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Descending from 1 to g^2

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NATURAL EXTENSIONS FOR NAKADA'S α-EXPANSIONS: DESCENDING FROM 1 TO g^2

JAAP DE JONGE AND COR KRAAIKAMP

ABSTRACT. By means of singularisations and insertions in Nakada's α-expansions, which involves the removal of partial quotients 1 while introducing partial quotients with a minus sign, the natural extension of Nakada's continued fraction map T_α is given for $(\sqrt{10} - 2)/3 \leq \alpha < 1$. From our construction it follows that Ω_α, the domain of the natural extension of T_α, is metrically isomorphic to Ω_g for $\alpha \in [g^2, g)$, where g is the small golden mean. Finally, although Ω_α proves to be very intricate and unmanageable for $\alpha \in [g^2, (\sqrt{10} - 2)/3)$, the α-Legendre constant $L(\alpha)$ on this interval is explicitly given.

1. INTRODUCTION

In 1981, Hitoshi Nakada introduced in [9] a family of continued fraction maps $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha)$, with $\alpha \in [0, 1]$, and obtained and studied their natural extensions for $\alpha \in [1/2, 1]$; see [9], [4] and [6]. Nakada defined these maps by

$$(1) \qquad T_\alpha(x) := \frac{\varepsilon(x)}{x} - a(x), \quad x \neq 0; \quad T_\alpha(0) := 0,$$

where $\varepsilon(x) = \mathrm{sign}(x)$ and $a : (-1, 1) \to \mathbb{N} \cup \{\infty\}$ is defined by

$$a(x) := \left\lfloor \left| \frac{1}{x} \right| + 1 - \alpha \right\rfloor, \quad x \neq 0; \quad a(0) := \infty.$$

In case $\alpha = 1$, we have the *regular* (or: *simple*) *continued fraction expansion* (rcf), while the case $\alpha = \frac{1}{2}$ is the *nearest integer continued fraction expansion* (nicf). In the case $\alpha = 0$, we have the *by-access continued fraction expansion*; see for instance [14].

For $\alpha \in [0, 1]$, if $T_\alpha^{n-1}(x) \neq 0$ for $n \in \mathbb{N}$, we define

$$\varepsilon_n = \varepsilon_n(x) := \varepsilon\left(T_\alpha^{n-1}(x)\right), \quad \text{and} \quad a_n = a_n(x) := \left\lfloor \left| \frac{1}{T_\alpha^{n-1}(x)} \right| + 1 - \alpha \right\rfloor.$$

Applying (1), we obtain

$$(2) \qquad x = T_\alpha^0(x) = \frac{\varepsilon_1}{a_1 + T_\alpha(x)} = \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + T_\alpha^2(x)}} = \cdots = \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + \cfrac{\varepsilon_3}{a_3 + \ddots}}} \quad ,$$

which we will throughout this paper write as $x = [0; \varepsilon_1 a_1, \varepsilon_2 a_2, \varepsilon_3 a_3, \dots]$; we will call the numbers a_i, $i \geq 1$, the *partial quotients* of the continued fraction expansion of x.

Let $\Omega_\alpha \in \mathbb{R}^2$ be the domain for the natural extension of T_α. We define $\mathcal{T}_\alpha : \Omega_\alpha \to \Omega_\alpha$ by

$$\mathcal{T}_\alpha(x, y) := \left(T_\alpha(x), \frac{1}{a(x) + \varepsilon(x)y} \right).$$

Omitting the suffix '(x)', we define the sequence of 'futures' $(t_n)_{n\geq 0}$ and the sequence of 'pasts' $(v_n)_{n\geq 0}$ of x by $v_0 = 0$, and

$$(t_n, v_n) := \mathcal{T}_\alpha^n(x, 0), \quad n \in \mathbb{N} \cup \{0\}.$$

We speak of 'futures' and 'pasts' because of the well-known relations (see for instance [4])

(3) $\qquad t_n = [0; \varepsilon_{n+1}a_{n+1}, \varepsilon_{n+2}a_{n+2}, \dots] \quad \text{and} \quad v_n = [0; a_n, \varepsilon_n a_{n-1}, \dots, \varepsilon_2 a_1].$

The sequence $(t_n, v_n)_{n\geq 0}$ plays a central role in the theory of approximation by continued fractions and is strongly connected with the sequence of convergents p_n/q_n of x. This sequence is given by $p_n/q_n = [0; \varepsilon_1 a_1, \varepsilon_2 a_2, \dots, \varepsilon_n a_n]$, $n \geq 1$. In [4], for instance, a lot of well-known properties concerning the sequence $(p_n, q_n)_{n\geq -1}$ can be found, such as $\gcd(p_n, q_n) = 1$, $n \geq -1$, and

(4)
$$\begin{aligned} p_{-1} &:= 1, \quad p_0 := a_0, \quad p_n = a_n p_{n-1} + \varepsilon_n p_{n-2}, \, n \geq 1; \\ q_{-1} &:= 0, \quad q_0 := 1, \quad q_n = a_n q_{n-1} + \varepsilon_n q_{n-2}, \, n \geq 1. \end{aligned}$$

The equation $v_n = q_{n-1}/q_n$, $n \geq -1$, serves as an example for the connection between the sequences $(t_n, v_n)_{n\geq 0}$ and p_n/q_n; the *approximation coefficients*

$$\theta_n(x) := q_n^2 |x - p_n/q_n|,$$

that are used to indicate the quality of the approximation of x by p_n/q_n, is another important example. Again from [4], we have, for instance (again omitting the suffix '(x)'):

$$\theta_{n-1} = \frac{v_n}{1 + t_n v_n} \quad \text{and} \quad \theta_n = \frac{\varepsilon_{n+1} t_n}{1 + t_n v_n},$$

the first of which we will use in the section on the α-Legendre constant at the end of this paper. Without giving Ω_α yet (the determination of which is the main topic of this paper), we note that – apart from a set of (Lebesgue) measure 0 – the map $\mathcal{T}_\alpha$ is bijective almost everywhere on Ω_α, with inverse map

$$\mathcal{T}_\alpha^{-1}(x, y) := \left(\frac{\varepsilon(x)}{x + a(x)}, \frac{1 - a(x)y}{\varepsilon(x)y} \right) = \left(\frac{\varepsilon(x)}{x + a(x)}, \frac{\varepsilon(x)}{y} - \varepsilon(x)a(x) \right);$$

note how the coordinates giving information about 'pasts' and 'futures' of the continued fraction expansion traded places with those of $\mathcal{T}_\alpha$.

A very helpful tool in the arithmetic of t_n and v_n is the well-known *Möbius transformation*: Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $a, b, c, d \in \mathbb{Z}$, $ad - bc = \pm 1$. The Möbius transformation $M : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$ associated with M^1 is defined by

$$M(x) := \frac{ax + b}{cx + d}, \quad x \neq \frac{-d}{c}, \infty \quad \text{and} \quad M\left(\frac{-d}{c}\right) = \infty; \quad M(\infty) = \frac{a}{c}.$$

So, using (3) and applying Möbius transformations, we can rephrase (2) as

$$x = \begin{pmatrix} 0 & \varepsilon_1 \\ 1 & a_1 \end{pmatrix}(t_1) = \begin{pmatrix} 0 & \varepsilon_1 \\ 1 & a_1 \end{pmatrix}\begin{pmatrix} 0 & \varepsilon_2 \\ 1 & a_2 \end{pmatrix}(t_2) = \begin{pmatrix} 0 & \varepsilon_1 \\ 1 & a_1 \end{pmatrix}\begin{pmatrix} 0 & \varepsilon_2 \\ 1 & a_2 \end{pmatrix}\begin{pmatrix} 0 & \varepsilon_3 \\ 1 & a_3 \end{pmatrix}\cdots.$$

Writing $M_n := \begin{pmatrix} 0 & \varepsilon_n \\ 1 & a_n \end{pmatrix}$, we also see that

$$t_n = M_{n+1}(t_{n+1}) \quad (\text{so } t_{n+1} = M_{n+1}^{-1}(t_n))$$

and generally,

$$t_n = M_{n+1} \cdots M_{n+k}(t_{n+k}) \quad (\text{and } t_{n+k} = M_{n+k}^{-1} \cdots M_{n+1}^{-1}(t_n)).$$

Likewise,

$$v_{n+1} = M_{n+1}^T(v_n)$$

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[1]Throughout this paper we will omit the distinction between a matrix and its associated Möbius transformation.

and generally
$$v_{n+k} = M_{n+k}^T \cdots M_{n+1}^T(v_n),$$
from which we derive the very useful fact that

(5) $$t_{n+k} = M(t_n) \quad \text{if and only if} \quad v_{n+k} = (M^T)^{-1}(v_n),$$

with M a Möbius transformation.

The main topic of this paper is the construction of all domains Ω_α with $\alpha \in [g^2, 1)$, where $g := (\sqrt{5}-1)/2 = 0.6180\ldots$ (while $G := (\sqrt{5}+1)/2 = 1.6180\ldots$; g and G are for the well-known *golden ratios*). We will show how to obtain these by transforming Ω_α into $\Omega_{\alpha'}$, with $\alpha' < \alpha$, starting with $\alpha = 1$. We will also show that for $\alpha \le (\sqrt{10}-2)/3$, Ω_α becomes rapidly extremely intricate, finally making it senseless if not impossible to give the associated set of pairs (t, v) explicitly when α approaches g^2. Our approach is based on *singularisations* and *insertions* in the continued fraction expansion of α; see [1] and also [6], where singularisations are used to define a suitable induced transformation. For $A, B \in \mathbb{N}$, $B \ge 2$, $\xi \in \mathbb{R}$ and $\varepsilon \in \{-1, 1\}$, the first of these operations is based on the equation

(6) $A + \dfrac{1}{1 + \dfrac{\varepsilon}{B + \xi}} = A + 1 - \dfrac{\varepsilon}{B + \varepsilon + \xi}$ (partial quotient 1 between A and B is *singularised*);

and the second one on

(7) $A + \dfrac{\varepsilon}{B + \xi} = A + \varepsilon - \dfrac{\varepsilon}{1 + \dfrac{1}{B - 1 + \xi}}$ (partial quotient 1 is *inserted* between A and B).

In terms of Möbius transformations, (6) is similar to

$$\begin{pmatrix} 0 & \varepsilon_n \\ 1 & a_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_{n+2} \\ 1 & a_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_n \\ 1 & a_n + 1 \end{pmatrix} \begin{pmatrix} 0 & -\varepsilon_{n+2} \\ 1 & a_{n+2} + \varepsilon_{n+2} \end{pmatrix}$$

and (7) to

$$\begin{pmatrix} 0 & \varepsilon_n \\ 1 & a_n \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_{n+1} \\ 1 & a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_n \\ 1 & a_n + \varepsilon_{n+1} \end{pmatrix} \begin{pmatrix} 0 & -\varepsilon_{n+1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_{n+1} \\ 1 & a_{n+1} - \varepsilon_{n+1} \end{pmatrix}.$$

Singularisations *shorten* the (regular) continued fraction expansion of numbers $x \in \mathbb{R}$. Considering (2), this implies a *loss* of convergents (as will be illustrated in the next section). Insertions obviously lengthen a continued fraction expansion, yielding so-called *mediants*, that are usually of lesser approximative quality compared to convergents. It be remarked, however, that inserting partial quotients 1 in a given continued fraction expansion involves the possibility of creating points (t, v) lying in a part of Ω_1 that had been removed in a previous stage of constructing. As long as we can compensate each insertion with a singularisation, we can avoid this. Applying this *compensated insertion* is actually the base of our construction of Ω_α, from $\alpha = g$ downwards, that we will gradually unfold in the next sections.

It is easy to see that compensation is only possible as long as an insertion involves the introduction of an extra 1, that is, if $B - 1$ in (7) equals 1, implying $B = 2$. In this case we deal with

$$A + \dfrac{\varepsilon_1}{2 + \dfrac{\varepsilon_2}{C + \xi}} = A + \varepsilon_1 - \dfrac{\varepsilon_1}{1 + \dfrac{1}{1 + \dfrac{\varepsilon_2}{C + \xi}}} = A + \varepsilon_1 - \dfrac{\varepsilon_1}{2 - \dfrac{\varepsilon_2}{C + \varepsilon_2 + \xi}}.$$

Of course, similar equations could be given in terms of Möbius transformations, but at this point we want to stress that either notation has its pros and cons as to readability and illuminating power. Depending on the context, for our computations we will use either continued fractions or Möbius transformations.

In the next sections we will step-by-step transform the domain $\Omega_1 := [0, 1) \times [0, 1]$ for the natural extension of the regular continued fraction to the domain Ω_α of the natural extension of any α-expansion for $\alpha \in [g^2, 1)$. Note that $t_n = 1$ only occurs in the case $x = 1$ and $n = 0$, associated with

the single point $(1,0)$, that is sent to $(0,1)$ under T_1. So, in view of (1), one should actually define $\Omega_1 := [0,1) \times [0,1] \cup \{(1,0)\}$. We will suppress subtleties like these in similar cases throughout this paper.

We will often use some well-known equations involving g and G, such as $g^2 = 1 - g$ and $\frac{1}{g} = G = g + 1$. Although we let α decrease continuously from 1 downwards, we speak of 'step-by-step', as the generic form of Ω_α varies over several intervals, according to feasability for singularisation (possible in Section 3 only) or compensated insertion. In Section 2 we will explain how to determine these intervals. We remark that the results for $\alpha \geq \sqrt{2} - 1$ are not new (see for instance [6] or [9]). However, our approach is best introduced by starting at $\alpha = 1$ and has some illuminative qualities. For one thing, we will show how our construction of Ω_α calls for the term *quilting*, as introduced in [7]. More importantly, some interesting results on α-expansions come with the determination of Ω_α. Since the construction of Ω_α will prove to be quite intricate, we will present these results mostly separately at the end of this paper. In Section 8 we will go into ergodic properties of $\mathcal{T}_\alpha$. Representing the collection of subsets of Ω_α by $\mathcal{B}$ and defining μ_α as the probability measure with density

$$\frac{1}{N_\alpha} \cdot \frac{1}{(1+tv)^2}$$

on $(\Omega_\alpha, \mathcal{B})$ (with N_α a normalising constant), we will also show that $(\Omega_\alpha, \mu_\alpha, \mathcal{T}_\alpha)$ is an ergodic dynamical system.

Finally, in Section 9 we will extend a result of Rie Natsui ([12]), based on the well-known theorem of Legendre in the theory of regular continued fractions; see for instance [2]:

Theorem 1. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$ and $p/q \in \mathbb{Q}$, with $\gcd(p,q) = 1$ and $q > 0$, such that $q^2|x - p/q| < 1/2$. Then there exists a non-negative integer n such that $p/q = p_n/q_n$, in words: p/q is a regular convergent of x. This constant $1/2$ is best possible in the sense that for every $\varepsilon > 0$ an irrational x and a rational number p/q, with $\gcd(p,q) = 1$ and $q > 0$, exist such that p/q is not a regular convergent and $q^2|x - p/q| < 1/2 + \varepsilon$.*

In [12], Natsui proves the existence of the α-*Legendre constant* for α-continued fractions, defined by

$$L(\alpha) := \sup\{c > 0 : q^2 \left| x - \tfrac{p}{q} \right| < c, \gcd(p,q) = 1 \Rightarrow \tfrac{p}{q} = \tfrac{p_n}{q_n}, n \geq 0\}$$

for $0 < \alpha \leq 1$; here p_n/q_n is an α-convergent of x. She expands the result given by Ito ([5]; see also [6]) for values $1/2 \leq \alpha \leq 1$ to giving explicit values of $L(\alpha)$ for $\sqrt{2} - 1 \leq \alpha < 1/2$. In Section 9 we will augment this result with the interval $[g^2, \sqrt{2} - 1)$ and the numbers $\alpha = 1/n$, for $n \in \mathbb{N}$, $n \geq 3$.

We will first concentrate on the construction of Ω_α for various values of α.

2. DETERMINING THE INTERVALS ACCORDING TO TYPE OF SINGULARISATION

One way to look at the domains Ω_α, $\alpha < 1$, is that in Ω_α no points (t,v) exist for which $\alpha \leq t \leq 1$. In combination with the definition of (t_n, v_n) as $\mathcal{T}_\alpha^n(x,0)$, $n \in \mathbb{N} \cup \{0\}$, for any $x \in [\alpha - 1, \alpha]$, this forms the basis of our approach for constructing the Ω_α with $\alpha \in [g^2, 1)$. We start with Ω_1 and fix α such that each time $\alpha \leq t_n \leq 1$, we replace the point (t_n, v_n) by a point (t_n^*, v_n^*) belonging to the continued fraction expansion of x after singularising. When fixing α, we use the continued fraction map T_1 of the associated ergodic system. Then we follow the same procedure, this time starting with Ω_α, fixing an $\alpha' < \alpha$ and applying T_α in order to construct $\Omega_{\alpha'}$. From Section 4 on, we will show how to make use of compensated insertion when $\alpha \leq g$.

We remark that under T_α the continued fraction expansion of numbers can vary considerably, according to the value of α. Denoting with x_α the α-expansion of $x \in (0,1)$, we have, for instance,

$g_1 = [0; \overline{1}]$ and $g_g = [0; 2, \overline{-3}]$, the bar indicating infinite repetition. Constructing $\Omega_{\alpha'}$ from Ω_α, the way points are replaced depends on the way x_α changes under singularisation or (compensated) insertion. As a start, in Section 3 we will remove all points (t, v) for which $t = [0; 1, \ldots]$, to which end mere singularisation suffices. In Sections 4 and 5 we will remove all points (t, v) for which $t = [0; 2, \ldots]$ by means of a single compensated insertion. In Section 6 we will show that removing points (t, v) for which $t = [0; 3, \ldots]$ (the ultimate one being $t = g^2$) is possible only in special cases, applying compensated insertion twice.

Not only the continued fraction expansion of a number x depends on which continued fraction map we use, the expansion of the associated numbers t_n does also. In order to determine the interval $[\alpha', \alpha]$, $\alpha' < \alpha \leq 1$, such that $t_n \in (\alpha', \alpha)$ implies $t_n = [0; 1, \ldots]$, we observe that on $(0, 1]$ both $\frac{1}{\alpha}$ and $1 - \alpha$ are decreasing functions of α. So, for $\alpha \in (\alpha', 1]$,

$$(8) \qquad 1 = \left\lfloor \frac{1}{1} + 1 - 1 \right\rfloor \leq \left\lfloor \frac{1}{\alpha} + 1 - \alpha \right\rfloor \leq a_1(\alpha'_\alpha) = \left\lfloor \frac{1}{\alpha'} + 1 - \alpha \right\rfloor \leq \left\lfloor \frac{1}{\alpha'} + 1 - \alpha' \right\rfloor,$$

from which we derive $a_1(\alpha'_\alpha) = 1$ if and only if $1/\alpha' + 1 - \alpha' < 2$, which is the case if and only if $g < \alpha' < 1$. This is why the next section is about the case $\alpha \in (g, 1]$. There (and more so in the sections following it) we will show that although replacing points by other points is in fact only part of the construction of the Ω_α, the heart of the construction is still the procedure sketched above, which is actually a way to determine α-*fundamental intervals* $\Delta_{n,\alpha} = \Delta_\alpha(i_1, i_2, \ldots, i_n)$ of rank n, that we define as follows:

$$\Delta_\alpha(i_1, i_2, \ldots, i_n) := \{\alpha \in [0, 1] : a_1(\alpha_\alpha) = i_1, a_2(\alpha_\alpha) = i_2, \ldots, a_n(\alpha_\alpha) = i_n\},$$

where $i_j \in \mathbb{Z} \setminus \{0\}$ for each $1 \leq j \leq n$. In the previous paragraph, for instance, we found $\Delta_\alpha(1) = \{\alpha \in [0, 1] : a_1(\alpha_\alpha) = 1\} = (g, 1]$.

3. THE CASE $\alpha \in (g, 1]$

We start with the natural extension domain for the regular continued fraction, the square Ω_1, and the classic *Gauss map* T_1, given by Nakada in [9]. Let $x \in [0, 1]$ and $n \geq 0$ be the smallest integer for which $t_n \in [\alpha, 1)$, i.e., for which $(t_n, v_n) \in R_\alpha := [\alpha, 1) \times [0, 1]$. We then have that the rcf of x satisfies

$$x = [0; a_1, \ldots, a_n, 1, a_{n+2}, a_{n+3}, \ldots];$$

observe that in Ω_1 we have $\varepsilon_k = 1$, $k \geq 1$. Singularising $a_{n+1} = 1$, we obtain

$$x = [0; a_1, \ldots, (a_n + 1), -(a_{n+2} + 1), a_{n+3}, \ldots].$$

Let the sequence $(t_n^*, v_n^*)_{n \geq 0}$ denote the futures and pasts of the singularised continued fraction expansion of x. Then

$$(t_n^*, v_n^*) = ([0; -(a_{n+2} + 1), a_{n+3}, \ldots], [0; a_n + 1, a_{n-1}, \ldots, a_1]),$$

while

$$(t_n, v_n) = ([0; 1, a_{n+2}, a_{n+3}, \ldots], [0; a_n, a_{n-1}, \ldots, a_1]).$$

Note that $v_n = 1/(a_n + v_{n-1})$, which is equivalent to $a_n + v_{n-1} = 1/v_n$, and so

$$v_n^* = \frac{1}{a_n + 1 + v_{n-1}} = \frac{1}{1 + \frac{1}{v_n}} = \frac{v_n}{1 + v_n}.$$

Applying (7) with $A = 0$, $\varepsilon = -1$, $B = a_n + 1$ and $\xi = t_{n+2}$, we get

$$t_n^* = \frac{-1}{a_n + 1 + t_{n+2}} = -1 + \frac{1}{1 + \frac{1}{a_n + t_{n+2}}} = t_n - 1.$$

Since $(t_n, v_n) \in R_\alpha$, we find that $(t^*, v^*) \in A_\alpha := [\alpha - 1, 0) \times [0, 1/2]$.

Note that the map $\mathcal{M} : R_\alpha \to A_\alpha$, defined by $\mathcal{M}(t, v) := (t-1, v/(1+v))$ is a bijection. In Section 8 we will use this to obtain ergodicity of the dynamical system $(\Omega_\alpha, \mathcal{B}, \mu_\alpha, \mathcal{T}_\alpha)$, $g \leq \alpha \leq 1$.

It is obvious that $t_{n+1}^* = t_{n+2}$, and since

$$v_{n+1}^* = \frac{1}{a_{n+2} + 1 - v_n^*} = \frac{1}{a_{n+2} + 1 - \frac{v_n}{1+v_n}} = \frac{1}{a_{n+2} + \frac{1}{1+v_n}} = \frac{1}{a_{n+2} + v_{n+1}} = v_{n+2},$$

we conclude that

(9) $$(t_{n+1}^*, v_{n+1}^*) = (t_{n+2}, v_{n+2}).$$

Having singularised the first partial quotient 1 in the continued fraction expansion of x and so obtained an alternative expansion, we repeat the procedure ad infinitum. When doing so, we actually not only remove R_a from Ω_1, but $\mathcal{T}_1(R_\alpha) := (0, (1-\alpha)/\alpha] \times [1/2, 1]$ as well, since $(t_{n+1}, v_{n+1}) = ((1 - t_n)/t_n, 1/(1 + v_n))$. Equation (9) implies $\mathcal{T}_1(A_\alpha) = \mathcal{T}_1^2(R_\alpha)$, so the loss of points is confined to R_α and $\mathcal{T}_1(R_\alpha)$. Removing from Ω_1 the strips R_α and $\mathcal{T}_1(R_\alpha)$ and adding A_α yields Ω_α; see Figures 1 and 2. Here we see a first glimpse of the similarity with the process of quilting as described in [7]. Expanding this metaphor, our construction in the next sections may be described as a form of recursive quilting.

We remark that Ω_α can be constructed from Ω_1 immediately, using $\mathcal{T}_1$, or from some $\Omega_{\alpha'}$, $\alpha < \alpha' < 1$, using $\mathcal{T}_{\alpha'}$. Indeed, in these cases we have something similar to (8):

$$\left\lfloor \frac{1}{\alpha} + 1 - \alpha' \right\rfloor < \left\lfloor \frac{1}{g} + 1 - g \right\rfloor = 2,$$

implying that changing "$t_n \in [\alpha, 1)$" in the beginning of this section into "$t_n \in [\alpha, \alpha')$" would leave the construction of Ω_α unaffected, save for the initial domain.

Rewriting $x = [0; a_1, \ldots, a_n, 1, a_{n+2}, \ldots]$ as $x = [0; a_1, \ldots, (a_n+1), -(a_{n+2}+1), \ldots]$ does *not* leave the sequence of convergents unaffected: since $p_k^* = p_k$ and $q_k^* = q_k$, $k < n$, we find, applying (4),

$$p_n^* = (a_n + 1)p_{n-1}^* + p_{n-2}^* = a_n p_{n-1}^* + p_{n-1}^* + p_{n-2}^* = a_n p_{n-1} + p_{n-2} + p_{n-1} = p_n + p_{n-1} = p_{n+1};$$
$$p_{n+1}^* = 2p_n^* - p_{n-1}^* = a_{n+2}p_{n+1} + p_{n+1} - p_{n-1} = a_{n+2}p_{n+1} + p_n = p_{n+2}.$$

Similarly, we find $q_n^* = q_{n+1}$ and $q_{n+1}^* = q_{n+2}$. We conclude that in the transformation from Ω_1 to Ω_α, $\alpha \in (g, 1)$, a convergent p_n/q_n is lost every time that $(t_n, v_n) \in R_\alpha$.

Note that $\lim_{\alpha \downarrow g}(1 - \alpha)/\alpha = (1 - g)/g = g = \lim_{\alpha \downarrow g} \alpha$, which is mirrored in the fact that for $\alpha = g$ the left boundary of R_α coincides with the right boundary of $\mathcal{T}_1(R_\alpha)$. We now have obtained the following result:

Theorem 2. *Let $\alpha \in (g, 1)$. Then $\Omega_\alpha = [\alpha - 1, 0) \times [0, \frac{1}{2}] \cup [0, \frac{1-\alpha}{\alpha}] \times [0, \frac{1}{2}) \cup (\frac{1-\alpha}{\alpha}, \alpha) \times [0, 1]$.*

This theorem was already obtained by Nakada in [9]; see also [6]. Figure 2 shows an example of the generic form of Ω_α, $\alpha \in (g, 1)$. In Figure 3 we have written $-g^2$ for the equivalent $g - 1$.

4. THE CASE $\alpha \in (\frac{1}{2}, g]$

An important implication of our construction of Ω_g in the previous section is that in continued fraction expansions associated to Ω_α, $\alpha \leq g$, the partial quotient 1 is non-existent. In the current and the following section we will similarly remove all partial quotients 2 (with plus sign). We determine the interval $(\alpha', \alpha]$, $\alpha' < \alpha \leq g$, such that $t_n \in (\alpha', \alpha]$ implies $t_n = [0; 2, \ldots]$, in a way similar to the one at the end of Section (2): if $x \in [0, g)$ and $\alpha \in (x, g]$, then

$$2 = \left\lfloor \frac{1}{g} + 1 - g \right\rfloor \leq \left\lfloor \frac{1}{\alpha} + 1 - \alpha \right\rfloor \leq a_1(\alpha_\alpha') = \left\lfloor \frac{1}{\alpha'} + 1 - \alpha \right\rfloor \leq \left\lfloor \frac{1}{\alpha'} + 1 - \alpha' \right\rfloor,$$
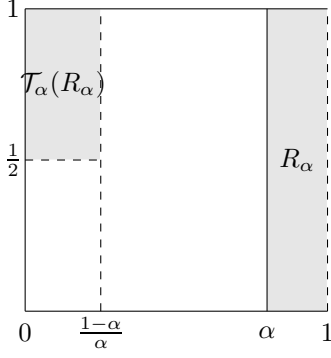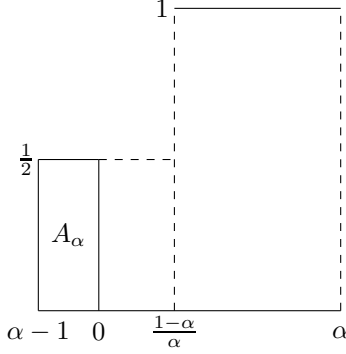
FIGURE 1.
Ω_1, $\alpha = 0.8$.

FIGURE 2.
Ω_α, $\alpha = 0.8$.

FIGURE 3. Ω_g.
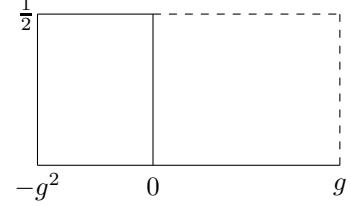
from which we derive $a_1(\alpha'_\alpha) = 2$ if and only if $1/\alpha' + 1 - \alpha' < 3$, which (given that $\alpha' \in [0, g)$) is the case if and only if $\sqrt{2} - 1 < \alpha' < g$; indeed $\sqrt{2} - 1_{\sqrt{2}-1} = [0; 3, \overline{-2, -4}]$. We conclude that $\Delta_\alpha(2) = (\sqrt{2} - 1, g]$, and therefore we will now investigate the case $\alpha \in (\sqrt{2} - 1, g]$.

In the current section we will confine ourselves to the case $\alpha \in (1/2, g]$. To make a cut at $\alpha = 1/2$ is because obviously $t \in (1/2, g]$ if and only if $t = [0; 2, -a, \ldots]$, with $a \in \mathbb{N}_{\geq 2}$, while $t \in (\sqrt{2} - 1, 1/2]$ if and only if $t = [0; 2, b, \ldots]$, with $b \in \mathbb{N}_{\geq 2}$. We will see how the distinction shows in different ways of transforming Ω_α.

Let $x \in [g - 1, g)$ and let $n \geq 0$ be the smallest integer for which $t_n \in [\alpha, g]$, i.e., for which $(t_n, v_n) \in R_\alpha := [\alpha, g] \times [0, 1/2)$. We already know that in this case $x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n a_n, 2, -a_{n+2}, \ldots]$. Since the continued fraction map of the related dynamical system is T_g, for $\alpha \in (1/2, g]$ we have

$$a_2(\alpha) = \left\lfloor \left| \frac{1}{T_g(\alpha)} \right| + 1 - g \right\rfloor = \left\lfloor \left| \frac{1}{\frac{1}{\alpha} - 2} \right| + 1 - g \right\rfloor \geq \left\lfloor \left| \frac{1}{\frac{1}{g} - 2} \right| + 1 - g \right\rfloor = G + 1 + 1 - g = 3,$$

so in this case

$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n a_n, 2, -a_{n+2}, \ldots], \quad a_{n+2} \geq 3.$$

To remove the partial quotient 2, we first insert -1, so as to write

$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n (a_n + 1), -1, 1, -a_{n+2}, \ldots].$$

Now we singularise 1 in this expansion and get

$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n (a_n + 1), -2, (a_{n+2} - 1), \ldots].$$

Having removed the first partial quotient 2 in the continued fraction expansion of x and so obtained the alternative expansion, we repeat the procedure ad infinitum. Note that the next $+2$ could be $a_{n+2} - 1$ in the rewritten expansion, since $a_{n+2} \geq 3$. In the current case we have $R_\alpha = [\alpha, g] \times [0, 1/2)$ and, considering that $(t_{n+1}, v_{n+1}) = ((1 - 2t_n)/t_n, 1/(2 + v_n))$, $\mathcal{T}_g(R_\alpha) = [-g^2, (1 - 2\alpha)/\alpha] \times (2/5, 1/2]$, knowing that $(1 - 2g)/g = -g^2$; see Figure 4, where we have taken $\alpha = 0.52$ for illustrative purposes. Having determined what to remove (in the sense of the previous section), we use similar calculations as before to find what to add. Using the same notation as in the previous case, we find

$$(t_n^*, v_n^*) = (t_n - 1, \tfrac{v_n}{1 + v_n}) \quad \text{(as in the previous case)},$$
$$(t_{n+1}^*, v_{n+1}^*) = (-\tfrac{t_{n+1}}{1 + t_{n+1}}, 1 - v_{n+1}) = (\tfrac{1 - 2t_n}{t_n - 1}, \tfrac{v_n + 1}{v_n + 2}) \text{ and}$$
$$(t_{n+2}^*, v_{n+2}^*) = (t_{n+2}, v_{n+2}).$$

So there are two regions to be added: $A_\alpha = [\alpha - 1, -g^2] \times [0, 1/3)$ and $\mathcal{T}_g(A_\alpha) = [(1 - 2\alpha)/(\alpha - 1), (1 - 2g)/(g - 1)] \times [1/2, 3/5)$. With regard to the last region, we note that $(1 - 2g)/(g - 1) = g$. Because of the latter, we have not yet established to construct Ω_α, since $\mathcal{T}_g(A_\alpha)$ contains points (t, v) that do not exist in Ω_α, i.e. those for which $t \in (\alpha, g]$; see the greyed region in Figure 5. We will call the newly constructed region $\Omega_{\alpha,1}^+$, defined by

$$\Omega_{\alpha,1}^+ := \Omega_g \cup A_\alpha \cup \mathcal{T}_g(A_\alpha) \setminus (R_\alpha \cup \mathcal{T}_g(R_\alpha))$$

Putting $R_{\alpha,1} := \mathcal{T}_g(A_\alpha) \cap [\alpha, g] \times [0, 3/5)$ as the part yet to be removed, we define

$$\Omega_{\alpha,1} := \Omega_{\alpha,1}^+ \setminus R_{\alpha,1}.$$

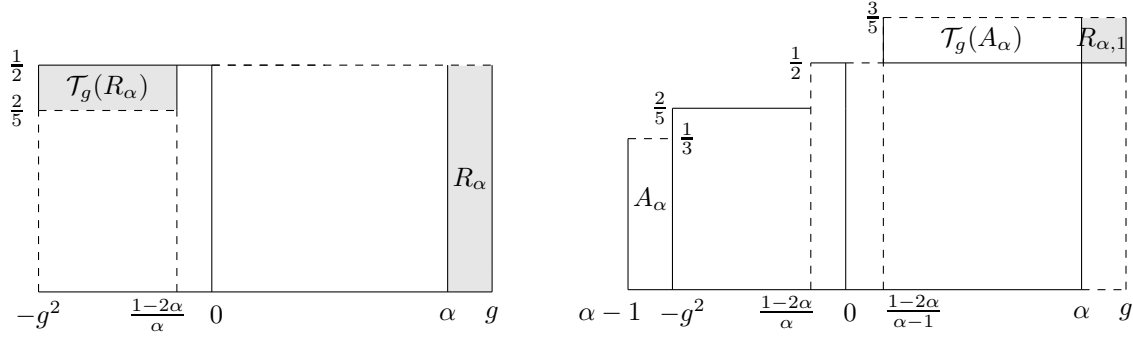We will follow the same procedure of removal and addition regarding $R_{\alpha,1}$ as in the case of R_α.

FIGURE 4. Ω_g with R_α and $\mathcal{T}_g(R_\alpha)$, $\alpha = 0.52$.

FIGURE 5. $\Omega_{\alpha,1}^+$, $\alpha = 0.52$.

Note that this renders the same t-coordinates as in the creation of $\Omega_{\alpha,1}$. To efficiently calculate the v-coordinates, it is convenient to use Möbius transformations. We saw that in the current construction of Ω_α the equation $v_n^* = v_n/(v_n + 1)$ holds, that we now write as $v_n^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}(v_n)$; similarly we write $v_{n+1}^* = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}(v_n)$. It follows that the v-coordinate of points in $A_{\alpha,1}$ are given by

$$(10) \qquad v_{n,1}^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}(v_n),$$

with v_n the v-coordinate of points in R_α; for the v-coordinate of points in $\mathcal{T}_g(A_{\alpha,1})$ we have

$$(11) \qquad v_{n+1,1}^* = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}(v_n).$$

For the construction of $\Omega_{\alpha,2}$ we also need to calculate $\mathcal{T}_g(R_{\alpha,1})$. Since $v_{n+1} = \frac{1}{v_n + 2} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}(v_n)$, we find that the v-coordinates of $\mathcal{T}_g(R_{\alpha,1})$ are given by
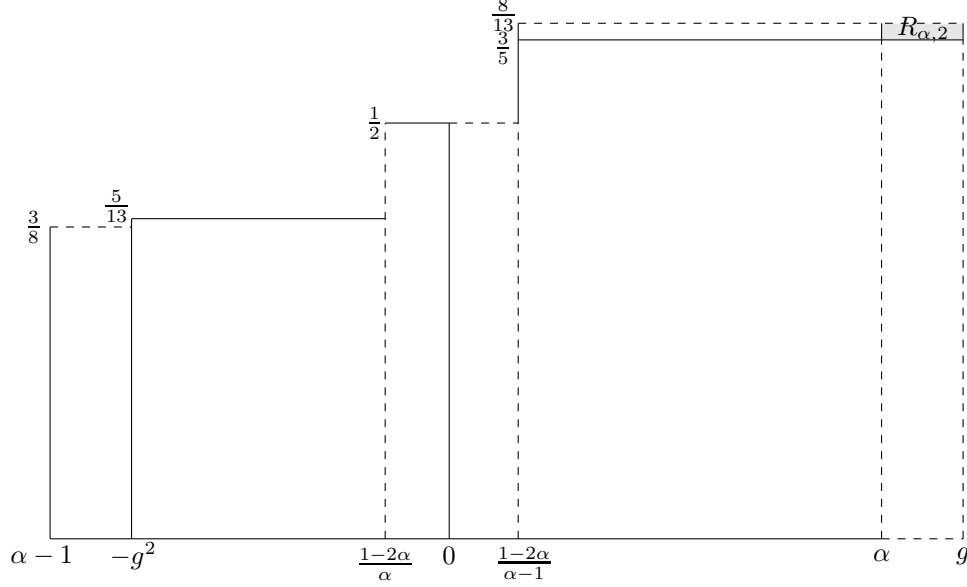
$$(12) \qquad v_{n+1,1} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}(v_n),$$

v_n still being the v-coordinate of points in R_α. Since $v \in [0, 1/2)$ for points in R_α, calculating the boundaries of $\Omega_{\alpha,2}$ is now straightforward. We have written the relevant values of t and v in Figure 6 and write

$$\Omega_{\alpha,2}^+ := \Omega_{\alpha,1}^+ \cup A_{\alpha,1} \cup \mathcal{T}_g(A_{\alpha,1}) \setminus (R_{\alpha,1} \cup \mathcal{T}_g(R_{\alpha,1}))$$

Defining $R_{\alpha,2} := \mathcal{T}_g(A_{\alpha,1}) \cap [\alpha, g] \times [3/5, 8/13)$ as the part yet to be removed, we define

$$\Omega_{\alpha,2} := \Omega_{\alpha,2}^+ \setminus R_{\alpha,2}.$$

FIGURE 6. $\Omega_{\alpha,2}$, $\alpha = 0.52$.

Proceeding this way, we construct the sequence $\Omega_{\alpha,k}$, $k = 1, 2, \ldots$, and bring forth Ω_α as being $\lim_{k\to\infty} \Omega_{\alpha,k}$. To do so, we need to calculate $\lim_{k\to\infty} v^*_{n,k}$, $\lim_{k\to\infty} v^*_{n+1,k}$ and $\lim_{k\to\infty} v_{n+1,k}$, using formulas associated with (10) through (12):

$$\begin{cases} v^*_{n,k} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^k (v_n), \ k \geq 1, \\[2mm] v^*_{n+1,k} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^k (v_n), \ k \geq 1, \quad \text{and} \\[2mm] v_{n+1,k} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^k (v_n), \ k \geq 1. \end{cases}$$

We note that $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ is the square of $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, the matrix that maps a pair of consecutive Fibonacci numbers (F_{i-1}, F_i) to (F_i, F_{i+1}); we put $F_0 = 0$ and $F_1 = 1$. It follows that

$$\lim_{k\to\infty} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^k (v_n) = \lim_{k\to\infty} \frac{F_{2k-1}v_n + F_{2k}}{F_{2k}v_n + F_{2k+1}}.$$

Now suppose $\lim_{k\to\infty} \frac{F_{2k-1}v_n + F_{2k}}{F_{2k}v_n + F_{2k+1}} = L$. Then

$$L = \lim_{k\to\infty} \frac{1}{\frac{F_{2k}v_n + F_{2k+1}}{F_{2k-1}v_n + F_{2k}}} = \lim_{k\to\infty} \frac{1}{1 + \frac{F_{2k-2}v_n + F_{2k-1}}{F_{2k-1}v_n + F_{2k}}} = \frac{1}{1+L},$$

from which we derive

$$\lim_{k\to\infty} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^k (v_n) = g,$$

whence

$$\begin{cases} \lim_{k\to\infty} v^*_{n,k} = \frac{g}{g+1} = g^2, \\[2mm] \lim_{k\to\infty} v^*_{n+1,k} = \frac{g+1}{g+2} = g \quad \text{and} \\[2mm] \lim_{k\to\infty} v_{n+1,k} = \frac{1}{g+2} = g^2. \end{cases}$$

We conclude that

$$\Omega_\alpha = \Omega_g \setminus \left([\alpha, g] \times [0, \tfrac{1}{2}) \bigcup_{k=1}^{\infty} [-g^2, \tfrac{1-2\alpha}{\alpha}] \times \left(\frac{F_{2k+1}}{F_{2k+3}}, \frac{F_{2k-1}}{F_{2k+1}} \right) \right)$$
$$\bigcup_{k=1}^{\infty} \left([\alpha - 1, -g^2] \times \left[\frac{F_{2(k-1)}}{F_{2k}}, \frac{F_{2k}}{F_{2(k+1)}} \right) \cup [\tfrac{1-2\alpha}{\alpha-1}, \alpha) \times \left[\frac{F_{2k}}{F_{2k+1}}, \frac{F_{2k+2}}{F_{2k+3}} \right) \right),$$

from which we derive the following results, again obtained by Nakada in [9] (cf. [6]), illustrated by Figures 7 and 8.

Theorem 3. *Let $\alpha \in (\tfrac{1}{2}, g]$. Then*

$$\Omega_\alpha = [\alpha - 1, -g^2) \times [0, g^2) \cup [-g^2, \tfrac{1-2\alpha}{\alpha}] \times [0, g^2]$$
$$\cup (\tfrac{1-2\alpha}{\alpha}, 0) \times [0, \tfrac{1}{2}] \cup [0, \tfrac{1-2\alpha}{\alpha-1}] \times [0, \tfrac{1}{2}) \cup (\tfrac{1-2\alpha}{\alpha-1}, \alpha) \times [0, g).$$

Moreover, we have

$$\Omega_{1/2} = [-\tfrac{1}{2}, 0] \times [0, g^2] \cup (0, \tfrac{1}{2}) \times [0, g).$$
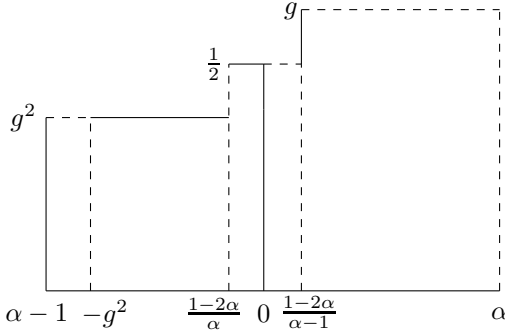

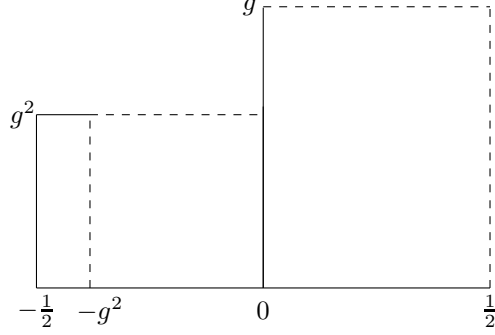FIGURE 7. Ω_α, $\alpha = 0.52$.

FIGURE 8. $\Omega_{\frac{1}{2}}$.

In order to find the effect of the current transformation on the sequence of convergents, we recall that we rewrote

$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n a_n, 2, -a_{n+2}, \ldots] \quad \text{as}$$
$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n (a_n + 1), -2, (a_{n+2} - 1), \ldots].$$

Since $p_k^* = p_k$ and $q_k^* = q_k$, $k < n$, we find, applying (4) and omitting the straightforward calculations,

$$\begin{cases} p_n^* = p_n + p_{n-1}; \\ p_{n+1}^* = p_{n+1}; \\ p_{n+2}^* = p_{n+2}. \end{cases}$$

Similarly, we find $q_n^* = q_n + q_{n-1}$, $q_{n+1}^* = q_{n+1}$ and $q_{n+2}^* = q_{n+2}$. It would seem that in the transformation from Ω_g to Ω_α, $\alpha \in [1/2, g)$, the convergent p_n/q_n is replaced by the so-called *mediant* $(p_n + p_{n-1})/(q_n + q_{n-1})$. We emphasise, however, that this is only a mediant with respect to a small interval with upper bound g; actually, this 'α-*mediant*' is a convergent of the regular continued fraction. This comes as no surprise: from [6] we know that regular convergents only disappear and are not replaced by mediants when moving from the regular expansion to the α-expansion for $1/2 \leq \alpha \leq g$. The regular convergent that replaces another one in the transformation

from Ω_g to Ω_α, $\alpha \in [1/2, g)$ had previously been 'singularised away' in the transformation from Ω_1 to Ω_g. We will illustrate this with an example.

Let $x = (\sqrt{17} - 3)/4 = 1/(3 + (\sqrt{17} - 3)/2) = 0.28077\ldots$; in the second, more intricate expression for x, the number $(\sqrt{17} - 3)/2$ is explicitly displayed for reasons that will show instantly. We distinguish

$$\begin{cases} x_\alpha = [0; \overline{3,1,1}], & \alpha \in (g, 1]; \\ x_\alpha = [0; 3, \overline{2, -4}], & \alpha \in ((\sqrt{17} - 3)/2, g]; \\ x_\alpha = [0; \overline{4, -2}], & \alpha \in (1/2, (\sqrt{17} - 3)/2]. \end{cases}$$

If we focus at the first partial quotients, we have

$$\begin{cases} \text{if } x_\alpha = [0; 3, 1, 1, 3, 1, 1 \ldots], & \text{then } (\frac{p_0}{q_0}, \ldots, \frac{p_6}{q_6}) = (\frac{0}{1}, \frac{1}{3}, \frac{1}{4}, \frac{2}{7}, \frac{7}{25}, \frac{9}{32}, \frac{16}{57}); \\ \text{if } x_\alpha = [0; 3, 2, -4, 2, \ldots], & \text{then } (\frac{p_0}{q_0}, \ldots, \frac{p_4}{q_4}) = (\frac{0}{1}, \frac{1}{3}, \frac{2}{7}, \frac{7}{25}, \frac{16}{57}); \\ \text{if } x_\alpha = [0; 4, -2, 4, -2, \ldots], & \text{then } (\frac{p_0}{q_0}, \ldots, \frac{p_4}{q_4}) = (\frac{0}{1}, \frac{1}{4}, \frac{2}{7}, \frac{9}{32}, \frac{16}{57}). \end{cases}$$

What happens is that we create $[0; 3, 2, -4, 2, \ldots]$ from $[0; 3, 1, 1, 3, 1, 1 \ldots]$ by singularising the second 1 from each pair $(1, 1)$, in which process some of the (regular) convergents get lost, in accordance with what we saw in Section 3. When creating $[0; 4, -2, 4, -2, \ldots]$ from $[0; 3, 2, -4, 2, \ldots]$, we first insert a 1 between each 2 followed by -4, so as to retrieve $[0; 3, 1, 1, 3, 1, 1]$ (and its convergents), and subsequently singularise the first 1 from each pair $(1, 1)$, so as to lose other convergents of the regular case. Generally, we conclude that in the transformation from Ω_g to Ω_α, $\alpha \in [1/2, g)$, some regular convergents p_n/q_n are replaced by others.

Although direct singularisation, as used in for instance [6], would be more straightforward, it is limited to $\alpha \geq 1/2$, while our approach enables a decrease beyond $\alpha = 1/2$, as we will show in the next sections. Moreover, it shows how to transform a domain Ω_α into another domain $\Omega_{\alpha'}$ in a uniform manner, where α' is smaller than but close to α, for any $\alpha \in (g^2, 1]$. On top of that, in Section 8 we will go into the isomorphism between Ω_α and Ω_g for $g^2 \leq \alpha < g$.

5. The case $\alpha \in (\sqrt{2} - 1, \frac{1}{2}]$

In this section we will show how to derive Ω_α from $\Omega_{1/2}$, with $\sqrt{2} - 1 \leq \alpha < 1/2$. Let $x \in [-1/2, 1/2)$ and let $n \geq 0$ be the smallest integer for which $t_n \in [\alpha, 1/2]$, i.e., for which $(t_n, v_n) \in R_\alpha := [\alpha, 1/2] \times [0, g)$. We already know that in this case $x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n a_n, 2, a_{n+2}, \ldots]$. Since the continued fraction map of the related dynamical system is $T_{1/2}$, for $\alpha \in (\sqrt{2} - 1, 1/2]$ we have

$$a_2(\alpha) = \left\lfloor \frac{1}{T_{1/2}(\alpha)} + 1 - \frac{1}{2} \right\rfloor = \left\lfloor \frac{1}{\frac{1}{\alpha} - 2} + \frac{1}{2} \right\rfloor \geq \left\lfloor \frac{1}{\frac{1}{\sqrt{2} - 1} - 2} + \frac{1}{2} \right\rfloor = \left\lfloor \sqrt{2} + \frac{3}{2} \right\rfloor = 2,$$

so in this case

$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n a_n, 2, a_{n+2}, \ldots], \ a_{n+2} \geq 2.$$

The difference with the case in the previous section is merely the sign of a_{n+2}, which shows in the great resemblance of equations that we need to construct Ω_α. Applying insertion and singularisation in a similar way, we find

$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n (a_n + 1), -2, -(a_{n+2} + 1), \ldots]$$

this time. The other equations are exactly the same as in the previous section: $(t_{n+1}, v_{n+1}) = ((1 - 2t_n)/t_n, 1/(2 + v_n))$, $(t_n^*, v_n^*) = (t_n - 1, v_n/(1 + v_n))$, $(t_{n+1}^*, v_{n+1}^*) = ((1 - 2t_n)/(t_n - 1), (v_n + 1)/(v_n + 2))$ and $(t_{n+2}^*, v_{n+2}^*) = (t_{n+2}, v_{n+2})$.

Using similar calculations as in the previous section, we find that $R_\alpha = [\alpha, 1/2] \times [0, g)$ implies $\mathcal{T}_{1/2}(R_\alpha) = (0, (1 - 2\alpha)/\alpha) \times (g^2, 1/2]$; see Figure 9, where we have taken $\alpha = 0.43$ for illustrative purposes. The first region to be added is: $A_\alpha = [\alpha - 1, -\frac{1}{2}] \times [0, g^2)$; the second is $\mathcal{T}_{1/2}(A_\alpha) =$

$[(1 - 2\alpha)/(\alpha - 1), 0] \times [1/2, g)$. An important difference with the previous case is that $(R_\alpha \cup \mathcal{T}_{1/2}(R_\alpha)) \cap ((A_\alpha \cup \mathcal{T}_{1/2}(A_\alpha)) = \emptyset$, so all we have to do is remove R_α and $\mathcal{T}_{1/2}(R_\alpha)$ from $\Omega_{1/2}$ and add A_α and $\mathcal{T}_{1/2}(A_\alpha)$ to it. Doing so, we find

Theorem 4. *Let* $\alpha \in (\sqrt{2} - 1, 1/2]$. *Then*

$$\Omega_\alpha = [\alpha - 1, \alpha) \times [0, g^2) \cup [\tfrac{1-2\alpha}{\alpha}, \alpha) \times [g^2, \tfrac{1}{2}) \cup [\tfrac{1-2\alpha}{\alpha-1}, \alpha) \times [\tfrac{1}{2}, g).$$

Since $(1 - 2\alpha)/\alpha$ equals α for $\alpha = \sqrt{2} - 1$, at $\alpha = \sqrt{2} - 2$ the domain Ω_α is split in two parts:

$$\Omega_{\sqrt{2}-1} = [\sqrt{2} - 2, \sqrt{2} - 1) \times [0, g^2) \cup [\tfrac{1}{2}\sqrt{2} - 1, \sqrt{2} - 1) \times [\tfrac{1}{2}, g);$$
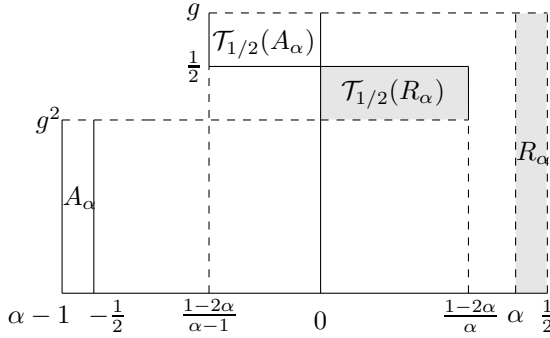
see Figure 10.

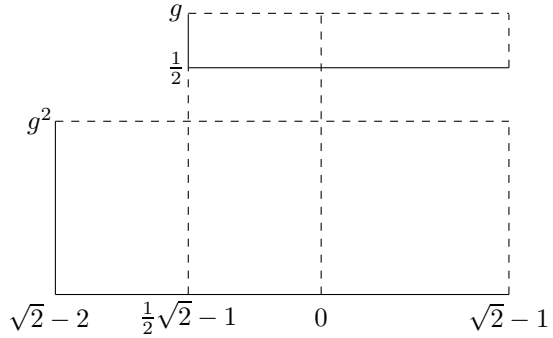FIGURE 9. $\Omega_{\frac{1}{2}}, \alpha = 0.43.$ FIGURE 10. $\Omega_{\sqrt{2}-1}.$

With a great similarity of calculations as in the previous section, we find that in this case also

$$\begin{cases} p_n^* = p_n + p_{n-1}; \\ p_{n+1}^* = p_{n+1}; \\ p_{n+2}^* = p_{n+2}. \end{cases}$$

The difference with the case $\alpha \in (1/2, g]$ is that now the convergent p_n/q_n is really replaced by the mediant $(p_n + p_{n-1})/(q_n + q_{n-1})$ in the sense of the regular case.

6. THE CASE $\alpha \in (\frac{\sqrt{10}-2}{3}, \sqrt{2} - 1]$

In Section 4 we noticed that $\sqrt{2} - 1_{\sqrt{2}-1} = [0; 3, \overline{-2, -4}]$; actually, $\sqrt{2} - 1$ is the largest $\alpha \in [0, 1]$ such that no partial quotients 1 or 2 (with plus sign) occur in α_α. In this section we want to remove points (t_n, v_n) with $t_n = [0; 3, \varepsilon_{n+2}a_{n+2}, \dots]$, that is, points associated with numbers $x = [0; \varepsilon_1 a_1, \dots, \varepsilon_n a_n, 3, \varepsilon_{n+2}a_{n+2}, \dots]$. Similar to the cases in Sections 4 and 5, the removal of a 3 requires compensated insertion, for which we need a partial quotient 2, as explained in Section 1. Since we removed all partial quotients 2 with a plus sign, we are dealing with the following numbers:

$$x = [0; \varepsilon_1 a_1, \dots, \varepsilon_n a_n, 3, -2, \varepsilon_{n+3}a_{n+3}, \dots].$$

Note that uncompensated insertion would leave us with partial quotients 1 or $+2$, involving points outside Ω_α. Because of the necessity of compensated insertion, the next interval to explore is not simply $\Delta_\alpha(3) = ((\sqrt{13} - 3)/2, \sqrt{2} - 1]$ but the much smaller $\Delta_\alpha(3, -2)$. To find the boundaries

of this interval, we determine $(\alpha', \alpha]$, $\alpha' < \alpha \leq \sqrt{2} - 1$, such that $t_n \in (\alpha', \alpha]$ implies $t_n = [0; 3, -2, \ldots]$. If $\alpha' \in [0, \sqrt{2} - 1)$ and $\alpha \in (\alpha', \sqrt{2} - 1]$, then

$$2 = \left\lfloor \frac{1}{3 - \frac{1}{\sqrt{2}-1}} + 2 - \sqrt{2} \right\rfloor \leq \left\lfloor \frac{1}{3 - \frac{1}{\alpha}} + 1 - \alpha \right\rfloor \leq a_2(\alpha'_\alpha)$$

$$= \left\lfloor \frac{1}{3 - \frac{1}{\alpha'}} + 1 - \alpha \right\rfloor \leq \left\lfloor \frac{1}{3 - \frac{1}{\alpha'}} + 1 - \alpha' \right\rfloor,$$

from which we derive $a_2(\alpha'_\alpha) = 2$ if and only if $1/(3-1/\alpha')+1-\alpha' < 3$. Given that $\alpha' \in [0, \sqrt{2}-1))$, this is the case if and only if $(\sqrt{10} - 2)/3 < \alpha' < \sqrt{2} - 1$; indeed, $(\sqrt{10} - 2)/3_{(\sqrt{10}-2)/3} = [0; 3, \overline{-3, -2, -3, -4}]$. So $\Delta_\alpha(3, -2) = ((\sqrt{10} - 2)/3, \sqrt{2} - 1]$. Since $\sqrt{2} - 1 = [0; 3, \overline{-2, -4}]$ and $\lim_{a \to \infty}[0; 3, -2, -a] = \frac{2}{5}$, we will first show how to construct Ω_α from $\Omega_{\sqrt{2}-1}$, with $2/5 \leq \alpha < \sqrt{2} - 1$, which will prove to be more complicated than the work we have done so far. After that, the step from $\Omega_{2/5}$ to $\Omega_{(\sqrt{10}-2)/3}$ will prove to be fairly small.

Let $\alpha \in (2/5, \sqrt{2} - 1]$, $x \in [\sqrt{2} - 2, \sqrt{2} - 1)$ and $n \geq 0$ be the smallest integer for which $t_n \in [\alpha, \sqrt{2} - 1)$, i.e., for which

(13) $$(t_n, v_n) \in R_\alpha := [\alpha, \sqrt{2} - 1] \times [0, g^2) \cup [\alpha, \sqrt{2} - 1] \times [\tfrac{1}{2}, g).$$

Considering the fact that $\sqrt{2} - 1_{\sqrt{2}-1} = [0; 3, \overline{-2, -4}]$, it is not hard to see that in this case

$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n a_n, 3, -2, -a_{n+3}, \ldots], \ a_{n+3} \geq 4.$$

First we insert 1 between 3 and -2, yielding

$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n a_n, 2, 1, 1, -a_{n+3}, \ldots], \ a_{n+3} \geq 4.$$

Next we insert -1 between a_n and 2, yielding

$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n (a_n + 1), -1, 1, 1, \mathbf{1}, -a_{n+3}, \ldots], \ a_{n+3} \geq 4.$$

Then we singularise the last $+1$, yielding

$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n (a_n + 1), -1, \mathbf{1}, 2, -2, a_{n+3} - 1, \ldots], \ a_{n+3} \geq 4.$$

Finally we singularise the remaining $+1$, so as to get

$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n (a_n + 1), -2, -3, a_{n+3} - 1, \ldots], \ a_{n+3} \geq 4.$$

For the first time we are confronted with the singularised expansion differing at four places from the original expansion, necessitating one more stage of calculating successive (t, v)-pairs (in the sense of the previous sections). Omitting the straightforward calculations, we find

$$(t_{n+1}, v_{n+1}) = (\tfrac{1-3t_n}{t_n}, \tfrac{1}{3+v_n}), \quad (t_{n+2}, v_{n+2}) = (\tfrac{2-5t_n}{3t_n-1}, \tfrac{v_n+3}{2v_n+5}),$$

$$(t_n^*, v_n^*) = (t_n - 1, \tfrac{v_n}{1+v_n}), \quad (t_{n+1}^*, v_{n+1}^*) = (\tfrac{1-2t_n}{t_n-1}, \tfrac{v_n+1}{v_n+2}),$$

$$(t_{n+2}^*, v_{n+2}^*) = (\tfrac{2-5t_n}{2t_n-1}, \tfrac{v_n+2}{2v_n+5}), \quad (t_{n+3}^*, v_{n+3}^*) = (t_{n+3}, v_{n+3}).$$

We see that in the current situation (i.e. for the current values of α), constructing Ω_α is much more complicated than in the previous sections. There is one more step of removing and adding regions, and these regions are all split in two disjoint parts. In particular, $R_{\alpha,1} := \mathcal{T}_{\sqrt{2}-1}^2(A_\alpha) \cap R_\alpha \neq \emptyset$, similar to Section 4. Since (13) holds, we can draw $\Omega_{\alpha,1}$, defined similarly as in Section 4. We will first sketch roughly what is removed from $\Omega_{\sqrt{2}-1}$ (see Figure 11; the removed parts are in grey), consisting of

$$R_\alpha = [\alpha, \sqrt{2} - 1] \times [0, g^2) \cup [\alpha, \sqrt{2} - 1] \times [\tfrac{1}{2}, g),$$

$$\mathcal{T}_{\sqrt{2}-1}(R_\alpha) = [\sqrt{2} - 2, \tfrac{1-3\alpha}{\alpha}] \times (\tfrac{1}{3+g^2}, \tfrac{1}{3}] \cup [\sqrt{2} - 2, \tfrac{1-3\alpha}{\alpha}] \times (\tfrac{1}{3+g}, \tfrac{2}{7}] \quad \text{and}$$

$$\mathcal{T}_{\sqrt{2}-1}^2(R_\alpha) = [\tfrac{1}{2}\sqrt{2} - 1, \tfrac{2-5\alpha}{3\alpha-1}] \times (\tfrac{3+g^2}{5+2g^2}, \tfrac{3}{5}] \cup [\tfrac{1}{2}\sqrt{2} - 1, \tfrac{1-3\alpha}{\alpha}] \times (\tfrac{3+g}{5+2g}, \tfrac{7}{12}].$$

The removal of R_α does not show in the resulting figure (it is 'cut off' $\Omega_{\sqrt{2}-1}$), but the four parts of which $\mathcal{T}_{\sqrt{2}-1}(R_\alpha)$ and $\mathcal{T}^2_{\sqrt{2}-1}(R_\alpha)$ consist, show as coves. In Figure 12 we see Figure 11 with the added regions; $R_{\alpha,1}$ is in grey:

$$A_\alpha = [\alpha-1, \sqrt{2}-2] \times [0, \tfrac{g^2}{1+g^2}) \cup [\alpha-1, \sqrt{2}-2] \times [\tfrac{1}{3}, g^2),$$

$$\mathcal{T}_{\sqrt{2}-1}(A_\alpha) = [\tfrac{1-2\alpha}{\alpha-1}, \tfrac{1}{2}\sqrt{2}-1] \times [\tfrac{1}{2}, \tfrac{1+g^2}{2+g^2}) \cup [\tfrac{1-2\alpha}{\alpha-1}, \tfrac{1}{2}\sqrt{2}-1] \times [\tfrac{3}{5}, g) \text{ and}$$

$$\mathcal{T}^2_{\sqrt{2}-1}(A_\alpha) = [\tfrac{2-5\alpha}{2\alpha-1}, \sqrt{2}-1] \times [\tfrac{2}{5}, \tfrac{2+g^2}{5+2g^2}) \cup [\tfrac{2-5\alpha}{2\alpha-1}, \sqrt{2}-1] \times [\tfrac{5}{12}, \tfrac{2+g}{5+2g}).$$

Similar to the construction in Section 4, we are now left with $R_{\alpha,1}$ to remove, consisting of the protuberant parts of two small rectangles between the parts that we already removed. Figure 13 shows a detailed example of $\Omega_{\alpha,1}$, $2/5 < \alpha < \sqrt{2}-1$. This first stage shows how Ω_α is not only characterised by more and more protuberant parts and coves as α decreases, but is also splitting in more and more disjoint regions. Nevertheless, the construction of Ω_α in the current case is very similar to the construction of Ω_α in Section 4: likewise, we construct the sequence $\Omega_{\alpha,k}$, $k = 1, 2, \ldots$, and bring forth Ω_α as being $\lim_{k\to\infty} \Omega_{\alpha,k}$.

We remark that fractions involving g or G can be represented in many ways. In this particular
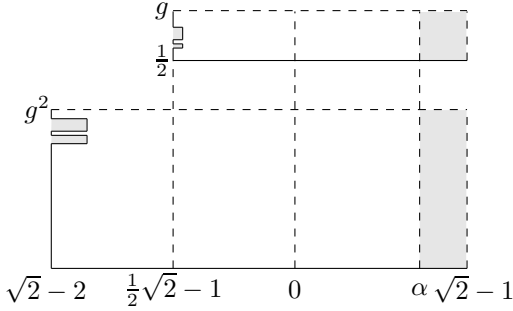

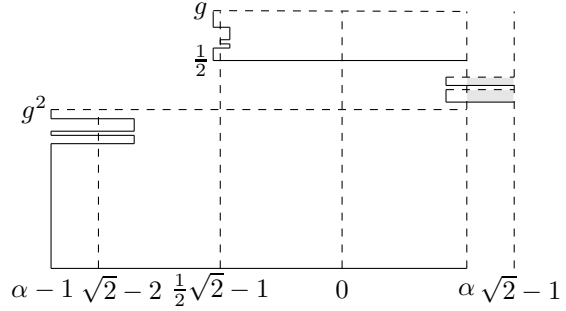FIGURE 11. $\Omega_{\alpha,1}$, removed parts, rough sketch.

FIGURE 12. $\Omega_{\alpha,1}$, removed and added parts, rough sketch.

case, substituting g for v_n in $1/(3+v_n)$ (associated with the v-bounds of $\mathcal{T}_{\sqrt{2}-1}(R_\alpha)$) yields $1/(3+g)$, which is equal to $g^2/(1+g^2)$; this is the fraction we find when we substitute g^2 for v_n in $v_n/(1+v_n)$, which is associated with the v-bounds of A_α; see the lower corner on the left side of Figure 13.

To obtain Ω_α as a 'limit' of the sequence $\Omega_{\alpha,k}$, we need to infinitely remove and add regions. Similar to the case $1/2 < \alpha \leq g$, only new v-values need to be calculated, to which end we use the Möbius transformations associated with v_n^*, v_{n+1}^*, v_{n+2}^*, v_{n+1} and v_{n+2}. The matrices involved are immediately derived from the way v_n^*, v_{n+1}^* et cetera are expressed in v_n. In the recurrent process of removing and adding regions, the matrix $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ - the one belonging to v_{n+2}^* - plays a central role, because it determines the domain of v-values in each new stage of constructing $\Omega_{\alpha,k}, k \geq 1$,
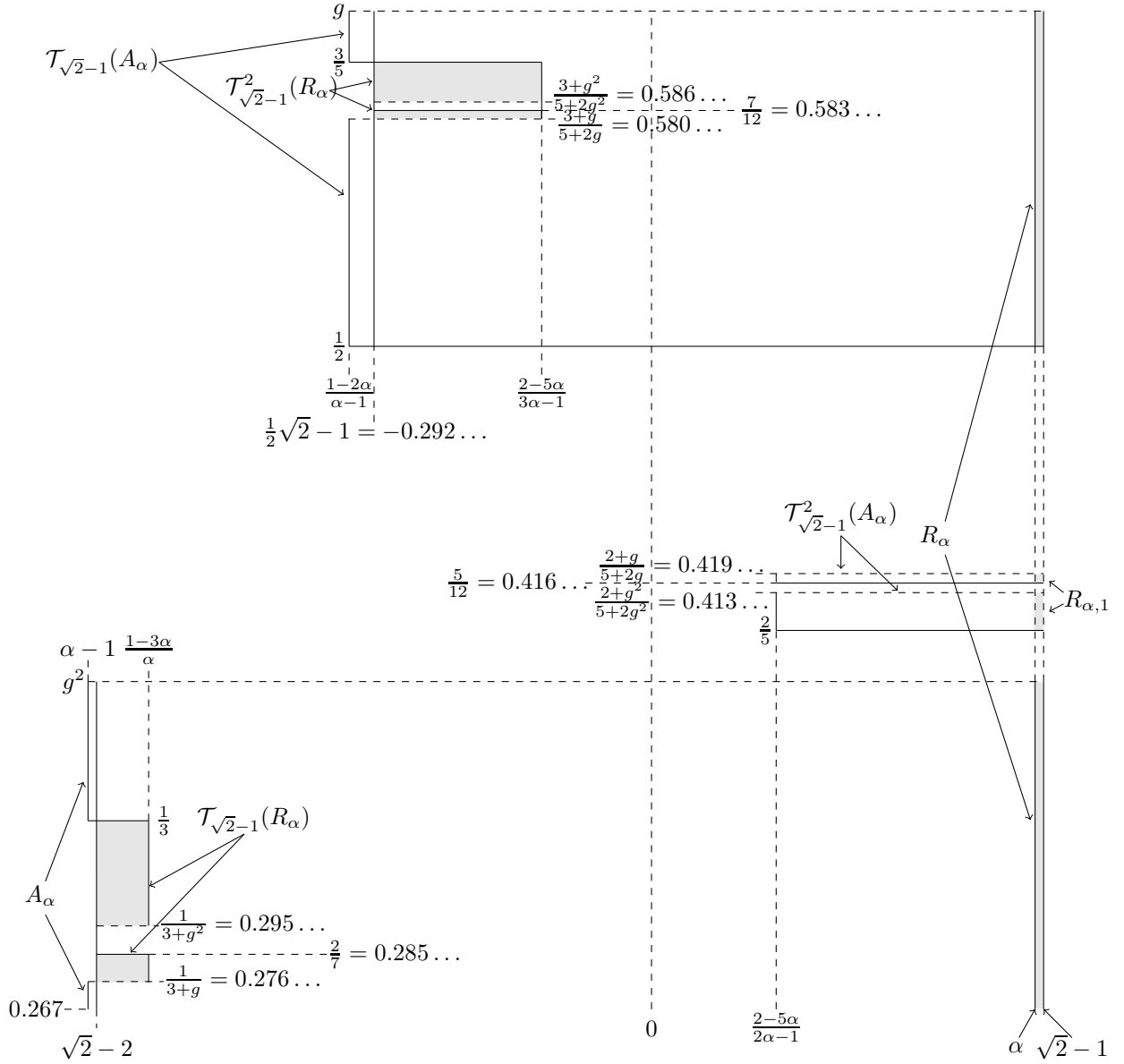
FIGURE 13. $\Omega_{\alpha,1}$, $\alpha = 0.405$. The lower rectangle $[\alpha - 1, \sqrt{2} - 1] \times [0, .267)$ has been omitted.

similar to $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ in Section 4. Using similar notations for new v-values also, we find

$$
\begin{cases}
v_{n,k}^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}^k (v_n), \ k \geq 1, \\[2mm]
v_{n+1,k}^* = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}^k (v_n), \ k \geq 1, \\[2mm]
v_{n+2,k}^* = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}^k (v_n), \ k \geq 1
\end{cases}
$$

and

$$\begin{cases} v_{n+1,k} = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}^k (v_n), \ k \geq 1, \quad \text{and} \\[2em] v_{n+2,k} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}^k (v_n), \ k \geq 1. \end{cases}$$

It be remarked that graphic representations of $\Omega_{\alpha,k}, k \geq 2$ would show merely very small changes compared to $\Omega_{\alpha,1}$. The six regions that are removed as well as the six regions that are added in each new stage are very small. We have, for example:

$$A_{\alpha,1} = [\alpha - 1, \sqrt{2} - 2] \times [\tfrac{2}{7}, \tfrac{2+g^2}{7+3g^2}) \cup [\alpha - 1, \sqrt{2} - 2] \times [\tfrac{5}{17}, \tfrac{2+g}{7+3g}),$$

while $\frac{2}{7} = 0.285714\ldots; \frac{2+g^2}{7+3g^2} = 0.292412\ldots; \frac{5}{17} = 0.294117\ldots$ and $\frac{2+g}{7+3g} = 0.295686\ldots;$

$$\mathcal{T}_{\sqrt{2}-1}(A_{\alpha,1}) = [\tfrac{1-2\alpha}{\alpha-1}, \tfrac{1}{2}\sqrt{2} - 1] \times [\tfrac{7}{12}, \tfrac{7+3g^2}{12+5g^2}) \cup [\tfrac{1-2\alpha}{\alpha-1}, \tfrac{1}{2}\sqrt{2} - 1] \times [\tfrac{17}{29}, \tfrac{7+3g}{12+5g}),$$

while $\frac{7}{12} = 0.583333\ldots; \frac{7+3g^2}{12+5g^2} = 0.585621\ldots; \frac{17}{29} = 0.586206\ldots$ and $\frac{7+3g}{12+5g} = 0.586746\ldots;$

$$\mathcal{T}^2_{\sqrt{2}-1}(A_{\alpha,1}) = [\tfrac{2-5\alpha}{2\alpha-1}, \sqrt{2} - 1] \times [\tfrac{12}{29}, \tfrac{12+5g^2}{29+12g^2}) \cup [\tfrac{2-5\alpha}{2\alpha-1}, \sqrt{2} - 1] \times [\tfrac{29}{70}, \tfrac{12+5g}{29+12g}),$$

while $\frac{12}{29} = 0.413793\ldots; \frac{12+5g^2}{29+12g^2} = 0.414185\ldots; \frac{29}{70} = 0.414285\ldots$ and $\frac{12+5g}{29+12g} = 0.414378\ldots;$

$$\mathcal{T}_{\sqrt{2}-1}(R_{\alpha,1}) = [\sqrt{2} - 2, \tfrac{1-3\alpha}{\alpha}] \times (\tfrac{5+2g^2}{17+7g^2}, \tfrac{5}{17}] \cup [\sqrt{2} - 2, \tfrac{1-3\alpha}{\alpha}] \times (\tfrac{5+2g}{17+7g}, \tfrac{12}{41}],$$

while $\frac{5}{17} = 0.294117\ldots; \frac{5+2g^2}{17+7g^2} = 0.292975\ldots; \frac{12}{41} = 0.292682\ldots$ and $\frac{5+2g}{17+7g} = 0.292412\ldots$ and

$$\mathcal{T}^2_{\sqrt{2}-1}(R_{\alpha,1}) = [\tfrac{1}{2}\sqrt{2} - 1, \tfrac{2-5\alpha}{3\alpha-1}] \times (\tfrac{17+7g^2}{29+12g^2}, \tfrac{17}{29}] \cup [\tfrac{1}{2}\sqrt{2} - 1, \tfrac{1-3\alpha}{\alpha}] \times (\tfrac{17+7g}{29+12g}, \tfrac{41}{70}],$$

while $\frac{17}{29} = 0.586206\ldots; \frac{17+7g^2}{29+12g^2} = 0.585814\ldots; \frac{41}{70} = 0.585714\ldots$ and $\frac{17+7g}{29+12g} = 0.585621\ldots.$
In Figure 14 the leftmost corner of $\Omega_{\alpha,2}$ is shown. Similar to Figure 13, we have omitted the lower rectangle (where $v < \frac{2}{7}$), in order not to waste too much blank space. Even more than Figure 14, Figure 15 illustrates the rapid decrease of the surface of the rectangles, the union of which finally builds Ω_α. In particular,

$$R_{\alpha,2} = [\alpha, \sqrt{2} - 1] \times [\tfrac{12}{29}, \tfrac{12+5g^2}{29+12g^2}) \cup [\alpha, \sqrt{2} - 1] \times [\tfrac{29}{70}, \tfrac{12+5g}{29+12g}),$$

the union of two very small rectangles between the two rectangles of $R_{\alpha,1}$. In the next step, we would find $R_{\alpha,3}$ consisting of two very small rectangles between the two rectangles of $R_{\alpha,2}$ - and so on.
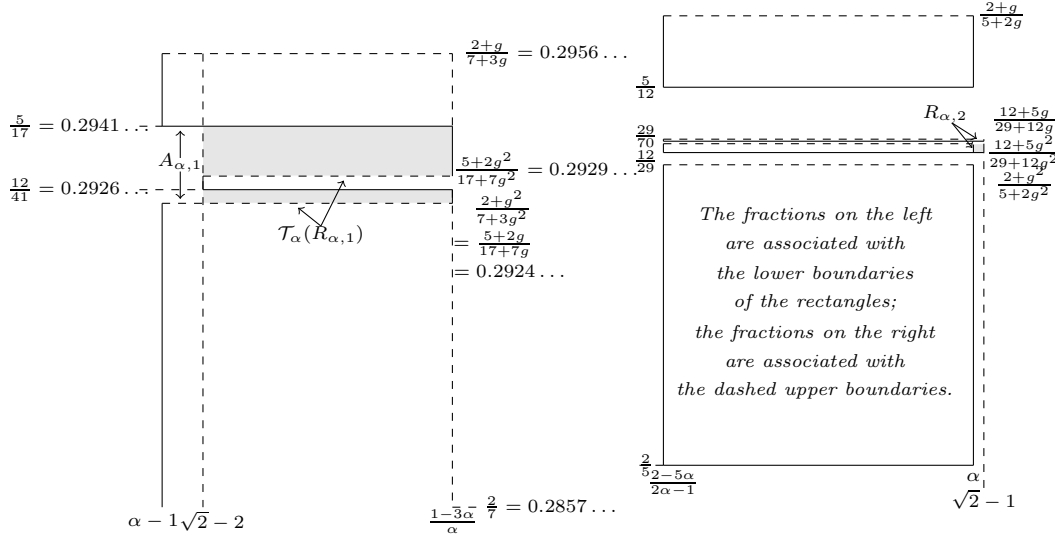Since $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}^2$, we see that at the base of our current construction lies a matrix that is almost equal to the one in Section 4 that renders consecutive Fibonacci numbers. In this case, we have a matrix that renders the recurrent sequence E_n given by $E_{n+1} = 2E_n + E_{n-1}, n \geq 0$, with $E_{-1} = 1$ and $E_0 = 0$. Now suppose $\lim_{k\to\infty} \frac{E_{2k-1}v_n + E_{2k}}{E_{2k}v_n + E_{2k+1}} = L$. Then

$$L = \lim_{k\to\infty} \frac{1}{\frac{E_{2k}v_n + E_{2k+1}}{E_{2k-1}v_n + E_{2k}}} = \lim_{k\to\infty} \frac{1}{2 + \frac{E_{2k-2}v_n + E_{2k-1}}{E_{2k-1}v_n + E_{2k}}} = \frac{1}{2 + L},$$

from which we derive

$$\lim_{k\to\infty} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^k (v_n) = \sqrt{2} - 1, \quad \text{whence}$$

$$
\begin{cases}
\lim_{k\to\infty} v_{n,k}^* = \frac{\sqrt{2}-1}{\sqrt{2}} = 1 - \tfrac{1}{2}\sqrt{2} = 0.292893\ldots, \\[4pt]
\lim_{k\to\infty} v_{n+1,k}^* = \frac{\sqrt{2}}{\sqrt{2}+1} = 2 - \sqrt{2} = 0.585786\ldots, \\[4pt]
\lim_{k\to\infty} v_{n+2,k}^* = \frac{\sqrt{2}+1}{2\sqrt{2}+3} = \sqrt{2} - 1 = 0.414213\ldots, \\[4pt]
\lim_{k\to\infty} v_{n+1,k} = \frac{1}{\sqrt{2}+2} = 1 - \tfrac{1}{2}\sqrt{2} \quad \text{and} \\[4pt]
\lim_{k\to\infty} v_{n+2,k} = \frac{\sqrt{2}+2}{2\sqrt{2}+3} = 2 - \sqrt{2}.
\end{cases}
$$

FIGURE 14. The leftmost corner of $\Omega_{\alpha,2}, \alpha = 0.405$.
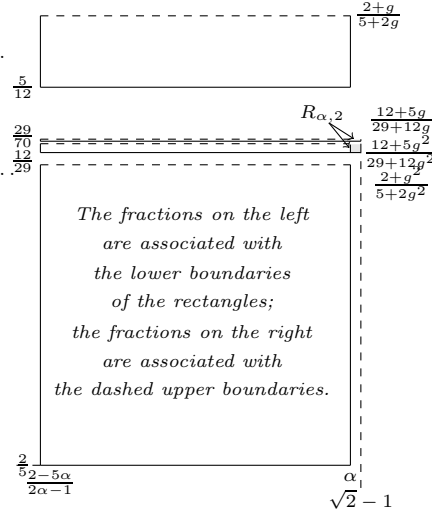
FIGURE 15. The rectangles between the upper and lower block of $\Omega_{\alpha,2}, \alpha = 0.405$.

Omitting straightforward calculations, we conclude that

$$
\Omega_\alpha = \Omega_{\sqrt{2}-1} \setminus \left(\bigcup_{k=0}^{\infty} R_{\alpha,k} \cup \mathcal{T}_{\sqrt{2}-1}(R_{\alpha,k}) \cup \mathcal{T}_{\sqrt{2}-1}^2(R_{\alpha,k}) \right)
$$

$$
\bigcup_{k=0}^{\infty} \left(A_{\alpha,k} \cup \mathcal{T}_{\sqrt{2}-1}(A_{\alpha,k}) \cup \mathcal{T}_{\sqrt{2}-1}^2(A_{\alpha,k})^* \right),
$$

with

$$
R_{\alpha,k} = [\alpha, \sqrt{2} - 1] \times [\tfrac{E_{2k}}{E_{2k+1}}, \tfrac{E_{2k}+g^2 E_{2k-1}}{E_{2k+1}+g^2 E_{2k}}) \cup [\alpha, \sqrt{2} - 1] \times [\tfrac{E_{2k+1}}{E_{2k+2}}, \tfrac{E_{2k}+g E_{2k-1}}{E_{2k+1}+g E_{2k}});
$$

$$
\mathcal{T}_{\sqrt{2}-1}(R_{\alpha,k}) = [\sqrt{2} - 2, \tfrac{1-3\alpha}{\alpha}] \times (\tfrac{E_{2k+1}+g^2 E_{2k}}{E_{2k+2}+E_{2k+1}+g^2(E_{2k+1}+E_{2k})}, \tfrac{E_{2k+1}}{E_{2k+2}+E_{2k+1}}]
$$

$$
\cup [\sqrt{2} - 2, \tfrac{1-3\alpha}{\alpha}] \times (\tfrac{E_{2k+1}+g E_{2k}}{E_{2k+2}+E_{2k+1}+g(E_{2k+1}+E_{2k})}, \tfrac{E_{2k+2}}{E_{2k+3}+E_{2k+2}}];
$$

$$
\mathcal{T}_{\sqrt{2}-1}^2(R_{\alpha,k}) = [\tfrac{1}{2}\sqrt{2} - 1, \tfrac{2-5\alpha}{3\alpha-1}] \times (\tfrac{E_{2k+2}+E_{2k+1}+g^2(E_{2k+1}+E_{2k})}{E_{2k+3}+g^2 E_{2k+2}}, \tfrac{E_{2k+2}+E_{2k+1}}{E_{2k+3}}]
$$

$$
\cup [\tfrac{1}{2}\sqrt{2} - 1, \tfrac{2-5\alpha}{3\alpha-1}] \times (\tfrac{E_{2k+2}+E_{2k+1}+g(E_{2k+1}+E_{2k})}{E_{2k+3}+g E_{2k+2}}, \tfrac{E_{2k+3}+E_{2k+2}}{E_{2k+4}}];
$$

$$
A_{\alpha,k} = [\alpha - 1, \sqrt{2} - 2] \times [\tfrac{E_{2k}}{E_{2k+1}+E_{2k}}, \tfrac{E_{2k}+g^2 E_{2k-1}}{E_{2k+1}+E_{2k}+g^2(E_{2k}+E_{2k-1})})
$$

$$
\cup [\alpha - 1, \sqrt{2} - 2] \times [\tfrac{E_{2k+1}}{E_{2k+2}+E_{2k+1}}, \tfrac{E_{2k}+g E_{2k-1}}{E_{2k+1}+E_{2k}+g(E_{2k}+E_{2k-1})});
$$

$$\mathcal{T}_{\sqrt{2}-1}(A_{\alpha,k}) = [\tfrac{1-2\alpha}{\alpha-1}, \tfrac{1}{2}\sqrt{2}-1] \times [\tfrac{E_{2k+1}+E_{2k}}{E_{2k+2}}, \tfrac{E_{2k+1}+E_{2k}+g^2(E_{2k}+E_{2k-1})}{E_{2k+2}+g^2 E_{2k+1}})$$
$$\cup [\tfrac{1-2\alpha}{\alpha-1}, \tfrac{1}{2}\sqrt{2}-1] \times [\tfrac{E_{2k+2}+E_{2k+1}}{E_{2k+3}}, \tfrac{E_{2k+1}+E_{2k}+g(E_{2k}+E_{2k-1})}{E_{2k+2}+g E_{2k+1}});$$
$$\mathcal{T}^2_{\sqrt{2}-1}(R_{\alpha,k})^* = [\tfrac{2-5\alpha}{2\alpha-1}, \alpha) \times [\tfrac{E_{2k+2}}{E_{2k+3}}, \tfrac{E_{2k+2}+g^2 E_{2k+1}}{E_{2k+3}+g^2 E_{2k+2}}) \cup [\tfrac{2-5\alpha}{2\alpha-1}, \alpha) \times [\tfrac{E_{2k+3}}{E_{2k+4}}, \tfrac{E_{2k+2}+g E_{2k+1}}{E_{2k+3}+g E_{2k+2}}),$$

where $\mathcal{T}^2_{\sqrt{2}-1}(R_{\alpha,k})^* := \mathcal{T}^2_{\sqrt{2}-1}(R_{\alpha,k}) \setminus R_{\alpha,k+1}$, $R_{\alpha,0} := R_\alpha$ and $A_{\alpha,0} := A_\alpha$.
In order to find the effect of the current transformation on the sequence of convergents, we recall that we rewrote

$$x = [0; \varepsilon_1 a_1, \dots, \varepsilon_n a_n, 3, -2, -a_{n+3}, \dots] \quad \text{as}$$
$$x = [0; \varepsilon_1 a_1, \dots, \varepsilon_n(a_n+1), -2, -3, a_{n+3}-1, \dots]$$

Since $p_k^* = p_k$ and $q_k^* = q_k$, $k < n$, we find, applying (4) and this time omitting the straightforward calculations,

$$\begin{cases} p_n^* = p_n + p_{n-1}; \\ p_{n+1}^* = p_{n+1} - p_n; \\ p_{n+2}^* = p_{n+2}; \\ p_{n+3}^* = p_{n+3}. \end{cases}$$

Of course, we find similar relations for q_n^* through q_{n+3}^*. We conclude that in the transformation from $\Omega_{\sqrt{2}-1}$ to Ω_α, $\alpha \in [2/5, \sqrt{2}-1)$, the convergent p_n/q_n is replaced by the mediant $(p_n + p_{n-1})/(q_n + q_{n-1})$ and that p_{n+1}/q_{n+1} is replaced by the mediant $(p_{n+1} - p_n)/(q_{n+1} - q_n)$.
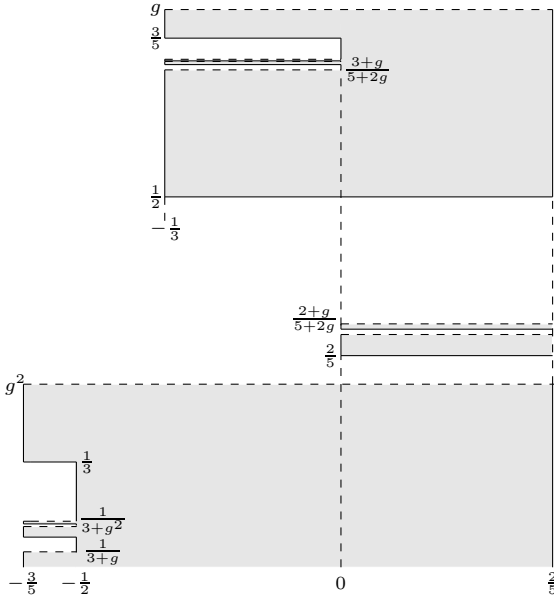

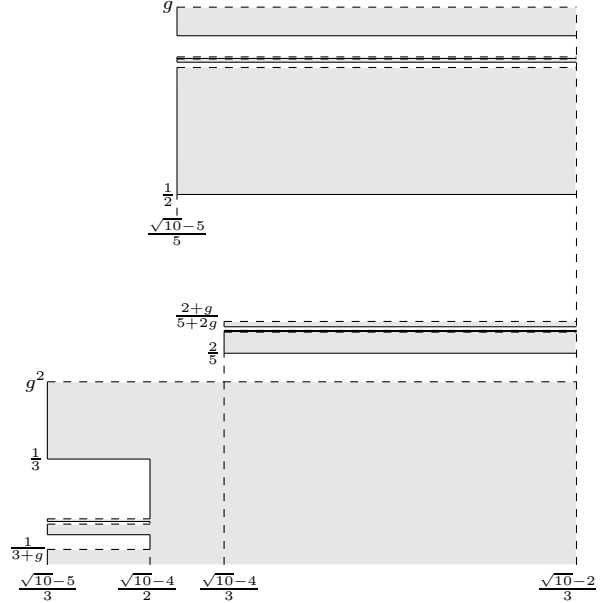FIGURE 16. $\Omega_{\frac{2}{5}}$, 'rough sketch'.

FIGURE 17. $\Omega_{\frac{\sqrt{10}-2}{3}}$, 'rough sketch'.

We will now show how to construct Ω_α from $\Omega_{2/5}$, with $\sqrt{10}-2)/3 < \alpha < 2/5$. Let $\alpha \in ((\sqrt{10}-2)/3, 2/5)$, $x \in [-3/5, 2/5)$ and $n \geq 0$ be the smallest integer for which $t_n \in [\alpha, 2/5]$, i.e., for which

$$(t_n, v_n) \in \bigcup_{k=0}^{\infty} \left([\alpha, \tfrac{2}{5}] \times \left([\tfrac{E_{2k}}{E_{2k+1}}, \tfrac{E_{2k}+g^2 E_{2k-1}}{E_{2k+1}+g^2 E_{2k}}) \cup [\tfrac{E_{2k+1}}{E_{2k+2}}, \tfrac{E_{2k}+g E_{2k-1}}{E_{2k+1}+g E_{2k}}) \right) \right).$$

We already know that in this case

$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n a_n, 3, -2, a_{n+3}, \ldots],$$

that we rewrite as

$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n(a_n + 1), -2, -3, -(a_{n+3} + 1), \ldots];$$

it is not hard to find that $a_{n+3} \geq 3$. Remember that in Section 5 we saw how much the case $\alpha \in (\sqrt{2} - 1, 1/2]$ resembled the case $\alpha \in (1/2, g]$. Here we find a similar resemblance between $\alpha \in ((\sqrt{10} - 2)/3, 2/5)$ and $\alpha \in [2/5, \sqrt{2} - 1]$. Of course, in the current case the continued fraction map is $T_{2/5}$ instead of $T_{\sqrt{2}-1}$, but straightforward calculations show that exactly the same equations for calculating the (t, v)-pairs for the regions R_α through $\mathcal{T}^2(A_\alpha)$ hold, which proves to be very convenient for the construction of Ω_α, $\alpha \in ((\sqrt{10} - 2)/3, 2/5)$. It appears that the transformation consists solely of expanding the regions already added and removed, in other words: both the protuberant parts as the coves are extended. Of special interest are the coves in the upper block: as α decreases, they scoop out the upper block until it splits in infinitely many rectangles at $t = (\sqrt{10} - 2)/3$; see Figure 17. Finally, extending our research from $[2/5, \sqrt{2} - 1]$ to $((\sqrt{10} - 2)/3, \sqrt{2} - 1]$ makes no difference for the way convergents are replaced by mediants as described earlier in this section.
We have now proved the following theorem:

Theorem 5. *Let $\alpha \in ((\sqrt{10} - 2)/3, \sqrt{2} - 1]$ and let the sequence E_n, $n \geq -1$, be defined by $E_{n+1} := 2E_n + E_{n-1}, n \geq 0$, with $E_{-1} := 1$ and $E_0 := 0$. Define*

$$V_{1,k} := [\tfrac{E_{2k+2}}{E_{2k+3}}, \tfrac{E_{2k+2}+g^2 E_{2k+1}}{E_{2k+3}+g^2 E_{2k+2}}) \cup [\tfrac{E_{2k+3}}{E_{2k+4}}, \tfrac{E_{2k+2}+g E_{2k+1}}{E_{2k+3}+g E_{2k+2}});$$

$$V_{2,k} := [\tfrac{E_{2k+1}+E_{2k}}{E_{2k+2}}, \tfrac{E_{2k+1}+E_{2k}+g^2(E_{2k}+E_{2k-1})}{E_{2k+2}+g^2 E_{2k+1}}) \cup [\tfrac{E_{2k+2}+E_{2k+1}}{E_{2k+3}}, \tfrac{E_{2k+1}+E_{2k}+g(E_{2k}+E_{2k-1})}{E_{2k+2}+g E_{2k+1}});$$

$$V_{3,k} := [\tfrac{E_{2k}}{E_{2k+1}+E_{2k}}, \tfrac{E_{2k}+g^2 E_{2k-1}}{E_{2k+1}+E_{2k}+g^2(E_{2k}+E_{2k-1})}) \cup [\tfrac{E_{2k+1}}{E_{2k+2}+E_{2k+1}}, \tfrac{E_{2k}+g E_{2k-1}}{E_{2k+1}+E_{2k}+g(E_{2k}+E_{2k-1})}).$$

Then

$$\Omega_\alpha = [\tfrac{1-3\alpha}{\alpha}, \alpha) \times [0, g^2) \cup [\tfrac{2-5\alpha}{3\alpha-1}, \alpha) \times [\tfrac{1}{2}, g)$$

$$\bigcup_{k=0}^{\infty} \left([\tfrac{2-5\alpha}{2\alpha-1}, \alpha) \times V_{1,k} \cup [\tfrac{1-2\alpha}{\alpha-1}, \tfrac{2-5\alpha}{3\alpha-1}) \times V_{2,k} \cup [\alpha - 1, \tfrac{1-3\alpha}{\alpha}) \times V_{3,k} \right).$$

7. THE LONG WAY DOWN TO g^2

In the previous section we rewrote

$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n a_n, 3, -2, \varepsilon_{n+3} a_{n+3}, \ldots]$$

as

$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n(a_n + 1), -2, -3, -\varepsilon_{n+3}(a_{n+3} + \varepsilon_{n+3}), \ldots].$$

In terms of Möbius transformations this is

$$
\begin{pmatrix} 0 & \varepsilon_n \\ 1 & a_n \end{pmatrix}
\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}
\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}
\begin{pmatrix} 0 & \varepsilon_{n+3} \\ 1 & a_{n+3} \end{pmatrix}
$$

(14)
$$
= \begin{pmatrix} 0 & \varepsilon_n \\ 1 & a_n + 1 \end{pmatrix}
\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}
\begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}
\begin{pmatrix} 0 & -\varepsilon_{n+3} \\ 1 & a_{n+3} + \varepsilon_{n+3} \end{pmatrix}
$$

$$
= \begin{pmatrix} 5\varepsilon_n & 5\varepsilon_n a_{n+3} + 3\varepsilon_n \varepsilon_{n+3} \\ 5a_n + 2 & 5a_n a_{n+3} + 3\varepsilon_{n+3} a_n \\ & +2a_{n+3} + \varepsilon_{n+3} \end{pmatrix};
$$

We can extend and then generalise this: just as

$$\begin{pmatrix} 0 & \varepsilon_n \\ 1 & a_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}}_{k-1 \text{ times}} \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_{n+k+2} \\ 1 & a_{n+k+2} \end{pmatrix}$$

$$(15) \quad = \begin{pmatrix} 0 & \varepsilon_n \\ 1 & a_n+1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}}_{k \text{ times}} \begin{pmatrix} 0 & -\varepsilon_{n+k+2} \\ 1 & a_{n+k+2}+\varepsilon_{n+k+2} \end{pmatrix}$$

$$= \begin{pmatrix} F_{2k+5}\varepsilon_n & F_{2k+5}\varepsilon_n a_{n+k+2}+F_{2k+4}\varepsilon_n\varepsilon_{n+k+2} \\ F_{2k+5}a_n+F_{2k+4} & F_{2k+5}a_n a_{n+k+2}+F_{2k+4}\varepsilon_{n+k+2}a_n \\ & +F_{2k+3}a_{n+k+2}+F_{2k+2}\varepsilon_{n+k+2} \end{pmatrix},$$

which can be derived from (14) by inserting $\begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}$-matrices at the right place, we have

$$[0; \varepsilon_1 a_1, \ldots, \varepsilon_n a_n, 3, \underbrace{-3, \ldots, -3}_{k-1 \text{ times}}, -2, \varepsilon_{n+k+2}a_{n+k+2}, \ldots]$$

$$=[0; \varepsilon_1 a_1, \ldots, \varepsilon_n(a_n+1), -2, \underbrace{-3, \ldots, -3}_{k \text{ times}}, -\varepsilon_{n+k+2}(a_{n+k+2}+\varepsilon_{n+k+2}), \ldots].$$

We observe that

$$\lim_{k\to\infty} [0; 3, \underbrace{-3, \ldots, -3}_{k-1 \text{ times}}] = g^2.$$

The good part of this is that it appears to be possible to apply compensated insertion beyond $(\sqrt{10}-2)/3$, by means of extending the $(3,-2)$-insertion by adding partial quotients -3 between 3 and -2. The bad part is that this possibility vanishes once we reach g^2. Taking the approach we took so far, the thing to do is letting α decrease from $(\sqrt{10}-2)/3$ to g^2 via the sequence $[0; 3, \underbrace{-3, \ldots, -3}_{k-1 \text{ times}}, -2, \varepsilon_{k+2}a_{k+2}, \ldots]$, starting with the case $k=2$, which is $[0; 3, -3, -2, \varepsilon_4 a_4, \ldots]$.

In the previous sections we have seen how our approach of singularisation and insertion gradually involved more details to process. We will show how things rapidly become even more complex, making it impossible to continue this approach much longer. So far, we have been able to construct Ω_α for all $\alpha \in [(\sqrt{10}-2)/3, 1)$. The next set of numbers to tackle would seem to consist of numbers $x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n a_n, 3, -3, -2, \varepsilon_{n+4}a_{n+4}, \ldots]$ and $t_n = [0; 3, -3, -2, \varepsilon_{n+4}a_{n+4}, \ldots]$, the largest of which is $(\sqrt{10}-2)/3 = [0; 3, \overline{-3, -2, -3, -4}]$. Let us first determine the interval (α', α), $\alpha' < \alpha \leq (\sqrt{10}-2)/3$, such that $t_n \in (\alpha', \alpha)$ implies $t_n = [0; 3, -3, -2, \ldots]$. We remark that

$$f_2(x) := \cfrac{1}{3 - \cfrac{1}{3 - \frac{1}{x}}},$$

where the index 2 indicates the number of successive 3s, is a decreasing function of x on $\mathbb{R} \setminus \{0\}$, so if $\alpha' \in [0, (\sqrt{10}-2)/3)$ and $\alpha \in (\alpha', (\sqrt{10}-2)/3]$, then

$$(16) \quad 2 = \left\lfloor f((\sqrt{10}-2)/3) + 1 - (\sqrt{10}-2)/3 \right\rfloor \leq a_3(\alpha'_\alpha) = \lfloor f(\alpha')+1-\alpha \rfloor \leq \lfloor f(\alpha')+1-\alpha' \rfloor,$$

from which we derive $a_3(\alpha'_\alpha) = 2$ if and only if $f(\alpha')+1-\alpha' < 3$. Given that $\alpha' \in [0, (\sqrt{10}-2)/3))$, this is the case if and only if $(\sqrt{65}-5)/8 < \alpha' < (\sqrt{10}-2)/3$; indeed

$$(\sqrt{65}-5)/8_{(\sqrt{65}-5)/8} = [0; 3, \overline{-3, -3, -2, -3, -3, -4}].$$

So the next step of our investigation would be the case $\alpha \in \Delta_\alpha(3, -3, -2) = ((\sqrt{65}-5)/8, (\sqrt{10}-2)/3]$. Proceeding in a similar way, a sequence of successive intervals to be investigated would

come into view, the boundaries of which are the positive roots R_k of

$$f_k(x) := 1/\underbrace{(3 - 1/(3 - \ldots 1/(3-1/x)\ldots))}_{k \text{ times a } 3} - x - 2.$$

Straightforward calculation shows that

$$R_k = \frac{\sqrt{F_{2k+1}F_{2k+3}} - F_{2k+1}}{F_{2k+2}}$$

and

$$\lim_{k\to\infty} R_k = \lim_{k\to\infty}\left(\sqrt{\frac{F_{2k+1}F_{2k+3}}{F_{2k+2}^2}} - \frac{F_{2k+1}}{F_{2k+2}}\right) = \sqrt{g \cdot G} - g = 1 - g = g^2,$$

as was to be expected. For our investigations we would also need that

$$R_{k\,R_k} = [0; 3, \underbrace{-3, \ldots, -3}_{k \text{ times}}, -2, \underbrace{-3, \ldots, -3}_{k \text{ times}}, -4].$$

Although this perspective would already promise a lot of intricacies, when we turn back to the case $\alpha \in ((\sqrt{65}-5)/8, (\sqrt{10}-2)/3]$, we are confronted with a problem we had not encountered yet: if we apply compensated insertion to numbers equal to or only slightly smaller than $(\sqrt{10} - 2)/3$, we get

$$\alpha_\alpha = [0; 3, -3, -2, -3, -a, \ldots] = [1; -2, -3, -3, 2, -a, \ldots], \ \ a \geq 4.$$

Of course, the occurrence of the partial quotient 2 is not allowed, which calls for one more compensated insertion, so as to get

$$\alpha_\alpha = [0; 3, -3, -2, -3, -a, \ldots] = [1; -2, -3, -3, 2, -a, \ldots] = [1; -2, -3, -4, -2, a - 1, \ldots], \ \ a \geq 4.$$

To determine the interval where these numbers occur, we have to solve

$$\cfrac{1}{2 - \cfrac{1}{3 - \cfrac{1}{3 - \cfrac{1}{\alpha}}}} + 1 - \alpha = 4.$$

The positive solution of this equation is

$$(5\sqrt{13} - 13)/13_{(5\sqrt{13}-13)/13} = [0; 3, \overline{-3, -2, -4, -2, -3, -4, -2, -4}] = 0.386750\ldots.$$

So the next case to investigate is actually $\alpha \in \Delta_\alpha(3, -3, -2, -3) = ((5\sqrt{13} - 13)/13, (\sqrt{10} - 2)/3]$. More precisely, let $x \in [(\sqrt{10} - 5)/3, (\sqrt{10} - 2)/3)$ and $n \geq 0$ be the smallest integer for which $t_n \in [\alpha, (\sqrt{10} - 2)/3]$, i.e., for which

$$(t_n, v_n) \in \bigcup_{k=0}^{\infty}\left\{[\alpha, (\sqrt{10} - 2)/3] \times \left([\tfrac{E_{2k}}{E_{2k+1}}, \tfrac{E_{2k}+g^2E_{2k-1}}{E_{2k+1}+g^2E_{2k}}) \cup [\tfrac{E_{2k+1}}{E_{2k+2}}, \tfrac{E_{2k}+gE_{2k-1}}{E_{2k+1}+gE_{2k}})\right)\right\}.$$

Above we saw that in this case

(17)
$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n a_n, 3, -3, -2, -3, -a_{n+5}, \ldots] \ , a_{n+5} \geq 4, \text{ that we rewrite as}$$
$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n(a_n + 1), -2, -3, -4, -2, a_{n+5} - 1, \ldots] \ , a_{n+5} \geq 4.$$

We now have $(t_n^*, v_n^*) = (t_n - 1, \frac{v_n}{v_n+1})$ and

$$\begin{aligned}
(t_{n+1}, v_{n+1}) &= (\tfrac{1-3t_n}{t_n}, \tfrac{1}{v_n+3}) & (t_{n+1}^*, v_{n+1}^*) &= (\tfrac{1-2t_n}{t_n-1}, \tfrac{v_n+1}{v_n+2}) \\
(t_{n+2}, v_{n+2}) &= (\tfrac{3-8t_n}{3t_n-1}, \tfrac{v_n+3}{3v_n+8}) & (t_{n+2}^*, v_{n+2}^*) &= (\tfrac{2-5t_n}{2t_n-1}, \tfrac{v_n+2}{2v_n+5}) \\
(t_{n+3}, v_{n+3}) &= (\tfrac{5-13t_n}{8t_n-3}, \tfrac{3v_n+8}{5v_n+13}) & (t_{n+3}^*, v_{n+3}^*) &= (\tfrac{7-18t_n}{5t_n-2}, \tfrac{2v_n+5}{7v_n+18}) \\
(t_{n+4}, v_{n+4}) &= (\tfrac{12-31t_n}{13t_n-5}, \tfrac{5v_n+13}{12v_n+31}) & (t_{n+4}^*, v_{n+4}^*) &= (\tfrac{12-31t_n}{18t_n-7}, \tfrac{7v_n+18}{12v_n+31}), \\
(t_{n+5}, v_{n+5}) &= (t_{n+5}^*, v_{n+5}^*)
\end{aligned}$$

Although it would still be possible to explicitly give Ω_α for $\alpha \in ((5\sqrt{13}-13)/13, (\sqrt{10}-2)/3]$, it may be clear that it becomes quite unmanageable. Determining the effect of the current transformation on the sequence of convergents is still easy, though. Since $p_k^* = p_k$ and $q_k^* = q_k$, $k < n$, we find, applying (4) and referring to (17) (while omitting the straightforward calculations),

$$\begin{cases} p_n^* = p_n + p_{n-1}; \\ p_{n+1}^* = p_{n+1} - p_n; \\ p_{n+2}^* = p_{n+2} - p_{n+1}; \\ p_{n+3}^* = p_{n+4} - p_{n+3}; \\ p_{n+4}^* = p_{n+4}; \\ p_{n+5}^* = p_{n+5}. \end{cases}$$

Of course, we find similar relations for q_n^* through q_{n+5}^*. We conclude that in the transformation from $\Omega_{(\sqrt{10}-2)/3}$ to Ω_α, $\alpha \in [(5\sqrt{13}-13)/13, (\sqrt{10}-2)/3)$, the convergent p_n/q_n is replaced by the mediant $(p_n+p_{n-1})/(q_n+q_{n-1})$, that p_{n+1}/q_{n+1} is replaced by the mediant $(p_{n+1}-p_n)/(q_{n+1}-q_n)$, p_{n+2}/q_{n+2} is replaced by the mediant $(p_{n+2}-p_{n+1})/(q_{n+2}-q_{n+1})$ and that p_{n+3}/q_{n+3} is replaced by the mediant $(p_{n+4}-p_{n+3})/(q_{n+4}-q_{n+3})$.

A substantial part of our construction of Ω_α, $(\sqrt{10}-2)/3 \leq \alpha < 1$, consisted in infinitely repeating the removal and addition of rectangles in $[(\sqrt{10}-5)/3, 1] \times [0,1]$, in the cases that part of the added regions overlapped part of the regions to be removed. In Section 4, for instance, we worked at the case $x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n a_n, 2, -a_{n+2}, \ldots], a_{n+2} \geq 3$, and rewrote it as

$$(18) \qquad x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n(a_n+1), -2, (a_{n+2}-1), \ldots].$$

The overlapping stems from the possibility that (18) can be written as

$$(19) \qquad x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n(a_n+1), -2, 2, \varepsilon_{n+3}a_{n+3}, \ldots], \ a_{n+3} \geq 3,$$

i.e. when $a_{n+2} = 3$. One could ask whether the work of infinite removal and addition could have possibly been avoided by immediately performing one more compensated insertion and so rewriting (19) as

$$x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n(a_n+1), -3, -2, -\varepsilon_{n+3}(a_{n+3}+\varepsilon_{n+3}), \ldots].$$

This, however, confronts us with some intricacies, among which $t_{n+2}^* \neq t_{n+2}$, that appear to at least cancel the benefits of this approach. Earlier in this section we showed that for $\alpha \leq (\sqrt{10}-2)/3$, this repeated compensated insertion is unavoidable, as is a further complication of the construction of Ω_α. One obvious problem is the growth of the number of regions to be removed and added in each step of construction, due to the lengthening of the continued fraction expansions of the numbers involved on the way down to $g^2 = 0.381966\ldots$, the first of which are (in decreasing order):

$$(5\sqrt{13}-13)/13_{(5\sqrt{13}-13)/13} = [0; 3, \overline{-3, -2, -4, -2, -3, -4, -2, -4}]$$
$$= 0.3867504\ldots,$$
$$5/13_{5/13} = [0; 3, -3, -2]$$
$$= 0.3846153\ldots,$$
$$(\sqrt{65}-5)/8_{(\sqrt{65}-5)/8} = [0; 3, \overline{-3, -3, -2, -3, -3, -4}]$$
$$= 0.3827822\ldots,$$

$$(\sqrt{14401} - 89)/81_{(\sqrt{14401}-89)/81} = [0; 3, -3, -3, -2, -3, -4, -2, -3, -3, -4, -3, -2, \ldots]$$
$$= 0.3827674\ldots,$$
$$(\sqrt{2210} - 34)/34_{(\sqrt{2210}-34)/34} = [0; 3, \overline{-3, -3, -2, -4, -2, -3, -3, -4, -2, -4}]$$
$$= 0.3826657\ldots,$$
$$13/34_{13/34} = [0; 3, -3, -3, -2]$$
$$= 0.3823529\ldots,$$
$$(\sqrt{442} - 13)/21_{(\sqrt{442}-13)/21} = [0; 3, \overline{-3, -3, -3, -2, -3, -3, -3, -4}]$$
$$= 0.3820855\ldots,$$
$$(\sqrt{670762} - 610)/547_{(\sqrt{670762}-610)/547} = [0; 3, -3, -3, -3, -2, -3, -3, -4, -2, -3, -3, -3\ldots]$$
$$= 0.3820852\ldots$$

The red thread in this list is the first appearance of the partial quotient 2 with minus sign in the continued fraction expansion α_α of the related decreasing numbers α, always preceded by the sequence $3, \underbrace{-3, \ldots, -3}_{k \text{ times}}$, $k > 0$. With k increasing, the number of partial quotients 3 with minus sign following this 2 before the first appearance of the partial quotient 4 with minus sign may be from 0 to k, each case calling for an α-fundamental interval to be investigated separately. On our way down from $(\sqrt{65} - 5)/8 = [0; 3, \overline{-3, -3, -2, -3, -3, -4}]$ to $(\sqrt{442} - 13)/21_{(\sqrt{442}-13)/21} = [0; 3, \overline{-3, -3, -3, -2, -3, -3, -3, -4}]$, for instance, we have to distinguish between the cases associated with the sequences starting with $3, -3, -3, -2, -3, -3, -4$, then $3, -3, -3, -2, -3, -4$, then $3, -3, -3, -2, -4$ and $3, -3, -3, -2$ before arriving at $3, -3, -3, -3, \ldots$.

In sections 3, 5 and 6 we saw how (compensated) insertion in most cases involves loss of convergents, although for $\alpha \leq 1/2$ each convergent is replaced by a mediant of the form $(p_k \pm p_{k-1})/(q_k \pm q_{k-1})$ for some $k \in \mathbb{N}$. As α decreases beyond $(5\sqrt{13} - 13)/13$, this will remain the case, which we can best illustrate by again using Möbius transformations. If p_{n-1} and p_n are two consecutive denominators of a convergent of some number $x = [0; \varepsilon_1 a_1, \ldots, \varepsilon_{n-1} a_{n-1}, \varepsilon_n a_n, \ldots]$, then

$$\begin{pmatrix} p_{n+1} \\ p_n \end{pmatrix} = \begin{pmatrix} a_{n+1} & \varepsilon_{n+1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix}.$$

The denominators p^*_{n+k}, $k \in \{0, \ldots, 5\}$ that we calculated in the previous section, can easily be derived by comparing the partial products of

$$\begin{pmatrix} p^*_{n+5} \\ p^*_{n+4} \end{pmatrix} = \begin{pmatrix} a_{n+5} - 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\cdot \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n + 1 & \varepsilon_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ p_{n-2} \end{pmatrix}$$

with those of

$$\begin{pmatrix} p_{n+5} \\ p_{n+4} \end{pmatrix} = \begin{pmatrix} a_{n+5} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\cdot \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n & \varepsilon_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ p_{n-2} \end{pmatrix},$$

associated with the sequence $\varepsilon_n(a_n + 1), -2, -3, -4, -2, a_{n+5} - 1$ as derived from the sequence $\varepsilon_n a_n, 3, -3, -2, -3, -a_{n+5}$ by applying compensated insertion.
On the interval $((\sqrt{14401} - 89)/81, (\sqrt{65} - 5)/8]$ we could similarly compare the partial products

of

$$\begin{pmatrix} p_{n+7}^* \\ p_{n+6}^* \end{pmatrix} = \begin{pmatrix} a_{n+7}-1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\cdot \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n+1 & \varepsilon_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ p_{n-2} \end{pmatrix}$$

with those of

$$\begin{pmatrix} p_{n+7} \\ p_{n+6} \end{pmatrix} = \begin{pmatrix} a_{n+7} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\cdot \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n & \varepsilon_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ p_{n-2} \end{pmatrix},$$

associated with the sequence $\varepsilon_n(a_n + 1), -2, -3, -3, -4, -3, -2, a_{n+7} - 1$ as derived from the sequence $\varepsilon_n a_n, 3, -3, -3, -2, -3, -3, -a_{n+7}$ by applying compensated insertion. It is not hard to see that as α decreases to g^2, it is just the number of lost convergents that increases, but not the way in which they are replaced by mediants as described above; the difference is merely the number of matrices $\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$ in both products, similar to the difference made by the number of matrices $\begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}$ in (15).

8. The ergodic systems $(\Omega_\alpha, \mathcal{B}, \mu_\alpha, \mathcal{T}_\alpha)$

Starting with Ω_1, we constructed domains $\Omega_{\alpha'}$ by removing sets of points from a given Ω_α, with $\alpha' < \alpha$, and adding other sets to it. Each stage of construction – corresponding with successive sections of this paper – had its own characteristic set of singularisations, such as simply singularising a partial quotient 1 for $\alpha' \in (g, 1]$. The first step of constructing consisted of fixing a subset $R_{\alpha'}$ of Ω_α, consisting of all points $(t, v) \in \Omega_\alpha$ for which $\alpha' \le t < \alpha$.
Now let the collection of subsets of Ω_α be denoted by $\mathcal{B}$ and μ_α be defined as the probability measure with density

$$\frac{1}{N_\alpha} \cdot \frac{1}{(1+tv)^2}$$

on $(\Omega_\alpha, \mathcal{B})$, where N_α is a normalising constant. In [9], Nakada showed that for $\alpha \in [1/2, 1]$ the dynamical systems $(\Omega_\alpha, \mathcal{B}, \mu_\alpha, \mathcal{T}_\alpha)$ are ergodic (he actually obtained stronger mixing properties); see also [6]. Here we will use this result for $\alpha = 1$ only, which was the starting point in [6] as well. Note that for $g^2 < \alpha < g$ our construction actually is a bijection between $\Omega_{\alpha'}$ and Ω_α, $\alpha' < \alpha$. Clearly this is not the case for $g < \alpha' < \alpha \le 1$, where one region is added to Ω_α while two are removed from it when constructing $\Omega_{\alpha'}$.

We will now show that μ_α is an invariant measure on Ω_α. Let $x_\alpha = [0; \varepsilon_1 a_1, \ldots, \varepsilon_n a_n, \varepsilon_{n+1} a_{n+1}, \ldots]$ and $D := [t_1, t_2] \times [v_1, v_2] \subset R_{\alpha'}$. Then

$$m(D) := \iint_D \frac{1}{(1+tv)^2} \, dt \, dv = \log \frac{(1+t_2 v_2)(1+t_1 v_1)}{(1+t_2 v_1)(1+t_1 v_2)}.$$

Let $D_k := \mathcal{T}_\alpha^k(D)$ and M_n as the Möbius transformation associated with the matrix $\begin{pmatrix} 0 & \varepsilon_n \\ 1 & a_n \end{pmatrix}$. We define $\widetilde{M}_k := M_{n+1} \cdots M_{n+k}$. From (5) we derive that

$$D_k = \left[\widetilde{M}_k^{-1}(t_1), \widetilde{M}_k^{-1}(t_2) \right] \times \left[\widetilde{M}_k^T(v_1), \widetilde{M}_k^T(v_2) \right],$$

from which it follows that

$$m(D_k) = \iint_{D_k} \frac{1}{(1+tv)^2} \, dt \, dv$$

$$= \log \frac{(1 + \widetilde{M}_k^{-1}(t_2)\widetilde{M}_k^T(v_2))(1 + \widetilde{M}_k^{-1}(t_1)\widetilde{M}_k^T(v_1))}{(1 + \widetilde{M}_k^{-1}(t_2)\widetilde{M}_k^T(v_1))(1 + \widetilde{M}_k^{-1}(t_1)\widetilde{M}_k^T(v_2))}$$

$$= \log \frac{|\widetilde{M}_k|(t_2v_2 + 1)|\widetilde{M}_k|(t_1v_1 + 1)}{|\widetilde{M}_k|(t_2v_1 + 1)|\widetilde{M}_k|(t_1v_2 + 1)}$$

$$= \log \frac{(1 + t_2v_2)(1 + t_1v_1)}{(1 + t_2v_1)(1 + t_1v_2)} = m(D). \qquad (*)$$

Rewriting

$$x_\alpha = [0; \varepsilon_1 a_1, \varepsilon_2 a_2, \dots, \varepsilon_n a_n, a_{n+1}, \varepsilon_{n+2} a_{n+2}, \dots] \text{ as}$$

$$x_\alpha = [0; \varepsilon_1 a_1, \varepsilon_2 a_2, \dots, \varepsilon_n(a_n + 1), -a_{n+1}^*, \varepsilon_{n+2}^* a_{n+2}^*, \dots],$$

as we have done throughout this paper, we define $D^* := [t_1 - 1, t_2 - 1] \times [v_1/(1+v_1), v_2/(1+v_2)] \in A_{\alpha'}$. With a straightforward calculation one finds that $m(D^*) = m(D)$. Similarly to what we have written above, we define $D_k^* := \mathcal{T}_\alpha^k(D^*)$ and let M_n^* be the Möbius transformation associated with the matrix $\begin{pmatrix} 0 & \varepsilon_n^* \\ 1 & a_n^* \end{pmatrix}$. We define $\widetilde{M}_k^* := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} M_{n+1}^* \cdots M_{n+k}^*$, where $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is associated with mapping $R_{\alpha'}$ to $A_{\alpha'}$. Again applying (5), we find that

$$D_k^* = \left[(\widetilde{M}_k^*)^{-1}(t_1), (\widetilde{M}_k^*)^{-1}(t_2) \right] \times \left[(\widetilde{M}_k^*)^T(v_1), (\widetilde{M}_k^*)^T(v_2) \right],$$

from which it follows that

$$m(D_k^*) = \iint_{D_k} \frac{1}{(1+tv)^2} \, dt \, dv$$

$$= \log \frac{(1 + (\widetilde{M}_k^*)^{-1}(t_2)(\widetilde{M}_k^*)^T(v_2))(1 + (\widetilde{M}_k^*)^{-1}(t_1)(\widetilde{M}_k^*)^T(v_1))}{(1 + (\widetilde{M}_k^*)^{-1}(t_2)(\widetilde{M}_k^*)^T(v_1))(1 + (\widetilde{M}_k^*)^{-1}(t_1)(\widetilde{M}_k^*)^T(v_2))}$$

$$= \log \frac{|\widetilde{M}_k^*|(t_2v_2 + 1)|\widetilde{M}_k^*|(t_1v_1 + 1)}{|\widetilde{M}_k^*|(t_2v_1 + 1)|\widetilde{M}_k^*|(t_1v_2 + 1)}$$

$$= \log \frac{(1 + t_2v_2)(1 + t_1v_1)}{(1 + t_2v_1)(1 + t_1v_2)} = m(D).$$

From this and (*) it follows that $\mu(D_k^*) = \mu(D)$ for any pair of α-measurable sets D_k and D. Since the dynamical system $(\Omega_1, \mathcal{B}, \mu_1, T_1)$ is ergodic, from our construction it easily follows that $(\Omega_\alpha, \mathcal{B}, \mu_\alpha, \mathcal{T}_\alpha)$ forms an ergodic system as well. Indeed, from our construction and regarding the above observations on m, it follows that for $\alpha \in [g^2, g)$, all Ω_α are isomorphic to Ω_g, so all dynamical systems $(\Omega_\alpha, \mathcal{B}, \mu_\alpha, \mathcal{T}_\alpha)$, $\alpha \in [g^2, 1)$, will 'inherit' ergodicity from $(\Omega_g, \mathcal{B}, \mu_g, \mathcal{T}_g)$. Obviously, for $\alpha \in [g, 1)$ the dynamical systems $(\Omega_\alpha, \mathcal{B}, \mu_\alpha, \mathcal{T}_\alpha)$ are *not* isomorphic to Ω_1, but from [6] we know they are isomorphic to an induced transformation of Ω_1, which is also clear from our construction; for $g < \alpha < 1$, let $I_\alpha \subseteq \Omega_\alpha$ be invariant under $\mathcal{T}_\alpha$. We define $I_{\alpha,1} := I_\alpha \cap A_\alpha$ and $I_{\alpha,2} := I_\alpha \setminus I_{\alpha,1}$. Then $I_{\alpha,2} \subseteq \Omega_1$. Furthermore, putting $I_{1,1} := \mathcal{M}^{-1}(I_{\alpha,1})$ (with $\mathcal{M}$ the map defined in Section 3), $I_{1,2} := \mathcal{T}_1(I_{1,1})$, and $I_1 := I_{\alpha,2} \cup I_{1,1} \cup I_{1,2}$, then $I_1 \subseteq \Omega_1$ is by construction $\mathcal{T}_1$-invariant, and therefore I_1 has μ_1-measure 0 or 1. Since $m(I_{\alpha,1}) = m(I_{1,1}) = m(I_{1,2})$, it follows that $m(I_\alpha) = m(I_1) - m(I_{1,1})$. If $\mu_1(I_1) = 0$, we are done. If $\mu_1(I_1) = 1$, we see that $m(I_\alpha) = \log 2(\mu_1(I_1) - \mu_1(I_{1,1})) = \log 2(1 - \mu_1(R_\alpha)) = \log 2 - m(R_\alpha) = N_\alpha$, and we see that

$$\mu_\alpha(I_\alpha) = \frac{1}{N_\alpha} m(I_\alpha) = 1;$$

so if $I_\alpha \subseteq \Omega_\alpha$ is a measurable $\mathcal{T}_\alpha$-invariant set, we find that $\mu_\alpha(I_\alpha) \in \{0, 1\}$, and we conclude that $\mathcal{T}_\alpha$ is ergodic.

Writing $h(\mathcal{T}_\alpha)$ for the *entropy* of $\mathcal{T}_\alpha$, in [9] Nakada also proved that

$$h(\mathcal{T}_\alpha) = \begin{cases} \frac{\pi^2}{6\log(1+\alpha)}, & \alpha \in (g, 1]; \\ \frac{\pi^2}{6\log(G)}, & \alpha \in [\frac{1}{2}, g]. \end{cases}$$

For $g < \alpha < 1$ this result is essentially due to the fact that Ω_α is isomorphic to an appropriate induced transformation of $(\Omega_1, \mathcal{B}, \mu_1, T_1)$ and Abramov's formula; see [6] for explicit details. This explains the increase of entropy as α decreases from 1 to g. In the previous sections, we have shown that for $\alpha \in (g^2, g]$, the loss of R_α is completely compensated by the addition of A_α, from which it follows that the systems $(\Omega_\alpha, \mathcal{B}, \mu_\alpha, \mathcal{T}_\alpha)$ are metrically isomorphic to $(\Omega_g, \mathcal{B}, \mu_g, \mathcal{T}_g)$. We conclude that $h(\mathcal{T}_\alpha) = \pi^2/(6\log G), \alpha \in (g^2, g]$, a result previously obtained also in [7]. For α smaller than g^2, the entropy map $h(\alpha)$ behaves behaves quite irregularly; see [8, 11] and Giulio Tiozzo's thesis [13] for further details.

9. THE α-LEGENDRE CONSTANT

In ([10]), Nakada obtains a very strong connection between two constants that play an important role in the study of continued fractions: the *Legendre constant* and the *Lenstra constant*.
The Legendre constant is associated with the theorem of Legendre, mentioned on page 4, where we also introduced the α-Legendre constant

$$L(\alpha) := \sup\{c > 0 : q^2 \left| x - \tfrac{p}{q} \right| < c, \gcd(p, q) = 1 \Rightarrow \tfrac{p}{q} = \tfrac{p_n}{q_n}, n \geq 0\}.$$

The Lenstra constant is associated with the following conjecture by W. Doeblin and Hendrik Lenstra on regular continued fractions, proved by Wieb Bosma, Hendrik Jager and Freek Wiedijk in [3]:

Theorem 6. *For almost all x and all $t \in [0, 1]$, the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{\infty} \{1 \leq j \leq n : \theta_j \leq t\}$$

exists, and is equal to the distribution function

$$F(t) = \begin{cases} \frac{t}{\log 2}, & 0 \leq t \leq 1/2, \\ \frac{1}{\log 2}(1 - t + \log 2t), & 1/2 \leq t \leq 1. \end{cases}$$

The number $1/2$, the upper bound of the interval where $F(t)$ is linear, is called the *Lenstra constant of regular continued fractions*; for α-expansions the Lenstra constant can be defined similarly.

Let $\tilde{g} := \dfrac{g - 2 + \sqrt{g^2 + 4}}{2g} = 0.57549\ldots$. In 1988, Shunji Ito ([5]; see also [6]) proved that

$$L(\alpha) = \begin{cases} \dfrac{\alpha}{\alpha + 1}, & g \leq \alpha \leq 1; \\ 1 - \alpha, & \tilde{g} \leq \alpha \leq g; \\ \dfrac{\alpha}{1 + g\alpha}, & \frac{1}{2} \leq \alpha \leq \tilde{g}. \end{cases}$$

In 2010, Nakada proved in [10] that for a continued fraction algorithm the Legendre constant equals the Lenstra constant whenever the Legendre constant exists; one year later Natsui ([12]) proved the existence of the α-Legendre constant for $0 < \alpha \leq 1$. She also extends Ito's result to

$$L(\alpha) = \frac{\alpha}{1 + g\alpha}, \quad \sqrt{2} - 1 \leq \alpha \leq \tilde{g}.$$

In this section we will in turn extend Natsui's result to $g^2 \leq \alpha < \tilde{g}$. Moreover, we will show how we can use a result of Laura Luzzi and Stefano Marmi ([8]) to make some observations on $L(\alpha)$ for $\alpha < g^2$.

From Section 7 it is clear that it is both very hard and unmanageable to give Ω_α explicitly for $\alpha \in [g^2, (\sqrt{10} - 2)/3]$. We can, however, give some useful characteristics for this case with regard to $L(\alpha)$. The first thing we note is that $0 \leq v < g$ for $(t, v) \in \Omega_\alpha$, where $\alpha \in [g^2, 1/2]$. This follows from the fact that in our method of constructing Ω_α from $\Omega_{\alpha'}$, with $\alpha < \alpha'$, any -2 that is introduced by insertion and singularisation is precedented by either a 3 with minus sign or at least a 4. Any second coordinate v in Ω_α will therefore be smaller than $1/(2 - g^2) = g$. From this it follows that the bottom side of the lowermost cove in Figure 18 is on the line $v = 1/(3 + g)$; see our calculations in Section 6. Figure 18 is an example of Ω_α with α near $(\sqrt{10} - 2)/3$, and is more or less the same as Figures 16 and 17. The difference between Figure 18 and the other two figures is the addition of curves associated with $\theta_{n-1} = \alpha/(1 + g\alpha)$ and $\theta_{n-1} > \alpha/(1 + g\alpha)$, the choice of which we will explain now.

Recall that in the introduction we defined $\theta_n(x) := q_n^2 |x - p_n/q_n|$ and mentioned the equations $\theta_{n-1} = v_n/(1 + t_n v_n)$ and $\theta_n = (\varepsilon_n t_n)/(1 + t_n v_n)$ (omitting the suffix '(x)'). In [6], $L(\alpha)$ is found by determining the largest C for which the curve $\theta_n = C$ contains no points in what is called a singularisation area: indeed, this area is associated with convergents that are replaced (when $1/2 < \alpha \leq g$) or lost (when $g < \alpha < 1$ or $\alpha \leq 1/2$), as we have shown in Sections 3 through 7. Note that this C is also the Lenstra constant for the α-expansion under consideration. In the current case ($g^2 \leq \alpha < \sqrt{2} - 1$), we find it convenient to consider curves $\theta_{n-1} = C$ rather than $\theta_n = C$, if only for the numerator of $\theta_n = \varepsilon_n t_n/(1 + t_n v_n)$, yielding two branches that we would have to take into account separately. Since curves associated with the equation $\theta_{n-1} = C$, C small enough, will be in the lower part of Ω_α, a first guess would be that $L(\alpha)$ is determined by either the points $M_\alpha := ((1 - 3\alpha)/\alpha, 1/(3 + g))$ or $N_\alpha := (\alpha, g^2)$. No matter how intricate Ω_α becomes for $\alpha \leq \sqrt{2} - 1$, from our earlier obeservations it follows that these v-coordinates $1/(3 + g)$ and g^2 do not change as α decreases to g^2. In Figure 18 we see how curves associated with $\theta_{n-1} > \alpha/(1 + g\alpha)$ contain points in the cove left and above M_α. We find $\theta_{n-1}(M_\alpha) = \alpha/(1 + g\alpha)$, while $\theta_{n-1}(N_\alpha) = 1/(G^2 + \alpha)$. Since

$$\min\{\alpha/(1 + g\alpha), 1/(G^2 + \alpha)\} = \alpha/(1 + g\alpha)$$

on $[g^2, \sqrt{2} - 1)$, we conclude that

$$L(\alpha) = \frac{\alpha}{1 + g\alpha}, \quad g^2 \leq \alpha < \sqrt{2} - 1,$$
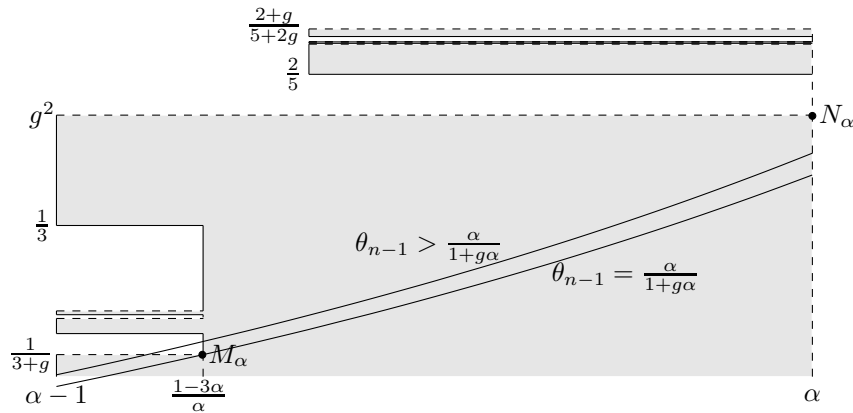
hence extending Natsui's result.

FIGURE 18. The lower part of Ω_α, with α near $\frac{\sqrt{10} - 2}{3}$.

For various reasons, this approach cannot be extended to values of $\alpha < g^2$. For one thing, g will cease to be the upper bound for the values of v, since insertions and singularisations in the case $\alpha < g^2$ involve strings of more than one -2, yielding values of v larger than g. This implies that points have to be removed with v-values under $1/(3 + g)$. Unfortunately, we cannot say much more

about Ω_α for general α smaller than g^2. Still, a result of Luzzi and Marmi (see [8], pages 27 and 28) enables us to at least find how $L(\alpha)$ evolves for these rational values of α. In [8], Luzzi and Marmi use the aforementioned strings of partial quotients 2 with minus sign to give a description of Ω_α for $\alpha = 1/r$, $r \in \mathbb{N}_{\geq 3}$. Although their description is not very explicit, it is not hard to find that it yields that $\Omega_{1/r}$ consists of the bottom part

$$[-1 + 1/r, 0] \times [0, (r + 1 - \sqrt{(r + 1)^2 - 4})/2] \cup [0, 1/r] \times [0, (\sqrt{(r + 1)^2 - 4} - r + 1)/(2r - 2)]$$

and a myriad of vertically disjunct rectangles above it; see Figure 19 for the bottom part. Taking the approach of considering curves $\theta_{n-1} = C$ as we did above, we see that the Lenstra constant for $\alpha = 1/r$, $L(1/r)$ is determined by the vertex $(0, (r + 1 - \sqrt{(r + 1)^2 - 4})/2)$ (which is the point $(0, (5 - \sqrt{21})/2)$ in Figure 19). Thanks to Nakada's result in [10] it follows that

$$(20) \qquad L\left(\tfrac{1}{r}\right) = \frac{r + 1 - \sqrt{(r + 1)^2 - 4}}{2}.$$

Although this formula is different from the one we found for $g^2 \leq \alpha \leq 1/2$, we remark that $L(1/2) = g^2$ fits both formulas. What's more, we suspect that with (20) a sharpening is possible of Natsui's result on the boundaries of $L(\alpha)$ in [12], which is:

Theorem 7. *Let* $r \in \mathbb{N}$. *If* $\frac{1}{r+1} < \alpha < \frac{1}{r}$, *then* $\frac{1}{r+2} < L(\alpha) < \frac{1}{r}$, *with* $r \in \mathbb{N}_{\geq 3}$.
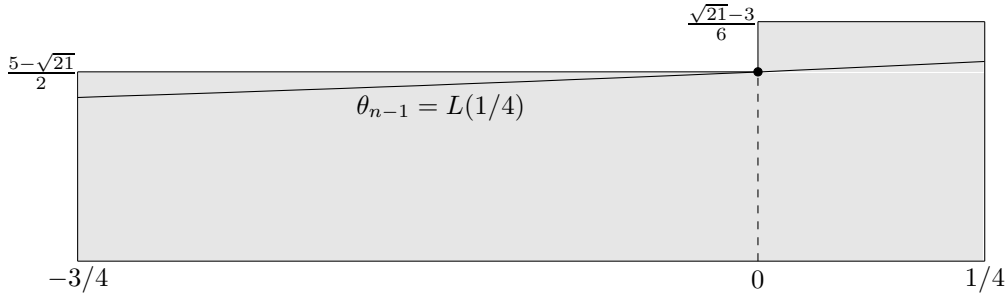

FIGURE 19.   The bottom part of $\Omega_{1/4}$.

Since (20) yields

$$(21) \qquad L(\tfrac{1}{r}) = \frac{r + 1 - \sqrt{(r + 1)^2 - 4}}{2} = \frac{1}{r + 1 - L(\frac{1}{r})},$$

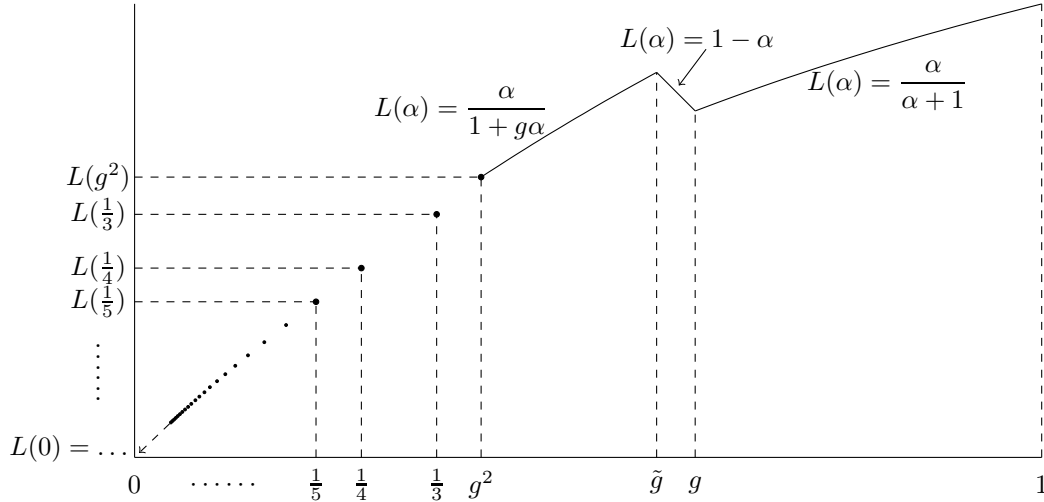we suspect that in fact (if $1/(r + 1) < \alpha < 1/r$)

$$\frac{1}{r + 2} < L(\alpha) < \frac{1}{r + 1 - L(\frac{1}{r})}.$$

Considering this figure, one would expect that $L(0) = 0$; this is actually true, and we will return on this in a forthcoming note.

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FIGURE 20. The graph of $L(\alpha)$ as it is currently known.

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