatively short options is small a tiny difference in price in absolute terms leads to a relatively large difference in relative price. So in this sense the steep curvature near the origin can be explained. On the other hand the calibrated Hull-White model seems to price relatively well the longer the expiry on the option and maturity of the swap. In Figure 5.1b we see the relative price difference for the second objective function (4.2). Because of the penalty for relative deviation the amount of deviation towards the short expirations on a swap with short maturity is far smaller, but as a result it now does not price the longer expiration and longer swaps in line with the market.

We now consider the market volatilities compared to the implied volatilities for the Hull-White prices, a surface plot can be found in Figure 5.2a. Now we see that by fitting the general shape of the market volatilities it does severely overstate the volatility of the swaptions with short expiration and short maturities for the swaps. We compare it with the surface under the relative objective function (4.2) illustrated in 5.2b. It seems that by trying so hard to limit the amount of relative deviation the shape of the Hull-White volatility surface it does not resemble the shape of the market volatilities at all. On the other hand the Hull-White volatility surface for the first criteria seems to follow the market volatilities surface quite well, but fails on the short end of expiration/maturities. We could argue that optimizing for relative deviation cripples the overall optimization process because of the big difference in market price between swaptions with long expiration/maturities versus short expirations/maturities. The penalty is so high for tiny absolute differences in price between low market prices and Hull-White prices that the optimization process is forced to fit the low prices as well as possible and thereby sacrifices the ability to fit the overall shape of the market volatility surface.



(a) volatility surface under (4.1) (b) volatility surface under (4.2)

Figure 5.2: surface plot of the market volatilities vs the implied volatilities from PriceHW

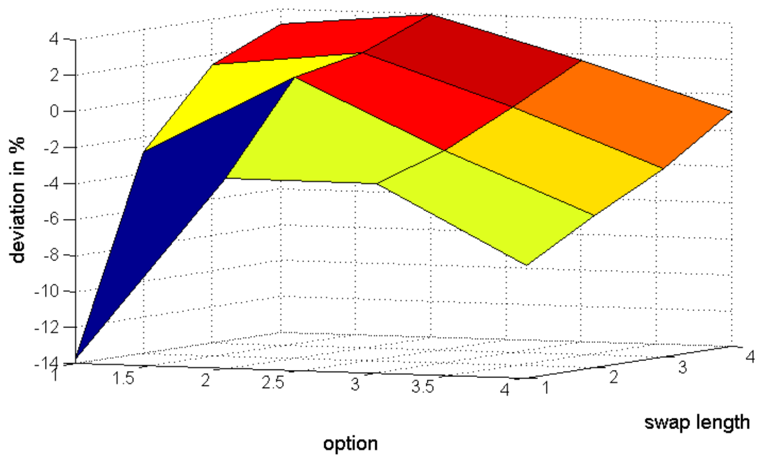


Figure 5.3: deviation in percentage between the Market price and Hull-White price

To find a mispriced swaption we will consider a sub matrix of 16 swaptions prices as calibration instruments, namely maturities 1Y, 2Y, 3Y and 4Y for option expiration with 1Y, 2Y, 3Y, and 4Y as the maturtiies of the underlying forwards swaps. The methodology is as followed, we leave out one swaption and calibrate the Hull-White model parameters on the remaining swaptions. With the fitted parameters we then price the left out swaption and compare it to the market price. We do this for all swaptions and the swaption that deviates the most from the market price percentage wise will be regarded as the most mispriced swaption. The result can be found in Figure 5.3.

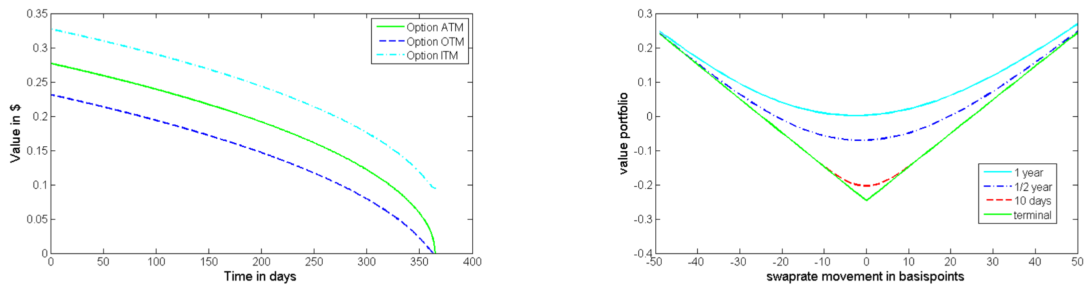
As you can see the swaption that expires one year from now on a one year swap is the most mispriced as it deviates approximately 14%. It is important to note that just like the calibrated model for the entire swaption matrix the one with the shortest expiration date and maturity of the swap is the most mispriced. This swaption is by far the cheapest compared to the other 15 market prices, so again a tiny price difference in absolute terms leads to a big difference in relative terms.

5.1 Trading Strategy

We found our mispriced swaptions, now what? How could we possibly profit from this perceived mispricing? The simple way of doing this would be just selling the overpriced swaption and wait, it is overpriced and we should be making money in the long run by selling it. Likewise if the swaption is underpriced we could buy the swaption and wait in similar fashion. However this strategy is not without risk since the market could turn against us. Ultimately we would like a strategy in which we profit from the mispricing without any further risk (if possible). The way to do this is through volatility arbitrage/trading. If the swaption is overpriced we in essence think that the actual future volatility of the underlying is lower than the implied volatility given by the price today. In order to benefit from this we could create a delta-neutral portfolio but not vega-neutral in the sense that the value of our portfolio changes when the volatility of the underlying forward swap rate changes. Traders would say: "we want to buy volatility low and sell high". When we state that we are long in a volatility trade we anticipate the realized volatility to be higher than the implied volatility by the market price. Vice versa when we are short in a volatility trade we anticipate the realized volatility to be lower than implied by the market price. There are infinite ways of trading volatility but we will outline some major ones and outline what the outcomes are for different scenarios. The swaptions considered in the trading strategies have all the same principal 100, and with movements in the forward swaprate we mean parallel movements of the entire yield curve.

5.1.1 Theta decay

In an ideal situation we would do our volatility trade and instantaneously the market realizes we are right and corrects the prices. If we would sell directly afterwards, we would make a profit equal to the difference between the initial market price and our theoretical price under the Hull-White model. In reality the market will take some time to adjust or might not adjust at all. In the mean time we lose value in our option through the passing of time. Recall that the greek associated with the time value of a swaption is called Θ, which is the derivative of the swaption value to time. To illustrate, consider an at-the-money (ATM), out-of-the-money (OTM) and in-the-money (ITM) swaption that has one year left till the option expiries, we calculate the value of this swaption everyday till it expires, ceteris paribus. The result is shown in Figure 5.4a.



(a) Value option as time till expiration decreases for ATM, ITM and OTM swaptions

(b) Value long straddle with different times left till expiry

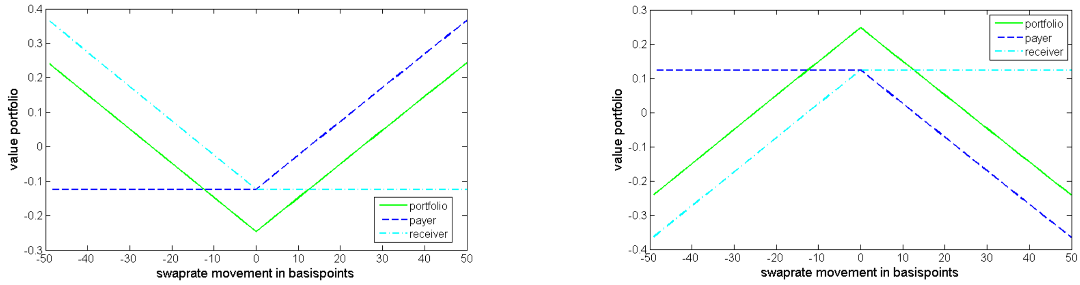
Figure 5.4: Theta decay for a receiver swaption on the left side, on the right side we see the value of a straddle that has different times till expiration left

Observe that the swaption slowly but steadily loses value as it comes closer to expiration, near expiration we see a very steep decline in value. Traders that are long in volatility talk of "bleeding through theta", as it represents a loss for holding the swaptions if the price of the underlying forward swap rate does not move in the desired direction. If we anticipate high volatility, we in essence expect large market moves to be perfectly in the realm of likely possibilities. Trading on the value of the forward swap rate till expiration is also possible and will be outlined in the following sections.

5.1.2 Straddle strategy long-short

If we think that the implied volatility of an ATM swaption is too low we could buy a long straddle. A long straddle is a financial strategy that consists of buying both an ATM call (payer) swaption and an ATM put (receiver) swaption with the same maturity. The ATM straddles are among the most liquid volatility instruments. The reason for this is that the combination of both a call (payer) and a put (receiver) gives a double exposure to the volatility and since we do not have a view on the direction of the forward swap rate we want exposure in both directions. The payoff as a function of movements in the swap rate can be seen in Figure 5.5a. We see that the long straddle has quite significant costs in terms of option prices, the move of the swap rate has to be quite big in either direction to offset our initial costs. On the upside we can have only a limited downside while having an unlimited upside if the swap rate moves significantly in either direction. To see how the value of a long straddle evolves till expiration look at Figure 5.4b.

Likewise if we think that the volatility of the swaption is too high we could buy a short straddle. The term buying a short straddle is a bit ambiguous, since a short straddle consists of selling both an ATM payer and receiver swaption with the same maturity. The payoff is exactly inverse that of the straddle long as can be seen in Figure 5.5b. The downside of this strategy is the possible unlimited loss while having only a limited upside. We need the swaprate to be in a narrow range of the initial forward swap rate for us to make a profit at the expiration date of the option. Please note that with this strategy we sell the swaptions so that we receive the initial option price instead of paying it as if we were to buy a long straddle.



(a) long straddle (b) short straddle

Figure 5.5: The payoff of a straddle as a function of the movement in the swap rate

For the swaption under the Black Formula the delta for the receiver swaption is equal to $N(-d_1)$, and for the payer $N(d_1)$ (see section 2.1.1). Recall that for ATM swaptions $d_1 = \sigma\frac{\sqrt{T}}{2}$. The total delta for a portfolio consisting of a payer and receiver swaption is equal to,

$$N(-d_1) - N(d_1) = 1 - 2N(d_1) = 1 - 2N(\sigma\frac{\sqrt{T}}{2}) \tag{5.2}$$

To have no exposure to the underlying swaprate we require $d_1 = 0$, which is not the case for ATM swaptions. We could accomplish $d_1 = 0$ by setting the strike of the straddle as,

$$s_k = s_0 e^{\sigma^2 \frac{T}{2}} \tag{5.3}$$

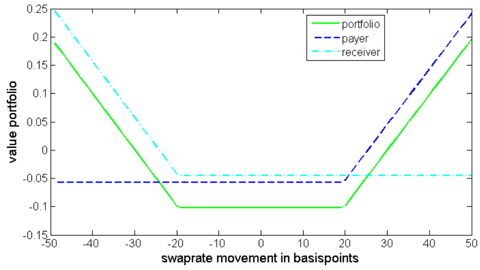
This can be easily seen by evaluating our formula for d_1,

$$\begin{aligned} d_1 &= \frac{ln\left(\frac{s_0}{s_0 e^{\sigma^2 \frac{T}{2}}}\right) + \sigma^2\frac{T}{2}}{\sigma\sqrt{T}} && (5.4)\\ &= \frac{-\sigma^2\frac{T}{2} + \sigma^2\frac{T}{2}}{\sigma\sqrt{T}} = 0 && (5.5)\end{aligned}$$

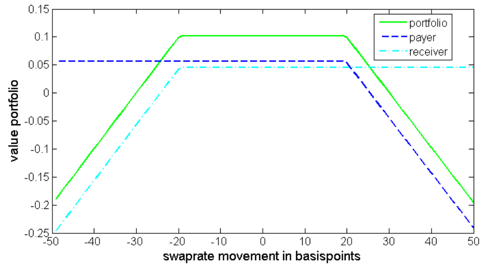
This straddle has exposure to the volatility without local exposure to the underlying.

5.1.3 Strangle strategy long-short

As a variation of the straddle we could consider a strangle. A long straddle consists of buying an OTM receiver swaption and an OTM payer swaption.



(a) long strangle



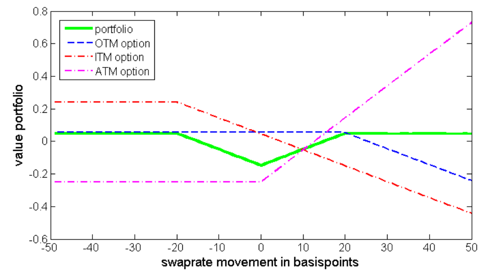
(b) short strangle

Figure 5.6: The payoff of a strangle as a function of the movement in the swap rate

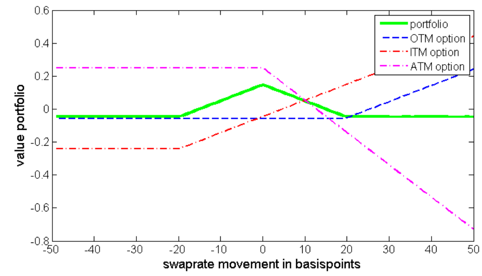
The strangle has a wider range of swap rates that give our portfolio a positive return compared to the straddle which has a positive value in a quite narrow range of swap rates. Further more the strangle has a lower initial cost because both options are slightly out of the money and therefore cheaper. In contrast to the short straddle, the short strangle has a smaller possible gain. For both the short strangle and short straddle the possible loss is unlimited. To overcome this we consider the butterfly strategy.

5.1.4 Butterfly strategy long-short

The name butterfly strategy originates form the shape of the payoff function as displayed in Figure 5.7 (green line). Creating a portfolio that replicates the long butterfly payoff can be done in multiple ways, for one you could buy an ITM payer swaption, sell two ATM payer swaptions and buy one OTM payer swaption. The short butterfly is the exact other way around and consists of selling an ITM payer swaption, buying two ATM payer swaptions and selling one OTM payer swaption. The initial costs are small compared to the straddle and it has limited downside for both long/short butterflies, the upside potential is also limited.



(a) long butterfly



(b) short butterfly

Figure 5.7: The payoff of a butterfly as a function of the movement in the swap rate

6 Conclusion

The Hull-White model assumes the short-rate to be normally distributed, where the Black-76 model assumes a log normal distribution. Obviously not both can be true at the same time, or even both can turn out to be false as empirical evidence suggests. There is no consensus among practitioners what the actual distribution of the short-rate is, but the option market is largely built around variations of Black's formula. Also traders prefer models that reproduce the exact market prices, something that the Hull-White model is not capable of. Both models have their merits, Black's Formula for being Black-Scholes-esque, something practicioners are very accustomed to as well as the few parameters that are needed to calculate the price. Hull-White among the short rate models for having analytic formula's for zero-coupon bonds and european options, as well as being able to perfectly fit today's term structure. In recent years the volatility smile effect has entered the interest rate market. This usually means that OTM volatilities are higher than the corresponding ITM ones. This makes the Black Formula flawed in the sense that it assumes the same volatility for all strike prices. This volatility smile directly affects volatilities associated with option contracts like swaptions.

7 Discussion

The question remains whether or not the deviating swaptions are truly mispriced and if so, does our proposed trading strategy produce significant returns. This could be the aim of further research. If we were to test volatility trading strategies we would be interested in which model produces the most realistic delta for the portfolio. Henrard [10] did a hedging contest where he created delta neutral portfolios consisting of swaptions and swaps based on the different models and measured the standard deviation of the returns. If hedging was done perfectly it would result in a standard deviation of zero. Henrard found in his research that the Hull-White model outperformed the Black model over the 3 year sample by approximately 30%. Both models were not able to produce a standard deviation of zero, meaning that the theoretical delta neutral portfolios under the assumed models didn't turn out to be empirically delta neutral portfolios.

Appendices

A Black-Scholes and Black Formula derivation

Black's Formula has its roots in the Black-Scholes Formula so we will start there. The Black and Scholes option pricing formula published in 1973 was of huge importance for the financial world. It gave an explicit formula for the fair value of an European option on a stock given the following "ideal market" assumptions for the stock and option,

- The risk-free rate earned on a riskless asset is constant through time.
- The stock price follows a random walk in continuous time with a variance proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is lognormal. The variance rate of the return on the stock is constant.
- The stock does not pay dividend or other distribution.
- The option can only be exercised at maturity.
- There are no transaction costs in buying or selling the stock or the option.
- There are no penalties to short selling. A seller wo does not own a security will simply accept the price of the security from a buyer, and will agree to settle with the buyer on some future date by paying him an amount equal tothe price of the security on that date.

We assume that the stock S follows the following stochastic differential equation,

$$dS = \mu S dt + \sigma S dW \tag{A.1}$$

Where $S(t)$ denotes the value at time t, and $W(t)$ is a Wiener process. Suppose V is the price of a call option contingent on S. The value of V at maturity T is known as $V = \max\left(S(T) - K, 0\right)$, with K the strike price. To find its value at an earlier time we need to find how V evolves as a function of S and t. Hence from Ito's Lemma,

$$dV = \left(\frac{\delta V}{\delta S}\mu S + \frac{\delta V}{\delta t} + \frac{1}{2}\frac{\delta^2 V}{\delta S^2}\sigma^2 S^2\right) dt + \frac{\delta V}{\delta S}\sigma S dW \tag{A.2}$$

We can get rid of the Wiener process by building a portfolio that consist of selling one call option and buying $\frac{\delta V}{\delta S}$ stock. This portfolio is essentially riskless (delta-neutral) and should therefore earn the riskfree rate, otherwise we would have an opportunity for arbitrage. Finally we find the Black-Scholes-Merton differential equation,

$$\frac{\delta V}{\delta t} + rS\frac{\delta V}{\delta S} + \frac{1}{2}\sigma^2\frac{\delta^2 V}{\delta S^2} = rV \tag{A.3}$$

The solution to this PDE, with terminal value $max\,(S(T)-K,0)$ is known as the Black-Scholes formula and is for the value of a call option at time t on stock S with strike price K given as,

$$S(t)N\,(d_1) - Ke^{-r(T-t)}N\,(d_2) \tag{A.4}$$

with,

$$d_1 = \frac{ln\left(\frac{S(t)}{Ke^{-r(T-t)}}\right) + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{A.5}$$

$$d_2 = \frac{ln\left(\frac{S(t)}{Ke^{-r(T-t)}}\right) + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{A.6}$$

The Black-Scholes formula can be used to value call options on futures. Assume $t < T < T_1$ and consider a derivative which at time T both gives a new future contract in the asset with delivery at the time T_1 and an amount of the size,

$$Y = max\left(0, S(T)e^{r(T_1-T)} - K\right) \tag{A.7}$$

This together with the fact that a future contract is valueless at the start we get,

$$Y = e^{r(T_1-T)}max\left(0, S(T) - Ke^{-r(T_1-T)}\right) \tag{A.8}$$

Define $\hat{S}_{T_1}(t) = S(t)e^{r(T_1-t)}$, the contract at time $t < T$ then has the value,

$$e^{r(T_1-T)}\left(S(t)N\,(d_1) - Ke^{-r(T_1-T)}e^{-r(T-t)}N\,(d_2)\right) \tag{A.9}$$

$$= e^{-r(T-t)}\left(\hat{S}_{T_1}(t)\,(d_1) - KN\,(d_2)\right) \tag{A.10}$$

$$\tag{A.11}$$

d_1 and d_2 are given by,

$$d_1 = \frac{ln\left(\frac{S(t)}{Ke^{-r(T_1-T)}}\right) + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = \frac{ln\left(\frac{\hat{S}_{T_1}(t)}{K}\right) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \tag{A.12}$$

$$d_2 = \frac{ln\left(\frac{S(t)}{Ke^{-r(T_1-T)}}\right) + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = \frac{ln\left(\frac{\hat{S}_{T_1}(t)}{K}\right) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \tag{A.13}$$

This result is known as the Black Formula and is used to price bond options like caps/floors and swaptions.

B Black Formula Example

Suppose that the LIBOR yield curve is flat at 5% per annum with continuous compounding. Consider a swaption that give the holder the right to pay a fixed rate s_k of $5,2\%$ in a 1-year swap starting 2 years from today. We assume that the swap rate volatility σ is equal to 15%. Payments are made semi-annually and the (notional) principal L is 100\$. Applying Black formula yields,

$$A = \frac{1}{2}[e^{-0.05*2.5} + e^{-0.05*3}] = 0.8716$$

the 5% per annum with continuous compounding translates to 5.063% with semi-annually compounding. We find $s_0 = 0.05063$, $s_k = 0.052$, $T = 2$ and $\sigma = 0.15$, we now have all the ingredients for the Black formula,

$$d_1 = \frac{ln(\frac{s_0}{s_k}) + \sigma^2\frac{T}{2}}{\sigma\sqrt{T}} = -0.0198$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.2319$$

From equation (2.7) the value of the payer swaption is,

$$100\$ \times 0.8716 \times [0.05063 \times N(-0.0198) - 0.052 \times N(-0.2319)] = 0.321\$$$

C Implied Black Volatility

Suppose that the LIBOR yield curve is flat at 5% per annum with semi-annual compounding. Consider an at-the-money swaption that gives the holder the right to pay a fixed rate s_k of 5% in a 1-year swap starting 2 years from today. We assume that the principal L is 100. The price derived from the Hull White model is 0.3$. The implied Black volatility turns out to be 12.22% through our root finding algorithm. To see this is indeed the case we evaluate the Black Formula,

$$A = \frac{1}{2}[e^{-0.05*2.5} + e^{-0.05*3}] = 0.8716$$

We find $s_0 = 0.05$, $s_k = 0.05$, $T = 2$ and $\hat{\sigma} = 0.1222$,

$$d_1 = \frac{ln(\frac{s_0}{s_k}) + \hat{\sigma}^2 \frac{T}{2}}{\hat{\sigma}\sqrt{T}} = 0.0864$$

$$d_2 = d_1 - \hat{\sigma}\sqrt{T} = -0.0864$$

From equation (2.7) the value of the payer swaption is,

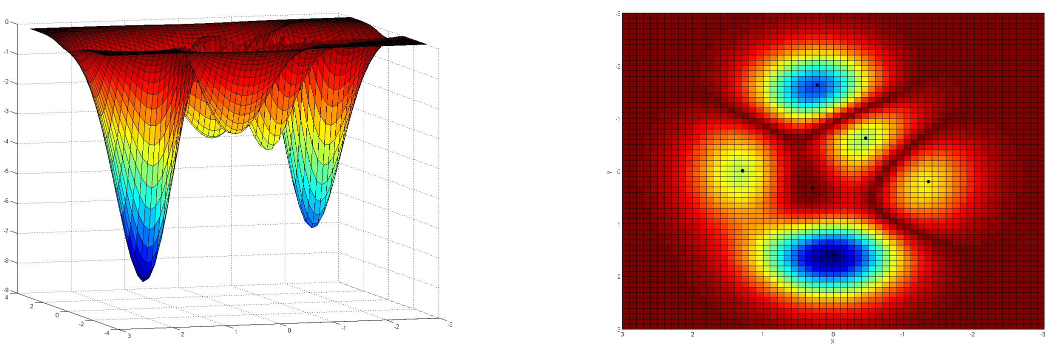
$$100\$ \times 0.8716 \times [0.05 \times N(0.0864) - 0.05 \times N(-0.0864)] = 0.3\$$$

D Optimization

In this thesis we are faced with the calibration of the Hull-White one factor model. To calibrate the model we minimize the pricing errors of swaptions compared to the theoretical price derived from the model parameters. In absence of an explicit formula to find the parameters that best fit our market prices, we need to use an optimization algorithm to find the values of the parameters that produce the best fit. While many algorithms exist for this purpose, all have their strengths and weaknesses. One issue that we must keep in mind is the basins of attraction of a specific optimization algorithm, which means that given a specific initial condition chosen from the solution space leads to the same attractor. For *deterministic algorithms* a given starting condition x_0 within the basin of attraction ψ, deterministic algorithms converge to the same local optimum. While this statement is rather abstract, we will show the concept by means of an example. We start by showing the basins of attraction for a deterministic algorithm, applied on the following formula,

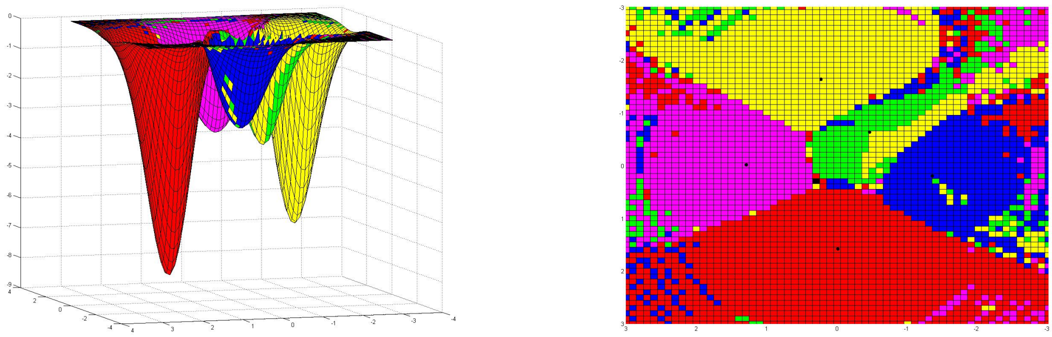
$$f(x, y) = -\left| 3(1-x)^2 e^{-x^2-(y+1)^2} - 10(\frac{x}{5} - x^3 - y^5) e^{-x^2-y^2} - \frac{1}{3} e^{-(x+1)^2-y^2} \right| \tag{D.1}$$

The function seems rather complex and has several local minima and we strive to map the corresponding basins of attractions for a deterministic algorithm that uses hessian approximation (BFGS) to move into the direction of steepest descent. We aim to visualize the several subspaces that lead to the same local minimum. In Figure D.1a we see the surface of the function we wish to minimize. To get a better feel for the function (D.1) we display the heat map. Figure D.1b contains the heat map, where warmer colors represent higher values of the objective function and colder colors represent lower values. To accurately display the different basins of attraction we divide the entire space into small squares and determine to what local minima it converges.



(a) surface plot (b) heatmap

Figure D.1: The 3D representation and heatmap of the objective function (D.1)



(a) 3D (b) 2D

Figure D.2: Basins of attraction (2D/3D) for the objective function (D.1)

The result can be found in Figure D.2a, it draws an accurate picture of the basins of attraction. We also make the following observation, some blocks have a deviating color compared to the blocks surrounding it. This anomaly, becomes even more apparent when we look at the two dimensional view perpendicular to the x-plane. As can be seen from Figure D.2b, there are clear borders between distinct basins of attractions. However, within each attraction basin there is an occasional exception, where squares differ from the surrounding blocks as noted before. This is a specific characteristic of the algorithm and each algorithm has its own anomalies, but the concept of basins of attraction applies to all kind of algorithms. The most important thing to take away from this example is that if we want to find the global minima (optimum), we need to find the corresponding basin of attraction. Typically researchers solve this by using multiple starting values through the entire solution space, or use hybrid algorithms, implementing robust probabilistic algorithms to find the basin of attraction and subsequently use the output as a starting value for a deterministic algorithm that utilizes the gradient to accurately find the optimum.

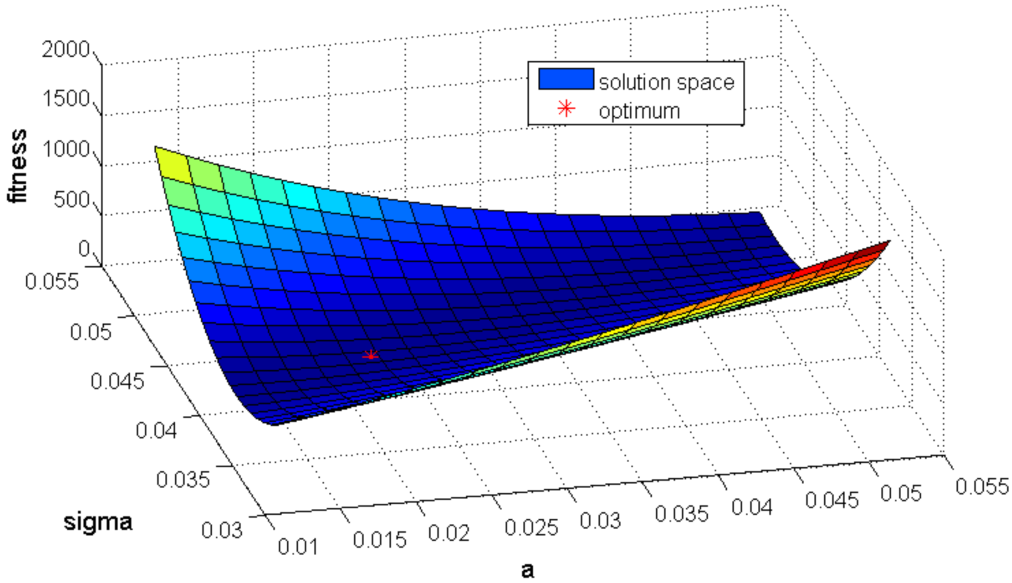


Figure D.3: solution space for the Hull-White parameters under objective function (4.1)

Now to the task at hand, calibrating the Hull-White model. In Figure D.3 you can see the (partial) solution space for our data under objective function (4.1). For $\hat{a} = 0.023$ and $\hat{\sigma} = 0.043$ it reaches its optimum, in the diagonal direction the objective function value looks flat, but by evaluating the matrix we conclude that this is not the case and that we indeed found an optimum. The surface of our solution space does not show any erratic behavior and looks well-behaved, this gives us faith that we indeed found the optimal estimates for a and σ.

E Tables & Figures

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	12Y
1M	0.19	0.62	1.17	1.65	2.14	2.47	2.77	3.07	3.34	3.58	4.14
3M	0.34	1.09	1.98	2.87	3.72	4.29	4.82	5.34	5.81	6.24	7.22
6M	0.59	1.67	2.89	4.17	5.25	6.05	6.82	7.55	8.20	8.81	10.17
1Y	0.82	2.17	3.64	5.11	6.39	7.36	8.31	9.19	9.97	10.73	12.35
2Y	1.07	2.61	4.25	5.83	7.26	8.39	9.50	10.50	11.39	12.28	14.10
3Y	1.98	4.09	6.04	7.79	9.53	11.07	12.49	13.77	15.05	16.23	18.50
4Y	2.41	4.79	6.90	8.91	10.76	12.51	14.11	15.68	17.13	18.42	21.02
5Y	2.59	5.00	7.28	9.41	11.38	13.25	15.08	16.79	18.34	19.80	22.32
6Y	2.62	5.10	7.45	9.64	11.64	13.65	15.57	17.31	19.00	20.58	22.99
7Y	2.67	5.19	7.55	9.74	11.84	13.88	15.75	17.56	19.30	20.71	23.14
8Y	2.61	5.11	7.42	9.64	11.73	13.70	15.60	17.45	18.99	20.43	22.86

Table E.1: Market prices

	1Y	**2Y**	**3Y**	**4Y**	**5Y**	**6Y**	**7Y**	**8Y**	**9Y**	**10Y**	**12Y**
1M	0.36	0.71	1.06	1.41	1.75	2.09	2.44	2.77	3.11	3.43	4.08
3M	0.61	1.22	1.82	2.42	3.01	3.60	4.19	4.77	5.34	5.91	7.02
6M	0.86	1.71	2.56	3.40	4.23	5.06	5.87	6.68	7.48	8.27	9.82
1Y	1.05	2.08	3.11	4.13	5.14	6.14	7.13	8.11	9.08	10.04	11.91
2Y	1.20	2.38	3.56	4.72	5.88	7.03	8.16	9.29	10.39	11.48	13.61
3Y	1.64	3.27	4.88	6.48	8.06	9.63	11.17	12.69	14.18	15.64	18.48
4Y	1.95	3.88	5.80	7.70	9.58	11.42	13.24	15.02	16.77	18.48	21.75
5Y	2.19	4.36	6.51	8.64	10.74	12.80	14.81	16.79	18.71	20.59	24.17
6Y	2.38	4.75	7.09	9.39	11.66	13.88	16.06	18.17	20.23	22.22	26.00
7Y	2.54	5.07	7.56	10.01	12.41	14.76	17.05	19.28	21.42	23.51	27.40
8Y	2.68	5.33	7.94	10.51	13.02	15.47	17.85	20.15	22.38	24.51	28.45

Table E.2: Hull-White prices under objective function 4.1

	1Y	**2Y**	**3Y**	**4Y**	**5Y**	**6Y**	**7Y**	**8Y**	**9Y**	**10Y**	**12Y**
1M	0.50	0.96	1.38	1.78	2.15	2.50	2.82	3.12	3.39	3.65	4.10
3M	0.85	1.64	2.37	3.05	3.68	4.28	4.82	5.33	5.80	6.24	7.01
6M	1.18	2.28	3.30	4.24	5.13	5.95	6.71	7.41	8.07	8.67	9.74
1Y	1.43	2.75	3.97	5.11	6.18	7.16	8.08	8.93	9.71	10.43	11.72
2Y	1.62	3.12	4.51	5.81	7.01	8.13	9.17	10.13	11.01	11.83	13.29
3Y	2.15	4.14	5.99	7.70	9.30	10.77	12.13	13.39	14.55	15.61	17.50
4Y	2.47	4.76	6.89	8.86	10.69	12.37	13.92	15.35	16.66	17.87	19.99
5Y	2.68	5.17	7.48	9.62	11.59	13.40	15.07	16.61	18.01	19.30	21.57
6Y	2.82	5.44	7.87	10.11	12.18	14.08	15.82	17.41	18.87	20.21	22.57
7Y	2.91	5.62	8.12	10.42	12.54	14.49	16.27	17.90	19.38	20.75	23.15
8Y	2.96	5.71	8.25	10.58	12.73	14.70	16.49	18.13	19.64 21.02	23.41	

Table E.3: Hull-White prices under objective function 4.2

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