STELLINGEN

behorende bij het proefschrift

Optimal Maintenance Decisions for Hydraulic Structures under Isotropic Deterioration

van

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I

Zij c_0 de initiële bouwkosten, $F(t)$ de cumulatieve kansverdeling van het continue vernieuwingstijdstip T en $c(t)$ de kosten die zijn verbonden aan een vernieuwing op tijdstip t, waarbij $c_0 > 0$, $T \geq 0$ en $c(t) > 0$. Met behulp van de vernieuwingstheorie zijn de verwachte gedisconteerde kosten over een oneindige tijdshorizon dan te schrijven als

$$c_0 + \lim_{\tau \to \infty} C_\alpha(\tau) = c_0 + \frac{\int_0^\infty \alpha^t c(t)\, dF(t)}{1 - \int_0^\infty \alpha^t \, dF(t)},$$

met α de discontofactor en $C_\alpha(\tau)$ de verwachte gedisconteerde onderhoudskosten in de eindige tijdsperiode $(0, \tau]$, waarbij $0 < \alpha < 1$ en $\tau > 0$.

Daar deze stelling uitermate geschikt is voor het vinden van een optimale balans tussen initiële bouwkosten enerzijds en toekomstige onderhoudskosten anderzijds ('life cycle costing'), zou zij in geen enkel leerboek op het gebied van de operationele analyse mogen ontbreken.

Deze stelling is analoog aan de discrete vernieuwingsstelling met gedisconteerde kosten in Hoofdstuk 4 van het proefschrift.

II

Stel de oneindige rij van niet-negatieve reële stochastische grootheden $\{X_i^\lambda : i \in \mathbb{N}\}$ is l_μ-isotropisch voor zekere $\lambda > 0$ en $\mu > 0$, d.w.z. de kansdichtheidsfunctie van $(Y_1, \ldots, Y_n) = (X_1^\lambda, \ldots, X_n^\lambda)$ kan - met betrekking tot de Lebesgue-maat - worden geschreven als

$$p(y_1, \ldots, y_n) = f_n \left(\textstyle\sum_{i=1}^n y_i^\mu \right)$$

voor elke $n \in \mathbb{N}$. Het is bekend dat de kansdichtheidsfunctie van $(X_1, \ldots, X_n)$ in dit geval een zogenaamd 'schaalmengsel' van gegeneraliseerde gamma-verdelingen is:

$$p(x_1, \ldots, x_n) = \int_0^\infty \prod_{i=1}^n \frac{\nu^{\frac{1}{\mu}} \lambda \mu}{\Gamma(\frac{1}{\mu})} x_i^{\lambda - 1} \exp\left\{ -\nu x_i^{\lambda\mu} \right\} I_{[0,\infty)}(x_i)\, dP(\nu).$$

De gegeneraliseerde gamma-verdeling met onzekere parameters λ, μ en ν kan worden gebruikt voor het schatten van de kans van optreden van extreme rivierafvoeren.

Zie: Hoofdstuk 3 van het proefschrift;

HKV 'Lijn in Water', Rijkswaterstaat RIZA en Waterloopkundig Laboratorium. Integrale Verkenning inrichting Rijntakken; Rapportnummer 12: Veiligheid, 1996.

III

Zij $0 < \alpha < 1$, $s > 0$ en $x > 0$, dan geldt

$$\lim_{\alpha \uparrow 1} \sum_{j=1}^{\infty} (1-\alpha)\alpha^j [x + j(1-\alpha)]^{-s} = e^x \Gamma(1-s, x),$$

waarbij $\Gamma(a, x) = \int_{t=x}^{\infty} t^{a-1} e^{-t}\, dt$ de incomplete gamma-functie is voor $x > 0$ en $a \in \mathbb{R}$.

Volgt uit een combinatie van Stelling 7 en Vergelijking (4.8), (4.11) en (4.14) in het proefschrift.

IV

Stel de stochastische vector $(X_1, \ldots, X_n)$ heeft reële coördinaten, zodanig dat zijn kansdichtheidsfunctie - met betrekking tot de Lebesgue-maat - kan worden geschreven als

$$p(x_1, \ldots, x_n) = f\left(\sum_{i=1}^n x_i\right) \prod_{i=1}^n I_{[0,y]}(x_i).$$

Voor elke $n \geq 2$ en $k < n$ geldt dan dat

$$p\left(x_1, \ldots, x_k \,\middle|\, \sum_{i=1}^n x_i = n\theta\right) = \\ = \frac{\Gamma(n)}{\Gamma(n-k)} \frac{\sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j \left[1 - \frac{\sum_{i=1}^k x_i + jy}{n\theta}\right]_+^{n-k-1}}{\sum_{j=0}^{n} \binom{n}{j} (-1)^j \left[1 - \frac{jy}{n\theta}\right]_+^{n-1}} \prod_{i=1}^k \frac{1}{n\theta} I_{[0,y]}(x_i)$$

voor $\theta > 0$, waarbij $[x]_+ = \max\{0, x\}$, alsmede $I_A(x) = 1$ voor $x \in A$ en $I_A(x) = 0$ voor $x \notin A$.

V

Zij $\alpha > 0$ en $\beta > 0$, dan volgt

$$\frac{\frac{1}{\beta} \mathrm{B}(\frac{1}{\beta}, \frac{1}{\beta})}{\alpha \mathrm{B}(\alpha, \alpha)} \geq \left[\frac{2\mathrm{B}(\alpha\beta + \alpha, \alpha)}{\mathrm{B}(\alpha, \alpha)}\right]^{-2/\beta},$$

waarbij $\mathrm{B}(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt$ de beta-functie is. Gelijkheid geldt alleen als $\alpha\beta = 1$.

VI

De benaming 'normale verdeling' suggereert ten onrechte, dat deze kansverdeling normaal gesproken zou moeten worden gebruikt. Om dit te voorkomen kan deze benaming beter worden vervangen door 'de Moivre-verdeling', zo genoemd naar haar ontdekker.

Abraham de Moivre. *Approximatio ad Summan Terminorum Binomii* $\overline{a+b}^n$ *in Serium Expansi,* 1733.

VII

Hoe meer onzekerheid in een beslissingsprobleem, des te eenvoudiger het beslissingsmodel.

VIII

Ook bestuurlijke fouten en politieke nalatigheden - zoals die bijvoorbeeld veelvuldig hebben plaatsgevonden voor en tijdens de overstromingsramp van 1953 - dienen in aanmerking te worden genomen bij het maken van een risico-analyse.

Zie: Kees Slager. *De Ramp: Een Reconstructie.* De Koperen Tuin, Goes, 1992.

IX

De schatting van de kans op extreme Noordzee-waterstanden zou niet alleen moeten worden gebaseerd op waarnemingen van de afgelopen honderd jaar, maar ook op die van ouder datum zoals de Allerheiligenvloed (1570), de Sint-Felix-vloed (1530) en de Sint-Elisabethsvloed (1421).

X

Nieuwbouw in het winterbed van de Maas moet worden verboden, tenzij men overgaat tot het bouwen van paalwoningen.

XI

Aangezien 'Die Haghe' de oudstbekende naam is van Den Haag, verdient de benaming 'Den Haag' de voorkeur boven het voor buitenlanders zo moeilijk uitspreekbare "s-Gravenhage'.

Zie: Jacob de Riemer. *Beschryving van 's Graven-Hage; Eerste Deels Eerste Stuk,* 1730.

Optimal Maintenance Decisions for Hydraulic Structures under Isotropic Deterioration

Optimale Onderhoudsbeslissingen voor Waterbouwkundige Constructies onder Isotropische Veroudering

PROEFSCHRIFT

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
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in het openbaar te verdedigen ten overstaan van een commissie,
door het College van Dekanen aangewezen,
op dinsdag 28 mei 1996 te 16.00 uur

door

Jan Maarten VAN NOORTWIJK

wiskundig ingenieur

geboren te Haarlem

Dit proefschrift is goedgekeurd door de promotor:

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Den Haag, April 1996 Jan van Noortwijk

Contents

Chapter 1

Introduction

The subject of this Ph.D. thesis is the determination of optimal maintenance decisions for hydraulic structures subject to deterioration. This introduction briefly describes the maintenance of hydraulic structures in The Netherlands, presents three cost-based criteria for comparing maintenance decisions under uncertain deterioration, and gives an overview of the thesis.

1.1 Maintenance of hydraulic structures

With storm-induced tides of some four metres above average sea level, the flood of February 1, 1953, caused a severe catastrophe in the south-west of The Netherlands. Almost 200,000 hectares of polderland flooded, 1,835 people and tens of thousands of animals were drowned, about 100,000 people had to be evacuated, and more than 46,000 buildings were destroyed or damaged. At the price-level of 1953, the flood damage totaled up to 1.5×10^9 Dutch guilders. Behind these huge figures there is a sorry story: on the basis of 200 eye-witness accounts, an impressive and a harrowing reconstruction of the 1953 flood has been made by Slager [110].

To protect the Dutch lowlands against flooding, a flood defence system has been constructed in which The Netherlands is subdivided into fifty-three dyke-ring areas (see Fig. 1.1). These areas are surrounded by dyke rings consisting of dykes (more than 2,500 km), dunes (254 km), water-retaining works (e.g. the Eastern-Scheldt storm-surge barrier), and higher ground. Acceptable inter-occurrence times of the exceedence of the water level that a dyke-ring component should withstand are laid down in the Dutch Flood Protection Act [117] and vary from 10,000 years (for dyke-ring areas subject to sea floods) to 1,250 years (for dyke-ring areas subject to river floods).

Each component of a dyke ring has to fulfill certain requirements in the areas of flood protection, environment, recreation, shipping access, road connection, transport, agriculture, fishery, and landscape. As soon as a component fails to meet its main requirements it should be repaired, preferably against minimal costs. For example,

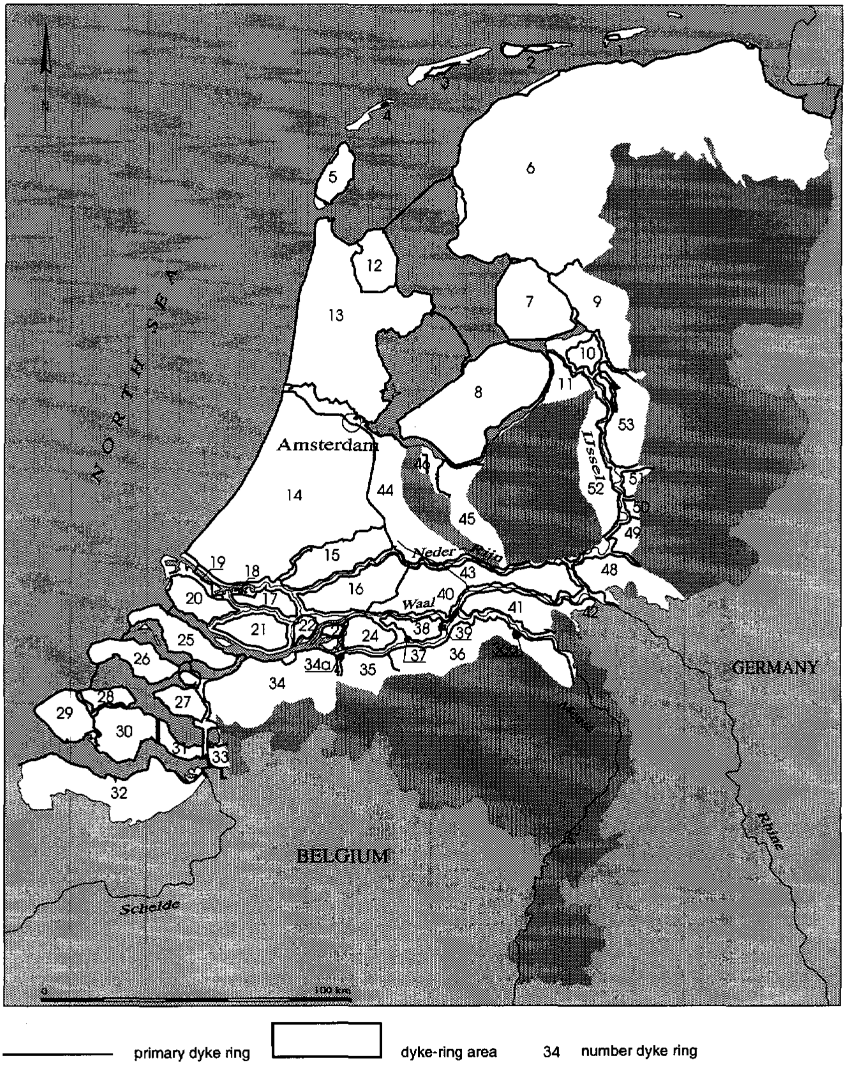

Figure 1.1: Dyke rings in The Netherlands.

when focussing on flood protection only, a distinction can often be made between a structure's resistance (e.g. dyke height) and its design stress (e.g. the maximal water level to be withstood). A failure may then be defined as the event in which the resistance drops below the stress.

Usually, *maintenance* is defined as a combination of actions carried out to restore a hydraulic structure to, or "renew" it to, its desired condition. Inspections, repairs, and replacements are possible maintenance actions. Kelly [72, Ch. 2] distinguishes two types of maintenance: *corrective maintenance* (after failure) and *preventive maintenance* (before failure). Failure-based corrective maintenance can best be chosen if the costs arising from failure are low; preventive maintenance if these costs are high (see Fig. 1.2). We can choose between two preventive maintenance strategies: (i) *time-based maintenance* carried out at regular intervals of time, operation, or use, and (ii) *condition-based maintenance* carried out at times determined by inspecting or monitoring a structure's condition. Time-based maintenance can be applied if the times to failure are almost known. Condition-based maintenance can be relied on if the deterioration is inspectable or monitorable. By contrast, a structure should not simply be maintained but redesigned or seriously improved when the cost of its failure is high, the rate of its deterioration is very uncertain, and its condition cannot be inspected or monitored.

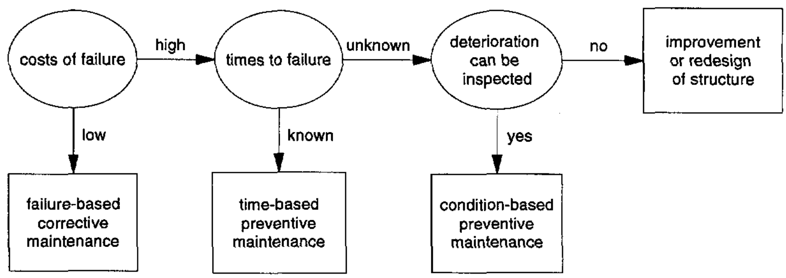

Figure 1.2: Decision diagram for corrective and preventive maintenance.

In hydraulic engineering, expensive condition-based preventive maintenance is mainly applied. Every year 7 million cubic metres sand must be supplied at beach locations subject to ongoing erosion against costs of about 70 million Dutch guilders. The costs of back maintenance of the Dutch water ways is estimated to be about 650 million Dutch guilders. An annual budget of over 500 million Dutch guilders is dedicated to the Dutch flood defence system. Since the Delta Plan will soon be completed, the attention is shifting from building structures to maintaining structures: for example, the annual costs of maintaining the water-retaining works is expected to increase from about 50 million Dutch guilders today to about 175 million Dutch guilders in 2040 (see Rijkswaterstaat [104]). The use of maintenance optimisation models will therefore be of considerable interest.

During the life cycle of a hydraulic structure we can roughly identify four phases: the design, the building, the use, and the demolition. There are mainly two phases in which it is worth applying maintenance optimisation techniques: the design phase and the use phase. In the design phase, one might obtain an optimum balance between building costs and maintenance costs. In the use phase, one might minimise the costs of inspection, repair, replacement, and failure. Problems of finding an optimum balance between initial and future costs belong to the area of *life cycle costing* (see Blanchard [10], Fabrycky & Blanchard [46], and Flanagan et al. [51]).

The area of optimising maintenance through mathematical models has been founded in the early sixties by researchers like Barlow, Proschan, Jorgenson, McCall, Radner, and Hunter. Their pioneering work is summarised in McCall [85] and Barlow & Proschan [5, 6]. Their interest in maintenance optimisation has been provoked by the high cost of military-industrial equipment like jet airliners, electronic computers, ballistic missiles, etc. Well-known models of this period are the age replacement model (replacement upon failure or upon reaching age T) and the block replacement model (replacement upon failure or periodically at the times $T, 2T, 3T, \ldots$). More recently, a large number of papers on maintenance optimisation, mainly focussing on the mathematical aspects, have been published. For an, inherently incomplete, overview see Pierskalla & Volker [96], Sherif & Smith [108], Sherif [109], Valdez-Flores & Feldman [118], and Cho & Parlar [12].

Most maintenance optimisation models are based on lifetime distributions or Markovian deterioration models. Unfortunately, only a few of them have been applied (see Dekker [28] and Pintelon & Gelders [97]). According to De Jonge, Kok & Van Noortwijk [25], there are two possible reasons for this poor applicability. First, from the theoretical point of view, there is often no interest in "details" that are of practical importance: a problem description is often lacking or even purely hypothetical. Second, from the practical point of view, there is little experience in using maintenance optimisation models and it is often hard to gather data for estimating either the parameters of a lifetime distribution or the transition probabilities of a Markov chain. Moreover, in case of well-planned preventive maintenance, complete lifetimes will be observed rarely.

In mechanical and electrical engineering, one often considers equipment which can assume at most two states: the failed state and the non-failed state. In hydraulic engineering, however, a structure can be in a range of states depending on its degrading resistance. Examples of stochastic deterioration processes are: ongoing coastal erosion of a beach section, crest-level decline of a dyke section, longshore rock transport of a berm breakwater, and current-induced scour erosion of a sea-bed protection.

For the most part, the maintenance optimisation literature deals with mechanical and electrical engineering. A few papers on optimum maintenance policies in structural reliability and hydraulic engineering were found. Tang & Yen [113, 114] have studied optimal inspection scheduling for dams on the basis of lifetime distributions (the gamma distribution). Deodatis et al. [39] and Ito et al. [66] have evaluated non-periodic inspection intervals for fatigue-sensitive structures by considering the time to crack initiation, the probability of crack detection, and the exponential rate of crack propagation to be random quantities. In several maintenance models, the deterioration process has been regarded as the celebrated Brownian motion with drift (a stochastic process with stationary independent decrements and increments having a normal distribution): see, e.g., Gijsbers [54], Hontelez, Burger & Wijnmalen [62], Kok [74], and Kuijper & Vrijling [80].

1.2 Optimal maintenance decisions

As mentioned before, a hydraulic structure is said to fail when it does not satisfy a pre-determined failure or design level (like the basal coastline of a beach section and the acceptable crest-level of a dyke section). Therefore, and because the maintenance action to be taken typically depends on the observed amount of deterioration (we deal with condition-based maintenance), we restrict ourselves to stochastic resistance and deterministic stress. This means that a structure fails when its resistance R is below a constant failure level s. Furthermore, we focus on failure of *one* hydraulic structure (e.g. a dyke section) with respect to *one* requirement (e.g. safety) due to *one* failure mode (e.g. crest-level decline). In a fault tree analysis, this failure mode can be combined with other failure modes, other structures, other dyke-ring components, and other requirements, which might be interdependent. It should be noted that there is a fundamental difference between failure and collapse: a failed dyke section only collapses when the applied water level exceeds its crest-level height. For an overview of maintenance aspects of the Dutch flood defence system, see De Quelerij & Van Hijum [26], Reij & Van der Toorn [99], Van der Toorn [122], and Van Noortwijk [123].

In this thesis, a methodology has been developed that might bridge the gap between theory and practice by modelling maintenance of hydraulic structures on the basis of the main uncertainty involved: the value of the limiting, non-negative, average rate of deterioration denoted by the random quantity Θ. To achieve this, and to account for most deterioration processes to proceed in one direction and in random jumps, these processes have been regarded as so-called *generalised gamma processes.*

A gamma process is a stochastic process with independent non-negative increments having a gamma distribution with known limiting average rate. A generalised gamma process is then defined as a mixture of gamma processes, where the mixture represents the uncertainty in the unknown limiting average rate of deterioration. Note that the Brownian motion with drift is often not applicable in a maintenance context, since we must require that increments of deterioration are non-negative. Moran [91] used gamma processes in his theory of the storage of water by dams. Reliability models based on the gamma process have been developed by Dykstra & Laud [45] and Wenocur [143].

A useful property of the generalised gamma process is that we can always find a unique uniform time-partition, in time-intervals of length Δ, for which the joint probability density function of the increments of deterioration is a mixture of exponentials, which are conditionally independent when the limiting average rate of deterioration is given. Let D_i be the increment of deterioration in unit time $((i-1)\Delta, i\Delta], i = 1, \ldots, n$, then

$$p_{D_1,\ldots,D_n}(\delta_1, \cdots, \delta_n) = \int_0^\infty \prod_{i=1}^n \frac{1}{\theta} \exp\left\{-\frac{\delta_i}{\theta}\right\} dP(\theta)$$

for $(\delta_1, \cdots, \delta_n) \in \mathbb{R}_+^n$ and zero otherwise, where $\mathbb{R}_+ = [0, \infty)$. The random quantity Θ represents the uncertainty about the limiting average rate of deterioration $\lim_{N\to\infty}[(\sum_{i=1}^N D_i)/N]$. The infinite sequence of random quantities $\{D_i : i \in \mathbb{N}\}$ is said to be l_1-isotropic (or l_1-norm symmetric), because its distribution can be written as a function of the l_1-norm. Note that the variance of the generalised gamma process determines the unit-time length for which the increments are l_1-isotropic: in fact, the smaller the variance, the smaller the corresponding unit-time length. Since we are equipped with the exponential likelihood function, we can express various probabilistic properties, such as the probability of exceedence of a failure level per unit time, in explicit form conditional on the limiting average rate of deterioration. Due to the exchangeability of the (l_1-isotropic) increments of deterioration, the expected cumulative amount of deterioration is linear in time. Relevant references for the notion of isotropy include Diaconis & Freedman [42], Misiewicz & Cooke [90], Mendel [89], Barlow & Mendel [3], and Hayakawa [60].

To make optimal maintenance decisions while taking account of the uncertainty in the limiting average rate of deterioration, we can use statistical decision theory (see DeGroot [27, Ch. 8] and Savage [106]). In a typical maintenance decision problem, a decision-maker must choose a maintenance decision d from all possible decisions $\mathcal{D}$, with the consequences of decision d depending on the unknown value of the limiting average rate of deterioration Θ. Let us assume that there exists a probability distribution P on Θ, where P may represent *a priori* beliefs about Θ, which can be updated with actual observations using Bayes' theorem. Let $L(\theta, d)$ be the *loss* when the decision-maker chooses decision d and when the limiting average rate of deterioration is given by θ, where the loss represents the monetary losses due to maintenance. The decision-maker can best choose, if possible, a maintenance decision d^* whose expected loss is minimal. A decision d^* is called an *optimal decision* or a *Bayes optimal decision* when

$$E(L(\Theta, d^*)) = \min_{d\in\mathcal{D}} E(L(\Theta, d)).$$

These optimal maintenance decision are also known as Bayes adaptive maintenance policies: they can be revised in the light of new observations (see McCall [85]). Bayes adaptive age and block replacement policies have been studied by Bassin [7], Fox [52, 53], and Mazzuchi & Soyer [83, 84]. Bayes adaptive maintenance policies for structures subject to deterioration that can be regarded as generalised gamma processes are new.

The maintenance of hydraulic structures can best be modelled as a renewal process, where the renewals are the maintenance actions restoring a structure to its desired condition. After each renewal we start, in a statistical sense, all over again. Since the planned lifetime of the Dutch dyke-rings is unbounded, maintenance decisions can best be compared over an unbounded time-horizon. According to Wagner [138, Ch. 11] there are basically three cost-based criteria that can serve as loss functions:

1. the *expected average costs per unit time*, which are determined by averaging the costs over an unbounded horizon;
2. the *expected discounted costs over an unbounded horizon*, which are determined by summing the (present) discounted values of the costs over an unbounded horizon under the assumption that the utility of money decreases in time; and
3. the *expected equivalent average costs per unit time*, which are determined by averaging the discounted costs.

These cost-based criteria can be computed using the discrete renewal theorem (see e.g. Feller [47, Ch. 13] and Karlin & Taylor [71, Ch. 3]) and are derived in Chapter 4. The notion of equivalent average costs relates the notions of average costs and discounted costs in the sense that the equivalent average costs per unit time approach the average costs per unit time, as the discount rate tends to zero, from above. Although in the literature most attention has been focussed on the criterion of average costs, the cost-based criteria of discounted costs and equivalent average costs are most suitable for optimally balancing the initial building costs against the future maintenance costs (life cycle costing). The criterion of average costs can be used in situations in which no large investments are made (like inspections) and in which the time value of money is of no consequence to us. Often, however, it is preferable to spread the costs of maintenance and failure over time and to use discounting.

In this thesis, generalised gamma processes have been used to model decision problems for optimising maintenance of various characteristic components of a dyke ring: a beach section, a dyke section, a berm breakwater, and a retaining work (the Eastern-Scheldt barrier). Furthermore, decisions that reduce flood damage along the river Meuse have been evaluated and compared by assuming, amongst others, l_1-isotropy in the flood damage (using the so-called principle of indifference of Mendel [89]).

1.3 Overview of the thesis

The thesis is organised as follows. The mathematical foundations for modelling deterioration processes as generalised gamma processes are laid in Chapters 2 and 3. Four case studies on optimal maintenance decisions are discussed in Chapter 4 (a beach section), Chapter 5 (a dyke section), Chapter 6 (a berm breakwater), and Chapter 7 (a sea-bed protection). They all represent tailor-made models for the maintenance problems at hand with cost of repair proportional to the degraded volume of sand, clay, or rock. A decision-theoretic application on river flooding, by using isotropy and discounting, is presented in Chapter 8. Finally, Chapter 9 describes a maintenance model for failure probabilities having a Dirichlet distribution, which is useful when both resistance and stress are stochastic. The chapters are self-contained and can be read separately.

Chapter 2: A Bayesian failure model based on isotropic deterioration.
A failure model is developed on the basis of the amount of deterioration averaged over a bounded or an unbounded time-horizon leading to, respectively, a finite or an infinite sequence of l_1-isotropic increments of deterioration. Bayes estimates of the probabilities of failure and preventive repair are expressed explicitly conditional on the average deterioration. The failure model is applied to the rock dumping of the Eastern-Scheldt barrier which is subject to current-induced rock displacement (the barrier connects dyke rings 26 and 28; see Figs. 1.1 and 7.1).

Chapter 3: A characterisation of generalised gamma processes.
In addition to the classical characterisation of gamma processes in terms of compound Poisson processes (see e.g. Gnedenko & Kolmogorov [55, Ch. 3 & 5], Lévy [81, pp. 173-180], Itô [65, Ch. 1], Ferguson & Klass [49], and de Finetti [23, Ch. 8]), this chapter presents two new mathematical characterisations of generalised gamma processes: (i) in terms of conditioning on sums of increments, serving as sufficient statistics for the unknown limiting average rate, and (ii) in terms of isotropy. The characterisation in terms of sufficiency extends results of Diaconis & Freedman [41] and Küchler & Lauritzen [78]. The characterisation in terms of isotropy originates from the work of Barlow & Mendel [3] and Misiewicz & Cooke [90].

Chapter 4: Optimal sand nourishment decisions.
This chapter studies optimal sand nourishment sizes for which the expected discounted costs over an unbounded horizon are minimal with respect to the probability distribution of the limiting average rate of ongoing coastal erosion. The decision model is applied to sand nourishment at Zwanenwater (a beach section that is part of dyke ring 13; see Figs. 1.1 and 4.5).

Chapter 5: Optimal maintenance decisions for dykes.
This chapter discusses optimal dyke heightenings for which the expected discounted costs over an unbounded horizon are minimal with respect to the probability distribution of the limiting average rate of crest-level decline (being a combination of settlement, subsoil consolidation, and relative sea-level rise). On the basis of a physical

law, crest-level decline has also been regarded as a stochastic process with expected decline being approximately logarithmic in time (using engineering probability: see Mendel & Chick [88] and Chick [11]). Both decision models are applied to the Dutch 'Oostmolendijk' (a dyke section that is part of dyke ring 17; see Fig. 1.1). The proposed models extend results of Van Dantzig [119], Vrijling & Van Beurden [134], and Kuijper [79].

Chapter 6: Optimal maintenance decisions for berm breakwaters.
This chapter examines optimal inspection intervals for berm breakwaters whose expected (equivalent) average costs per unit time are minimal with respect to the probability distribution of the limiting average rate of longshore rock transport. The model that is proposed is a two-phase inspection model in which the first phase represents the event of no breach of the armour layer and the second phase represents the event of longshore rock transport initiated by an armour breach. It is a special case of the delay-time model as studied by Christer & Waller [14, 15]: the time lapse from the occurrence of an armour breach until the time of failure, due to longshore rock transport, can be interpreted as the discrete delay time of a failure. These discrete delay times are assumed to be distributed according to a mixture of geometrics. The inspection model has been applied to a hypothetical berm breakwater; it extends results of Vrijling & Van Gelder [135].

Chapter 7: Optimal maintenance decisions for a sea-bed protection.
In this chapter, cost-optimal inspection intervals for two components of the sea-bed protection of the Eastern-Scheldt barrier are obtained: (i) the block mats and (ii) the rock dumping. The decision model for the block mats is a two-phase inspection model: the inter-occurrence times of scour holes are distributed according to a mixture of Poisson processes and the scour holes develop according to a generalised gamma process. The decision model for the rock dumping is based on the failure model of Chapter 2.

Chapter 8: Optimal decisions that reduce flood damage along the Meuse.
Optimal decisions that might reduce future losses due to flooding of the river Meuse are investigated. When the loss is defined as the expected discounted costs of decisions minus the yields of decisions plus the remaining mean flood damage over an unbounded horizon, optimal decisions can be obtained with respect to the following three decision criteria: the criterion of minimal expected loss, the criterion of minimal uncertainty in the loss, and the criterion of maximal safety. By using dependent Monte Carlo simulation, the present situation and five strategies (combinations of decisions) are analysed. It should be noted that, strictly speaking, this chapter is beyond the scope of this thesis: although it does not deal with maintenance optimisation, it is included because it nicely illustrates the usefulness of isotropy and discounting. The research

is performed within in the framework of the project *Investigation of the 1993 Meuse Flood* (see [29, 37]).

Chapter 9: Optimal maintenance decisions on the basis of expert judgment.
A maintenance model is studied in which the failure probabilities are Dirichlet distributed. This model is useful when failure probabilities cannot be expressed in explicit form: for example, when both the structure's resistance and the applied stress must be regarded as stochastic. Another application is when, due to a lack of data, maintenance optimisation models have to be initialised on the basis of subjective, discretised, lifetime distributions (see Van Noortwijk et al. [128]).

For all the deterioraton processes that have been considered, the unit time establishing l_1-isotropy is assumed to be smaller than, or equal to, the possible inspection intervals (in Chapter 4, this unit time is equal to the inspection interval). In case of a unit time larger than an inspection interval, one might argue whether the structure's resistance is too uncertain to be left unimproved (see also Fig. 1.2). The applications described in Chapters 5 and 7 show that the optimality of a maintenance decision hardly depends on the l_1-isotropic grid that is chosen. In determining the expected deterioration at the end of unit time j, while exceedence of the failure level occurs in unit time n, the following mathematical interrelations between the theorems of Chapters 4, 5, and 6 can be identified: Theorems 5, 8, and 14 treat the cases "$j = n$", "$j \geq n$", and "$j < n$", respectively, for $j, n \in \mathbb{N}$.

Chapter 2

A Bayesian Failure Model based on Isotropic Deterioration

Jan M. van Noortwijk, Roger M. Cooke, and Matthijs Kok

Abstract. For the purpose of modelling the maintenance of hydraulic structures, a failure model has been developed that is based on the only information that is commonly available: the amount of deterioration averaged over a bounded or an unbounded time-horizon. By introducing a prior density on the average deterioration per unit time, we can properly account for the uncertainty in a maintenance decision problem. The advantages of our Bayesian approach are that the failure model is based on a physical observable quantity, the deterioration, and that the probabilities of preventive repair and failure can be expressed explicitly conditional on the average deterioration. One illustration from the field of hydraulic engineering is studied. (This chapter has previously been published as [126].)

Keywords. Bayesian inference, isotropy, maintenance, stochastic processes, structural reliability.

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2.1 Introduction

The aim of this chapter is to develop a failure model for hydraulic structures based on observable deterioration characteristics. This model can be further used for maintenance optimisation purposes. Although the model has been developed to solve maintenance decision problems in hydraulic engineering, it can be applied in the fields of mechanical and electrical engineering as well.

In hydraulic engineering, a failure is often defined as the event in which a structure's resistance drops below the applied stress. We focus on a stochastic resistance and a deterministic stress. Roughly, preventive maintenance consists of inspection and repair. A repair can either be a preventive repair (before failure) or a corrective repair (after failure), where both return the system to the "as good as new state". A preventive repair is executed when the resistance falls below a stipulated preventive repair level. Exceedence of the preventive repair level can only be noted through inspection. If inspection is scheduled too late, the structure passes into the failed state. By inspecting more frequently, we can reduce the probability of failure and increase the likelihood of a preventive repair.

Most existing failure models are based on lifetime distributions or Markovian deterioration models. In practice, however, it is often hard to gather data for estimating either the parameters of a lifetime distribution or the transition probabilities of a Markov chain. Moreover, in case of well planned preventive maintenance, lifetimes will be observed very rarely. A recent literature review of Dekker [28] illustrates the difficulties in quantifying such models; inspite of voluminous theoretical researches, very few applications of maintenance optimisation were found.

In the field of hydraulic engineering, the deterioration process is often assumed to be the Brownian motion with drift, i.e. the amounts of deterioration in a given time-interval are assumed to be independent and to have a normal distribution with both the mean and the variance linear in the time-interval length (see for example Gijsbers [54]). This stochastic process, however, entails a significant probability that the structure will spontanuously improve. This is especially so if the uncertainty in the deterioration process is large, as is often the case in hydraulic engineering. Moreover, for the Brownian motion with drift, simple explicit expressions for the probabilities of preventive repair and failure are not available (see Kok [74] for a discussion).

Our model is based on non-negative increments of deterioration having a joint probability density function that is a mixture of exponentials, which are conditionally independent given the unknown average rate of deterioration. The uncertainty in the deterioration process is thus represented by a probability distribution on the average deterioration per unit time; a random quantity about which engineers are able to have a subjective opinion. Abstract quantities, like shape and scale parameters of a lifetime distribution are avoided; instead, our degree of belief about (more frequently) observable magnitudes is quantified and possibly updated with field data. Conditional on the average deterioration, the amounts of deterioration are said to be isotropic. The notion of isotropy will be discussed in Sec. 2.3 and was first applied to decision problems by Mendel [87, 89].

Explicit expressions for the probabilities of preventive repair and failure are given in Sec. 2.4. The failure model has been applied to the rock-fill top-layer of the sea-bed protection of the Storm-Surge barrier in the Dutch Eastern-Scheldt. All technical proofs can be found in an appendix or, in more detail, in Van Noortwijk [124].

2.2 Notation

D_h, δ_h	Amount of deterioration in the hth unit time.
Δ	Unit-time length.
$N\Delta$	Time-horizon.
p_{norm}	Safety probability norm.
R	Resistance or strength.
r_0	Resistance at time zero.
R_n	Resistance in unit time n.
s	Deterministic stress or failure level.
S	Stress, load or action effect.
$R - S < 0$	Failure.
X_n	Cumulative amount of deterioration in $(0, n\Delta]$: $\sum_{h=1}^{n} D_h$.
Θ_N, Θ, θ	Average amount of deterioration per unit time.
ρ	Preventive repair level.

2.3 Assumptions and definitions

In the fields of mechanical and electrical engineering, one often considers lifetime distributions to model the occurrence of failure in a system, e.g. a motor or switch is working or not. In the area of hydraulic engineering, failure is often defined as comparing a structure's *resistance* or *strength* R with its *stress*, *load* or *action effect* S (see e.g. Ang & Tang [1]). A structure is said to *fail* if its resistance is below the stress, i.e. if the so-called *performance function* $R - S$ is negative. In general, both resistance and stress are unknown functions of time and, moreover, are not necessarily independent. We restrict ourselves to stochastic resistance and deterministic stress. This means that a structure will fail if its resistance R is below a constant failure level s.

The strength of a structure will degrade in time due to deterioration. Let us subdivide the time-axis $(0, \infty)$ into units of time of length Δ: $((n-1)\Delta, n\Delta]$, with $n \in \mathbb{N}$. Often a structure is planned to function for a bounded time-horizon, say $(0, N\Delta]$, where N is some fixed integer, $N = 2, 3, \ldots$ When the structure has completed its mission it will be replaced at $N\Delta$. Suppose that in unit time n the structure's resistance suffers a stochastic deterioration D_n, $D_n \in [0, \infty)$, and assume the resistance at time zero equals r_0, where $r_0 > s$ (see Fig. 2.1). Consequently, the resistance in unit time n can be written as

$$R_n = r_0 - \sum_{h=1}^{n} D_h = r_0 - X_n, \quad n = 1, \ldots, N. \tag{2.1}$$

In practice, especially in the field of hydraulic engineering, a lack of deterioration data is common at the outset. As a consequence, most maintenance decisions are only based on subjective ideas about the average rate of deterioration. Although deterioration can accelerate in time, the acceleration is often difficult, sometimes even impossible, to determine. Moreover, the probability of failure only depends on the accumulated deterioration that results in failure. Therefore, we assume that the available prior information concerns the beliefs about the average decrease in the resistance until failure occurs.

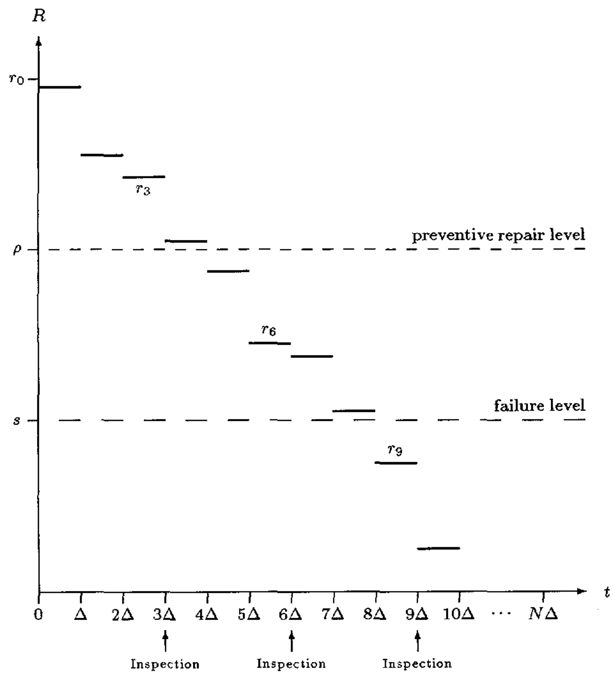

Figure 2.1: A sample path of the resistances $\mathbf{R}_N = (R_1, \cdots, R_N)$ with inspection interval length 3Δ. At the first inspection (t=3Δ): $r_3 \geq \rho$, no preventive repair takes place. At the second inspection: $s \leq r_6 < \rho$, no failure has occurred, but a preventive repair is needed and will be carried out. If the inspection interval length were 9Δ, a failure would occur before inspection.

Our probabilistic modelling of the deterioration process, given the average deterioration, is now straightforward and is based on two assumptions (where the second implies the first):

- The order in which the amounts of deterioration $D_1, \ldots, D_N$ appear is irrelevant. Hence, their joint probability measure is invariant to their order: the amounts of deterioration are *exchangeable.*

- Given the average amount of deterioration per unit time, the decision-maker is indifferent to the way this average is obtained or, in other words, all combinations leading to the same average have the same degree of belief for our decision-maker: the amounts of deterioration are l_1*-isotropic.*

Note that we do not assume independence, but we use the more general notion of exchangeability; independence is a special case. Furthermore, the choice of the unit-time length Δ has influence on the isotropy. In Chapter 3, a general class of isotropic monotone continuous-time processes is defined. For each process in this class, it is possible to choose Δ such that the corresponding amounts of deterioration per unit time are l_1-isotropic. For now, we restrict ourselves to the assumption of l_1-isotropy.

Examples of isotropic deterioration processes (at least in approximation) in the field of hydraulic engineering are coastal erosion (under the continuous action of waves and wind), settlement of a dyke, corrosion of underwater pipelines, and rock displacement and scour erosion near underwater footings of a hydraulic structure.

Recall that the random vector $\mathbf{D}_N = (D_1, \cdots, D_N)$ of N uncertain amounts of deterioration is assumed to be *exchangeable.* This can be interpreted as that the joint probability density function is invariant under all $N!$ permutations of the coordinates, i.e.

$$(D_1, \cdots, D_N) \stackrel{\mathrm{d}}{=} \left(D_{\pi(1)}, \cdots, D_{\pi(N)}\right), \tag{2.2}$$

where π is any permutation of $1, \ldots, N$. An infinite sequence of random quantities $\{D_i : i \in \mathbb{N}\}$ is exchangeable if $\mathbf{D}_n$ is exchangeable for each $n \in \mathbb{N}$.

Suppose the decision-maker has subjective information about the average amount of deterioration determined with regard to the time-horizon $(0, N\Delta]$ and this information can be expressed as a probability density function over $(\sum_{h=1}^{N} D_h)/N = \theta$. Since the decision-maker has no other information, we adopt a uniform distribution over all deterioration vectors $(\delta_1, \cdots, \delta_N)$ having the same average. That is, the distribution of the finite sequence of random quantities $D_1, \ldots, D_N$ is uniform on the simplices

$$\left\{(\delta_1, \cdots, \delta_N) \in \mathbb{R}_+^N : \sum_{i=1}^{N} \delta_i = N\theta\right\}, \tag{2.3}$$

$\theta \in \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$. The random vector $\mathbf{D}_N$ is now said to be l_1-*isotropic*[1] (see Mendel [89] for details). An infinite sequence of random quantities $\{D_i : i \in \mathbb{N}\}$ is l_1-isotropic if $\mathbf{D}_n$ is l_1-isotropic for each $n \in \mathbb{N}$.

The likelihood function of $\delta_1, \ldots, \delta_n$, $1 \leq n < N$, given that $\mathbf{D}_N$ is l_1-isotropic, can be obtained by integrating the uniform distribution out on the simplex $\sum_{h=1}^{N} D_h = N\theta$ over the $(n+1)$th through the Nth amount of deterioration. This can be achieved by applying the Dirichlet integral and results in the Dirichlet likelihood function:

$$l_N(\delta_1, \cdots, \delta_n | \theta) = \frac{\Gamma(N)}{\Gamma(N-n)} \left[1 - \frac{\sum_{h=1}^{n} \delta_h}{N\theta}\right]_+^{N-n-1} \prod_{h=1}^{n} \left[\frac{1}{N\theta}\right] I_{[0,\infty)}(\delta_h), \tag{2.4}$$

where $[x]_+ = \max\{0, x\}$, and $I_A(x) = 1$ for $x \in A$ and $I_A(x) = 0$ for $x \notin A$. For a proof of Eq. (2.4), see Mendel [89].

As N approaches infinity, we obtain a product of n exponentials:

$$l_N(\delta_1, \cdots, \delta_n | \theta) \to l(\delta_1, \cdots, \delta_n | \theta) = \prod_{h=1}^{n} \frac{1}{\theta} \exp\left\{-\frac{\delta_h}{\theta}\right\} I_{[0,\infty)}(\delta_h) = \prod_{h=1}^{n} l(\delta_h | \theta) \tag{2.5}$$

as $N \to \infty$. The convergence is uniform for $(\delta_1/\theta, \cdots, \delta_n/\theta)$ in compact sets. This result coincides with the classical Bayesian exponential model, where the coordinates of $\mathbf{D}_n$ are identically distributed exponential random quantities that are conditionally independent when the mean θ is given.

Let $\Theta_N \stackrel{\mathrm{d}}{=} (\sum_{h=1}^{N} D_h)/N$ have the prior distribution P_N and Θ the prior distribution P. If $E(D_1) < \infty$, then P_N converges weakly to P as $N \to \infty$ or, in other words, Θ_N converges in distribution to Θ as $N \to \infty$ (see Mendel [89]). As an immediate consequence, the joint probability density function of $D_1, \ldots, D_n$ satisfies

$$\begin{aligned} p_N(\delta_1, \cdots, \delta_n) &= \int_0^\infty l_N(\delta_1, \cdots, \delta_n | \theta)\, dP_N(\theta) \to \\ &\to \int_0^\infty l(\delta_1, \cdots, \delta_n | \theta)\, dP(\theta) = \int_0^\infty \prod_{h=1}^{n} l(\delta_h | \theta)\, dP(\theta), \end{aligned} \tag{2.6}$$

as $N \to \infty$, which can be recognised as de Finetti's representation theorem (see de Finetti [22] and Barlow & Mendel [3]). The available information about the average amount of deterioration can be represented by the prior distribution P.

[1] l_p-isotropic random quantities $D_1, \ldots, D_N$ have a uniform distribution on $\sum_{h=1}^{N} D_h^p = N\theta$ (see Mendel [89]). l_p-isotropy implies exchangeability, on the other hand, an exchangeable measure that is l_p-isotropic in two coordinates is l_p-isotropic.

2.4 Deterioration, failure and inspection

A hydraulic structure has to fullfill safety requirements. As soon as a structure's resistance exceeds its failure level, it is unsafe and failed. Based on the assumption of l_1-isotropic deterioration, we shall determine the probability of failure per unit time. Moreover, if a preventive repair is possible when the inspected resistance is in between its preventive repair level and its failure level, a reduction factor can be identified in the mathematical expression for the probability of failure. Recall that, for notational convenience, $X_n = \sum_{h=1}^{n} D_h$ for $n = 1, \ldots, N$.

2.4.1 Distribution of cumulative deterioration

Using Theorem 2 from the appendix (with $x = y$) and the law of total probability, the probability distribution of the cumulative deterioration in time-interval $(0, n\Delta]$, conditional on the average deterioration $(\sum_{h=1}^{N} D_h)/N = X_N/N = \theta$, is the beta distribution in $y/N\theta$ with parameters n and $N-n$:

$$\Pr\{X_n \le y \mid X_N = N\theta\} = 1 - \sum_{h=1}^{n} \binom{N-1}{h-1} \left[1 - \frac{y}{N\theta}\right]_+^{N-h} \left[\frac{y}{N\theta}\right]^{h-1}, \qquad (2.7)$$

for $y \ge 0$ and zero otherwise, $n = 1, \ldots, N-1$. Note that the beta distribution (2.7) coincides with the probability distribution of the nth order statistic of $N-1$ independent and identically distributed random quantities with uniform distribution on $[0, N\theta]$.

The beta distribution (2.7) converges to the gamma distribution with parameters n and θ (for $y \ge 0$):

$$\Pr\{X_n \le y \mid X_N = N\theta\} \to 1 - \sum_{h=1}^{n} \frac{1}{(h-1)!} \left[\frac{y}{\theta}\right]^{h-1} \exp\left\{-\frac{y}{\theta}\right\}, \qquad (2.8)$$

as $N \to \infty$, uniformly, for y/θ in bounded intervals, $n \in \mathbb{N}$. As was to be expected, the gamma distribution (2.8) is the distribution of the sum of n identically distributed exponential random quantities that are conditionally independent when the mean θ is given.

For the conditional expectation of the cumulative amount of deterioration in time-interval $(0, n\Delta]$, given the average deterioration, we have

$$E(X_n | X_N = N\theta) = n\theta \quad \text{for} \quad 0 \le n < N \quad \text{and} \quad N = 2, 3, \ldots$$

Hence, the expected deterioration increases linearly in time. For a bounded horizon, the conditional variance given the average deterioration can be written as

$$\mathrm{Var}(X_n | X_N = N\theta) = \frac{N-n}{N+1} n\theta^2 \quad \text{for} \quad 0 \le n < N \quad \text{and} \quad N = 2, 3, \ldots$$

For an unbounded horizon, the conditional variance $\mathrm{Var}(X_n | X_N = N\theta)$ approaches $n\theta^2$, as $N \to \infty$.

2.4.2 Probability of failure without inspection

Often, the probability of failure in any unit time n should not exceed a given safety probability norm p_{norm}, or using Eq. (2.1)

$$\Pr\{R_{n-1} \geq s, R_n < s\} < p_{norm}, \ \forall n \in \mathbb{N} \tag{2.9}$$

Hence, a preventive repair (or an inspection) should be carried out just before that unit time n in which the probability of failure exceeds the safety probability norm first.

For finite N, using Eqs. (2.1) and (2.9), and Theorem 2 from the appendix (with $x = y$), the conditional probability of failure in unit time n given the average deterioration has a binomial distribution with parameters $(r_0 - s)/N\theta$ and N:

$$\Pr\{X_{n-1} \leq r_0 - s, X_n > r_0 - s \mid X_N = N\theta\} = \\ = \binom{N-1}{n-1} \left[1 - \frac{r_0 - s}{N\theta}\right]_+^{N-n} \left[\frac{r_0 - s}{N\theta}\right]^{n-1}, \tag{2.10}$$

for $s \leq r_0$, $n = 1, \ldots, N-1$.

This discrete lifetime probability function converges to a Poisson distribution with parameter $(r_0 - s)/\theta$ (for $s \leq r_0$):

$$\Pr\{X_{n-1} \leq r_0 - s, X_n > r_0 - s \mid X_N = N\theta\} \to \\ \to \frac{1}{(n-1)!} \left[\frac{r_0 - s}{\theta}\right]^{n-1} \exp\left\{-\frac{r_0 - s}{\theta}\right\}, \tag{2.11}$$

as $N \to \infty$, uniformly, for $(r_0 - s)/\theta$ in bounded intervals, $n \in \mathbb{N}$. The mean life is $1 + [(N-1)/N](r_0 - s)/\theta$ for a bounded horizon and $1 + (r_0 - s)/\theta$ for an unbounded horizon.

2.4.3 Choice of a prior density for the average deterioration

The uncertainty in the average amount of deterioration per unit time can be expressed in the prior density $\pi_N(\theta)$. In the presence of data, we can derive the posterior density of the average amount of deterioration per unit time when $D_i = \delta_i$, $i = 1, \ldots, m$, by applying Bayes' theorem:

$$\pi_N(\theta|\delta_1, \cdots, \delta_m) = \frac{l_N(\delta_1, \cdots, \delta_m|\theta)\pi_N(\theta)}{\int_0^\infty l_N(\delta_1, \cdots, \delta_m|\theta)\pi_N(\theta)\, d\theta}. \tag{2.12}$$

A proper choice for the prior density, however, is not easy. On the one hand, the prior density should be taken as a quantification of what is known about the deterioration process, on the other hand, it would be convenient if the posterior density (2.12) can be evaluated easily. To choose a prior density, we follow the treatments of Raiffa &

Schlaifer [98] and DeGroot [27]. The statistic $\sum_{i=1}^{m} D_i$ is a sufficient statistic of fixed dimension for every sample size m for the exponential likelihood function $l(\delta_1, \cdots, \delta_m|\theta)$. Therefore, there exists a simple so-called *conjugate* family of probability density functions of Θ such that (i) for any sample size m and any observations $\delta_1, \ldots, \delta_m$ the likelihood function $l(\delta_1, \cdots, \delta_m|\theta)$, regarded as a function of θ, is proportional to one of the probability density functions in the family, and (ii) the family is closed under sampling (or closed under multiplication). Note that for the Dirichlet likelihood function the statistic $\sum_{i=1}^{m} D_i$ is only sufficient for $1 \leq m \leq N-1$. Furthermore, the smallest family of probability density functions to which the Dirichlet likelihood function $l_N(\delta_1, \cdots, \delta_m|\theta)$ is proportional, is not closed under sampling. Therefore, it is not possible to find a useful conjugate family of distributions for the Dirichlet likelihood function.

For observations from an exponential distribution with unknown mean Θ, the family of inverted gamma distributions is a conjugate family:

$$\pi(\theta) = \frac{\mu^{\nu}}{\Gamma(\nu)} \theta^{-(\nu+1)} \exp\left\{-\frac{\mu}{\theta}\right\} I_{(0,\infty)}(\theta), \tag{2.13}$$

for $\mu > 0$, $\nu > 2$. The parameters μ, ν are based on subjective opinions of experts, and the prior mean and variance are $E(\Theta) = \mu/(\nu-1)$ and $\text{Var}(\Theta) = E^2(\Theta)/(\nu-2)$, respectively. The posterior density when $D_i = \delta_i$, $i = 1, \ldots, m$, is also an inverted gamma distribution with parameters $\mu + \sum_{h=1}^{m} \delta_h$ and $\nu + m$. Furthermore, the posterior mean is a convex combination of the sample mean and the prior mean. For detailed calculations, see Cooke, Misiewicz & Mendel [16] who also deal with censored data. We get this kind of data in the case of *imperfect inspection*, i.e. the deterioration is observed with uncertainty.

In brief, the family of inverted gamma distributions is: (i) *analytically tractable*, (ii) *rich* (the decision-maker's prior information and beliefs can be expressed conveniently), and (iii) *interpretable* (it represents the uncertainty in the average rate of deterioration).

For an unbounded horizon, the predictive probability of failure in unit time n, Eq. (2.9), can now be obtained explicitly (for $s \leq r_0$):

$$\int_0^{\infty} \Pr\{X_{n-1} \leq r_0 - s, X_n > r_0 - s \mid X_N/N = \theta\} \pi(\theta)\, d\theta \to$$

$$\to \binom{\nu + n - 1 - 1}{n-1} \left[\frac{r_0 - s}{r_0 - s + \mu}\right]^{n-1} \left[\frac{\mu}{r_0 - s + \mu}\right]^{\nu}, \tag{2.14}$$

as $N \to \infty$, $n \in \mathbb{N}$. This probability function is the negative binomial distribution with parameters $(r_0 - s)/(r_0 - s + \mu)$ and ν, and mean life time $1 + (r_0 - s)\nu/\mu$ and variance $(r_0 - s)(r_0 - s + \mu)\nu/\mu^2$.

Of course, beliefs about the average deterioration may not be properly described with the family of inverted gamma distributions. Fortunately, as Diaconis & Ylvisaker [44] have pointed out, any prior density can be arbitrarily well approximated by a mixture of conjugate prior densities.

2.4.4 Probability of failure with inspection

In hydraulic engineering, corrective repairs (after failure) and the collateral losses are prohibitively high and should be avoided by well planned inspections and preventive repairs. If an inspection reveals that the structure's resistance fell below a given preventive repair level, while no failure occurred, a preventive repair must be performed. With the aid of an appropriate inspection schedule, we may reduce the probability of failure. In this section, we derive expressions for the probabilities of failure and preventive repair under inspection.

Let us assume that we deal with *perfect inspection*, i.e. that the actual resistance R can be determined without uncertainty. Furthermore, there is an inspection at time $j\Delta$ that takes negligible time and does not degrade the structure. A preventive repair takes place if the structure's resistance is below a preventive repair level, denoted by ρ and given by the decision-maker, where $s < \rho < r_0$. Hence, a preventive repair will be executed at time $j\Delta$ if $s \leq R_j < \rho$. No action will be taken if $R_j \geq \rho$ (see Fig. 2.1).

Using Theorem 2 (see the appendix), the probability of failure in unit time n and no preventive repair in unit time j, conditional on the average deterioration, can be written as

$$\begin{aligned}
&\Pr\{R_j \geq \rho, R_{n-1} \geq s, R_n < s \mid X_N = N\theta\} = \\
&= \Pr\{X_j \leq r_0 - \rho, X_{n-1} \leq r_0 - s, X_n > r_0 - s \mid X_N = N\theta\} \qquad (2.15)\\
&= \Pr\{U_{j:n-1} \leq r_0 - \rho\} \times \Pr\{X_{n-1} \leq r_0 - s, X_n > r_0 - s \mid X_N = N\theta\}
\end{aligned}$$

for $j < n$, $j, n = 1, \ldots, N-1$, $N = 2, 3, \ldots$, and $0 \leq r_0 - \rho \leq r_0 - s$, where $U_{j:n-1}$ denotes the jth order statistic of $n-1$ independent and identically distributed random quantities with uniform distribution on $[0, r_0 - s]$. The cumulative distribution function of $U_{j:n-1}$, at $r_0 - \rho$, represents the reduction in the probability of failure in unit time n due to inspection in unit time j. This reduction factor has the form

$$\Pr\{U_{j:n-1} \leq r_0 - \rho\} = \sum_{i=j+1}^{n} \binom{n-1}{i-1} \left[1 - \frac{r_0 - \rho}{r_0 - s}\right]^{n-i} \left[\frac{r_0 - \rho}{r_0 - s}\right]^{i-1}. \qquad (2.16)$$

By the law of total probability and Eqs. (2.10) and (2.15), the probability of preventive repair at time $m\Delta$ when the structure has not been preventively repaired due to the previous inspection at time $j\Delta$, conditional on the average deterioration, can be written as

$$\begin{aligned}
&\Pr\{R_j \geq \rho, s \leq R_m < \rho \mid X_N = N\theta\} = \\
&= \sum_{n=j+1}^{m} \Big(\Pr\{R_{n-1} \geq \rho, R_n < \rho \mid X_N = N\theta\} - \qquad (2.17)\\
&\qquad - \Pr\{R_j \geq \rho, R_{n-1} \geq s, R_n < s \mid X_N = N\theta\} \Big),
\end{aligned}$$

where $j < m$, $j, n = 1, \ldots, N-1$, and $0 \leq r_0 - \rho \leq r_0 - s$, for $N = 2, 3, \ldots$

2.4.5 Illustration: the Eastern-Scheldt barrier

A major protection against flooding by storm surges in The Netherlands is the Eastern-Scheldt storm-surge barrier. To preserve the natural salt-water environment, the barrier remains open during normal weather and hydraulic conditions. The barrier will be closed in case of a severe storm-surge. The hydraulic structure has been erected in three closure gaps and it has 62 steel gates each with a span of nearly 42 metres. To provide for the long-term stability, the piers are embedded with several layers of rock and an adjoining sea-bed protection has been constructed on both sides of the barrier. Millions of tons of rock rubble were placed at the sea-bed protection to prevent sand to be washed out. However, the sea-bed protection is subject to rock displacement in extreme conditions. Furthermore, the sea-bed protection can deteriorate due to a variety of factors, like anchoring and extreme tidal current. The deterioration has to be monitored and, if necessary, has to be repaired. If too many stones were removed, scour holes would develop, which might cause progressive scour and might endanger the stability of the piers. In this situation, inspection intervals and cost of repair have to be optimised. For a detailed description of the technical aspects of the Eastern-Scheldt storm-surge barrier, see the summary of Watson & Finkl [140].

The resistance of the upper rock layer, R, is defined to be a function of the number of stones removed. At time zero, the resistance equals zero ($r_0 = 0$). Scale model experiments have shown that the preventive repair level and corresponding failure level for one particular steelgate section are $\rho = -50$ and $s = -70$, respectively (in other words, 50 and 70 removed stones, respectively). It has been determined that the average deterioration over an approximately unbounded horizon ($N \to \infty$) is about 3.5 stones per unit time, with unit time length $\Delta = 5$ years. However, extrapolating from a scale model to the real rock layer is rather difficult. Although one may have an indication how the average deterioration will develop, still many uncertainties are involved. Based on physical models, properly qualified with expert opinion, the prior mean and variance can be taken as $E(\Theta) = 3.5$ and $\mathrm{Var}(\Theta) = 1.2$ (see Kok [74]).

The (discrete) lifetime probability function is now given by the negative binomial distribution (2.14) with $\mu \approx 39.2$ and $\nu \approx 12.2$, and mean life 22.8 units of time (see Fig. 2.2). Suppose we introduce an inspection at time 20Δ, with a possible preventive repair, then the probability of failure just after this inspection drops down considerably (see Eqs. (2.14) and (2.15) that are displayed in Fig. 2.2). By increasing the rate of inspection, the probability of failure can be decreased further. Suppose we have an inspection at the end of each unit time, i.e. at $n\Delta$, $n \in \mathbb{N}$, then the probabilities of failure and the probabilities of preventive repair are given by Eqs. (2.15) and (2.17), respectively (see Fig. 2.3).

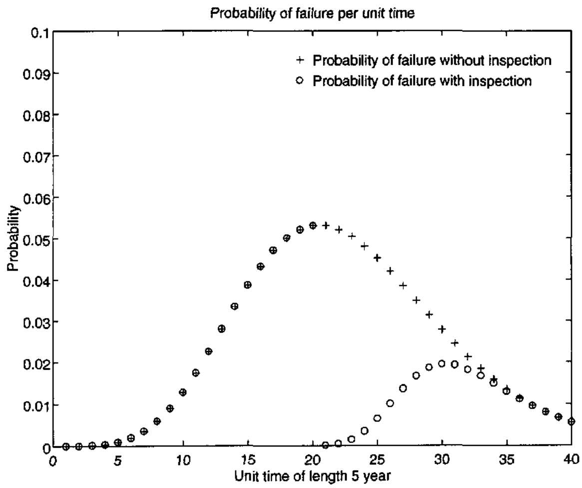

Figure 2.2: The (discrete) lifetime probability function in case of no inspection (+) and inspection at time 20Δ (o) for the rock-fill top-layer of the sea-bed protection of the Eastern-Scheldt barrier.

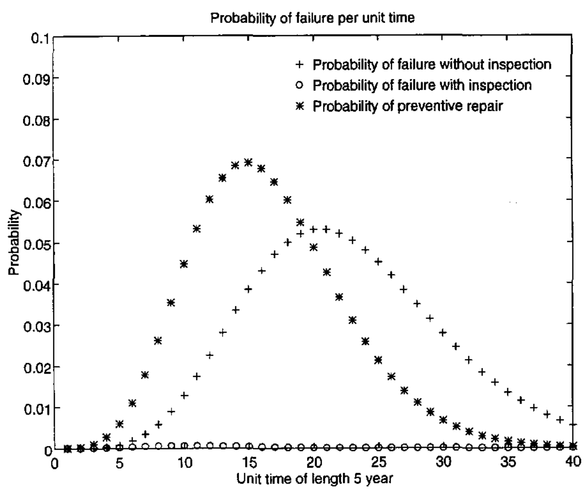

Figure 2.3: The probability of failure per unit time in case of no inspections (+) and inspections at the end of every unit time (at times $n\Delta$, $n \in \mathbb{N}$) (o), and the probability of preventive repair per unit time (*) for the rock-fill top-layer of the sea-bed protection of the Eastern-Scheldt barrier.

2.5 Conclusions

For the purpose of maintenance optimisation, isotropic properties of the deterioration process were used to build a Bayesian failure model. The model is based on the only (subjective) information that is commonly available: the average amount of deterioration per unit time with regard to a bounded or an unbounded time-horizon. If this information is all the decision-maker has, it is not relevant to analyse how the cumulative sum is realised, and the decision-maker is indifferent in what way this total sum is gathered. Based on this knowledge, it follows that the amounts of deterioration per unit time are l_1-isotropic. A consequence of the l_1-isotropy is that the expected accumulated deterioration is linear in time. Prior beliefs about the average deterioration per unit time are encoded in a prior density and can be updated using (censored or uncensored) inspection data.

Conditional on the average deterioration, the probabilities of preventive repair and failure can be expressed in closed-form results. The failure modelling approach is illustrated with the lifetime probability function of the rock-fill top-layer of the sea-bed protection of the Eastern-Scheldt storm-surge barrier.

2.6 Appendix: Proofs of theorems

Theorem 1 *Suppose* $\delta_i \in \mathbb{R}_+$, $i = 1, \ldots, n$, *and* $\sum_{h=1}^{j} \delta_h \leq x$, $\sum_{h=1}^{n} \delta_h \leq y$ *then*

$$\begin{aligned} J(j,n,x,y) &= \int_{\delta_1=0}^{x} \cdots \int_{\delta_j=0}^{x-\sum_{h=1}^{j-1}\delta_h} \int_{\delta_{j+1}=0}^{y-\sum_{h=1}^{j}\delta_h} \cdots \int_{\delta_n=0}^{y-\sum_{h=1}^{n-1}\delta_h} 1 d\delta_n \cdots d\delta_1 \\ &= \frac{1}{n!}\left[y^n - \sum_{i=1}^{j} \binom{n}{i-1} (y-x)^{n-i+1} x^{i-1}\right], \end{aligned} \quad (2.18)$$

for $j < n$, $j, n \in \mathbb{N}$, *and* $x \leq y$, $x, y \in \mathbb{R}_+$.

Proof:
The integral can be determined by successively integrating out the variables δ_n, δ_{n-1}, ..., δ_1.

$$\begin{aligned} &J(j,n,x,y) = \\ &= \int_{\delta_1=0}^{x} \cdots \int_{\delta_j=0}^{x-\sum_{h=1}^{j-1}\delta_h} \int_{\delta_{j+1}=0}^{y-\sum_{h=1}^{j}\delta_h} \cdots \int_{\delta_n=0}^{y-\sum_{h=1}^{n-1}\delta_h} 1 d\delta_n \cdots d\delta_1 \\ &= \int_{\delta_1=0}^{x} \cdots \int_{\delta_j=0}^{x-\sum_{h=1}^{j-1}\delta_h} \int_{\delta_{j+1}=0}^{y-\sum_{h=1}^{j}\delta_h} \cdots \int_{\delta_{n-2}=0}^{y-\sum_{h=1}^{n-3}\delta_h} \frac{1}{2!}\left[y - \sum_{h=1}^{n-2}\delta_h\right]^2 d\delta_{n-2} \cdots d\delta_1 \\ &= \cdots = \int_{\delta_1=0}^{x} \cdots \int_{\delta_j=0}^{x-\sum_{h=1}^{j-1}\delta_h} \frac{1}{(n-j)!}\left[y - \sum_{h=1}^{j}\delta_h\right]^{n-j} d\delta_j \cdots d\delta_1 \\ &= -\int_{\delta_1=0}^{x} \cdots \int_{\delta_{j-1}=0}^{x-\sum_{h=1}^{j-2}\delta_h} \frac{1}{(n-j+1)!}\left[y-x\right]^{n-j+1} d\delta_{j-1} \cdots d\delta_1 + \\ &\quad + \int_{\delta_1=0}^{x} \cdots \int_{\delta_{j-1}=0}^{x-\sum_{h=1}^{j-2}\delta_h} \frac{1}{(n-j+1)!}\left[y - \sum_{h=1}^{j-1}\delta_h\right]^{n-j+1} d\delta_{j-1} \cdots d\delta_1. \end{aligned}$$

By using the classical multi-dimensional Dirichlet-integral, the first integral can be solved and becomes

$$-\frac{(y-x)^{n-j+1} x^{j-1}}{(n-j+1)!(j-1)!}. \quad (2.19)$$

By successively solving the second integral, we get

$$-\frac{(y-x)^{n-j+2} x^{j-2}}{(n-j+2)!(j-2)!} \cdots - \frac{(y-x)^{n-1} x^1}{(n-1)!1!} + \int_{\delta_1=0}^{x} \frac{1}{(n-1)!}\left[y-\delta_1\right]^{n-1} d\delta_1, \quad (2.20)$$

where we used the Dirichlet-integral $(j-2)$ times. The last integral leads to

$$\int_{\delta_1=0}^{x} \frac{1}{(n-1)!}\left[y-\delta_1\right]^{n-1} d\delta_1 = -\frac{(y-x)^n}{n!} + \frac{y^n}{n!}. \quad (2.21)$$

Summing (2.19), (2.20) and (2.21) completes the proof. □

Theorem 2 *Suppose the non-negative random quantities $D_1, \ldots, D_N$ are l_1-isotropic then*

$$\Pr\left\{ \sum_{h=1}^{j} D_h \leq x, \sum_{h=1}^{n-1} D_h \leq y, \sum_{h=1}^{n} D_h > y \,\middle|\, \frac{1}{N}\sum_{h=1}^{N} D_h = \theta \right\} = \tag{2.22}$$

$$= \sum_{i=j+1}^{n} \binom{n-1}{i-1} \left[1-\frac{x}{y}\right]^{n-i} \left[\frac{x}{y}\right]^{i-1} \binom{N-1}{n-1} \left[1-\frac{y}{N\theta}\right]_+^{N-n} \left[\frac{y}{N\theta}\right]^{n-1},$$

for $j < n$, $j, n = 1, \ldots, N-1$, and $x \leq y$, $x, y \in \mathbb{R}_+$.

Proof:
In evaluating Eq. (2.22), we use the likelihood $l_N(\delta_1, \cdots, \delta_n|\theta)$ derived in (2.4). The integral can be determined by first integrating out the variable δ_n. In the last step but one, we use $J(j, n-1, x, y)$, following Theorem 1. For $\delta_i \in \mathbb{R}_+$, $i = 1, \ldots, n$:

$$\Pr\left\{ \sum_{h=1}^{j} D_h \leq x, \sum_{h=1}^{n-1} D_h \leq y, \sum_{h=1}^{n} D_h > y \,\middle|\, \frac{1}{N}\sum_{h=1}^{N} D_h = \theta \right\} =$$

$$= \int_{\delta_1=0}^{x} \cdot\cdot \int_{\delta_j=0}^{x-\sum_{h=1}^{j-1}\delta_h} \int_{\delta_{j+1}=0}^{y-\sum_{h=1}^{j}\delta_h} \cdot\cdot \int_{\delta_n=y-\sum_{h=1}^{n-1}\delta_h}^{N\theta-\sum_{h=1}^{n-1}\delta_h} \frac{\Gamma(N)}{\Gamma(N-n)} \left[1-\frac{\sum_{h=1}^{n}\delta_h}{N\theta}\right]_+^{N-n-1} \times$$
$$\times \left[\frac{1}{N\theta}\right]^{n} d\delta_n \cdots d\delta_1$$

$$= \int_{\delta_1=0}^{x} \cdot\cdot \int_{\delta_j=0}^{x-\sum_{h=1}^{j-1}\delta_h} \int_{\delta_{j+1}=0}^{y-\sum_{h=1}^{j}\delta_h} \cdot\cdot \int_{\delta_{n-1}=0}^{y-\sum_{h=1}^{n-2}\delta_h} \frac{\Gamma(N)}{\Gamma(N-n+1)} \left[1-\frac{y}{N\theta}\right]_+^{N-n} \times$$
$$\times \left[\frac{1}{N\theta}\right]^{n-1} d\delta_{n-1} \cdots d\delta_1$$

$$= \frac{\Gamma(N)}{\Gamma(N-n+1)} \left[1-\frac{y}{N\theta}\right]_+^{N-n} \left[\frac{1}{N\theta}\right]^{n-1} \times$$
$$\times \frac{1}{(n-1)!} \left\{ y^{n-1} - \sum_{i=1}^{j} \binom{n-1}{i-1} (y-x)^{n-i} x^{i-1} \right\}$$

$$= \binom{N-1}{n-1} \left[1-\frac{y}{N\theta}\right]_+^{N-n} \left[\frac{y}{N\theta}\right]^{n-1} \left\{ 1 - \sum_{i=1}^{j} \binom{n-1}{i-1} \left[1-\frac{x}{y}\right]^{n-i} \left[\frac{x}{y}\right]^{i-1} \right\},$$

for $j < n$, $j, n = 1, \ldots, N-1$, and $x \leq y$, $x, y \in \mathbb{R}_+$. □

Chapter 3

A Characterisation of Generalised Gamma Processes in Terms of Sufficiency and Isotropy

Jan M. van Noortwijk, Roger M. Cooke, and Jolanta K. Misiewicz

Abstract. To optimise maintenance of deteriorating structures, we need to model the event of failure and the process of deterioration. Due to the common lack of data, there is often only (subjective) information on the limiting average rate of deterioration. Furthermore, most deterioration processes proceed in one direction and in random jumps. In order that stochastic processes with non-negative exchangeable increments be based on the unknown limiting average rate of deterioration, they can best be regarded as generalised gamma processes. In this chapter, two new characterisations of generalised gamma processes are given: (i) in terms of conditioning on sums of increments being sufficient statistics and (ii) in terms of isotropy. (This chapter has previously been published as [127].)

Keywords. gamma processes, deterioration processes, sufficient statistics, isotropy, Brownian motion.

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3.1 Introduction

In designing hydraulic structures such as dykes, bridges, dams, and other water barriers, a good design ensures that a structure's resistance exceeds the applied stress, at least with a high probability during its service life. Due to uncertain deterioration, the resistance decreases in time creating the need for regular, possibly expensive, maintenance actions. In order to optimise maintenance, we need to model the event of failure and the process of deterioration. Most maintenance models are based on lifetime distributions or Markovian deterioration models. Unfortunately, it is often hard to gather data for estimating either the parameters of a lifetime distribution or the transition probabilities of a Markov chain. Moreover, in case of well-planned preventive maintenance, complete lifetimes will be observed rarely.

In practice, there is often only information in terms of probability distributions on uncertain limiting average rates of deterioration. To give five examples: (i) the limiting average rates of rock displacement for optimising maintenance of the rock dumping of the Eastern-Scheldt barrier (Chapters 2 and 7); (ii) the limiting average rates of current-induced scour erosion for optimising maintenance of the block mats of the Eastern-Scheldt barrier (Chapter 7); (iii) the limiting average rates of ongoing coastal erosion for optimising sand nourishment (Chapter 4); (iv) the limiting average rates of rock displacement for optimising maintenance of berm breakwaters (Chapter 6); and (v) the limiting average rates of settlement for optimising maintenance of dykes (Chapter 5).

Furthermore, most deterioration processes proceed in one direction and in random jumps. Common practice nowadays is modelling deterioration as a Brownian motion with drift (see e.g. Karlin & Taylor [71, Ch. 7], Hontelez, Burger & Wijnmalen [62], and Pettit [95]). Unfortunately, this process implies the existence of periods in which a structure's resistance actually improves, which does not fit unless the structure undergoes maintenance. Since the Brownian motion has continuous sample paths, it also does not properly model the jumps that occur when the structure is subject to random shocks.

Instead, an adequate deterioration model should have non-negative increments and, due to the lack of data, should have increments that are judged to be exchangeable for every uniform time-partition. In order that stochastic processes with non-negative exchangeable increments be based on the unknown limiting average rate of deterioration, they can best be regarded as generalised gamma processes.

In this chapter, two new characterisations of generalised gamma processes are given: (i) in terms of conditioning on sums of increments being sufficient statistics (Sec. 3.3) and (ii) in terms of isotropy (Sec. 3.4). The classical characterisation of gamma processes in terms of Poisson processes is briefly reviewed in Sec. 3.2. The characterisation in terms of sufficiency extends results of Diaconis & Freedman [40, 41] and Küchler & Lauritzen [78]. The characterisation in terms of isotropy originates from the work of

Barlow & Mendel [3] and Misiewicz & Cooke [90]. The proofs of these characterisations can be found in an appendix. The aim of this chapter is to present the mathematical framework for modelling deterioration via generalised gamma processes - other chapters report on its hydraulic engineering applications (see Chapters 2, 4, 5, 6, and 7).

3.2 Generalised gamma processes

Before we characterise scale mixtures of gamma processes, called generalised gamma processes, in terms of sufficiency and isotropy, we briefly review the classical characterisation of gamma processes in terms of compound Poisson processes (for an explanation of notation see Appendix A).

Definition 1 (Gamma process.) *The gamma process with shape parameter $a > 0$ and scale parameter $b > 0$ is a continuous-time stochastic process $\{Y(t) : t \geq 0\}$ with the following properties:*

1. $Y(0) = 0$ *with probability one;*
2. $Y(\tau) - Y(t)$ *has a gamma distribution* $\mathrm{Ga}(a(\tau - t), b)$ *for all* $\tau > t \geq 0$*;*
3. $Y(t)$ *has independent increments.*

Since the finite-dimensional joint probability density function of the increments is consistent and uniquely defined, Kolmogorov's Extension Theorem assures us that the gamma process exists. By the infinitely divisibility of the the gamma distribution, the gamma process is a Lévy process. Every Lévy process may be written as a sum of a Brownian motion, a deterministic part (linear in time), and an integral of compound Poisson processes, where all the contributing processes are mutually independent. The sample paths of a Lévy process are discontinuous with probability one if the process is monotone, because such a process can be decomposed into a linear part plus an integral of compound Poisson processes. The sample paths of a Brownian motion are continuous with probability one. For details, see Gnedenko & Kolmogorov [55, Ch. 3 & 5], Lévy [81, pp. 173-180], Itô [65, Ch. 1], Ferguson & Klass [49], and de Finetti [23, Ch. 8]. In particular, the characteristic function of the gamma distribution $\mathrm{Ga}(a, b)$, which is given by

$$\phi(u) = [b/(b - iu)]^a = \exp\left\{\int_0^\infty \left(e^{iux} - 1\right)\, dM(x)\right\}$$

where $M(x) = -a\int_x^\infty (e^{-by}/y)\,dy$ for $x > 0$, shows us that the gamma process is an integral of compound Poisson processes with jump intensity $M(x)$ (see Gnedenko & Kolmogorov [55, pp. 86-87]). Thus, the gamma process is a pure jump process.

Moran [91] used gamma processes in his theory of the storage of water by dams. Reliability models based on the gamma process have been developed, amongst others, by Dykstra & Laud [45] and Wenocur [143]. Since the Dirichlet distribution can be defined as the probability distribution of a random vector with independent gamma distributed coordinates (with equal scale parameters) divided by their total sum, Ferguson [48] defined the Dirichlet process in terms of the gamma process in a similar way. He used the Dirichlet process for solving Bayesian nonparametric estimation problems.

In structural reliability, it is useful to obtain the cumulative distribution function of the time to failure T, i.e. the time at which a structure's resistance $r_0 - Y(t)$ crosses a fixed stress or failure level s (with r_0 the resistance at time zero):

$$\Pr\{T \leq t\} = \Pr\{Y(t) \geq r_0 - s\} = \int_{r_0 - s}^{\infty} \mathrm{Ga}(x|\, at, b)\, dx = \frac{\Gamma(at, b[r_0 - s])}{\Gamma(at)},$$

where $\Gamma(a,x) = \int_{t=x}^{\infty} t^{a-1} e^{-t}\, dt$ is the incomplete gamma function for $x \geq 0$ and $a > 0$.

3.3 Characterisation in terms of sufficiency

The purpose of this section is characterising generalised gamma processes in terms of the only (subjective) information that is commonly available: the limiting average rate of deterioration. Let us denote the deterioration process by $\{X(t) : t \geq 0\}$, where $X(t)$ represents the cumulative deterioration at time t and $X(0) = 0$ with probability one. For every uniform time-partition in time-intervals of length $\tau > 0$, we assume $D_i(\tau) = X(i\tau) - X([i-1]\tau) \geq 0$ for $i \in \mathbb{N}$. We derive the generalised gamma process via assumptions on exchangeability and sufficiency by using results of Diaconis & Freedman [41] and Küchler & Lauritzen [78].

The exchangeability assumption means that the order in which the infinite sequence of increments $\{D_i(\tau) : i \in \mathbb{N}\}$ occur is judged to be irrelevant. In mathematical terms, this can be interpreted as that the probability density function of the random vector $(D_1(\tau), \ldots, D_n(\tau))$ is invariant under all $n!$ permutations of the coordinates, i.e.

$$p_{D_1(\tau),\ldots,D_n(\tau)}\left(\delta_1, \ldots, \delta_n\right) = p_{D_1(\tau),\ldots,D_n(\tau)}\left(\delta_{\pi(1)}, \ldots, \delta_{\pi(n)}\right), \tag{3.1}$$

where π is any permutation of $1, \ldots, n$ for all $n \in \mathbb{N}$ and $\tau > 0$. The notion of exchangeability is weaker than the notion of independence; it can best be utilised in situations with a lack of data.

The sufficiency assumption means that, for every $\tau > 0$, and all $n \geq 2$ and $k < n$, the conditional probability density function of k increments of deterioration, when the sum of n increments is given, can be expressed as

$$p_{D_1(\tau),\ldots,D_k(\tau)|X(n\tau)}(\delta_1, \ldots, \delta_k|\, x) = \frac{\left[\prod_{i=1}^{k} h(\delta_i, \tau)\right] h^{(n-k)}\left(x - \sum_{i=1}^{k} \delta_i, \tau\right)}{h^{(n)}(x, \tau)}, \tag{3.2}$$

where $h(x,\tau)$ is differentiable and non-negative, $h^{(n)}(x,\tau)$ is the n-fold convolution in x of $h(x,\tau)$ with itself, and $c(\theta)$ is defined by

$$\int_0^{\infty} h(x,\tau) c(\theta) \exp\{-\,x/\theta\}\, dx = \int_0^{\infty} l(x|\theta)\, dx = 1 \tag{3.3}$$

for $\theta \in (0, \infty)$. In addition, the probability model should be independent of the scale of measurement (e.g. being either inches or centimetres). In other words, the likelihood function $l(x|\theta)$ should be a scale density:

$$l(x|\theta) = f(x/\theta)/\theta \quad \text{for } x, \theta \in (0, \infty). \tag{3.4}$$

If Eqs. (3.2-3.4) are satisfied for all $\tau > 0$, it follows from Theorem 3 (see the appendix) that

$$h(x, \tau) = x^{a\tau - 1}/\Gamma(a\tau) \tag{3.5}$$

for some constant $a > 0$. As a consequence, the joint probability density function of the increments $D_1(\tau), \ldots, D_n(\tau)$ can be written as a mixture of conditionally independent gamma densities:

$$p_{D_1(\tau),\ldots,D_n(\tau)}(\delta_1, \ldots, \delta_n) = \int_0^\infty \prod_{i=1}^n \frac{\delta_i^{a\tau-1}}{\Gamma(a\tau)} \left[\frac{a\tau}{\theta}\right]^{a\tau} \exp\left\{-\frac{a\tau\delta_i}{\theta}\right\} dP_{\Theta(\tau)}(\theta) \tag{3.6}$$

for some constant $a > 0$ with

$$\begin{aligned} E(X(n\tau)) &= E(n\Theta(\tau)), \\ \text{Var}(X(n\tau)) &= \left[1 + \frac{1}{na\tau}\right] E([n\Theta(\tau)]^2) - [E(n\Theta(\tau))]^2 \end{aligned} \tag{3.7}$$

for all $\tau > 0$, provided the first and the second moment of the probability distribution of $\Theta(\tau)$ exist. A stochastic process for which the increments are distributed according to Eq. (3.6) is called a *generalised gamma process*. By substituting Eq. (3.5) in Eq. (3.2), conditioning on sums of increments leads to a transformed Dirichlet distribution:

$$p_{D_1(\tau),\ldots,D_k(\tau)|X(n\tau)}(\delta_1, \ldots, \delta_k | \, x) = \text{Di}_k\left(\frac{\delta_1}{x}, \ldots, \frac{\delta_k}{x} \,\middle|\, a\tau, \ldots, a\tau, (n-k)a\tau\right) \left[\frac{1}{x}\right]^k.$$

The generalised gamma process has three useful properties.

First, the probability distribution $P_{\Theta(\tau)}$ on the random quantity $\Theta(\tau)$, with possible values $\theta \in (0, \infty)$, represents the uncertainty in the unknown limiting average amount of deterioration per time-interval of length τ: $\lim_{n\to\infty}[(\sum_{i=1}^n D_i(\tau))/n]$. By the strong law of large numbers for exchangeable random quantities, the average converges with probability one if $E(D_1(\tau)) < \infty$ (see Chow & Teicher [13, p. 227]). In applications of decision theory, the probability distribution of the limiting average rate of deterioration can be the prior distribution, which can be updated in the light of actual data by Bayes' theorem.

Second, the summarisation of the n random quantities $D_1(\tau), \ldots, D_n(\tau)$ in terms of the statistic $[n, \sum_{i=1}^n D_i(\tau)]$ is sufficient for the unknown limiting average rate of deterioration $\Theta(\tau)$. In fact, the characterisation in terms of conditioning on sums of random quantities is motivated by sufficiency ideas, since, by sufficiency, the resulting conditional probability density function does not depend on θ:

$$p_{D_1(\tau),\ldots,D_k(\tau)|X(n\tau),\Theta(\tau)}(\delta_1,\ldots,\delta_k|\,x,\theta) = p_{D_1(\tau),\ldots,D_k(\tau)|X(n\tau)}(\delta_1,\ldots,\delta_k|\,x)$$

for $k < n$. Moreover, since a sum of increments is a single sufficient statistic for the scale parameter, classical results establish, under various regularity conditions, that the scale density in Eq. (3.4) belongs to the exponential family (see e.g. Koopman [76] and Huzurbazar [63]).

Third, the mixture of gamma's in Eq. (3.6) transforms into a mixture of exponentials if $\tau = a^{-1}$. The infinite sequence of random quantities $\{D_i(a^{-1}) : i \in \mathbb{N}\}$ is said to be l_1-isotropic (or l_1-norm symmetric), since its distribution can be written as a function of the l_1-norm.

The unit time for which the increments of deterioration are l_1-isotropic can be obtained using the conditional probability density function of the first increment, when the sum of the first and the second increment is given, being a transformed beta distribution with both parameters equal to $a\tau$, i.e.

$$p_{D_1(\tau)|X(2\tau)}(\delta_1|\,x) = \frac{\Gamma(2a\tau)}{[\Gamma(a\tau)]^2}\,\frac{\delta_1^{a\tau-1}[x-\delta_1]^{a\tau-1}}{x^{2a\tau-1}}\,I_{[0,x]}(\delta_1) = \text{Be}\left(\frac{\delta_1}{x}\middle|\,a\tau,a\tau\right)\frac{1}{x} \tag{3.8}$$

for some constant $a > 0$ with

$$\begin{aligned} E(D_1(\tau)|X(2\tau)=x) &= x/2, \\ \text{Var}(D_1(\tau)|X(2\tau)=x) &= [x/2]^2/(2a\tau+1). \end{aligned}$$

Hence, for fixed $\tau > 0$, the smaller the unit-time length for which the increments are l_1-isotropic, i.e. the smaller $\Delta = a^{-1}$, the more deterministic the deterioration process. An alternative way to obtain the unit time for which l_1-isotropy holds is by assessing $\text{Var}(X(n\tau))$ in Eq. (3.7). This variance approaches $\text{Var}(n\Theta(\tau))$, from above, as $\Delta = a^{-1}$ tends to 0, from above.

For the unit-time length $\Delta = a^{-1}$, many probabilistic properties of the stochastic deterioration process, like the probability of exceedence of a failure level, can be expressed in explicit form conditional on the limiting average deterioration (see e.g. Chapter 2). Note that specifying the l_1-isotropic grid of the generalised gamma process is similar to specifying the precision of the Brownian motion with drift.

In conclusion, we advocate regarding stochastic deterioration processes as generalised gamma processes with probability distribution on the limiting average rate of deterioration.

3.4 Characterisation in terms of isotropy

Even though it is quite reasonable to derive stochastic deterioration processes based on sums of increments that are sufficient statistics for the unknown limiting average rate, a weaker characterisation might be of interest. Weaker conditions can be established by allowing sums of increments to the power, rather than only sums of increments, to serve as sufficient statistics for the scale parameter. To achieve this, we assume the infinite sequence of increments to be isotropic for every uniform time-partition.

The random vector $\mathbf{D}_n = (D_1, \ldots, D_n)$ is said to be *l_β-isotropic* (or l_β-norm symmetric) if its distribution can be written as a function of the statistic $\sum_{i=1}^n D_i^\beta$ where $\beta > 0$; i.e. its distribution is uniform on the l_β-spheres in $\mathbb{R}_+^n$, where $\mathbb{R}_+ = [0, \infty)$. The infinite sequence of random quantities $\{D_i : i \in \mathbb{N}\}$ is l_β-isotropic if $\mathbf{D}_n$ is l_β-isotropic for each $n \in \mathbb{N}$. Mendel [89] and Misiewicz & Cooke [90], amongst others, proved that if the infinite sequence of random quantities $\{D_i : i \in \mathbb{N}\}$ is l_β-isotropic then there exists a probability distribution P_Θ of Θ such that the probability density function of $(D_1, \ldots, D_n)$ is

$$p_{D_1,\ldots,D_n}(\delta_1, \ldots, \delta_n) = \int_0^\infty \prod_{i=1}^n \frac{\beta}{\Gamma(\frac{1}{\beta})} \left[\frac{1}{\theta}\right]^{\frac{1}{\beta}} \exp\left\{-\frac{\delta_i^\beta}{\theta}\right\} dP_\Theta(\theta) = f_n\left(\textstyle\sum_{i=1}^n \delta_i^\beta\right) \qquad (3.9)$$

for $(\delta_1, \ldots, \delta_n) \in \mathbb{R}_+^n$ and zero otherwise. If $\beta = 1$, we have a mixture of n conditionally independent exponentials. If $\beta = 2$, we have a mixture of n conditionally independent normals truncated at zero. Note that isotropy preserves exchangeability and that the statistic $[n, \sum_{i=1}^n D_i^\beta]$ is sufficient for Θ.

The characterisation of generalised gamma processes in terms of isotropy is the following. For every uniform time-partition in time-intervals of length $\tau > 0$, let there be positive continuous functions $\alpha(\tau)$ and $\beta(\tau)$ such that the infinite sequence of powers of increments, $\{D_i(\tau)^{\alpha(\tau)} : i \in \mathbb{N}\}$, is $l_{\beta(\tau)}$-isotropic; that is, such that

$$p_{D_1(\tau),\ldots,D_n(\tau)}(\delta_1, \ldots, \delta_n) = f_n\left(\textstyle\sum_{i=1}^n \delta_i^{\alpha(\tau)\beta(\tau)}\right) \prod_{i=1}^n \alpha(\tau)\delta_i^{\alpha(\tau)-1}$$

for all $n \in \mathbb{N}$. If, in addition, the mixing distribution (in Eq. (3.9)) has finite moments then Theorem 4 from the appendix entails: $\alpha(\tau) = a\tau$ and $\alpha(\tau)\beta(\tau) = 1$ for some constant $a > 0$; Eq. (3.6) follows accordingly. The theorem has been proved by achieving consistency in the sense that probability distributions of increments and those of sums of increments belong to the same family of distribution.

3.5 Conclusions

As Barlow & Mendel [3] have argued that appropriate lifetime distributions conditional on the limiting average lifetime are the generalised gamma distributions, we have argued that appropriate deterioration processes conditional on the limiting average rate of deterioration are the generalised gamma processes.

In addition to the classical characterisation of gamma processes in terms of compound Poisson processes, we have presented two new characterisations of generalised gamma processes: (i) in terms of conditioning on sums of increments, serving as sufficient statistics for the unknown limiting average rate, and (ii) in terms of isotropy. A useful property of generalised gamma processes is that we can always find a uniform time-partition for which the joint probability density function of the increments can be written as a mixture of exponentials.

In The Netherlands, generalised gamma processes have been used to model decision problems for optimising maintenance of the sea-bed protection of the Eastern-Scheldt barrier, beaches, berm breakwaters, and dykes (see Chapters 2 and 7, 4, 6, and 5, respectively).

3.6 Appendix: Proofs of theorems

Theorem 3 *Let $\{X(t) : t \geq 0\}$ be a non-decreasing continuous-time stochastic process with $X(0) = 0$, with probability one, such that for every $\tau > 0$ the infinite sequence of non-negative real-valued increments $D_i(\tau) = X(i\tau) - X([i-1]\tau)$, $i \in \mathbb{N}$, is exchangeable. Moreover, for every $\tau > 0$, and all $n \geq 2$ and $k < n$, the joint conditional probability density function of the increments $D_1(\tau), \ldots, D_k(\tau)$, when $X(n\tau) = x$ is given, can be represented by Eqs. (3.2-3.4). Then there exists a constant $a > 0$ such that the joint probability density function of the increments $D_1(\tau), \ldots, D_n(\tau)$ is given by Eq. (3.6).*

Proof:
By Diaconis & Freedman [41] and Küchler & Lauritzen [78], there exists a probability distribution $P_{\Theta(\tau)}$ such that

$$p_{D_1(\tau),\ldots,D_n(\tau)}(\delta_1, \ldots, \delta_n) = \int_0^\infty \prod_{i=1}^n h(\delta_i, \tau) c(\theta) \exp\{-\delta_i/\theta\}\, dP_{\Theta(\tau)}(\theta)$$

for every $\tau > 0$ and all $n \in \mathbb{N}$. Since the likelihood function $l(x|\theta)$ is a scale density, the function $h(x, \tau)$ satisfies the functional equation

$$l(x|\theta) = h(x, \tau) c(\theta) \exp\{-x/\theta\} = f(x/\theta)/\theta$$

or, with $g(x/\theta) = \exp\{x/\theta\} f(x/\theta)$, $\phi_1(x) = h(x, \tau)$, and $\phi_2(\theta) = \theta c(\theta)$:

$$g(x/\theta) = \phi_1(x)\phi_2(\theta).$$

This functional equation can be recognised as an extension of one of the four well-known Cauchy equations in which $g(x) = \phi_1(x) = \phi_2(1/x)$ (see Huzurbazar [63, p. 204]). Its general solution is $g(x) = c_1 x^{c_2}$, where c_1 and c_2 are arbitrary constants. Hence, using Eq. (3.3), the functions $h(x, \tau)$ and $c(\theta)$ have the form

$$h(x, \tau) = x^{\alpha(\tau)}/\Gamma(\alpha(\tau) + 1), \qquad c(\theta) = \theta^{-\alpha(\tau)-1},$$

respectively, where $\alpha(\tau) > -1$ is a differentiable function. In turn, $\alpha(\tau)$ satisfies another Cauchy functional equation: $\alpha(n\tau) = n\alpha(\tau)$ for all $\tau > 0$ and $n \in \mathbb{N}$. This functional equation is generated by Eq. (3.2), i.e.

$$p_{D_1(n\tau)|X(2n\tau)}(\delta|\, x) = p_{D_1(\tau)+\ldots+D_n(\tau)|X(2n\tau)}(\delta|\, x), \tag{3.10}$$

when dividing both sides of Eq. (3.10) by $\delta^{n\alpha(\tau)}$ and letting δ approach zero from the right. The general solution is $\alpha(\tau) = a\tau + b$ for some constants $a > 0$ and $b \geq -1$. Since $X(0) = 0$, with probability one, we have $b = -1$.
Eq. (3.6) follows by replacing θ with $\theta/(a\tau)$, which proves the theorem. □

Theorem 4 *Let $\{X(t) : t \geq 0\}$ be a non-decreasing continuous-time stochastic process with $X(0) = 0$, with probability one, such that for every $\tau > 0$ there are positive continuous functions $\alpha(\tau)$ and $\beta(\tau)$ for which the infinite sequence of random quantities $\{D_i(\tau)^{\alpha(\tau)} : i \in \mathbb{N}\}$ is $l_{\beta(\tau)}$-isotropic with respect to a mixing distribution with finite moments, where $D_i(\tau) = X(i\tau) - X([i-1]\tau)$, $i \in \mathbb{N}$. Then there exists a constant $a > 0$ such that $\alpha(\tau) = a \cdot \tau$ and $\beta(\tau) = (a \cdot \tau)^{-1}$, and the joint probability density function of the increments $D_1(\tau), \ldots, D_n(\tau)$ is given by Eq. (3.6).*

Proof:
Fix $\tau > 0$. On the one hand, there are functions $\lambda = \alpha(2\tau)$ and $\mu = \beta(2\tau)$ such that the infinite sequence of random quantities $\{X_i^\lambda : i \in \mathbb{N}\}$ is l_μ-isotropic, where $X_i = X(2i\tau) - X(2[i-1]\tau)$, $i \in \mathbb{N}$. By applying Eq. (3.9) to the probability density function of the random vector $(X_1^\lambda, \ldots, X_n^\lambda)$ and transforming back to $(X_1, \ldots, X_n)$, there exists a probability distribution $\tilde{P}(\varphi)$ such that the joint probability density function of $X_1, \ldots, X_n$ can be written as

$$\tilde{p}(x_1, \ldots, x_n) = \int_{\varphi=0}^{\infty} \prod_{i=1}^{n} \frac{\mu}{\Gamma(\frac{1}{\mu})} \left[\frac{1}{\varphi}\right]^{\frac{1}{\mu}} \lambda x_i^{\lambda-1} \exp\left\{-\frac{x_i^{\lambda\mu}}{\varphi}\right\} d\tilde{P}(\varphi). \tag{3.11}$$

On the other hand, there are functions $\alpha = \alpha(\tau)$ and $\beta = \beta(\tau)$ such that the infinite sequence of random quantities $\{D_i^\alpha : i \in \mathbb{N}\}$ is l_β-isotropic, where $D_i = X(i\tau) - X([i-1]\tau)$, $i \in \mathbb{N}$. Then, there exists a probability distribution $P(\theta)$ for which the probability density function of the random vector $(D_1, \ldots, D_{2n})$ can be obtained from Eq. (3.11) by replacing $(\lambda, \mu, n, 2\tau)$ with $(\alpha, \beta, 2n, \tau)$. In turn, we get the joint probability density function of the subsums $X_i = D_{2i-1} + D_{2i}$, $i = 1, \ldots, n$, by applying the one-to-one transformation $\delta_{2i-1} = t_i x_i$, $\delta_{2i} = (1 - t_i)x_i$, $i = 1, \ldots, n$, with Jacobian $\prod_{i=1}^n x_i$. Without loss of generality, we shall focus on the case $n = 1$. The probability density function of $X = X_1$ follows by integrating out the variable $t = t_1$:

$$p(x) = \int_{\theta=0}^{\infty} \frac{\alpha^2\beta^2 x^{2\alpha-1}}{\Gamma^2(\frac{1}{\beta})\theta^{2/\beta}} \int_{t=0}^{1} [t(1-t)]^{\alpha-1} \exp\left\{-\frac{x^{\alpha\beta}[t^{\alpha\beta} + (1-t)^{\alpha\beta}]}{\theta}\right\} dt\, dP(\theta).$$

By applying the mean value theorem of the integral calculus and using the beta integral, there exists a constant ξ such that

$$p(x) = \int_{\theta=0}^{\infty} \frac{\alpha^2\beta^2 x^{2\alpha-1}}{\Gamma^2(\frac{1}{\beta})\theta^{2/\beta}} \mathrm{B}(\alpha, \alpha) \exp\left\{-\frac{x^{\alpha\beta}\xi}{\theta}\right\} dP(\theta),$$

where $\min\{2^{1-\alpha\beta}, 1\} \leq \xi \leq \max\{2^{1-\alpha\beta}, 1\}$.

The probability density functions $\tilde{p}(x)$ and $p(x)$ must be equal for all $x > 0$. With the existence of both $E(\Phi^{-r})$ and $E(\Theta^{-r})$ for all $r > 0$, we can prove in two stages that $\lambda = 2\alpha$ and $\lambda\mu = \alpha\beta$.

First, multiply both sides of the equation $\tilde{p}(x) = p(x)$ by $x^{1-2\alpha}$, where $x > 0$. As x approaches zero, from the right, we have

$$a_0 = \lim_{x \downarrow 0} p(x) x^{1-2\alpha} = \lim_{x \downarrow 0} \tilde{p}(x) x^{1-2\alpha} = b_0 \lim_{x \downarrow 0} x^{\lambda - 2\alpha},$$

where $a_0, b_0 > 0$. The left-hand side is a constant greater than zero, whereas the limit on the right-hand side equals zero for $\lambda > 2\alpha$ and tends to infinity for $\lambda < 2\alpha$: both leading to a contradiction. Thus $\lambda = 2\alpha$.

Second, we show $\lambda\mu = \alpha\beta$ by subsequently dividing both sides of $\tilde{p}(x) = p(x)$ by $x^{2\alpha-1}$, taking the derivative with respect to x, multiplying by $x^{1-\alpha\beta}$, and letting x tend to zero from above. Then, we have

$$-a_1 = -b_1 \lim_{x \downarrow 0} x^{\lambda\mu - \alpha\beta},$$

where $a_1, b_1 > 0$. The left-hand side is a constant smaller than zero, whereas the limit on the right-hand side equals zero for $\lambda\mu > \alpha\beta$ and tends to minus infinity for $\lambda\mu < \alpha\beta$: both leading to a contradiction. Hence $\lambda\mu = \alpha\beta$.

As a consequence, we may rewrite the equation $\tilde{p}(x) = p(x)$ in the form of two Laplace transforms:

$$\int_0^\infty \tilde{s}^{2/\beta} \exp\left\{-x^{\alpha\beta}\tilde{s}\right\} d\tilde{P}(\tilde{s}) = \int_0^\infty s^{2/\beta} \exp\left\{-x^{\alpha\beta}s\right\} dP(s) \, \frac{\alpha \mathrm{B}(\alpha,\alpha)}{\frac{1}{\beta}\mathrm{B}(\frac{1}{\beta},\frac{1}{\beta})} \, \xi^{-\frac{2}{\beta}},$$

where we have applied the transformations $\tilde{s} = 1/\varphi$ and $s = \xi/\theta$. Hence, we have

$$d\tilde{P}(s) = dP(s) \, \frac{\alpha \mathrm{B}(\alpha,\alpha)}{\frac{1}{\beta}\mathrm{B}(\frac{1}{\beta},\frac{1}{\beta})} \, \xi^{-\frac{2}{\beta}},$$

using the uniqueness of the Laplace transform. By integrating with respect to s on both sides, we can solve for ξ and substitute its value into the equation that was derived from the mean value theorem. Then,

$$\frac{1}{\mathrm{B}(\alpha,\alpha)} \int_{t=0}^1 [t(1-t)]^{\alpha-1} \exp\left\{-y[t^{\alpha\beta} + (1-t)^{\alpha\beta}]\right\} dt = \exp\left\{-y\left[\frac{\alpha \mathrm{B}(\alpha,\alpha)}{\frac{1}{\beta}\mathrm{B}(\frac{1}{\beta},\frac{1}{\beta})}\right]^{\frac{\beta}{2}}\right\},$$

where, for notational convenience, $y = x^{\alpha\beta}/\theta$. By expanding the exponential function with a Taylor series, we are now able to prove that $\alpha\beta = 1$.

The right-hand side results in

$$\tilde{h}(y) = \sum_{j=0}^\infty \frac{1}{j!} \left[\frac{\alpha \mathrm{B}(\alpha,\alpha)}{\frac{1}{\beta}\mathrm{B}(\frac{1}{\beta},\frac{1}{\beta})}\right]^{\frac{\beta}{2}j} (-y)^j,$$

while the left-hand side can be expanded in a similar way. With the Weierstrass M test, using the upper bound $t^{\alpha\beta} + (1-t)^{\alpha\beta} \leq \max\{2^{1-\alpha\beta}, 1\}$ for all $t \in [0,1]$, we may interchange the order of summation and integration through which the left-hand side can be rewritten as

$$h(y) = \sum_{j=0}^{\infty} \frac{1}{j!} E\left(\left[T^{\alpha\beta} + (1-T)^{\alpha\beta}\right]^j\right)(-y)^j,$$

where $T \sim \mathrm{Be}(\alpha, \alpha)$. Since the power series $\tilde{h}(y)$ and $h(y)$ must be equal for all $y > 0$, all their coefficients must coincide. Equating the $j = 1$ and $j = 2$ terms of both series yields the following relation in $\alpha\beta$:

$$E\left(\left[T^{\alpha\beta} + (1-T)^{\alpha\beta}\right]^2\right) = \left[E\left(T^{\alpha\beta} + (1-T)^{\alpha\beta}\right)\right]^2.$$

Applying Jensen's inequality, we see that $\alpha\beta = 1$.

In conclusion, we have for all $\tau > 0$:

$$\begin{aligned} \alpha(2\tau) &= 2\alpha(\tau), \\ \alpha(2\tau)\beta(2\tau) &= \alpha(\tau)\beta(\tau) = 1. \end{aligned}$$

The solutions of these functional equations are $\alpha(\tau) = a \cdot \tau$ and $\beta(\tau) = (a \cdot \tau)^{-1}$ for some constant $a > 0$.

Note that the above arguments also hold for subdividing $(0, k\tau]$ into k equal time-intervals instead of subdividing $(0, 2\tau]$ into 2 equal time-intervals, except that, roughly, k-dimensional Dirichlet integrals replace the 2-dimensional beta integrals.

Eq. (3.6) follows by replacing θ with $\theta/(a \cdot \tau)$, which proves the theorem. □

Chapter 4

Optimal Sand Nourishment Decisions

Jan M. van Noortwijk and E. Bart Peerbolte

Abstract. To maintain the Dutch coastline, every year millions of cubic metres of sand must be supplied at locations subject to ongoing erosion. A decision model has been developed to obtain optimal sand nourishment decisions whose expected costs are minimal with respect to the only information that is available: the probability distribution of the limiting average rate of ongoing erosion. In order that the stochastic erosion process be based on this uncertain limiting average, we consider it as a generalised gamma process.

There are three cost-based criteria for comparing sand nourishment decisions over an unbounded time-horizon: the average costs, the discounted costs, and the equivalent average costs. From these three criteria, only the last two are appropriate to obtain optimal sand nourishment decisions. In a case study, the decision model has been applied to sand nourishment at Zwanenwater, The Netherlands.

Although the decision model has been developed for the purpose of sand nourishment, it can be applied to other fields of engineering to solve many problems in the area of life cycle costing. (This chapter has previously been published as [131].)

Keywords. coastal management, sand nourishment, maintenance, gamma processes, decision theory, life cycle costing, renewal theory.

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4.1 Introduction

To prevent the Dutch coastline from receding, the government has decided to carry out sand nourishments at those beach locations at which a certain reference line is crossed (see Rijkswaterstaat [100, 101]). In this chapter, we present a probabilistic model that enables us to make nourishment decisions for which the expected costs over an unbounded horizon are minimal. The model generalises results from Wind & Peerbolte [144]. Since the sand nourishment programme is based on the uncertain average rates of ongoing erosion, we have based our probabilistic model on these averages as well. To achieve this, the process of ongoing erosion has been regarded as a generalised gamma process. In The Netherlands, generalised gamma processes have also been used to model decision problems for optimising maintenance of the sea-bed protection of the Eastern-Scheldt barrier, berm breakwaters, and dykes (see Chapters 2 and 7, 6, and 5, respectively).

Using the discrete renewal theorem, three cost-based criteria for comparing decisions over unbounded horizons can be determined: the average costs, the discounted costs, and the equivalent average costs. Although the criterion of the average costs is often used for maintenance optimisation in mechanical and electrical engineering (see e.g. Barlow & Proschan [5]), it is not useful for nourishment optimisation. Instead, the criteria of the discounted costs and the equivalent average costs should be used to find an optimum balance between initial costs and future costs, which is the area of life cycle costing (see e.g. Flanagan et al. [51]). Even though the discrete renewal theorem is well-known, it has not been often applied to solve problems in life cycle costing.

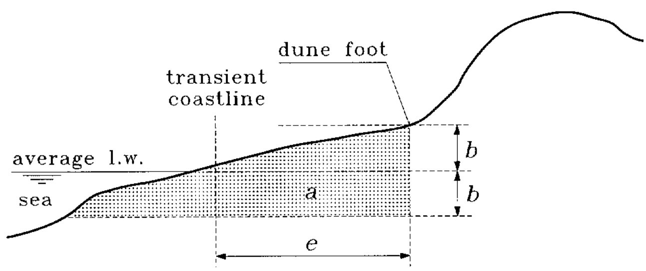
The chapter is composed as follows. We describe the problem of sand nourishment in Sec. 4.2. Next, in Sec. 4.3, optimal sand nourishment decisions are defined in terms of the minimal expected monetary loss. The costs of carrying out one sand nourishment are obtained in Sec. 4.4. In Sec. 4.5, these costs are then used to obtain the expected loss of an infinite sequence of sand nourishments using the above cost-based criteria. The sand nourishment decision model is applied to nourishment at Zwanenwater, The Netherlands, in Sec. 4.6. Finally, Sec. 4.7 presents some conclusions. Necessary definitions and theorems can be found in an appendix.

4.2 Sand nourishment in The Netherlands

The Dutch North-Sea coast is made up of dunes, dykes, and barriers (like the Eastern-Scheldt barrier). Together they protect the areas that are below sea level (over one third of The Netherlands) against flooding. The main part of the Dutch coastal defence line consists of dunes (254 km out of 353 km), varying in width from less than one hundred metres to several kilometres. Under the influence of nature, the dunes are continuously moving: advancing at one location, receding at another. As stated in Rijkswaterstaat [101], the patterns of coastal accretion (sand sedimentation) and coastal erosion are well known.

If we gave nature a free hand, hundreds of hectares of dune would disappear into the sea due to erosion. The safety of the polders and other interests, like recreation and the supply of drinking water, would then be endangered. For this reason, the Dutch Parliament adopted the policy of preserving the coastline at its position on January 1, 1990 (see Rijkswaterstaat [100]). As soon as this reference coastline, the so-called *basal coastline*, is crossed due to ongoing coastal erosion, preventive maintenance has to be carried out by adding sand to the coastal system (sand nourishment).

Every year, the position of the transient coastline is measured through inspection and is compared with the basal coastline. The shoreface (shallow sea floor) is inspected by means of depth measurements using a sounding system. The beach and the foredune (first line of the dunes) are inspected by means of height measurements using stereo photogrammetry. The coastal measurements are stored in the so-called JARKUS file. In Fig. 4.1, the transient coastline is defined in terms of the position of the dune foot, the average low waterline, and the vertical distance b in between. This volumetric approach is based on the area of sand that is enclosed by the vertical line through the dune foot, the horizontal line at a distance of $2b$ under the dune foot, and the measured beach profile.



a = area for volumetric integration (in m^2)

b = height difference dune foot–average low waterline (in m)

$e = \frac{a}{2b}$ (in m)

Figure 4.1: The definition of the transient coastline in terms of the position of the dune foot and the average low waterline.

Measurements over a period of years may serve to obtain average rates of sedimentation and erosion. These rates are used to plan the annual sand nourishment programme: per year about 7 million cubic metres of sand has to be supplied against costs of about 70 million Dutch guilders.

There are two kinds of coastal erosion: the ongoing long-term erosion and the incidental short-term erosion. The ongoing erosion is the most serious one and has to be compensated with sand nourishments. Although the incidental erosion caused by incidental severe storms can result in cut dunefaces, the sand that was extracted from the dunes during these storms has been temporarily transported to the beach and the shoreface. During calm weather, the waves return this sand partly back to the dunes. Therefore, the policy of coastline management is to accept the incidental erosion, but to combat the ongoing erosion. Verhagen [133] recommends using coastal measurements (properly qualified with expert judgment) for dealing with the irregular coastal morphology when determining sand nourishments.

At those beach locations where the coast is receding, sand is added by removing it from the bottom of the sea by trailing suction hopper dredgers. The most important advantages of sand nourishments are that they

- improve the safety while maintaining wide recreational beaches;
- are relatively cheap in comparison with dykes;
- fit in with the natural character of the Dutch coast;
- are flexible in the sense that they can be utilised almost everywhere and that they allow spreading of costs.

The above problem description of sand nourishment in The Netherlands is taken from Rijkswaterstaat [100, 101] and Verhagen [133].

4.3 Optimal sand nourishment decisions

The planning of sand nourishments in The Netherlands is mainly based on the average rates of ongoing erosion. Although yearly coastal measurements are available, the natural process of erosion is uncertain. This uncertainty has the following sources:

- The coastal measurements are confounded. Since the only erosion that matters for the decision problem is the ongoing erosion, one has to determine what part of the erosion is ongoing and what part is incidental.
- The transient coastline is a schematisation of the "real" coastline.
- The coastal monitoring is subject to measurement errors.
- New coastal structures, changes in tidal prisms of estuaries and changes in up-river sediment supply to the coastal system may influence the rate of erosion in a systematic way.

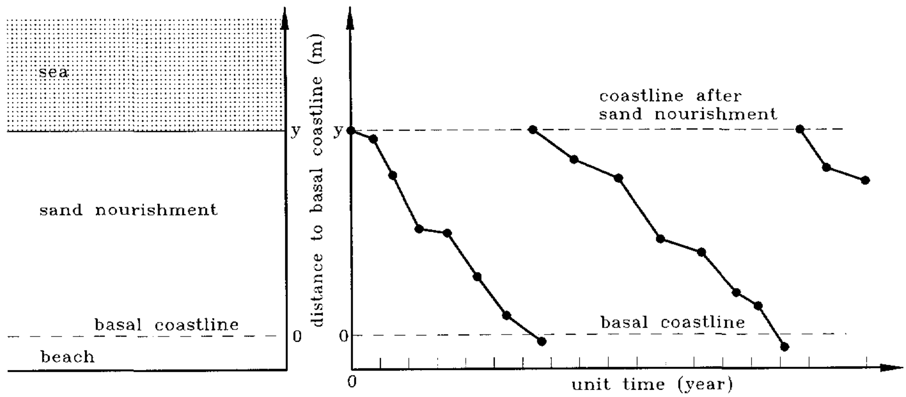


Figure 4.2: View from above of sand nourishment up to a distance of y metres to the basal coastline (left) and the process of ongoing erosion regarded as a renewal process (right).

Since the sand nourishment programme depends on the uncertain average rates of ongoing erosion, our probabilistic model should be based on these rates as well. In addition, we adopt the assumption of Verhagen [133] that the rate of erosion before nourishment equals the rate of erosion after nourishment. From now on, when we use the word "erosion", we shall mean ongoing erosion. Because sand nourishment optimisation is only useful for locations at which the coastline is receding quickly, we focus on non-negative rates of erosion. Let us denote the non-decreasing erosion process by $\{X(t) : t \geq 0\}$, where $X(t)$ represents the cumulative erosion at time t and $\Pr\{X(0) = 0\} = 1$. For every uniform time-partition in time-intervals of length $\tau > 0$, we write $D_i(\tau) = X(i\tau) - X([i-1]\tau)$, $i \in \mathbb{N}$. Furthermore, we judge the infinite sequence of increments $\{D_i(\tau) : i \in \mathbb{N}\}$ to be *exchangeable*, i.e. the order in which the increments occur is irrelevant. In mathematical terms, this can be interpreted as that the probability density function of the random vector $(D_1(\tau), \ldots, D_n(\tau))$ is invariant under all $n!$ permutations of the coordinates, i.e.

$$p_{D_1(\tau),\ldots,D_n(\tau)}\left(\delta_1, \ldots, \delta_n\right) = p_{D_1(\tau),\ldots,D_n(\tau)}\left(\delta_{\pi(1)}, \ldots, \delta_{\pi(n)}\right), \tag{4.1}$$

where π is any permutation of $1, \ldots, n$, for all $n \in \mathbb{N}$ and $\tau > 0$. We remark that the notion of exchangeability is weaker than the notion of independence.

In order that a stochastic erosion process with non-negative exchangeable increments be based on the unknown limiting average rate, we have shown in Chapter 3 that we can best regard it as a generalised gamma process. For this process, the joint probability density function of the increments of erosion $D_1(\tau), \ldots, D_n(\tau)$ is given by a mixture of conditionally independent gamma densities:

$$p_{D_1(\tau),\ldots,D_n(\tau)}(\delta_1,\ldots,\delta_n) = \int_0^\infty \prod_{i=1}^n \frac{\delta_i^{a\tau-1}}{\Gamma(a\tau)} \left[\frac{a\tau}{\theta}\right]^{a\tau} \exp\left\{-\frac{a\tau\delta_i}{\theta}\right\} dP_{\Theta(\tau)}(\theta) \qquad (4.2)$$

with mean $E(X(n\tau)) = nE(\Theta(\tau))$ for some constant $a > 0$ and all $\tau > 0$. The probability distribution $P_{\Theta(\tau)}$ on the random quantity $\Theta(\tau)$, with possible values $\theta \in (0,\infty)$, represents the uncertainty in the unknown limiting average amount of erosion per time-interval of length τ: $\lim_{n\to\infty}[(\sum_{i=1}^n D_i(\tau))/n]$. By the strong law of large numbers for exchangeable random quantities, the average converges with probability one if $E(D_1(\tau)) < \infty$ (see e.g. Chow & Teicher [13, p. 227]).

A useful property of the generalised gamma process is that the mixture of gamma's in Eq. (4.2) transforms into a mixture of exponentials if $\tau = a^{-1}$:

$$p_{D_1(a^{-1}),\ldots,D_n(a^{-1})}(\delta_1,\ldots,\delta_n) = \int_0^\infty \prod_{i=1}^n \frac{1}{\theta}\exp\left\{-\frac{\delta_i}{\theta}\right\} dP_\Theta(\theta) = f_n(\textstyle\sum_{i=1}^n \delta_i) \qquad (4.3)$$

for $(\delta_1,\ldots,\delta_n) \in \mathbb{R}_+^n$ and zero otherwise, $\mathbb{R}_+ = [0,\infty)$. The infinite sequence of random quantities $\{D_i(a^{-1}) : i \in \mathbb{N}\}$ is said to be l_1-isotropic (or l_1-norm symmetric), since its distribution can be written as a function of the l_1-norm (for details, see Barlow & Mendel [3] and Misiewicz & Cooke [90]). As we have shown in Chapter 3, the smaller the unit-time length Δ for which the increments are l_1-isotropic, i.e. the smaller $\Delta = a^{-1}$, the more deterministic the erosion process.

Owing to the annual monitoring of the transient coastline, a reasonable unit time for which the increments of erosion are assumed to be l_1-isotropic is one year. If Δ were smaller than one year, the fluctuations due to seasonal influences should be taken into account. If Δ were larger than one year, our model would differ from results obtained by Wind & Peerbolte [144]: in their opinion, the annual increments of erosion are independent, identically distributed, exponential random quantities with known mean. From now on, we consider increments of erosion that are l_1-isotropic with respect to the units of time $\{([i-1]\Delta, i\Delta] : i \in \mathbb{N}\}$, where $\Delta = 1$. For notational convenience, let $D_i = D_i(\Delta)$, $X_i = \sum_{j=1}^i D_j$ for all $i \in \mathbb{N}$, and let Θ represent the uncertainty in the limiting average rate of erosion $\lim_{n\to\infty}[(\sum_{i=1}^n D_i)/n]$.

To make optimal sand nourishment decisions in uncertainty, we can use statistical decision theory (see DeGroot [27, Ch. 8]). At beach locations subject to erosion, the decision problem is to choose a sand nourishment decision y, from $(0,\infty)$, such that the nourished coastline is located y metres seaward of the basal coastline, where the consequences of the decision depend on the unknown value of the limiting average erosion per year Θ. Let $L(\theta, y)$ be the *loss* when the decision-maker chooses decision y and when the limiting average erosion Θ has the value θ, where the loss represents the monetary losses due to sand nourishment. The decision-maker can best choose, if possible, a nourishment decision y^* whose expected loss is minimal. A decision y^* is called an *optimal decision* when $E(L(\Theta, y^*)) = \min_{y\in(0,\infty)} E(L(\Theta, y))$.

4.4 The costs of one sand nourishment

For each sand nourishment, the associated costs can be subdivided into the fixed cost c_f (cost of mobilisation) and the variable cost c_v (cost per cubic metre sand). Furthermore, we assume that the volume of sand to be supplied is a function of the sand nourishment width w (in metres) in the following way:

$$v(w) = lh \cdot w + \frac{l\sin(\psi)}{2}\left[\cos(\psi) + \frac{\sin(\psi)}{\tan(\varphi-\psi)}\right] \cdot w^2 = v_1 \cdot w + v_2 \cdot w^2 \qquad (4.4)$$

in cubic metres $[m^3]$, where l is the sand nourishment length, h is the sand nourishment height, φ is the angle of the beach-profile slope, and ψ is the angle of the sea floor (see Fig. 4.3). Although we restrict ourselves to a polynomial of degree 2, the decision model can deal with polynomials of any other degree (see Theorem 5 from the appendix).

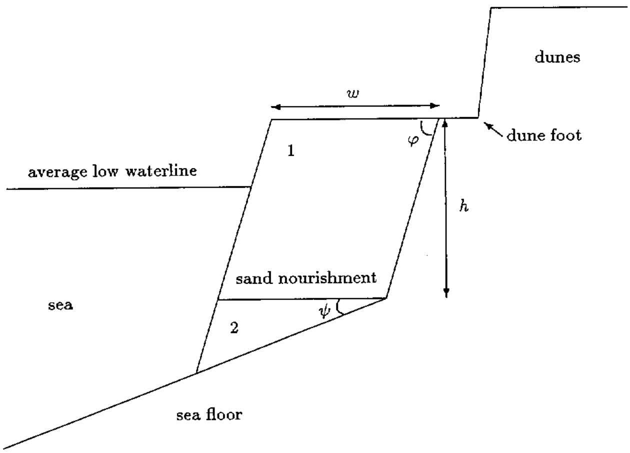


Figure 4.3: The compressed cross-section of a sand nourishment (1: area $v_1 w$; 2: area $v_2 w^2$).

Due to the mixture of exponentials in Eq. (4.3), we can express various probabilistic properties in explicit form when the limiting average erosion θ is given (for other illustrations, see Chapter 2). For the purpose of optimal sand nourishment, two probabilistic properties are useful: (i) the probability of exceedence of the basal coastline in unit time i and (ii) the expected costs of sand nourisment due to exceedence of the basal coastline in unit time i. These two properties are derived in Theorem 5 (see the appendix).

First, the conditional probability of exceedence of the basal coastline in unit time i, when the limiting average erosion is θ and when the decision-maker chooses the coastline after nourishment to be located y metres away from the basal coastline, can be written as:

$$p_i(\theta, y) = \Pr\{X_{i-1} \leq y, X_i > y | \theta\} = \frac{1}{(i-1)!}\left[\frac{y}{\theta}\right]^{i-1} \exp\left\{-\frac{y}{\theta}\right\} \tag{4.5}$$

for $i = 1, 2, \ldots,$ and $\theta, y > 0$. This discrete probability function is the Poisson distribution with mean lifetime $1 + (y/\theta)$ and variance y/θ.

Second, the expected costs of sand nourisment due to exceedence of the basal coastline in unit time i, when the limiting average erosion is θ and when the decision-maker chooses the coastline after nourishment to be located y metres away from the basal coastline, can be written as (using Eq. (4.4)):

$$\begin{aligned} &E\left(\left[c_f + c_v\left\{v_1 X_i + v_2 X_i^2\right\}\right] I_{[0,y]}(X_{i-1}) I_{(y,\infty)}(X_i) \Big| \theta\right) = \\ &= \left[c_f + c_v\left\{v_1(y+\theta) + v_2\left[(y+\theta)^2 + \theta^2\right]\right\}\right] \times \frac{1}{(i-1)!}\left[\frac{y}{\theta}\right]^{i-1} \exp\left\{-\frac{y}{\theta}\right\} \\ &= c_i(\theta, y) p_i(\theta, y) \end{aligned} \tag{4.6}$$

for $i = 1, 2, \ldots,$ and $\theta, y > 0$.

Using Eqs. (4.5) and (4.6), the costs $c_i(\theta, y)$ are the expected costs of a sand nourishment resulting in a nourished coastline at a distance of y metres to the basal coastline *given* that the basal coastline is exceeded in unit time i. The costs of nourishment do not depend on the unit time at which the basal coastline is exceeded: i.e. $c_i(\theta, y) \equiv c(\theta, y)$ for all $i \in \mathbb{N}$.

4.5 The costs of sequences of sand nourishments

Until now, we have studied the probability of occurrence and the expected costs of just one sand nourishment. A sand nourishment programme, however, consists of a series of consecutive nourishments. In this section, we derive three cost-based criteria to compare infinite sequences of sand nourishments over unbounded horizons.

4.5.1 Types of cost-based criteria

Wagner [138, Ch. 11] gave two reasons for comparing decisions over unbounded instead of bounded time-horizons. First, in making repeated investment decisions it is better to employ an unbounded horizon model than to simply ignore the future. Second, as we will see later, the mathematical models are less complex, while they still provide reasonable answers in practice. However, since maintenance costs over an unbounded horizon are infinite in most cases, we need models that can handle an infinite accumulation of costs. For this purpose, Wagner [138, Ch. 11] distinguishes three cost-based criteria for comparing decisions over unbounded horizons: the average costs per unit time, the discounted costs over an unbounded horizon, and the equivalent average costs per unit time.

These costs can be determined by formulating the process of sand nourishments as a discrete renewal process. A discrete renewal process $\{N(n) : n \in \mathbb{N}\}$ is a non-negative integer-valued stochastic process that registers the successive renewals in the time-interval $(0, n]$. In coastal management, the renewals are the sand nourishment actions carried out to move the transient coastline y metres seaward from the basal coastline (see Fig. 4.2). Conditional on θ, the renewal times $T_1, T_2, \ldots,$ are non-negative, independent, identically distributed, Poisson random quantities having the probability function (4.5), i.e. $\Pr\{T_k = i|\theta\} = p_i(\theta, y)$, $i \in \mathbb{N}$, when the limiting average erosion is θ and when the decision-maker chooses decision y. The above cost-based criteria will be discussed in more detail in the following subsections.

4.5.2 The average costs per unit time

The expected average costs per unit time are determined by simply averaging the costs over an unbounded horizon. They follow from the expected costs $C(n, \theta, y)$, over the bounded horizon $(0, n]$, that solve the equation

$$C(n, \theta, y) = \sum_{i=1}^{n} p_i(\theta, y)[c_i(\theta, y) + C(n - i, \theta, y)] \tag{4.7}$$

for $n \in \mathbb{N}$ and $C(0, \theta, y) \equiv 0$. To obtain this equation, we condition on the values of the first renewal time T_1 and apply the law of total probability. The costs associated with occurrence of the event $T_1 = i$ are $c_i(\theta, y)$ (see Eq. (4.6)) plus the expected additional costs during the interval $(i, n]$, $i = 1, \ldots, n$.

Using the discrete renewal theorem (see Feller [47, Ch. 12 & 13] and Karlin & Taylor [71, Ch. 3]), the expected *average costs per unit time* are

$$\lim_{n \to \infty} \frac{C(n, \theta, y)}{n} = \frac{\sum_{i=1}^{\infty} c_i(\theta, y) p_i(\theta, y)}{\sum_{i=1}^{\infty} i p_i(\theta, y)} = \frac{c(\theta, y)}{1 + (y/\theta)} = C(\theta, y). \tag{4.8}$$

(see Theorem 6 from the appendix). Let a renewal cycle be the time-period between two renewals, then we recognise the numerator as the expected cycle costs and the denominator as the expected cycle length. Eq. (4.8) is a well-known result from renewal reward theory (see e.g. Ross [105]). If $c(\theta, y) \equiv 1$ in Eq. (4.8), then the expected *average number of renewals per unit time* is:

$$\lim_{n \to \infty} \frac{M(n, \theta, y)}{n} = \frac{1}{\sum_{i=1}^{\infty} i p_i(\theta, y)} = \frac{1}{1 + (y/\theta)} = \frac{\theta}{y + \theta}, \tag{4.9}$$

being the reciprocal of the mean lifetime.

4.5.3 The discounted costs over an unbounded horizon

Discounting expected costs over an unbounded horizon is based on the assumption that the utility of a certain amount of money decreases in time from the standpoint of the present. Using the criterion of the discounted costs, it is possible to compare the value of money at different dates while taking into account the idea that "a dollar today is worth more than a dollar a year from today". In fact, the more money we have available now, the better off we are, for the sooner we can earn more money with it. Formally, the *(present) discounted value* of the costs c_n in unit time n is defined to be $\alpha^n c_n$ with $\alpha = [1 + (r/100)]^{-1}$ the *discount factor* per unit time and $r\%$ the *discount rate* per unit time, where $r > 0$. The decision-maker is indifferent between the costs c_n at time n and the costs $\alpha^n c_n$ at time 0. Therefore, the higher the discount rate, the better it is to postpone expensive sand nourishment actions. What discount rate is to be taken depends on the decision problem.

The expected discounted costs over a bounded time-horizon can be obtained with a recursive formula similar to that for the expected number of renewals in Eq. (4.7). Again, we condition on the values of the first renewal time T_1 and apply the law of total probability. In this case, however, we want to account for the discounted value of the renewal costs $c_i(\theta, y)$ plus the additional expected discounted costs in time-interval $(i, n]$, $i = 1, \ldots, n$. Hence, the expected discounted costs over the bounded horizon $(0, n]$ can be written as

$$C_\alpha(n, \theta, y) = \sum_{i=1}^{n} \alpha^i p_i(\theta, y) \left[c_i(\theta, y) + C_\alpha(n - i, \theta, y)\right] \tag{4.10}$$

for $n \in \mathbb{N}$, and $C_\alpha(0, \theta, y) \equiv 0$.

By using Feller [47, Ch. 13], the expected *discounted costs over an unbounded horizon* $C_\alpha(\theta, y)$ can be written as

$$\begin{aligned} C_\alpha(\theta, y) &= \lim_{n\to\infty} C_\alpha(n, \theta, y) = \frac{\sum_{i=1}^{\infty} \alpha^i c_i(\theta, y) p_i(\theta, y)}{1 - \sum_{i=1}^{\infty} \alpha^i p_i(\theta, y)} = \\ &= \frac{\alpha \exp\{-(1-\alpha)y/\theta\}}{1 - \alpha \exp\{-(1-\alpha)y/\theta\}} \cdot c(\theta, y). \end{aligned} \tag{4.11}$$

(see Theorem 6 from the appendix). We recognise the numerator of $C_\alpha(\theta, y)$ as the discounted cycle costs, while the denominator can be interpreted as the probability that the renewal process terminates due to discounting. Such a renewal process is called a *terminating renewal process* since infinite inter-occurrence times can cause the renewals to cease. The inter-occurrence times $Z_1, Z_2, \ldots$, of our imaginary terminating renewal process have the distribution $\Pr\{Z_k = i\} = \alpha^i p_i(\theta, y)$, $i \in \mathbb{N}$, and $\Pr\{Z_k = \infty\} = 1 - \sum_{i=1}^{\infty} \alpha^i p_i(\theta, y)$. The expected number of imaginary "discounted renewals" over an unbounded time-horizon is

$$\lim_{n\to\infty} M_\alpha(n,\theta,y) = \frac{\sum_{i=1}^{\infty}\alpha^i p_i(\theta,y)}{1-\sum_{i=1}^{\infty}\alpha^i p_i(\theta,y)} = \frac{\Pr\{Z_k<\infty\}}{\Pr\{Z_k=\infty\}}. \tag{4.12}$$

4.5.4 The equivalent average costs per unit time

The expected equivalent average costs per unit time relate the two notions of average costs and discounted costs. To determine this relation, we construct a new infinite stream of identical costs with the same present discounted value as the discounted costs over an unbounded time-horizon $C_\alpha(\theta,y)$. This can easily be achieved by defining an infinite stream of costs appearing at times $i=0,1,2,\ldots$, which are all equal to $(1-\alpha)C_\alpha(\theta,y)$. Using the geometric series, we can write

$$\textstyle\sum_{i=0}^{\infty}\alpha^i(1-\alpha)C_\alpha(\theta,y) = C_\alpha(\theta,y) \tag{4.13}$$

for $\theta,y>0$, and $0<\alpha<1$. We call $(1-\alpha)C_\alpha(\theta,y)$ the *equivalent average costs per unit time.* As α tends to 1, from below, the equivalent average costs approach the average costs per unit time:

$$\lim_{\alpha\uparrow 1}(1-\alpha)C_\alpha(\theta,y) = C(\theta,y), \tag{4.14}$$

for $\theta,y>0$, using L'Hôpital's rule.

4.5.5 Choice of cost-based criteria

For a cost-optimal sand nourishment programme, we are interested in finding an optimum balance between the initial costs and the future costs, being the area of life cycle costing.

Let the transient coastline at time 0 be the basal coastline. Then, the first sand nourishment has to be carried out at a costs of $c_0(y)=c_f+c_v v(y)$ Dutch guilders. The monetary losses over an unbounded horizon are the sum of the initial costs and the expected discounted future costs,

$$L_\alpha(\theta,y) = c_0(y) + C_\alpha(\theta,y), \tag{4.15}$$

when the decision-maker chooses decision y, the limiting average erosion is θ, and the discount factor is α. For the purpose of sand nourishment, we cannot use the criterion of the expected average costs per unit time,

$$L(\theta,y) = C(\theta,y), \tag{4.16}$$

because the contribution of the initial costs to the average costs is completely ignored: $\lim_{n\to\infty} c_0(y)/n = 0$. In conclusion, we recommend to choose an optimal sand nourishment decision y^* satisfying $E(L_\alpha(\Theta,y^*)) = \min_{y\in(0,\infty)} E(L_\alpha(\Theta,y))$.

4.6 Case study: Zwanenwater, The Netherlands.

The developed sand nourishment decision model has been applied to Zwanenwater, a beach section with a length of 4300 m in the north-west of The Netherlands (see Fig. 4.5). In the opinion of the contractor, in 1987 a sand nourishment of $1.85 \times 10^6\ m^3$ was carried out, including a unique dune strengthening of $1.55 \times 10^5\ m^3$. However, as pointed out in Rijkswaterstaat [102], probably less sand was supplied. Indeed, the coastal measurements of 1986, 1987 and 1988 from the JARKUS file show an increase of volume of about 1 million m^3 only. The sand nourishment of 1987 was intended to assure that the basal coastline would not be crossed for a period of more than 20 years, with an initial distance to the basal coastline of about 21 metres. The data on the beach profile and the costs of sand nourishment are given in Table 4.1 (using Rijkswaterstaat [102] and the 1986, 1987, and 1988 JARKUS measurements). To illustrate, one beach profile at Zwanenwater is shown in Fig. 4.4.

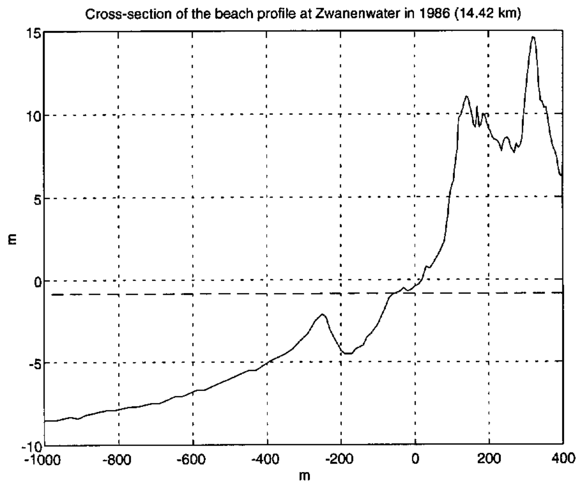


Figure 4.4: A cross-section of the beach profile at Zwanenwater, The Netherlands, from the JARKUS file before the sand nourishment was carried out. The average low waterline is located at -0.84 m.

For economic reasons, the decision-maker can best choose a sand nourishment whose expected discounted costs over an unbounded horizon $E(L_\alpha(\Theta, y))$ are minimal, where y is the distance between the nourished coastline and the basal coastline. Since the limiting average rate of ongoing erosion at Zwanenwater is about 1 m/year, we use an inverted gamma distribution with a 5%-percentile of 0.5 m/year, a 95%-percentile of 2 m/year, and a mean of 1 m/year (see Fig. 4.6): i.e. $\Theta \sim \text{Ig}(\theta | \nu, \mu)$. Although we assume there is only coastal erosion (i.e. $\theta > 0$), we can include coastal accretion as well

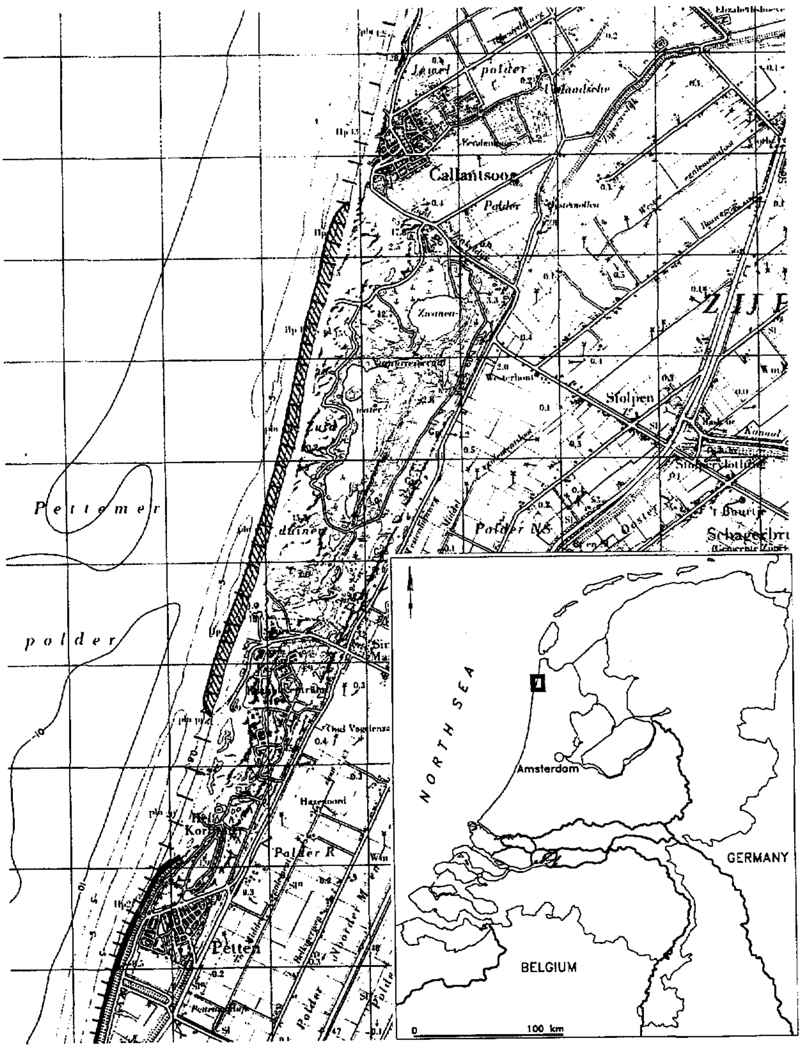


Figure 4.5: The location of the 1987 sand nourishment at Zwanenwater, The Netherlands.

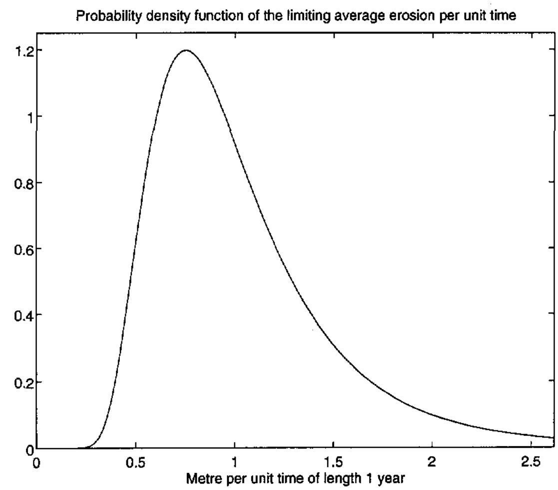


Figure 4.6: The probability density function of the limiting average rate of ongoing erosion [m/year] at Zwanenwater, The Netherlands.

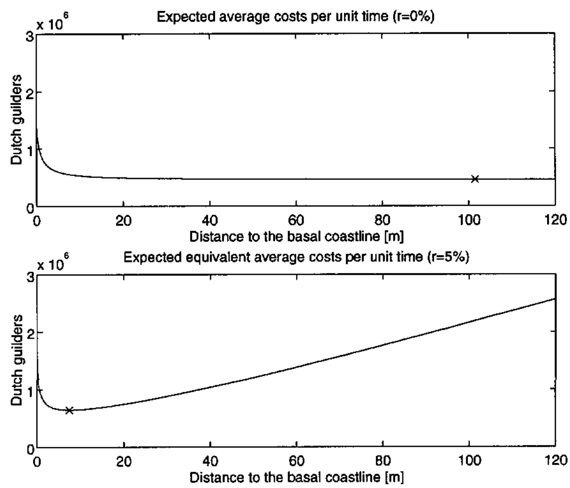


Figure 4.7: The expected average costs of sand nourishment per unit time and the expected equivalent average costs of sand nourishment per unit time at Zwanenwater, The Netherlands.

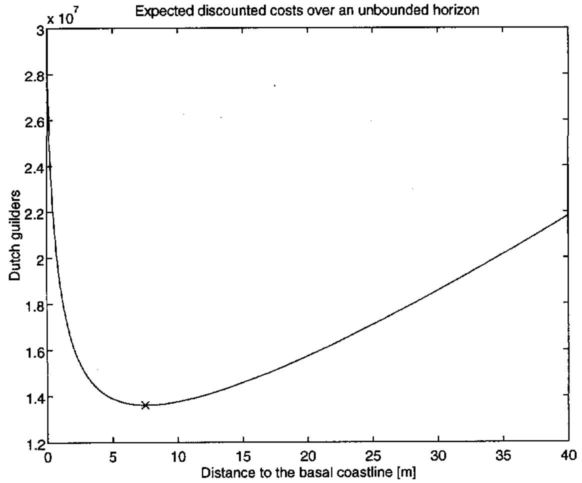


Figure 4.8: The expected discounted costs of sand nourishment over an unbounded time-horizon at Zwanenwater, The Netherlands.

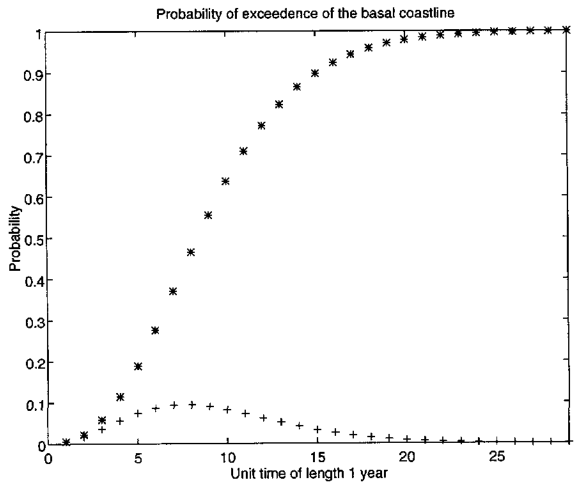


Figure 4.9: The probability of exceedence of the basal coastline per unit time at Zwanenwater, The Netherlands, when the distance between the nourished coastline and the basal coastline is optimal, i.e. is 7.5 m. Both the probability function and the cumulative distribution function are shown.

by choosing other probability distributions in combination with a zero (or negative) loss function.

In Theorem 7 (see the appendix), two expressions for the expected loss are obtained: $E(L(\Theta, y))$, the expected average costs per unit time (4.8), and $E(L_\alpha(\Theta, y))$, the expected discounted costs over an unbounded horizon (4.11). From the latter, the equivalent average costs per unit time can be easily obtained, being $E((1-\alpha)L_\alpha(\Theta, y))$. In Fig. 4.7, the expected average costs and the expected equivalent average costs are shown as a function of the distance y between the nourished coastline and the basal coastline. As α tends to 1, from below, the expected equivalent average costs approach the expected average costs (by interchanging the order of the operations of expectation and passing to the limit through Lebesgue's Theorem of Dominated Convergence; see e.g. Weir [141, Ch. 5]).

Recall that for the purpose of sand nourishment the criterion of average costs should not be used, since the costs of the first sand nourishment are neglected. Indeed, the

Table 4.1: The parameters of the sand nourishment model for Zwanenwater, The Netherlands.

parameter	description	value	dimension
Δ	unit time	1	year
n	time-horizon	∞	year
r	discount rate per year	5	%
α	discount factor per year	0.9524	
Θ	uncertain limiting average rate of erosion	$(0, \infty)$	m/year
$\theta_{0.05}$	5%-percentile average rate of erosion	0.5	m/year
$\theta_{0.95}$	95%-percentile average rate of erosion	2.0	m/year
ν	shape parameter average rate of erosion	6.1	
μ	scale parameter average rate of erosion	5.3	
$E(\Theta)$	mean of the average rate of erosion	1.0	m/year
c_f	fixed cost	9.2×10^5	Dfl
c_v	variable cost	9	Dfl/m^3
l	sand nourishment length	4300	m
h	sand nourishment height	11	m
w	sand nourishment width	$(0, \infty)$	m
-	location dune foot	6	m +NAP
φ	angle of the beach-profile slope	3.8×10^{-2}	radials
ψ	angle of the sea floor	4.0×10^{-3}	radials
y	distance to the basal coast line	$(0, \infty)$	m
y^*	optimal distance to the basal coast line	7.5	m
$v(y^*)$	optimal sand nourishment volume	3.5×10^5	m^3

cost figures in Fig. 4.7 show that the expected average costs per unit time are minimal for a sand nourishment that is unrealistically large. Actually, the average costs are minimal for $y = 102\ m$ and $v(y) = 4.9 \times 10^6\ m^3$, resulting in average costs of 4.6×10^5 Dutch guilders per unit time, but equivalent average costs of 2.2×10^6 Dutch guilders per unit time.

The expected discounted costs over an unbounded time-horizon are minimal when the decision-maker chooses the distance between the nourished coastline and the basal coastline to be $y^* = 7.5\ m$ (see Figs. 4.7 and 4.8). The corresponding optimal sand nourishment volume is $v(y^*) = 3.5 \times 10^5\ m^3$. For the optimal decision, the expected discounted costs are 1.4×10^7 Dutch guilders, where the expected equivalent average costs are 6.5×10^5 Dutch guilders per unit time. No less important than obtaining a unique optimal decision, however, is obtaining a range of nearly optimal decisions.

If the distance between the nourished coastline and the basal coastline is the optimal distance y^* and if the sand nourishment is carried out at time 0, then we can derive the expected probability that the basal coastline will be crossed in a particular unit time. This (discrete) probability function is the negative binomial distribution. It follows by integrating Eq. (4.5) over the limiting average erosion Θ and it is graphically shown in Fig. 4.9. The mean time between two sand nourishments is 10 units of time, with a standard deviation of 5 (for the mathematics, see Chapter 2). The average number of sand nourishments per unit time is 0.12, which can be obtained with Eqs. (4.9) and (4.21) (see Theorem 7 from the appendix).

4.7 Conclusions

In this chapter, we have presented a sand nourishment decision model which enables the decision-maker to optimise nourishment programmes. As decision criterion, we recommend to apply the expected discounted costs over an unbounded time-horizon for finding an optimum balance between initial costs and future costs. An important starting point is the probability distribution of the limiting average rate of ongoing erosion, which is assumed to be unaffected by sand nourishment. The probability distribution can be given *a priori* and can later be refined on the basis of measurements.

4.8 Appendix: Proofs of theorems

Theorem 5 *Suppose the infinite sequence of random quantities* $\{D_i : i \in \mathbb{N}\}$ *is* l_1*-isotropic and* $X_n = \sum_{i=1}^n D_i$ *for all* $n \in \mathbb{N}$*, then*

$$E\left([X_n]^m I_{[0,y]}(X_{n-1}) I_{(y,\infty)}(X_n) \Big| \theta\right) =$$
$$= \left\{\sum_{i=0}^m \frac{m!}{(m-i)!} y^{m-i}\theta^i\right\} \times \frac{1}{(n-1)!}\left[\frac{y}{\theta}\right]^{n-1} \exp\left\{-\frac{y}{\theta}\right\}, \tag{4.17}$$

for $n = 1, 2, \ldots$, $m = 0, 1, 2, \ldots$, $y \in (0, \infty)$, *where* $I_A(x) = 1$ *if* $x \in A$ *and* $I_A(x) = 0$ *if* $x \notin A$.

Proof:
Since $X_n = \sum_{i=1}^n D_i$ for all $n \in \mathbb{N}$, it follows that the integration bounds are determined by $0 \leq X_1 \leq \ldots \leq X_{n-1} \leq X_n$. Moreover, $X_{n-1} \leq y$ and $X_n > y$, and the Jacobian equals one. Hence, we may write

$$E\left([X_n]^m I_{[0,y]}(X_{n-1}) I_{(y,\infty)}(X_n) \Big| \theta\right) =$$
$$= \int_{x_1=0}^{y} \cdots \int_{x_{n-1}=x_{n-2}}^{y} \int_{x_n=y}^{\infty} x_n^m \frac{1}{\theta^n} \exp\left\{-\frac{x_n}{\theta}\right\} dx_n dx_{n-1} \cdots dx_1.$$

By applying the transformation $t = (x_n - y)/\theta$, and by using the binomial formula and the gamma function, we can integrate out the variable x_n:

$$\int_{x_n=y}^{\infty} x_n^m \exp\left\{-\frac{x_n}{\theta}\right\} dx_n = \sum_{i=0}^m \frac{m!}{(m-i)!} y^{m-i}\theta^{i+1} \exp\left\{-\frac{y}{\theta}\right\}. \tag{4.18}$$

The remaining integral is the Dirichlet integral:

$$\int_{x_1=0}^{y} \cdots \int_{x_{n-1}=x_{n-2}}^{y} 1\, dx_{n-1} \cdots dx_1 = \frac{y^{n-1}}{(n-1)!}. \tag{4.19}$$

Combining Eqs. (4.18) and (4.19) leads to the desired result. □

Theorem 6 (The discrete renewal theorem.) *Let $\{q_i\}$, $\{u_i\}$, $\{b_i\}$ be sequences indexed by $i = 0, 1, 2, \ldots$ with $q_i \geq 0$ for all i, and $\sum_{i=0}^{\infty} |b_i| < \infty$. Suppose the renewal equation*

$$u_n = b_n + \textstyle\sum_{i=0}^{n} q_i u_{n-i} \tag{4.20}$$

is satisfied for $n = 0, 1, 2, \ldots$ by a bounded sequence $\{u_i\}$ of real non-negative numbers. Then, (a) if $\sum_{i=0}^{\infty} q_i = 1$, $q_1 > 0$,

$$\lim_{n\to\infty} u_n = \frac{\sum_{i=0}^{\infty} b_i}{\sum_{i=0}^{\infty} i q_i};$$

and (b) if $\sum_{i=0}^{\infty} q_i < 1$, u_n converges to 0, as $n \to \infty$, at such a rate that

$$\lim_{n\to\infty} \textstyle\sum_{i=0}^{n} u_i = \dfrac{\sum_{i=0}^{\infty} b_i}{1 - \sum_{i=0}^{\infty} q_i}.$$

Proof:
See Feller [47, Ch. 13] and Karlin & Taylor [71, Ch. 3]. □

Corollary 1 *Let $u_i \equiv C(i, \theta, y) - C(i-1, \theta, y)$, $q_i \equiv p_i(\theta, y)$, $b_i \equiv c_i(\theta, y) p_i(\theta, y)$ for all $i = 1, 2, \ldots$, $\theta, y > 0$, and $u_0 \equiv q_0 \equiv b_0 \equiv 0$, then the recurrence relation (4.7) can be written as the renewal equation (4.20). With Cauchy's first limit theorem, the expression for the expected average costs per unit time, Eq. (4.8), follows from part (a) of the discrete renewal theorem.*

Corollary 2 *Let $u_i \equiv C_\alpha(i, \theta, y) - C_\alpha(i-1, \theta, y)$, $q_i \equiv \alpha^i p_i(\theta, y)$, $b_i \equiv c_i(\theta, y)\alpha^i p_i(\theta, y)$ for all $i = 1, 2, \ldots$, $\theta, y > 0$, $0 < \alpha < 1$, and $u_0 \equiv q_0 \equiv b_0 \equiv 0$, then the recurrence relation (4.10) can be written as the renewal equation (4.20). The expression for the expected discounted costs over an unbounded horizon, Eq. (4.11), follows from part (b) of the discrete renewal theorem.*

Theorem 7 *Let* $\Theta \sim \mathrm{Ig}(\nu, \mu)$. *Then (a)*

$$E\left([y+\Theta]^{-1}\Theta^{m+1}\right) = E\left(\Theta^m\right) e^{\mu/y}[\mu/y]^{\nu-m}\Gamma(1-\nu+m, \mu/y), \tag{4.21}$$

and (b)

$$E\left(\frac{\alpha \exp\{-(1-\alpha)y/\Theta\}}{1-\alpha\exp\{-(1-\alpha)y/\Theta\}} \cdot \Theta^m\right) = E\left(\Theta^m\right)\sum_{j=1}^{\infty}\alpha^j\left[\frac{\mu}{\mu+j(1-\alpha)y}\right]^{\nu-m} \tag{4.22}$$

for $0 \le m < \nu$, *where* $E\left(\Theta^m\right) = \mu^m\Gamma(\nu-m)/\Gamma(\nu)$, *and* $\Gamma(a,x) = \int_{t=x}^{\infty} t^{a-1}e^{-t}\,dt$ *is called the incomplete gamma function for* $x > 0$ *and* $a \in \mathbb{R}$.

Proof:
(a) The proof follows by applying the transformation $t = y/\theta$ and using the following formula from Nielsen [92, pages 210-211]:

$$e^x\Gamma(a)\Gamma(1-a,x) = \int_{t=0}^{\infty}\frac{t^{a-1}e^{-xt}}{1+t}\,dt,$$

where $x > 0$ and $a > 0$.
(b) The series in Eq. (4.22) can be obtained by using the geometric series and by interchanging the order of the operations of expectation and summation through the Monotone Convergence Theorem (see e.g. Weir [141, Ch. 5]). □

Chapter 5

Optimal Maintenance Decisions for Dykes

Lennaert J.P. Speijker, Jan M. van Noortwijk, Matthijs Kok, and Roger M. Cooke

Abstract. To protect the Dutch polders against flooding, more than 2,500 km of dykes have been constructed. Due to settlement, subsoil consolidation, and relative sea-level rise, these dykes slowly sink "away into the sea" and should therefore be heightened regularly (at present every 50 years). In this respect, one is interested in safe and cost-optimal dyke heights for which the sum of the initial costs of investment and the future (discounted) costs of maintenance are minimal.

For optimisation purposes, a maintenance model has been developed for dykes subject to uncertain crest-level decline. On the basis of data and engineering knowledge, crest-level decline has been modelled as a monotone stochastic process with expected decline being either linear or non-linear (i.e. linear after transformation) in time. For both models, and for a particular unit time, the increments are distributed according to mixtures of exponentials.

In a case study, the maintenance decision model has been applied to the problem of heightening the Dutch 'Oostmolendijk'. (This chapter has previously been published as [111].)

Keywords. maintenance, dykes, engineering probability, gamma processes, relative sea-level rise, settlement, subsoil consolidation, renewal theory.

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5.1 Introduction

To protect the Dutch polders against flooding, a network of dykes, dams, and barriers has been constructed. For the largest part, this network consists of dykes at a total length of more than 2,500 km. In order to provide for the long-term safety, these dykes have to be maintained on the basis of five-yearly inspections as laid down in the Dutch Flood Protection Act [117]. Unfortunately, due to settlement, subsoil consolidation, and relative sea-level rise, the dykes slowly sink "away into the sea" and should therefore be heightened and strengthened regularly. In this chapter, we present a probabilistic model that enables us to determine safe dyke heights for which the costs of maintenance are minimal.

Up to the beginning of this century, the crest of a dyke was built about one metre above the highest water level hitherto observed at that place. Since then, however, statistical considerations about the frequency of occurrence of maximum water levels have been introduced (see Wemelsfelder [142]). With storm-induced tides of some four metres above average sea level, the flood of February 1, 1953, caused a severe catastrophe in the south-west of The Netherlands. Almost 200,000 hectares of polderland flooded, 1,835 people drowned, and the flood damage totaled 1.5×10^9 Dutch guilders (in 1953). To avoid future losses due to floods like the one in 1953, the Dutch parliament adopted the so-called Delta Plan. This plan called for raising the dykes and for closing the main tidal estuaries and inlets by a network of dams and barriers. An economic analysis was carried out to balance the investments in heightening the dykes against the expected losses of flooding (see the Delta Commission [38] and Van Dantzig [119]).

In The Netherlands fifty-three so-called dyke-ring areas can be identified, which are areas surrounded by a ring of dykes, dunes, retaining works (e.g. the Eastern-Scheldt barrier), and higher ground. The acceptable probability of flooding for the dyke-ring area Central Holland was set by the Delta Commission at 8×10^{-6} per year. Acceptable inter-occurrence times of the exceedence of the water level that dyke-ring components should withstand are laid down in the Dutch Flood Protection Act [117] and vary from 10,000 years (for dyke-ring areas subject to sea floods) to 1,250 years (for dyke-ring areas subject to river floods).

We focus on failure of a dyke section due to settlement, subsoil consolidation, and relative sea-level rise (denoted by *crest-level decline*), under a condition-based preventive maintenance strategy. In a fault tree analysis, this failure mode can be combined with other failure modes like overflowing, wave overtopping, instability, piping, sliding, and erosion. Note that a dyke ring is a series system of dyke sections with probabilities of failure that might be dependent.

Given the acceptable probability of failure of a dyke section, its crest height is nowadays determined on the basis of the so-called "design water level" (the acceptable maximum water level) plus a safety margin (needed to cope with possible wave runup, gust and squall oscillations, seiches, and crest-level decline during fifty years). The present dyke design in The Netherlands prescribes dykes to be heightened every fifty years, which, however, might not be economical. For details on the Dutch flood protection programme, see, e.g., TAW [115, 116], CUR & TAW [21], Peerbolte [93], Vrijling [136], and Vrouwenvelder & Struik [137].

In this chapter, we present a new probabilistic model for determining safe dyke heights that optimally balance the initial costs of investment against the future costs of maintenance. The basic idea behind our model comes from Van Dantzig [119] and differs from the latter in the sense that we regard crest-level decline as a stochastic process rather than as a deterministic number. Moreover, we consider condition-based preventive maintenance (carried out at times determined by five-yearly dyke inspections) rather than time-based preventive maintenance (carried out at pre-determined repair times). Van Dantzig's economic model was extended by Vrijling & Van Beurden [134], who assumed the average rate of crest-level decline to be uncertain, and Kuijper [79], who assumed the process of crest-level decline to be a stochastic process.

To account for possible crest-level decline in a period of fifty years, dyke heights are often designed on the basis of uncertain average rates of crest-level decline. In order that the stochastic process of crest-level decline be based on its uncertain limiting average rate, we consider it as a generalised gamma process. A gamma process is a stochastic process with independent non-negative increments having gamma distributions with given scale parameters and shape parameters proportional to the length of the time-interval over which the increments are taken. A generalised gamma process is then defined as a scale mixture of gamma processes, where the scale parameter can be interpreted as the unknown limiting average rate of crest-level decline. Note that the Brownian motion with drift (a stochastic process with stationary independent decrements and increments having a normal distribution) is not applicable in this context, since we must require that the increments are non-negative.

In solving the economic problem of dyke heightening, Kuijper [79] also employed the gamma process. He applied approximations to determine, for example, the probability of exceeding a failure level in a particular unit time. Instead, we give analytical expressions for many useful probabilistic properties of the gamma process. Generalised gamma processes have also been used to model decision problems for optimising maintenance of the sea-bed protection of the Eastern-Scheldt barrier, beaches, and berm breakwaters (see Chapters 2 and 7, 4, and 6, respectively).

Although the uncertainty in the limiting average rate of crest-level decline is often large, the question arises whether the above assumptions still hold when much data is available. In addition to the generalised gamma process, with expected decline being linear in time, we study a monotone stochastic process with expected decline being non-linear in time. The latter process has been derived from a physical law which is well accepted by engineers in soil mechanics: the law of settlement and subsoil consolidation of Terzaghi & Koppejan [77]. In doing so, we base our probabilistic model on well-known engineering knowledge, an approach which has recently been proposed by Mendel & Chick [88] and Chick [11].

For the purpose of finding an optimum balance between the initial costs and the future costs, which is the area of life cycle costing (see e.g. Flanagan et al. [51]), we can best use the criterion of the expected discounted costs over an unbounded time-horizon. These costs can be determined by applying the discrete renewal theorem, where the renewals are the events at which a dyke is heightened.

The chapter is organised as follows. The costs of one dyke heightening, proportional to the increase in dyke volume, are presented in Sec. 5.2. In Secs. 5.3 and 5.4, analytical expressions are derived for the expected discounted costs over an unbounded horizon under linear and non-linear crest-level decline, respectively. The maintenance model is applied to the Dutch 'Oostmolendijk' in Sec. 5.5. Sec. 5.6 ends with some conclusions. Necessary definitions and theorems can be found in an appendix.

5.2 The costs of one dyke heightening

In modelling the maintenance of dykes, we make a distinction between the initial dyke heightening and the future dyke heightenings due to crest-level decline. The initial dyke heightening entails heightening the crest level *and* broadening the base, whereas the future dyke heightenings leave the base unchanged (see Fig. 5.1). Determining the costs of heightening dykes is the subject of study in this section.

For each dyke heightening, the costs can be subdivided into the fixed cost c_f (cost of mobilisation and road reconstruction) and the variable cost c_v (cost per cubic metre dyke volume). By using the schematised cross-section in Fig. 5.1, the dyke volume is a quadratic function of the dyke height h (in metres) in the following way:

$$v(h) = wl \cdot h + \frac{l}{2}\left[\frac{1}{\tan(\varphi)} + \frac{1}{\tan(\omega)}\right] \cdot h^2 = v_1 \cdot h + v_2 \cdot h^2 \tag{5.1}$$

in cubic metres $[m^3]$, where h is the crest level, w is the crest width, l is the length of the dyke section, and φ and ω are the angles of the inner slope and the outer slope, respectively.

The costs of initially heightening the dyke from h_0 up to h metres and changing the base width accordingly, where $h_0 < h$, are simply

$$c_0(h) = c_f + c_v\left[v(h) - v(h_0)\right] \tag{5.2}$$

(see Fig. 5.1). Van Dantzig [119] approximated Eq. (5.1) by a linear function of h, though he acknowledged that his approximation is not valid for large h: indeed, the higher a dyke, the broader a base that is required.

Similarly, the costs of future heightenings to annul a crest-level decline of x metres, while keeping the base width unchanged, can be written as a linear function of x (see

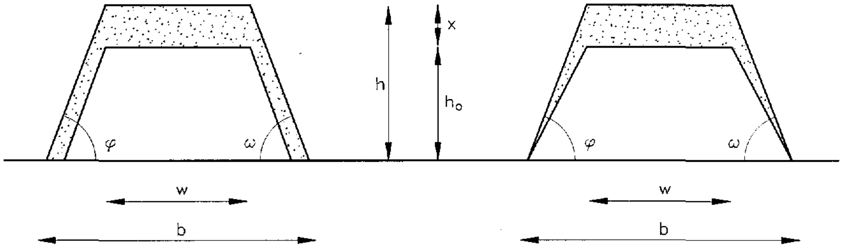

Figure 5.1: The cross-section of a dyke before and after heightening: the initial dyke heightening from h_0 up to h metres (left) and a future dyke heightening of x metres (right).

Fig. 5.1):

$$c_f + c_v\tilde{v}(h,x) = c_f + c_v\left\{wl + \frac{hl}{2}\left[\frac{1}{\tan(\varphi)} + \frac{1}{\tan(\omega)}\right]\right\}\cdot x = c_f + c_v\tilde{v}_1(h)\cdot x, \quad (5.3)$$

where h, w, l, φ, and ω are defined as in Eq. (5.1).

5.3 Linear crest-level decline

5.3.1 The stochastic process of crest-level decline

In this subsection, we present a probabilistic model for the process of crest-level decline based on the unknown limiting average rate. Let us consider the non-decreasing stochastic process $\{X(t) : t \geq 0\}$, where $X(t)$ represents the cumulative amount of crest-level decline at time t and $X(0) = 0$ with probability one. For every uniform time-partition in time-intervals of length $\tau > 0$, we write $D_i(\tau) = X(i\tau) - X([i-1]\tau)$, $i \in \mathbb{N}$. Furthermore, due to the lack of data, we judge the infinite sequence of increments $\{D_i(\tau) : i \in \mathbb{N}\}$ to be *exchangeable*, i.e. the order in which the increments occur is irrelevant. In mathematical terms, this means that the probability density function of the random vector $(D_1(\tau), \ldots, D_n(\tau))$ is invariant under all $n!$ permutations of the coordinates, i.e.

$$p_{D_1(\tau),\ldots,D_n(\tau)}(\delta_1, \ldots, \delta_n) = p_{D_1(\tau),\ldots,D_n(\tau)}\left(\delta_{\pi(1)}, \ldots, \delta_{\pi(n)}\right), \quad (5.4)$$

where π is any permutation of $1, \ldots, n$, for all $n \in \mathbb{N}$ and $\tau > 0$.

In order that a stochastic deterioration process with non-negative exchangeable increments be based on its unknown limiting average rate, we have argued in Chapter 3 that we can best regard it as a *generalised gamma process*. For this process, the joint probability density function of the increments $D_1(\tau), \ldots, D_n(\tau)$ is given by a mixture of conditionally independent gamma densities:

$$p_{D_1(\tau),\ldots,D_n(\tau)}(\delta_1, \ldots, \delta_n) = \int_0^\infty \prod_{i=1}^n \frac{\delta_i^{a\tau-1}}{\Gamma(a\tau)} \left[\frac{a\tau}{\theta}\right]^{a\tau} \exp\left\{-\frac{a\tau\delta_i}{\theta}\right\} dP_{\Theta(\tau)}(\theta) \qquad (5.5)$$

for some constant $a > 0$ with

$$\begin{aligned} E(X(n\tau)) &= E(n\Theta(\tau)), \\ \mathrm{Var}(X(n\tau)) &= \left[1 + \frac{1}{na\tau}\right] E([n\Theta(\tau)]^2) - [E(n\Theta(\tau))]^2 \end{aligned} \qquad (5.6)$$

for all $\tau > 0$, provided the first and the second moment of the probability distribution of $\Theta(\tau)$ exist. By the strong law of large numbers for exchangeable random quantities, the probability distribution $P_{\Theta(\tau)}$ of the random quantity $\Theta(\tau)$ represents the uncertainty in the unknown limiting average amount of crest-level decline per time-interval of length τ: $\lim_{n\to\infty}[(\sum_{i=1}^n D_i(\tau))/n]$.

A useful property of the generalised gamma process is that the mixture of gamma's in Eq. (5.5) transforms into a mixture of exponentials if $\tau = a^{-1}$:

$$p_{D_1(a^{-1}),\ldots,D_n(a^{-1})}(\delta_1, \ldots, \delta_n) = \int_0^\infty \prod_{i=1}^n \frac{1}{\theta} \exp\left\{-\frac{\delta_i}{\theta}\right\} dP_\Theta(\theta) = f_n(\textstyle\sum_{i=1}^n \delta_i) \qquad (5.7)$$

where $(\delta_1, \ldots, \delta_n) \in \mathbb{R}_+^n$ and zero otherwise, $\mathbb{R}_+ = [0, \infty)$. The infinite sequence of random quantities $\{D_i(a^{-1}) : i \in \mathbb{N}\}$ is said to be l_1-isotropic (or l_1-norm symmetric), since its distribution can be written as a function of the l_1-norm (see Misiewicz & Cooke [90]).

The unit time for which the increments of crest-level decline are l_1-isotropic can be obtained, for example, by specifying the conditional probability density function of the first increment, when the sum of the first and the second increment is given. This probability density function is a transformed beta distribution with both parameters equal to $a\tau$ (the equality of these parameters is due to the exchangeability of the two increments), i.e.

$$p_{D_1(\tau)|X(2\tau)}(\delta_1 | x) = \frac{\Gamma(2a\tau)}{[\Gamma(a\tau)]^2} \frac{\delta_1^{a\tau-1}[x-\delta_1]^{a\tau-1}}{x^{2a\tau-1}} I_{[0,x]}(\delta_1) = \mathrm{Be}\left(\frac{\delta_1}{x}\middle| a\tau, a\tau\right)\frac{1}{x} \qquad (5.8)$$

for some constant $a > 0$ with

$$\begin{aligned} E(D_1(\tau)|X(2\tau){=}x) &= x/2, \\ \mathrm{Var}(D_1(\tau)|X(2\tau){=}x) &= [x/2]^2/(2a\tau + 1). \end{aligned}$$

Hence, for fixed $\tau > 0$, the smaller the unit-time length for which the increments are l_1-isotropic, i.e. the smaller $\Delta = a^{-1}$, the more deterministic the deterioration process. An alternative way to obtain the unit time for which l_1-isotropy holds is assessing $\text{Var}(X(n\tau))$ in Eq. (5.6). This variance approaches $\text{Var}(n\Theta(\tau))$, from above, as $\Delta = a^{-1}$ tends to 0, from above. As we shall see in Sec. 5.3.2, for this unit-time length, denoted by $\Delta = a^{-1}$, many probabilistic properties of the stochastic process, like the probability of exceedence of a failure level, can be expressed in explicit form conditional on the limiting average.

In conclusion, we advocate regarding the stochastic process of crest-level decline as a generalised gamma process with probability distribution on the limiting average rate of crest-level decline. To keep the mathematics of the decision model tractable, we impose the property of *posterior linearity* introduced by Diaconis & Ylvisaker [43], i.e. $E(X(2\tau)|\, D_1(\tau) = \delta_1) = c_1\delta_1 + c_2$ for some constants $c_1, c_2 > 0$ and $\tau > 0$. Note that, due to exchangeability, before observing D_1, $E(D_2) = E(D_1)$. If posterior linearity holds, then the mixing distribution in Eq. (5.5) is an inverted gamma distribution (see Diaconis & Ylvisaker [43]).

From now on, we consider increments of crest-level decline that are l_1-isotropic with respect to the units of time $\{([i-1]\Delta, i\Delta] : i \in \mathbb{N}\}$. For notational convenience, let $D_i = D_i(\Delta)$, $X_n = \sum_{i=1}^{n} D_i$ for all $n \in \mathbb{N}$, and let Θ represent the uncertainty in the limiting average rate of crest-level decline, $\lim_{n\to\infty}[(\sum_{i=1}^{n} D_i)/n]$, with probability density function $\text{Ig}(\theta|\nu, \mu)$ (see Definition 3 of Appendix A). The summarisation of the n random quantities $D_1, \ldots, D_n$ in terms of the statistic $[n, \sum_{i=1}^{n} D_i]$ is sufficient for Θ. The mean and the variance of X_n are $E(X_n) = nE(\Theta)$ and $\text{Var}(X_n) = E(n\Theta^2) + \text{Var}(n\Theta)$, respectively. Note that the unit time for which the increments of crest-level decline are l_1-isotropic can also be obtained by assessing $\text{Var}(X_n)$, where $X_n \sim \text{Gg}(\nu, \mu, n)$ (see Definition 4 of Appendix A).

5.3.2 The expected discounted costs of dyke heightening

As stated in the introduction, the Dutch Flood Protection Act [117] prescribes the dykes to be inspected every 5 years. For this reason, we assume the dyke section to be periodically inspected at times $\{jk\Delta : j \in \mathbb{N}\}$ for fixed $k \in \mathbb{N}$, where $k\Delta = 5$ years. Each heightening brings the dyke section back into its "as good as new state". Therefore, we may consider the maintenance process as a renewal process, where each renewal cycle ends at an inspection time $jk\Delta$ when the inspection reveals that the dyke section should be heightened (for some $j \in \mathbb{N}$). We assume that inspection of the dyke takes negligible time and does not degrade the dyke.

As proposed by Speijker [112], we define the failure level s as the "design water level" plus a safety margin needed to cope with wave runup, oscillations, seiches, and crest-level decline during an inspection interval of five years (not fifty years as in the present dyke design). Failure is then defined as the event in which a dyke height drops below the failure level s; it can only be noted through inspection. When inspection reveals that the crest level of a dyke section is lower than the failure level s, it should be heightened. Note that a failure need not imply a collapse: a failed dyke section only collapses when the actual water level exceeds the actual dyke height. We study "design water levels" rather than actual water levels.

Let $y = h - s$ and let the times at which the failure level is first crossed be conditionally independent random quantities having a discrete probability function $p_i(\theta, y)$ and associated repair cost $c_i(\theta, y)$, with respect to the units of time $\{([i-1]\Delta, i\Delta] : i \in \mathbb{N}\}$, when the limiting average rate of crest-level decline is θ and the decision-maker chooses the dyke to be y metres higher than its failure level s.

To obtain optimal maintenance decisions in uncertainty, we can use statistical decision theory (see DeGroot [27, Ch. 8]). Let $L_\alpha(\theta, h)$ be the (monetary) *loss* when the decision-maker chooses dyke height h and the limiting average rate of crest-level decline is given by θ. The decision-maker can best choose a dyke height h^* whose expected loss is minimal. A decision h^* is called an *optimal decision* when

$$E\left(L_\alpha(\Theta, h^*)\right) = \min_{h \in (h_0, \infty)} E\left(L_\alpha(\Theta, h)\right). \tag{5.9}$$

Since determining optimal dyke heights actually means balancing the initial cost against the future cost, the criterion of discounted costs can best serve as a loss function (for a discussion, see Chapter 4). The expected *discounted costs over an unbounded horizon* can be determined by summing the expected discounted values of the costs over an unbounded horizon, where the discounted value of the costs c_n in unit time n is defined to be $\alpha^n c_n$ with discount factor $\alpha = [1 + (r/100)]^{-1}$ and discount rate $r\%$ $(r > 0)$:

$$L_\alpha(\theta, h) = c_0(h) + \frac{\sum_{j=1}^{\infty} \alpha^{jk} \sum_{i=(j-1)k+1}^{jk} c_i(\theta, y) p_i(\theta, y)}{1 - \sum_{j=1}^{\infty} \alpha^{jk} \sum_{i=(j-1)k+1}^{jk} p_i(\theta, y)}, \tag{5.10}$$

where $y = h - s$ and the initial cost $c_0(h)$ stems from Eq. (5.2). Eq. (5.10) follows from the discrete renewal theorem (see Chapter 4).

The form of Eq. (5.7) enables us to express various probabilistic properties in explicit form when θ is given. For the purpose of optimal dyke heightening, two probabilistic properties are useful: (i) the probability of exceedence of the failure level in unit time i and (ii) the expected costs of dyke heightening due to exceedence of the failure level in unit time i. These two properties are derived in Theorem 8 (see the appendix), which generalises Theorem 5 (see Chapter 4).

First, the conditional probability of exceedence of the failure level in unit time i, when the limiting average crest-level decline is θ and when the decision-maker chooses the dyke to be h metres high, can be written as:

$$p_i(\theta, y) = \Pr\{X_{i-1} \le y, X_i > y \,|\, \theta\} = \frac{1}{(i-1)!} \left[\frac{y}{\theta}\right]^{i-1} \exp\left\{-\frac{y}{\theta}\right\} \tag{5.11}$$

for $i = 1, 2, \ldots, \theta > 0$, and $y = h - s > 0$. This discrete probability function can be recognised as the Poisson distribution[1] with mean lifetime $1 + (y/\theta)$ and variance y/θ.

Second, the expected costs of dyke heightening due to exceedence of the failure level in unit time i, when the limiting average crest-level decline is θ and when the decision-maker chooses the dyke to be h metres high, can be written (using Eq. (5.3)) as:

$$E\left(\left[c_f + c_v\tilde{v}_1(h)\cdot X_{jk}\right] I_{[0,y]}(X_{i-1}) I_{(y,\infty)}(X_i) \middle| \theta\right) =$$
$$= \left[c_f + c_v\tilde{v}_1(h)(y + [jk - i + 1]\theta)\right]\cdot p_i(\theta, y) = c_i(\theta, y)p_i(\theta, y) \qquad (5.12)$$

for $i = (j-1)k + 1, \ldots, jk$, where $j, k = 1, 2, \ldots$; $\theta > 0$ and $y = h - s > 0$.

It should be noted that, in contrast with the model of Van Dantzig [119], our model only includes the costs of maintenance, not the costs of possible flooding. Since these latter are very difficult to assess, we have introduced a safety margin instead: the higher the costs of flooding, the larger we can choose the safety margin. In essence, Van Dantzig optimises "design water levels", whereas we optimise dyke heightenings when the "design water levels" are given.

In conclusion, the expected discounted costs over an unbounded horizon can be obtained by substituting Eqs. (5.11) and (5.12) into Eq. (5.10) and by taking the expectation with respect to the probability distribution of Θ. The optimal dyke height follows from Eq. (5.9) (for an example, see Sec. 5.5).

5.4 Non-linear crest-level decline

5.4.1 The stochastic process of crest-level decline

Although the assumption of expected crest-level decline being linear in time is quite reasonable when data is lacking, the question arises how to proceed when data gives evidence to an expected decline being non-linear in time. In order to investigate the sensitivity of the optimal dyke height to different rates of crest-level decline, we also consider stochastic processes with non-negative, but non-exchangeable, increments.

Engineering knowledge suggests the expected crest-level decline to be a logarithmic function of time. Recall that the process of crest-level decline is a combination of settlement, subsoil consolidation, and relative sea-level rise.

Settlement and subsoil consolidation has thoroughly been studied by Koppejan [77]. With emperical experiments he showed that, a large time t after increasing the stress from p_1' to p_2', the thickness of a compressed layer of sand or clay behaves according to

[1]Kuijper [79] approximated Equation (5.11) by $\Pr\{X_{i-1} \leq y, X_i > y | \theta\} \approx \Pr\{X_{i-1} \leq y | \theta\}$ $\Pr\{X_i > y | \theta\}$.

the so-called formula of Terzaghi & Koppejan (see also TAW [115, Ch. 8]):

$$z(t) = z_0 \left[\frac{1}{C_p} + \frac{1}{C_s} \frac{\ln(t)}{\ln(10)} \right] \ln \left(\frac{p_2'}{p_1'} \right), \tag{5.13}$$

where:

z_0 = the initial thickness of the layer [m]
$z(t)$ = the thickness of the compressed layer at time t [m]
C_p = primary compression constant [-]
C_s = secondary compression constant [-]
t = time [s]
p_1' = initial stress [N/m^2]
p_2' = increased stress [N/m^2].

Relative sea-level rise has probably the following causes: melting of glaciers, changes in the Greenland and the Antarctic icecaps, thermal expansion of the oceans, and, for The Netherlands, readjustment of the earthcrust due to the melting away of the Fennoscandian icecap about 10,000 years ago. The estimates of the relative sea-level rise for the next century vary between 20 cm and 120 cm, with a best estimate of 60 cm (see Van Dantzig [119], Vrijling & Van Beurden [134], Peerbolte [94], and Hesselmans & Peerbolte [61]).

In order to preserve the mathematical tractability in determining the expected discounted costs, when transforming linear decline into non-linear decline, we link up with a stochastic process having l_1-isotropic increments in the following way. Let us consider an infinite sequence of random quantities $\{D_i : i \in \mathbb{N}\}$ that is transformed l_1-isotropic in the sense that the probability density function of the random vector $(D_1, \ldots, D_n)$ can be written as a function of the statistic $\sum_{i=1}^n D_i/\beta_i$ for all $n \in \mathbb{N}$:

$$p_{D_1,\ldots,D_n}(\delta_1, \ldots, \delta_n) = \int_0^\infty \prod_{i=1}^n \frac{1}{\beta_i \theta} \exp\left\{ -\frac{\delta_i}{\beta_i \theta} \right\} dP_\Theta(\theta) = f_n\left(\textstyle\sum_{i=1}^n \delta_i/\beta_i \right) \tag{5.14}$$

for $(\delta_1, \ldots, \delta_n) \in \mathbb{R}_+^n$ and zero otherwise, where $\beta_i > 0$ for $i = 1, \ldots, n$ and $\beta_i \neq \beta_j$ unless $i = j$. The conditional cumulative distribution function of the sum $X_n = \sum_{i=1}^n D_i$, when θ is given, is known as the general Erlang distribution and can be found in Theorem 9 (see the appendix). It has been used in theories of radioactive decay, queuing, reliability, and psychology (see e.g. Jensen [68] and McGill & Gibbon [86]).

The mixture of conditionally independent exponentials with different means, Eq. (5.14), converges to the mixture of conditionally independent exponentials with equal means, Eq. (5.7), as $\beta_i \to 1$ for all $i = 1, \ldots, n$. Therefore, it is convenient that the increments D_i, $i \in \mathbb{N}$, are defined with respect to the same units of time (of length Δ) as for the l_1-isotropic increments in Sec. 5.3.1. By Theorem 10 (see the appendix), the probability distribution P_Θ on the random quantity Θ, in Eq. (5.14), represents the uncertainty in the unknown limiting weighted average $\lim_{n\to\infty}[(\sum_{i=1}^{n} D_i/\beta_i)/n]$. The summarisation of the n random quantities $D_1, \ldots, D_n$ in terms of the statistic $[n, \sum_{i=1}^{n} D_i/\beta_i]$ is sufficient for Θ. The mean and the variance of X_n are $E(X_n) = \sum_{i=1}^{n} \beta_i E(\Theta)$ and $\text{Var}(X_n) = E(\text{Var}(X_n|\Theta)) + \text{Var}(E(X_n|\Theta))$, respectively.

Our next aim is finding patterns of β_i's in Eq. (5.14) such that the expected deterioration is a logarithmic function of time (at least for large values of time). The so-called *digamma function* or *Euler's psi function* (see Nielsen [92, § 5]),

$$\psi(x) = \frac{d}{dx}\{\ln\Gamma(x)\} = \frac{\Gamma'(x)}{\Gamma(x)} = \lim_{n\to\infty}\left[\ln(n) - \sum_{i=1}^{n}\frac{1}{x+i-1}\right], \tag{5.15}$$

suggests setting $\beta_i = a/(b+i-1)$ for all $i \in \mathbb{N}$, where $a, b > 0$. Indeed, for large n, i.e. for large time t in Eq. (5.13), the expected crest-level decline conditional on θ can then be written as

$$E(X_n|\,\theta) = \left[\sum_{i=1}^{n}\frac{a}{b+i-1}\right]\cdot\theta \approx a\left[\ln(n) - \psi(b)\right]\cdot\theta, \tag{5.16}$$

as $n \to \infty$. Note that $\psi(b)$ can be negative[2].

5.4.2 The expected discounted costs of dyke heightening

In a similar way as was done for linear expected crest-level decline (Sec. 5.3), we can determine two important probabilistic properties for non-linear expected crest-level decline: (i) the probability of exceedence of the failure level in unit time i and (ii) the expected costs of dyke heightening due to exceedence of the failure level in unit time i. Conditional on the limiting weighted average Θ, these two properties are derived in Theorem 11 (see the appendix).

First, the conditional probability of exceedence of the failure level in unit time i, when the limiting weighted average is θ and when the decision-maker chooses the dyke to be h metres high, can be written as:

$$p_i(\theta, y) = \sum_{m=1}^{i}\frac{\beta_i/\beta_m}{\prod_{h=1,\,h\neq m}^{i}[1-\beta_h/\beta_m]}\exp\left\{-\frac{y}{\beta_m\theta}\right\} \tag{5.17}$$

for $i = 1, 2, \ldots$, $\theta > 0$, and $y = h - s > 0$. When $\beta_i = a/(b+i-1)$ for all $i \in \mathbb{N}$, this discrete probability function simplifies to the negative binomial distribution with parameters $1 - \exp\{-y/(a\theta)\}$ and b (see Jensen [68]):

$$p_i(\theta, y) = \binom{b+i-1-1}{i-1}\left[1 - \exp\left\{-\frac{y}{a\theta}\right\}\right]^{i-1}\left[\exp\left\{-\frac{y}{a\theta}\right\}\right]^{b}, \tag{5.18}$$

[2] For the digamma function holds: $\psi(x) \leq 0$ for $0 < x \leq x_0$ and $\psi(x) > 0$ for $x > x_0$, where $x_0 \approx 1.462$.

for $i = 1, 2, \ldots, \theta > 0$, and $y = h - s > 0$, with mean life time $1 + b[\exp\{y/(a\theta)\} - 1]$ and variance $b\exp\{y/(a\theta)\}[\exp\{y/(a\theta)\} - 1]$.

Second, the expected costs of dyke heightening due to exceedence of the failure level in unit time i, when the limiting weighted average is θ and when the decision-maker chooses the dyke to be h metres high, can be written as (by using Eq. (5.3), and Theorems 11 and 12 in the appendix):

$$E\left([c_f + c_v\tilde{v}_1(h) \cdot X_{jk}]\, I_{[0,y]}(X_{i-1}) I_{(y,\infty)}(X_i) \Big| \theta \right) =$$

$$= \left[c_f + c_v\tilde{v}_1(h)\left(y + \textstyle\sum_{h=i}^{jk} \beta_h \theta \right) \right] \cdot p_i(\theta, y) = c_i(\theta, y) p_i(\theta, y) \qquad (5.19)$$

for $i = (j-1)k + 1, \ldots, jk$, where $j, k = 1, 2, \ldots$, and $\theta > 0$ and $y = h - s > 0$.

In conclusion, the expected discounted costs over an unbounded horizon can be obtained by substituting Eqs. (5.18) and (5.19) into Eq. (5.10) and by taking the expectation with respect to $\text{Ig}(\theta|\nu, \mu)$. The optimal dyke height follows from Eq. (5.9).

5.5 Case study: The Dutch 'Oostmolendijk'

The above decision model for optimal dyke heightening has been applied to the 'Oostmolendijk', a dyke section with a length of 1000 m in the west of The Netherlands. The 'Oostmolendijk' is located between the towns Ridderkerk and Hendrik-Ido-Ambacht, along the river Noord, and belongs to the dyke ring IJsselmonde. In the last decades, the 'Oostmolendijk' has been subject to extreme settlement and subsoil consolidation: about 0.60 m in the period 1969-1981 and about 0.15 m in the period 1981-1989. In 1969, its crest-level height was about 5.20 m +NAP (normal Amsterdam level), whereas the last dyke heightening, in 1991, resulted in a crest-level of 4.90 m +NAP (the difference is due to reduction of the "design water level" by the storm-surge barrier in the 'Nieuwe Waterweg'). With respect to the ground level, being 1 m +NAP, the dyke height h was 3.90 m (see Fig. 5.1). Speijker [112] proposed a failure level of $s = 3.44$ m, allowing the probability of exceedence of this failure level during an inspection interval to be at most 0.1.

For obtaining an optimal dyke height for the 'Oostmolendijk', we use the parameters in Table 5.1. The probability density function of Θ, the limiting average rate of crest-level decline per unit time, is shown in Fig. 5.2. Since experts perform better when assessing the 5%- and 95%-percentiles of a probability density function than when assessing its mean and variance, the parameters of the inverted gamma distribution have been derived from $\theta_{0.05}$ and $\theta_{0.95}$ in Table 5.1 by using Theorem 13 (see the appendix). The expected crest-level decline in a period of fifty years is 1.30 m: 1.00 m is due to settlement and subsoil consolidation; 0.30 m is due to relative sea-level rise.

The unit time for which the increments of crest-level decline are distributed as mixtures of exponentials ($\Delta = 5/3$ year) has been determined by specifying the conditional

Table 5.1: The parameters of the dyke heightening model for the Dutch 'Oostmolendijk'.

parameter	description	value	dimension
Δ	unit time	5/3	year
k	inspection-interval length	3	unit time
$k\Delta$	inspection-interval length	5	year
r	discount rate per year	5	%
α	discount factor per unit time	0.9219	
c_f	fixed cost	1.8×10^6	Dfl
c_v	variable cost	30	Dfl/m^3
Θ	limiting average crest-level decline	$(0, \infty)$	m/unit time
$\theta_{0.05}$	5%-percentile average crest-level decline	0.033	m/unit time
$\theta_{0.50}$	50%-percentile average crest-level decline	0.043	m/unit time
$\theta_{0.95}$	95%-percentile average crest-level decline	0.057	m/unit time
ν	shape parameter of $\text{Ig}(\theta\|\nu, \mu)$	35.86	
μ	scale parameter of $\text{Ig}(\theta\|\nu, \mu)$	1.511	
$E(\Theta)$	mean of average crest-level decline	0.043	m/unit time
$E(\Theta/\Delta)$	mean of average crest-level decline	0.026	m/year
$\text{Var}(\Theta)$	variance of average crest-level decline	5.5×10^{-5}	
a	parameter non-linear crest-level decline	8.95	
b	parameter non-linear crest-level decline	3	
h_0	crest-level height before heightening	3.44	m
h	crest-level height of the dyke section	(h_0, ∞)	m
l	length of the dyke section	1000	m
w	crest width of the dyke section	7	m
s	failure level of the dyke section	3.44	m
y	$h - s$	$(0, \infty)$	m
-	ground level (or terrain level)	1	m +NAP
φ	the angle of the inner slope (1:3)	0.32	radials
ω	the angle of the outer slope (1:3)	0.32	radials

Table 5.2: Optimal dyke heightenings and the corresponding mean times between dyke heightenings for different expected average rates of linear crest-level decline.

$E(\Theta/\Delta)$	0.50	1.00	1.50	2.00	2.60	$\times 10^{-2}$ m/year
optimal dyke heightening y^*	0.31	0.49	0.63	0.74	0.86	m
mean time to dyke heightening	84	71	49	42	38	year

probability density function of the amount of crest-level decline in a period of 25 years when the decline in a period of 50 years is given to be 1 metre (using Eq. (5.8) shown in Fig. 5.3).

The expected crest-level decline with sums of increments being linear or non-linear in time and under a condition-based maintenance strategy are displayed in Fig. 5.4: when a five-yearly inspection reveals that the dyke section has failed, it is heightened up to 4.30 m.

For economic reasons, the decision-maker can best choose a dyke height h whose expected discounted costs over an unbounded horizon, $E(L_\alpha(\Theta, h))$, are minimal. In Fig. 5.5, the expected discounted costs over an unbounded horizon are shown as a function of y, where $y = h - s$, for expected crest-level decline being linear and non-linear in time. The optimal decision, satisfying Eq. (5.9) under *linear* crest-level decline, is $y^* = 0.86$ m, or equivalently $h^* = 4.30$ m, with expected discounted costs over an unbounded horizon of 3.09×10^6 Dutch guilders. The optimal decision, satisfying Eq. (5.9) under *non-linear* crest-level decline, is $y^* = 1.08$ m, or equivalently $h^* = 4.52$ m, with expected discounted costs over an unbounded horizon of 3.05×10^6 Dutch guilders. The main reason the optimal dyke height is larger for non-linear decline than for linear decline is because the variable cost of dyke heightening depend on the rate of crest-level decline (which is smaller in the event of non-linear decline after exceeding the failure level: see Fig. 5.4).

For practical purposes, no less important than obtaining a unique optimal decision, however, is obtaining a range of nearly cost-optimal decisions. In this respect, the values of the loss function $L_\alpha(\theta, h)$ at the 5%-, 50%-, and 95%-percentile of the limiting average rate of crest-level decline are of interest: they are displayed in Figs. 5.6 and 5.7 (with $y = h - s$), describing expected crest-level decline being linear and non-linear in time, respectively.

The sensitivity of the optimal dyke height to the choice of the unit time Δ is investigated in Fig. 5.8: h^* hardly depends on Δ. Furthermore, from Table 5.2, we see that the smaller the expected average rate of linear crest-level decline, the smaller the optimal dyke heightening y^* and the larger the mean time between two dyke heightenings. From these results, we can conlude that the present practice of heightening the Dutch dykes every 50 years has sense. Note that the average rate of crest-level decline in The Netherlands is about 0.5 to 0.7 cm per year in the lower river area and about 0.3 to 0.5 cm per year in the upper river area (see TAW [116, Ch. 6]).

When $y = 0.86$ m at time 0, the expected probabilities of failure per unit time can be determined by integrating Eqs. (5.11) and (5.18) over Θ. These discrete probability functions are shown in Fig. 5.9. The mean time between two dyke heightenings is 22.5 units of time (37.5 years) for linear decline and 30.4 units of time (50.6 years) for non-linear decline (while taking into account that dyke heightenings can only take place at times of inspection).

5.6 Conclusions

In this chapter, we have presented a decision model for determining safe dyke heights that optimally balance the initial costs of investment against the future costs of maintenance. As decision criterion, we have used the expected discounted costs over an unbounded time-horizon. An important starting point is the probability distribution of the rate of crest-level decline (a combination of settlement, subsoil consolidation, and relative sea-level rise).

We have investigated two types of monotone crest-level decline: expected decline being linear in time (when there is a lack of data) and expected decline being non-linear in time (when there is abundant data and engineering knowledge can be used). For linear decline, we have regarded the deterioration process as a generalised gamma process for which we can always find a uniform time-partition such that the joint probability density function of the increments is a mixture of conditionally independent exponentials with equal means. For non-linear, strictly monotone, decline, we have similary regarded the joint probability density function of the increments as a mixture of conditionally independent exponentials with different means.

With respect to the case study on the Dutch 'Oostmolendijk', we can conclude that the value of the optimal dyke height is sensitive to the rate of crest-level decline (also whether being linear or non-linear in time), but insensitive to the unit time for which the increments are distributed according to a mixture of exponentials. Although a high 'Oostmolendijk' is economically optimal, to cope with a rather extreme crest-level decline, there might be other reasons (e.g. of preserving the landscape or a road connection) to choose the dyke to be lower.

The maintenance models that are presented in this chapter have the following advantages: they enable optimal dyke heightening decisions to be determined under uncertainty, they estimate how much money is needed for the future maintenance of dykes, they do not assume that dykes may rise (as in the case of the Brownian motion with drift model), they are based on random quantities that can be observed (viz. increments of crest-level decline), and they can be expressed in explicit form when the limiting average rate of crest-level decline is given.

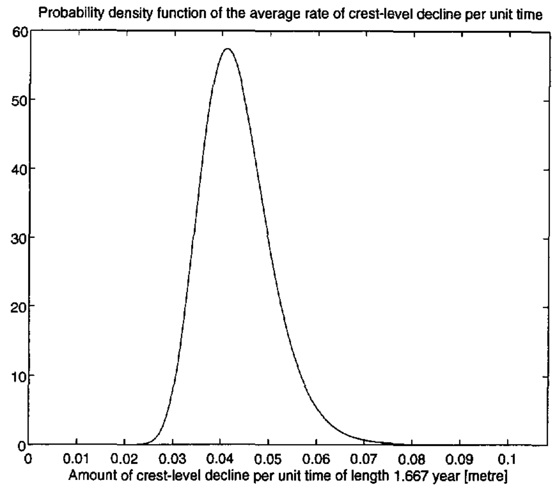

Figure 5.2: The probability density function of the limiting average rate of crest-level decline per unit time, $\mathrm{Ig}(\theta|35.86, 1.511)$, with mean $E(\Theta) = 0.043$.

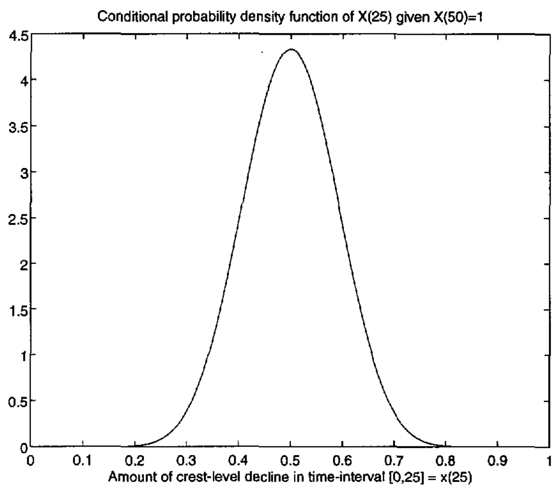

Figure 5.3: The conditional probability density function of the amount of crest-level decline in a period of 25 years, $X(25)$, when $X(50) = 1$ metre.

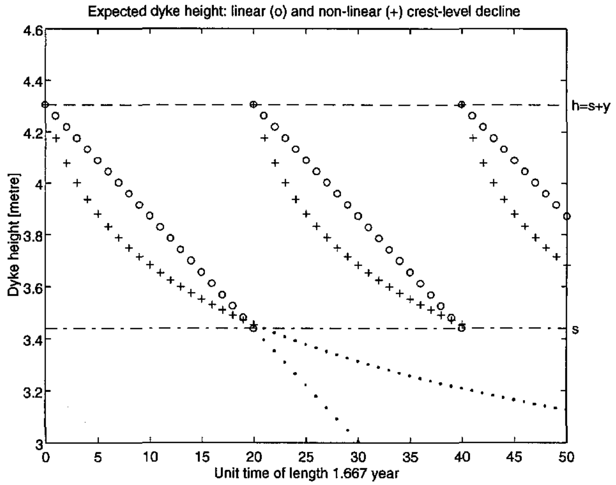

Figure 5.4: The expected dyke height in the event of expected crest-level decline being linear and non-linear in time: each dyke heightening brings the crest level back to $h = 4.30$ metre.

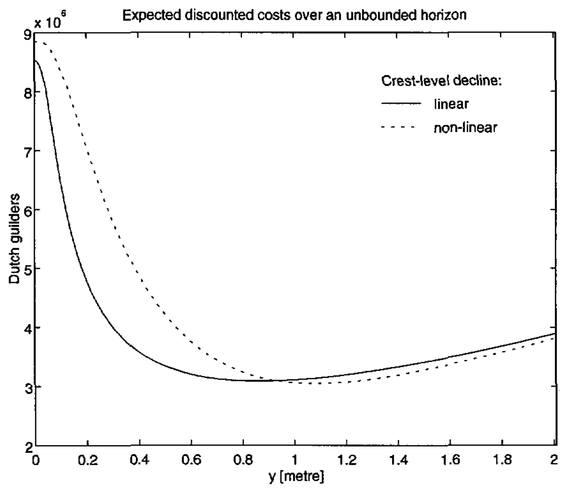

Figure 5.5: The expected discounted costs of dyke heightening over an unbounded time-horizon in the event of expected crest-level decline being linear and non-linear in time for the Dutch 'Oostmolendijk'.

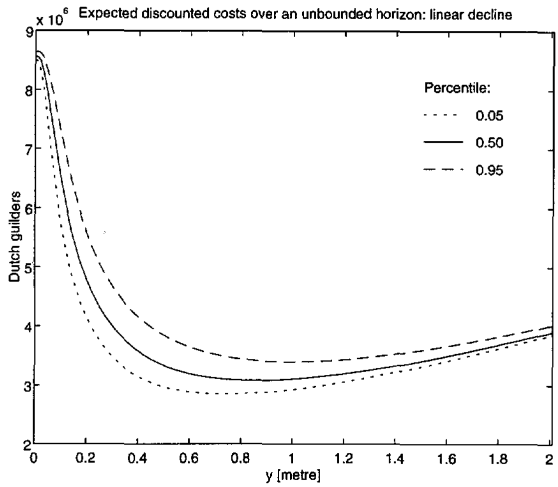

Figure 5.6: The expected discounted costs of dyke heightening over an unbounded time-horizon at the 5%-, 50%-, and 95%-percentile of the limiting average rate of crest-level decline in the event of expected decline being linear in time for the Dutch 'Oostmolendijk'.

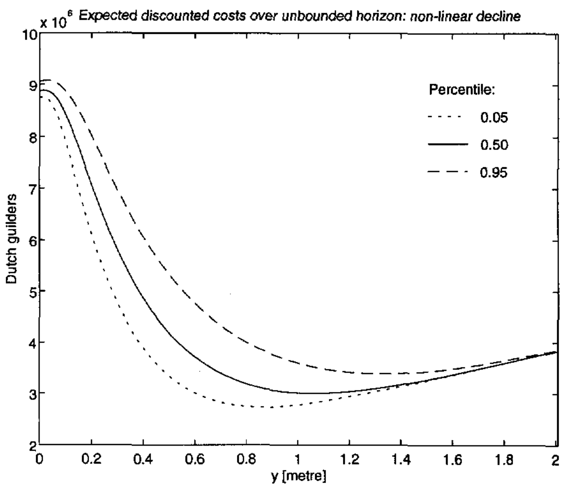

Figure 5.7: The expected discounted costs of dyke heightening over an unbounded time-horizon at the 5%-, 50%-, and 95%-percentile of the limiting average rate of crest-level decline in the event of expected decline being non-linear in time for the Dutch 'Oostmolendijk'.

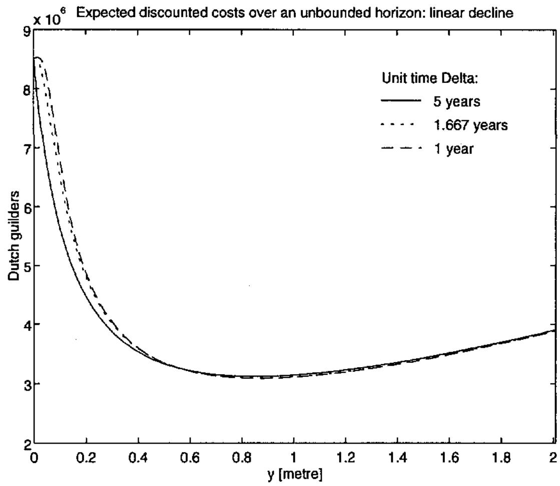

Figure 5.8: The expected discounted costs of dyke heightening over an unbounded time-horizon for different units of time Δ in the event of expected crest-level decline being linear in time.

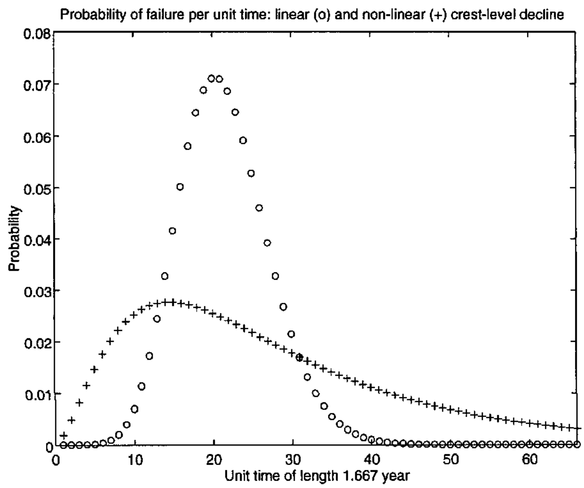

Figure 5.9: The probability of failure per unit time in the event of expected crest-level decline being linear and non-linear in time when $y = 0.86$ m and $h = 4.30$ m at time 0.

5.7 Appendix: Proofs of theorems

Theorem 8 *Suppose the infinite sequence of random quantities* $\{D_i : i \in \mathbb{N}\}$ *is* l_1*-isotropic and* $X_n = \sum_{i=1}^n D_i$ *for all* $n \in \mathbb{N}$*, then*

$$E\left([X_j]^m I_{[0,y]}(X_{n-1}) I_{(y,\infty)}(X_n) \middle| \theta \right) = \tag{5.20}$$
$$= \left\{ \sum_{i=0}^{m} \frac{m!}{(m-i)!} \binom{j-n+i}{j-n} y^{m-i}\theta^i \right\} \times \frac{1}{(n-1)!} \left[\frac{y}{\theta}\right]^{n-1} \exp\left\{-\frac{y}{\theta}\right\},$$

for $j, n = 1, 2, \ldots,$ $j \geq n$, $m = 0, 1, 2, \ldots,$ $y \in (0, \infty)$*, where* $I_A(x) = 1$ *if* $x \in A$ *and* $I_A(x) = 0$ *if* $x \notin A$.

Proof:
Since $X_n = \sum_{i=1}^n D_i$ for all $n \in \mathbb{N}$, it follows that the integration bounds are determined by $X_j \geq X_{j-1} \geq \ldots \geq X_1 \geq 0$. Moreover, $X_{n-1} \leq y$ and $X_n > y$, where $n \leq j$, and the Jacobian equals one. Hence, we may write

$$E\left([X_j]^m I_{[0,y]}(X_{n-1}) I_{(y,\infty)}(X_n) \middle| \theta \right) = \tag{5.21}$$
$$= \int_{x_j=y}^{\infty} \int_{x_{j-1}=y}^{x_j} \cdots \int_{x_n=y}^{x_{n+1}} \int_{x_{n-1}=0}^{y} \cdots \int_{x_2=0}^{x_3} \int_{x_1=0}^{x_2} \frac{x_j^m}{\theta^j} \exp\left\{-\frac{x_j}{\theta}\right\} dx_1 \cdots dx_j.$$

This multiple integral can be solved in the following way. The Dirichlet integral gives

$$\int_{x_{n-1}=0}^{y} \cdots \int_{x_2=0}^{x_3} \int_{x_1=0}^{x_2} 1 \, dx_1 \cdots dx_{n-1} = \frac{y^{n-1}}{(n-1)!} \tag{5.22}$$

and

$$\int_{x_{j-1}=y}^{x_j} \cdots \int_{x_n=y}^{x_{n+1}} 1 \, dx_n \cdots dx_{j-1} = \frac{[x_j - y]^{j-n}}{(j-n)!}. \tag{5.23}$$

By applying the transformation $t = (x_j - y)/\theta$, and using the binomial formula and the gamma function, we obtain

$$\int_{x_j=y}^{\infty} x_j^m [x_j - y]^{j-n} \exp\left\{-\frac{x_j}{\theta}\right\} dx_j =$$
$$= \sum_{i=0}^{m} \binom{m}{i} y^{m-i}\theta^{i+j-n+1}\Gamma(i+j-n+1) \exp\left\{-\frac{y}{\theta}\right\}. \tag{5.24}$$

Finally, combining Eqs. (5.21-5.24) proves the theorem. □

Theorem 9 *Suppose the random quantities* $\{D_i : i \in \mathbb{N}\}$ *are exponentially distributed with mean* $\beta_i\theta$*, where* $\beta_i > 0$ *for* $i = 1, \ldots, n$ *and* $\beta_i \neq \beta_j$ *unless* $i = j$*, and conditionally independent when* $\theta > 0$ *is given. Let* $X_n = \sum_{i=1}^{n} D_i$ *for all* $n \in \mathbb{N}$*, then*

$$\Pr\{X_n \leq x | \theta\} = 1 - \sum_{i=1}^{n} \frac{1}{\prod_{j=1, j\neq i}^{n} [1 - \beta_j/\beta_i]} \exp\left\{-\frac{x}{\beta_i\theta}\right\}. \tag{5.25}$$

This probability distribution is called the general Erlang or general gamma distribution (see e.g. Jensen [68] and McGill & Gibbon [86]).

Proof:
See McGill & Gibbon [86]. □

Theorem 10 *Let the random vector* $(D_1, \ldots, D_N)$ *have a uniform distribution on the hyperplane*

$$\left\{(\delta_1, \cdots, \delta_N) \in \mathbb{R}_+^N : \sum_{i=1}^{N} \frac{\delta_i}{\beta_i} = N\theta\right\}, \tag{5.26}$$

where $\theta > 0$*, and* $\beta_i > 0$ *for* $i = 1, \ldots, n$ *and* $\beta_i \neq \beta_j$ *unless* $i = j$*. Furthermore, for convenience, let* $X_n = \sum_{i=1}^{n} D_i$ *for all* $n \in \mathbb{N}$*, then*

$$\Pr\left\{X_n \leq x \,\middle|\, \frac{1}{N}\sum_{i=1}^{N} \frac{D_i}{\beta_i} = \theta\right\} = 1 - \sum_{i=1}^{n} \frac{1}{\prod_{j=1, j\neq i}^{n} [1 - \beta_j/\beta_i]} \left[1 - \frac{x}{N\beta_i\theta}\right]_+^{N-1} \tag{5.27}$$

for $x \geq 0$ *and zero otherwise,* $n = 1, \ldots, N-1$*, where* $[x]_+ = \max\{0, x\}$*. The probability distribution (5.27) converges uniformly to the probability distribution (5.25), as* $N \to \infty$*.*

Proof:
With the Dirichlet integral, the variables δ_{n+1} through δ_N can be integrated out and we find (see Mendel [87]):

$$l_N(\delta_1, \cdots, \delta_n | \theta) = \left[1 - \sum_{i=1}^{n} \frac{\delta_i}{N\beta_i\theta}\right]_+^{N-n-1} \prod_{i=1}^{n} \frac{(N-i)}{N\beta_i\theta}. \tag{5.28}$$

Eq. (5.27) can now be proved by conditioning on the values of D_n, employing the law of total probability, and using complete induction. Note that the likelihood function (5.28) converges uniformly to the likelihood function in Eq. (5.14), as $N \to \infty$. □

Theorem 11 *Suppose the random quantities $\{D_i : i \in \mathbb{N}\}$ are exponentially distributed with mean $\beta_i\theta$, where $\beta_i \neq \beta_j$ unless $i = j$, and conditionally independent when θ is given. Let $X_n = \sum_{i=1}^n D_i$ for all $n \in \mathbb{N}$, then*

$$E\left([X_j]^m I_{[0,y]}(X_{n-1}) I_{(y,\infty)}(X_n) \Big| \theta\right) = \tag{5.29}$$

$$= \sum_{i=n}^{j} \frac{1}{\prod_{h=n,\, h\neq i}^{j} [1 - \beta_h/\beta_i]} \sum_{k=0}^{m} \binom{m}{k} y^{m-k} (\beta_i\theta)^k \times \Pr\{X_{n-1} \leq y, X_n > y \,|\, \theta\},$$

where

$$\Pr\{X_{n-1} \leq y, X_n > y \,|\, \theta\} = \sum_{i=1}^{n} \frac{\beta_n/\beta_i}{\prod_{h=1,\, h\neq i}^{n} [1 - \beta_h/\beta_i]} \exp\left\{-\frac{y}{\beta_i\theta}\right\}, \tag{5.30}$$

for $j, n = 1, 2, \ldots,$ $j \geq n$, $m = 0, 1, 2, \ldots,$ $y \in (0, \infty)$, where $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$.

Proof:
Since $X_n = \sum_{i=1}^n D_i$ for all $n \in \mathbb{N}$, it follows that the integration bounds are determined by $X_j \geq X_{j-1} \geq \ldots \geq X_1 \geq X_0 = 0$. Moreover, $X_{n-1} \leq y$ and $X_n > y$, where $n \leq j$, and the Jacobian equals one. By using

$$\sum_{i=1}^{j} \frac{D_i}{\beta_i} = \frac{X_j}{\beta_j} + \sum_{i=1}^{j-1} \left[\frac{1}{\beta_i} - \frac{1}{\beta_{i+1}}\right] X_i, \tag{5.31}$$

we may write

$$E\left([X_j]^m I_{[0,y]}(X_{n-1}) I_{(y,\infty)}(X_n) \Big| \theta\right) =$$

$$= \int_{x_j=y}^{\infty} \int_{x_{j-1}=y}^{x_j} \cdots \int_{x_n=y}^{x_{n+1}} \int_{x_{n-1}=0}^{y} \cdots \int_{x_2=0}^{x_3} \int_{x_1=0}^{x_2} \frac{x_j^m}{\beta_j\theta} \exp\left\{-\frac{x_j}{\beta_j\theta}\right\} \times$$

$$\times \prod_{i=1}^{j-1} \frac{1}{\beta_i\theta} \exp\left\{-\left[\frac{1}{\beta_i} - \frac{1}{\beta_{i+1}}\right] \frac{x_i}{\theta}\right\} dx_1 \cdots dx_j. \tag{5.32}$$

This multiple integral can be solved in four steps.
First, by applying the transformation $\delta_i = x_i - x_{i-1}$, $i = 1, \ldots, n-1$, and subsequently using Eq. (5.31) and Theorem 9, we find

$$\int_{x_{n-1}=0}^{y} \cdots \int_{x_2=0}^{x_3} \int_{x_1=0}^{x_2} \prod_{i=1}^{n-1} \frac{1}{\beta_i\theta} \exp\left\{-\left[\frac{1}{\beta_i} - \frac{1}{\beta_{i+1}}\right] \frac{x_i}{\theta}\right\} dx_1 \cdots dx_{n-1} =$$

$$= \int_{\delta_1=0}^{y} \cdots \int_{\delta_{n-1}=0}^{y-\sum_{h=1}^{n-2}\delta_h} \prod_{i=1}^{n-1} \frac{1}{\beta_i\theta} \exp\left\{-\left[\frac{1}{\beta_i} - \frac{1}{\beta_n}\right] \frac{\delta_i}{\theta}\right\} d\delta_{n-1} \cdots d\delta_1$$

$$= \left(\prod_{i=1}^{n-1} \frac{1}{[1-\beta_i/\beta_n]}\right) \cdot \left(1 - \sum_{i=1}^{n-1} \prod_{h=1,\, h\neq i}^{n-1} \left[\frac{1-\beta_h/\beta_n}{1-\beta_h/\beta_i}\right] \exp\left\{-\left[\frac{1}{\beta_i} - \frac{1}{\beta_n}\right]\frac{y}{\theta}\right\}\right)$$

$$= \sum_{i=1}^{n} \frac{\beta_n/\beta_i}{\prod_{h=1,\, h\neq i}^{n}[1-\beta_h/\beta_i]} \exp\left\{-\left[\frac{1}{\beta_i} - \frac{1}{\beta_n}\right]\frac{y}{\theta}\right\}. \tag{5.33}$$

Second, by applying the transformation $z_i = x_i - y$, $i = n, \ldots, j-1$, and using Eq. (5.33), we obtain

$$\int_{x_{j-1}=y}^{x_j} \cdots \int_{x_n=y}^{x_{n+1}} \prod_{i=n}^{j-1} \frac{1}{\beta_i\theta} \exp\left\{-\left[\frac{1}{\beta_i} - \frac{1}{\beta_{i+1}}\right]\frac{x_i}{\theta}\right\} dx_n \cdots dx_{j-1} = \tag{5.34}$$

$$= \int_{z_{j-1}=0}^{z_j} \cdots \int_{z_n=0}^{z_{n+1}} \prod_{i=n}^{j-1} \frac{1}{\beta_i\theta} \exp\left\{-\left[\frac{1}{\beta_i} - \frac{1}{\beta_{i+1}}\right]\frac{z_i+y}{\theta}\right\} dz_n \cdots dz_{j-1}$$

$$= \sum_{i=n}^{j} \frac{\beta_j/\beta_i}{\prod_{h=n,\, h\neq i}^{j}[1-\beta_h/\beta_i]} \exp\left\{-\left[\frac{1}{\beta_i} - \frac{1}{\beta_j}\right]\frac{z_j}{\theta}\right\} \exp\left\{-\left[\frac{1}{\beta_n} - \frac{1}{\beta_j}\right]\frac{y}{\theta}\right\}$$

$$= \sum_{i=n}^{j} \frac{\beta_j/\beta_i}{\prod_{h=n,\, h\neq i}^{j}[1-\beta_h/\beta_i]} \exp\left\{-\left[\frac{1}{\beta_i} - \frac{1}{\beta_j}\right]\frac{x_j}{\theta}\right\} \exp\left\{-\left[\frac{1}{\beta_n} - \frac{1}{\beta_i}\right]\frac{y}{\theta}\right\}.$$

Third, by applying the transformation $t_i = (x_i - y)/(\beta_i\theta)$, $i = n, \ldots, j$, and using the binomial formula and the gamma function, the one-dimensional integral over x_j can be written as

$$\sum_{i=n}^{j} \frac{\exp\{-y/(\beta_n\theta)\}}{\prod_{h=n,\, h\neq i}^{j}[1-\beta_h/\beta_i]} \int_{x_j=y}^{\infty} \frac{x_j^m}{\beta_i\theta} \exp\left\{-\frac{x_j-y}{\beta_i\theta}\right\} dx_j =$$

$$= \sum_{i=n}^{j} \frac{\exp\{-y/(\beta_n\theta)\}}{\prod_{h=n,\, h\neq i}^{j}[1-\beta_h/\beta_i]} \sum_{k=0}^{m} \binom{m}{k} y^{m-k}(\beta_i\theta)^k. \tag{5.35}$$

Finally, combining Eqs. (5.32-5.35) proves the theorem. □

Theorem 12 *Let $\beta_i > 0$ for $i = 1, \ldots, n$ and $\beta_i \neq \beta_j$ unless $i = j$, $n \in \mathbb{N}$, then*

$$\sum_{i=n}^{j} \frac{\beta_i}{\prod_{h=n,\, h\neq i}^{j} [1 - \beta_h/\beta_i]} = \sum_{i=n}^{j} \beta_i. \tag{5.36}$$

Proof:
Suppose that $D_n, \ldots, D_j$ are independent random quantities and that D_i has an exponential distribution with mean $\beta_i\theta$, $i = n, \ldots, j$. On the one hand, we have simply $\sum_{i=n}^{j} E(D_i|\theta) = \sum_{i=n}^{j} \beta_i\theta$. On the other hand, we can use Eq. (5.25) to write

$$E\left(\sum_{i=n}^{j} D_i \Big| \theta\right) = \int_{x=0}^{\infty} \Pr\left\{\sum_{i=n}^{j} D_i > x \Big| \theta\right\} dx = \sum_{i=n}^{j} \frac{\beta_i\theta}{\prod_{h=n,\, h\neq i}^{j} [1 - \beta_h/\beta_i]}.$$

□

Theorem 13 *Let θ_i be such that $\int_0^{\theta_i} \mathrm{Ig}(\theta|\nu,\mu)\, d\theta = \epsilon_i$, $i = 1, 2$, where $0 < \theta_1 < \theta_2$. Then, approximations to ν and μ can be obtained by solving the following equations:*

$$\begin{aligned} 0 &= 9(t-1)\nu + 3(u_{\epsilon_1} - u_{\epsilon_2} t)\sqrt{\nu} - (t-1), \\ \mu &= \nu \left[u_{\epsilon_i}\sqrt{\tfrac{1}{9\nu}} + 1 - \tfrac{1}{9\nu}\right]^3 \theta_i, \end{aligned}$$

where $t = (\theta_2/\theta_1)^{1/3}$ and $\int_{-\infty}^{u_\epsilon} (2\pi)^{-1/2} \exp\{-u^2/2\}\, du = \epsilon$. The so-obtained approximate values for ν and μ are accurate unless ν is small, or at least one ϵ_i is near to zero or one.

Proof:
The approximations to ν and μ are based on a well-known approximation to the percentile points of the chi-square distribution (see Johnson & Kotz [69, page 176]). □

Chapter 6

Optimal Maintenance Decisions for Berm Breakwaters

Jan M. van Noortwijk and Pieter H.A.J.M. van Gelder

Abstract. To prevent coastal lines of defence from being affected by severe hydraulic loadings from the sea, berm breakwaters can be used. Although berm breakwaters are dynamically stable in the sense that they allow for some rock displacement, they can fail due to severe longshore rock transport. To avoid this type of failure, berm breakwaters have to be inspected and, if necessary, have to be repaired. A decision model is presented enabling cost-optimal maintenance decisions to be determined while taking account of the (possibly large) uncertainties in: (i) the limiting average rate of occurrence of breaches in the armour layer and (ii), given a breach has occurred, the limiting average rate of longshore rock transport. The stochastic process of rock displacement is modelled by a modified generalised gamma process, enabling us to explicitly take account of the uncertainty in these limiting averages. (This chapter has previously been published as [132].)

Keywords. maintenance, gamma processes, berm breakwaters, decision theory, renewal theory.

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6.1 Introduction

In order to protect coastal lines of defence from being damaged by severe hydraulic loadings from the sea, berm breakwaters can be used (see e.g. Van der Meer [120], Van der Meer & Veldman [121], and Vrijling & Van Gelder [135]). The characteristic feature of berm breakwaters is that the original dynamically stable profile becomes statically stable under certain wave conditions. However, oblique wave attack can initiate longshore transport of stones along the center line of the berm breakwater. To avoid failure due to severe longshore rock transport, berm breakwaters have to be inspected and, if necessary, have to be repaired. The purpose of this chapter is to develop a decision model for obtaining optimal inspection intervals whose expected maintenance costs are minimal.

Although most maintenance optimisation models are based on lifetime distributions or Markovian deterioration models, it is often hard to gather data for estimating their parameters. Moreover, in case of well-planned preventive maintenance, complete lifetimes will be observed rarely. In practice, there is often only (subjective) information available on limiting average rates of deterioration: for berm breakwaters, the stochastic deterioration process can be characterised by (i) the limiting average rate of occurrence of breaches in the armour layer and (ii), given a breach has occurred, the limiting average rate of longshore rock transport. In order that the processes of the occurrence of breaches and of longshore rock transport be based on their limiting average rates, they have been regarded as a mixture of geometrics and a scale mixture of gamma's (a generalised gamma process), respectively.

According to Chapter 4, three cost-based criteria can be used to compare maintenance decisions over an unbounded time-horizon: (i) the expected average costs per unit time, (ii) the expected discounted costs over an unbounded horizon, and (iii) the expected equivalent average costs per unit time. These costs can be determined with the aid of renewal theory, where the renewals are either preventive repairs (before failure) or corrective repairs (after failure).

The maintenance decision model which is proposed extends the model of Vrijling & Van Gelder [135] in the sense that the renewals not only take place at fixed preventive repair intervals but also upon failure, the costs are determined with respect to an unbounded horizon, and the dependence between the probability of preventive repair and the expected repair costs is taken into account. The parameters of the uncertainty distributions of the above two limiting averages are assessed using the simulation results of Vrijling & Van Gelder [135].

In The Netherlands, generalised gamma processes have also been used to model decision problems for optimising maintenance of the sea-bed protection of the Eastern-Scheldt barrier, beaches, and dykes (see Chapters 2 and 7, 4, and 5, respectively). Note that the developed decision model can be viewed as a delay-time model as studied by Christer & Waller [14, 15]. In fact, the time lapse from an armour breach until the time of failure due to longshore rock transport can be interpreted as the discrete delay time of a failure. Therefore, generalised gamma processes can be employed to assess delay-time distributions based on the physics of deterioration.

The chapter is organised as follows. A brief description on berm breakwaters is given in Sec. 6.2. In Sec. 6.3, we model the event of failure due to severe longshore rock transport. We present the maintenance decision model for minimising the costs of inspection, repair, and failure in Sec. 6.4. Some necessary definitions and theorems are presented in two appendices.

6.2 Berm breakwaters

The main function of breakwaters is to prevent coastal lines of defence (e.g. sand dunes or cliffs) from being affected by severe hydraulic loadings from the sea. Recently, the attention has been shifted from statically stable rubble-mount breakwaters to dynamically stable berm breakwaters (see e.g. Van der Meer [120], Van der Meer & Veldman [121], and Vrijling & Van Gelder [135]). The profiles of statically stable structures are not permitted to change under severe wave conditions, whereas the profiles of dynamically stable structures (such as berm breakwaters and beaches of sand, gravel, shingle, and rock) may change according to the wave climate.

The main components of a berm breakwater are the core (with stones of diameter 0.5 m) and the armour layer (with stones of diameter 0.8 m) (see Fig. 6.1). A berm breakwater is said to be dynamically stable when the net cross-shore transport of stones is zero and its profile has reached an equilibrium under certain wave conditions. In fact, the originally built profile becomes dynamically stable when wave attack moves rock in the berm partly upward to the crest and partly downward to the toe and the sand; thus reshaping the seaward slope into a (more) statically stable S-shape profile (see Fig. 6.1).

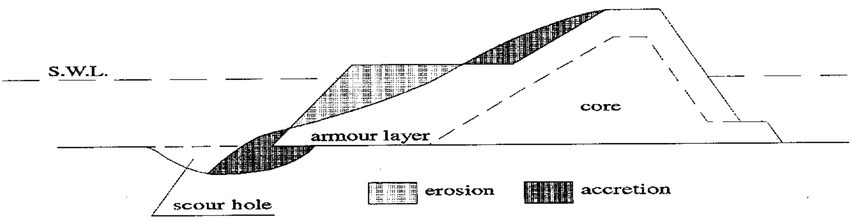

Figure 6.1: The cross-section of a berm breakwater: the originally built (dynamically stable) profile and the S-shape (statically stable) profile. S.W.L. means still-water level: the surface of the water if all wave and wind action were to cease.

A berm breakwater can fail due to longshore transport of the armour elements, which occurs when oblique wave attack results in wave forces parallel to the alignment of the structure. We define a failure as the event at which the stones of the armour layer have been displaced to such a degree that the core is instable and needs to be reconstructed. To reveal possible longshore transport of stones, berm breakwaters have to be inspected and, if necessary, have to be repaired. In this chapter, we develop a decision model that enables cost-optimal maintenance decisions to be determined while taking account of the main uncertainties in the stochastic process of rock displacement.

6.3 The stochastic process of rock displacement

In modelling the maintenance of berm breakwaters on the basis of the stochastic process of rock displacement, there are mainly two uncertainties involved (see also Fig. 6.2): (i) the probability of an initial breach of the armour layer and (ii), given a breach has occurred, the limiting average rate of longshore rock transport. In fact, an armour breach initiates longshore rock transport. Next, we study these two deterioration characteristics, which are judged to be independent, in more detail.

6.3.1 The stochastic process of longshore rock transport

In this subsection, we derive a probabilistic model for the process of longshore rock transport based on the unknown limiting average rate. Let us denote the non-decreasing stochastic process of longshore rock transport by $\{X(t) : t \geq 0\}$, where $X(t)$ represents the cumulative amount of transported rock at time t and $X(0) = 0$ with probability one. For every uniform time-partition in time-intervals of length $\tau > 0$, we write $D_i(\tau) = X(i\tau) - X([i-1]\tau)$, $i \in \mathbb{N}$. Furthermore, due to the lack of data, we judge the infinite sequence of increments $\{D_i(\tau) : i \in \mathbb{N}\}$ to be *exchangeable*, i.e. the order in which the increments occur is irrelevant. In mathematical terms, this can be interpreted as that the probability density function of the random vector $(D_1(\tau), \ldots, D_n(\tau))$ is invariant under all $n!$ permutations of the coordinates, i.e.

$$p_{D_1(\tau),\ldots,D_n(\tau)}\left(\delta_1, \ldots, \delta_n\right) = p_{D_1(\tau),\ldots,D_n(\tau)}\left(\delta_{\pi(1)}, \ldots, \delta_{\pi(n)}\right), \tag{6.1}$$

where π is any permutation of $1, \ldots, n$, for all $n \in \mathbb{N}$ and $\tau > 0$.

In order that a stochastic deterioration process with non-negative exchangeable increments be based on the unknown limiting average rate, we have argued in Chapter 3 that it can best be regarded as a *generalised gamma process*. For this process, the joint probability density function of the increments $D_1(\tau), \ldots, D_n(\tau)$ is given by a mixture of conditionally independent gamma densities:

$$p_{D_1(\tau),\ldots,D_n(\tau)}\left(\delta_1, \ldots, \delta_n\right) = \int_0^\infty \prod_{i=1}^n \frac{\delta_i^{a\tau-1}}{\Gamma(a\tau)} \left[\frac{a\tau}{\theta}\right]^{a\tau} \exp\left\{-\frac{a\tau\delta_i}{\theta}\right\} dP_{\Theta(\tau)}(\theta) \tag{6.2}$$

for some constant $a > 0$ with

$$\begin{aligned} E(X(n\tau)) &= E(n\Theta(\tau)), \\ \mathrm{Var}(X(n\tau)) &= \left[1 + \frac{1}{na\tau}\right] E([n\Theta(\tau)]^2) - [E(n\Theta(\tau))]^2 \end{aligned} \tag{6.3}$$

for all $\tau > 0$, provided the first and the second moment of the probability distribution of $\Theta(\tau)$ exist. By the strong law of large numbers for exchangeable random quantities, the probability distribution $P_{\Theta(\tau)}$ on the random quantity $\Theta(\tau)$ represents the uncertainty in the unknown limiting average amount of longshore rock transport per time-interval of length τ: $\lim_{n\to\infty}[(\sum_{i=1}^n D_i(\tau))/n]$.

A useful property of the generalised gamma process is that the mixture of gamma's in Eq. (6.2) transforms into a mixture of exponentials if $\tau = a^{-1}$:

$$p_{D_1(a^{-1}),\ldots,D_n(a^{-1})}(\delta_1,\ldots,\delta_n) = \int_0^\infty \prod_{i=1}^n \frac{1}{\theta}\exp\left\{-\frac{\delta_i}{\theta}\right\} dP_\Theta(\theta) = f_n(\textstyle\sum_{i=1}^n \delta_i) \tag{6.4}$$

for $(\delta_1,\ldots,\delta_n) \in \mathbb{R}_+^n$ and zero otherwise, where $\mathbb{R}_+ = [0,\infty)$. The infinite sequence of random quantities $\{D_i(a^{-1}) : i \in \mathbb{N}\}$ is said to be l_1-isotropic (or l_1-norm symmetric), since its distribution can be written as a function of the l_1-norm. The unit time for which the increments of longshore rock transport are l_1-isotropic can be obtained, amongst others, by specifying the variance of the generalised gamma process in Eq. (6.3). For fixed $\tau > 0$, the smaller the unit-time length for which the increments are l_1-isotropic, i.e. the smaller $\Delta = a^{-1}$, the more deterministic the deterioration process. As we shall see in Sec. 6.4, for this unit-time length, denoted by $\Delta = a^{-1}$, many probabilistic properties of the stochastic process, like the probability of exceedence of a failure level, can be expressed in explicit form conditional on the limiting average.

In conclusion, we advocate regarding the stochastic process of longshore rock transport as a generalised gamma process with probability distribution on the limiting average rate of longshore rock transport. To keep the mathematics of the decision model tractable, we impose the property of *posterior linearity* introduced by Diaconis & Ylvisaker [43], i.e. $E(X(2\tau)| D_1(\tau) = \delta_1) = c_1\delta_1 + c_2$ for some constants $c_1, c_2 > 0$ and $\tau > 0$. If this property holds, then the mixing distribution in Eq. (6.2) is an inverted gamma distribution.

From now on, we consider increments of longshore rock transport that are l_1-isotropic with respect to the units of time $\{([i-1]\Delta, i\Delta] : i \in \mathbb{N}\}$. For notational convenience, let $D_i = D_i(\Delta)$, $X_n = \sum_{i=1}^n D_i$ for all $n \in \mathbb{N}$, and let Θ represent the uncertainty in the limiting average rate of longshore rock transport, $\lim_{n\to\infty}[(\sum_{i=1}^n D_i)/n]$, with probability density function $\mathrm{Ig}(\theta|\nu,\mu)$ (for the definition of the inverted gamma distribution, see Appendix A).

6.3.2 The stochastic process of armour breaches

Beside the uncertainty in the limiting average rate of longshore rock transport, the process of the occurrence of breaches in the armour layer must be specified. Let us denote the times at which armour breaches occur by the infinite sequence of non-negative discrete random quantities $T_1, T_2, \ldots$ (which are defined with respect to units of time of length Δ). Since rocks start to move due to external causes (i.e. severe waves), at a rate that does not change in time, we may judge the random quantities $T_1, T_2, \ldots$ to be exchangeable and to exhibit the "lack of memory" property. The "lack of memory" property means that the discrete probability function of the remaining time until the occurrence of the first armour breach does not depend on the fact that no breach has occurred yet.

Under the above two assumptions, we can write the joint probability function of $T_1, \ldots, T_n$ as a mixture of conditionally independent geometric distributions (see Diaconis & Freedman [42]):

$$\Pr\{T_1 = t_1, \ldots, T_n = t_n\} = \int_0^1 \prod_{i=1}^{n} \phi(1-\phi)^{t_i} \, d\Pr\{\Phi \leq \phi\} \tag{6.5}$$

for $t_i = 0, 1, 2, \ldots$, $i = 1, \ldots, n$, and zero otherwise. The conditional mean can be written as $E(T_i|\phi) = \phi^{-1} - 1$, $i = 1, \ldots, n$. By the strong law of large numbers for exchangeable random quantities, the random quantity Φ may be interpreted as the limiting relative frequency of the times at which initial armour breaches occur per unit time of length Δ, i.e. $\lim_{n\to\infty}[n/(n + \sum_{i=1}^{n} T_i)]$. If, in addition, the property of posterior linearity holds then the mixing distribution in Eq. (6.5) is a beta distribution, say $\mathrm{Be}(\phi|a, b)$ (see Diaconis & Ylvisaker [43]).

6.4 Optimal maintenance decisions

6.4.1 The maintenance model

To obtain optimal maintenance decisions under uncertainty, we can use statistical decision theory (see DeGroot [27, Ch. 8]). Let a berm breakwater be inspected at times $\{jk\Delta : j \in \mathbb{N}\}$ for fixed $k \in \mathbb{N}$. Let $L(\phi, \theta, k)$ be the (monetary) *loss* when the decision-maker chooses inspection interval k, the limiting relative frequency of armour breaches is ϕ, and the limiting average rate of longshore rock transport is θ. The decision-maker can best choose an inspection interval k^* whose expected loss is minimal. A decision k^* is called an *optimal decision* when

$$E\left(L(\Phi, \Theta, k^*)\right) = \min_{k \in 1,2,\ldots} E\left(L(\Phi, \Theta, k)\right). \tag{6.6}$$

The resistance of the berm breakwater, denoted by R, is defined as the number of stones belonging to the armour layer, where r_0 is the resistance of the originally built profile. When an initial breach of the armour layer occurs at time $(i-1)\Delta$, i.e. when $T_1 = i-1$, the longshore rock transport at the inspection time $k\Delta$, $i \le k$, is represented by the random quantity X_{k-i+1}, where $i, k = 1, 2, \ldots$ and $X_n \sim \text{Ga}(n, 1/\theta)$ for all $n \in \mathbb{N}$ (for the definition of the gamma distribution, see Appendix A). The expected longshore rock transport at time $k\Delta$, given ϕ and θ, can be obtained by conditioning on the possible values of T_1 and using Eqs. (6.4), (6.5), and (6.28):

$$\begin{aligned} &\textstyle\sum_{i=1}^{k} \Pr\{T_1 = i-1 \,|\, \phi\} E\left(X_{k-i+1} \,|\, T_1 = i-1, \theta\right) = \\ &= \textstyle\sum_{i=1}^{k} \phi(1-\phi)^{i-1}[k-(i-1)]\,\theta = \xi_{k,k}(\phi)\,\theta. \end{aligned} \tag{6.7}$$

Since repairs bring the berm breakwater into the "as good as new state", we may regard the maintenance process as a renewal process (see Fig. 6.2). Each renewal cycle ends either upon a failure or at an inspection time $jk\Delta$ when the inspection reveals that a preventive repair should be carried out (for some $j \in \mathbb{N}$). A failure is defined as the event in which the resistance R drops below the failure level s: $R < s$. A preventive repair is defined as the event at which inspection reveals that longshore rock transport has taken place but no failure has occurred: $s \le R < r_0$. Inspection takes a negligible amount of time, does not degrade the berm breakwater, and entails fixed costs c_I. The costs of failure are c_F (costs of reconstructing the core and of possible damage to the coastline), while the costs of repair consist of the fixed costs c_P (costs of mobilisation) and the variable costs c_V (costs per rock). Let the renewal times be conditionally independent random quantities having a discrete probability function $p_i(\phi, \theta, k)$, $i \in \mathbb{N}$, when the limiting average rates are ϕ and θ, and the decision-maker chooses inspection decision k. The costs associated with a renewal at time $i\Delta$ are denoted by $c_i(\phi, \theta, k)$, $i \in \mathbb{N}$.

Since berm breakwaters are planned to function for a very long time, maintenance decisions can best be compared over an unbounded time-horizon. According to Chapter 4, there are basically three cost-based criteria that can serve as loss functions in Eq. (6.6): (i) the expected average costs per unit time, (ii) the expected discounted costs over an unbounded horizon, and (iii) the expected equivalent average costs per unit time. These cost-based criteria can be obtained using the discrete renewal theorem (see Feller [47, Ch. 13] and Karlin & Taylor [71, Ch. 3]).

First, the expected *average costs per unit time* are determined by averaging the expected costs over an unbounded horizon:

$$L(\phi, \theta, k) = \lim_{n \to \infty} \frac{C(n, \phi, \theta, k)}{n} = \frac{\sum_{i=1}^{\infty} c_i(\phi, \theta, k) p_i(\phi, \theta, k)}{\sum_{i=1}^{\infty} i p_i(\phi, \theta, k)}, \tag{6.8}$$

where $C(n, \theta, k)$ are the expected costs in time-interval $(0, n\Delta]$. Eq. (6.8) is a well-known result from renewal reward theory (see e.g. Ross [105]).

Second, the expected *discounted costs over an unbounded horizon* are determined by summing the expected discounted values of the costs over an unbounded horizon,

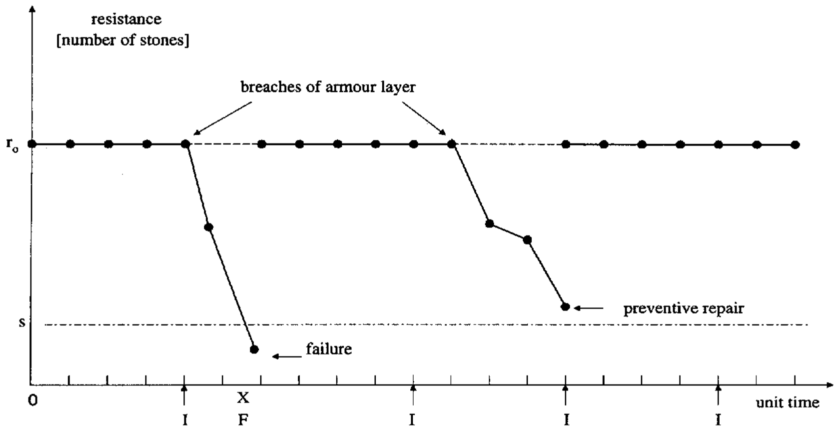

Figure 6.2: The deterioration process of a berm breakwater regarded as a renewal process: each renewal cycle ends either upon an inspection (I), revealing that a preventive repair should be carried out, or upon a failure (F). The inspection interval is taken to be $k = 4$.

where the discounted value of the costs c_n in unit time n is defined to be $\alpha^n c_n$ with discount factor $\alpha = [1 + (r/100)]^{-1}$ and discount rate $r\%$ $(r > 0)$:

$$L_\alpha(\phi, \theta, k) = \lim_{n\to\infty} C_\alpha(n, \phi, \theta, k) = \frac{\sum_{i=1}^{\infty} \alpha^i c_i(\phi, \theta, k) p_i(\phi, \theta, k)}{1 - \sum_{i=1}^{\infty} \alpha^i p_i(\phi, \theta, k)}, \tag{6.9}$$

where $C_\alpha(n, \phi, \theta, k)$ are the expected discounted costs in time-interval $(0, n\Delta]$.

Third, the expected *equivalent average costs per unit time* are determined by averaging the discounted costs. The notion of equivalent average costs relates the notions of average costs and discounted costs in the sense that the equivalent average costs per unit time approach the average costs per unit time, as α tends to 1, from below:

$$\lim_{\alpha\uparrow 1} (1 - \alpha) L_\alpha(\phi, \theta, k) = L(\phi, \theta, k). \tag{6.10}$$

Before deriving the above cost-based criteria, we need to express the failure probability of a berm breakwater in terms of the limiting averages ϕ and θ. By summing over the possible values of the first time at which an armour breach occurs, T_1, the probability of failure due to longshore rock transport in unit time i becomes

$$\begin{aligned} v_i(\phi, \theta) &= \Pr\{\text{failure in } ([i-1]\Delta, i\Delta] \mid \phi, \theta\} \\ &= \sum_{h=1}^{i} \Pr\{T_1 = i - h \mid \phi\} \Pr\{X_{h-1} \leq y, X_h > y \mid \theta\} \end{aligned} \tag{6.11}$$

for $i = 1, 2, \ldots$ and $y = r_0 - s$, where

$$\Pr\{X_{h-1} \leq y, X_h > y \mid \theta\} = \frac{1}{(h-1)!}\left[\frac{y}{\theta}\right]^{h-1} \exp\left\{-\frac{y}{\theta}\right\} = q_h(\theta), \tag{6.12}$$

$h = 1, 2, \ldots$, is a Poisson distribution (see Chapter 2). Using Eq. (6.5), the probability of failure in unit time i can be rewritten as a recursive formula in the following way:

$$v_i(\phi, \theta) = \sum_{h=1}^{i} \phi(1-\phi)^{i-h} q_h(\theta) = (1-\phi)\, v_{i-1}(\phi, \theta) + \phi\, q_i(\theta), \tag{6.13}$$

$i = 1, 2, \ldots$ and $v_0(\phi, \theta) = 0$, with mean lifetime and variance given by

$$\begin{aligned} E(T_1|\phi) + E(H|\theta) &= (1-\phi)/\phi + 1 + (y/\theta), \\ \mathrm{Var}(T_1|\phi) + \mathrm{Var}(H|\theta) &= (1-\phi)/\phi^2 + (y/\theta), \end{aligned} \tag{6.14}$$

where T_1 has a geometric distribution with parameter $0 < \phi < 1$ and H has a Poisson distribution with parameter $y/\theta > 0$ (see Eqs. (6.5) and (6.12), respectively). By Eqs. (6.23-6.29) of Appendix 6.7, the expected average costs per unit time, Eq. (6.8), are

$$\begin{aligned} L(\phi, \theta, k) &= \frac{c_I + c_P\left[1 - (1-\phi)^k\right] + \sum_{i=1}^{k}\left[c_F - (c_I + c_P)\right] v_i(\phi, \theta)}{k + \sum_{i=1}^{k}[i-k]\, v_i(\phi, \theta)} + \\ &\quad \frac{c_V\left[\xi_{k,k}(\phi) - \sum_{h=1}^{k+1} \xi_{k-h+2,k}(\phi) q_h(\theta)\right]\theta}{k + \sum_{i=1}^{k}[i-k]\, v_i(\phi, \theta)}, \end{aligned} \tag{6.15}$$

whereas the expected discounted costs over an unbounded horizon, Eq. (6.9), are

$$\begin{aligned} L_\alpha(\phi, \theta, k) &= \\ &= \frac{\alpha^k\left\{c_I + c_P\left[1 - (1-\phi)^k\right]\right\} + \sum_{i=1}^{k}\left[\alpha^i c_F - \alpha^k(c_I + c_P)\right] v_i(\phi, \theta)}{[1-\alpha^k] - \sum_{i=1}^{k}[\alpha^i - \alpha^k]\, v_i(\phi, \theta)} + \\ &\quad \frac{\alpha^k c_V\left[\xi_{k,k}(\phi) - \sum_{h=1}^{k+1} \xi_{k-h+2,k}(\phi) q_h(\theta)\right]\theta}{[1-\alpha^k] - \sum_{i=1}^{k}[\alpha^i - \alpha^k]\, v_i(\phi, \theta)}. \end{aligned} \tag{6.16}$$

In conclusion, we recommend choosing an optimal inspection interval k^* for which the expected average costs per unit time, $E(L(\Phi, \Theta, k^*))$, or the expected equivalent average costs per unit time, $E((1-\alpha)L_\alpha(\Phi, \Theta, k^*))$, are minimal. The choice for one or the other depends on the application and no general recommendation can be given. The expected costs are taken with respect to the probability distributions of the unknown random quantities Φ and Θ which are judged to be independent. As α tends to 1, from below, the expected equivalent average costs approach the expected average costs (by Lebesgue's Theorem of Dominated Convergence, we may interchange the order of the operations of expectation and passing to the limit).

6.4.2 Numerical results

Next, we apply the above maintenance model to the data obtained by Vrijling & Van Gelder [135] (see Table 6.1). We consider a hypothetical harbour (in India) which is protected by a berm breakwater, and focus on one breakwater section with an armour layer having a rock volume of about 2500 stones. The costs of failure not only consist of the costs of reconstructing the berm breakwater, but also of possible damage due to wave disturbance in the harbour basin and of resulting downtime in ship handling.

The unit time for which the increments of longshore rock transport are distributed as mixtures of exponentials ($\Delta = 1$ year) and the parameters of the probability distributions on the limiting averages rates Φ and Θ have been assessed such that they fit the data of Vrijling & Van Gelder [135] (see Table 6.1). In doing so, the probabilities of no armour breach per unit time, i.e. $E([1-\Phi]^i)$, $i = 1, \ldots, 50$, are given by Fig. 6.3 and the probability of failure in time-interval $(0, 50]$ has the value 0.22. When using the parameters of Table 6.1 and applying Monte Carlo integration (number of samples: 10,000), the average costs per year and the equivalent average costs per year are represented by the curves in Fig. 6.4. The optimal decision with respect to the criterion of average costs is $k^* = 4$ with expected average costs per unit time of 6371 Dfl, whereas the optimal decision with respect to the criterion of equivalent average costs is $k_\alpha^* = 5$ with expected equivalent average costs per unit time of 5676 Dfl.

In Table 6.2, the optimal inspection intervals are presented for different costs of failure (for a discussion on determining these costs of failure, see Hauer et al. [59]): the higher the costs of failure, the smaller the optimal inspection interval. Also, we have investigated the sensitivity of the optimum to the variances of the probability distributions of the limiting averages rates Φ and Θ (while keeping the means unchanged). In Figs. 6.5 and 6.6, the expected (equivalent) average costs per unit time are shown for, respectively, $\mathrm{Var}(\Phi)$ and $\mathrm{Var}(\Theta)$ having a value one thousand times smaller than the ones in Table 6.1. It can be concluded that the set of (nearly) optimal decisions is more sensitive to the uncertainty in the limiting average rate of longshore rock transport than to the uncertainty in the limiting average rate of occurrence of breaches in the armour layer. All in all, the larger the uncertainty in the stochastic process of rock displacement, the smaller the optimal inspection interval.

6.5 Conclusions

In this chapter, we have presented a decision model which enables the decision-maker to optimise maintenance of berm breakwaters. The model has been derived on the basis of the probability distributions of (i) the limiting average rate of occurrence of breaches in the armour layer and (ii), given a breach has occurred, the limiting average rate of longshore rock transport. As decision criteria, we have used the expected average costs per unit time (no discounting) and the expected equivalent average costs per unit

Table 6.1: The parameters of the maintenance model for the berm breakwater.

parameter	description	value	dimension
Δ	unit time	1	year
Θ	average rate longshore rock transport	$(0, \infty)$	stones/unit time
ν	shape parameter of $\text{Ig}(\theta\|\nu, \mu)$	2.13	
μ	scale parameter of $\text{Ig}(\theta\|\nu, \mu)$	55.36	
$E(\Theta)$	mean	49	stones/unit time
$\text{Var}(\Theta)$	variance	18420	
Φ	rel. frequency of armour breaches	$(0, 1)$	breaches/unit time
a	parameter of $\text{Be}(\phi\|a, b)$	0.5	
b	parameter of $\text{Be}(\phi\|a, b)$	0.5	
$E(\Phi)$	mean	0.5	breaches/unit time
$\text{Var}(\Phi)$	variance	0.125	
r	discount rate per unit time	5	%
α	discount factor per unit time	0.9524	
c_I	costs of inspection	10^3	Dfl
c_P	fixed costs of preventive repair	10^4	Dfl
c_V	variable costs of preventive repair	10^2	Dfl/stone
c_F	costs of failure	2.5×10^6	Dfl
r_0	initial resistance	2500	stones
s	failure level	0	stones
k	inspection-interval length	$\mathbb{N}$	unit time

Table 6.2: Optimal inspection intervals for different costs of failure.

c_F	0.75	1.00	2.50	5.00	7.50	10.0	25.0	$\times 10^6$ Dfl
k^*	7	6	4	3	2	2	1	year
k^*_α	24	11	5	3	3	2	1	year

time (discounting). The stochastic process of rock displacement has been regarded as a modified generalised gamma process.

The maintenance models that are presented in this chapter have the following advantages: they enable optimal inspection intervals to be determined under uncertainty, they estimate how much money is needed for the future maintenance of berm breakwaters, and they can be expressed in explicit form when the limiting average rates of rock displacement are given.

Even though the decision model has been used for obtaining optimal inspection intervals only, it can also be applied for determining the optimal resistance of a berm breakwater (in terms of the number of stones). The decision model is a delay-time model. It can be applied to many fields of engineering to solve problems in maintenance optimisation and life cycle costing.

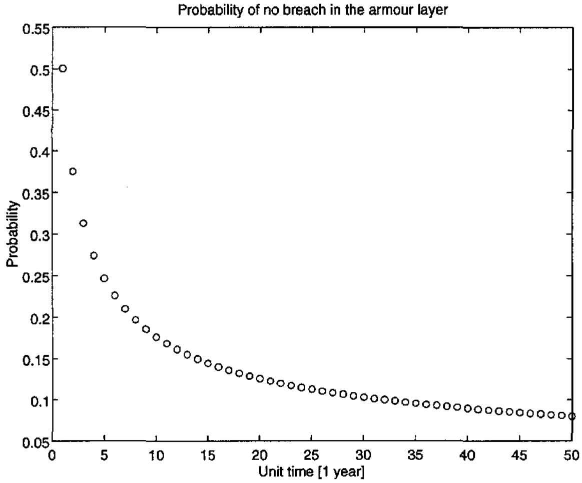

Figure 6.3: The probability of no breach in the armour layer per unit time: i.e. $E([1-\Phi]^i)$, $i = 1, \ldots, 50$, for $\Phi \sim \mathrm{Be}(0.5, 0.5)$ with $E(\Phi) = 0.5$ and $\mathrm{Var}(\Phi) = 1.25 \times 10^{-1}$.

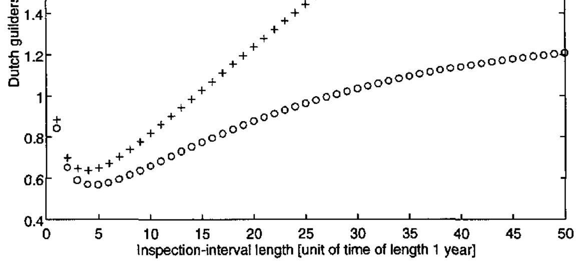

Figure 6.4: The expected average costs per unit time and the expected equivalent average costs per unit time for $\Phi \sim \mathrm{Be}(0.5, 0.5)$ and $\Theta \sim \mathrm{Ig}(2.13, 55.36)$ with $E(\Phi) = 0.5$, $\mathrm{Var}(\Phi) = 1.25 \times 10^{-1}$, $E(\Theta) = 49$, and $\mathrm{Var}(\Theta) = 1.84 \times 10^4$.

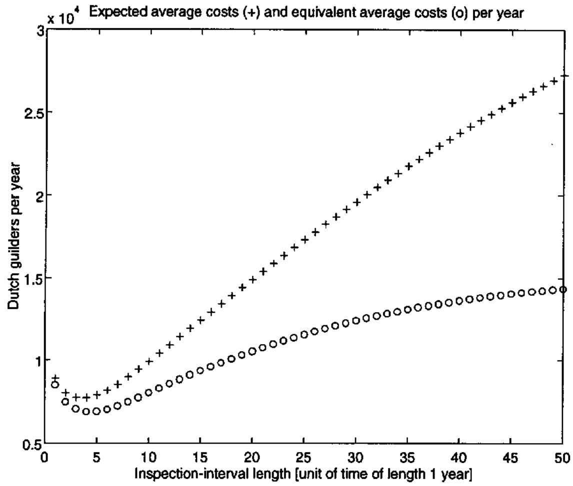

Figure 6.5: The expected average costs per unit time and the expected equivalent average costs per unit time for $\Phi \sim \text{Be}(1000, 1000)$ and $\Theta \sim \text{Ig}(2.13, 55.36)$ with $E(\Phi) = 0.5$, $\text{Var}(\Phi) = 1.25 \times 10^{-4}$, $E(\Theta) = 49$, and $\text{Var}(\Theta) = 1.84 \times 10^{4}$.

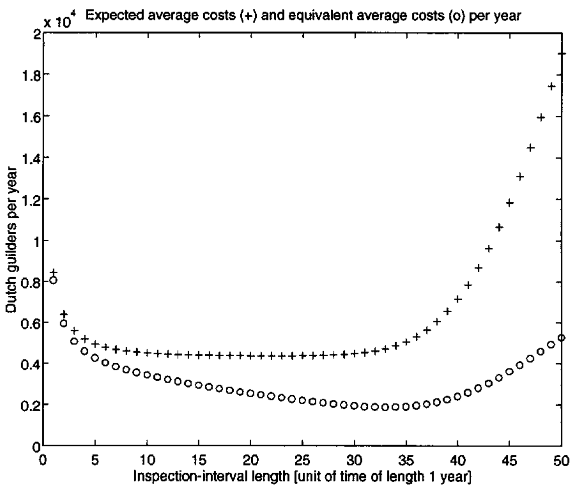

Figure 6.6: The expected average costs per unit time and the expected equivalent average costs per unit time for $\Phi \sim \text{Be}(0.5, 0.5)$ and $\Theta \sim \text{Ig}(132.2, 6428)$ with $E(\Phi) = 0.5$, $\text{Var}(\Phi) = 1.25 \times 10^{-1}$, $E(\Theta) = 49$, and $\text{Var}(\Theta) = 1.84 \times 10^{1}$.

6.6 Appendix: Proof of theorem

Theorem 14 *Suppose the infinite sequence of random quantities* $\{D_i : i \in \mathbb{N}\}$ *is* l_1*-isotropic and* $X_n = \sum_{i=1}^n D_i$ *for all* $n \in \mathbb{N}$, *then*

$$E\left([X_j]^m \, I_{[0,y]}(X_{n-1}) I_{(y,\infty)}(X_n) \middle| \theta\right) =$$

$$= \left\{\frac{(j)_m}{(n)_m} \cdot y^m\right\} \times \frac{1}{(n-1)!} \left[\frac{y}{\theta}\right]^{n-1} \exp\left\{-\frac{y}{\theta}\right\}, \qquad (6.17)$$

for $j, n = 1, 2, \ldots, j < n, m \geq 0, y \in (0, \infty)$, *where* $I_A(x) = 1$ *if* $x \in A$ *and* $I_A(x) = 0$ *if* $x \notin A$. *The quantity* $(n)_m = \Gamma(n+m)/\Gamma(n)$ *is known as Pochhammer's symbol.*

Proof:
Since $X_n = \sum_{i=1}^n D_i$ for all $n \in \mathbb{N}$, it follows that the integration bounds are determined by $0 \leq X_1 \leq \ldots \leq X_{n-1} \leq X_n$. Moreover, $X_{n-1} \leq y$ and $X_n > y$, and the Jacobian equals one. Hence, we may write

$$E\left([X_j]^m \, I_{[0,y]}(X_{n-1}) I_{(y,\infty)}(X_n) \middle| \theta\right) =$$

$$= \int_{x_1=0}^{y} \cdots \int_{x_j=x_{j-1}}^{y} \int_{x_{j+1}=x_j}^{y} \cdots \int_{x_{n-1}=x_{n-2}}^{y} \int_{x_n=y}^{\infty} \frac{x_j^m}{\theta^n} \exp\left\{-\frac{x_n}{\theta}\right\} dx_n \cdots dx_1.$$

This multiple integral can be solved in four steps.
First, integrating out the variable x_n gives:

$$\int_{x_n=y}^{\infty} \left[\frac{1}{\theta}\right]^n \exp\left\{-\frac{x_n}{\theta}\right\} dx_n = \left[\frac{1}{\theta}\right]^{n-1} \exp\left\{-\frac{y}{\theta}\right\}. \qquad (6.18)$$

Second, using the Dirichlet integral entails:

$$\int_{x_{j+1}=x_j}^{y} \cdots \int_{x_{j+(n-j-1)}=x_{j+(n-j-2)}}^{y} 1 \, dx_{j+1} \cdots dx_{j+(n-j-1)} = \frac{[y-x_j]^{n-j-1}}{(n-j-1)!}. \qquad (6.19)$$

Third, reversing the order of integration and applying the beta integral leads to:

$$\int_{x_1=0}^{y} \int_{x_2=x_1}^{y} \cdots \int_{x_j=x_{j-1}}^{y} x_j^m \, [y-x_j]^{n-j-1} \, dx_j \cdots dx_2 dx_1 =$$

$$= \int_{x_j=0}^{y} \int_{x_{j-1}=0}^{x_j} \cdots \int_{x_2=0}^{x_3} \int_{x_1=0}^{x_2} x_j^m \, [y-x_j]^{n-j-1} \, dx_1 dx_2 \cdots dx_{j-1} dx_j$$

$$= \frac{1}{(j-1)!} \int_{x_j=0}^{y} x_j^{m+j-1} \, [y-x_j]^{n-j-1} \, dx_j$$

$$= \frac{1}{(j-1)!} \, y^{m+n-1} \mathrm{B}(m+j, n-j). \qquad (6.20)$$

Finally, combining Eqs. (6.18-6.20) proves the theorem. □

6.7 Appendix: The expected maintenance costs

In order to compare maintenance decisions over unbounded horizons for berm breakwaters, we need to determine two cost-based criteria: (i) the expected average costs per unit time, Eq. (6.8), and (ii) the expected discounted costs over an unbounded horizon, Eq. (6.9). These costs can be computed using renewal theory, where the renewals are the maintenance actions restoring a berm breakwater to its "originally built" profile. Defining a renewal cycle as the time-period between two renewals, we can derive explicit expressions for the expected cycle costs, the expected cycle length, the expected discounted cycle costs, and the expected "discounted cycle length". Recall that inspections are scheduled at times $\{jk\Delta : j \in \mathbb{N}\}$ with inspection interval $k \in \mathbb{N}$. The costs of inspection are c_I, the costs of failure are c_F, the fixed costs of preventive repair are c_P, and the variable costs of preventive repair are c_V. For convenience, let $\Delta = 1$ and $c_V = 0$; the case $c_V > 0$ is considered in the last subsection.

The probabilities of failure and preventive repair per inspection interval.
By using Eqs. (6.11-6.13) and reversing the order of summation, the conditional probability of failure in time-interval $(0, k]$, given ϕ and θ, has the following forms

$$\Pr\{\text{failure in } (0,k] \,|\, \phi, \theta\} = \tag{6.21}$$

$$= \sum_{i=1}^{k} v_i(\phi,\theta) = \sum_{h=1}^{k} \left[1-(1-\phi)^{k-h+1}\right] q_h(\theta) = \sum_{i=1}^{k} \phi(1-\phi)^{i-1} \left[\sum_{h=1}^{k-i+1} q_h(\theta)\right].$$

Similarly, the conditional probability of preventive repair in time-interval $(0, k]$, given ϕ and θ, can be written as

$$\Pr\{\text{preventive repair at } k \,|\, \phi, \theta\} = \tag{6.22}$$

$$= \sum_{i=1}^{k} \phi(1-\phi)^{i-1} \left[1 - \sum_{h=1}^{k-i+1} q_h(\theta)\right] = \left[1-(1-\phi)^k\right] - \sum_{i=1}^{k} v_i(\phi,\theta).$$

The expected cycle costs ($c_V = 0$).
On the basis of Eqs. (6.21-6.22), the expected cycle costs can be decomposed into the expected cycle costs due to inspection, preventive repair ($c_V = 0$), and failure:

$$\sum_{i=1}^{\infty} c_i(\phi,\theta,k) p_i(\phi,\theta,k) =$$

$$= \sum_{j=1}^{\infty} [jc_I + c_P] \Pr\{\text{preventive repair at } jk \,|\, \phi, \theta\} +$$

$$
\begin{aligned}
&\sum_{j=1}^{\infty} \sum_{n=(j-1)k+1}^{jk} [(j-1)c_I + c_F] \Pr\{\text{failure in } (n-1,n] \mid \phi, \theta\} \\
&= \sum_{j=1}^{\infty} [jc_I + c_P](1-\phi)^{(j-1)k} \Pr\{\text{preventive repair at } k \mid \phi,\theta\} + \\
&\quad \sum_{j=1}^{\infty} [(j-1)c_I + c_F](1-\phi)^{(j-1)k} \sum_{i=1}^{k} \Pr\{\text{failure in } (i-1,i] \mid \phi,\theta\} \\
&= \left\{ \frac{c_I}{[1-(1-\phi)^k]^2} + \frac{c_P}{1-(1-\phi)^k} \right\} \left\{ \left[1-(1-\phi)^k\right] - \sum_{i=1}^{k} v_i(\phi,\theta) \right\} + \\
&\quad \left\{ \frac{c_I(1-\phi)^k}{[1-(1-\phi)^k]^2} + \frac{c_F}{1-(1-\phi)^k} \right\} \sum_{i=1}^{k} v_i(\phi,\theta) \\
&= \frac{c_I + c_P\left[1-(1-\phi)^k\right] + \sum_{i=1}^{k} [c_F - (c_I + c_P)] v_i(\phi,\theta)}{1-(1-\phi)^k}, \qquad (6.23)
\end{aligned}
$$

where the second step follows from the "lack of memory" property of the geometric distribution (in Eq. (6.5)).

The expected cycle length.
Since each renewal cycle ends either upon a failure or at a preventive repair, the expected cycle length can be written as

$$
\begin{aligned}
\sum_{i=1}^{\infty} i p_i(\phi,\theta,k) &= \sum_{j=1}^{\infty} jk \Pr\{\text{preventive repair at } jk \mid \phi,\theta\} + \\
&\quad \sum_{j=1}^{\infty} \sum_{n=(j-1)k+1}^{jk} n \Pr\{\text{failure in } (n-1,n] \mid \phi,\theta\} \\
&= \sum_{j=1}^{\infty} jk(1-\phi)^{(j-1)k} \Pr\{\text{preventive repair at } k \mid \phi,\theta\} + \\
&\quad \sum_{j=1}^{\infty} \sum_{i=1}^{k} [(j-1)k + i](1-\phi)^{(j-1)k} \Pr\{\text{failure in } (i-1,i] \mid \phi,\theta\} \\
&= \sum_{j=1}^{\infty} jk(1-\phi)^{(j-1)k} \left\{ \left[1-(1-\phi)^k\right] - \sum_{i=1}^{k} v_i(\phi,\theta) \right\} +
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{j=1}^{\infty}(j-1)k\,(1-\phi)^{(j-1)k}\sum_{i=1}^{k} v_i(\phi,\theta)+\sum_{j=1}^{\infty}(1-\phi)^{(j-1)k}\sum_{i=1}^{k} i v_i(\phi,\theta) \\
= \; & \frac{k+\sum_{i=1}^{k}[i-k]\,v_i(\phi,\theta)}{1-(1-\phi)^k}. \qquad (6.24)
\end{aligned}
$$

The expected discounted cycle costs $(c_V = 0)$.
The expected discounted cycle costs consist of the expected discounted costs due to inspection, preventive repair $(c_V = 0)$, and failure:

$$
\begin{aligned}
& \sum_{i=1}^{\infty}\alpha^i c_i(\phi,\theta,k)p_i(\phi,\theta,k) = \\
= \; & \sum_{j=1}^{\infty}\left[\left(\sum_{h=1}^{j}\alpha^{hk}\right)c_I+\alpha^{jk}c_P\right]\Pr\{\text{preventive repair at } jk\,|\,\phi,\theta\}+ \\
& \sum_{j=1}^{\infty}\sum_{n=(j-1)k+1}^{jk}\left[\left(\sum_{h=1}^{j-1}\alpha^{hk}\right)c_I+\alpha^{n}c_F\right]\Pr\{\text{failure in } (n-1,n]\,|\,\phi,\theta\} \\
= \; & \sum_{j=1}^{\infty}\left[\left(\sum_{h=1}^{j}\alpha^{hk}\right)c_I+\alpha^{jk}c_P\right](1-\phi)^{(j-1)k}\Pr\{\text{preventive repair at } k\,|\,\phi,\theta\}+ \\
& \sum_{j=1}^{\infty}\sum_{i=1}^{k}\left[\left(\sum_{h=1}^{j-1}\alpha^{hk}\right)c_I+\alpha^{(j-1)k+i}c_F\right](1-\phi)^{(j-1)k}\Pr\{\text{failure in } (i-1,i]\,|\,\phi,\theta\} \\
= \; & \sum_{j=1}^{\infty}\left[\left(\sum_{h=1}^{j}\alpha^{hk}\right)c_I+\alpha^{jk}c_P\right](1-\phi)^{(j-1)k}\left\{\left[1-(1-\phi)^k\right]-\sum_{i=1}^{k}v_i(\phi,\theta)\right\}+ \\
& \sum_{j=1}^{\infty}\left(\sum_{h=1}^{j-1}\alpha^{hk}\right)(1-\phi)^{(j-1)k}c_I\sum_{i=1}^{k}v_i(\phi,\theta)+\sum_{j=1}^{\infty}[\alpha(1-\phi)]^{(j-1)k}c_F\sum_{i=1}^{k}\alpha^i v_i(\phi,\theta) \\
= \; & \sum_{j=1}^{\infty}\left[\left(\frac{1-\alpha^{jk}}{1-\alpha^k}\right)\alpha^k c_I+\alpha^{jk}c_P\right](1-\phi)^{(j-1)k}\left[1-(1-\phi)^k\right]- \\
& \sum_{j=1}^{\infty}(c_I+c_P)\,\alpha^k\,[\alpha(1-\phi)]^{(j-1)k}\sum_{i=1}^{k}v_i(\phi,\theta)+\frac{c_F\sum_{i=1}^{k}\alpha^i v_i(\phi,\theta)}{1-[\alpha(1-\phi)]^k} \\
= \; & \frac{\alpha^k c_I}{1-\alpha^k}\left\{1-\frac{\alpha^k\left[1-(1-\phi)^k\right]}{1-[\alpha(1-\phi)]^k}\right\}+\frac{\alpha^k c_P\left[1-(1-\phi)^k\right]}{1-[\alpha(1-\phi)]^k}-
\end{aligned}
$$

$$\frac{\alpha^k (c_I + c_P) \sum_{i=1}^k v_i(\phi,\theta)}{1-[\alpha(1-\phi)]^k} + \frac{c_F \sum_{i=1}^k \alpha^i v_i(\phi,\theta)}{1-[\alpha(1-\phi)]^k}$$

$$= \frac{\alpha^k \left\{c_I + c_P \left[1-(1-\phi)^k\right]\right\} + \sum_{i=1}^k \left[\alpha^i c_F - \alpha^k (c_I+c_P)\right] v_i(\phi,\theta)}{1-[\alpha(1-\phi)]^k}. \tag{6.25}$$

The expected "discounted cycle length".
Similarly, the expected "discounted cycle length" becomes

$$\begin{aligned}
\sum_{i=1}^{\infty} \alpha^i p_i(\phi,\theta,k) &= \sum_{j=1}^{\infty} \alpha^{jk} \Pr\{\text{preventive repair at } jk \mid \phi,\theta\} + \\
&\quad \sum_{j=1}^{\infty} \sum_{n=(j-1)k+1}^{jk} \alpha^n \Pr\{\text{failure in } (n-1,n] \mid \phi,\theta\} \\
&= \sum_{j=1}^{\infty} \alpha^{jk} (1-\phi)^{(j-1)k} \Pr\{\text{preventive repair at } k \mid \phi,\theta\} + \\
&\quad \sum_{j=1}^{\infty} \sum_{i=1}^{k} \alpha^{(j-1)k+i} (1-\phi)^{(j-1)k} \Pr\{\text{failure in } (i-1,i] \mid \phi,\theta\} \\
&= \sum_{j=1}^{\infty} \alpha^{jk} (1-\phi)^{(j-1)k} \left\{\left[1-(1-\phi)^k\right] - \sum_{i=1}^{k} v_i(\phi,\theta)\right\} + \\
&\quad \sum_{j=1}^{\infty} [\alpha(1-\phi)]^{(j-1)k} \sum_{i=1}^{k} \alpha^i v_i(\phi,\theta) \\
&= \frac{\alpha^k \left[1-(1-\phi)^k\right] + \sum_{i=1}^k \left[\alpha^i - \alpha^k\right] v_i(\phi,\theta)}{1-[\alpha(1-\phi)]^k}.
\end{aligned} \tag{6.26}$$

The expected variable costs of a preventive repair per cycle ($c_V > 0$).
The costs that remain to be determined are the expected variable costs of a preventive repair at time jk, where $j,k \in \mathbb{N}$. Let $j=1$. If $T_1 = i-1$, $1 \le i \le k$, then the variable repair costs are proportional to the amount of transported rock in the remaining time up to k not inducing a failure: i.e. $X_{k-i+1} \le y$. By taking the expectation of X_{k-i+1} (subject to $X_{k-i+1} \le y$), summing over all possible values for i, and using Theorem 14, we obtain

$$E\,(\text{amount of transported rock in } (0,k] \text{ with preventive repair at } k \mid \phi,\theta) =$$

$$= \sum_{i=1}^{k} \phi(1-\phi)^{i-1} \left[\sum_{h=k-i+2}^{\infty} E\left(X_{k-i+1} I_{[0,y]}(X_{h-1}) I_{(y,\infty)}(X_h) \Big| \, \theta \right) \right] =$$

$$= \sum_{i=1}^{k} \phi(1-\phi)^{i-1} \left[\sum_{h=k-i+2}^{\infty} \left(\frac{k-i+1}{h} \right) \cdot y \cdot \frac{1}{(h-1)!} \left[\frac{y}{\theta} \right]^{h-1} \exp\left\{ -\frac{y}{\theta} \right\} \right]$$

$$= \sum_{i=1}^{k} [k-i+1] \phi(1-\phi)^{i-1} \theta \left[\sum_{h=k-i+2}^{\infty} \frac{1}{h!} \left[\frac{y}{\theta} \right]^{h} \exp\left\{ -\frac{y}{\theta} \right\} \right]$$

$$= \sum_{i=1}^{k} [k-(i-1)] \phi(1-\phi)^{i-1} \theta \left[1 - \sum_{h=1}^{k-i+2} q_h(\theta) \right]$$

$$= \left[\sum_{i=1}^{k} [k-(i-1)] \phi(1-\phi)^{i-1} - \sum_{h=1}^{k+1} \sum_{i=1}^{k-h+2} [k-(i-1)] \phi(1-\phi)^{i-1} q_h(\theta) \right] \theta$$

$$= \left[\xi_{k,k}(\phi) - \sum_{h=1}^{k+1} \xi_{k-h+2,k}(\phi) q_h(\theta) \right] \theta, \tag{6.27}$$

where we have used the finite sum

$$\xi_{n,k}(\phi) = \sum_{i=1}^{n} [k-(i-1)] \phi(1-\phi)^{i-1}$$

$$= k - \frac{1-\phi}{\phi} \left\{ 1 - [1 + (n-k-1)\phi](1-\phi)^{n-1} \right\} \tag{6.28}$$

for $k, n = 1, 2, \ldots$. Since the geometric distribution possesses the "lack of memory" property, the expected (discounted) variable costs of a preventive repair per renewal cycle are

$$E\,(\text{variable preventive repair costs per renewal cycle} \,|\, \phi, \theta) =$$

$$= \frac{\alpha^k c_V \left[\xi_{k,k}(\phi) - \sum_{h=1}^{k+1} \xi_{k-h+2,k}(\phi) q_h(\theta) \right] \theta}{1 - [\alpha(1-\phi)]^k} \tag{6.29}$$

for $0 < \alpha \leq 1$.

Chapter 7

Optimal Maintenance Decisions for the Sea-Bed Protection of the Eastern-Scheldt Barrier

Jan M. van Noortwijk, Matthijs Kok, and Roger M. Cooke

Abstract. To prevent The Netherlands from flooding, a flood defence system has been constructed, which must be inspected and, when needed, repaired. Therefore, one might be interested in obtaining cost-optimal rates of inspection, i.e. rates of inspection for which the expected maintenance costs are minimal and for which the flood defence system is safe.

For optimisation purposes, maintenance models have been developed for two components of the sea-bed protection of the Eastern-Scheldt barrier: (i) the block mats and (ii) the rock dumping. These models enable optimal maintenance decisions to be determined on the basis of (possibly large) uncertainties in the limiting average rates of deterioration. The modelling assumption that the stochastic processes of scour erosion and rock displacement depend just on limiting averages, leads us to regard them as generalised gamma processes. (This chapter has previously been published as [130].)

Keywords. maintenance, gamma processes, renewal theory, decision theory, Eastern-Scheldt barrier, rock displacement, scour erosion.

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7.1 Introduction

In this chapter, we consider the problem of inspecting the sea-bed protection of the Eastern-Scheldt storm-surge barrier. Since the barrier is planned to function for a period of 200 years, it is inspected to reveal possible deterioration that might endanger the stability of the barrier. Therefore, one might be interested in obtaining cost-optimal rates of inspection, i.e. rates of inspection for which the expected maintenance costs are minimal and for which the barrier is safe.

A large number of papers have been published on the subject of optimising maintenance through mathematical models. For an, inherently incomplete, overview see Barlow & Proschan [5], McCall [85], Pierskalla & Volker [96], Sherif & Smith [108], Sherif [109], Valdez-Flores & Feldman [118], and Cho & Parlar [12]. Most maintenance optimisation models are based on lifetime distributions or Markovian deterioration models. Unfortunately, only a few of them have been applied (see Dekker [28]). According to De Jonge, Kok & Van Noortwijk [25], there are two possible reasons for this poor applicability. First, from the theoretical point of view, there is often no interest in "details" that are of practical importance: a problem description is often lacking or even purely hypothetical. Second, from the practical point of view, there is little experience in using maintenance optimisation models and it is often hard to gather data for estimating either the parameters of a lifetime distribution or the transition probabilities of a Markov chain. Moreover, in case of well-planned preventive maintenance, complete lifetimes will be observed rarely. The authors hope to develop a methodology that might bridge the gap between theory and practice by modelling maintenance on the basis of the main uncertainties involved: the values of the limiting average rates of deterioration. To achieve this, deterioration processes can best be regarded as generalised gamma processes. In The Netherlands, generalised gamma processes have also been used to model decision problems for optimising maintenance of beaches, berm breakwaters, and dykes (see Chapters 4, 6, and 5, respectively).

The chapter is organised as follows. A brief description on the Eastern-Scheldt barrier is given in Sec. 7.2. In Secs. 7.3 and 7.4, we present maintenance models for two components of the barrier: the block mats and the rock dumping, respectively. Some necessary definitions and theorems are presented in two appendices.

7.2 The Eastern-Scheldt barrier

With storm-induced tides of some 4 metres above average sea level, the flood of February 1, 1953, caused a severe catastrophe in Zeeland, The Netherlands. Almost 200,000 hectares of polderland flooded, resulting in huge losses of life and property. In the south-west of The Netherlands, 1,835 people and tens of thousands of animals were drowned. To avoid future losses due to floods like the one in 1953, the Dutch parliament adopted the so-called Delta Plan. The greater part of this plan called for raising the dykes and for closing the main tidal estuaries and inlets by a network of dams and barriers. Since the Delta Plan will soon be completed, the attention is shifting from building structures to maintaining structures. Hence, the use of maintenance optimisation models is of considerable interest.

This chapter is devoted to modelling preventive maintenance of the most expensive and the most complicated structure of the Delta Works: the Eastern-Scheldt storm-surge barrier. The design of this lift-gate barrier is complex for it has to satisfy requirements in the following areas: (i) safety (flood protection during severe storm-surges when the gates are closed), (ii) environment (preservation of the natural salt-water environment during normal weather and hydraulic conditions when the gates are open), and (iii) transport (shipping access to the North-Sea and a road-connection).

The Eastern-Scheldt barrier has been built in three closure gaps separated by two

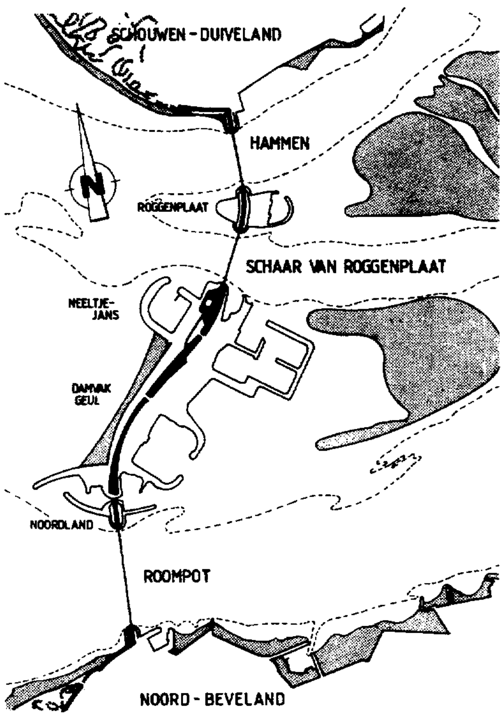

Figure 7.1: The map of the Eastern-Scheldt estuary showing the two artificial dredge-improved islands ('Roggenplaat' and 'Neeltje Jans') and the three tidal channels controlled by lift-gate barriers ('Hammen', 'Schaar', and 'Roompot').

artificial islands (see Fig. 7.1). It has 62 pier-supported steel gates each with a span of nearly 42 metres and a height varying from 6 to 12 metres. To provide for the long-term stability of the barrier, the supporting concrete piers are embedded with several layers of rock and an adjoining sea-bed protection has been constructed with a width of about 500 metres on either side of the center line of the barrier. This sea-bed protection consists of asphalt mastic and block mats in the outer periphery, and graded-filter mattresses under the piers (see Fig. 7.2). Since the protection can be damaged, it is monitored for the appearence of scour holes. In this situation, the rates of inspection and the costs of maintenance have to be optimised. For brief summaries on the technical aspects and the maintenance aspect of the Eastern-Scheldt barrier, see Rijkswaterstaat [103] and Watson & Finkl [140], and De Jonge, Kok & Van Noortwijk [25], respectively.

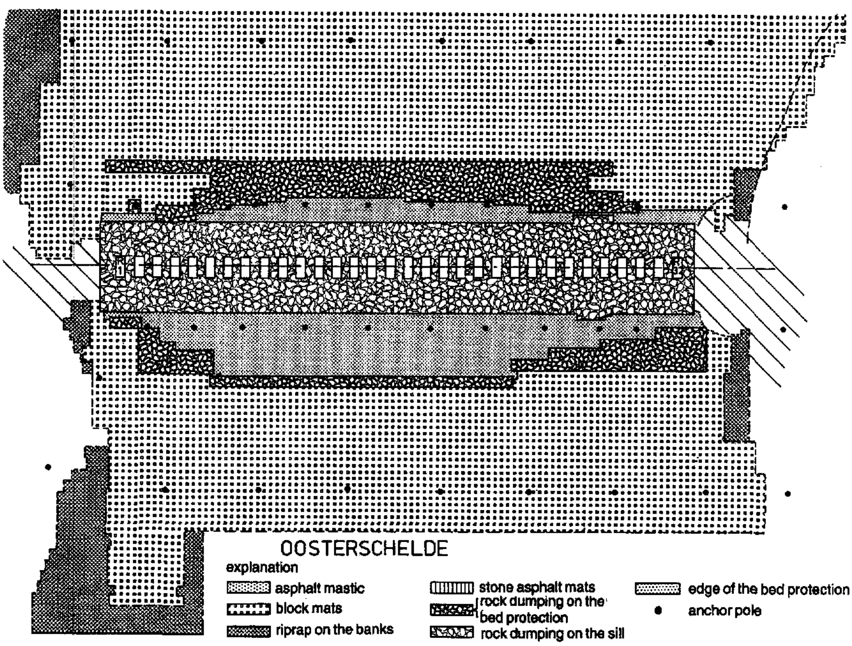

Figure 7.2: View from above of the sea-bed protection of the Eastern-Scheldt barrier at the tidal channel 'Roompot' (from Rijkswaterstaat (1994)).

7.3 Maintenance of the block mats

The block mats consist of synthetic material to which small concrete-blocks (with a height of 17 cm) are attached in a regular pattern. The purpose of this section is to obtain safe and cost-optimal rates of inspection for these mats.

7.3.1 Inspection and repair of scour holes

Due to extreme tidal currents or ship anchorings, the block mats may be damaged in such a way that sands wash away and scour holes appear. To detect possible scour, the block mats are inspected by means of acoustic measurements. If acoustic inspection reveals a scour hole, then a visual dive inspection will be carried out, followed by a repair. Scour holes can only be detected when they are deeper than about 2 metres. By approximation, we therefore assume the probability of detection to be equal to one when the scour hole is deeper than $z = 2$ and zero otherwise, where z is the detectability level. To confirm the statement that there is often a lack of data in practice: up to now, no scour holes have been detected!

Since preventive maintenance is based on the condition of the block mats, we are dealing with so-called condition-based preventive maintenance. Apart from condition-based repairs, it might be economic to perform (aperiodic) condition-based inspections as well. In practice, however, periodic inspections are often to be preferred since the necessary manpower and budget can be anticipated and scheduled well beforehand. Furthermore, we assume that inspection of the whole block mats takes negligible time, does not degrade the block mats, and entails a cost c_I.

A repair is defined as placing graded rip-rap on a scour-hole surface approximately being a hemisphere of radius h, where h is the scour hole depth (in metres). The costs of repairing one scour hole can be subdivided into the fixed cost c_f (cost of mobilisation like shipping transport) and the variable cost c_v (cost per square metre rip-rap). Hence, the costs of repairing one scour hole, which is h metres deep, are

$$c(h) = c_f + 2\pi h^2 c_v. \tag{7.1}$$

Although we assume possible repairs to be carried out during inspections, the decision model may be extended with delay-times between detecting holes and repairing holes.

In summary, the block mats must be inspected to avoid instability of the barrier due to the following uncertain deterioration characteristics: (i) the average rate of occurrence of scour holes at the whole block mats and (ii) the average rate of current-induced scour erosion given there is a scour hole. Because there is no deterioration data available, we have to rely on prior expert judgment. Moreover, since the decision problem of obtaining safe and cost-optimal inspection rates is characterised by the above two uncertainties, the decision model should be based on them as well. These uncertainties can best be represented by probability distributions, where Bayes' theorem can be used to update subjective prior opinion with actual observations.

7.3.2 The rate of occurrence of scour holes

The lack of observations has prompted us to assume that the rate of occurrence of scour holes does not depend on the location and the time. Although this assumption may be criticised for being unrealistic, the probabilistic models representing location- and time-dependent inter-occurrence times are inherently more complex and, not the least important, require many observations to estimate their parameters. Recall that no scour holes have been observed yet and that, as a consequence, even subjective opinion on *average* rates is hard to obtain.

Given the above problem description, we shall derive the probability model of the scour-hole inter-occurrence times by making two reasonable judgments: the inter-occurrence times (i) are exchangeable and (ii) exhibit the "lack of memory" property. These are explained below. Let us denote the successive times between occurrences of scour holes by the infinite sequence of non-negative real-valued random quantities $T_1, T_2, \ldots$.

First, the inter-occurrence times are assumed to be *exchangeable*: i.e. the order in which the scour holes occur is judged to be irrelevant. In mathematical terms, this can be interpreted as that the probability density function of the random vector $\mathbf{T}_n = (T_1, \ldots, T_n)$ is invariant under all $n!$ permutations of the coordinates, i.e.

$$p_{T_1,\ldots,T_n}(t_1, \ldots, t_n) = p_{T_1,\ldots,T_n}\left(t_{\pi(1)}, \ldots, t_{\pi(n)}\right), \tag{7.2}$$

where π is any permutation of $1, \ldots, n$. The infinite sequence of random quantities $\{T_i : i \in \mathbb{N}\}$ is said to be exchangeable if $\mathbf{T}_n$ is exchangeable for each $n \in \mathbb{N}$. The assumption of exchangeability is weaker than the assumption of independence.

Second, the inter-occurrence times are assumed to exhibit the *"lack of memory" property*: i.e. the probability distribution of the remaining time until the occurrence of the first scour hole does not depend on the fact that no scour hole has appeared yet since the completion of the barrier in 1986 (for a formal definition, see Theorem 16 of the appendix). Another explanation of the "lack of memory" property is the following. Suppose the second scour hole has occurred at time t, i.e. $t_1 + t_2 = t$. If the occurrence time of the first scour hole, t_1, could be any time in the interval $[0, t]$, then the "lack of memory" property holds. Actually, $p_{T_1|T_1+T_2}(t_1|\, t) = t^{-1} I_{[0,t]}(t_1)$, being the uniform distribution on $[0, t]$.

Under the assumptions that the infinite sequence $T_1, T_2, \ldots$ is exchangeable and satisfies the "lack of memory" property for all $n \in \mathbb{N}$, we can write the joint probability density function of $T_1, \ldots, T_n$ as a mixture of conditionally independent exponentials (using Theorem 16 or 17 from the appendix):

$$p_{T_1,\ldots,T_n}(t_1, \ldots, t_n) = \int_0^\infty \prod_{i=1}^n \frac{1}{\lambda} \exp\left\{-\frac{t_i}{\lambda}\right\} dQ_\Lambda(\lambda) = f_n(\textstyle\sum_{i=1}^n t_i) \tag{7.3}$$

for $(t_1, \ldots, t_n) \in \mathbb{R}_+^n$ and zero otherwise, where $\mathbb{R}_+ = [0, \infty)$. The infinite sequence of random quantities $\{T_i : i \in \mathbb{N}\}$ is said to be l_1-*isotropic* (or l_1-norm symmetric), since its distribution can be written as a function of the l_1-norm. In general, an infinite sequence of random quantities is said to be *l_p-isotropic*, if its distribution can be written as a function of the l_p-norm (see Misiewicz & Cooke [90]). The random quantity Λ, with probability distribution Q_Λ, describes the uncertainty about the limiting average inter-occurrence time of scour holes: $\lim_{n\to\infty}[(\sum_{i=1}^{n} T_i)/n]$ (see e.g. Barlow & Mendel [3] and Chapter 2). Note that the information about the unknown parameter λ, contained in $T_1, \ldots, T_n$, is summarised by the statistic $[n, \sum_{i=1}^{n} T_i]$ which is sufficient for λ. The characterisation of Eq. (7.3) in terms of the "lack of memory" property is due to Diaconis & Freedman [42] (see Theorem 16 in the appendix) and the characterisation of Eq. (7.3) in terms of conditioning on the sufficient statistic $[n, \sum_{i=1}^{n} T_i]$ is due to Diaconis & Freedman [41] (see Theorem 17 in the appendix). For an overview on statistical modelling using exchangeability and sufficiency, see Bernardo & Smith [9, Ch. 4].

For modelling the occurrences of scour holes, only the probability distribution of the average inter-occurrence time remains to be determined. To keep the mathematics of the decision model tractable, we impose the property of *posterior linearity* introduced by Diaconis & Ylvisaker [43, 44], i.e. $E(T_2|\,T_1 = t_1) = at_1 + b$ for some constants $a, b > 0$. Remark that, due to exchangeability, before observing T_1, $E(T_2) = E(T_1)$. If posterior linearity holds, then the mixing distribution Q_Λ is the inverted gamma distribution $\text{Ig}(\lambda|\alpha, \beta)$ (see also Theorem 16 from the appendix). The mathematical tractability is especially useful if one wants to update the prior distribution $\text{Ig}(\lambda|\alpha, \beta)$ with actual observations $t_1, \ldots, t_n$. In fact, using Bayes' theorem, the posterior distribution is $\text{Ig}(\lambda|\alpha + n, \beta + \sum_{i=1}^{n} t_i)$. Owing to the fact that the posterior mean can be written as a linear combination of the prior mean and the sample mean, the property of posterior linearity has been satisfied.

7.3.3 The rate of current-induced scour erosion

Beside the uncertainty in the average rate of occurrence of scour holes, we have to take into account the uncertainty in the average rate of current-induced scour erosion. This average is taken over all possible scour holes deeper than z metres and over all possible locations at the block mats. For each scour hole, erosion is measured in terms of the scour-hole depth h, where $h > z$ and $z = 2$.

The purpose of this subsection is to characterise the stochastic process of scour erosion in terms of the only (subjective) information that is available: the probability distribution of the average rate of scour erosion. In doing so, we shall adopt the following two assumptions: with respect to any uniform time-partition and any location at the block mats, the increments of erosion (i) are non-negative and exchangeable; and (ii) have a joint conditional probability distribution, given their sum, which can be represented by a multi-dimensional beta distribution (Dirichlet distribution). Let us denote the erosion process by $\{X(t) : t \geq 0\}$: a non-decreasing continuous-time stochastic process with $\Pr\{X(0)=0\}=1$ and $X(t)$ representing the cumulative erosion, per scour hole, at time t.

The first assumption means: for every uniform time-partition in time-intervals of length $\tau>0$, the infinite sequence of random increments of erosion, $D_i(\tau)=X(i\tau)-X([i-1]\tau)$, $i\in\mathbb{N}$, is assumed to be exchangeable, where $D_i(\tau)\geq 0$ for all i.

The second assumption means: for all $\tau>0$, the conditional probability density function of the first increment of erosion, when the sum of the first and the second increment is given, can be expressed as a transformed beta distribution with both parameters equal to $a\tau$, i.e.

$$p_{D_1(\tau)|X(2\tau)}(\delta_1|\,x)=\frac{\Gamma(2a\tau)}{[\Gamma(a\tau)]^2}\,\frac{\delta_1^{a\tau-1}[x-\delta_1]^{a\tau-1}}{x^{2a\tau-1}}\,I_{[0,x]}(\delta_1)=\text{Be}\left(\frac{\delta_1}{x}\middle|\,a\tau,a\tau\right)\frac{1}{x} \qquad (7.4)$$

for some constant $a>0$ with

$$\begin{aligned} E(D_1(\tau)|X(2\tau)=x) &= x/2, \\ \text{Var}(D_1(\tau)|X(2\tau)=x) &= [x/2]^2/(2a\tau+1). \end{aligned}$$

We now indicate how Eq. (7.4) is derived. To begin with, if D_1 were not symmetrically distributed about its mean, $x/2$, then the random quantities D_1 and D_2 would not be exchangeable. Hence, the parameters of the beta distribution should be equal. Moreover, in Chapter 3 we actually have derived Eq. (7.4) in two ways: by conditioning on sums of increments and by invoking isotropy. The characterisation in terms of conditioning on sums of increments extends the results of Diaconis & Freedman [41] (Theorem 17 from the appendix) by achieving consistency in the sense that probability distributions of increments and sums of increments belong to the same family of distributions and by assuming the probability model to be independent of the scale of measurement (i.e. to be a scale mixture). The characterisation in terms of isotropy is the following: if, for all $\tau>0$, the infinite sequences of powers of increments, $\{D_i(\tau)^{\alpha(\tau)}: i\in\mathbb{N}\}$, are $l_{\beta(\tau)}$-isotropic for some positive continuous functions $\alpha(\tau)$ and $\beta(\tau)$, then Eq. (7.4) follows.

By characterising exchangeable erosion processes in terms of conditional distributions given sums of increments, i.e. in terms of Eq. (7.4), for all $\tau>0$, we so characterise the *generalised gamma process*. Indeed, it follows from Theorem 17 (see the appendix) with $h(y)=y^{a\tau-1}/\Gamma(a\tau)$ that, for all $\tau>0$, the joint probability density function of the increments $D_1(\tau),\ldots,D_n(\tau)$ can be written as a mixture of conditionally independent gamma densities:

$$p_{D_1(\tau),\ldots,D_n(\tau)}(\delta_1,\ldots,\delta_n)=\int_0^\infty\prod_{i=1}^n\frac{\delta_i^{a\tau-1}}{\Gamma(a\tau)}\left[\frac{a\tau}{\theta}\right]^{a\tau}\exp\left\{-\frac{a\tau\delta_i}{\theta}\right\}dP_{\Theta(\tau)}(\theta) \qquad (7.5)$$

for some constant $a>0$ with

$$E(X(n\tau)) \;=\; E(n\Theta(\tau)),$$

$$\text{Var}(X(n\tau)) = \left[1 + \frac{1}{na\tau}\right] E([n\Theta(\tau)]^2) - [E(n\Theta(\tau))]^2$$

for all $\tau > 0$, provided the first and the second moment of the probability distribution of $\Theta(\tau)$ exist.

The generalised gamma process has three useful properties.

First, the probability distribution $P_{\Theta(\tau)}$ on the random quantity $\Theta(\tau)$, with possible values $\theta \in (0, \infty)$, represents the uncertainty in the unknown limiting average amount of erosion per time-interval of length τ: $\lim_{n\to\infty}[(\sum_{i=1}^{n} D_i(\tau))/n]$. By the strong law of large numbers for exchangeable random quantities, the average converges with probability one if $E(D_1(\tau)) < \infty$ (see Chow & Teicher [13, p. 227]).

Second, the summarisation of the n random quantities $D_1(\tau), \ldots, D_n(\tau)$ in terms of the statistic $[n, \sum_{i=1}^{n} D_i(\tau)]$ is sufficient for the unknown limiting average erosion $\Theta(\tau)$. In fact, the characterisation in terms of conditioning on sums of random quantities is motivated by sufficiency ideas, since, by sufficiency, the conditional probability density function $p_{D_1(\tau)|X(2\tau),\Theta(\tau)}(\delta_1 | x, \theta)$ does not depend on θ.

Third, the mixture of gamma's in Eq. (7.5) transforms into a mixture of exponentials if $\tau = a^{-1}$. As we shall see in Secs. 7.3 and 7.4, for this unit-time length, denoted by $\Delta = a^{-1}$, many probabilistic properties of the stochastic process, like the probability of exceedence of a failure level, can be expressed in explicit form conditional on the limiting average. The unit time for which the increments of erosion are distributed according to a mixture of exponentials, i.e. are l_1-isotropic, follows directly from the characterisation in terms of conditioning on sums of increments in Eq. (7.4). From this equation (for fixed $\tau > 0$), it can be seen that the smaller the unit-time length for which the increments are l_1-isotropic, i.e. the smaller $\Delta = a^{-1}$, the more deterministic the erosion process.

Note that specifying the l_1-isotropic grid of the generalised gamma process is similar to specifying the precision of the Brownian motion with drift (see e.g. Karlin & Taylor [71, Ch. 7]). This stochastic process is often used to model stochastic deterioration. Unfortunately, the Brownian motion allows for a probability of "negative deterioration", especially when large uncertainties are involved.

In conclusion, we advocate regarding the stochastic erosion process as a generalised gamma process with probability distribution on the limiting average rate of erosion. This process does not entail extra difficulties, but has the advantage that it models realistic non-negative deterioration rather than unrealistic real-valued deterioration. As for the limiting average inter-occurrence time, we impose the property of posterior linearity and assume the probability distribution of the limiting average erosion, per unit time of length Δ, to be the inverted gamma distribution $\text{Ig}(\nu, \mu)$.

7.3.4 The maintenance decision model

Based on the two stochastic processes of the occurrences and the development of scour holes, which are judged to be independent, we can formulate the maintenance optimisation model. From now on, we choose our units of time so that the increments of scour erosion per unit time are l_1-isotropic (with respect to $\{([n-1]\Delta, n\Delta] : n \in \mathbb{N}\}$). For notational convenience, let $D_n = D_n(\Delta)$, $X_n = \sum_{h=1}^{n} D_h$ for all $n \in \mathbb{N}$, and let Θ represent the uncertainty in the limiting average rate of scour erosion $\lim_{n\to\infty}[(\sum_{i=1}^{n} D_i)/n]$. Subsequently, we determine the expected number of scour holes that occur per unit time and the expected cumulative amount of detectable scour erosion.

When the limiting average inter-occurrence time has the value λ, then the scour holes arrive according to a Poisson process with arrival rate λ^{-1}. It is well-known (see e.g. Karlin & Taylor [71, pp. 173-175]) that the number of scour holes occurring in unit time i follows a Poisson distribution with parameter Δ/λ:

$$\Pr\{\text{number of holes in } i\text{th unit time} = j \mid \lambda\} = \frac{(\Delta/\lambda)^j e^{-\Delta/\lambda}}{j!} \tag{7.6}$$

for $j = 0, 1, 2, \ldots$. The expected number of scour holes that occur in unit time i can simply be written as $s_i(\lambda) = \Delta/\lambda$ and does not depend on i (due to the "lack of memory" property).

When the limiting average rate of scour erosion per unit time has the value θ, then a scour hole that first could have been detected in unit time i, but is inspected in unit time k, entails the following expected costs of repair (using Eq. (7.1)):

$$u_i(\theta, k) = c_f + 2\pi c_v E\left(\left[z + X_{k-i+1}\right]^2 \,\middle|\, \theta\right) \tag{7.7}$$

where $1 \leq i \leq k$ and X_n having a gamma distribution $\mathrm{Ga}(n, 1/\theta)$ for all $n \in \mathbb{N}$ (for the definition of the gamma distribution, see Appendix A).

Our main interest is to determine an inspection interval of length $k\Delta$, $k \in \mathbb{N}$, for which the expected maintenance costs are minimal and the barrier is safe, where inspections are carried out at times $\{jk\Delta : j \in \mathbb{N}\}$. Let $L(\lambda, \theta, k)$ be the monetary *loss* when the decision-maker chooses inspection interval k, $k \in \mathbb{N}$, and when the limiting averages λ and θ are given. Under the requirement that the barrier is safe, the decision-maker can best choose the inspection interval k^* whose expected loss, $E(L(\Lambda, \Theta, k^*))$, is minimal. The decision k^* is called an *optimal decision* (see e.g. DeGroot [27, Ch. 8]).

The best choice for the loss is the expected *average costs per year* (see Sec. 7.4), which can be determined by averaging the maintenance costs over an unbounded time-horizon. Since we have renewal cycles of length $k\Delta$, the average costs per year becomes:

$$L(\lambda, \theta, k) = \frac{c_I + \sum_{i=1}^{k} u_i(\theta, k) s_i(\lambda)}{k\Delta}, \tag{7.8}$$

where the numerator consists of the costs of inspection c_I plus the costs of repairing k possible scour holes, summed over all units of time in which they may occur, times the expected number of occurrences. Note that since the expected discounted costs over an unbounded horizon (see Sec. 7.4) approach zero, from above, for unbounded inspection-intervals, the criterion of discounted costs is not useful for solving this decision problem.

The evaluation of the average costs in Eq. (7.8) is straightforward: by induction and using the gamma integral, we get

$$\sum_{i=1}^{k} E\left(X_{k-i+1}^{m} \,\middle|\, \theta\right) = m!\,\theta^{m} \sum_{i=1}^{k} \binom{m+k-i}{k-i} = m!\,\theta^{m} \binom{m+k}{m+1} \tag{7.9}$$

for $m = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots$. Substitution of Eqs. (7.7) and (7.9) into Eq. (7.8) yields

$$L(\lambda, \theta, k) = \frac{c_I}{k\Delta} + \left\{c_f + 2\pi c_v \left[z^2 + z\theta(k+1) + \theta^2(k+1)(k+2)/3\right]\right\} \frac{1}{\lambda}. \tag{7.10}$$

Up to now, we did not incorparate the possibility of a severe failure of the block mats such that the barrier is unsafe. Strictly speaking, costs of failure due to potential unsafe situations should be incorporated as well. However, what is "unsafe" and what are the costs of failure involved? The costs of failure not only consist of costs due to damaged block mats, but also of possible costs due to instability of the barrier, and, when there is a severe storm-surge, of possible costs due to flooding. Unfortunately, these costs are very hard to determine or to assess. For this reason, we have chosen to leave out the failure costs, but to introduce an upper bound for the inspection interval with the following property: when this upperbound is crossed, the block mats are said to be unsafe in the sense that there is at least one scour hole deeper than a certain failure level, say y. The probability of this event should be smaller than a predefined norm probability which itself is a function of the inspection-interval length $k\Delta$. For example: $1 - (1 - p_{norm})^{k\Delta}$, where p_{norm} is the annual norm probability.

In mathematical terms, the probability of failure of the block mats can be expressed as follows. By assuming the scour holes being independent given θ and by rewriting the probability of the event "in $(0, k\Delta]$ at least one scour hole occurs that is deeper than y metres" as one minus the probability of the event "in $(0, k\Delta]$ no scour holes occur that are deeper than y metres", we get

$$\begin{aligned} v_k(\lambda, \theta) &= \Pr\{\text{in } (0, k\Delta] \text{ at least one hole is deeper than } y \mid \lambda, \theta\} \\ &= 1 - \prod_{i=1}^{k} \sum_{j=0}^{\infty} \frac{(\Delta/\lambda)^j e^{-\Delta/\lambda}}{j!} \left[\Pr\{z + X_{k-i+1} \le y \mid \theta\}\right]^j \\ &= 1 - \exp\left\{-\frac{\Delta}{\lambda} \sum_{i=1}^{k} \sum_{h=1}^{k-i+1} \Pr\{X_{h-1} \le y - z, X_h > y - z \mid \theta\}\right\}, \end{aligned} \tag{7.11}$$

where the probability of failure of one scour hole follows the Poisson distribution:

$$\Pr\{X_{h-1} \leq x, X_h > x \,|\, \theta\} = \frac{1}{(h-1)!}\left[\frac{x}{\theta}\right]^{h-1} \exp\left\{-\frac{x}{\theta}\right\} = q_h(\theta, x) \tag{7.12}$$

for $h = 1, 2, \ldots$, $\theta > 0$, and $x = y - z$ (see Chapter 2).

In summary, the decision-maker can best choose the inspection-interval k^* whose expected long-term average costs of maintenance are minimal and whose expected probability of failure is safe:

$$\begin{aligned} E(L(\Lambda, \Theta, k^*)) &= \min_{k \in \mathcal{D}} E(L(\Lambda, \Theta, k)), \text{ where} \\ \mathcal{D} &= \left\{k : k \in \mathbb{N}; E(v_k(\Lambda, \Theta)) < 1 - (1 - p_{norm})^{k\Delta}\right\}. \end{aligned} \tag{7.13}$$

Although explicit computation of $E(v_k(\Lambda, \Theta))$ is not possible, reasonably sharp lower and upper bounds have been found (see Theorem 15 of the appendix).

For obtaining optimal inspection and repair decisions for the block mats, we use the parameters in Table 7.1. The unit time for which the increments of scour erosion are distributed as mixtures of exponentials has been determined by specifying the conditional probability density function of the amount of scour erosion in a period of six months when the amount of erosion in a period of one year is given to be 10 metres (using Eq. (7.4) shown in Fig. 7.3). The optimal decision k^*, satisfying Eq. (7.13), is

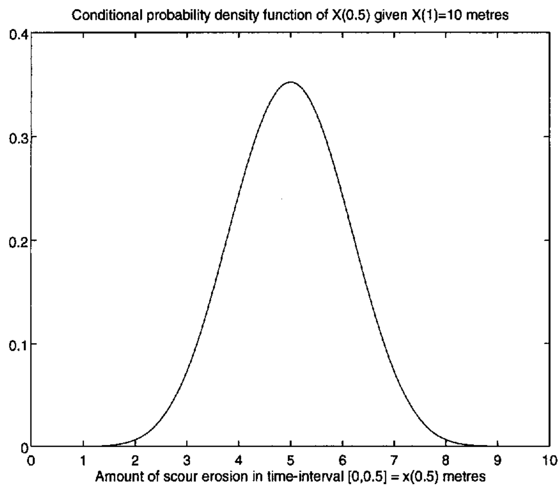

Figure 7.3: The conditional probability density function of the amount of scour erosion in a period of six months, $X(0.5)$, when the amount of erosion in a period of one year is $X(1) = 10$ metres.

Table 7.1: The parameters of the maintenance model for the block mats.

parameter	description	value	dimension
Δ	unit time	0.05	year
Θ	average rate of scour erosion per unit time	$(0, \infty)$	m/unit time
$\theta_{0.05}/\Delta$	[5%-percentile of $\text{Ig}(\theta\|\nu, \mu)$]$/\Delta$	7	m/year
$\theta_{0.95}/\Delta$	[95%-percentile of $\text{Ig}(\theta\|\nu, \mu)$]$/\Delta$	13	m/year
ν	shape parameter of $\text{Ig}(\theta\|\nu, \mu)$	28.7	
μ	scale parameter of $\text{Ig}(\theta\|\nu, \mu)$	13.3	
$E(\Theta)$	mean of the average rate of scour erosion	0.5	m/unit time
$E(\Theta/\Delta)$	mean of the average rate of scour erosion	10	m/year
Λ	average scour-hole inter-occurrence time	$(0, \infty)$	year
$\lambda_{0.05}$	5%-percentile of $\text{Ig}(\lambda\|\alpha, \beta)$	1	year
$\lambda_{0.95}$	95%-percentile of $\text{Ig}(\lambda\|\alpha, \beta)$	10	year
α	shape parameter of $\text{Ig}(\lambda\|\alpha, \beta)$	2.46	
β	scale parameter of $\text{Ig}(\lambda\|\alpha, \beta)$	5.46	
$E(\Lambda)$	mean of the average inter-occurrence time	4	year
c_I	costs of inspection	125,000	Dfl
c_f	fixed costs of repairing one scour hole	100,000	Dfl
c_v	variable cost of rip-rap	159	Dfl/m^2
h	scour hole depth	$(0, \infty)$	m
z	scour-hole detectability level	2	m
y	scour-hole failure level	15	m
p_{norm}	annual norm probability of failure	0.01	
k	inspection-interval length	$\mathbb{N}$	unit time
k^*	optimal inspection-interval length	20	unit time
$k^*\Delta$	optimal inspection-interval length	1	year

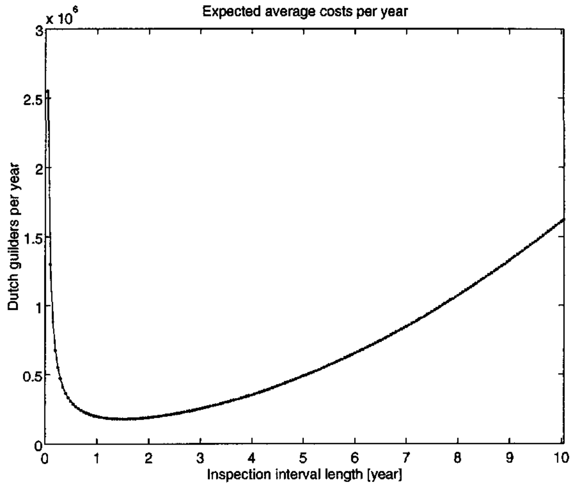

Figure 7.4: The expected average costs per year for the block mats.

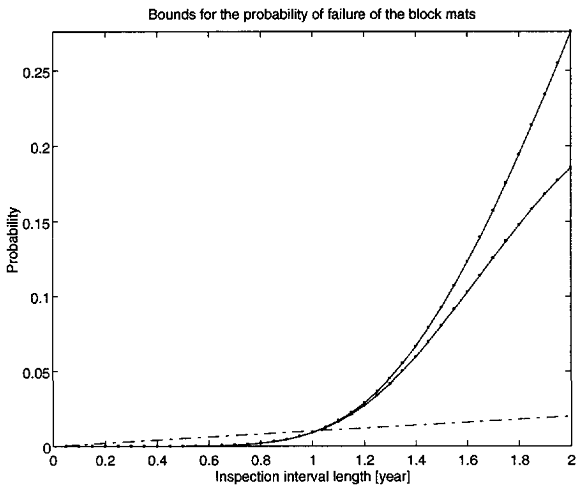

Figure 7.5: The lower and upper bounds for the expected probability of failure of the block mats crossed by the norm probability of failure.

an inspection interval of one year whose expected average costs per year are 2×10^5 Dutch guilders. Although the maintenance costs are minimal for an inspection interval of 1.5 years (see Fig. 7.4), the barrier is unsafe for inspection intervals larger than 1 year (see Fig. 7.5). For practical purposes, no less important than obtaining a unique optimal decision, however, is obtaining a range of nearly cost-optimal and safe decisions. The decision-maker can find an optimum balance between cost and safety using the curves in Figs. 7.4 and 7.5.

7.4 Maintenance of the rock dumping

Using a similar method to that used for the block mats, we can obtain cost-optimal rates of inspection for the rock dumping of the barrier. Millions of tons of rock rubble were placed at the sea-bed protection near the center line of the barrier (see Fig. 7.2). This protection is subject to current-induced rock displacement, which has to be monitored by means of acoustic measurements and, if necessary, has to be repaired. The inspection problem is due to Kok [73, 74]; it also has been studied in Chapter 2. Because it was not our purpose to determine optimal maintenance decisions in Chapter 2, we revisit this inspection problem and use the failure model of Chapter 2 to determine optimal rates of inspection.

By the same reasoning as for the process of scour erosion, we can best regard the stochastic process of rock displacement as a generalised gamma process with an inverted gamma distribution as mixing measure (see Sec. 7.3.3). The infinite sequence of l_1-isotropic increments of rock displacement is denoted by $\{D_i : i \in \mathbb{N}\}$. The resistance of the upper rock layer of the rock dumping, R, is defined as the number of stones removed (at time zero: $r_0 = 0$). Due to the stochastic process of rock displacement, the resistance in unit time n can be written as

$$R_n = r_0 - \textstyle\sum_{h=1}^{n} D_h = r_0 - X_n, \;\; n \in \mathbb{N}. \tag{7.14}$$

We consider *one* steel gate section and assume perfect inspection in the sense that the actual resistance can be determined without uncertainty.

Let the rock dumping be inspected at times $\{jk\Delta : j \in \mathbb{N}\}$ for $k \in \mathbb{N}$. Furthermore, inspection takes negligible time, does not degrade the rock dumping, and entails fixed costs c_I. We may regard the maintenance process as a renewal process, where renewals bring the rock dumping into the "as good as new state". Each renewal cycle ends either upon failure or at an inspection time $jk\Delta$ when the inspection reveals that a preventive repair should be carried out (for some $j \in \mathbb{N}$). A failure is defined as the event in which the resistance R drops below the failure level s: $R < s$. A preventive repair is defined as the event at which inspection reveals that the resistance has crossed the preventive repair level ρ while no failure has occurred: $s \leq R < \rho$, where $s < \rho < r_0$. A failure costs c_F Dutch guilders, while a preventive repair costs c_P Dutch guilders. Let the renewal times be conditionally independent random quantities having a discrete probability function $p_i(\theta, k)$, $i \in \mathbb{N}$, when the limiting average rate of rock displacement θ is given and the decision-maker chooses inspection decision k. The costs associated with a renewal at time $i\Delta$ are denoted by $c_i(\theta, k)$, $i \in \mathbb{N}$.

Since the planned lifetime of the barrier is very large, maintenance decisions can best be compared over an unbounded time-horizon. As we have pointed out in Chapter 4, there are basically three cost-based criteria that can serve as loss functions: (i) the expected average costs per unit time, (ii) the expected discounted costs over an unbounded time-horizon, and (iii) the expected equivalent average costs per unit time. These cost-based criteria can be obtained using the discrete renewal theorem (see e.g. Feller [47, Ch. 13] and Karlin & Taylor [71, Ch. 3]).

First, the expected *average costs per unit time* are determined by averaging the expected costs over an unbounded horizon:

$$L(\theta, k) = \lim_{n\to\infty} \frac{C(n, \theta, k)}{n} = \frac{\sum_{i=1}^{\infty} c_i(\theta, k) p_i(\theta, k)}{\sum_{i=1}^{\infty} i p_i(\theta, k)}, \tag{7.15}$$

where $C(n, \theta, k)$ are the expected costs in time-interval $(0, n\Delta]$. Eq. (7.15) is a well-known result from renewal reward theory (see e.g. Ross [105]).

Second, the expected *discounted costs over an unbounded horizon* are determined by summing the expected discounted values of the costs over an unbounded horizon, where the discounted value of the costs c_n in unit time n is defined to be $\alpha^n c_n$ with discount factor $\alpha = [1 + (r/100)]^{-1}$ and discount rate $r\%$, where $r > 0$:

$$L_\alpha(\theta, k) = \lim_{n\to\infty} C_\alpha(n, \theta, k) = \frac{\sum_{i=1}^{\infty} \alpha^i c_i(\theta, k) p_i(\theta, k)}{1 - \sum_{i=1}^{\infty} \alpha^i p_i(\theta, k)}, \tag{7.16}$$

where $C_\alpha(n, \theta, k)$ are the expected discounted costs in time-interval $(0, n\Delta]$.

Third, the expected *equivalent average costs per unit time* are determined by averaging the discounted costs over a "discounted" unbounded horizon (of length $1/(1-\alpha)$). In fact, the notion of equivalent average costs relates the notions of average costs and discounted costs in the sense that the equivalent average costs per unit time approach the average costs per unit time, as α tends to 1, from below:

$$\lim_{\alpha\uparrow 1} (1-\alpha) L_\alpha(\theta, k) = L(\theta, k). \tag{7.17}$$

The unit time for which the increments of rock displacement are distributed as mixtures of exponentials has been determined by specifying the conditional probability density function of the amount of rock displacement in a period of 50 years when the amount of erosion in a period of 100 years is given to be 70 displaced stones (using Eq. (7.4) shown in Fig. 7.6). When using the parameters of Table 7.2 (from Kok [74]) and applying numerical integration, the average costs per year and the equivalent average costs per year are represented by the curves in Fig. 7.7; the necessary expressions for deriving these costs can be found in Appendix 7.7. The optimal decision with respect to the criterion of average costs is $k^*\Delta = 10$ years, whereas the optimal decision with respect to the criterion of equivalent average costs is $k^*\Delta = 30$ years. Fortunately, the optimal inspection interval does not depend so much on the choice of the unit time (see Fig. 7.8). Recall that the maintenance costs are determined with respect to *one* of the 124 steel gate sections.

7.5 Conclusions

In this chapter, we have presented two maintenance models that enable optimal inspection and repair decisions to be determined for two components of the Eastern-Scheldt barrier: the block mats and the rock dumping. Since there are only (subjective) probability distributions available on the average rates of scour erosion and rock displacement, we have based our models on these averages. In fact, we have shown that only with generalised gamma processes is one able to model stochastic deterioration processes with non-negative, exchangeable, real-valued increments when their average rates are uncertain. Two case studies have been carried out to show the usefulness of the maintenance models.

Table 7.2: The parameters of the maintenance model for the rock dumping.

parameter	description	value	dimension
Δ	unit time	5	year
Θ	average rate of rock displacement	$(0, \infty)$	stones/unit time
ν	shape parameter of $\text{Ig}(\theta\|\nu, \mu)$	12.2	
μ	scale parameter of $\text{Ig}(\theta\|\nu, \mu)$	39.2	
$E(\Theta)$	mean	3.5	stones/unit time
$\text{Var}(\Theta)$	variance	1.2	
r	discount rate per year	5	%
α	discount factor per unit time	0.7835	
c_I	costs of inspection	1,000	Dfl
c_P	costs of preventive repair	10,000	Dfl
c_F	costs of failure	120,000	Dfl
r_0	initial resistance	0	stones
ρ	preventive repair level	-50	stones
s	failure level	-70	stones
k	inspection-interval length	$\mathbb{N}$	unit time

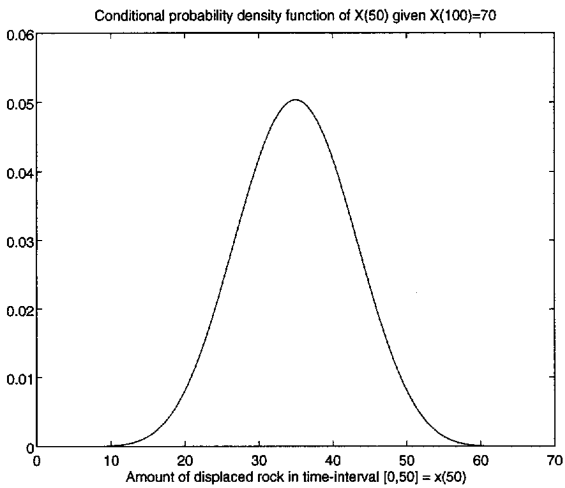

Figure 7.6: The conditional probability density function of the amount of rock displacement in a period of 50 years, $X(50)$, when the amount of rock displacement in a period of 100 years is $X(100) = 70$ stones.

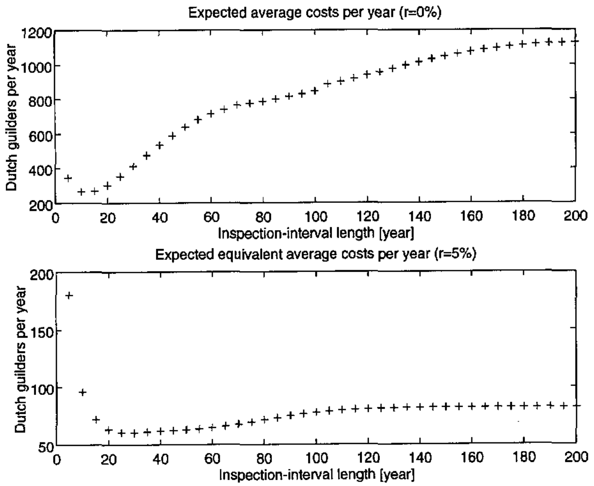

Figure 7.7: The expected average costs per year and the expected equivalent average costs per year for the rock dumping.

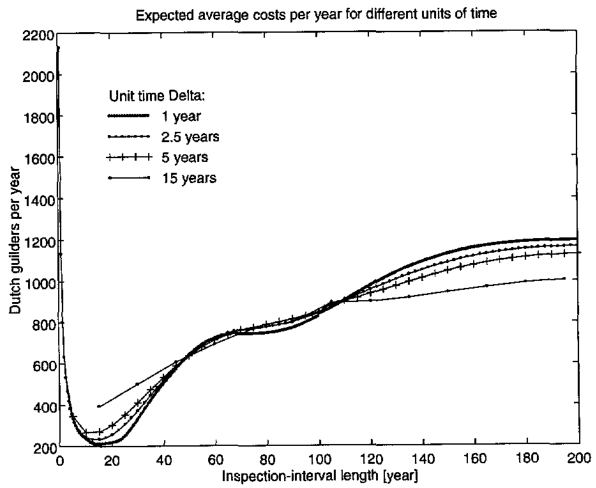

Figure 7.8: The expected average costs per year for different units of time Δ.

7.6 Appendix: Proofs of theorems

Theorem 15 *Let Λ and Θ be independent inverted gamma distributed random quantities such that $\Lambda \sim \mathrm{Ig}(\alpha,\beta)$ and $\Theta \sim \mathrm{Ig}(\nu,\mu)$ for $\alpha,\nu > 2$ and $\beta,\mu > 0$. Furthermore, let $(Y_1, Y_2) \sim \mathrm{Nm}(y/(\mu+2y), y/(\mu+2y), \nu)$ for $y > 0$ (implying the marginal distribution of Y_1 to have a negative binomial distribution: $Y_1 \sim \mathrm{Nm}(y/(\mu+y),\nu)$). Then,*

$$\xi_{k1} - \xi_{k2} < 1 - E\left(\exp\left\{-\frac{\Delta}{\Lambda}\sum_{i=1}^{k}\sum_{h=1}^{k-i+1}\frac{1}{(h-1)!}\left[\frac{y}{\Theta}\right]^{h-1}\exp\left\{-\frac{y}{\Theta}\right\}\right\}\right) < \xi_{k1}, \quad (7.18)$$

where

$$\xi_{km} = \frac{1}{m}E\left(\left[\frac{\Delta}{\Lambda}\right]^m\right)\left[\sum_{i=1}^{k}\sum_{h=1}^{k-i+1}\left(\Pr\{Y_1 = h, Y_m = h\}\right)^{\frac{1}{m}}\right]^m$$

for $m = 1, 2$, $k \in \mathbb{N}$ and $\Delta > 0$.

Proof:
The upper bound in Eq. (7.18), ξ_{k1}, can be derived by using the inequality $1 - e^{-x} < x$ for $x > 0$ and the gamma integral.

The lower bound in Eq. (7.18), $\xi_{k1} - \xi_{k2}$, can be derived by using the inequality $1 - e^{-x} > x - \frac{1}{2}x^2$ for $x > 0$, Minkowski's inequality for integrals, and the gamma integral. □

Theorem 16 *Let $\{Y_i : i \in \mathbb{N}\}$ be an infinitely exchangeable sequence of positive real-valued random quantities such that*

$$\Pr\{(Y_1,\ldots,Y_n) \in A\} = \Pr\{(Y_1,\ldots,Y_n) \in A + \mathbf{x}\}$$

for all $n \in \mathbb{N}$ and any Borel set $A \in \mathbb{R}_+^n$ with $\mathbf{x} \in \mathbb{R}^n$ satisfying $\sum_{i=1}^n x_i = 0$ and $A + \mathbf{x} \in \mathbb{R}_+^n$. Then, the joint probability density function of $(Y_1,\ldots,Y_n)$ is a scale mixture of exponentials for all $n \in \mathbb{N}$. If in addition, $E(Y_2|Y_1 = y_1) = ay_1 + b$ for some constants $a, b > 0$, then the mixing measure is a gamma distribution.

Proof:
See Diaconis & Freedman [42] and Diaconis & Ylvisaker [44]. □

Theorem 17 *Let $\{Y_i : i \in \mathbb{N}\}$ be an infinitely exchangeable sequence of real-valued random quantities such that, for all $n \geq 2$ and $k < n$,*

$$p(y_1,\ldots,y_k|\textstyle\sum_{i=1}^n y_i = t) = \frac{\left[\prod_{i=1}^k h(y_i)\right] h^{(n-k)}\left(t - \sum_{i=1}^k y_i\right)}{h^{(n)}(t)}, \quad (7.19)$$

where $h^{(n)}$ *is the n-fold convolution of the (positive and continuous) function h with itself and*

$$c(\theta) = \int h(y) \exp\{\theta y\}\, dy < \infty \quad \textit{for } \theta \in \mathbb{R}.$$

Then, there exists a probability distribution P such that

$$p(y_1, \ldots, y_n) = \int \prod_{i=1}^{n} [h(y_i) \exp\{\theta y_i\}/c(\theta)]\, dP(\theta).$$

Proof:
See Diaconis & Freedman [41]. □

7.7 Appendix: The expected maintenance costs

In order to compare maintenance decisions over an unbounded horizon for the rock dumping of the Eastern-Scheldt barrier, we need to determine two cost-based criteria: (i) the expected average costs per unit time, Eq. (7.15), and (ii) the expected discounted costs over an unbounded horizon, Eq. (7.16). For this purpose, expressions have been derived for the expected cycle costs, the expected cycle length, the expected discounted cycle costs, and the expected "discounted cycle length". They can be determined using the failure model of Chapter 2. For notational convenience, let $x = r_0 - \rho$, $y = r_0 - s$, and

$$\psi_{j,n} = \sum_{i=j+1}^{n} \binom{n-1}{i-1} \left[1 - \frac{x}{y}\right]^{n-i} \left[\frac{x}{y}\right]^{i-1}$$

for $j < n$. Recall that inspections are scheduled at times $\{jk\Delta : j \in \mathbb{N}\}$ with inspection interval $k \in \mathbb{N}$. The costs of inspection are c_I, the costs of failure are c_F, and the costs of preventive repair are c_P.

The expected cycle costs.
The expected cycle costs can be written as the sum of the expected costs due to inspection, preventive repair, and failure (using Eq. (7.12)):

$$\sum_{i=1}^{\infty} c_i(\theta, k) p_i(\theta, k) =$$

$$= \sum_{j=1}^{\infty} [jc_I + c_P] \Pr\left\{R_{(j-1)k} \geq \rho, s \leq R_{jk} < \rho \,\middle|\, \theta\right\} +$$

$$\sum_{j=1}^{\infty} [(j-1)c_I + c_F] \Pr\left\{R_{(j-1)k} \geq \rho, R_{jk} < s \,\middle|\, \theta\right\}$$

$$= \sum_{j=1}^{\infty} \sum_{n=(j-1)k+1}^{jk} \left\{[jc_I + c_P]\, q_n(\theta, x) + [c_F - c_P - c_I]\, \psi_{(j-1)k,n} q_n(\theta, y)\right\}.$$

The expected cycle length.
Similarly, the expected cycle length can be written as

$$\sum_{i=1}^{\infty} i p_i(\theta, k) =$$

$$= \sum_{j=1}^{\infty} jk \Pr\left\{R_{(j-1)k} \geq \rho, s \leq R_{jk} < \rho \,\middle|\, \theta\right\} +$$

$$\sum_{j=1}^{\infty} \sum_{n=(j-1)k+1}^{jk} n \Pr\left\{ R_{(j-1)k} \geq \rho, R_{n-1} \geq s, R_n < s \,\middle|\, \theta \right\}$$

$$= \sum_{j=1}^{\infty} \sum_{n=(j-1)k+1}^{jk} \left\{ jkq_n(\theta, x) + (n - jk)\psi_{(j-1)k,n} q_n(\theta, y) \right\}.$$

The expected discounted cycle costs.

The expected discounted cycle costs can be written as the discounted value of the expected costs due to inspection, preventive repair, and failure:

$$\sum_{i=1}^{\infty} \alpha^i c_i(\theta, k) p_i(\theta, k) =$$

$$= \sum_{j=1}^{\infty} \left[\left(\sum_{h=1}^{j} \alpha^{hk} \right) c_I + \alpha^{jk} c_P \right] \Pr\left\{ R_{(j-1)k} \geq \rho, s \leq R_{jk} < \rho \,\middle|\, \theta \right\} +$$

$$\sum_{j=1}^{\infty} \sum_{n=(j-1)k+1}^{jk} \left[\left(\sum_{h=1}^{j-1} \alpha^{hk} \right) c_I + \alpha^n c_F \right] \Pr\left\{ R_{(j-1)k} \geq \rho, R_{n-1} \geq s, R_n < s \,\middle|\, \theta \right\}$$

$$= \sum_{j=1}^{\infty} \sum_{n=(j-1)k+1}^{jk} \left[\left(\frac{1 - \alpha^{jk}}{1 - \alpha^k} \right) \alpha^k c_I + \alpha^{jk} c_P \right] q_n(\theta, x) +$$

$$\sum_{j=1}^{\infty} \sum_{n=(j-1)k+1}^{jk} \left(\alpha^n c_F - \alpha^{jk} [c_I + c_P] \right) \psi_{(j-1)k,n} q_n(\theta, y).$$

The expected "discounted cycle length".

Similarly, the expected "discounted cycle length" can be written as:

$$\sum_{i=1}^{\infty} \alpha^i p_i(\theta, k) =$$

$$= \sum_{j=1}^{\infty} \alpha^{jk} \Pr\left\{ R_{(j-1)k} \geq \rho, s \leq R_{jk} < \rho \,\middle|\, \theta \right\} +$$

$$\sum_{j=1}^{\infty} \sum_{n=(j-1)k+1}^{jk} \alpha^n \Pr\left\{ R_{(j-1)k} \geq \rho, R_{n-1} \geq s, R_n < s \,\middle|\, \theta \right\}$$

$$= \sum_{j=1}^{\infty} \sum_{n=(j-1)k+1}^{jk} \left\{ \alpha^{jk} q_n(\theta, x) + \left(\alpha^n - \alpha^{jk} \right) \psi_{(j-1)k,n} q_n(\theta, y) \right\}.$$

Chapter 8

Optimal Decisions that Reduce Flood Damage along the Meuse: an Uncertainty Analysis

Jan M. van Noortwijk, Matthijs Kok, and Roger M. Cooke

Abstract. In December 1993, the river Meuse flooded and caused a damage of about 250 million Dutch guilders. This prompted the Dutch government to initiate a project to investigate and compare strategies that can reduce future losses due to flooding on the basis of several criteria, including some concerning uncertainties.

To obtain decisions that reduce flood damage, one should account for the following uncertainties: the river discharge, the flood damage given the discharge, the downstream water level given the discharge, the costs and the yields of extracting sand and gravel, and the costs of constructing embankments. These uncertainties can best be represented by probability distributions, where Bayes' theorem can be used to update subjective prior information with observations.

When the loss is defined as the net present discounted value of the costs of decisions minus the yields of decisions plus the remaining mean flood damage over an unbounded time-horizon, decision theory can be used to obtain optimal decisions with respect to the following three decision criteria: the criterion of minimal expected loss, the criterion of minimal uncertainty in the loss, and the criterion of maximal safety. By using simulation, the present situation and five strategies have been analysed. The strategy with minimal expected loss and maximal safety is based on widening the summer bed in the south of Limburg, lowering the summer bed in the middle and north of Limburg, and constructing 62 km of embankments around the remaining bottlenecks along the Meuse. (This chapter has also been published as [129].)

Keywords. Bayesian inference, flooding, uncertainty analysis, decision theory, river management.

8.1 Introduction

Around Christmas 1993, the Dutch river Meuse flooded due to an extreme discharge at Borgharen (near the Dutch-Belgian border) of 3120 m^3/s (see Fig. 8.1). Discharges at Borgharen larger than 3120 m^3/s have a probability of occurrence of about once in 150 years. In Limburg, the flood caused a damage of about 250 million Dutch guilders, a flooded area of about 18,000 hectares, an evacuation of about 8,000 people, and, therefore, raised emotions.

To investigate and compare decisions that reduce future losses due to flooding, the Dutch Minister of Transport, Public Works and Water Management initiated the project *Investigation of the Meuse Flood.* The project was carried out in the period march-november 1994 by about 90 researchers, mainly from Delft Hydraulics and the Dutch Ministry of Transport, Public Works and Water Management (Rijkswaterstaat), and had a budget of 6 million Dutch guilders. The study was supervised by a committee, named after its chairman Dr. B.C. Boertien, which had the following members: two from the local waterboards, three from the province Limburg, two from the Rijkswaterstaat, one mayor, and one representative from Belgium.

Roughly, the Dutch Meuse can be subdivided into two parts: (i) the upstream area in Limburg without dykes, but with small embankments that were all overtopped during the 1993 flood, and (ii) the downstream area with dykes that were high and strong enough to prevent the protected polders from flooding. The subject of study is the upstream area of the Meuse in Limburg. The results[1] of the project are reported in one main report [29] and 14 subreports; this chapter summarises Subreport 14 on the uncertainty analysis [37]. For a summary of the main report, see Kok [75].

Since decisions that reduce flood damage must be made under uncertainty, Sec. 8.2 presents three types of criteria to compare decisions in uncertainty: the criterion of minimal expected loss, the criterion of minimal uncertainty in the loss, and the criterion of maximal safety. For flooding of the river Meuse, the most important uncertainties are the river discharge at Borgharen (Sec. 8.3.1), the flood damage given the discharge (Sec. 8.3.2), the downstream water levels along the Meuse given the discharge (Sec. 8.3.3), and the costs and yields of decisions (Sec. 8.3.4).

After representing the above uncertainties with probability distributions, the uncertainty in the loss due to flooding remains to be determined. For flooding of the Meuse, the loss is defined as the net present discounted value of the costs of decisions minus the yields of decisions plus the mean flood damage over an unbounded time-horizon (Sec. 8.3.5). Sec. 8.4 reports the results of an uncertainty analysis in which the uncertainties are determined in the mean flood damage and in the loss, for the present situation and for five strategies (combinations of decisions). In Sec. 8.5, we

[1] In the beginning of 1995, the high water levels on the Dutch rivers were world news. Also the Meuse flooded due to a maximum discharge at Borgharen of 2870 m^3/s. The 1995 data, however, are not incorporated into the present uncertainty analysis.

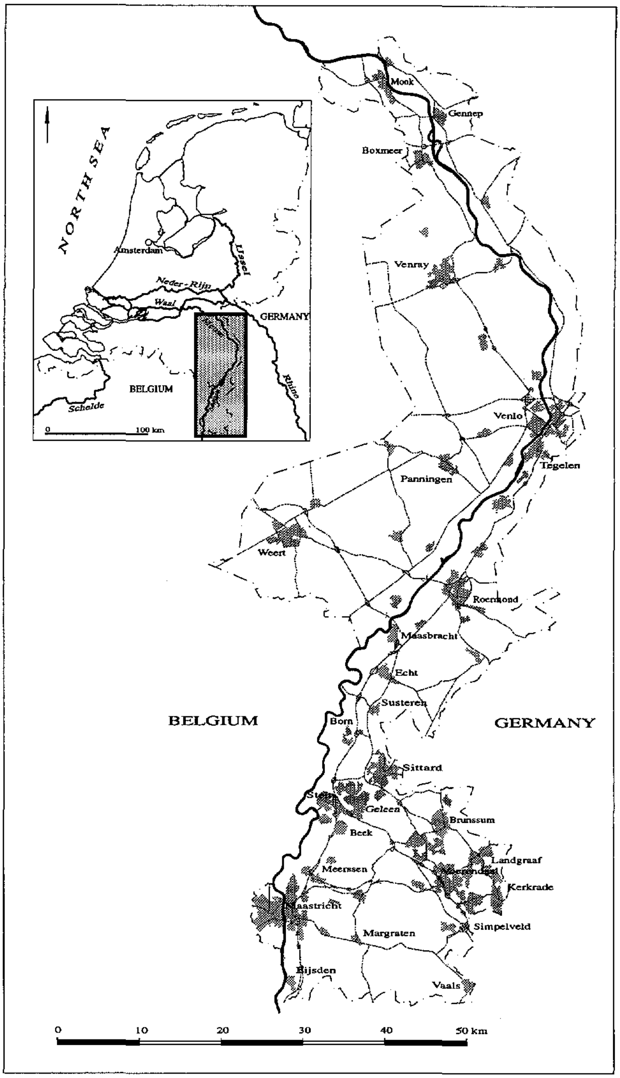

Figure 8.1: The Meuse river in Limburg, The Netherlands.

discuss which strategy is optimal according to each decision criterion. Some definitions of probability distributions can be found in Appendix A.

8.2 Decision making under uncertainty

8.2.1 Optimal decisions

Optimal decisions that reduce flood damage must be made under uncertainty and can be obtained using decision theory. Following the treatments of DeGroot [27] and Savage [106], a decision problem is a problem in which the decision-maker has to choose a decision d (or a combination of decisions) from the set of all possible decisions $\mathcal{D}$, where the consequences of decision d depend on the unknown value w of the state of the world W (for example, the discharge of the Meuse at Borgharen). Optimal decisions can be defined with respect to the following three decision criteria: the criterion of minimal expected loss, the criterion of minimal uncertainty in the loss, and the criterion of maximal safety.

The set of possible decisions.
Five strategies that reduce future losses due to flooding have been selected for further investigation in [29, 36]. These strategies were developed during a careful screening process of all measures that might result in a reduction of flood damage and in new natural development of the Meuse. Also, measures proposed by society were considered, but none of them were attainable. Finally, a limited number of measures were selected and combined in strategies. Roughly, the strategies can be subdivided into three categories (see Table 8.1) on the basis of: lowering the summer bed (Strategy 1), natural development of the Meuse in the south of Limburg (Strategy 2abc), and the construction of embankments and dykes only (Strategy 3). Note that all strategies cover the construction of embankments and dykes around the remaining bottlenecks along the Meuse.

Decisions with minimal expected loss.
Let $L(w, d)$ be the loss when the decision-maker chooses decision d and when the value of W is w. In flooding, the loss function equals the costs of decision d, say $c(w, d)$, minus the yields of decision d, say $y(w, d)$, plus the remaining flood damage, say $s(w, d)$. Hereby, $c(w, d)$ represents the costs of extracting sand and gravel from the river bed (to lower or widen it) and of constructing embankments and dykes; $y(w, d)$ represents the yields of extracting sand and gravel. Hence, the *loss* can be written as

$$L(w, d) = c(w, d) - y(w, d) + s(w, d) \tag{8.1}$$

with $c(w, \emptyset) = y(w, \emptyset) = 0$ (no decisions are made). For any decision $d \in \mathcal{D}$, the expected loss is given by $E(L(W, d))$. The decision-maker can best choose, if possible,

Table 8.1: The investigated strategies with the measures included (+).

Measure		Strategy				
		1	2a	2b	2c	3
Lowering the summer bed:	*north of Limburg*	+	+	+	−	−
	middle of Limburg	+	+	+	−	−
	south of Limburg	+	−	−	−	−
Lowering the winter bed:	*north of Limburg*	−	+	−	−	−
Widening the summer bed:	*south of Limburg*	−	+	+	+	−
Constructing embankments and dykes [km]:		55	58	62	128	137

the decision d^* whose expected loss is minimal. A decision d^* is called an *optimal decision* when $E(L(W, d^*)) = \min_{d \in \mathcal{D}} E(L(W, d))$.

Decisions with minimal uncertainty in the loss.
Uncertainty in the loss can be important when a decision-maker has to choose between two decisions with equal expected loss. A possible decision rule could be to choose the decision with minimal uncertainty in the loss.

Decisions with maximal safety.
Instead of minimising the loss, one might prefer maximising the safety (or the utility). Some people living along the Meuse are interviewed about their subjective feelings of safety under different flooding conditions (see [30]). Since minimising loss and maximising safety cannot both be achieved, a possible decision rule is choosing the decision for which an acceptable safety level will yet be attained.

8.2.2 The expected loss of a decision

In general, decision problems are based on the consequences of the uncertain state of the world W. For flooding of the Meuse, the state of the world is characterised by the following five random quantities: the river discharge at Borgharen, the downstream water levels along the Meuse given the Borgharen discharge, the flood damage given the water level, the costs of decisions, and the yields of decisions. Although more uncertainties can be identified, these five are the most relevant, for small uncertainties pale into insignificance beside large uncertainties.

The main aim of this chapter is to compute the probability distribution of the loss function, Eq. (8.1), and its expected value. We determine the joint probability density function of the above five random quantities by formulating the decision problem in terms of an influence diagram. For a brief introduction to influence diagrams, we refer to Barlow & Pereira [4] and Jae & Apostolakis [67].

To obtain an influence diagram for flooding of the Meuse, we split the joint probability density function up into conditional probability distributions that can be easily assessed (see Fig. 8.2). The main source of uncertainty in the event of a Meuse flood is the maximal river discharge Q at Borgharen in m^3/s (in Fig. 8.2, Q is displayed as a chance node). To avoid calculational burden, we discretise the probability distribution of the discharge Q into the intervals $(q_{i-1}, q_i]$, $i = 1, \ldots, 9$ (see Table 8.2). Given

Table 8.2: List of discharges at Borgharen $[m^3/s]$ according to which the probability distribution of the discharge has been discretised.

	List of discharges $[m^3/s]$									
i	0	1	2	3	4	5	6	7	8	9
q_i	2000	2120	2500	2750	2990	3120	3305	3545	3860	∞

a particular discharge at Borgharen, we can obtain the downstream water level with the one-dimensional physical model ZWENDL developed by Rijkswaterstaat. The water level at a given location mainly depends on the discharge and the river geometry. Since we can measure the river geometry, we may regard it as a known quantity resulting in the flood depth (a deterministic node). Given the water level and the geometry, the damage assessment model (see [33] and De Jonge, Kok & Hogeweg [24]) determines whether, and to what extent, immovables will be damaged due to flooding. For a number of flood depths, this model estimates the number of flooded houses, industries, farms, green houses [hectares], roads [km], and government agencies.

To reduce damage due to flooding, we can identify three types of decisions (decision nodes in Fig. 8.2). First, structural decisions to be taken upstream in Germany, Belgium, and France, which effect the Borgharen discharges. Second, structural decisions to be taken downstream in The Netherlands which effect the water levels via changes in the geometry and in the hydraulic roughness. Third, nonstructural decisions which effect the flood damage via heightening of meter-cupboards, making the furniture water resistant and improving the water-level predictions. Given the costs of decisions, the yields of decisions, and the flood damage, we can now obtain the loss.

8.3 Modelling uncertainty

In modelling the main uncertainties in our decision problem, being the discharge, the flood damage, the water level, the costs, the yields, and the loss, we use the arguments of Barlow [2]: (i) in the presence of a large amount of data, summarisation is useful; (ii) a decision-theoretic tool should consist of a small number of parameters; (iii) possible

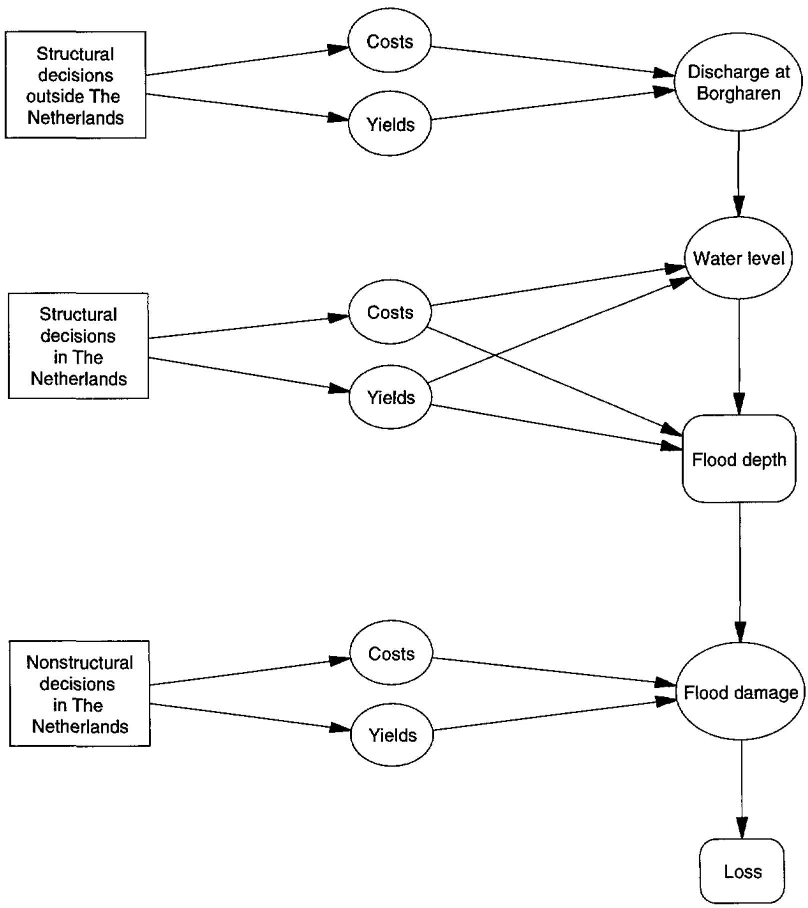

Figure 8.2: Influence diagram of decisions that reduce flood damage along the Meuse river.

prior information should be used for analysing data; and (iv) uncertainty can best be represented by probability distributions.

8.3.1 Uncertainty in the discharge

To derive a probabilistic model for the Borgharen discharge, we extend the frequentist approach of [31] with uncertainty distributions. In essence, the frequentist approach has resulted in a probability of exceedence of a discharge q of the form

$$\Pr\{Q > q|\phi,\theta\} = \Pr\{Q > q_0|\phi\}\Pr\{Q > q|Q > q_0,\theta\} = \phi\exp\{-(q-q_0)/\theta\} \quad (8.2)$$

for $q > q_0$ and $\phi, \theta > 0$. Rather than treating ϕ and θ as known parameters, an uncertainty analysis requires regarding them as unknown random quantities. The choice for the threshold q_0 has been motivated by the decision problem: q_0 is the largest discharge for which the Meuse does not exceed the summer bed and for which no flood damage occurs, i.e. $q_0 = 2000\ m^3/s$ (see [33]).

The probability that flood damage occurs

For determining the probability of flood damage, i.e. of the event $Q > q_0$, we need to know the relative frequency of years in which flood damage occurs in a potentially infinite sequence of hydrological years. Suppose that the order in which the floods occur is irrelevant or, in other words, the years are exchangeable. Furthermore, we define the random quantity V_i as follows: $V_i = 1$, if flood damage occurs in year i, and $V_i = 0$, if flood damage does not occur in year i, where $i \in \mathbb{N}$. By de Finetti's representation theorem [22], there exists a unique probability distribution P such that the joint probability density function of $V_1, \ldots, V_n$ can be written as a mixture of conditionally independent Bernouilli trials:

$$p(v_1,\ldots,v_n) = \int_0^1 \prod_{i=1}^n \phi^{v_i}(1-\phi)^{1-v_i}\,dP(\phi) = \int_0^1 \prod_{i=1}^n l(v_i|\phi)\,dP(\phi), \qquad (8.3)$$

where $l(v_i|\phi)$ is the likelihood function of the observation v_i. The random quantity Φ may be interpreted as the limiting relative frequency of discharges larger than q_0, i.e. $\lim_{n\to\infty}[(\sum_{i=1}^n V_i)/n]$, and the probability distribution P as the representation of beliefs about Φ (for details, see Cooke [17, Ch. 7] and Bernardo & Smith [9, Ch. 4]).

As soon as data comes available, the prior distribution P on Φ can be updated to the posterior distribution by Bayes' theorem. For an uncertainty analysis, it is convenient when the prior distribution enables the posterior distribution to be expressed in explicit form. It is well-known that the beta distribution $\mathrm{Be}(\phi|a,b)$ has this property. With a Bernoulli likelihood function, both the prior distribution and the posterior distribution are a beta distribution: i.e. $\mathrm{Be}(\phi|a,b)$ and $\mathrm{Be}(\phi|a+\sum_{i=1}^n v_i, b+n-\sum_{i=1}^n v_i)$, respectively. The beta distribution is said to be conjugate with respect to the Bernoulli likelihood function (DeGroot [27, Ch. 9]). Note that the (prior) predictive limiting relative frequency of discharges larger than q_0 is given by

$$\Pr\{Q > q_0\} = \int_0^1 \phi \, \mathrm{Be}(\phi|a,b)\, d\phi = a/[a+b]. \tag{8.4}$$

The probability of exceedence of a discharge

The last step in determining the probability distribution of the discharge Q is to obtain the conditional probability $\Pr\{Q > q|Q > q_0\}$. Define the random quantity Q_j to be the maximal Borgharen discharge in hydrological year j, where $j \in \mathbb{N}$. If the discharge causes flood damage, i.e. if $Q_j > q_0$, let $X_j = Q_j - q_0 \in \mathbb{R}_+$ for $j \in \mathbb{N}$, where $\mathbb{R}_+ = [0, \infty)$. Furthermore, we assume that the joint probability density function of $X_1, \ldots, X_m$ can be written as a function of the statistic $\sum_{j=1}^m X_j$. Then, with Mendel [89] and Misiewicz & Cooke [90], we have a mixture of exponentials:

$$p(x_1, \ldots, x_m) = \int_0^\infty \prod_{j=1}^m \frac{1}{\theta} \exp\left\{-\frac{x_j}{\theta}\right\} dP(\theta) = f_m\left(\sum_{j=1}^m x_j\right), \tag{8.5}$$

where the random quantity Θ may be interpreted as the limiting average discharge larger than q_0, i.e. $\lim_{m\to\infty}[(\sum_{j=1}^m X_j)/m]$ (provided $E(X_j) < \infty$ for $j \in \mathbb{N}$). The probability distribution P represents the uncertainty in Θ. If the joint probability density function of $X_1, \ldots, X_m$ satisfies Eq. (8.5), then $X_1, \ldots, X_m$ are called l_1-isotropic.

The probability distribution that is conjugate with respect to the exponential likelihood function is the inverted gamma distribution $\mathrm{Ig}(\theta|\nu,\mu)$. By updating this prior distribution with the observations $x_1, \ldots, x_m$, the posterior distribution is the inverted gamma distribution $\mathrm{Ig}(\theta|\nu + m, \mu + \sum_{j=1}^m x_j)$. The (prior) predictive conditional probability that $Q > q$, given $Q > q_0$, is called the gamma-gamma distribution (see e.g. Bernardo & Smith [9, Ch. 3]):

$$\Pr\{Q > q|Q > q_0\} = \int_0^\infty \exp\{-(q-q_0)/\theta\}\, \mathrm{Ig}(\theta|\nu,\mu)\, d\theta = \left[\frac{\mu}{\mu + q - q_0}\right]^\nu \tag{8.6}$$

for $q > q_0$.

Let us assume the random quantities Φ and Θ, representing the uncertainties in the limiting relative frequency of discharges larger than q_0 and the limiting average discharge larger than q_0, respectively, to be independent. In conclusion, we can easily obtain the (prior) predictive probability of exceedence of a discharge q by using Eqs. (8.4) and (8.6):

$$\Pr\{Q > q\} = \int_0^\infty \int_0^1 \phi \exp\{-(q-q_0)/\theta\}\, \mathrm{Be}(\phi|a,b)\mathrm{Ig}(\theta|\nu,\mu)\, d\phi\, d\theta \tag{8.7}$$

for $q > q_0$. As was to be required, Eq. (8.7) extends Eq. (8.2) with the uncertainty distributions $\text{Be}(\phi|a,b)$ and $\text{Ig}(\theta|\nu,\mu)$. Actually, we have approximated the tail of the probability distribution of the discharge with a mixture of exponentials. In the presence of m observations which are larger than q_0, i.e. for $x_j = q_j - q_0$, $j = 1,\ldots,m$, the parameters a, b, μ and ν must be replaced by $a+m$, $b+n-m$, $\mu + \sum_{j=1}^{m} x_j$ and $\nu + m$, respectively, where $m = \sum_{i=1}^{n} v_i$.

Prior information and observed discharges

Next, we shall determine the parameters of the two uncertainty distributions that we have chosen. For this purpose, we consider two types of distributions: a non-informative and an informative prior distribution.

Non-informative prior distribution.
There is no prior information to be taken into account: i.e. the prior uncertainty is very large. To express "very large uncertainty" in probabilistic terms, we can use several non-informative prior distributions (see Berger [8, Ch. 3] and Bernardo & Smith [9, Ch. 5]). In our opinion, we can best use a non-informative prior distribution whose posterior mean equals the sample mean. Under this condition, the non-informative prior densities for ϕ and θ are $\phi^{-1}(1-\phi)^{-1}I_{[0,1]}(\phi)$ and $\theta^{-2}I_{(0,\infty)}(\theta)$, respectively.

Informative prior distribution.
There is prior information to be taken into account: the data on floods that occurred between 1400 and 1910. This data has been interpreted from Gottschalk [56, 57] and the KNMI (see [37]). Although the real historical discharges are unknown and the river geometry is changed, these references do mention whether there was flood damage or a catastrophe attended with drowned people, dyke-bursts, flooded polders, carried-away houses, and collapsed bridges. Probably, the most serious flood of the river Meuse, up to now, is the flood of 1643. On the basis of the amount of flood damage, we assume for the Borgharen discharge q that (using Table 8.2):

- $q \le q_0$, if the references do not mention any flood damage;
- $q_0 < q \le q_2$, if the references mention flood damage, but no catastrophe;
- $q > q_2$, if the references mention a catastrophe.

From the historical data of the period 1400-1910, we have derived the following estimates: $\Pr\{Q > q_0\} = 0.1373$ and $\Pr\{Q > q_2\} = 0.0275$. Hence, it follows that $\Pr\{Q > q_2 | Q > q_0\} = 0.2003$. Since the historical data are probably less reliable for determining $\Pr\{Q > q_0\}$ than for determining $\Pr\{Q > q_2|Q > q_0\}$ and since the weight of the historical data in comparison with the data of the years 1911-1993 should not be too large, we assume that the prior information of the years 1400-1910:

- may not be used for determining the probability distribution of the probabiliy that the discharge is larger than q_0 (i.e. of Φ). Hence, let the prior density of Φ be non-informative: $\phi^{-1}(1-\phi)^{-1}I_{[0,1]}(\phi)$.
- may be used as 5 imaginary data for determining the prior distribution of the average discharge larger than q_0 (i.e. of Θ). Hence, let the prior density of Θ be informative: $\mathrm{Ig}(\theta|5,\mu)$ where μ follows from Eq. (8.6) if $q=q_2$.

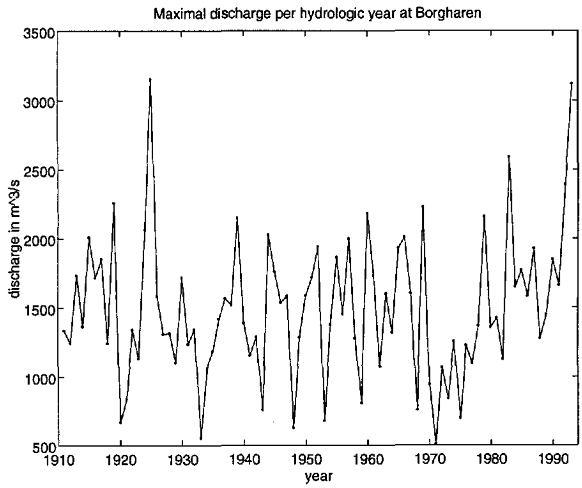

Figure 8.3: Maximal discharge per hydrological year at Borgharen in 1911-1993.

In turn, Bayes' theorem can be applied to update the historical prior information with the observations of the years 1911-1993. These 83 observations are displayed in Fig. 8.3 and can be found in [37]: 13 observations are larger than q_0 and 3 are larger than q_2. The posterior distribution of Φ, the limiting relative frequency of discharges larger than q_0, given the observations is the beta distribution $\mathrm{Be}(\phi|13,70)$. The posterior distribution of the limiting average discharge Θ is the inverted gamma distribution $\mathrm{Ig}(\theta|18,5630.12)$. The (posterior) predictive probability of exceedence of a discharge q, Eq. (8.7), is shown in Fig. 8.4. It slightly differs from the frequentist result [31] in the sense that the probabilities of exceedence of very extreme discharges are larger, just because the uncertainty is taken into account (see [37]).

8.3.2 Uncertainty in the flood damage

Beside the uncertainty in the occurrence frequency of the Borgharen discharges, the uncertainty in the flood damage is also important. Since the Meuse flood in December

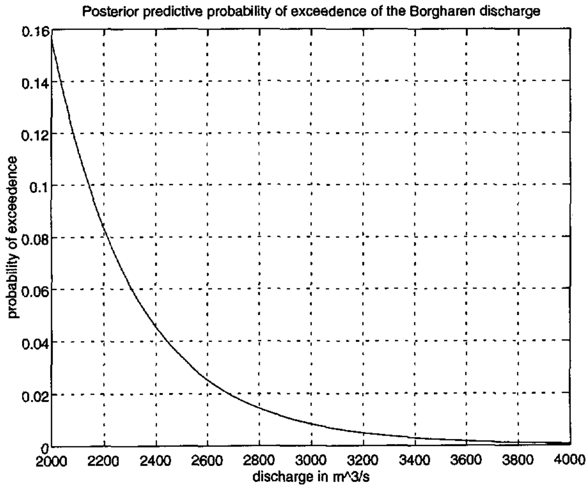

Figure 8.4: Posterior predictive probability of exceedence of the Borgharen discharge q.

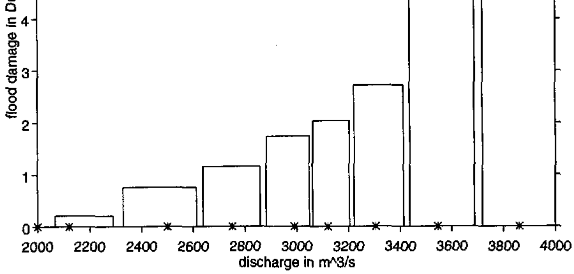

Figure 8.5: Expected flood damage in Limburg given the Borgharen discharges (*) for the present situation.

1993 caused much damage, there is much data available (at least on damage caused by flood depths up to about 1 metre). For example, about 5600 houses were flooded. For about 4600 houses, damage data is known with an average damage of about 15,600 Dfl per house (see Table 8.3). The damage assessment was done by experts of the insurance companies by order of the Dutch government, because the government partly covered the flood damage. The methodology to determine the uncertainty in the flood damage, will be explained in the light of the damage category "houses".

Table 8.3: The average flood damage per house [Dutch guilders] and the number of flooded houses against the flood depth [cm] due to the Meuse flood in December 1993.

Flood depth [cm]	0-10	10-17.5	17.5-25	25-50	50-75	> 75
Average flood damage [Dfl]	7,500	10,500	12,200	15,200	18,600	18,800
Number of flooded houses	249	578	374	1262	1052	1059

To deal with a large amount of observations (e.g. 4600 flooded houses), it is necessary to summarise them. As we have argued in Sec. 8.2.2, the probability distributions of the flood damage can best be assessed conditional on disjunct classes of flood depths (see Table 8.3). The conditional probability distribution of the flood damage, given a flood depth class, must satisfy the following three criteria:

- flood damage is non-negative (the assumption that the damage has a normal distribution can cause inconsistencies, especially if we are interested in the tails);
- per flood depth, a proper summarising statistic is the number of flooded houses and the corresponding sum of flood damages to these houses;
- the predictive flood damage equals the outcome of the damage assessment model MAAS-GIS in which the Meuse has been subdivided into 200,000 "pieces" (see [33] and De Jonge, Kok & Hogeweg [24]).

Let the random quantity $Z_i \geq 0$ be the flood damage to house i for a particular flood depth class, where $i \in \mathbb{N}$. For the decision-maker only the number of flood damages and the sum of flood damages are important. Therefore, we may assume that the decision-maker is indifferent to the way this sum is composed. In other words, we may assume that the joint probability density function of $Z_1, \ldots, Z_n$ can be written as a function of the statistic $\sum_{i=1}^{n} Z_i$: i.e. $p(z_1, \ldots, z_n) = f_n\left(\sum_{i=1}^{n} z_i\right)$ (see Mendel [89]). Hence, per flood depth class, $Z_1, \ldots, Z_n$ are l_1-isotropic and there exists a probability distribution of the average flood damage per house, Λ, which can be updated with the observations in Table 8.3 using Bayes' theorem. Analogous to the discharges, we use the inverted gamma distribution $\text{Ig}(\lambda|\nu, \mu)$ to represent the uncertainty in the average flood damage per house Λ. As a consequence, the sum of flood damages to n houses, $U_n = \sum_{i=1}^n Z_i$, has a gamma-gamma distribution:

$$\mathrm{Gg}(u_n|\,\nu,\mu,n) = \textstyle\int_0^\infty \mathrm{Ga}(u_n|\,n,1/\lambda)\mathrm{Ig}(\lambda|\,\nu,\mu)\,d\lambda \tag{8.8}$$

with mean $E(U_n) = n \cdot [\mu/(\nu-1)]$ (see Bernardo & Smith [9, Ch. 3]). Because the expected flood damage to n houses has to be equal to the outcome of the damage assessment model, we choose the non-informative prior density $\lambda^{-2}I_{(0,\infty)}(\lambda)$. In the presence of the observations $z_1,\ldots,z_m$, this results in the inverted gamma distribution $\mathrm{Ig}(\lambda|m+1,\sum_{i=1}^m z_i)$. Indeed, $E(U_n) = n \cdot [(\sum_{i=1}^m z_i)/m]$. Note that the larger the predictive number of houses, the larger the absolute uncertainty, but the smaller the relative uncertainty.

Given the flood depth class, we can best summarise the observations by the number of houses, industries, farms, green houses [hectares], roads [km], government agencies, and the corresponding sum of flood damages to these objects. The unit of area for green houses [hectare] and the unit of length for roads [km] are chosen in such a way that the uncertainties per damage category are of the same order of magnitude (for the underlying mathematics, see Chapter 3).

Up to now, we have considered the ideal case, i.e. the case for which the probability distribution of the flood damage can be determined for every flood depth class separately. Eventually, the number of flood depth classes for other categories than houses appeared to be so large (e.g. for industries: 60) that it was rather impossible to assess the number of flooded objects for every flood depth class separately. With the damage assessment model, we can only determine the number of flooded objects and the corresponding flood damage aggregated per damage category. On the basis of the number of objects and the flood damage per category, and the detailed information on houses, we may approximate the probability distributions of the flood damage to other categories than houses. From [37], it appears that the flood damages summed over all flood depth classes can be approximated by the gamma-gamma distibution $\mathrm{Gg}(\nu,\mu,n)$ for which:

$$\begin{aligned} n &= \text{“number of flooded objects from damage assessment model”} \\ \nu &= 0.5 \times \text{“number of flooded objects in December 1993”} \\ \mu &= [(\nu-1)/n] \times \text{“total damage to flooded objects”}. \end{aligned}$$

The number of flooded objects in December 1993 determines the uncertainty in the average damage per object. The larger the amount of data of 1993, the smaller the uncertainty in the average damage. Finally, the probability distribution of the total damage summed over all damage categories can be approximated by a gamma distribution with equal first and second moments (see [37]). In Fig. 8.5, we present the expected flood damage summed over all categories for every discharge in Table 8.2, calculated by the damage assessment model for the present situation (see [33]).

8.3.3 Uncertainty in the water level

Given a discharge at Borgharen, we can obtain the water level at locations downstream with the physical model ZWENDL (see [32]). Beside the uncertainty in the discharge, the added uncertainty in the water level is also important. Conditional on the Borgharen discharge, the uncertainty in the water level is twofold:

- the uncertainty in *the shape of the waterwave* at Borgharen (see [32]);
- the uncertainty in *the physical model* with which the water levels have been determined.

The two types of uncertainty are complementary: the larger the discharge at Borgharen, the smaller the variation in the shape of the waterwave, but the larger the uncertainty in the physical model.

On the basis of the two types of uncertainty in the water levels, the uncertainty in the frequency of occurrence of the water levels have to be determined. It is well-known that there exists an approximate power law between the discharge (e.g. at Borgharen) and the downstream water levels (see e.g. Shaw [107, Ch. 6]):

$$q - q_0 = a(h - h_0)^b, \qquad q > q_0 > 0, \quad h > h_0 > 0 \tag{8.9}$$

with q the upstream discharge $[m^3/s]$ and h the downstream water level $[m]$.

Since the power law between discharge and water level is just an approximation, the quantities a and b are unknown. Hence, we are interested in the joint probability density function of the random quantities A and B, where A and B may be dependent. For just this situation, Cooke [18] has developed a technique to obtain the marginal distributions of A and B, and their correlation, on the basis of the uncertainties in the (logarithm of the) water level given the (logarithm of the) discharge.

Although the transformation from discharges at Borgharen to downstream water levels introduces extra uncertainties, they are not so large that they should be taken into account (for details, see [37]).

8.3.4 Uncertainty in the costs and the yields of decisions

The probability distributions of the costs and the yields of decisions are based on [29, 34, 35]. They are all assumed to be gamma distributed with parameters that are assessed using informal expert judgment (for details, see [37]).

Costs of extracting sand and gravel.
As the 5%- and the 95%-percentile of the probability distribution of the costs of extracting sand and gravel in the north and the middle of Limburg, we use the mean in [37] plus or minus a deviation of 10%. Since there is more knowledge about the

price of extracting sand and gravel in the south of Limburg, this deviation from the mean is taken to be plus or minus 5%.

Yields of extracting sand and gravel.
Since the yields of extracting sand and gravel are just a matter of supply and demand, the uncertainty in the yields is probably larger than the uncertainty in the costs. Hence, we assume the 5%- and the 95%-percentile to be the mean plus or minus a deviation of 17%.

Costs of constructing embankments and dykes.
For the construction of embankments and dykes, one usually expresses the uncertainty in the costs by an uncertainty factor F, where $F \approx 2.5$. Instead, we regard F as a random quantity that satisfies the following two requiremensts:

$$\Pr\{F \leq 2\} = 0.1 \quad \cap \quad \Pr\{F > 2.5\} = 0.95. \tag{8.10}$$

Next, we scale the gamma distribution satisfying these two equalities with the expected costs in [37].

Dependencies between costs and yields of decisions.
Since the yields of sand and gravel do not depend so much on the locations of extraction, we assume the yields per region (north, middle and south) to be dependent with a rank correlation of 0.75. The costs of extracting sand and gravel are dependent, but less than the yields, since the costs are partly determined by the location. Hence, we assume a rank correlation between the costs per region of 0.5. Furthermore, the costs and yields both depend on the same price-level: by assumption, a rank correlation of 0.2.

8.3.5 Uncertainty in the loss

On the basis of the probability distribution of the discharge, we can derive an approximate expression for the loss function in Eq. (8.1): the costs of decisions minus the yields of decisions plus the remaining flood damage. Since we have to find an optimum balance between the initial investment costs and the future flood damage costs, we can best compare decisions over an unbounded time-horizon by using the discounted cost criterion (see Chapter 4). By discounting the future costs, yields, and flood damage, the loss can be approximated by (see [33]):

$$\begin{aligned} L(w,d) &= \sum_{j=0}^{k} \tilde{\alpha}^j \left[c_j(d) - y_j(d)\right] + \\ &\quad + \frac{\alpha}{1-\alpha} \sum_{i=1}^{9} \Pr\left\{q_{i-1} < Q \leq q_i | \phi, \theta\right\} \times \frac{\left[s_{i-1}(d) + s_i(d)\right]}{2}, \end{aligned} \tag{8.11}$$

where $\Pr\{Q > q|\phi,\theta\} = \phi\exp\{-(q-q_0)/\theta\}$ and the vector $w = (\mathbf{c}(d), \mathbf{y}(d), \mathbf{s}(d), \phi, \theta)$ is uncertain. The loss function in Eq. (8.11) represents the net present discounted value of the costs minus the yields plus the remaining mean flood damage over an unbounded horizon, where

d	=	decision or strategy (Sec. 8.2.1)
k	=	15 years (time-horizon for carrying out decision d; see [37])
$c_j(d)$	=	costs of decision d in year j (Sec. 8.3.4)
$y_j(d)$	=	yields of decision d in year j (Sec. 8.3.4)
q_i	=	discharge at Borgharen in m^3/s, $i = 0, 1, \ldots, 9$ (Table 8.2)
ϕ	=	limiting relative frequency of discharges larger than q_0 (Sec. 8.3.1)
θ	=	limiting average discharge larger than q_0 (Sec. 8.3.1)
$s_i(d)$	=	total flood damage to houses, industries, farms, green houses, roads, and government agencies per discharge q_i when making decision d, where $s_0(d) = 0$ and $s_9(d) = s_8(d)$, $i = 0, 1, \ldots, 9$ (Sec. 8.3.2)
r	=	discount rate per year (5%)
g	=	growth rate of capital in the flood plain per year (1%)
$\tilde{\alpha}$	=	$[1 + (r/100)]^{-1}$
α	=	$[1 + ((r-g)/100)]^{-1}$

As has been discussed in Sec. 8.2, an optimal decision d^* with respect to the probability distribution of the random vector W is the decision whose expected net present discounted value of the costs minus the yields plus the remaining mean flood damage, over an unbounded time-horizon, is minimal. Since the discount rate r and the growth rate of captital g are essentially based on an agreement on comparing decisions over a long time-horizon, we assume these to be constant. The expected values $E(\Pr\{Q > q|\Phi,\Theta\})$ and $E(S_i(\emptyset))$, $i = 0, 1, \ldots, 8$, are shown in Figs. 8.4 and 8.5, respectively. We have determined both the expected loss and the uncertainty in the loss by using Monte Carlo simulation.

8.4 The results of the uncertainty analysis

When the uncertainties in the discharge, the flood damage, and the costs and the yields of decisions are given, the uncertainty in the loss can be determined. We consider the present situation and the five selected strategies of Sec. 8.2.1. The results of the uncertainty analysis have been obtained by the Monte Carlo simulation program UNICORN (see Cooke [19]). On the basis of the probability distributions of the random quantities $C_j(d)$, $Y_j(d)$, $j = 1, \ldots, k$, $S_i(d)$, $i = 1, \ldots, 9$, Φ and Θ, and their correlations, the UNICORN program approximates the probability distribution of an analytic function of these random quantities, like the loss function, by performing dependent Monte Carlo sampling as described in Cooke & Waij [20]. The sample size was 200,000 for the present situation and 100,000 for the five strategies; the number of random quantities was 37 per strategy (for details, see [37]).

8.4.1 Uncertainty in the mean flood damage

In Figs. 8.6 and 8.7, some probabilistic characteristics of the damage due to flooding of the Meuse are displayed.

In Fig. 8.6, the approximated probability density function is shown of $s(w, \emptyset)$, the present discounted value of the mean flood damage over an unbounded horizon when no decisions are taken.

In Fig. 8.7, the results of the uncertainty analysis are displayed in the form of 80%-probability bars (i.e. probability masses between the 10%-percentile and the 90%-percentile). For the present situation and for the five strategies, these bars represent the uncertainty in $s(w, d)$: the present discounted value of the remaining mean flood damage over an unbounded horizon. As should be expected, all strategies result in a smaller expected mean flood damage, and a smaller uncertainty, than for the present situation.

8.4.2 Uncertainty in the loss

In Figs. 8.8 and 8.9, the uncertainties are displayed in the costs of decisions plus the remaining mean flood damage, the benefits of decisions, and the loss. The costs of decisions, i.e. $c(w, d)$, consist of the present discounted value of the costs of extracting sand and gravel, the costs of constructing embankments and dykes, the future cost of cleaning up the polluted summer bed in the north of Limburg, and the remaining costs, over a bounded horizon of 15 years (see Secs. 8.3.4 and 8.3.5). The benefits of decisions, i.e. $y(w, d) + s(w, \emptyset) - s(w, d)$, consist of the present discounted value of the yields of extracting sand and gravel over a bounded horizon of 15 years and the reduction in the mean flood damage over an unbounded horizon when choosing decision d.

8.5 Conclusions

In this chapter, we have compared five strategies (combinations of decisions) that reduce damage due to flooding of the Dutch river Meuse. Because we seek an optimum balance between initial investment costs and future flood damage costs, we have used the cost-based criterion of the (discounted) loss, where the loss is defined as the net present discounted value of the costs of decisions minus the yields of decisions plus the mean flood damage over an unbounded time-horizon. The strategies have been compared with respect to three types of decision criteria: the criterion of minimal expected loss, the criterion of minimal uncertainty in the loss, and the criterion of maximal safety. On the basis of the results in Sec. 8.4, we can conclude the following:

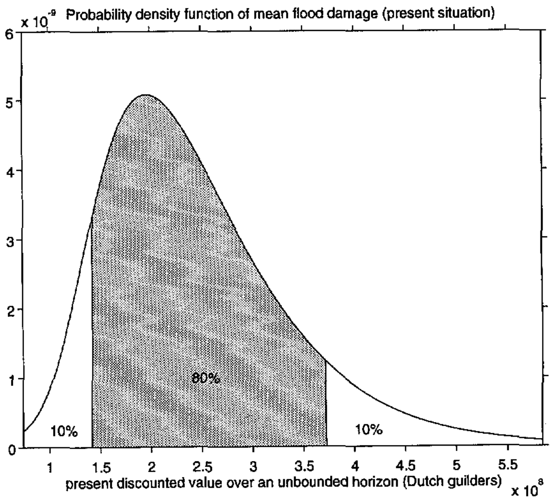

Figure 8.6: The probability density function, and its 10%- and 90%-percentiles, of the present discounted value of the mean flood damage over an unbounded horizon when no decisions are made.

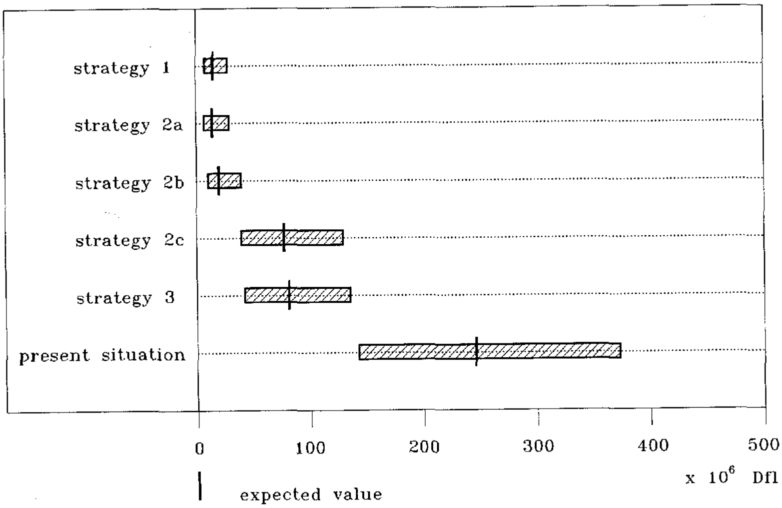

Figure 8.7: The present discounted value of the mean flood damage over an unbounded horizon (in millions of Dutch guilders). For the present situation and for five strategies, the 80%-probability bars and the expected values are displayed.

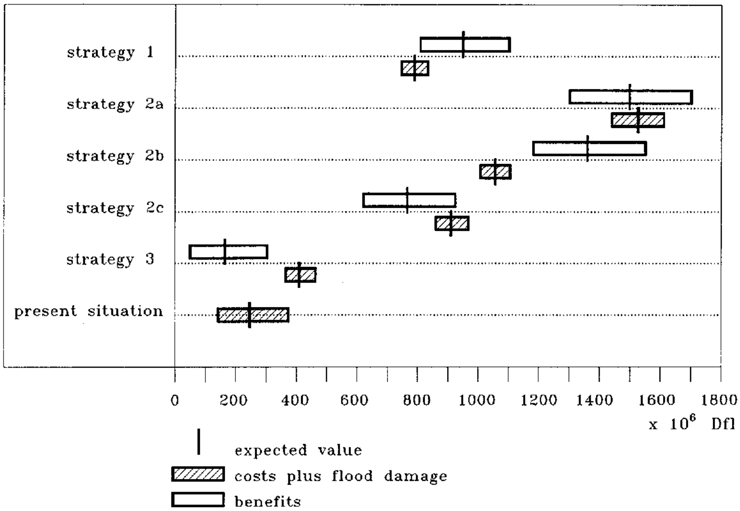

Figure 8.8: The present discounted value of the costs plus the remaining mean flood damage and of the benefits over an unbounded horizon (in millions of Dutch guilders). For the present situation and for five strategies, the 80%-probability bars and the expected values are displayed.

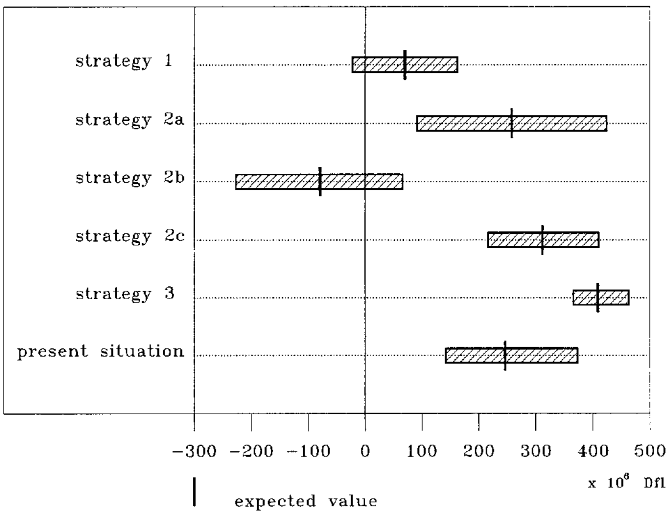

Figure 8.9: The net present discounted value of the loss: the costs minus the yields plus the remaining mean flood damage over an unbounded horizon (in millions of Dutch guilders). For the present situation and for five strategies, the 80%-probability bars and the expected values are displayed.

- In the present situation, there is a large uncertainty in the present discounted value of the mean flood damage over an unbounded horizon.
- The uncertainties in the losses are relatively large for all strategies.
- The expected present discounted value of the mean flood damage over an unbounded horizon is "minimal" for the Strategies 1, 2a, and 2b. These strategies are "optimal" with respect to the criterion of maximal safety and have a small uncertainty in the flood damage.
- The classification of the strategies on the basis of the expected loss, from a small loss to a large loss, is: Strategy 2b - Strategy 1 - present situation - Strategy 2a - Strategy 2c - Strategy 3.
- The classification of the strategies on the basis of the uncertainty in the loss, from a small uncertainty to a large uncertainty, is: Strategy 3 - Strategy 1 - Strategy 2c - present situation - Strategy 2b - Strategy 2a.
- The strategy with minimal uncertainty in the loss is the most expensive strategy: Strategy 3 (the construction of embankments and dykes only).
- The extraction of sand and gravel is very uncertain.
- From the viewpoint of both the expected loss and the uncertainty in the loss, even the present situation is to be preferred to Strategy 2a.
- The uncertainties in the benefits are larger than the uncertainties in the costs of decisions plus the remaining mean flood damage.
- In comparison with the present situation, the strategy with minimal expected loss (Strategy 2b) has a large uncertainty in the loss. For Strategy 2b, the uncertainty, and therefore the risks, will be "shifted" from private persons and industries dealing with flood damage to the government.

Chapter 9

Optimal Maintenance Decisions over Unbounded Horizons on the Basis of Expert Judgment

Jan M. van Noortwijk

Abstract. Due to a lack of data, many maintenance optimisation models have to be initialised on the basis of expert jugdment. Rather than eliciting the parameters of a continuous lifetime distribution, experts give more reliable answers when assessing a discrete lifetime distribution. If the prior uncertainty in the probabilities of failure per unit time is expressed in terms of a Dirichlet distribution, Bayes estimates can be obtained of three cost-based criteria to compare maintenance decisions over unbounded time-horizons: (i) the expected average costs per unit time, (ii) the expected discounted costs over an unbounded horizon, and (iii) the expected equivalent average costs per unit time. The maintenance model presented here can be easily implemented and is illustrated by determining optimal age replacement and life cycle costing policies. (This chapter has previously been published as [125].)

Keywords. maintenance, expert judgment, Dirichlet distribution, age replacement, life cycle costing, renewal theory.

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9.1 Introduction

Although many maintenance optimisation models can be found in the literature, only a few of them have been applied (see Dekker [28]). Probably the main reason for this is that these models often presume the availability of data which does not exist. In many cases there is not enough data to determine, for example, the parameters of a lifetime distribution (see Van Noortwijk et al. [128] and Pintelon & Gelders [97]). In this situation, the parameters must be initialised on the basis of expert judgment.

Requiring experts to provide subjective estimates of abstract (scale and shape) parameters of a lifetime distribution is a lot to ask even from statistically trained experts: they give more reliable answers when assessing histograms (see Kabus [70] and Ibrekk & Morgan [64]). For this reason, Van Noortwijk et al. [128] have proposed to elicit discrete lifetime distributions, where the prior uncertainty in the probabilities of failure per unit time is expressed in terms of a Dirichlet distribution. As data becomes available, this prior distribution can be updated to the posterior distribution using Bayes' theorem. Wang [139] has used this approach to estimate discrete delay time distributions.

Both Van Noortwijk et al. [128] and Wang [139] have suggested fitting continuous parametric lifetime distributions to the Bayes estimates (posterior means) of the failure probabilities per unit time. In turn, such continuous lifetime distributions can be used to determine the expected costs of maintenance in, for example, an age replacement model. However, rather than introducing errors due to loss of information through statistical fitting, the elicited discrete lifetime distributions can better be preserved as starting points for further maintenance modelling.

This chapter focusses on Bayes estimates of the costs of maintenance when the probabilities of failure are Dirichlet distributed. In Sec. 9.2, Bayes estimates are obtained of three cost-based criteria to compare maintenance decisions over unbounded time-horizons: (i) the expected average costs per unit time, (ii) the expected discounted costs over an unbounded horizon, and (iii) the expected equivalent average costs per unit time. In Sec. 9.3, an illustrative example is given to determine optimal age replacement policies with respect to the above three criteria. In fact, a new model for the purpose of life cycle costing is developed in order to find an optimum balance between the initial costs of investment and the future costs of age replacement. Some definitions of probability distributions can be found in Appendix A.

9.2 The expected costs of maintenance

Most maintenance processes can be regarded as renewal processes, where the renewals are the repairs that bring a component back into its "as good as new state". Since the purpose of maintenance is reducing the number of failures, failure data is rarely observed, and expert judgment of discrete probability functions can best be used instead. Let the domain of lifetimes $(0,\infty)$ be divided into n units of time, $([i-1]\tau, i\tau]$, $i = 1,\ldots,n-1$, and $([n-1]\tau,\infty)$ for $\tau > 0$. Furthermore, we assume the renewal times to be non-negative independent random quantities having a discrete probability function $p_i(d)$, $i = 1,\ldots,n$, with $\sum_{i=1}^{n} p_i(d) = 1$, where $p_i(d)$ represents the probability of a renewal in unit time i when the decision-maker chooses maintenance decision d. We denote the costs associated with a renewal in unit time i by $c_i(d)$, $i = 1,\ldots,n$. For convenience, let $\tau = 1$

In order to obtain optimal maintenance decisions against minimal costs, we define $L(\mathbf{p}(d), d)$ to be the (monetary) *loss* when the decision-maker chooses maintenance decision d, where $d \in \mathcal{D}$. The decision-maker can best choose a maintenance decision d^* whose expected loss with respect to the probability distribution of the random vector $\mathbf{P}(d) = (P_1(d),\ldots,P_{n-1}(d))$ is minimal. A decision d^* is called an *optimal* or a *Bayes optimal* decision when (see DeGroot [27, Ch. 8])

$$E\left(L(\mathbf{P}(d^*), d^*)\right) = \min_{d\in\mathcal{D}} E\left(L(\mathbf{P}(d), d)\right). \tag{9.1}$$

According to Chapter 4, there are basically three cost-based criteria that can serve as loss functions: (i) the average costs per unit time, (ii) the discounted costs over an unbounded horizon, and (iii) the equivalent average costs per unit time. These cost-based criteria can be obtained using the discrete renewal theorem (see Feller [47, Ch. 13] and Karlin & Taylor [71, Ch. 3]). For notational convenience, we write $p_i(d) = p_i$ and $c_i(d) = c_i$, $i = 1,\ldots,n$.

First, the expected *average costs per unit time* are determined by averaging the expected costs over an unbounded horizon:

$$C(\mathbf{p}) = \lim_{m\to\infty} \frac{C(m,\mathbf{p})}{m} = \frac{\sum_{i=1}^{n} c_i p_i}{\sum_{i=1}^{n} i p_i} = \frac{c_n - \sum_{i=1}^{n-1}[c_n - c_i]\,p_i}{n - \sum_{i=1}^{n-1}[n-i]\,p_i}, \tag{9.2}$$

where $C(m,\mathbf{p})$ are the expected costs in time-interval $(0,m]$. Eq. (9.2) is a well-known result from renewal reward theory (see e.g. Ross [105]).

Second, the expected *discounted costs over an unbounded horizon* are determined by summing the expected discounted values of the costs over an unbounded horizon, where the discounted value of the costs c_n in unit time n is defined to be $\alpha^n c_n$ with discount factor $\alpha = [1 + (r/100)]^{-1}$ and discount rate $r\%$ $(r > 0)$:

$$\begin{aligned} C_\alpha(\mathbf{p}) &= \lim_{m\to\infty} C_\alpha(m,\mathbf{p}) = \frac{\sum_{i=1}^{n} \alpha^i c_i p_i}{\sum_{i=1}^{n}(1-\alpha^i)p_i} \\ &= \frac{\alpha^n c_n - \sum_{i=1}^{n-1}[\alpha^n c_n - \alpha^i c_i]\,p_i}{(1-\alpha^n) - \sum_{i=1}^{n-1}\left[(1-\alpha^n) - (1-\alpha^i)\right]p_i}, \end{aligned} \tag{9.3}$$

where $C_\alpha(m,\mathbf{p})$ are the expected discounted costs in time-interval $(0,m]$ (see Chapter 4).

Third, the expected *equivalent average costs per unit time* are determined by averaging the discounted costs over an unbounded horizon: i.e. by dividing the expected discounted costs over an unbounded horizon by $1/(1-\alpha)$. In fact, the notion of equivalent average costs relates the notions of average costs and discounted costs in the sense that the equivalent average costs per unit time approach the average costs per unit time, as α tends to 1, from below:

$$\lim_{\alpha \uparrow 1} (1-\alpha) C_\alpha(\mathbf{p}) = C(\mathbf{p}). \tag{9.4}$$

As Van Noortwijk et al. [128] have proposed, let the random vector of probabilities of failure per unit time, i.e. $\mathbf{P} = (P_1, \ldots, P_{n-1})$, have a Dirichlet distribution such that $\mathbf{P} \sim \mathrm{Di}_{n-1}(a_1, \ldots, a_n)$. According to Theorem 18 (see the appendix), we have

$$E\left(\frac{v_n - \sum_{i=1}^{n-1} [v_n - v_i] P_i}{b_n - \sum_{i=1}^{n-1} [b_n - b_i] P_i}\right) =$$
$$= \sum_{j=1}^{n} \frac{v_j a_j}{\sum_{i=1}^{n} a_i} E\left((1 + [b_j - 1] X)^{-1} \prod_{i=1}^{n} (1 + [b_i - 1] X)^{-a_i} \right), \tag{9.5}$$

where $X \sim \mathrm{Be}(1, \sum_{i=1}^{n} a_i)$ for $a_i, b_i, v_i > 0$, $i = 1, \ldots, n$, and $\sum_{i=1}^{n} a_i > 0$. The expected average costs per unit time $E(C(\mathbf{P}))$, Eq. (9.2), follow immediately from Eq. (9.5) when putting $v_i = c_i$ and $b_i = i$ for $i = 1, \ldots, n$. Likewise, the expected discounted costs over an unbounded horizon $E(C_\alpha(\mathbf{P}))$, Eq. (9.3), follow when putting $v_i = \alpha^i c_i$ and $b_i = 1 - \alpha^i$ for $i = 1, \ldots, n$.

When the renewal costs do not depend on the renewal time, i.e. when $c_i = c$ for all $i = 1, \ldots, n$, then the expected average costs per unit time reduce (see Theorem 18 from the appendix) to

$$E(C(\mathbf{P})) = E\left(\frac{c}{n - \sum_{i=1}^{n-1} [n-i] P_i}\right) =$$
$$= cE\left(\prod_{i=1}^{n} (1 + [i-1] Y)^{-a_i}\right), \tag{9.6}$$

where $Y \sim \mathrm{Be}(1, \sum_{i=1}^{n} a_i - 1)$ for $\sum_{i=1}^{n} a_i > 1$. Similarly, the expected discounted costs over an unbounded horizon can be written as

$$E(C_\alpha(\mathbf{P})) = E\left(\frac{c}{[1-\alpha^n] - \sum_{i=1}^{n-1} [\alpha^i - \alpha^n] P_i}\right) - c =$$
$$= c\left[E\left(\prod_{i=1}^{n} (1 - \alpha^i Y)^{-a_i}\right) - 1\right], \tag{9.7}$$

where $Y \sim \mathrm{Be}(1, \sum_{i=1}^{n} a_i - 1)$ for $\sum_{i=1}^{n} a_i > 1$.

9.3 The expected costs of age replacement

In this section, we consider the problem of specifying an age replacement policy that balances both the failure costs against the preventive repair costs and the initial costs against the future costs.

Under an age replacement policy, a repair is carried out at age k (preventive repair) or at failure (corrective repair), whichever occurs first, where $1 \leq k < n$. A preventive repair entails a cost c_P, whereas a corrective repair entails a cost c_F, where $0 < c_P \leq c_F$. By Eq. (9.2), the expected average costs per unit time under age replacement are

$$C(\mathbf{p}, k) = \frac{\sum_{i=1}^{k} c_F\, p_i + \sum_{i=k+1}^{n} c_P\, p_i}{\sum_{i=1}^{k} i p_i + \sum_{i=k+1}^{n} k p_i}, \tag{9.8}$$

while, by Eq. (9.3), the expected discounted costs over an unbounded horizon are

$$C_\alpha(\mathbf{p}, k) = \frac{\sum_{i=1}^{k} \alpha^i c_F\, p_i + \sum_{i=k+1}^{n} \alpha^k c_P\, p_i}{\sum_{i=1}^{k}(1-\alpha^i)p_i + \sum_{i=k+1}^{n}(1-\alpha^k)p_i}. \tag{9.9}$$

Next, Eq. (9.5) can be used to obtain $E\left(C(\mathbf{P}, k)\right)$, when putting

$$\begin{array}{rcrcr} v_i & = & c_F\, I_{1,\ldots,k}(i) & + & c_P\, I_{k+1,\ldots,n}(i), \\ b_i & = & i\, I_{1,\ldots,k}(i) & + & k\, I_{k+1,\ldots,n}(i), \end{array} \tag{9.10}$$

and $E\left(C_\alpha(\mathbf{P}, k)\right)$, when putting

$$\begin{array}{rcrcr} v_i & = & \alpha^i c_F\, I_{1,\ldots,k}(i) & + & \alpha^k c_P\, I_{k+1,\ldots,n}(i), \\ b_i & = & (1-\alpha^i)\, I_{1,\ldots,k}(i) & + & (1-\alpha^k)\, I_{k+1,\ldots,n}(i), \end{array} \tag{9.11}$$

where $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$.

Suppose a decision-maker has to choose between two components: Component 2 is twice as reliable, but twice as expensive, as Component 1, where the consequences of failure are the same for both components (for the cost parameters, see Table 9.1). The decision that selects component h is denoted by decision d_h, $h = 1, 2$. For both components, the Bayes estimates of the probabilities of failure per unit time, i.e. $E(P_j) = a_j / (\sum_{i=1}^{n} a_i)$, $j = 1, \ldots, n$, are shown in Fig. 9.1. In Figs. 9.2 and 9.3, the expected average costs per unit time, $E\left(C(\mathbf{P}, k)\right)$, and the expected equivalent average costs per unit time, $E\left((1-\alpha)C_\alpha(\mathbf{P}, k)\right)$, are represented for Component 1 and 2, respectively, for age replacement interval k, $k = 1, \ldots, n$. Recall that, as α tends to 1, from below, the expected equivalent average costs approach the expected average costs (by Lebesgue's Theorem of Dominated Convergence, the order of the operations of expectation and passing to the limit may be interchanged).

For Component 1, the optimal age replacement interval k^* is 4 units of time with expected average costs per unit time of 176 Dfl and expected equivalent average costs per unit time of 156 Dfl. For Component 2, the optimal age replacement interval k^* is 11 units of time with expected average costs per unit time of 106 Dfl and expected equivalent average costs per unit time of 88 Dfl.

The component that optimally balances the initial investment costs, c_P, against the future maintenance costs, $E\left(C_\alpha(\mathbf{P},k)\right)$, can only be obtained by using a cost-based criterion that takes account of the time value of money: viz., the discounted costs or the equivalent average cost. The criterion of average costs is not useful since the initial costs are neglected by averaging them out over an unbounded horizon. Including the initial investment costs, the criterion of equivalent average costs has the form

$$E\left(L_\alpha(\mathbf{p},k)\right) = (1-\alpha)\left[c_P + E\left(C_\alpha(\mathbf{p},k)\right)\right]. \tag{9.12}$$

This criterion can be used to solve many problems in the area of life cycle costing. As an illustration, the optimal decision is choosing Component 2 under an age replacement policy of $k = 11$ units of time with minimal expected equivalent average costs per unit time of 126 Dfl (see Fig. 9.4).

When failure and maintenance data comes available, the prior distribution on the probabilities of failure per unit time, a Dirichlet distribution with parametric vector $(a_1,\ldots,a_n)$, can be updated by using Bayes' theorem. In the case of actual failure data, say $(x_1,\ldots,x_n)$, the likelihood function is given by $\prod_{i=1}^n p_i^{x_i}$ and the posterior distribution is also a Dirichlet distribution with parametric vector $(a_1+x_1,\ldots,a_n+x_n)$ (see e.g. DeGroot [27, Ch. 9]). We can interpret the sum $a_0 = \sum_{i=1}^n a_i$ as the (subjective) number of virtual observations as opposed to the (objective) number of actual observations $\sum_{i=1}^n x_i$. Note that, although the value of a_0 serves as a measure of the variability in the elicited failure probabilities, numerical experiments have shown that the optimal decisions obtained above remain unchanged for different values of a_0. In the case of right censored data (see Lochner [82] and Van Noortwijk et al. [128]), the expressions for the expected costs cannot be reduced to one-dimensional integrals, like Eq. (9.5), and cannot be easily solved numerically either.

Table 9.1: The parameters of the age replacement model.

parameter	description	value	dimension
r	discount rate per unit time	5	%
α	discount factor per unit time	0.9524	
$c_P(d_1)$	costs of preventive repair Component 1	500	Dfl
$c_P(d_2)$	costs of preventive repair Component 2	1000	Dfl
c_F	costs of failure	10000	Dfl
n	maximum lifetime of a component	40	unit time
$\sum_{i=1}^{n} ip_i(d_1)$	mean lifetime Component 1	10	unit time
$\sum_{i=1}^{n} ip_i(d_2)$	mean lifetime Component 2	20	unit time
$\sum_{i=1}^{n} a_i(d_1)$	number of virtual observations Component 1	100	
$\sum_{i=1}^{n} a_i(d_2)$	number of virtual observations Component 2	100	

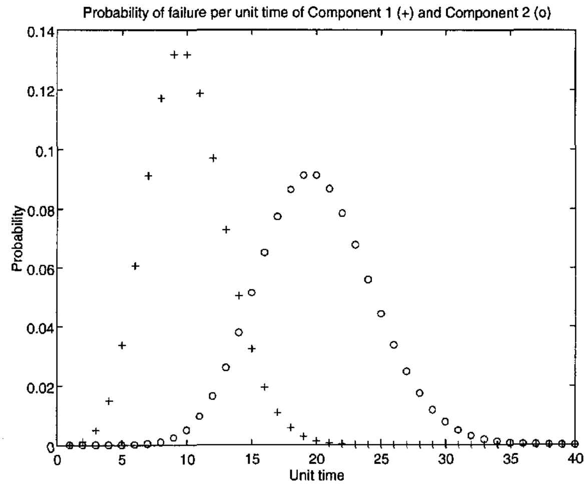

Figure 9.1: The probability of failure per unit time for Component 1 and Component 2.

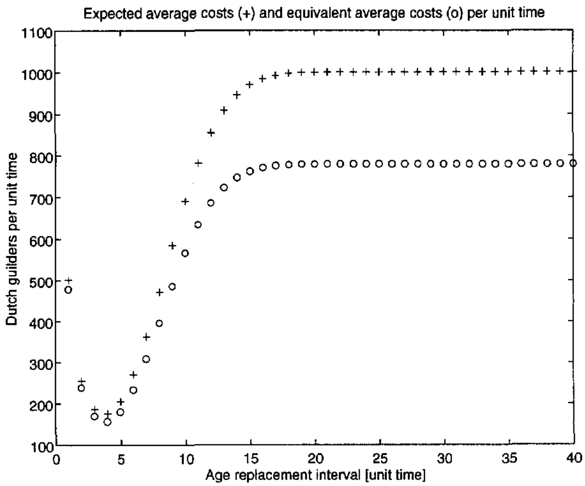

Figure 9.2: The expected average costs per unit time and the expected equivalent average costs per unit time for Component 1.

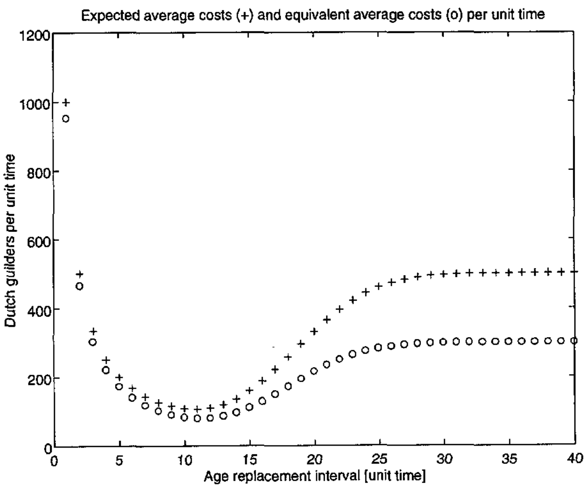

Figure 9.3: The expected average costs per unit time and the expected equivalent average costs per unit time for Component 2.

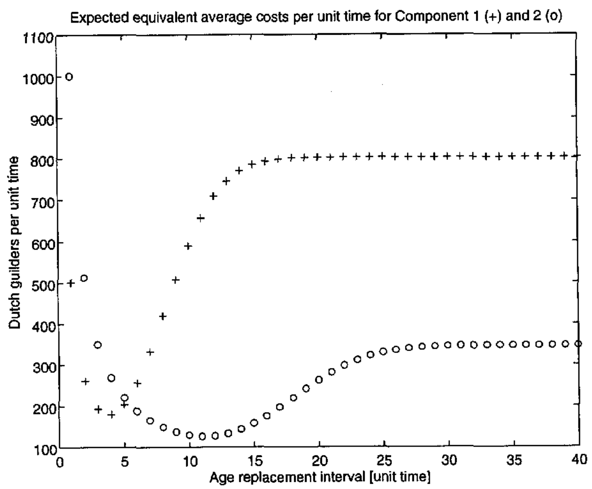

Figure 9.4: The expected equivalent average costs per unit time for Component 1 and Component 2.

9.4 Appendix: Proof of theorem

Theorem 18
Let $(P_1,\ldots,P_{n-1}) \sim \mathrm{Di}_{n-1}(a_1,\ldots,a_n)$ *and* $Y \sim \mathrm{Be}(q, \sum_{i=1}^n [a_i + r_i] - q)$. *Then,*

$$E\left(\frac{\left[1-\sum_{i=1}^{n-1} P_i\right]^{r_n} \prod_{i=1}^{n-1} P_i^{r_i}}{\left(b_n - \sum_{i=1}^{n-1}[b_n - b_i]\, P_i\right)^q}\right) =$$

$$= \frac{\Gamma(\sum_{i=1}^n a_i)}{\prod_{i=1}^n \Gamma(a_i)} \frac{\prod_{i=1}^n \Gamma(a_i + r_i)}{\Gamma(\sum_{i=1}^n a_i + r_i)} E\left(\prod_{i=1}^n (1 + [b_i - 1]\, Y)^{-[a_i+r_i]}\right) \tag{9.13}$$

for $a_i, b_i > 0$, *and* $r_i \geq 0$, *for all* $i = 1,\ldots,n$, *and* $\sum_{i=1}^n [a_i + r_i] > q > 0$.

Proof:
Eq. (9.13) can be obtained by solving the n-dimensional integral

$$H_n = \underset{\substack{x_1 \geq 0 \quad x_n \geq 0 \\ x_1 + \ldots + x_n \leq 1}}{\int \cdots \int} \frac{\prod_{i=1}^n x_i^{a_i + r_i - 1}}{(\sum_{i=1}^n b_i x_i)^q}\, dx_1 \ldots dx_n$$

in two ways. On the one hand, making the transformation $t = \sum_{i=1}^n x_i$ and $p_i = x_i/t$, $i = 1,\ldots,n-1$, with Jacobian t^{n-1}, results in

$$H_n = \underset{\substack{p_1 \geq 0 \quad p_{n-1} \geq 0 \\ p_1 + \ldots + p_{n-1} \leq 1}}{\int \cdots \int} \int_{t=0}^1 \frac{\left[t - \sum_{i=1}^{n-1} t p_i\right]^{a_n + r_n - 1} \prod_{i=1}^{n-1} (t p_i)^{a_i + r_i - 1}}{\left(b_n t - \sum_{i=1}^{n-1}[b_n - b_i]\, t p_i\right)^q} t^{n-1}\, dt dp_1 \ldots dp_{n-1}$$

$$= \underset{\substack{p_1 \geq 0 \quad p_{n-1} \geq 0 \\ p_1 + \ldots + p_{n-1} \leq 1}}{\int \cdots \int} \frac{\left[1 - \sum_{i=1}^{n-1} p_i\right]^{a_n + r_n - 1} \prod_{i=1}^{n-1} p_i^{a_i + r_i - 1}}{(\sum_{i=1}^n [a_i + r_i] - q)\left(b_n - \sum_{i=1}^{n-1}[b_n - b_i]\, p_i\right)^q}\, dp_1 \ldots dp_{n-1}.$$

On the other hand, due to a classical result of Fichtenholz [50, pp. 402-403] (see also Gradshteyn & Ryzhik [58, p. 624]), we can write

$$H_n = \frac{\prod_{i=1}^n \Gamma(a_i + r_i)}{\Gamma(\sum_{i=1}^n [a_i + r_i] - q + 1)\, \Gamma(q)} \int_{u=0}^{\infty} \frac{u^{q-1}}{\prod_{i=1}^n (1 + b_i u)^{a_i + r_i}}\, du$$

$$= \frac{\prod_{i=1}^n \Gamma(a_i + r_i)}{\Gamma(q)\, \Gamma(\sum_{i=1}^n [a_i + r_i] - q + 1)} \int_{y=0}^1 \frac{y^{q-1}(1-y)^{\sum_{i=1}^n [a_i + r_i] - q - 1}}{\prod_{i=1}^n (1 + [b_i - 1]\, y)^{a_i + r_i}}\, dy,$$

where $u = y/(1-y)$. Combining of the above two results leads to Eq. (9.13). □

Summary

To protect the Dutch lowlands against flooding, a flood defence system has been constructed in which The Netherlands is subdivided into fifty-three dyke-ring areas. These areas are surrounded by dyke rings consisting of dunes, dykes, special water-retaining structures (e.g. the Eastern-Scheldt storm-surge barrier), and higher ground. Each component of a dyke ring has to fulfill certain requirements in the areas of flood protection, environment, recreation, shipping access, road connection, transport, agriculture, fishery, and landscape. As soon as a component deteriorates to such a degree that it fails to meet its main requirements, it should be maintained; preferably against minimal costs. The Ph.D. thesis is devoted to determining cost-optimal maintenance decisions for hydraulic structures subject to deterioration.

Maintenance is defined as a combination of actions carried out to restore a hydraulic structure to, or "renew" it to, its desired condition. In hydraulic engineering, expensive condition-based preventive maintenance, i.e. maintenance based on inspecting or monitoring a structure's condition, is mainly applied. In The Netherlands, the attention is shifting from building structures to maintaining structures and the use of maintenance optimisation models is therefore of considerable interest.

There are two phases of a structure's life cycle in which it is economic to apply maintenance optimisation techniques: the design phase and the use phase. In the design phase, one might obtain an optimum balance between the initial costs of building and the future costs of maintenance and failure (being the area of life cycle costing). In the use phase, one might minimise the costs of inspection, repair, replacement, and failure. A large number of papers on maintenance optimisation models, mainly focussing on the mathematical aspects, have been published. Unfortunately, since the use of these models is restricted to situations in which abundant data is available, only a few of them have been applied.

In hydraulic engineering, a distinction can often be made between a structure's resistance (e.g. the crest-level of a dyke) and its design stress (e.g. the maximal water level to be withstood). A failure may then be defined as the event in which - due to deterioration - the resistance drops below the stress. Since deterioration is uncertain, it can best be regarded as a stochastic process.

Even though it is common to model a deterioration process mathematically as a so-called 'Brownian motion with drift' (a stochastic process with stationary independent

decrements and increments having a normal distribution), the 'Brownian motion' is inadequate in describing the deterioration of hydraulic structures. To illustrate, a dyke whose height is subject to a Brownian deterioration can, according to the model, spontaneously rise up, which cannot occur in practice. Furthermore, in most applications there is only information available in terms of a probability distribution (uncertainty distribution) of the *average* rate of deterioration.

In order that a stochastic deterioration process has the desired properties, we consider it as a so-called 'generalised gamma process'. A gamma process is a stochastic process with independent non-negative increments (e.g. the increments of crest-level decline of a dyke) having a gamma distribution with known (certain) average rate. A generalised gamma process is then defined as a so-called 'mixture' of gamma processes, where the mixture represents the uncertainty in the unknown average rate. In addition to the classical characterisation of gamma processes in terms of compound Poisson processes, the thesis presents two new mathematical characterisations of generalised gamma processes: (i) in terms of conditional probability distributions (given a cumulative amount of deterioration which serves as a summarising, sufficient, statistic for the unknown average rate) and (ii) in terms of isotropic probability distributions (an l_p-istropic probability distribution can be written as a function of the l_p-norm).

A useful property of the generalised gamma process is that various probabilistic properties, such as the probability of exceedence of a failure level per unit time, can be expressed in explicit form when the average rate of deterioration is given. In mathematical terms, this means that we can always find units of time of equal length for which the joint probability density function of the increments of deterioration can be written as a mixture of exponential probability densities. This mixture represents the uncertainty in the unknown average rate of deterioration. Since the probability density function of any finite sequence of increments can then be written as a function of the sum of the increments (i.e. the l_1-norm of the increments), the infinite sequence of increments is said to be l_1-isotropic or l_1-norm symmetric. Due to the exchangeability of the l_1-isotropic increments of deterioration, the expected cumulative amount of deterioration is linear in time.

To make optimal maintenance decisions while explicitly taking account of the uncertainty in the average rate of deterioration, statistical decision theory can be used. A decision-maker can then best choose a maintenance decision whose expected monetary loss (in terms of the expected costs of maintenance and failure) is minimal; such a decision is called an optimal decision. The expected loss is determined with respect to the probability distribution of the average rate of deterioration, which can be updated with (new) observations using Bayes' theorem.

The maintenance of hydraulic structures can best be modelled as a so-called 'renewal process', where the renewals are the maintenance actions restoring a structure to its desired condition. After each renewal we start, in a statistical sense, all over again. Since the planned lifetime (including maintenance) of the Dutch dyke rings

is essentially unbounded, maintenance decisions can best be compared over an unbounded time-horizon. There are basically three cost-based criteria that can serve as loss functions:

1. the expected average costs per unit time (which are determined by averaging the costs over an unbounded horizon);

2. the expected discounted costs over an unbounded horizon (which are determined by summing the present discounted values of the costs over an unbounded horizon); and

3. the expected equivalent average costs per unit time (which are determined by averaging the discounted costs).

These cost-based criteria can be computed using renewal theory (the discrete renewal theorem). The notion of equivalent average costs relates the notions of average costs and discounted costs. Although in the literature most attention has been focussed on the criterion of average costs, the cost-based criteria of discounted costs and equivalent average costs are most suitable for optimally balancing the initial building costs against the future maintenance costs.

On the basis of generalised gamma processes, tailor-made models have been built and implemented to enable optimal maintenance decisions to be determined for four characteristic components of a dyke ring:

Beach section: optimal sand nourishment sizes for which the expected discounted costs over an unbounded horizon are minimal with respect to the probability distribution of the average rate of ongoing coastal erosion.

Dyke section: optimal dyke heightenings for which the expected discounted costs over an unbounded horizon are minimal with respect to the probability distribution of the average rate of crest-level decline (being a combination of settlement, subsoil consolidation, and relative sea-level rise). On the basis of a physical law, crest-level decline has been regarded as a stochastic process with expected decline being linear or non-linear (approximately logarithmic) in time.

Berm breakwater: optimal inspection intervals for berm breakwaters whose expected (equivalent) average costs per unit time are minimal with respect to the probability distribution of the average rate of rock displacement (due to so-called 'longshore rock transport'). The model that is proposed is a two-phase inspection model in which the first phase represents the event of no rock displacement and the second phase represents the event of rock displacement (initiated by an armour breach). The occurrence times of armour breaches are assumed to be distributed according to a mixture of geometric densities.

Special water-retaining structure: optimal inspection intervals for two components of the sea-bed protection of the Eastern-Scheldt barrier: (i) the block mats and (ii) the rock dumping. The decision model for the block mats is a two-phase inspection model: the inter-occurrence times of scour holes are distributed according to a mixture of Poisson processes and the scour holes develop according to a generalised gamma process. The decision model for the rock dumping is based on the probability distribution of the average rate of current-induced rock displacement.

The models for sand nourishment and dyke heightening are examples of maintenance optimisation in the design phase, whereas the two inspection models are examples of maintenance optimisation in the use phase.

In the last part of the thesis, two decision models have been presented which are not directly based on deterioration processes: one model for evaluating and comparing decisions that reduce flood damage along the Meuse river (by using l_1-isotropy and discounting) and one model for optimising maintenance when the uncertainty in failure probabilities can be expressed in terms of a Dirichlet distribution (this model is useful when both resistance and stress are stochastic).

Although the decision models in the thesis have primarily been developed for the maintenance of beaches, dykes, berm breakwaters, and the Eastern-Scheldt barrier, they can also be applied to other hydraulic structures and other engineering systems for solving many decision problems in maintenance optimisation and life cycle costing.

Appendix A

Definitions of probability distributions

Definition 2 (Gamma distribution.) *A random quantity X has a gamma distribution with shape parameter $a > 0$ and scale parameter $b > 0$ if its probability density function is given by:*

$$\mathrm{Ga}(x|\,a,b) = [b^a/\Gamma(a)]\, x^{a-1} \exp\{-bx\}\, I_{(0,\infty)}(x).$$

Definition 3 (Inverted gamma distribution.) *A random quantity Y has an inverted gamma distribution with shape parameter $a > 0$ and scale parameter $b > 0$ if $X = Y^{-1} \sim \mathrm{Ga}(a,b)$. Hence, the probability density function of Y is:*

$$\mathrm{Ig}(y|\,a,b) = [b^a/\Gamma(a)]\, y^{-(a+1)} \exp\{-b/y\}\; I_{(0,\infty)}(y).$$

Definition 4 (Gamma-gamma distribution.) *A random quantity X has a gamma-gamma distribution with parameters $a, b > 0$ and $n = 1, 2, \ldots$ if its probability density function is given by:*

$$\begin{aligned} \mathrm{Gg}(x|\,a,b,n) &= \int_0^\infty \mathrm{Ga}(x|\,n,\lambda)\mathrm{Ga}(\lambda|\,a,b)\; d\lambda \\ &= \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n)} \left[\frac{x}{b+x}\right]^{n-1} \left[1 - \frac{x}{b+x}\right]^{a-1} \frac{b}{(b+x)^2}\, I_{(0,\infty)}(x). \end{aligned}$$

Definition 5 (Beta distribution.) *A random quantity X has a beta distribution with parameters $a, b > 0$ if its probability density function is given by:*

$$\mathrm{Be}(x|a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\, x^{a-1}(1-x)^{b-1}\, I_{[0,1]}(x).$$

Definition 6 (Dirichlet distribution.) *A random vector* $\mathbf{Y} = (Y_1, \ldots, Y_{n-1})$ *has a Dirichlet distribution with parameters* $a_1, \ldots, a_n > 0$ *if* $\mathbf{Y}$ *has a probability density function given by:*

$$\mathrm{Di}_{n-1}(\mathbf{y} \,|\, a_1, \ldots, a_n) = \frac{\Gamma(\sum_{i=1}^{n} a_i)}{\prod_{i=1}^{n} \Gamma(a_i)} \left[1 - \sum_{i=1}^{n-1} y_i\right]_+^{a_n - 1} \prod_{i=1}^{n-1} y_i^{a_i - 1} \, I_{[0,1]}(y_i),$$

where $[x]_+ = \max\{0, x\}$.

Definition 7 (Negative multinomial distribution.) *An* n*-dimensional random vector* $\mathbf{Y}$*, where* $\mathbf{Y} = (Y_1, \ldots, Y_n)$*, has a negative multinomial distribution with parameters* $\mathbf{p} = (p_1, \ldots, p_n)$ *and* ν *if* $\mathbf{Y}$ *has a probability function given by:*

$$\mathrm{Nm}(\mathbf{y}|\mathbf{p}, \nu) = \frac{\Gamma\left(\nu + \sum_{i=1}^{n} y_i - n\right)}{\Gamma(\nu) \prod_{i=1}^{n} \Gamma(y_i)} \left(1 - \sum_{i=1}^{n} p_i\right)^{\nu} \prod_{i=1}^{n} p_i^{y_i - 1},$$

$y_i = 1, 2, \ldots$ *for* $i = 1, \ldots, n$, $\nu > 0$, $p_i > 0$ *for all* i, *and* $\sum_{i=1}^{n} p_i < 1$.

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Samenvatting

Om Nederland te beschermen tegen overstromingen is een stelsel van waterkeringen aangelegd. Er zijn drie-en-vijftig dijkringgebieden te onderscheiden die worden omsloten door een ring van duinen, dijken, bijzondere waterkerende constructies (b.v. de stormvloedkering in de Oosterschelde) en hoge gronden. Elke component van een dijkring moet bepaalde functies vervullen op het gebied van veiligheid, natuur, recreatie, scheepvaart, verkeer, transport, landbouw, visserij en landschap. Als een component van een dijkring - tengevolge van functieverlies - zodanig veroudert dat het niet meer voldoet aan zijn belangrijkste functies, dan moet onderhoud worden uitgevoerd; en dit liefst tegen minimale kosten. Het proefschrift is gewijd aan het bepalen van kosten-optimale onderhoudsbeslissingen voor waterbouwkundige constructies die onderhevig zijn aan veroudering.

Onderhoud wordt gedefinieerd als een combinatie van activiteiten die worden uitgevoerd om een waterbouwkundige constructie terug te brengen in of te 'vernieuwen' naar de gewenste toestand. In de waterbouwkunde is er meestal sprake van kostbaar toestandsafhankelijk preventief onderhoud, d.w.z. van onderhoud dat is gebaseerd op het inspecteren van de toestand van een constructie. In Nederland verschuift de aandacht van het bouwen van constructies naar het onderhouden van constructies en het gebruik van modellen voor onderhoudsoptimalisatie is daarom van groot belang.

Gedurende de levenscyclus van een constructie zijn er twee fasen, waarin het economisch aantrekkelijk is om onderhoudsoptimalisatie toe te passen, namelijk de ontwerpfase en de gebruiksfase. In de ontwerpfase kan een optimale balans worden gevonden tussen initiële bouwkosten enerzijds en toekomstige onderhouds- en faalkosten anderzijds ('life cycle costing'). In de gebruiksfase kan de som van inspectie-, reparatie-, vervangings- en faalkosten worden geminimaliseerd. Van het grote aantal modellen voor onderhoudsoptimalisatie dat in de literatuur is gepubliceerd, concentreren de meeste zich op de wiskundige aspecten. Omdat deze modellen alleen kunnen worden gebruikt als er een overvloed van gegevens beschikbaar is, wordt slechts een klein deel hiervan toegepast.

In de waterbouwkunde wordt vaak een onderscheid gemaakt tussen de sterkte van een constructie (b.v. de kruinhoogte van een dijk) en haar ontwerpbelasting (b.v. het maximale waterniveau dat moet worden weerstaan). Falen kan dan worden gedefinieerd als de gebeurtenis waarbij de sterkte - tengevolge van veroudering of functieverlies -

beneden de belasting zakt. Aangezien veroudering (of functieverlies) onzeker is, kan zij het beste worden beschouwd als een stochastisch proces.

Hoewel het gebruikelijk is om een verouderingsproces wiskundig te modelleren als een zogenaamde 'Brownse beweging met drift' (een stochastisch proces met stationaire onafhankelijke afnemende en toenemende verouderingen die een normale verdeling hebben), is dit model niet geschikt om het verouderingsgedrag van waterbouwkundige constructies te beschrijven. Zo kan bij een normaal verdeelde veroudering een dijk spontaan omhoog komen, hetgeen in de praktijk niet gebeurt. Verder is er bij de meeste toepassingen slechts informatie aanwezig in de vorm van een kansverdeling (onzekerheidsverdeling) van de *gemiddelde* verouderingssnelheid.

Om te bewerkstelligen dat een stochastisch verouderingsproces de gewenste eigenschappen heeft, beschouwen we het als een zogenaamd 'gegeneraliseerd gamma-proces'. Een gamma-proces is een stochastisch proces met onafhankelijke, niet-negatieve aangroeiingen (b.v. toenemende verouderingen in de vorm van kruinhoogtedaling van een dijk) die een gamma-verdeling hebben met een bekende (zekere) gemiddelde veroudering. Een gegeneraliseerd gamma-proces wordt dan gedefinieerd als een zogenaamd 'mengsel' van gamma-processen, waarbij het mengsel de onzekerheid representeert in de onbekende (onzekere), gemiddelde veroudering. Als aanvulling op de klassieke karakterisering van gamma-processen in de vorm van samengestelde Poisson-processen, worden in het proefschrift twee nieuwe wiskundige karakteriseringen van gegeneraliseerde gamma-processen gepresenteerd: (i) in de vorm van conditionele kansverdelingen (gegeven een cumulatieve veroudering die dient als een samenvattende, uitputtende steekproefgrootheid voor de gemiddelde verouderingssnelheid) en (ii) in de vorm van isotropische kansverdelingen (een l_p-isotropische kansverdeling kan worden geschreven als een functie van de l_p-norm).

Een 'handige' eigenschap van gegeneraliseerde gamma-processen is dat verschillende probabilistische karakteristieken, zoals de kans op overschrijden van een faalgrens in een bepaalde tijdseenheid, expliciet kunnen worden uitgedrukt bij een gegeven gemiddelde verouderingssnelheid. In wiskundige termen betekent dit, dat we altijd tijdseenheden van gelijke lengte kunnen vinden, waarvoor de gezamenlijke kansdichtheidsfunctie van de aangroeiingen kan worden geschreven als een mengsel van exponentiële kansdichtheden. Dit mengsel representeert de onzekerheid in de onbekende gemiddelde verouderingssnelheid. Omdat de kansdichtheidsfunctie van iedere eindige rij van aangroeiingen in dit geval kan worden geschreven als een functie van de som van de aangroeiingen (d.w.z. de l_1-norm van de aangroeiingen), wordt zo'n oneindige rij van aangroeiingen l_1-isotropisch of l_1-norm-symmetrisch genoemd. Vanwege de verwisselbaarheid van de l_1-isotropische aangroeiingen is de verwachte cumulatieve veroudering lineair in de tijd.

Om optimale onderhoudsbeslissingen te bepalen, waarbij expliciet rekening wordt gehouden met de onzekerheid in de gemiddelde verouderingssnelheid, kan gebruik worden gemaakt van statistische beslissingstheorie. Een beslisser kan in zo'n geval een onderhoudsbeslissing nemen waarvoor het verwachte financiële verlies (in de vorm van de verwachte onderhouds- en faalkosten) minimaal is; een dergelijke beslissing wordt een optimale beslissing genoemd. Het verwachte verlies wordt berekend met betrekking tot de kansverdeling van de gemiddelde verouderingssnelheid, welke kansverdeling overigens kan worden bijgewerkt met (nieuwe) waarnemingen met behulp van de stelling van Bayes.

Onderhoud van waterbouwkundige constructies kan het beste worden gemodelleerd door middel van een zogenaamd 'vernieuwingsproces', waarbij de vernieuwingen onderhoudsacties voorstellen, die een constructie terugbrengen in de gewenste toestand. Statistisch gezien beginnen we na iedere vernieuwing weer opnieuw. Omdat de geplande levensduur (inclusief onderhoud) van de Nederlandse dijkringen in principe oneindig is, kunnen onderhoudsbeslissingen het beste worden vergeleken over een oneindige tijdshorizon. Er bestaan feitelijk drie kostencriteria, die kunnen dienen als verliesfunctie:

1. de verwachte gemiddelde kosten per tijdseenheid (die worden bepaald door de kosten te middelen over een oneindige tijdshorizon);

2. de verwachte gedisconteerde kosten over een oneindige tijdshorizon (die worden bepaald door de gedisconteerde kosten, d.w.z. de contante waarden van de kosten, te sommeren over een oneindige tijdshorizon); en

3. de verwachte equivalente gemiddelde kosten per tijdseenheid (die worden bepaald door de gedisconteerde kosten te middelen).

Deze kostencriteria kunnen worden berekend met behulp van de vernieuwingstheorie (de discrete vernieuwingsstelling). Door het begrip 'equivalente gemiddelde kosten' relateren de begrippen 'gemiddelde kosten' en 'gedisconteerde kosten' aan elkaar. Hoewel er in de literatuur tot nu toe de meeste aandacht wordt geschonken aan het begrip 'gemiddelde kosten', zijn de criteria van gedisconteerde kosten en equivalente gemiddelde kosten het meest geschikt om een optimale balans te vinden tussen initiële bouwkosten en toekomstige onderhoudskosten.

Op basis van gegeneraliseerde gamma-processen, zijn probabilistische modellen op maat gemaakt om optimale onderhoudsbeslissingen te onderbouwen voor de volgende kenmerkende componenten van een dijkring:

Strandsectie: optimale zandsuppletie-groottes, waarvoor de verwachte gedisconteerde kosten over een oneindige tijdshorizon minimaal zijn met betrekking tot de kansverdeling van de gemiddelde structurele achteruitgang van de kustlijn.

Dijksectie: optimale dijkophogingen, waarvoor de gedisconteerde kosten over een oneindige tijdshorizon minimaal zijn met betrekking tot de kansverdeling van de gemiddelde kruinhoogtedaling (een combinatie van zetting, samendrukking van de ondergrond en relatieve zeespiegelstijging). Op basis van een fysische wet, wordt kruinhoogtedaling beschouwd als een stochastisch proces met een verwachte kruinhoogtedaling die lineair of niet-lineair (bij benadering logaritmisch) in de tijd is.

Dynamisch-stabiele golfbreker: optimale inspectie-intervallen voor dynamisch-stabiele golfbrekers, waarvoor de verwachte (equivalente) gemiddelde kosten per tijdseenheid minimaal zijn met betrekking tot de kansverdeling van de gemiddelde snelheid van het verdwijnen van stenen (het zogenaamde 'langs-transport' van stenen). Het voorgestelde model is een twee-fasen inspectie-model, waarbij in de eerste fase geen stenen verdwijnen en in de tweede fase wel. Er wordt aangenomen dat de tijdstippen, waarop voor het eerst stenen verdwijnen, verdeeld zijn volgens een mengsel van geometrisch verdeelde stochastische grootheden.

Bijzondere waterkerende constructie: optimale inspectie-intervallen voor twee componenten van de bodemverdediging van de Oosterscheldekering: (i) de blokkenmatten en (ii) de steenbestorting. Het beslissingsmodel voor de blokkenmatten is een twee-fasen inspectie-model: de 'tussen-aankomsttijden' van ontgrondingskuilen zijn verdeeld volgens een mengsel van Poisson-processen en de ontgrondingskuilen ontwikkelen zich volgens een gegeneraliseerd gamma-proces. Het beslissingsmodel voor de steenbestorting is gebaseerd op de kansverdeling van de gemiddelde snelheid van het verplaatsen van stenen als gevolg van stroming.

De modellen voor zandsuppletie en dijkophoging zijn voorbeelden van onderhoudsoptimalisatie in de ontwerpfase, terwijl de twee inspectie-modellen voorbeelden zijn van onderhoudsoptimalisatie in de gebruiksfase.

In het laatste gedeelte van het proefschrift worden twee beslissingsmodellen gepresenteerd die niet direct uitgaan van verouderingsprocessen: één model voor het evalueren en vergelijken van beslissingen, die overstromingsschade langs de Maas kunnen reduceren (met behulp van l_1-isotropie en gedisconteerde kosten) en één model voor het optimaliseren van onderhoud, indien de onzekerheid in faalkansen kan worden uitgedrukt als een Dirichlet-verdeling. Wanneer zowel de sterkte als de belasting stochastisch zijn, zal met name het laatstgenoemde model in aanmerking komen om te worden toegepast.

Hoewel de beslissingsmodellen in het proefschrift primair zijn ontwikkeld voor het onderhoud van stranden, dijken, dynamisch-stabiele golfbrekers en de Oosterscheldekering, kunnen ze ook worden toegepast op andere waterbouwkundige constructies en andere technische systemen voor het oplossen van vele beslissingsproblemen in onderhoudsoptimalisatie en 'life cycle costing'.

Curriculum Vitae

De samensteller van dit proefschrift, Jan Maarten van Noortwijk, werd op 16 november 1961 te Haarlem geboren. In 1980 behaalde hij het diploma Atheneum-B aan het Christelijk College 'Marnix van St. Aldegonde' te Haarlem. Aansluitend studeerde hij Technische Wiskunde aan de Technische Universiteit Delft (TUD), alwaar hij met goed gevolg het Propaedeutisch en Kandidaatsexamen aflegde, respectievelijk in december 1983 en mei 1986. Het afstudeerwerk werd in 1988 uitgevoerd bij het Koninklijke/Shell-Laboratorium te Amsterdam (KSLA) onder leiding van dr R.M. Cooke (TUD), dr R. Dekker (KSLA) en dr T.A. Mazzuchi (KSLA). In januari 1989 verscheen zijn afstudeerverslag 'Use of Expert Opinion for Maintenance Optimisation' en in dezelfde maand behaalde hij - met lof - de titel van ingenieur bij prof dr F.A. Lootsma (Leerstoel Operationele Analyse).

Na zijn studie was hij tot en met augustus 1990 als technisch wetenschappelijk medewerker verbonden aan het Dr. Neher Laboratorium van PTT Research te Leidschendam, met name op het gebied van statistische verkeerskarakteristieken van telecommunicatie-diensten en modellering van het proces voor de capaciteitsplanning van de telecommunicatie-infrastructuur.

Van september 1990 tot en met juni 1995 was Van Noortwijk achtereenvolgens Assistent In Opleiding en Toegevoegd Onderzoeker bij de Faculteit der Technische Wiskunde en Informatica van de Technische Universiteit Delft. Het promotieonderzoek, mede gefinancierd door het Waterloopkundig Laboratorium 'De Voorst' (WLV) te Marknesse, werd begeleid door prof dr R.M. Cooke (TUD) en dr ir M. Kok (WLV). Naast het promotieonderzoek over optimaal onderhoud van waterbouwkundige werken was hij gedurende 10 maanden betrokken bij een onzekerheidsanalyse in het kader van het 'Onderzoek Watersnood Maas' en verder werden door hem bijdragen geleverd aan een aantal andere WLV-projecten. Het promotieonderzoek is niet alleen in Delft en Marknesse uitgevoerd, maar heeft ook plaatsgevonden in de Verenigde Staten; en wel bij de University of California te Berkeley (bij prof dr R.E. Barlow en prof dr ir M.B. Mendel) en de George Washington University te Washington D.C. (bij prof dr T.A. Mazzuchi).

Per 1 oktober 1995 aanvaardde hij een betrekking als adviseur risico en veiligheid bij het advies- en onderzoeksbureau HKV 'Lijn in Water' te Lelystad.