The fallacy of edge elements

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Abstract—The present paper critically investigates the use of edge elements for computing electromagnetic fields. The application of edge elements in methods based on the use of vector potentials as well as in methods that compute electric and/or magnetic fields directly will be covered. In particular the popular idea that edge elements eliminate spurious solutions will be refuted. This erroneous idea is replaced by the insight that spurious solutions can be eliminated only by a proper finite-element formulation. A reference is made to alternative approaches, one of them introducing a new type of element, the so-called generalized Cartesian element, that combines the advantages of the classical Cartesian (nodal) elements with the ability of edge elements to allow the representation of discontinuities.

Keywords—edge elements, spurious modes, spurious solutions.

I. INTRODUCTION

Over the past decade edge elements have earned an explosive growth in attention in the electromagnetic finite-element community and this rapid development still seems to be continuing unhampered. Although functions of the edge type were first used by McMahon [1], who referred to them as pyramid vector fields, they gained their first popularity only after the fundamental theoretical paper by Nédélec [2] and the application of these elements, first by Bossavit and Verité [3] and subsequently by so many others that we have to refrain from an attempt at referencing them.

The main reasons for the success of edge elements seem to be the following:

1. Edge elements can be used for representing fields with continuous tangential components while leaving the normal component free to jump. (In the present paper we use exactly these properties of edge elements as their definition.) Because of these properties edge elements can, contrary to the standard nodal elements, be used for representing electric and magnetic fields in media with discontinuous medium properties.

2. Edge elements can be, and usually are, designed such that they are free of divergence. Among other reasons, this freedom of divergence has motivated the hope, and even conviction [4]–[11], of many that solutions of field problems obtained by using edge elements will be free of divergence and, consequently, free of the spurious solutions that haunt many finite-element codes for electromagnetic field computations. This hope is best illustrated by quoting Nédélec [2] who concluded his famous first paper on mixed finite elements with the statement: "The main advantage of these finite elements is the possibility of approximating Maxwell's equations while exactly verifying one of the physical law[s]."

Nédélec's paper was followed by many other papers proposing ever new types of edge elements, that sometimes were given new names such as, for instance, tangential vector elements. We mention only the new types that are relevant in the context of the present paper.

In 1985 Mur and de Hoop [12] introduced the so-called consistently linear edge elements. Contrary to the mixed edge elements mentioned above they provide a linear approximation of each component of the field in each Cartesian direction. Consistently linear edge elements, or more generally edge elements that are consistent of any polynomial order, are not free of divergence. In [13] Nédélec presented a very learned and general discussion on edge elements of this type.

In the present paper the validity of the various claims that are made regarding edge elements is analysed. A few additional properties of edge elements are also discussed.

II. EDGE ELEMENTS DO ALLOW SPURIOUS SOLUTIONS

A very simple and explicit example demonstrating that *edge elements do allow spurious solutions* in driven problems was given by Mur [14] and it is a trivial exercise for the reader to construct a similar example for eigenvalue problems. Since the example seems to have escaped the attention of most colleagues it is repeated here in a slightly modified and simplified version.

In the example we assume a computational domain $\mathcal{D}$ with outer boundary $\partial \mathcal{D}$ in which we have an electromagnetic field $\{E(r,t), H(r,t)\}$ that varies in time. For the three-dimensional domain $\mathcal{D}$, with a two-dimensional illustration as in Fig. 1, we choose the cube $0 \leq x \leq 1$,

$0 \leq y \leq 1$, $-0.5 \leq z \leq 0.5$. The lossy medium in this cube *is assumed to be homogeneous* with permittivity $\varepsilon = \varepsilon_r \varepsilon_0$, permeability $\mu = \mu_r \mu_0$ and conductivity σ, where $\varepsilon_r \geq 1$, $\mu_r \geq 1$ and $\sigma > 0$. We assume the outer boundary $\partial \mathcal{D}$ to be divided into two parts (that may themselves be subdivided into a number of subdomains) viz. $\partial \mathcal{D}_E$ and $\partial \mathcal{D}_H$, where $\partial \mathcal{D} = \partial \mathcal{D}_E \cup \partial \mathcal{D}_H$ and $\partial \mathcal{D}_E \cap \partial \mathcal{D}_H = \emptyset$. $\partial \mathcal{D}_E$ consists of those parts of $\partial \mathcal{D}$ that are located on either of the planes $z = -0.5$ and $z = 0.5$, while $\partial \mathcal{D}_H$ is defined as the remaining part of $\partial \mathcal{D}$, i.e. $\partial \mathcal{D}_H = \partial \mathcal{D} \setminus \partial \mathcal{D}_E$.
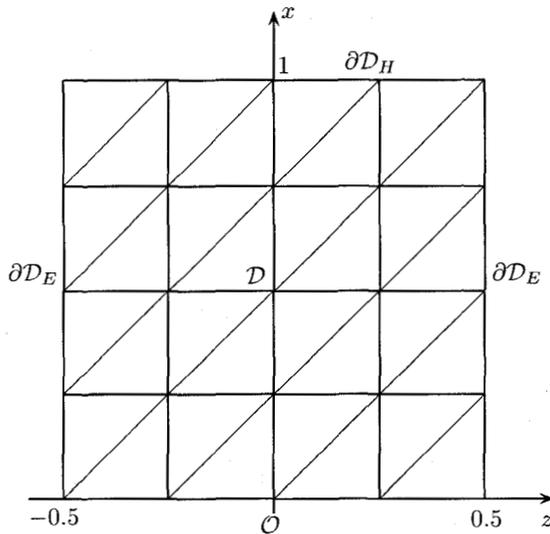

Fig. 1. Cube subdivided in bricks that in turn may be subdivided in tetrahedra or prisms, side view

We now assume that the tangential components of the electric field strength $\boldsymbol{E}(\boldsymbol{r}, t)$ are known functions of space and time on $\partial \mathcal{D}_E$, and that the tangential components of the magnetic field strength $\boldsymbol{H}(\boldsymbol{r}, t)$ are known functions of space and time on $\partial \mathcal{D}_H$. In addition, we assume that the source distributions and the initial conditions $\boldsymbol{E}(\boldsymbol{r}, t_0)$ and $\boldsymbol{H}(\boldsymbol{r}, t_0)$ are known functions of both the space (and the time) coordinates in $\mathcal{D}$. With these data we have defined an electromagnetic field problem with a unique solution [15]. For generating a finite-element solution to this problem, the domain of computation is discretized by using a uniform mesh consisting of identical bricks of side length 0.25m (see Fig. 1) each of which may be subdivided into smaller domains (prisms or tetrahedra) depending on the type of edge element used.

As regards the problem to be solved we assume that at the time $t = t_{\mathrm{end}} > t_0$, the exact solution for the electric field strength equals

$$\boldsymbol{E}(\boldsymbol{r}, t) = E_0 \boldsymbol{i}_z. \tag{1}$$

Since this solution is a constant function of the spatial variables, it can be represented exactly using edge elements of any kind or any degree, and there is of course no doubt that many methods using edge expansions will find this solution with the highest possible degree of accuracy. The point we want to stress here is that when a correct solution is found its correctness can only be attributable to the finite-element formulation used and not to the use of edge elements. This claim is most easily confirmed by verifying that the "solution"

$$\boldsymbol{E}(\boldsymbol{r}, t) = \begin{cases} E_0 \boldsymbol{i}_z, & \text{for } -0.50 < z < -0.25, \\ 0\boldsymbol{i}_z, & \text{for } -0.25 < z < 0.25, \\ E_0 \boldsymbol{i}_z, & \text{for } 0.25 < z < 0.50, \end{cases} \tag{2}$$

for $t = t_{\mathrm{end}}$, is wrong or "spurious". However, since the errors in the above wrong solution consist of jumps in its normal component across the inter-element boundaries at the planes $|z| = 0.25$ it is an admissible solution when *judging this from the properties of the edge elements*. At the inter-element planes the normal component of the *exact* solution is continuous. Only a correct formulation of the problem could have prevented the unwanted discontinuity in the normal components from entering into the "solution".

Finally note the following:

1. The fact that the example uses a problem the solution of which is a simple, uniform field is immaterial. The example was chosen for the sake of utmost clarity and simplicity. It is a trivial matter to construct other examples instead of the one given in (1) and (2). Accurate, non-spurious, solutions can only be guaranteed by making the continuity of the normal component of the flux between edge elements a part of the formulation of the finite-element method or, more generally, by choosing a correct formulation of the problem to be solved [16], [17].

2. The example is such that the properties edge elements have by definition are used for constructing the demonstration of their failure in preventing the occurence of spurious solutions. The conclusion that edge elements do *not* eliminate spurious solutions applies in general to all edge elements as defined above since we have not referred to dimensionality, type, (mixed) order or shape of edge element in the example. *The strength of edge elements turns out to be their very weakness.*

In summary: An example was presented demonstrating the fact that the appearance of spurious solutions of finite-element problems in electromagnetics cannot be ruled out by using edge elements.

III. More complaints about edge elements

Without making an attempt to be exhaustive we now catalogue a number of additional disadvantages and prob-

lems one may encounter when using edge elements:

1. Edge elements are known to be less efficient, both as regards storage requirements and computation time, than the classical Cartesian (nodal) elements because of requiring much more unknowns for obtaining the same accuracy [14], [17]. Contrary to what is claimed by some authors, this disadvantage is not offset by the sparser matrices edge elements generate.

2. The condition of the representation of a field using vectorial finite elements depends, among other aspects, on the bases of the reference frames used in those elements. In edge elements those bases are not always easy to distinguish but they are often related to the vectorial orientations of the faces meeting at the vertices of the element. These faces usually are not mutually perpendicular which will degrade the condition of the representation of the expanded vector field [14], especially in configurations containing elements that are elongated.

3. Most types of edge element have a zero divergence. Because of this they can be applied only to solving problems the solution of which is a priori known to be free of divergence.

4. Under specific circumstances, the description of which is beyond the scope of the present paper, the use of edge elements may result in linearly dependent algebraic equations, i.e. in singular stiffness matrices [18].

5. Plots of solutions of field problems obtained by using edge elements often seem to be rather "rough". This is a natural consequence of the absence of an explicit normal continuity condition across the inter-element interfaces.

6. The use of edge elements seems to be incompatible with the least-squares minimization of the error in the modeling of the field.

IV. EDGE ELEMENTS AND RE-ENTRANT CORNERS

Edge elements are often mentioned as a method to eliminate the large errors that are made when using Cartesian elements near re-entrant corners in, for instance, a perfectly electrically conducting outer boundary. Obviously edge elements, which are polynomials, cannot be expected to accurately model the singular behaviour of the field near a re-entrant corner. The reason that the error observed when using edge elements near a re-entrant corner seems to be small lies in the fact that edge elements allow the normal component of the field across interfaces to be discontinuous and, consequently, the direction of the field near the corner is allowed to change abruptly as required. Nevertheless the local error near the re-entrant corner remains unbounded.

Similar results can be obtained when using (generalized) Cartesian elements provided the nodes at the corner are treated as multiple nodes. In this way we again allow the direction of the field to change abruptly across interfaces but we still have the same unbounded errors near the corner due to the fact that Cartesian elements also are polynomials. As compared with the use of edge elements the advantage of the latter approach could be that Cartesian bases are used which, in turn, yields a better condition of the representation of the field as mentioned above and simper logic.

It will be clear that accurate solutions near re-entrant corners can be obtained only by using expansion functions having the proper degree of singularity and it seems only natural to develop (generalized) Cartesian expansion functions, to be intruduced below, for that purpose.

V. EDGE ELEMENTS AND POTENTIALS

Edge and Cartesian elements are frequently used in finite element methods for solving (electro)magnetic field problems using vector potentials [19]. Our comments regarding the properties of edge elements can be applied directly to their use in vector potential methods. Vector potential methods are known to have the disadvantage that for computing electric and/or magnetic fields from them they require numerical differentiations with the accompanying loss of (one order of) accuracy and the consequent large loss of efficiency. The use of edge elements that are free of divergence causes a loss of efficiency in the representation of the unknown being represented. When using the latter elements in a vector potential method, however, they do not cause a further degradation of the efficiency of the method, i.e. assuming it is used for computing electric and/or magnetic field strengths.

VI. DO WE HAVE ALTERNATIVES?

The shopping list of problems encountered when using edge elements cries out for an alternative. Fortunately a number of methods is available that are expected to relieve us of (the disadvantages of) edge elements and the frequent confusion caused by them. We mention the following:

1. The first alternative, and in the opinion of this author the most promising one because of its efficiency, is provided by a new class of vectorial finite elements, the generalized Cartesian elements. These elements can accurately model fields that are discontinuous across interfaces as well as fields in homogeneous subdomains and will be presented at COMPUMAG97 [20]. Generalized Cartesian elements are also described in [21] and [22].

2. A second class of alternatives may be provided by so-called "dual" or "complementary" formulations. In

this type of formulations, two "complementary" vectorial field quantities are chosen that *together* allow a consistent representation of electromagnetic field inside the domain of computation. Dual approaches assume a subset of the Maxwell's equations to be exactly satisfied, i.e. the local curl equations as in [23],[24] or the domain-integrated curl *and* divergence equations as in [25], while imposing the remaining field equations in a weak form. Although requiring a larger number of degrees of freedom, dual approaches have the advantage of satisfactorily modeling *both* the field equations *and* the relevant compatibility relations.

3. Finally one has the possibility of retaining edge elements (for which there does not seem to be much reason left) and eliminating the possibility of obtaining spurious modes and spurious solutions by choosing a formulation that includes both the field equations and all relevant compatibility relations [16], [17].

VII. CONCLUSIONS

A critical discussion of the properties of edge elements was presented. The claim that edge elements eliminate spurious solutions was refuted and a series of additional disadvantages of edge elements were discussed. For coping with the resulting difficulty in formulating reliable and efficient finite-element methods for electromagnetics a number of alternatives for (the naive use of) edge elements were indicated.

REFERENCES

[1] J. McMahon, "Lower bounds for the electrostatic capacity of a cube," *Proc. Royal Irish Acad.*, Vol. 55, Sect. A, pp. 133-167, 1953.

[2] J.C. Nédélec, "Mixed finite elements in R3," *Numer. Math.*, Vol. 35, 1980, pp. 315-341.

[3] A. Bossavit and J.C. Verité, "The 'TRIFOU' code: Solving the 3-D eddy currents problem by using H as the state-variable," Digest of summaries of the COMPUMAG conference, Genoa, Italy, 1983, pp. 286-290.

[4] C.W. Crowley, P.P. Silvester and H. Hurwitz, "Covariant projection elements for 3D vector field problems," *IEEE Transactions on Magnetics*, Vol. 24, pp. 397-400, Jan. 1988.

[5] M. Hano, "Vector finite-element solution of anisotropic waveguides using novel triangular elements," *Electronics and Communications in Japan*, Part 2, Vol. 71, pp. 71-80, 1988.

[6] A. Bossavit and I. Mayergoyz, "Edge-elements for scattering problems," *IEEE Transactions on Magnetics*, Vol. 25, pp. 2816-2821, July 1989.

[7] J.-F. Lee, "Analysis of passive microwave devices by using three-dimensional tangential vector finite elements," *International Journal of Numerical Modelling*, Vol. 3, 1990, pp. 235-246.

[8] Z.J. Cendes, "Vector finite elements for electromagnetic field computation," *IEEE Transactions on Magnetics*, Vol. 27, pp. 3958-3966, Sept. 1991.

[9] J.P. Webb, "Edge elements and what they can do for you," *IEEE Transactions on Magnetics*, Vol. 29, pp. 1460-1465, Nov. 1993.

[10] B.M. Dillon, P.T.S. Liu and J.P. Webb, "Spurious modes in quadrilateral and triangular edge elements," *COMPEL*, Vol. 13, Supplement A, 1994, pp. 311-316.

[11] R. Dyczij-Edlinger and O. Biro, "A Joint vector and scalar potential formulation for driven high frequency problems using hybrid edge and nodal finite elements," *IEEE Transactions on Microwave Theory and Techniques*, Vol. 44, pp. 15-23, 1996.

[12] G. Mur and A.T. de Hoop, "A finite-element method for computing three-dimensional electromagnetic fields in inhomogeneous media," *IEEE Transactions on Magnetics*, Vol. 21, pp. 2188-2191, Nov. 1985.

[13] J.C. Nédélec, "A new family of mixed finite elements in R3," *Numer. Math.*, Vol. 50, 1986, pp. 57-81.

[14] G. Mur, "Edge elements, their advantages and their disadvantages," *IEEE Transactions on Magnetics*, Vol. 30, pp. 3552-3557, Sept. 1994.

[15] J.A. Stratton, *Electromagnetic Theory*, McGraw-Hill Book Company, New York, 1941,p. 486.

[16] G. Mur, "Compatibility relations and the finite-element formulation of electromagnetic field problems," *IEEE Transactions on Magnetics*, Vol. 30, pp. 2972-2975, Sept. 1994.

[17] B.-N. Jiang, J. Wu and L.A. Povinelli, "The origin of spurious solutions in computational electromagnetics," *Journal of Computational Physics*, Vol. 125, pp. 104-123, 1996.

[18] I.E. Lager, *Finite element modelling of static and stationary electric and magnetic fields*, PhD dissertation, Delft University Press, 1996.

[19] K. Preis, I. Bardi, O. Biro, C. Mangele, G. Vrisk, and K. R. Richter, "Different finite element formulations of 3D magnetostatic fields," *IEEE Trans. on Magnetics*, vol. MAG-28, no. 2, pp. 1056-1059, March 1992.

[20] I.E. Lager and G. Mur, "Least squares minimising finite element formulation for static and stationary electric and magnetic fields," invited for presentation at COMPUMAG97, paper I-5.

[21] G. Mur and I. E. Lager, "The finite-element modelling of electromagnetic fields," Proceedings of the International Conference on Electromagnetics in Advanced Applications, ICEAA97, Sept. 15-18, 1997, Torino, Italy, pp. 171-174.

[22] I.E. Lager and G. Mur, "Generalised Cartesian finite elements," accepted for publication in *IEEE Transactions on Magnetics*. Also: Report #EMG–002–97–08, Schlumberger-Doll Research, Old Quarry Road, Ridgefield, CT 06877-4108, USA, available upon request.

[23] N.A. Golias, T.D. Tsiboukis and A. Bossavit, "Constitutive inconsistency: rigorous solution of Maxwell equations based on a dual approach," *IEEE Transactions on Magnetics*, Vol. 30, No. 5, pp. 3586–3589, 1994.

[24] M.-F. Wong, O. Picon and V.F. Hanna, "A finite element method based on Whitney forms to solve Maxwell equations in the time domain," *IEEE Transactions on Magnetics*, Vol. 31, No. 3, pp. 1618–1621, 1995.

[25] A.T. de Hoop and I. E. Lager, "Static magnetic field computation - an approach based on the domain-integrated field equations," accepted for presentation at COMPUMAG97, paper PF1-11.