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--Manuscript Draft--

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The total word count limit is 7500 words including tables. Each table equals 250 words and must be included in your count. Papers exceeding the word limit may be rejected. My word count is:	6984
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TRAFFIC ASSIGNMENT OF MULTI-MODAL TRANSPORT NETWORKS BASED ON AN AUGMENTED LINK-BASED SUPER-NETWORK MODEL

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ABSTRACT

Multi-modal transport is getting more popular due to the emergence of new traffic modes. The increase of modes also adds complexity for the transport researchers. This paper proposes an augmented link-based super-network approach for modeling multi-modal transport networks, addressing the scalability and versatility issues of conventional methods. This approach is used to calculate the user equilibrium for urban transport networks traffic assignment with multiple traffic modes, a difficult problem due to the intractable enumeration of feasible paths between origin-destination pairs and restricted transfers between different traffic modes. In the super-network representation of multi-modal transport networks, the travel cost of any feasible route between the origin and destination is formulated as the sum of cost functions of the augmented links, thus avoiding the enumeration of feasible paths. Additionally, restrictions on traffic mode transfers can be embedded in the link-based model by excluding infeasible transfer links or adding penalties for undesired transfers. The user equilibrium of the augmented link-based super-network model is formulated as a variational inequality problem, solved using the extra-gradient algorithm. A multi-modal transport network is considered in the case study. Simulation results validate the effectiveness of the proposed model, demonstrating its scalability and versatility in addressing complex multi-modal transport networks with diverse traffic modes. We anticipate that our method can serve as an efficient modeling approach for more general and complex multi-modal transport networks, facilitating traffic management and network design.

Keywords: multi-modal traffic, traffic assignment, user equilibrium, augmented link-based model

INTRODUCTION

Multi-modal transport networks consist of various traffic modes, including private cars, public transit, active mobility, shared mobility, and more. Passengers can use a single mode or a combination of multiple modes, transferring between different modes to complete a trip. However, with ongoing urbanization and the growing size of cities, it becomes increasingly difficult to complete a trip using a single mode. From an individual traveler's perspective, the emergence of new traffic modes offers more choices for their daily commute and can meet the diverse travel demands of different types of passengers. From a collective perspective, multi-modal transport can facilitate achieving seamless travel and maximizing traffic capacity at the network level.

User equilibrium (UE) is a traffic flow assignment model, which has received considerable attention, since it is essential for managing and designing transport networks as well as evaluating traffic system performance. In a user equilibrium, all the users choose their best routes selfishly. An equilibrium is reached eventually, where the users sharing the same origin and destination will experience the same travel cost, irrespective of the chosen route (*1*). Note that "cost" includes monetary costs, as well as other disutilities (travel time, discomfort, etc.). In this study, the UE result is used to estimate the behavior of users with given demands and network settings, and determine the flows on each link and the optimal routes in multi-modal transport networks.

Adequate traffic network supply modelling is essential to estimating the collective behaviour of users, and thereby the impact that different measures, e.g. supply management or pricing, might have on traffic performance. There are extensive researches on the modelling of traffic network supply. One of the earliest examples is the Bureau of Public Roads (BPR) function (*2*), which estimates the travel cost of road links based on traffic flows. BPR and BPR-type functions have been widely used to approximate UE in traffic assignment processes, for instance in (*3–6*). BPR functions have not only been employed for estimating the travel time on road links, but also for approximating the waiting time for public transit in multi-modal transport networks. Other methods for modelling multi-modal traffic networks also exist. Pi et al. (*7*) studied a general formulation for multi-modal dynamic traffic assignment considering multi-class vehicles, public transit and parking. In this formulation, the dynamics of each traffic mode and the interactions among different modes are explicitly considered, which allows for the calculation for dynamic user equilibrium and the study of passenger route choices among different modes. Lo et al. (*3*) explicitly considered the restrictions on transfers among different modes in multi-modal transport networks. They proposed a state-augmented multi-modal (SAM) network model to encode the probable transfer rules as well as the maximum number of transfers. In addition, they constructed a direct in-vehicle link between each pair of connected nodes in each modal sub-network, in order to address non-linear and non-additive transit fares.

However, most of the existing works on multi-modal transport modelling relies on calculation of path travel costs in their assignment procedure, which means the enumeration of feasible paths for any OD pair is necessary in order to calculate UE. This is mainly due to the presence of travel cost components which are dependent on the travel information of the entire path (that is, path-additive) (*8*). For example, the restrictions on feasible transfers among different modes or, similarly, behavioural considerations w.r.t. the maximum number of acceptable transfers, represent non-separable path-based cost dynamics.

In large scale networks, this will result in combinatorial explosion of the number of possible paths, severely impacting computation times for enumeration schemes. Thus the scalability of path-based modelling methods can be significantly limited. Szeto and Jiang (*9*) proposed a link-based method to address the scalability issue of traffic assignment. But the method is dedicated to transit assignment. *To the best of our knowledge, there is not an efficient and general modelling method for multi-modal transport networks yet.*

Therefore, in this work we propose an augmented link-based super-network model to address this issue. Through employing the super-network framework and the augmented link concept, the path costs can be formulated as a sum of the augmented link costs. In addition, the proposed method can encode the feasible transfers among different traffic modes. Thus the enumeration of feasible paths is avoided. the method is not only scalable to larger traffic networks, but also can accommodate a wide range of traffic modes. The contributions of the paper include the following:

- Formulating a novel augmented link-based super-network model for multi-modal transport networks, which is scalable and versatile to complex networks and diverse traffic modes.
- Formulating the UE problem of the augmented link-based super-network model as a variational inequality problem, and providing a solution for the generated variational inequality problem based on an extra-gradient method.
- Validating the effectiveness of the proposed method and showing the equivalence to the conventional path-based method through a case study; showing the versatility of the method by considering diverse traffic modes in the case study.

The rest of this paper is organized as follows. In Section 3, the augmented link-based super-network model is presented to formulate the multi-modal transport network with an illustrative example, and the UE is formulated as a variational inequality problem which is solved with the extra-gradient algorithm. A case study is conducted based on a multi-modal transport network in Section 4 to show the efficiency and versatility of the proposed method. Finally, in Section 5 we draw some concluding remarks.

METHODOLOGY AND FORMULATION

This section will give the presentation of multi-modal traffic networks, and introduce the notation system used to formulate the model. After that, the augmented link-based super-network model is presented, and the assumptions used throughout the paper are given.

Multi-modal transport super-network

We aim to consider a general multi-modal traffic network that can accommodate a wide range of traffic modes, in which passengers can board on and alight from public transit and transfer among different modes. A widely-used expression of transport network is the graph network that consists of links and nodes, in which the nodes represent origin nodes, destination nodes, and activity nodes where passengers can park cars or transfer between modes, while the links represent the physical traffic links connecting the nodes. An illustrative example is shown in Figure 1, in which there are two origin nodes O_1, O_2 and one destination node D, and two intermediate nodes A and B where passengers can park their private cars. In addition to the road network, there are three public transit modes, i.e., bus, tram, and metro, which occupy different links of the network. In this multi-modal network, passengers can choose one single or multiple modes to complete their trips. For instance, one can take a bus or tram to travel directly from O_1 to D, or park and transfer to bus or tram at the intermediate node.

However, the transfers between modes cannot be generally modelled through a graph whose topology is a 1:1 representation of physical topology. For example, some temporal-related transfers cannot be captured by a physical link. A super-network (also known as hyper-network) is a generalization of a conventional graph, where additional links and nodes are employed to represent relationships between portions of the original graph that are not physical in nature, but rather behavioral. In our adopted approach, we consider a concatenation of single modal networks interconnected by transfer links that capture the mode transfer behaviours, through which all the feasible transfers can be explicitly modelled (*10, 11*).

A super-network description of the multi-modal network is given in Figure 2, where all the transfer links are explicitly depicted that describe the transfer relationships and restrictions between modes. For example, it is possible to transfer between bus and tram at the intermediate node bidirectionally; however this cannot be the case when transferring from private car to public transport. Carlier et al. (*8*) used a super-network to model a multi-modal transport network and embedded the transfer costs in the transfer links. However, in this approach the travel cost is still estimated based on path route.

The path-dependent transfer cost also requires the path information to estimate travel costs. For instance, the egress time of link O_1A on private car sub-network depends on the downstream link which is path information. Similarly, the last-mile travel cost from parking node D to the real activity position depends on the upstream link, i.e., the costs are very different depending on whether the user's previous traffic mode is private car or transit. Furthermore, the waiting time for public transit at one node can be influenced by the flow on the upstream link. The parking cost at one node depends on the car flow entering the parking lot instead of the arriving car flow. This is why path enumeration is necessary in conventional modelling methods.

In the next section, based on the super-network description of multi-modal network, an augmented link-based super-network model is developed to address the scalability issue by modelling the travel costs with augmented link cost functions.

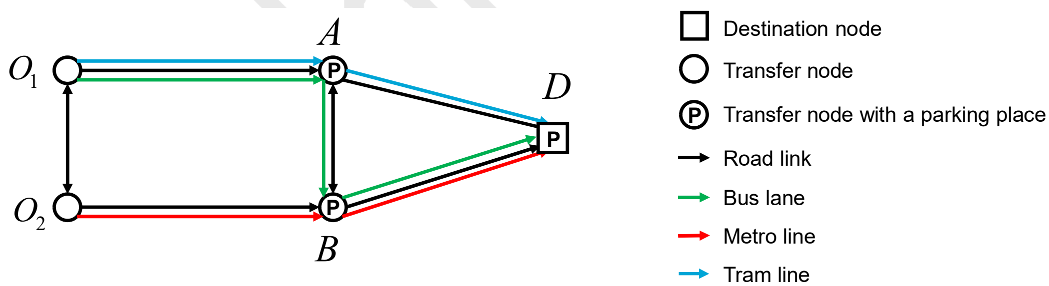

FIGURE 1: Illustrative multi-modal transport network

Augmented link-based super-network model

An augmented link-based super-network graph of a multi-modal transport network is shown in Figure 3. In this graph, each layer represents a single traffic mode network, in which all the mode-specific nodes are defined accordingly. All the layers share the same origin and destination nodes. We further extend the concept of conventional transfer links that connect two mode by defining the *augmented transfer link*. For example, the links connecting the origin/destination nodes and the traffic nodes belong to the augmented transfer link set, too. Therefore, in this model, links are categorized in two types: in-vehicle links and augmented transfer links. The in-vehicle links refer to the links connecting two nodes in the same sub-network; the augmented transfer links

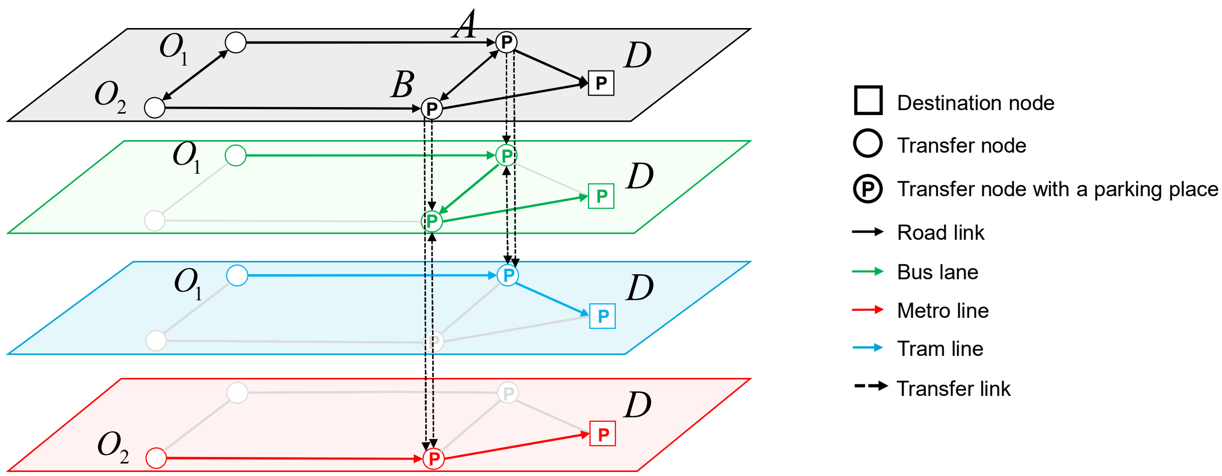

FIGURE 2: Super-network of a multi-modal transport network

include the links connect two nodes from different sub-networks and the links that connect the origin/destination nodes with from a traffic sub-network.

Compared to conventional multi-modal transport models, the proposed augmented link-based super-network model computes the total route travel cost as sum of augmented-link travel costs, without the need of the entire path information, under adequate regularity assumptions. This is achieved by encoding all the path-dependent travel cost in the augmented transfer link cost functions, while the in-vehicle links of each mode model the travel costs that are independent of paths, including link travel time, the fuel cost on private car links, and distance-related transit fares. Each type of link cost captures certain travel costs, which is independent of path. The detailed link cost functions of each link type are defined in next subsection.

Link cost functions

In order to formulate the travel cost functions, the notations used throughout the paper are first introduced in Table 1. Note that, in this paper, the time cost (i.e., time multiplied with a value of time) and monetary cost of users are considered, and the sum of both costs is taken as the perceived cost of users, given as:

$$C_a(\boldsymbol{f}) = \omega_t t_a(\boldsymbol{f}) + \omega_c c_a(\boldsymbol{f}). \tag{1}$$

In-vehicle link cost function

The cost function of in-vehicle links captures the time cost in travel between nodes within one traffic mode and the corresponding monetary cost. The in-vehicle travel time and monetary cost may vary according to the traffic mode, and these cost functions are defined exclusively dependent on the link only.

For private car mode link, the travel time on the link can be approximated by the BPR function, which is a monotone function of the flow on the link:

$$t_{a_v,\text{main}}(f_{a_v}) = t_0 \left(1 + \alpha_1 \left(\frac{f_{a_v}}{C}\right)^{\beta_1}\right), \tag{2}$$

in which t_0 is the free-flow travel time, C is the link capacity (veh/hour) and α_1, β_1 are parameters.

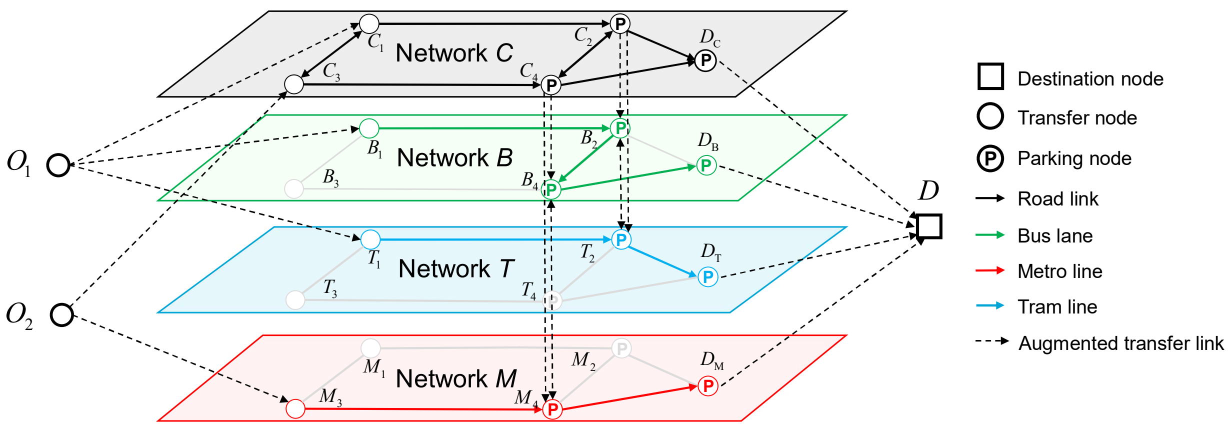

FIGURE 3: Augmented link-based super-network graph for a multi-modal transport network

The monetary cost can be calculated using the function:

$$c_{a_v}(\boldsymbol{f}) = c_{\text{fuel}} l_{a_v}. \tag{3}$$

Note that the monetary cost can also be approximated by a monotone nonlinear function dependent on the link length and link travel time.

For public transit mode links, the travel time on the link can be calculated based on the link length and transit speed, assuming no congestion occurs on the transit links:

$$t_a(\boldsymbol{f}) = \frac{l_a}{s_{m_t}}, m_t \in M_t. \tag{4}$$

The monetary cost of public transport is the transit fare, which usually consists of a base fare and a distance-based fare (see (*12, 13*)). In the monetary cost function, only the distance-related part is considered:

$$c_a(\boldsymbol{f}) = c_{m,\text{dis}} l_a, \tag{5}$$

which is a function of the link attribute only. The base fare part, however, is considered in the augmented transfer link cost functions as described next. For the bus sub-network, the travel time can be different depending on the infrastructure. If there are dedicated lanes for bus, then the travel time can be estimated with equation (4). Otherwise, the bus will share the lanes with private cars, and the travel time can also be estimated with the BPR function.

Augmented transfer link cost function

In general, the transfer links between different modes can capture the access cost and egress cost. However, in this proposed method, the cost function of augmented transfer link is used to model all the cost that cannot be captured by the in-vehicle links. The cost function of each augmented transfer link is specific-defined according to its connecting traffic modes.

For the augmented transfer link from car mode to transit mode, e.g., links C_2B_2, C_2T_2, C_4B_4, C_4M_4 in Figure 3, the travel time cost consists of three parts: egress time and access time from car to transit, the time needed for parking car, and the waiting time for transit. The function is given as:

$$t_{a_t}(\boldsymbol{f}) = t_{a_t,\text{egress}}(\boldsymbol{f}) + t_{a_t,\text{access}}(\boldsymbol{f}) + t_{a_t,\text{park}}(\boldsymbol{f}) + t_{a_t,\text{wait}}(\boldsymbol{f}), \tag{6}$$

TABLE 1: Notations

Sets	Description	**Indices**	Description
A	Set of links	a	Indices of links
A_v	Set of in-vehicle links	a_v	Indices of in-vehicle links
A_t	Set of augmented transfer links	a_t	Indices of augmented transfer links
N	Set of nodes	w	Indices of OD pairs
W	Set of OD pairs	p_w	Indices of paths between w
P	Set of paths in the network	i, j, k	Indices of nodes
P_w	Set of paths between $w \in W$	m	Indices of modes
Φ	Set of all feasible link flows	m_t	Indices of transit modes
M	Set of all traffic modes	p	Indices of paths
M_t	Set of all transit modes		
Variables	Description	**Functions**	Description
$\boldsymbol{f}$	Vector of link flows	$t_a(\boldsymbol{f})$	Total travel time on link a
f_a	Flow on link a	$t_{a,\text{access}}(\boldsymbol{f})$	Access time on link a
$\boldsymbol{x}$	Vector of path flows	$t_{a,\text{egress}}(\boldsymbol{f})$	Egress time on link a
x_p	Flow on path p	$t_{a,\text{wait}}(\boldsymbol{f})$	Waiting time on link a for transit
Parameters	Description	$t_{a,\text{main}}(f_a)$	Main in-vehicle travel time
d_w	Demand of OD pair w	$t_{a,\text{park}}(\boldsymbol{f})$	Time cost on parking on link a
l_{a_v}	Length of in-vehicle link a_v (km)	$c_a(\boldsymbol{f})$	Monetary cost on link a
c_{fuel}	Fuel/recharge cost for cars (euro/km)	$C_a(\boldsymbol{f})$	Total weighted cost on link a
c_{park}	Parking cost for cars (euro/h)	$C_p(\boldsymbol{x})$	Total travel cost on path p
$c_{m,\text{base}}$	Base fare of mode m (euro)	$\boldsymbol{C}(\boldsymbol{f})$	The vector of all the link costs
$c_{m,\text{dis}}$	Distance-dependent fare of mode m (euro/km)		
ω_t	Weight on time cost		
ω_c	Weight on monetary cost		
ν_{m_t}	Frequency of transit m_t (veh/h)		
s_m	Service speed of mode m (km/h)		
$\delta_{a,p}$	Link-path incidence matrix		

where $t_{a_t,\text{egress}}(\boldsymbol{f})$ and $t_{a_t,\text{access}}(\boldsymbol{f})$ are usually assumed constant. Note that in this model, the access and egress time can be considered together, since both describe the time needed to leave from one node to the other node. For the time to parking we also use a BPR function (*11*):

$$t_{a_t,\text{park}}(\boldsymbol{f}) = t_{0,\text{park}} \left(1 + \alpha_2 \left(\frac{\boldsymbol{f}}{C_{\text{cap}}} \right)^{\beta_2} \right), \tag{7}$$

where $t_{0,\text{park}}$ is the parking time when the parking lot is empty, C_{cap} is the capacity of the parking lot, $\boldsymbol{f}$ refers to the links flows that share the same parking place. For example, the parking lot at node C_2 accommodate cars from both transfer links C_2T_2 and C_2B_2. Thus the flows on both links should be considered when calculating the parking time. Equation (7) means that the parking time will increase with more cars parking in the parking lot, and the time will increase significantly when the parking lot is approaching its maximal capacity. Meanwhile, the waiting time cost for transit can be approximated with the function:

$$t_{a_t,\text{wait}}(\boldsymbol{f}) = \frac{\alpha_3}{\nu_{m_t}} + \beta_3 \boldsymbol{f}, \tag{8}$$

where α_3,β_3 are calibrated parameters, ν_{m_t} is the frequency of transit mode m_t, and $\boldsymbol{f}$ contains the flows that influence the waiting time, which includes the transfer flows and the prior passenger volume before the transfer node. For example, the waiting time on transfer link C_2B_2 depends not only on the bus frequency and traffic flow on link C_2B_2, T_2B_2, but also on the passenger flow on the in-vehicle link B_1B_2.

As for the monetary cost of this augmented transfer link, the cost function includes the parking fee and the base transit fare that the passenger is transferring to, which is given as:

$$c_{a_t}(\boldsymbol{f}) = c_{\text{park}} \Delta T + c_{m_t,\text{base}}, \tag{9}$$

where ΔT is the time duration of parking, and $c_{m_t,\text{base}}$ is the base part of the transit fare of mode m_t that the transfer link leads to. By encoding the parking time/monetary cost and base fare in the transfer cost function, these costs can be automatically considered whenever the transfer link is included in a path.

For other types of augmented transfer links, the cost function is defined in a similar way. For instance, the links connecting the origins and traffic network, also known as connectors, have a time cost function including the access time and waiting time if it links to a transit sub-network, and have a monetary cost function including the base fare if reaching a transit node. Meanwhile, the links approaching the destination can capture the parking time and parking fee, as well as the access/egress time. As for the transfer links between different transit modes, the time cost function only includes the egress/access time and waiting time, and the monetary cost equals to the base transit fare of the subsequent transit mode. In addition, the infeasible transfer links are explicitly excluded in the model. For example, the transfer link from transit to private car is not possible for the network in Figure 3, and thus can be excluded during the modelling stage.

Note that an additional comfort cost can be added in the transfer cost function for undesired transfers. For example, passengers would not prefer to transfer from one mode to another frequently, and adding a comfort cost can model this behaviour. Therefore, the path with multiple transfers can be excluded in the route choices through this way. Furthermore, in the illustrative example network, only private car and several transit modes are considered. However, the model can also be extended to accommodate a wide range of traffic modes by introducing extra layers in the super-network, including share mobility, active mobility, and so on. There examples are not presented in detail due to limited space. In the case study, we consider an extra mode of ride-hailing, and show the versatility of our method.

By defining the cost functions of all the links in the super-network in prior, all the relevant costs are detached from the path and encoded in the corresponding individual links. These costs are summed up automatically to calculate the total path cost for user equilibrium, without the need to enumerate all the feasible paths. These statements are based on the assumptions given in the next section.

Assumptions

The following assumptions are made throughout this paper to support the main conclusions:

1. The path costs are assumed to be separable, which is able to approximate most of the

travel costs.

2. Only time and monetary costs are considered by the passengers. Other costs, such as comfort cost, can also be included in the model easily.
3. The passengers have perfect rationality, and they therefore always select the route with the lowest cost.
4. Transit operates at a fixed speed on the in-vehicle links. Other types of travel time cost of transit can also be considered in this model.
5. The transit fare is assumed to include a base fare and a distance-related fare. Experiments can be conducted to analyse the influences of non-linear fare and study the difference between two fare settings.
6. The real-world traffic networks can be described by the multi-modal transport network as in Figure 3. This can be achieved with the pre-processing method from Najmi et al. (*14*), which basically includes identifying the important nodes and links, neglecting unnecessary information, and connecting nodes with aggregated links.
7. The travel demand between each origin–destination (OD) pair in the system is assumed to be known and fixed. It is reasonable for strategic planning considering day-to-day recurrent traffic scenarios.
8. Only one type of passengers is considered. This model can be extended easily to address multiple user types that perceive different weights on the travel costs.

User equilibrium and solution algorithm

In this sub-section we solve the traffic assignment problem. In order to do so, the user equilibrium (UE) of multi-modal transport network is formulated as a variational inequality (VI) problem, based on the proposed augmented link-based super-network model, and an extra-gradient solution algorithm is provided to solve the VI problem in an iterative way.

Variational inequality

Variation inequality has been widely used in the literature to describe user equilibrium (e.g., (*15, 16*)). According to the definition of UE, for all OD pairs w, and for $\forall p \in P_w$, the path flow pattern is said to be in equilibrium if the following condition holds:

$$C_p(\boldsymbol{f}) \begin{cases} = \lambda_w, & \text{if } x_p^* > 0, \\ \geq \lambda_w, & \text{if } x_p^* = 0, \end{cases} \tag{10}$$

where λ_w is an indicator of the least path travel cost, whose value is unknown a priori. In other words, all utilized paths connecting the same OD pair have equal and minimal travel costs. Following Theorem 1 from Nagurney (*17*), the condition (10) can be written as the link-based VI formulation:

$$\langle \boldsymbol{C}(\boldsymbol{f}^*), (\boldsymbol{f} - \boldsymbol{f}^*) \rangle \geq 0, \quad \forall \boldsymbol{f} \in \Phi. \tag{11}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in Euclidean space.

The feasible set Φ of link flows can be obtained according to the constraints. The problem can be reformulated in terms of path-link incidence, and assuming conservation of flows at nodes. First, the relationship between link flows and path flows is:

$$f_a = \sum_{p \in P} x_p \delta_{a,p}, \quad \forall i, \forall a, \tag{12}$$

where $\delta_{a,p} = 1$, if link a is contained in path p, and 0, otherwise. In addition, the path flows should satisfy the OD pair demand constraints:

$$d_w = \sum_{p \in P_w} x_p, \quad \forall w. \tag{13}$$

By combining (12) and (13) and eliminating the path flow x_p, a constraint on link flow $\boldsymbol{f}$ is obtained. In addition, extra constraints on link flows need to be considered for our model, that is, conservation of flows at nodes. For every node j in the super-network, the sum of entering flows should be equal to the sum of leaving flows:

$$\sum_{i \in N_{j,\text{in}}} f_{a_{i,j}} = \sum_{k \in N_{j,\text{out}}} f_{a_{j,k}}, \quad \forall j \in N, \tag{14}$$

where $N_{j,\text{in}}$ is the set of upstream nodes that derive a link flow to node j, while $N_{j,\text{out}}$ is the set of downstream nodes that receive a link flow from node j. Note that when applying (14) to the origin nodes and destination nodes with taking OD pair demand into account, it is identical to the constraint (12)-(13).

Extra-gradient algorithm solution

In general, the extra-gradient method iteratively utilises two projection operators to make predictions and corrections until a convergence criterion is satisfied (*18*). More specifically, the first step is a prediction projection, and the value of the gradient in a new "extrapolated" point is used as the direction for the next step, i.e., the correction projection. Hence, this method is referred to as the double projection method. The solution procedures are given as follows:

Step 0: Initialize parameters and generate an initial solution $\boldsymbol{f}^k$ by assigning all of the demand of each OD pair to the topologically least cost path, where the superscript k refers to the solution obtained in iteration k. Set the iteration counter, $k = 0$; set the initial step size of the prediction projection γ that satisfies $0 < \gamma < \frac{1}{L}$ where L is the Lipschitz constant of link cost function $\boldsymbol{C}(\boldsymbol{f})$; set the acceptable error ε.

Step 1: Perform the prediction projection. Calculate: $\bar{\boldsymbol{f}}^k = P_\Phi(\boldsymbol{f}^k - \gamma \boldsymbol{C}(\boldsymbol{f}^k))$.

Step 2: Perform the correction projection: $\boldsymbol{f}^{k+1} = P_\Phi(\boldsymbol{f}^k - \gamma \boldsymbol{C}(\bar{\boldsymbol{f}}^k))$.

Step 3: Check if $\left| \boldsymbol{f}^{k+1} - \boldsymbol{f}^k \right| \leq \varepsilon$; if so, stop; otherwise, $k = k + 1$, and go to **Step 1**.

Note that $P_\Phi(\cdot)$ denotes the orthogonal projection onto feasible set Φ:

$$y = P_\Phi(\boldsymbol{f}) = \arg \min_{\forall z \in \Phi} \| \boldsymbol{f} - z \|. \tag{15}$$

Thus the calculation in Step 1 and Step 2 can be formulated as a quadratic problem. For convergence of the solution and equivalence of this method to the conventional path-based VI formulation, the interested reader can refer to (*16, 18*).

CASE STUDY

In this section, a numerical case study is conducted on the multi-modal transport network in Figure 3, in order to validate the proposed modelling method. First, in case 1, we showed that the augmented link-based super-network model can generate exactly the same result as the conventional path-based, given the same network setting. Then, in case 2, we introduce an extra traffic mode, i.e., ride hailing, which makes the network more complex, and demonstrate the scalability and versatility of our method to address a more complicated network and accommodate diverse traffic modes.

Case 1

Fixed demands for pairs O_1D and O_2D are considered, which are 800 pax/h and 1000 pax/h, respectively. This is an estimation of the morning rush hours when the passengers travel from residence to the office area, and it is assumed that the average number of passengers per car is 1, without loss of generality. The parameters of the network in Figure 3 are given in Table 2. In this case study, the buses share the same lane with private cars and have the same free flow speed, and the link lengths of the car sub-network defined in Table 2 also apply to the other sub-networks. Note that there are two values for c_{park}, in which 2.2 €/h is the parking fee at nodes C_2, C_4 while the price is 5.0 €/h at node D_C. This means that parking is more expensive near the city center. All the parking lots in the network have the same capacity.

TABLE 2: Parameters of multi-modal network in the case study

C [veh/h]	s_{car} [km/h]	s_{metro} [km/h]	s_{bus} [km/h]	s_{tram} [km/h]	α_1	β_1	α_2
1000	60	40	60	20	0.5	4	2.5
d_{O_1D} [pax/h]	d_{O_2D} [pax/h]	$l_{C_1C_2}$ [km]	$l_{C_3C_4}$ [km]	$l_{C_1C_3}$ [km]	β_2	α_3	β_3
800	1000	10	10	6	2	0.5	0.001
c_{fuel} [€/km]	c_{park} [€/h]	$l_{C_2C_4}$ [km]	$l_{C_4D_c}$ [km]	$l_{C_2D_c}$ [km]	ω_t	ω_c	C_{cap} [veh]
0.27	2.2/5.0	3	5	5	1000	10	250

The transit system setting is given in Table 3, including transit frequency and the distance-based fares for all the three transit modes.

TABLE 3: Public transit system setting in case 1

Metro			Bus			Tram		
$c_{m,\text{base}}$	$c_{m,\text{dis}}$	v_{m_t}	$c_{m,\text{base}}$	$c_{m,\text{dis}}$	v_{m_t}	$c_{m,\text{base}}$	$c_{m,\text{dis}}$	v_{m_t}
2.20 €	0.12 €	12 veh/h	2.00 €	0.10 €	6 veh/h	1.50 €	0.08 €	6 veh/h

In this case, a set of feasible paths are selected in advance for each OD pair, where 5 paths for O_1D and 4 paths for O_2D as shown in Table 4. Path 1,2,3,6,7 contain only one traffic mode, while the other paths include two modes. This implies that only the transfer links $C_2B_2, C_2T_2, C_4B_4, C_4M_4$, i.e., from are to transit, are considered. Therefore, in the augmented link-based model, the other transfer links ($B_2T_2, T_2B_2, B_4M_4, M_4B_4$) are imposed an extra inconvenience penalty cost to approximate the path-based method.

By enumerating all the paths, it is possible to formulate the path travel costs as functions of the path flows. Using the algorithm from Szeto and Jiang (*9*), a path-based UE can be obtained. The results of UE path flows and costs are presented in Table 5, which shows that three paths are selected for O_1D (Path 2,3,5) with the same least path cost, while two paths (Path 6,7) are chosen by the passengers from origin O_2 to destination D. The UE implies that direct routes without transfers are preferred. The only transfer adopted is from private car to tram in Path 5. The corresponding link flows are presented in Table 6.

Then the UE calculation is conducted with the augmented link-based super-network model. There are 32 links in total, each of which is modelled with a link cost function. With the defined link cost function, the augmented link-based UE can be obtained using the algorithm in Section 3.5.

TABLE 4: The selected feasible paths for the two OD pairs

O_1D	Path 1	$O_1 - C_1 - C_2 - D_C - D$
	Path 2	$O_1 - B_1 - B_2 - B_4 - D_B - D$
	Path 3	$O_1 - T_1 - T_2 - D_T - D$
	Path 4	$O_1 - C_1 - C_2 - B_2 - B_4 - D_B - D$
	Path 5	$O_1 - C_1 - C_2 - T_2 - D_T - D$
O_2D	Path 6	$O_2 - C_3 - C_4 - D_C - D$
	Path 7	$O_2 - M_3 - M_4 - D_M - D$
	Path 8	$O_2 - C_3 - C_4 - B_4 - D_B - D$
	Path 9	$O_2 - C_3 - C_4 - M_4 - D_M - D$

TABLE 5: Equilibrium path flows and path costs obtained by path-based model

	Path 1	Path 2	Path 3	Path 4	Path 5	Path 6	Path 7	Path 8	Path 9
Flow [pax/h]	0	575.39	30.30	0	194.31	390.43	609.57	0	0
Cost	1108.5	1045.6	1045.6	1231.0	1045.6	1111.5	1111.5	1119.3	1123.1

TABLE 6: Equilibrium link flows and link costs obtained by path-based model

Link	O_1C_1	C_1C_2	C_1C_3	C_3C_1	C_3C_4	C_2C_4	C_4C_2	C_2D_C
Flow [pax/h]	194.31	194.31	0	0	390.43	0	0	0
Link	C_4D_C	D_CD	O_1B_1	B_1B_2	B_2B_4	B_4D_B	D_BD	C_2B_2
Flow [pax/h]	390.43	390.43	575.39	575.39	575.39	575.39	575.39	0
Link	C_4B_4	O_1T_1	T_1T_2	T_2D_T	D_TD	B_2T_2	T_2B_2	C_2T_2
Flow [pax/h]	0	30.3	30.3	224.61	224.61	0	0	194.31
Link	O_2C_3	O_2M_3	M_3M_4	M_4D_M	C_4M_4	B_4M_4	M_4B_4	D_MD
Flow [pax/h]	390.43	609.57	609.57	609.57	0	0	0	609.57

The iteration process of the augmented links is presented in Figure 4 and Figure 5, respectively. It is shown that the link flows converge efficiently after 400 iterations with a tolerance of $\varepsilon = 10^{-5}$ veh/h. Figure 5 shows that only the transfer link C_2T_2 is selected. This is consistent with the UE result of the path-based method. The equilibrium links flows obtained with the proposed method are presented in Figure 6, which is identical to the results in Table 6 obtained by the path-based method, which validates the effectiveness of our method and the equivalence of the augmented link-based VI and the path-based VI.

Case 2

In Case 2, we further introduce an extra traffic mode based on the multi-modal network in Figure 3, i.e., a ride-hailing mode, as shown in Figure 7. This mode shares the same road network with the private car mode, and thus the link travel times of both modes are the same. It can be seen that the extra ride-hailing sub-network introduces more transfer links, and subsequently the feasible path set explodes compared to the original network, which make it difficult to enumerate all the

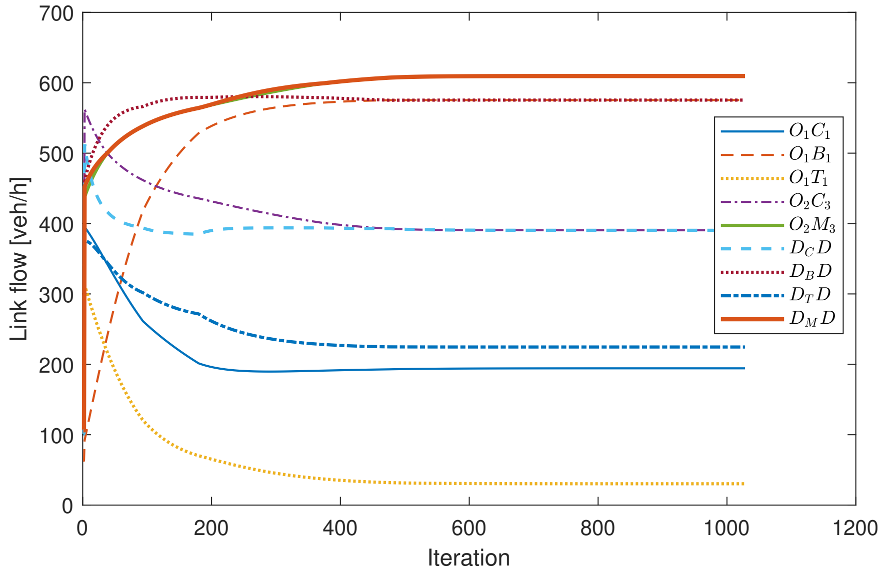

FIGURE 4: User equilibrium convergence process of transfer link flows between origin/destination and modes

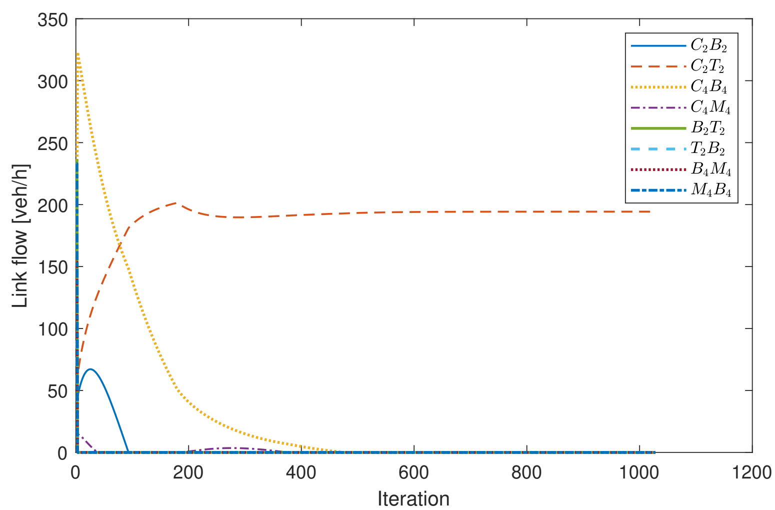

FIGURE 5: User equilibrium convergence process of transfer link flows between different modes

paths. More specifically, the number of feasible paths increases exponentially from 34 to 336, when extending the network in Case 1 by adding one more mode. Therefore, in this case, we will show the scalability of our model to address a more complicated network, and demonstrate its ability to accommodate diverse traffic modes.

The passengers can choose to hail a ride from the origin until the destination, or transfer to other transit modes in the middle. They can also take a taxi/Uber midway when leaving from other modes. The cost functions of this mode can be formulated with the method introduced in the previous section. For example, the time cost of using the ride-hailing links includes the waiting

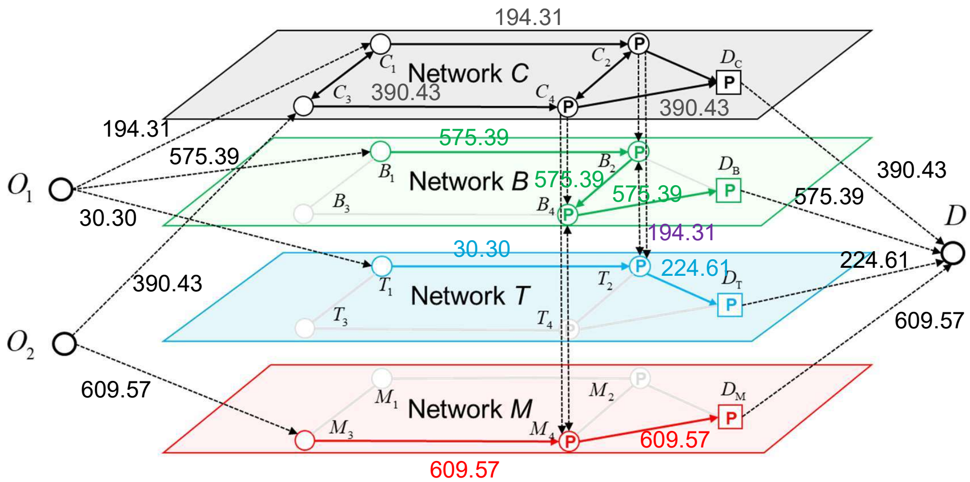

FIGURE 6: User equilibrium result of the multi-modal network using the augmented link-based super-network model

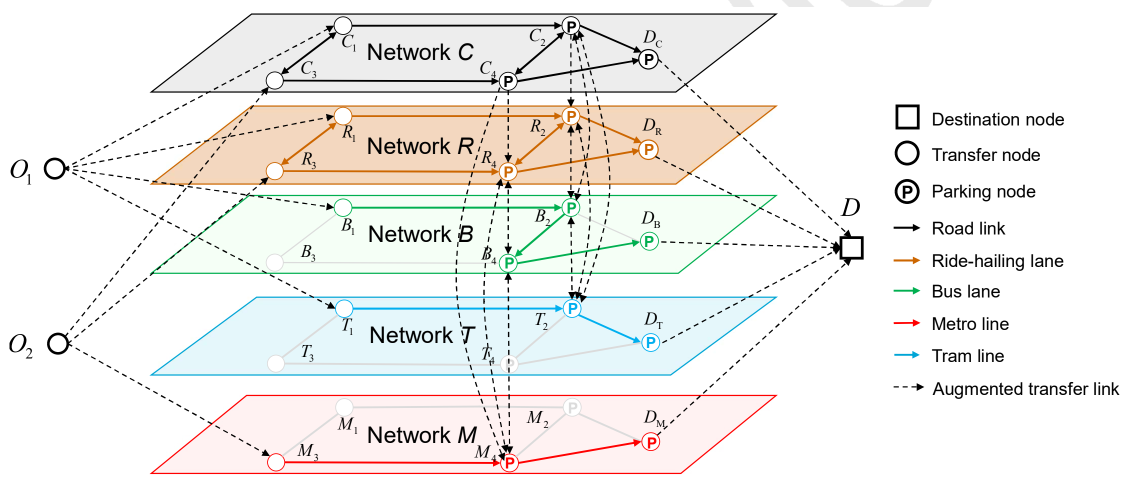

FIGURE 7: The multi-modal transport network with a ride-hailing mode

time for taxi and in-vehicle travelling time, while the monetary cost is only the fare. For simplicity, the average waiting time for a taxi can be estimated by a BPR function considering the taxi service capacity is limited (*11*) as Equation (7), with $t_0 = 0.05$ h, $\alpha = 4$, $\beta = 2$, and $C_{\text{cap}} = 150$ veh/h. Note that this time cost is encoded in the cost function of the transfer link that points to the ride-hailing mode, and $\boldsymbol{f}$ are the corresponding transfer link flows that take the taxi at the same node. The in-vehicle travel time of this mode is combined with the private car mode and bus mode, and is included in the in-vehicle link cost function. The fare of taxi is calculated based on the function:

$$c_a(\boldsymbol{f}) = c_{m,\text{base}} + c_{m,\text{dis}} l_a,$$

where $c_{m,\text{base}} = 3$ € included in the corresponding transfer link, and $c_{m,\text{dis}} = 1.2$ € included in the in-vehicle link, which is the same as the transit mode cost function. Then the UE is calculated with all the well-defined link cost function. The link flows of the UE results are shown in Figure 8, in which each color corresponds to each mode and the purple numbers refer to the transfer link flows.

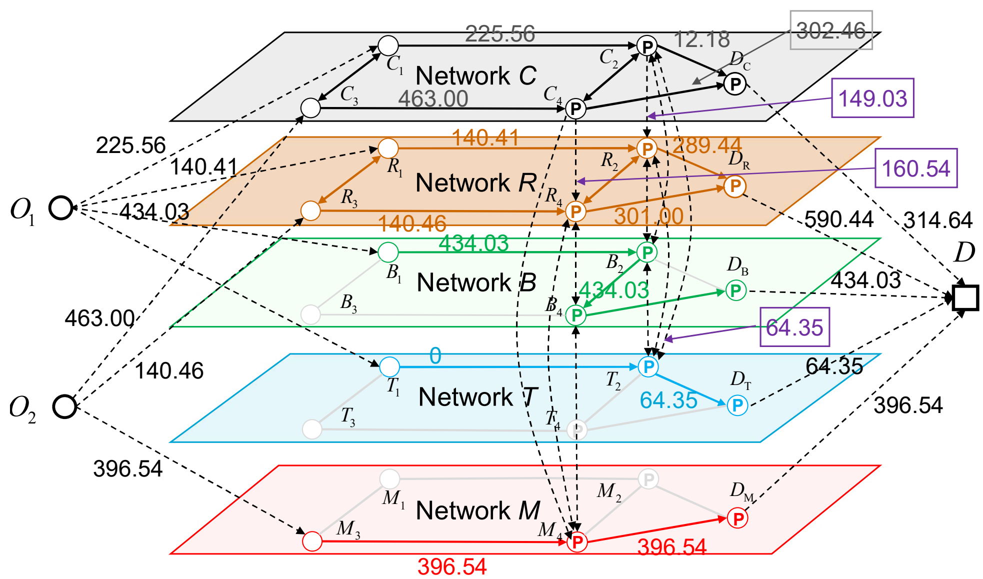

FIGURE 8: Link flows of user equilibrium results of the multi-modal transport network with a ride-hailing mode

With our method, it is only required to specify the cost of the links, which are 53 in total, compared to 32 links of the original network. This underscores the scalability of our method. Figure 8 shows the equilibrium link flows. It can be seen that fewer passengers take public transit, since the introduced raid-hailing mode provides a route with less travel time. Although the monetary cost of ride-hailing is higher, it requires shorter travel time thanks to its higher speed and the convenience of no parking requirements. Also notice that considerable users drive their cars and park at node C_2, and then hail a ride to the destination. This route could save their money since they do not need to pay the high parking fee at node D_C, and with this amount of money they can access the destination more quickly by taking a taxi, which is also more efficient than transferring to tram as in the previous case. Overall, taking a single mode to finish the entire trip is still preferred by the users, e.g., the pure private car route, bus mode, and metro mode. Transfers are not favored unless they can provide significant benefits in terms of time or money.

CONCLUSION

This paper develops an augmented link-based super-network method for modeling multi-modal transport networks. This approach addresses the scalability issues of conventional multi-modal traffic models, which require path enumeration—a process that becomes intractable in large, complex networks. The proposed method introduces augmented transfer links to decouple path information from travel cost, encoding path cost within individual links. This approach eliminates the need for enumerating feasible paths. By explicitly modeling link costs, infeasible or undesired transfers can be excluded a priori. Moreover, the proposed method is versatile, capable of being applied to complex multi-modal transport networks of large size and diverse traffic modes.

The model is used to calculate the UE of multi-modal traffic networks, formulated as a VI problem, which can be solved using the extra-gradient method. Case studies are conducted to demonstrate the effectiveness of our approach. In the first case study, we validate the efficiency and equivalence of our method compared to the path-based method using a multi-modal transport network that includes multiple transit modes and a private car mode. In the second case study, an additional traffic mode is introduced, highlighting the scalability and versatility of our method.

This method can be further applied to real-world multi-modal transport scenarios. Additionally, the generated UE results can be utilized for strategic multi-modal traffic management and network design.

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