Mathematical Modelling of a flute

by

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to obtain the degree of Bachelor of Science
at the Delft University of Technology,
to be defended publicly on Monday August 22nd, 2022 at 13:00 AM.

Student number:	4908872	
Project duration:	May 1, 2022 – August 22nd, 2012	
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Abstract

The transverse flute can be studied on many aspects. In this thesis the sound waves that can be generated inside the flute are studied on their shape and frequencies. To achieve this, two mathematical models of the sound waves inside a flute are developed. To develop these models, solutions to the second dimensional wave equation are calculated analytically and numerically. The exact model is found within a rectangular parallelepipid region with homogeneous boundary conditions. By the method of separation of variables the second-dimensional wave equation is solved and a function of the air pressure inside the flute is calculated. This model can be used to visualise the waves inside the flute. It allows approximate true frequencies of measured musical notes in the flute. The second model, the numerical model is found in the same domain as the exact model, but with non-homogeneous boundary conditions.These are more realistic boundary conditions and therefore this model might be more accurate than the exact model. By the method of finite difference the numerical second-dimensional wave equation is solved. A system of two numerical equations is found. One of the equations calculates the pressure of the air inside the flute by inserting the previous pressure and velocity of the wave inside the flute. The other one calculates the velocity of the sound wave inside the flute by inserting the previous pressure of the sound wave inside the flute. The visualisation of this model however is not yet an accurate model to visualise waves inside the transverse flute. Therefore a conclusion is made that the exact model best describes the sound waves that can be generated inside the transverse flute.

Contents

1 Introduction

The transverse flute, part of the flute family which originates in the late Neolithic period about 12,000 years ago as mentioned by Boerner [1], is a popular flute. Since the modern transverse flute is very common nowadays it is often referred to as the flute. This has not always been the case. Boerner states: ”Until the end of the Baroque, the end-blown recorder was called the flute and was more common” [1]. The modern transverse flute as we know it has developed over the years and gone from a simple ”hollow tube with carved-out holes for the fingers to cover played in a vertical plane.” [7]; to a complex cylindrical tube with 16 holes that can be closed by one player using certain mechanisms.

The transverse flute can be studied on many aspects. In this thesis we will study the sound waves that can be generated inside the flute and what these sound waves look like. To achieve this we will develop two mathematical models of the sound waves, generated inside the flute when a player starts to blow into the flute. First we describe how the flute works in a general setting and then we introduce how sound waves in the flute can be manipulated by opening and closing two different types of holes. This knowledge is needed when modelling the sound waves inside the flute with the help of the first model. Secondly we derive the wave equation inside the flute which we solve to find the first model. The first model is the exact solution of the two-dimensional wave equation with a simple domain and homogeneous boundary conditions. In reality the domain of the flute is not simple and the boundary conditions are not homogeneous. Therefore a second model is calculated numerically with slightly more complicated boundary conditions. Both methods to model the flute are analysed and a conclusion is made on which method best describes the sound waves inside the flute.

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How does a flute work?

When we mention the flute in this report we are talking about the transverse flute shown in figure 2.1. A flute is a essentially a cylindrical tube with two open ends and many holes in between. Whenever air is blown correctly into the flute a sound wave can originate. The possible sound waves inside the flute depend on different characteristics of the flute and the way in which the holes in the tube of the flute are opened and closed. In this chapter we will take a closer look on how sound waves can be generated by the player of the flute. After this we will look at how the player of the flute can manipulate the generated sound waves.

Figure 2.1: My transverse flute

2.1. Generating sound waves

A player generates sound waves by blowing into the flute in a special way. The player's lips are placed on the side of the first hole in the tube, not covering the entire hole as can be seen in figure 2.2. The player creates an air jet moving forward, by slightly blowing over the hole instead of inside the hole. Since the air jet is moving forward it is split by the further edge of the hole. The air then starts to move in and out of the flute at the further edge of the hole.

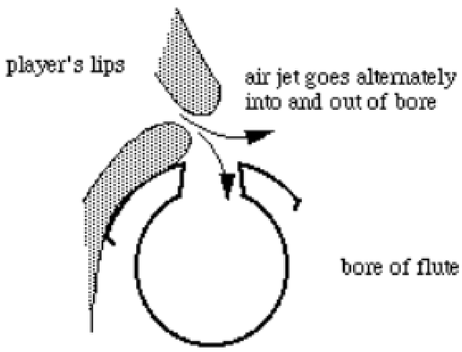


Figure 2.2: Air jet initiating a sound wave [8]

This oscillation of the jet leads to oscillation of the air inside the tube, which essentially is a sound wave inside the tube of the flute. The produced sound wave is a standing wave and consists of a set of combined harmonics, which are waves with only one frequency. The harmonic with the lowest frequency induces the height of the musical note heard by the human ear and therefore it is called the fundamental frequency, whereas the other harmonics create the timbre that is specific to the flute.

2.2. Manipulating sound waves

Of course it is not desirable to only play one musical note on the flute. Therefore the player of the flute can manipulate the frequencies of the harmonics in the generated sound wave by blowing a different air jet into the flute. This can be a harder, softer or differently placed air jet. However, the effect that a different air jet has on the generated sound waves inside the flute is not studied in this thesis. The player can also manipulate the generated sound waves by opening or closing specific holes of the flute. In Wolfe's article [8] a distinction is made between two types of holes of a flute: tone holes and register holes.

Tone holes are holes that change the maximal possible wave length of any generated wave, and thus the fundamental frequency of the sound wave. This is a result of essentially shortening or elongating the length of the tube as shown in figure 2.3. This is also supported by Joly's article [3] which states that opening an tone hole of the flute is roughly equivalent to cutting the tube at the place of the open hole. This is the case for "flutes with large holes as the modern transverse flute, except that one must add a small correction and the length ã of the equivalent tube is slightly larger than a." [3].

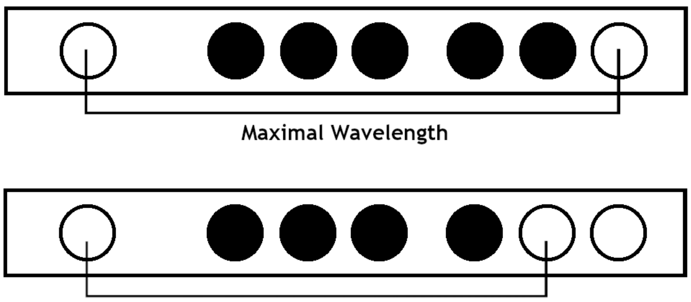
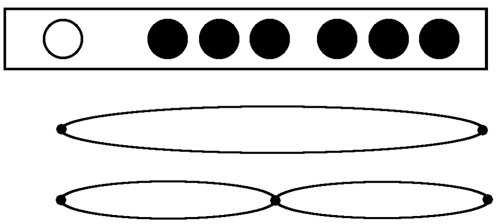
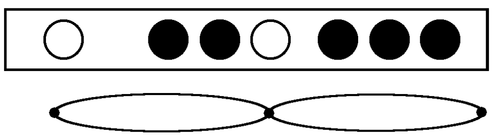


Figure 2.3: Opening a tone hole and the effect on the maximal wavelength. Open holes are white and closed holes are black.

Register holes affect certain harmonics of a musical node which do not have a node at the position of the hole. These harmonics are filtered and no longer produced when an air jet enters the flute as shown in figure 2.4.



(a) All harmonics are generated



(b) Some harmonics are filtered

Figure 2.4: Harmonics without a node at the place of an opened register hole are no longer generated

3

Wave equation

To really understand what the waves in the flute look like, we need to solve the wave equation inside the flute with the right boundary conditions. In this chapter we will derive this equation and hereby show that the air inside the flute indeed follows the wave equation.
We picture the flute as a cylindrical tube with two open ends as mentioned in chapter 2. The flute is filled with air and therefore has a starting pressure, p_0, with corresponding density, ρ_0 and a starting velocity **u** equal to zero. We will use the following three equations from the book of D.F. Parker [6] to derive the wave equation inside the flute.

Euler's equation:

$$\rho\Big[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big] = -\nabla p \tag{3.1}$$

The law of mass conservation:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = \mathbf{0} \tag{3.2}$$

The constitutive relation between pressure and density:

$$\rho = \rho(p) \tag{3.3}$$

We also note that: $p = p_0 + p_c$, $\rho = \rho_0 + \rho_c$, $\mathbf{u} = \mathbf{0} + \mathbf{u}_c$, where $p_c, \rho_c, \mathbf{u}_c$ are the small changes in the pressure, density and velocity of the air in the flute respectively.
Equation 3.1 can be written as:

$$\rho_0\Big[\frac{\partial \mathbf{u}_c}{\partial t}\Big] = -\nabla p_c, \tag{3.4}$$

Since the second term in equation 3.1 and ρ_c are both small.
Equation 3.2 can, with the same motivation, be written as:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho_0\mathbf{u}_c) = \mathbf{0} \tag{3.5}$$

When we take the divergence of equation 3.4 and derive equation 3.5 with respect to t we get:

$$\rho_0\cdot\Big[\frac{\partial(\mathbf{u}_c)}{\partial t}\Big] = -\nabla^2 p_c,$$

and

$$\frac{\partial^2 \rho_c}{\partial t^2} + \rho_0\nabla\cdot\Big[\frac{\partial(\mathbf{u}_c)}{\partial t}\Big] = \mathbf{0}$$

Substituting the first equation into the second one we get:

$$\frac{\partial^2 \rho_c}{\partial t^2} = \nabla^2 p_c \tag{3.6}$$

From the constitutive relation between pressure and density we can write the second derivative of ρ_c with respect to as the second derivative of p_c with respect to t. By linearising the constitutive equation $\rho(p)$ about the equilibrium pressure p_0 and hence assuming the variations in the pressure to be small, we arrive at

$$\rho = \rho_0 + \rho_c = \rho(p_0) + \frac{d\rho}{dp}(p_0)p_c, \tag{3.7}$$

where $rho_0 = \rho(p_0)$. By defining

$$c = 1/\sqrt{\frac{d\rho}{dp}(\rho_0)}$$

, and substitution of this expression in Eq. 3.6, we find

$$\frac{\partial^2 p_c}{\partial t^2} = c^2 \cdot \nabla^2 p_c \tag{3.8}$$

4

Soundwaves inside the flute

As mentioned in chapter 2 an initial air jet is needed to generate a sound wave inside of a flute. To find out what this sound wave looks like we need to solve the wave equation 3.8 derived in chapter 3 with the right boundary conditions. We will first determine the boundary conditions and domain of our problem and then find solutions to the wave equation. These solutions are then analysed for their accuracy.

4.1. Determining boundary conditions

The flute is a complex instrument which also makes the boundary conditions complex. Therefore we first look at a simplified model of the flute as shown in figure 4.1. We choose a closed rectangular parallelepiped tube in two dimensions with two open ends. The tube has a height of h meters and a length of L meters.

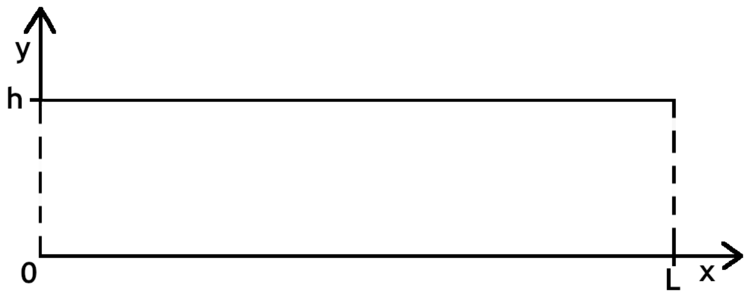


Figure 4.1: Simplified model of the flute

The problem that we solve in this chapter is similar, but different to the problem solved in the article of Romain Joly [3]. The purpose of the article is ”to obtain an explicit one-dimensional mathematical model for the flute with an open hole.”. Whereas in this thesis we want to find an explicit and numerical mathematical model for the sound waves that can be generated inside the flute. However the boundary conditions stated in the article can be used in this thesis. Joly states: ”On the inner surface of the tube, the pressure satisfies homogeneous Neumann boundary conditions. Where the tube is open to the exterior, we assume that the pressure is equal to the exterior pressure which may be assumed to be zero without loss of generality.” [3]. Therefore the boundary conditions for the pressure of the air inside the flute are homogenous Dirichlet conditions on the dotted lines of figure 4.1 and homogeneous Neumann conditions on the other two lines. So we have:

$$p(0, y, t) = 0 \tag{4.1}$$

$$p(L, y, t) = 0 \tag{4.2}$$

$$\frac{\partial p}{\partial y}(x, 0, t) = 0 \tag{4.3}$$

$$\frac{\partial p}{\partial y}(x,h,t) = 0 \tag{4.4}$$

4.2. Finding solution of the wave equation

The wave equation that we derived in chapter 3 is a homogeneous and linear partial differential equation. The boundary conditions are also linear and homogeneous, which is why we can use the method of separation of variables to solve the wave equations with boundary conditions 4.1 to 4.4, following the steps in the book of Richard Haberman [2]. We start with seperating the wave equation by writing:

$$p(x,y,t) = X(x) \cdot Y(y) \cdot T(t)$$

We can than write the wave equation 3 as follows:

$$X(x)Y(y)\frac{\partial^2 T}{\partial t^2}(t) = c^2 T(t)\Big(Y(y)\frac{\partial^2 X}{\partial x^2}(x) + Y(y)\frac{\partial^2 Y}{\partial y^2}(y)\Big)$$

When we devide by $X(x) \cdot Y(y) \cdot T(t)$ on each side we get:

$$\frac{T''(t)}{T(t)} = c^2\Big(\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}\Big) = \lambda,$$

where lambda is a constant since both sides of the equation contain different variable dependencies. Now we can split the right side of the equation again and we get:

$$\frac{X''(x)}{X(x)} = \frac{\lambda}{c^2} - \frac{Y''(y)}{Y(y)} = \mu$$

We now have three ordinary differential equations for the functions X, Y and T:

$$T''(t) = \lambda T(t) \tag{4.5}$$

$$X''(x) = \mu X(x) \tag{4.6}$$

$$Y''(y) = \Big(\frac{\lambda}{c^2} - \mu\Big)Y(y) \tag{4.7}$$

To solve these equations we need to look at 9 different cases, since lambda and mu can be negative, equal to zero or positive. Most of the cases lead to finding the trivial solution $p(x,y,t) = 0$. Only in the case where lambda and mu are both strictly negative we can find two different solutions:

$$p_n(x,y,t) = a_{1_n}\sin\Big(\frac{n\pi x}{L}\Big)\Big(a_{2_n}\cos\Big(\frac{n\pi ct}{L}\Big) + a_{3_n}\sin\Big(\frac{n\pi ct}{L}\Big)\Big) \tag{4.8}$$

, with $\lambda_n = -\Big(\frac{n\pi c}{L}\Big)^2$ and $\mu_n = -\Big(\frac{n\pi}{L}\Big)^2$

$$p_{n,m}(x,y,t) = \sin\Big(\frac{n\pi x}{L}\Big)\cos\Big(\frac{m\pi y}{h}\Big)\Big(a_{1_{n,m}}\cos\Big(\frac{\pi ct}{hL}\sqrt{n^2h^2 + m^2L^2}\Big) + a_{2_{n,m}}\sin\Big(\frac{\pi ct}{hL}\sqrt{n^2h^2 + m^2L^2}\Big)\Big) \tag{4.9}$$

, with $\lambda_{n,m} = -\Big(\frac{\pi c}{hL}\Big)^2(n^2h^2 + m^2L^2)$ and $\mu_n = -\Big(\frac{n\pi}{L}\Big)^2$

The derivation of these solution can be found in the Appendix A. If we look at the solutions 4.8 and 4.9 we can see that the first solution is included in the second one if we take $m = 0$. Therefore we will continue only with solution 4.9. By superposition we can add all possible solutions of the wave equation to find a general solution of the wave equation:

$$p_g(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sin\Big(\frac{n\pi x}{L}\Big)\cos\Big(\frac{m\pi y}{h}\Big)\Big(a_{1_{n,m}}\cos\Big(\frac{\pi ct}{hL}\sqrt{n^2h^2 + m^2L^2}\Big) + a_{2_{n,m}}\sin\Big(\frac{\pi ct}{hL}\sqrt{n^2h^2 + m^2L^2}\Big)\Big) \tag{4.10}$$

This solution contains the fundamental frequency and all harmonics of the sound wave inside a flute. To analyse these solutions we need to take into consideration the two different types of holes in the flute mentioned in section 2.2. Therefore we will first analyse the solutions looking at the first octave of musical notes that can be played on the flute. These musical notes are generated by opening tone holes. This octave corresponds to the fourth octave of the piano and will further be referenced by the fourth octave. Furthermore we will look at the fifth octave where register holes are first used to play different musical notes.

4.3. The fourth octave

The fourth octave, o4 of the flute is the most straight forward octave to play and analyse. The lowest note c4 is played by closing all holes of the flute. If the player wants to play a higher note, the furthest hole that is closed should be opened. Therefore we can conclude that all musical notes in the fourth octave are produced by opening an tone hole. When analysing the general solution of the wave equation inside the flute we can calculate the frequencies of all harmonics generated and the shape of the sound wave. First we will take a look at the frequencies since these implicate which musical note is being played. To analyse the individual solutions $p_{n,m}(x, y, t)$ of the previous section we define $h = 0.02m$ and $L = 0.33m$. These values correspond to the height of my flute and maximal wavelength described in chapter 2 when playing note a in the fourth octave as shown in figure 4.2.

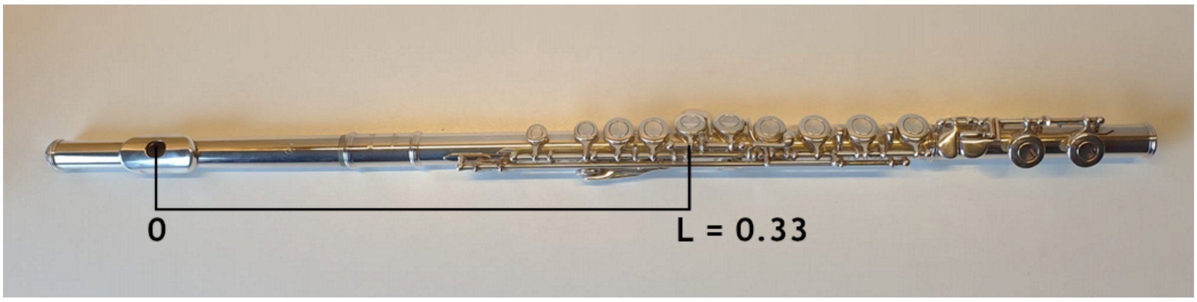


Figure 4.2: Length of the tube of a flute when playing musical note A in the fourth octave.

4.3.1. Fundamental frequencies

Now we can calculate the fundamental frequency of the wave inside the flute when musical note a4 is played and compare it to the true funcamental frequency from Mottola's website [4]. Looking at the general solution 4.10 we can see that the frequency of any wave inside the flute is equal to:

$$\frac{\sqrt{\lambda}}{2\pi} = \frac{c}{2hL}\sqrt{(nh)^2 + (mL)^2} \tag{4.11}$$

For c we assume the same speed of sound $c = 340.29m/s$ as Mottola also assumes [4]. The fundamental frequency of the flute when note a4 is played is the lowest possible frequency. We find this frequency when $n = 1$ and $m = 0$. Since h is much smaller than L, this gives the minimum of the square root $\sqrt{(nh)^2 + (mL)^2}$. The fundamental frequency of musical note a4 is then equal to:

$$\frac{c}{2L} \approx 515.59Hz$$

The true fundamental frequency however is equal to 440 Hz[4]. The fundamental frequency of 515 Hz should correspond to the c5 in the fifth octave approximately which is two notes higher than a4. When plugging in the wave length noted on Mottola's website [4] we do however get the correct frecuency. Therefore we suspect the formula to be correct but the measurement of the maximal wavelength incorrect. In section 2.2 we mentioned that a small correction should be added to the maximal possible wavelength L when assuming that the tube of the flute is cut off when an tone hole is opened [3] . Hence when calculating the fundamental frequencies of the musical notes in the fourth octave we can add 4.5 cm to the maximal wavelength to potentially correct the wave length. This correction is not arbitrary since it is the distance between the blowing hole of the flute and the closed left end of the flute. When implementing this correction and calculating the fundamental frequency of the musical note a4 we get:

$$\frac{c}{2L} \approx 453.72Hz$$

This is much closer to the true fundamental frequency of note a4. Naturally we need to check if this is the case for the other musical notes of the fourth octave too.

Figure 4.3 shows all calculated frequencies of the fourth octave and the corrected frequencies by adding 4.5 cm to the maximal wavelength. We can see that the distance between the blue and green line increases when higher notes of the fourth octave are played. So the necessary correction of length L could explain the errors in figure 4.3 that get bigger when the tube length gets smaller, considering the length correction is less significant with higher tube lengths. The orange line represents the frequencies calculated by adding 4.5 cm to the maximal wavelength and this line is much closer to the true values of the fundamental frequencies of the musical notes in the fourth octave. For this reason we will continue our calculations using the correction of 4.5 cm.

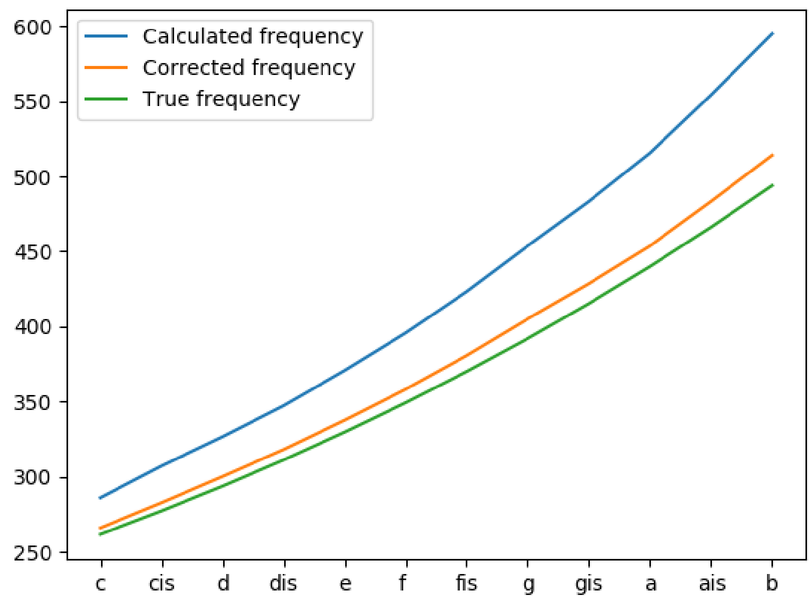


Figure 4.3: Frequencies corresponding to different musical notes in the fourth octave.

4.3.2. Harmonics

The fundamental frequencies of the fourth octave are not the only frequencies produced when a sound wave is generated inside the flute. The harmonics with higher frequencies than the fundamental frequency create the timbre of the flute. We can find the next harmonic by increasing n by 1 in equation 4.11 and keeping m equal to 0. We can then calculate the frequencies of the harmonics with the following formula: $\frac{n \cdot c}{2L}$. In table 4.1 the fundamental frequencies and the first four harmonics of all the musical notes in the fourth octave are shown.

If we take m=1 in equation 4.11 we suddenly get a very high frequency of around 8000 Hz. So when increasing m we skip a lot of harmonics that can be found when increasing n. Therefore we keep m=0 and only increase n to find more harmonics.

	L	Fundamental frequency	First harmonic n=2, m=0	Second harmonic n=3, m=0	Third harmonic n=4, m=0	Fourth harmonic n=5, m=0
c	0.64	265.85	531.70	797.55	1063.41	1329.26
cis	0.602	282.63	565.27	847.90	1130.53	1413.16
d	0.567	300.08	600.16	900.24	1200.32	1500.40
dis	0.535	318.03	636.06	954.08	1272.11	1590.14
e	0.504	337.59	675.18	1012.77	1350.36	1687.95
f	0.475	358.2	716.4	1074.6	1432.8	1791
fis	0.447	380.64	761.28	1141.91	1522.55	1903.19
g	0.42	405.11	810.21	1215.32	1620.43	2025.54
gis	0.397	428.58	857.15	1285.73	1714.31	2142.88
a	0.375	453.72	907.44	1361.16	1814.88	2268.6
ais	0.352	483.37	966.73	1450.10	1933.47	2416.83
b	0.331	514.03	1028.07	1542.10	2056.13	2570.17

Table 4.1: Fundamental frequencies and the first four harmonics of the musical notes in the fourth octave.

4.3.3. Visualisation

Now our goal was to find out what the sound waves inside a flute can look like. Since we have a formula for all the waves that are generated when playing a certain musical note and an understanding of the formula, we can visualise our solution with the help of python. When visualising our solution we need to take into account that p depends on 3 variables, which would result in a 4d plot. Naturally this is not possible and for this reason when plotting the solution in python we will set one of the 3 variables to a constant. We will also take all unknown constants $a_{1_{n,m}}$, $a_{2_{n,m}}$ to be equal to 1. The python-code used to plot the solution is included in appendix C. In figure 4.4 and 4.5 the initial wave, so the wave at t=0, is shown when playing note a4. The difference between the two figures is that figure 4.4 consists of the fundamental frequency and the first four harmonics and figure 4.5 consists of the fundamental frequency and the first 8 harmonics of musical note a4. When adding more harmonics to figure 4.5 the shape of the wave stays more or less the same.

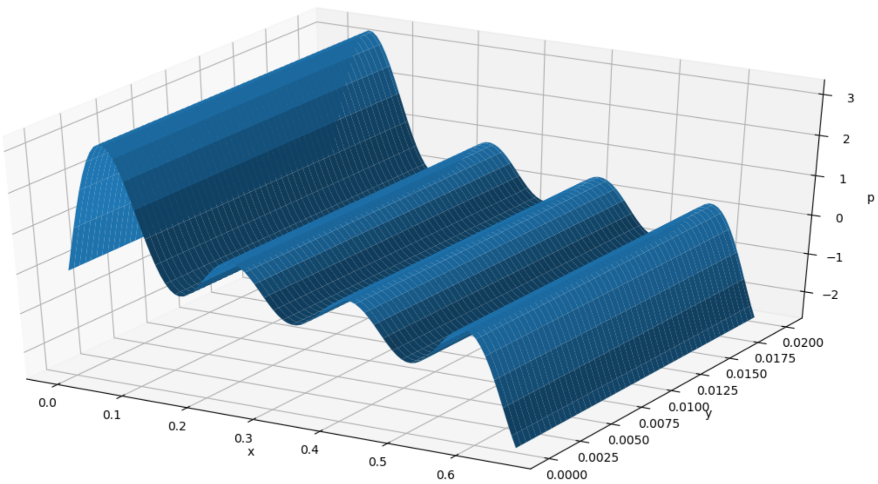


Figure 4.4: Initial sound wave when playing note a4, containing 4 harmonics

We can see that at t=0 the pressure p is highest when x is low. This is quite logical since at t=0, the player blows an air jet inside the flute which creates a pressure increase at x=0. We can also see that the pressure takes negative values. This not a problem since the pressure plotted is actually the pressure change, p_c, and so it can be negative when the pressure decreases. In both figures we can see that there is no dependence on the y variable. This is a result from setting m equal to 0, since m is also a factor inside of the cosine term that contains the y variable in the general solution of our wave equation 4.10.

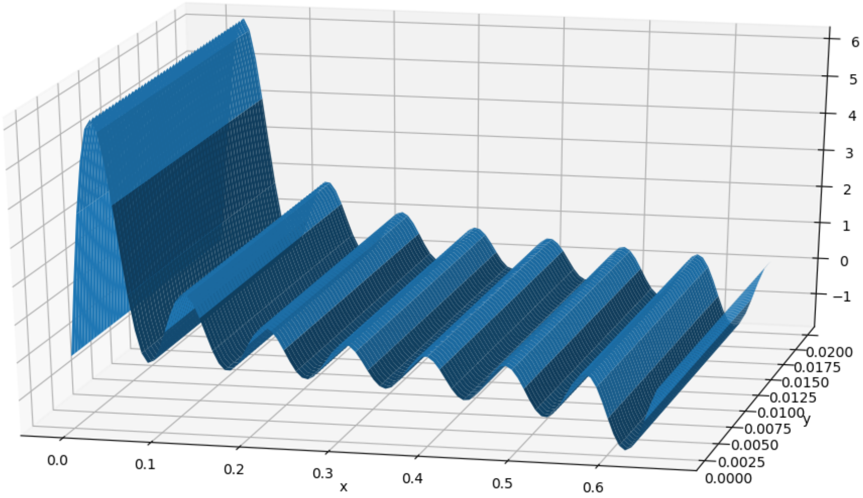


Figure 4.5: Initial sound wave when playing note a4, containing 8 harmonics

If we do include m by calculating for each value of m, the next 8 harmonics found by increasing n by 1, and increasing m eight times as well, we get figure 4.6. We can still see the same principle of a pressure increase where the air jet enters the flute.

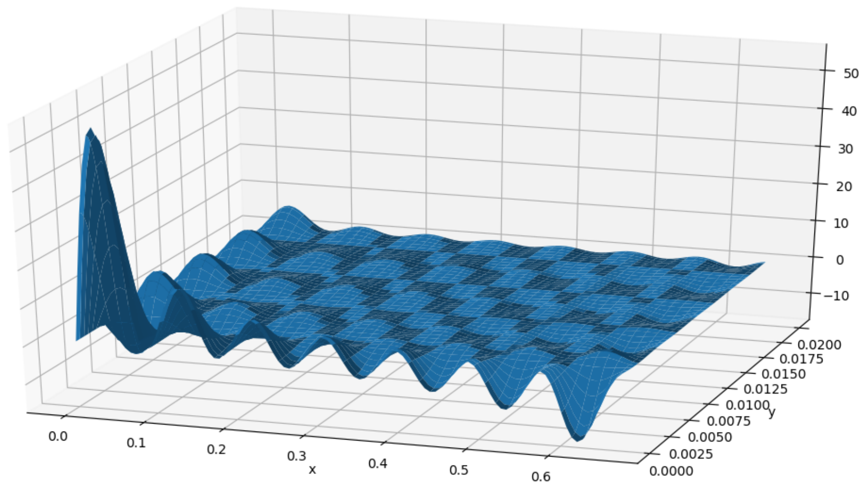


Figure 4.6: Initial sound wave when playing note a4, containing 64 harmonics

Of course it is also interesting to see how the wave behaves through time. Unfortunately the plots where we include the time axis and erase the y-axis by setting y to a constant, are very unreadable. Therefore we will analyse how the wave behaves through time by first plotting the pressure in the flute through time. This will give a view of the frequencies of the wave inside the flute. After this we will plot the shape of the wave inside the flute at different times. In figure 4.7 only the time variable is shown. We again plot the fundamental frequency and the first four harmonics. The left plot in figure 4.7 seems to imply that the wave of musical note a4 repeats its self after 6 seconds. This is incorrect since the period of the wave should be equal to one divided by the fundamental frequency. However when we increase the time steps taken in the code of our program we get the right plot in figure 4.7. This figure shows a much smaller period in which the wave is repeating its self, namely a period of 1/4 of a second.

Since we know the fundamental frequency of musical note a4, we can of course calculate what the period should be namely: $\frac{1}{453.72} \approx 0.0022s$. Therefore we can plot the sound wave again from t=0 to t=0.01 as in figure 4.8. In this figure we can clearly see the fundamental frequency and the first four harmonics.

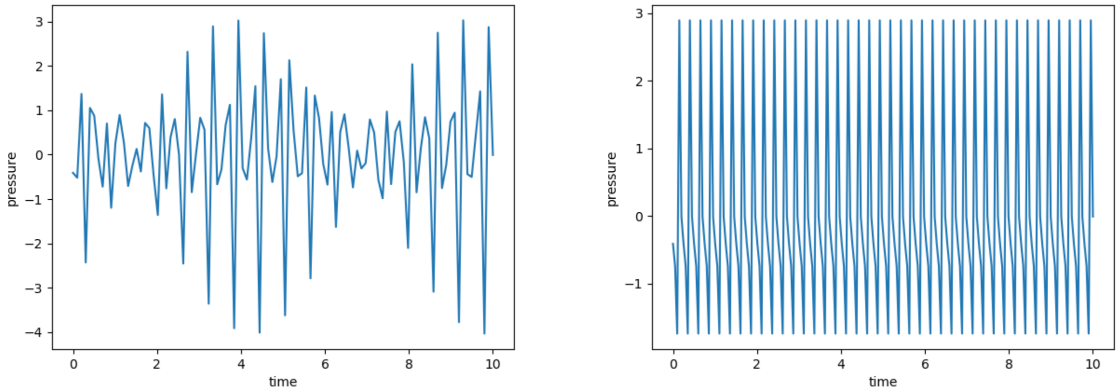


Figure 4.7: Pressure inside the flute plotted against time in second

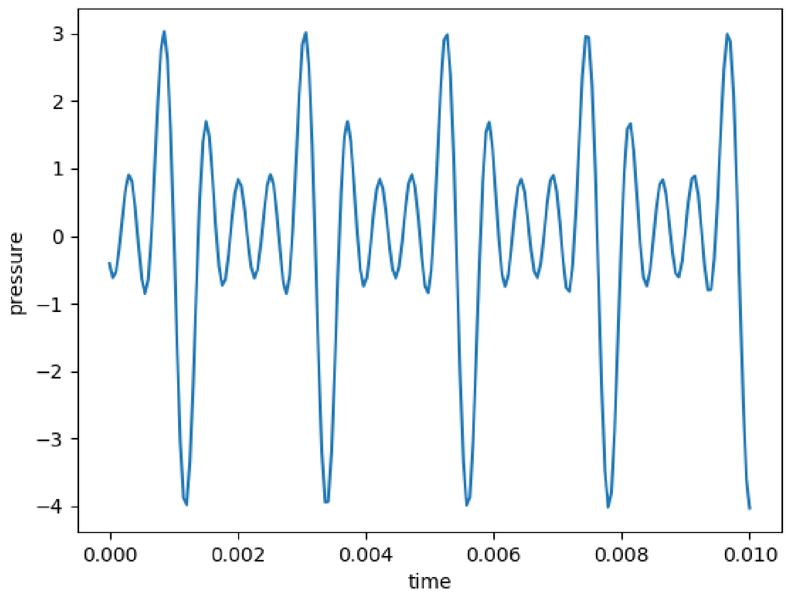


Figure 4.8: Pressure inside the flute at y=0.01 and x = 0.33 throughout 0.01 seconds

In figure 4.9 the shape of the sound wave of musical note a4 is plotted at eighth different time points. We can see that the wave starts with a high amplitude at low x-values and then the high amplitude travels further into the flute, which is expected. Whenever the highest amplitude peak leaves the flute a new amplitude peak starts at low x-values again. On a molecular level we can explain this as particles moving from high pressure regions to lower pressure regions.

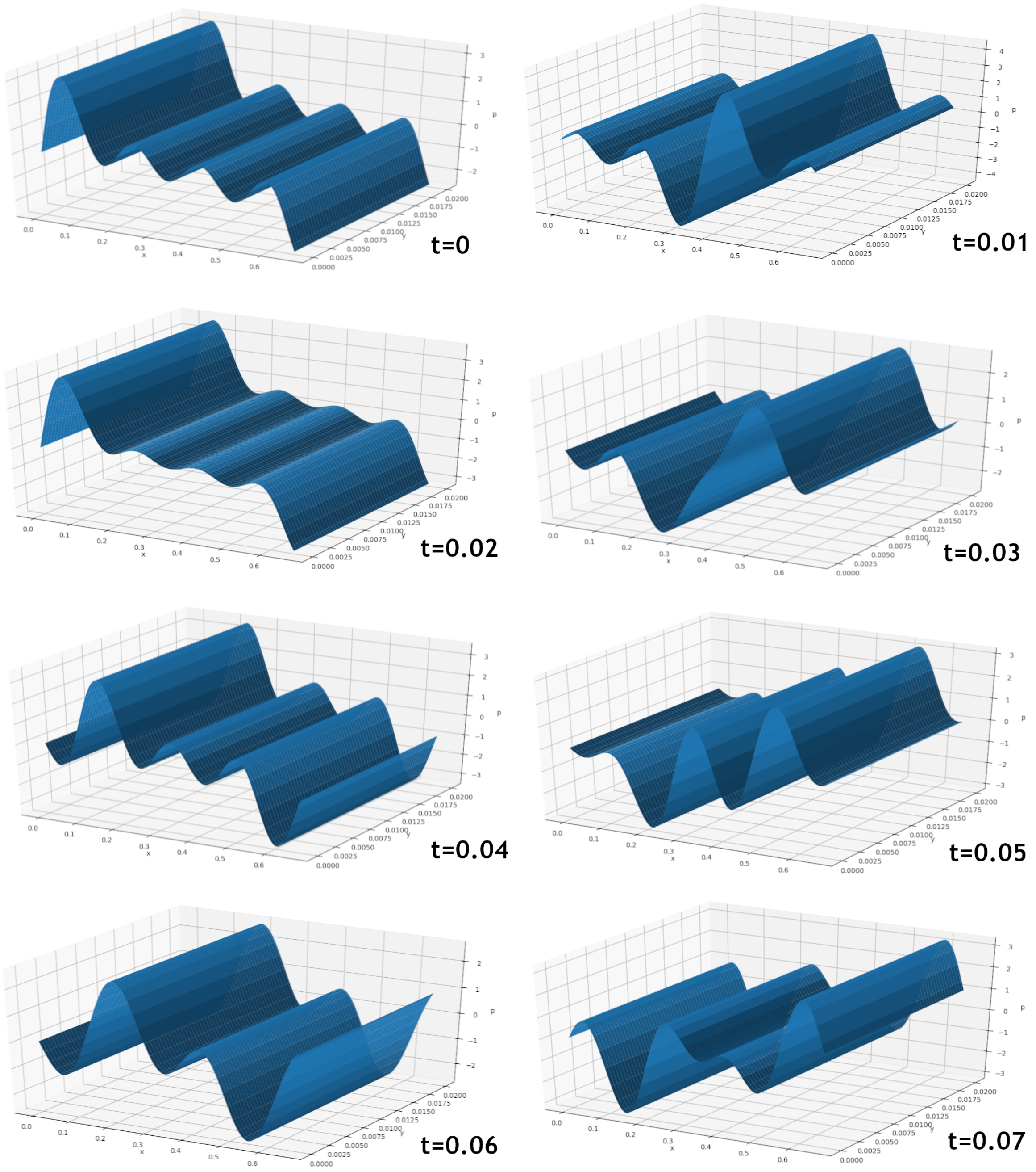


Figure 4.9: Sound waves inside the flute at different time points

4.4. The fifth octave

The fifth octave is played very differently than the fourth one. When playing the second note of the fifth octave the player of the flute needs to open a register hole for the first time, since the first note is still achieved by opening the next tone hole. In section 2.2 we discussed that opening a register hole filters some harmonics. In this section we will take a closer look as to which harmonics are filtered and what the effect is on the fundamental frequency. We will also look at different visualisations of the sound waves generated when playing the fifth octave and analyse the potential differences with the fourth octave.

4.4.1. Fundamental frequencies and harmonics

We will start by analysing the note d5. To play this note we need to close all holes that are closed when playing note d4, but we also need to open a register hole at length $L = 0.28m$. When closing the same holes as d4 we would get a maximum wavelength of $L = 0.567m$, so the register hole that is opened to play d5 is in the middle of the closed holes. Therefore all harmonics of d4 where n is a multiple of 2, describe the frequencies that are still generated when playing a d5.
In table 4.1 we can see that when a d4 is played and n is equal to two, we get a frequency of 600.16 Hz. This is thus our calculated fundamental frequency of musical note d5, while the true frequency is 587.33 Hz [4].
When playing the next note in the fourth octave, dis5, we need to close the same holes as with dis4, and open the register hole at $L = 0.28$ again. This is again more or less in the middle and hence we assume all harmonics of dis4 where n is a multiple of 2, describe the frequencies that are still generated when playing a dis5. This would imply the fundamental frequency of dis5 to be equal to 636.06 Hz as shown in table 4.1. The true fundamental frequency is equal to 622.25 Hz, which is again quite close to our calculation.
The musical notes that follow dis5 are are generated differently than we have seen up till now. The musical notes in the fifth octave starting from the e5 are played by opening exactly the same holes as their counterparts in the fourth octave. The player can influence the octave by blowing a different air jet into the hole. Unfortunately this is not researched in this thesis so we will not look more into these musical notes.

4.4.2. Visualisation

In this section we will visualise note d5 and analyse the differences with the visualisations of note a4. In figure 4.10 the musical note d5 is plotted at time t=0, with four harmonics. Similarly to the note a4 we can see an increase at x=0. Curiously there is also a big change in pressure from x=0.5. This could be caused by the open hole at L=0.567. However this is only based on speculation and should be further researched to draw a conclusion.

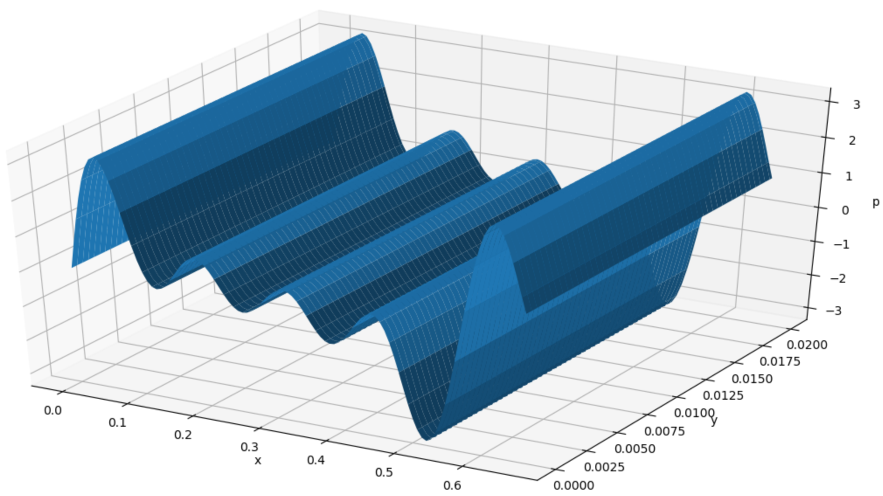


Figure 4.10: Musical note d5 at t=0, with 4 harmonics

Since d5 has a higher fundamental frequency we expect the period in the time plot of note d5 to be smaller than the period of a4, namely: $\frac{1}{600.16} \approx 0.0016$. Indeed figure 4.11 shows a slightly shorter period of the sound wave. Generally we do not see big differences between this time plot of musical note d5 and the time plot of musical note a4.

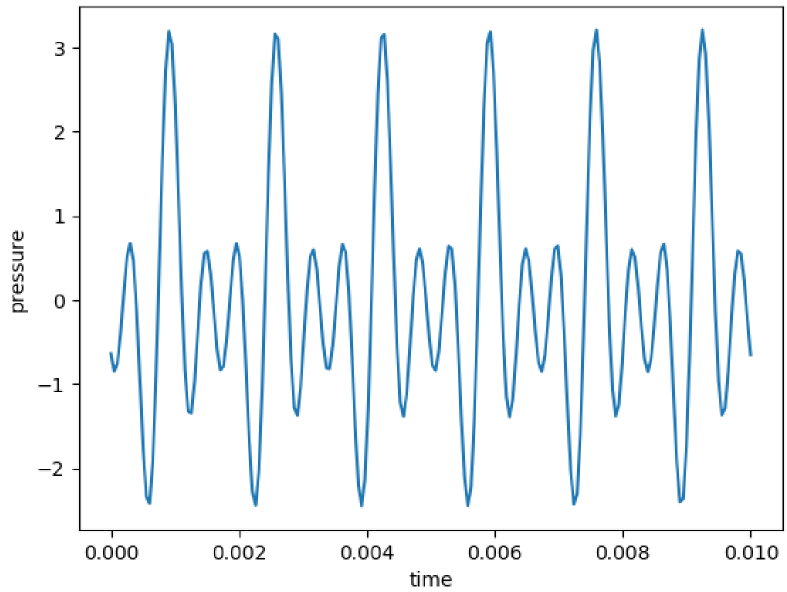


Figure 4.11: Pressure inside the flute at y=0.01 and x = 0.33, when playing d5 throughout 0.01 seconds

5

Numerical approximation

In the previous chapter we have found solutions to the wave equation in two dimensions and analysed these solutions. We have seen that solutions were not always accurate and some assumptions were made to calculate frequencies of musical notes in the flute produced by opening register holes. Exact solutions of differential equations are more desirable since it is easy to calculate the value of a solution. The disadvantage of exact solutions is that they can only be found with for example simple boundary conditions. The flute however, is a very complicated instrument and the sound waves inside the flute can potentially be better approximated numerically. Hence in this chapter we will assume more complicated boundary conditions that are more in line with the true boundary conditions of the flute. We will first define these boundary conditions and then formulate a new equation that approximates the wave equation 3.8. Following that we will formulate the initial conditions and boundary conditions numerically as well.

5.1. Numerical wave equation

To define our problem numerically we need to transform the earlier defined domain and wave equation 3.8 of the exact problem to a numerical domain and system of partial differential equations. We do this following the finite difference approximation method for partial derivatives in the book of Haberman [2]. We still take the domain defined in figure 4.1, but we split the x-axis in d equal parts and the y-axis in b equal parts as shown in figure 5.1. We also split the t axis in equal parts of length Δt. Furthermore we adjust the boundary conditions to better approximate the true boundary conditions of the flute. We now have a closed end on the right side and an open end on the left side of our domain. We also have an open part of the boundary at $y_b = h$ from 0.038m to 0.052m. This is the same placement as my own flute.

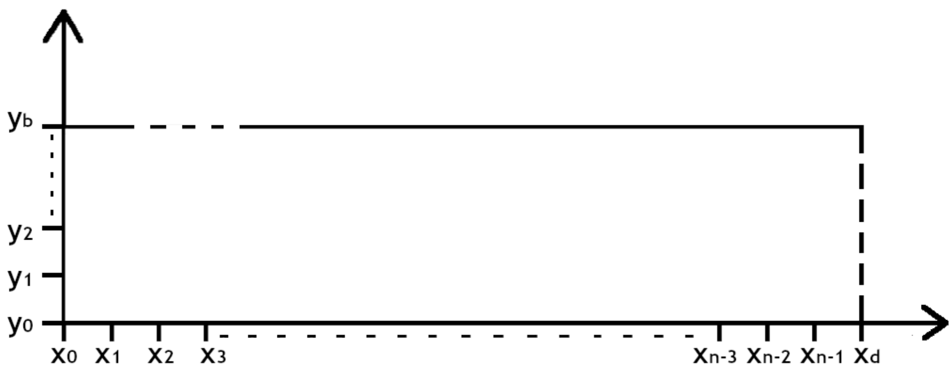


Figure 5.1: Numerical domain for the wave equation

We define:

$$x_i = i \cdot \Delta x, \text{ where } x_d = L$$
$$y_j = j \cdot \Delta y, \text{ where } y_b = h$$
$$t_k = k \cdot \Delta t$$

Furthermore we define the pressure and the velocity of the air inside the flute p and u respectively on space (x_i, y_j) at time t_k as follows:

$$p_{i,j}^k := p(x_i, y_j, t_k)$$
$$u_{i,j}^k := u(x_i, y_j, t_k)$$

By the centered difference approximation for second derivatives [2] we get:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \approx \frac{p_{i+1,j}^k - 2p_{i,j}^k + p_{i-1,j}^k}{(\Delta x)^2} + \frac{p_{i,j+1}^k - 2p_{i,j}^k + p_{i,j-1}^k}{(\Delta y)^2} \tag{5.1}$$

From chapter 3 we know that $\frac{\partial^2 p}{\partial t^2} = \rho_0 \frac{\partial u}{\partial t}$ and by the forward difference approximation method [2] we can approximate:

$$u_{i,j}^k = u(x_i, y_j, t_k) = \frac{1}{\rho_0}\frac{\partial p}{\partial t}(x_i, y_j, t_k) \approx \frac{1}{\rho_0}\frac{p_{i,j}^{k+1} - p_{i,j}^k}{\Delta t} \tag{5.2}$$

$$\implies p_{i,j}^{k+1} \approx \Delta t \cdot \rho_0 \cdot u_{i,j\,k} + p_{i,j}^k \tag{5.3}$$

$$\frac{\partial^2 p}{\partial t^2}(x_i, y_j, t_k) = \rho_0 \frac{\partial u}{\partial t}(x_i, y_j, t_k) \approx \rho_0 \frac{u_{i,j}^{k+1} - u_{i,j}^k}{\Delta t} \tag{5.4}$$

$$\implies u_{i,j}^{k+1} \approx \Delta t \cdot \frac{1}{\rho_0} \cdot \frac{\partial^2 p}{\partial t^2}(x_i, y_j, t_k) + u_{i,j}^k \tag{5.5}$$

We then get the following numerical system of equations:

$$\begin{cases} p_{i,j}^{k+1} \approx \Delta t \cdot \rho_0 \cdot u_{i,j}^k + p_{i,j}^k \\ u_{i,j}^{k+1} \approx \Delta t \cdot \frac{1}{\rho_0}\left(\frac{p_{i+1,j}^k - 2p_{i,j}^k + p_{i-1,j}^k}{(\Delta x)^2} + \frac{p_{i,j+1}^k - 2p_{i,j}^k + p_{i,j-1}^k}{(\Delta y)^2}\right) + u_{i,j}^k \end{cases} \tag{5.6}$$

We can solve this system numerically if we specify the boundary conditions and initial conditions of the differential equations.

5.1.1. Boundary conditions

We will first specify the boundary conditions. The domain that we specified in this chapter is more similar to the domain defined in Joly's article [3] than the domain specified in section 4.1. In the article Joly uses the 3-dimensional version of the domain in figure 5.1 with one extra hole added in the same plane as the blowing hole. Joly's article was used in section 4.1 to select realistic boundary conditions for our analytic problem. As in section 4.1 we again assume Dirichlet conditions on the dotted lines and Neumann conditions on the other lines. The Dirichlet boundary conditons can easlily be described numerically. We have: $p(x, y, t) = 0 \implies p_{i,j}^k = 0$.

The Neumann conditions however need to be approximated in our numerical problem. For this we use the centered difference approximation [2]. The boundary conditions of our numerical problem can be written as:

$$p_{i,b}^k = 0 \text{ for all } i \text{ such that } 0.038m < x_i < 0.052m \tag{5.7}$$

$$p_{d,j}^k = 0 \tag{5.8}$$

$$p_{i,1}^k = p_{i,-1}^k \tag{5.9}$$

$$p_{i,b+1}^k = p_{i,b-1}^k \text{ for all } i \text{ such that } x_i \leq 0.038m \text{ or } x_i \geq 0.052 \tag{5.10}$$

$$p_{1,j}^k = p_{-1,j}^k \tag{5.11}$$

5.1.2. Initial conditions

We will now specify the initial conditions of our problem. To start we define:

$$p(x_i, y_j, 0) = f_(x_i, y_j) := f_{i,j} \tag{5.12}$$

$$\frac{\partial p}{\partial t}(x_i, y_j, 0) = g(x_i, y_j) := g_{i,j} \tag{5.13}$$

$$\frac{\partial u}{\partial t}(x_i, y_j, 0) = h(x_i, y_j) := h_{i,j} \tag{5.14}$$

, where f, g and h are unknown arbitrary functions. The first initial condition is easily transformed into a numerical initial condition namely: $p^0_{i,j} = f_{i,j}$, whereas for the second and third initial condition we again need to make use of the forward difference approximation [2] to approximate the derivatives $\frac{\partial p}{\partial t}$ and $\frac{\partial u}{\partial t}$ as shown in appendix B.

We can conclude with the following initial conditions:

$$p^0_{i,j} = f_{i,j} \tag{5.15}$$

$$p^1_{i,j} = \Delta t \cdot g_{i,j} + p^0_{i,j} \tag{5.16}$$

$$u^1_{i,j} = \Delta t \cdot h_{i,j} + u^0_{i,j} \tag{5.17}$$

5.2. Solutions of the numerical approximation

When calculating solutions in python we need to decide on the values of Δx, Δy, and Δt and we need to define the initial conditions. For the initial conditions we define all functions f, g, and h to be the same function, namely:

$$f_{i,j} = g_{i,j} = h_{i,j} = \begin{cases} 1 \text{ for } 0.038 < x_i < 0.052 \\ 0 \text{ otherwise} \end{cases} \tag{5.18}$$

These initial conditions simulate a player inserting an air jet in the flute, which first enters the flute between 0.038 < x_i < 0.052. The size of the x-intervals and y-intervals can be chosen arbitrarily, but they do influence what we can choose our time interval to be. In a lecture Natalia K. Nikolova [5] explains that numerical algorithms can easily become unstable when certain criteria like the stability criterion, called the Courant-Firedrich-Levy criterion, are not satisfied. Nikolova states: "Often, this is observed as an exponential increase." [5], which we have also seen in the plots generated by our program, before implementing the CFL criterion. The CFL criterion for a 2-dimensional wave equation is stated below:

$$(c\Delta t)^2 \leq \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)^{-1} \tag{5.19}$$

, where c is again the speed of sound assumed to have the value $c = 340.29m/s$, as defined on Mottola's website [4]. In the program used to generate plots of the numerical solution we have implemented this CFL condition by taking time steps of size:

$$\Delta t = \frac{1}{2c} \cdot \sqrt{\left(\frac{1}{0.0165^2} + \frac{1}{0.001^2}\right)^{-1}} \approx 1.467 \cdot 10^{-6}$$

These time steps are very small and therefore we need to take a lot of time steps to show what a generated sound wave in a flute looks like.

5.2.1. Visualisation of numerical solutions

Now that all necessary values are specified we can run our program in python to visualise the sound wave inside the flute when no holes are opened. This corresponds to musical note c4. The python code used to generate the plots of this subsection can be found in appendix C. In appendix D a comparison between the analytic solution and the numerical solution is made with equal boundary conditions at equal times. Figure 5.2 shows the plot of the sound wave inside the flute while playing musical note c4 at approximately t = 0.00015s. This plot was generated by taking k=100, but the size of the time step is small so we still only see the shape of the initial condition of the wave. In figure 5.3 more time steps have been taken, namely 10^4 time steps, such that the sound wave at approximately t = 0.015s is visualised. The initial condition has not yet caused a change of the air pressure inside the flute, but the wave is moving forward in the positive x-direction. To find out what the wave will look like in the whole flute we need to take even more time steps.

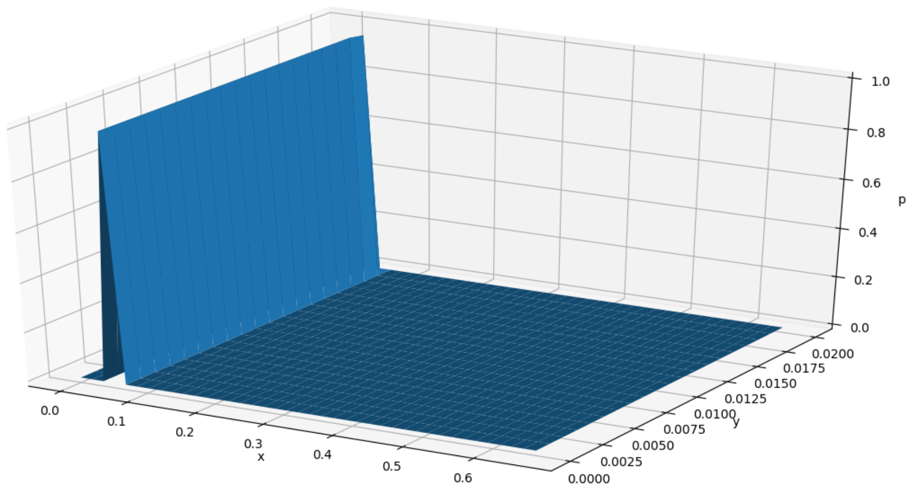


Figure 5.2: Sound wave of musical note c4 inside flute calculated numerically at time $t \approx 0.00015s$

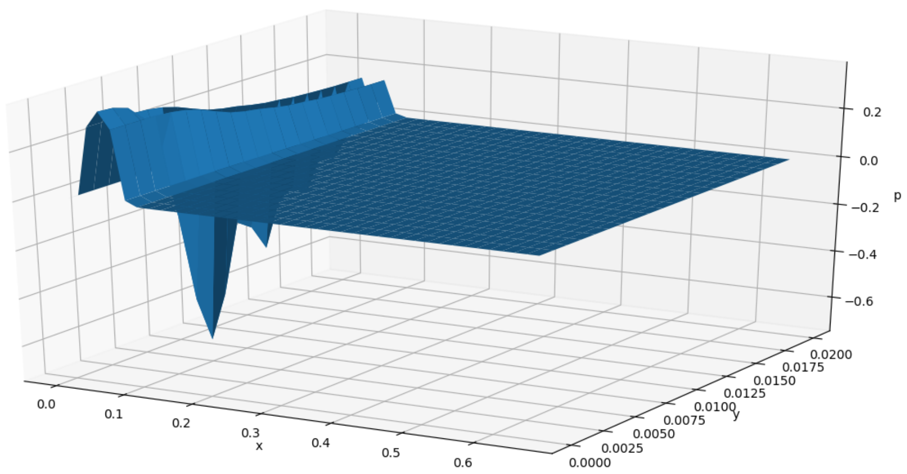


Figure 5.3: Sound wave of musical note c4 inside flute calculated numerically at time $t \approx 0.015s$

Figure 5.4 shows the sound wave of note c4 inside the flute at approximately t = 1.467s. Now we can see that the pressure inside the flute has been influenced by the initial condition everywhere in the flute. We can still see higher peeks of pressure at low values of x and then the amplitude of the sound wave does not change that much anymore. On the other hand if we look at the p-axis of the plot in figure 5.4, we can see that the whole wave does not have high pressure changes. In the book Field, flows and waves, D.F. Parker states: "In acoustics, the fractional change in pressure is very small (the human ear experiencing pain if $\Delta p/p_0 > 0.2\%$)"[6]. This could indicate that the numeric solution is more realistic than the exact solution if p_0 is not that large. Considering that the atmospheric air pressure on earth is approximately equal to $1.01 \cdot 10^5 Pa$ we can conclude that both solutions have a very small fractional change in pressure and both solutions are realistic when it comes to fractional change.

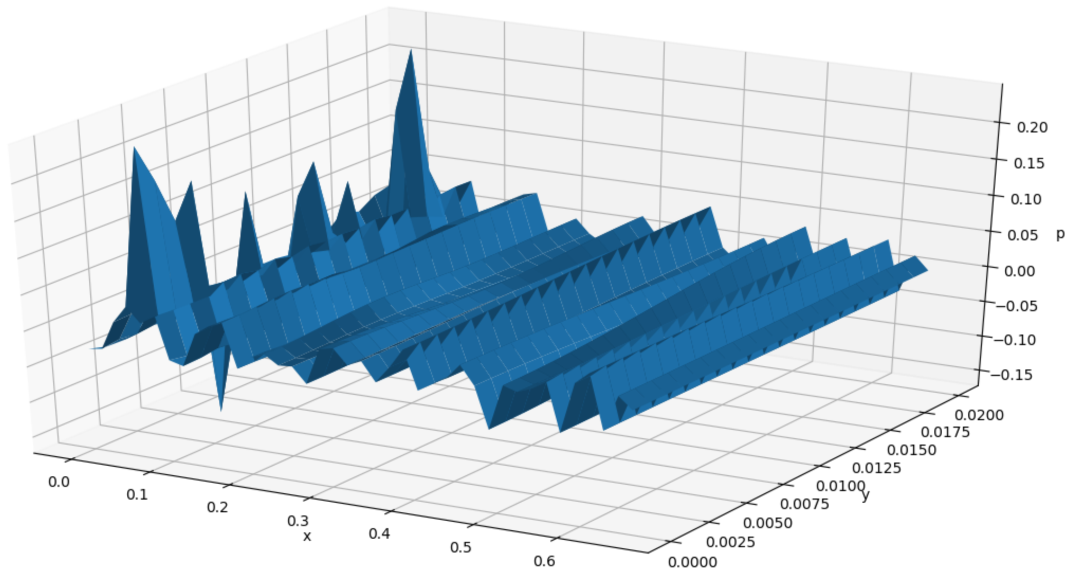


Figure 5.4: Sound wave of musical note c4 inside flute calculated numerically at time $t \approx 1.467s$

As in chapter 4 we will plot the pressure against only the time variable to take a closer look at the frequency of musical note c4. Figure 5.5 shows the pressure change when playing musical note c5 throughout approximately 1.467 seconds. As in chapter 4 we need to zoom in to truly see what the numerically calculated frequency of musical note c5 is equal to. From Mottola's website [4] we know that musical note c4 has a true frequency of 261.626 HZ, so we should plot the pressure throughout approximately 0.0038 seconds to visualise one period of the sound wave. Figure 5.6 shows the pressure throughout approximately 0.004 seconds, but we cannot see one period of the sound wave and therefore we can conclude that the numerical solution does not calculate the correct frequencies of the musical notes generated inside the flute.

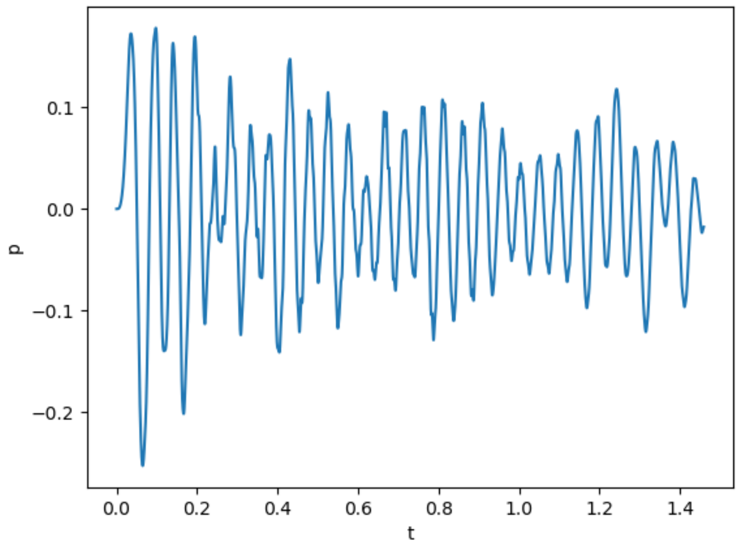


Figure 5.5: Pressure change when playing musical note c5 throughout approximately 1.467s

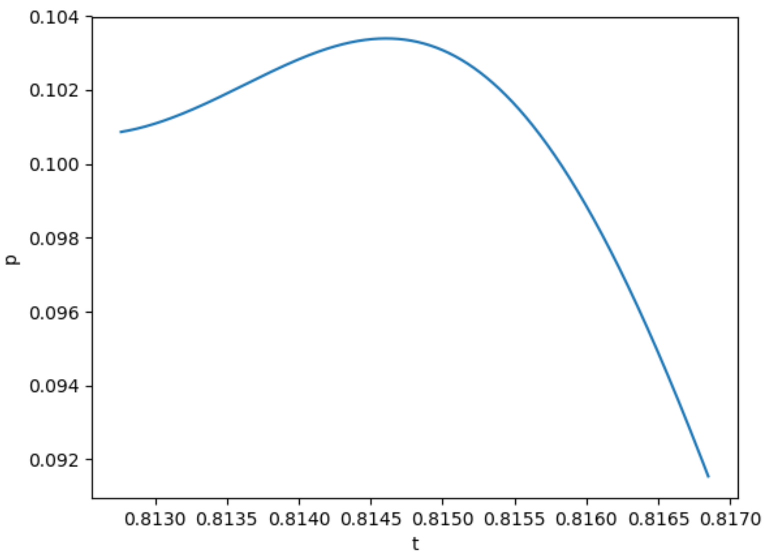


Figure 5.6: Pressure change when playing musical note c5 throughout approximately 0.0038 seconds

5.3. Visualisation of numerical solution of musical note d5

The advantage of the numerical approximation of the 2-dimensional wave equation is that we can add more holes in the domain shown in figure 5.1. For example if we would model musical note d5 numerically we need to add three more holes to our domain. If we add these holes in our earlier numerical domain we need to adjust boundary condition equations 5.7 and 5.10 to:

$p_{i,b}^k = 0$ for all $i,$ such that
$0.038 < x_i < 0.052$ or $0.298 < x_i < 0.306$ or $0.581 < x_i < 0.597$ or $0.614 < x_i < 0.63$
$p_{i,b+1}^k = p_{i,b-1}^k$ for all $i,$ such that
$x_i \leq 0.038$ or $0.052 < x_i < 0.298$ or $0.306 < x_i < 0.581$ or $0.597 < x_i < 0.614$ or $x_i \geq 0.63$

Figure 5.7 shows the numerical solution when musical note d5 is played. Unfortunately figure 5.7 looks almost exactly the same as figure 5.4. Because the holes added to our domain are really small, the fact that the numerical solution of d5 looks like the numerical solution of note c4, can be explained by the program not being precise enough. If only few x-coördinates are influenced by the new boundary conditions we will not see many differences with the earlier plot. Therefore we should plot the solution with higher precision in the x-axis. Plotting the numerical solution with even higher precision unfortunately takes a lot of time, since all previous time steps need to be calculated before obtaining a solution at a desired time. Therefore we cannot include a more precise plot in this thesis. Another viewing angle is that the initial conditions have a big influence on the numerical solutions and therefore sould not be the same for two different musical notes. To find out what initial conditions are realistic for certain musical notes further research need to be done.

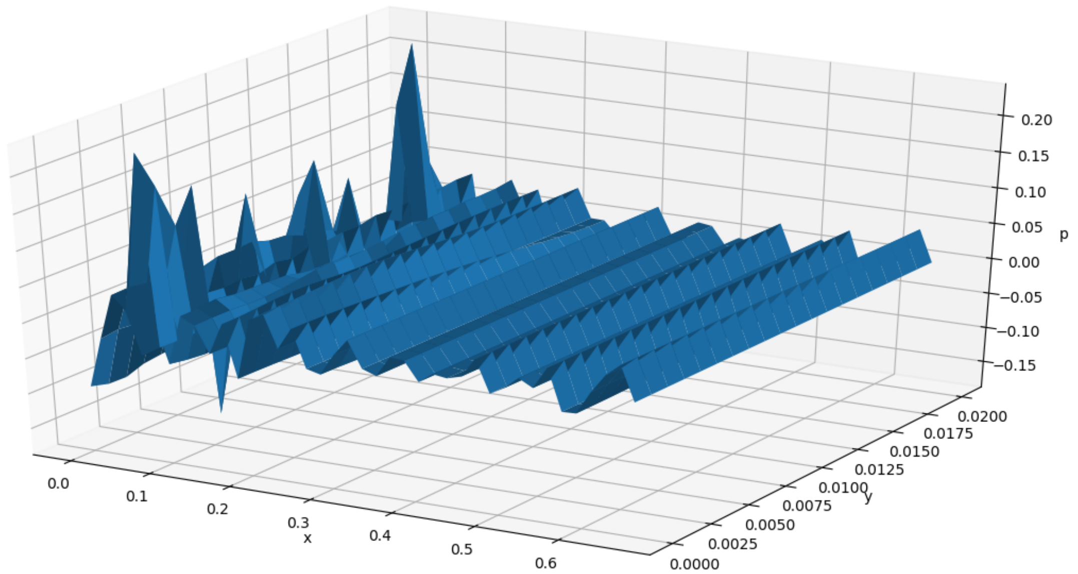


Figure 5.7: Numerical solution of musical note d5 at time $t \approx 1.467s$

6

Conclusion and discussion

In the previous two chapters we have modelled the sound waves inside the flute with two different calculation methods. Now we would like to make a conclusion on which model best describes the sound waves inside the flute.

In chapter 4 an analytic solution of the second-dimensional wave equation is found by the method of separation of variables. This solution is analysed and plotted to check its accuracy. The solution forms a good approximation on the frequency of musical notes and the shape of the calculated waves can also be logically explained. On the other hand the boundary conditions are not completely realistic. In chapter 5 a numerical solution of the second-dimensional wave equation is found by the method of finite difference. This solution does include more realistic boundary conditions but the shape of the waves inside the flute do not differ much between different musical notes. This could be a precision issue or it could indicate that the initial conditions assumed in the calculations are not realistic. The numerical model also takes more time to calculate than the analytic model and it does not find the right frequencies of the musical notes modelled. Therefore we can conclude that the analytic model best describes the sound waves inside the flute.

Nonetheless in future work the numerical model could be improved by increasing the precision of the program calculating the numerical solutions and researching what initial conditions should be used when a sound wave is generated inside the flute. This could result in a more realistic model of the sound waves inside the flute, since the boundary conditions in the analytic model are not completely realistic.

Bibliography

[1] Victoria Boerner. Replacement of the recorder by the transverse flute during the baroque and classical periods. *Udergraduate Awards*, page 17, December 2018.

[2] Richard Haberman, editor. *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*. Pearson Education Limites, 2014.

[3] Romain Joly. How opening a hole affects the sound of a flute. *Journal of Spectral Theory*, page 19, May 2011.

[4] R.M. Mottola. Table of musical notes and their frequencies and wavelengths, Jan 2020.

[5] Natalia K Nikolova. The finite-difference time-domain (fdtd) method - part i.

[6] D. F. Parker. *Fields, Flows and Waves*, volume 6 of *Springer Undergraduate Mathematics Series*. Springer-Verlag London, London, second edition, 2003.

[7] Sarah E. Vitullo. *The history and process of the development of the modern flute*. PhD thesis, Liberty University, 2013.

[8] Joe Wolfe. Flute acoustics: An introduction to how a flute works, 2012.

A

Finding solution of the wave equation

Following the method of separation of variables we assumed: $p(x,y,t) = X(x) \cdot Y(y) \cdot T(t)$ From this we found three ordinary differential equations:

$$T''(t) = \lambda T(t) \tag{A.1}$$

$$X''(x) = \mu X(x) \tag{A.2}$$

$$Y''(y) = \left(\frac{\lambda}{c^2} - \mu\right) Y(y) \tag{A.3}$$

With boundary conditions:

$$p(0,y,t) = 0 \tag{A.4}$$

$$p(L,y,t) = 0 \tag{A.5}$$

$$\frac{\delta p}{\delta y}(x,0,t) = 0 \tag{A.6}$$

$$\frac{\delta p}{\delta y}(x,h,t) = 0 \tag{A.7}$$

We can solve these by looking at nine different cases since λ and μ can be smaller tha0n, equal to, and bigger than zero.

A.1. Case 1: $\lambda = \mu = 0$

$$\begin{aligned}
&T''(t) = 0 \implies T(t) = a_1 t + a_2 \\
&X''(x) = 0 \implies X(x) = c_1 x + c_2 \\
&Y''(y) = 0 \implies Y(y) = b_1 y + b_2
\end{aligned}$$

$$\begin{aligned}
&p(0,y,t) = 0 \implies X(0) = 0 \implies c_2 = 0 \\
&p(L,y,t) = 0 \implies X(L) = 0 \implies c_1 \cdot L + c_2 = 0 \implies c_1 = 0
\end{aligned}$$

$X(x) = 0 \implies p(x,y,t) = 0 \implies$ trivial solution

A.2. Case 2: $\lambda = 0$ and $\mu = -w^2 < 0$

$$T''(t) = 0 \implies T(t) = a_1 t + a_2$$
$$X''(x) = -w^2 X(x) \implies X(x) = c_1 \cos wx + c_2 \sin wx$$
$$Y''(y) = W^2 Y(y) \implies Y(y) = b_1 e^{wy} + b_2 e^{-wy}$$

$$\frac{\delta p}{\delta y}(x,0,t) = 0 \implies \frac{b_1 n\pi}{L} - \frac{b_2 n\pi}{L} = 0 \implies b_1 = b_2$$
$$\frac{\delta p}{\delta y}(x,h,t) = 0 \implies \frac{b_1 n\pi}{L}(e^{\frac{n\pi h}{L}} - e^{\frac{-n\pi h}{L}}) = 0 \implies b_1 = 0$$

$Y(y) = 0 \implies p(x,y,t) = 0 \implies$ trivial solution

A.3. Case 3: $\lambda = 0$ and $\mu = w^2 > 0$

$$T''(t) = 0 \implies T(t) = a_1 t + a_2$$
$$X''(x) = w^2 X(x) \implies X(x) = c_1 e^{wx} + c_2 e^{wx}$$
$$Y''(y) = -W^2 Y(y) \implies Y(y) = b_1 \cos wy + b_2 \sin -wy$$

$$p(0,y,t) = 0 \implies X(0) = 0 \implies c_1 + c_2 = 0 \implies c_1 = -c_2$$
$$p(L,y,t) = 0 \implies X(L) = 0 \implies c_1 e^{wL} - c_1 e^{wL} = 0 \implies c_1 = 0$$

$X(x) = 0 \implies p(x,y,t) = 0 \implies$ trivial solution

A.4. Case 4: $\lambda = -w_1^2 < 0$ and $\mu = -w_2^2 < 0$

$$T''(t) = -w_1^2 T(t) \implies T(t) = a_1 \cos w_1 t + a_2 \sin w_1 t$$
$$X''(x) = -w_2^2 X(x) \implies X(x) = c_1 \cos w_2 x + c_2 \sin w_2 x$$
$$Y''(y) = (-\frac{w_1^2}{c^2} + w_2^2) Y(y)$$

$$p(0,y,t) = 0 \implies X(0) = 0 \implies c_1 = 0$$
$$p(L,y,t) = 0 \implies X(L) = 0 \implies c_2 \sin w_2 L = 0 \implies w_2 = \frac{n\pi}{L}$$

A.4.1. Case 4.1: $(-\frac{w_1^2}{c^2} + w_2^2) = 0$

$$\frac{w_1^2}{c^2} = \frac{n\pi}{L}^2 \implies w_1 = \frac{n\pi c}{L}$$
$$Y(y) = b_1 y + b_2$$
$$\frac{\delta p}{\delta y}(x,0,t) = 0 \implies \frac{\delta Y}{\delta y}(0) = 0 \implies b_1 = 0$$
$$p(x,y,t) = b \sin \frac{n\pi}{L} x \left(a_1 \cos \frac{n\pi c}{L} t + a_2 \sin \frac{n\pi c}{L} t \right)$$

A.4.2. Case 4.2: $(-\frac{w_1^2}{c^2} + w_2^2) = -w_3^2 < 0$

$$Y(y) = b_1 \cos w_3 y + b_2 \sin w_3 y$$
$$\frac{\delta p}{\delta y}(x,0,t) = 0 \implies \frac{\delta Y}{\delta y}(0) = 0 \implies b_2 w_3 = 0 \implies b_2 = 0$$
$$\frac{\delta p}{\delta y}(x,h,t) = 0 \implies \sin(w_3 h) = 0 \implies w_3 = \frac{m\pi}{h}$$
$$\frac{w_1^2}{c^2} = \frac{n\pi}{L}^2 + \frac{m\pi}{h}^2 \implies w_1 = \left(\frac{\pi ct}{hL}\sqrt{n^2h^2 + m^2L^2}\right)$$
$$p(x,y,t) = b \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{h}\right)\left(a_1 \cos\left(\frac{\pi ct}{hL}\sqrt{n^2h^2 + m^2L^2}\right) + a_2 \sin\left(\frac{\pi ct}{hL}\sqrt{n^2h^2 + m^2L^2}\right)\right)$$

A.4.3. Case 4.3: $(-\frac{w_1^2}{c^2} + w_2^2) = w_3^2 > 0$

$$Y(y) = b_1 e^{w_3 y} + b_2 e^{-w_3 y}$$
$$\frac{\delta p}{\delta y}(x,0,t) = 0 \implies \frac{\delta Y}{\delta y}(0) = 0 \implies b_1 w_3 - b_2 w_3 = 0 \implies b_1 = b_2$$
$$\frac{\delta p}{\delta y}(x,h,t) = 0 \implies \frac{\delta Y}{\delta y}(h) = 0 \implies b_1 w_3 (e^{w_3 h} - e^{-w_3 h}) = 0 \implies b_1 = 0$$

$Y(0) \implies p(x,y,t) = 0 \implies$ trivial solution

A.5. Case 5: $\lambda = -w^2 < 0$ and $\mu = 0$

$$T''(t) = -w^2 T(t) \implies T(t) = a_1 \cos(wt) + a_2 \sin(wt)$$
$$X''(x) = 0 \implies X(x) = c_1 x + c_2$$
$$Y''(y) = -\frac{w^2}{c^2} Y(y) \implies Y(y) = b_1 \cos(\frac{w}{c} y) + b_2 \sin(\frac{w}{c} y)$$

$$p(0,y,t) = 0 \implies X(0) = 0 \implies c_2 = 0$$
$$p(L,y,t) = 0 \implies X(L) = 0 \implies c_1 \cdot L + c_2 = 0 \implies c_1 = 0$$

$X(x) = 0 \implies p(x,y,t) = 0 \implies$ trivial solution

A.6. Case 6: $\lambda = -w_1^2 < 0$ and $\mu = w_2^2 > 0$

$$T''(t) = -w_1^2 T(t) \implies T(t) = a_1 \cos(w_1 t) + a_2 \sin(w_1 t)$$
$$X''(x) = w_2^2 X(x) \implies X(x) = c_1 e^{w_2 x} + c_2 e^{-w_2 x}$$
$$Y''(y) = (-\frac{w_1^2}{c^2} - w_2^2) Y(y) = -w_3^2 Y(y) \implies Y(y) = b_1 \cos(w_3 y) + b_2 \sin(w_3 y)$$

$$p(0,y,t) = 0 \implies X(0) = 0 \implies c_1 + c_2 = 0 \implies c_1 = -c_2$$
$$p(L,y,t) = 0 \implies X(L) = 0 \implies c_1\left(e^{w_2 L} + e^{-w_2 L}\right) = 0 \implies c_1 = 0$$

$X(x) = 0 \implies p(x,y,t) = 0 \implies$ trivial solution

A.7. Case 7: $\lambda = w_1^2 > 0$ and $\mu = -w_2^2 < 0$

$$T''(t) = w_1^2 T(t) \implies T(t) = a_1 e^{w_1 t} + a_2 e^{w_1 t}$$
$$X''(x) = -w_2^2 X(x) \implies X(x) = c_1 \cos w_2 x + c_2 \sin w_2 x$$
$$Y''(y) = (\frac{w_1^2}{c^2} + w_2^2) Y(y) = w_3^2 Y(y) \implies Y(y) = b_1 e^{w_3 y} + b_2 e^{-w_3 y}$$

$$\frac{\delta p}{\delta y}(x, 0, t) = 0 \implies \frac{\delta Y}{\delta y}(0) = 0 \implies b_1 w_3 - b_2 w_3 = 0 \implies b_1 = b_2$$
$$\frac{\delta p}{\delta y}(x, h, t) = 0 \implies \frac{\delta Y}{\delta y}(h) = 0 \implies b_1 w_3 (e^{w_3 h} - e^{-w_3 h}) = 0 \implies b_1 = 0$$

$Y(y) = 0 \implies p(x, y, t) = 0 \implies$ trivial solution

A.8. Case 8: $\lambda = w^2 > 0$ and $\mu = 0$

$$T''(t) = w^2 T(t) \implies T(t) = a_1 e^{wt} + a_2 e^{wt}$$
$$X''(x) = 0 \implies X(x) = c_1 x + c_2$$
$$Y''(y) = \frac{w^2}{c^2} Y(y) \implies Y(y) = b_1 e^{\frac{w}{c} y} + b_2 e^{\frac{w}{c} y}$$

$$p(0, y, t) = 0 \implies X(0) = 0 \implies c_2 = 0$$
$$p(L, y, t) = 0 \implies X(L) = 0 \implies c_1 \cdot L + c_2 = 0 \implies c_1 = 0$$

$X(x) = 0 \implies p(x, y, t) = 0 \implies$ trivial solution

A.9. Case 9: $\lambda = w_1^2 > 0$ and $\mu = w_2^2 > 0$

$$T''(t) = w_1^2 T(t) \implies T(t) = a_1 e^{w_1 t} + a_2 e^{w_1 t}$$
$$X''(x) = w_2^2 X(x) \implies X(x) = c_1 e^{w_2 x} + c_2 e^{-w_2 x}$$
$$Y''(y) = (\frac{w^2}{c^2} - w_2^2) Y(y)$$

$$p(0, y, t) = 0 \implies X(0) = 0 \implies c_1 + c_2 = 0 \implies c_1 = -c_2$$
$$p(L, y, t) = 0 \implies X(L) = 0 \implies c_1 \left(e^{w_2 L} + e^{-w_2 L} \right) = 0 \implies c_1 = 0$$

$X(x) = 0 \implies p(x, y, t) = 0 \implies$ trivial solution

B
Numerical approximation of the initial conditions

$$g(x,y) = \frac{\delta p}{\delta t}(x_i, y_j, 0) \approx \frac{p_{i,j}^1 - p_{i,j}^0}{\Delta t} \tag{B.1}$$

$$\implies p_{i,j}^1 \approx \Delta t g(x,y) + p_{i,j}^0 \tag{B.2}$$

$$h(x,y) = \frac{\delta u}{\delta t}(x_i, y_j, 0) \approx \frac{u_{i,j}^1 - u_{i,j}^0}{\Delta t} \tag{B.3}$$

$$\implies u_{i,j}^1 \approx \Delta t h(x,y) + u_{i,j}^0 \tag{B.4}$$

C

Python code of the solutions

C.1. Exact solution visualisation

```
import matplotlib.pyplot as plt
import numpy as np
from mpl_toolkits import mplot3d

#Solution of the wave equation
def p(x, y, t):
    flist = list()
    for m in range(0,2):
        for n in range(0,5):
            L=0.375
            h=0.02
            pi = np.pi
            w = (pi*340.29*np.sqrt(n*n*h*h + m*m*L*L))/(h*L)
            flist.append(np.sin((n*pi*x)/L)*np.cos((m*pi*y)/h)*(np.cos(w*t)+np.sin(w*t)))
    return sum(flist)


#Creating dataset without t
x = np.linspace(0, 0.66, 20)
y = np.linspace(0, 0.02, 20)
X, Y = np.meshgrid(x, y)
Z = p(X , Y, 2)

#Creating figure
ax = plt.axes(projection="3d", xlabel = "x", ylabel = "y", zlabel = "p")

#Creating plot
ax.plot_surface(X,Y,Z)

#Showing plot
plt.show()
```

C.2. Numerical solution visualisation

C.2.1. Musical note c4

```
import matplotlib.pyplot as plt
import numpy as np
from mpl_toolkits import mplot3d
```

```
#Spatial precision of our numerical calculations
precx = 40
precy = 20
dx = 0.66 / precx
dy = 0.02 / precy
d = precx - 1
b = precy - 1


#Making sure CFL stability criterion is satisfied
cfl = 1/(1/(dx*dx) + 1/(dy*dy))
dt = 0.5*np.sqrt(cfl)*(1/340.29)

#Choosing amount of time steps and setting up four 2-dimensional matrices to fill wit
timesteps: int = int(10e5)
mempold = np.zeros((precx, precy))
memuold = np.zeros((precx, precy))
mempnew = np.zeros((precx, precy))
memunew = np.zeros((precx, precy))

# Defining the initial conditions of our problem and some supporting functions for ou
count = 0
def inc():
    global count
    count += 1

def prep(x_prep, y_prep):
    return x_prep * y_prep


def f(i,j):
    if 0.038 < x[i] < 0.052:
        return 1
    else:
        return 0


def g(i,j):
    if 0.038 < x[i] < 0.052:
        return 1
    else:
        return 0


def h(i,j):
    if 0.038 < x[i] < 0.052:
        return 1
    else:
        return 0


#Defining the functions for the pressure and the velocity inside the flute.
#The boundary conditions are incorporated in the functions.
def p(i, j, k):
    global dt, b, d
    rho0 = 1
```

```
    if j == b and 0.038 < x[i] < 0.052:
        return 0
    if i == d:
        return 0
    if j == -1:
        return p(i, 1, k)
    if j == b+1 and (x[i]<=0.038 or x[i] >= 0.052):
        return p(i,b-1,k)
    if i == -1:
        return p(1,j,k)
    else:
        if k == 0:
            return f(i,j)
        if k == 1:
            return dt*g(i,j) + f(i,j)
        else:
            return dt*rho0* memuold[i,j] + mempold[i, j]

def u(i, j, k):
    global dt, dx, dy, b, d
    rho0 = 1
    if k == 0:
        return (p(i, j, 1) - p(i, j, 0))/(dt*rho0)
    if k == 1:
        return dt * h(i, j) + u(i,j,k-1)
    else:
        failsafejmax = j
        failsafeimin = i
        failsafejmin = j
        #When j is too large b+1 cannot be called and by the boundary conditions b-1 shoul
        if j == b:
            failsafejmax = failsafejmax - 2
        #When i or are too small -1 cannot be calles and by the boundary conditions 1 shou
        if i == 0:
            failsafeimin = failsafeimin + 2
        if j == 0:
            failsafejmin = failsafejmin + 2
        t1 = (mempold[i+1, j]-2*mempold[i, j]+ mempold[failsafeimin-1, j])/(dx*dx)
        t2 = (mempold[i, failsafejmax+1]-2*mempold[i, j]+ mempold[i, failsafejmin-1])/(dy*
        return dt * (1/rho0) * (t1 + t2) + memuold[i, j]


# Creating a grid on which we can calculate the pressure inside the flute.
# It is not necessary for x and y to be of the same length.
x = np.linspace(0, 0.66, precx)
y = np.linspace(0, 0.02, precy)
X, Y = np.meshgrid(x, y, indexing="ij")

# Creating index grid to make calculations with the functions easier.
iax = np.arange(0, precx)
jax = np.arange(0, precy)

#Filling in mem-matrices with solutions in an iterative way.
for k_global in range(timesteps):
    if k_global % int(timesteps/100) == 0:
        print("k : ", k_global)
```

```
    for i_global in iax:
        for j_global in jax:
            mempnew[i_global, j_global] = p(i_global, j_global, k_global)
            if i_global != precx-1:
                memunew[i_global, j_global] = u(i_global, j_global, k_global)
    mempold = mempnew
    memuold = memunew


#Choosing the last time step in mem-matrix for plot
Z = prep(X,Y)
for i_global in iax:
    for j_global in jax:
        Z[i_global, j_global] = mempnew[i_global, j_global]

# Plotting the first time step -> intital condition
ax = plt.axes(projection="3d", xlabel="x", ylabel="y", zlabel="p")

# Creating plot
ax.plot_surface(X, Y, Z)

# Showing plot
plt.show()
```

C.3. Musical note d5

```
import matplotlib.pyplot as plt
import numpy as np
from mpl_toolkits import mplot3d

#precision of our numerical calculations
precx = 60
precy = 5
dx = 0.66 / precx
dy = 0.02 / precy
d = precx - 1
b = precy - 1


#Making sure CFL stability criterion is satisfied
cfl = 1/(1/(dx*dx) + 1/(dy*dy))
dt = 0.5*np.sqrt(cfl)*(1/340.29)
#dt == 4.981701738334858e-07

#Choosing amount of time steps and setting up a 3-dimensional matrix to fill with sol
timesteps: int = int(10e5)

#Store values of p and u in a 3-dimenional matrix
mempold = np.zeros((precx, precy))
memuold = np.zeros((precx, precy))
mempnew = np.zeros((precx, precy))
memunew = np.zeros((precx, precy))

# Defining the initial conditions of our problem and some supporting functions for ou
count = 0
def inc():
    global count
```

```
    count += 1

def prep(x_prep, y_prep):
    return x_prep * y_prep


def f(i,j):
    if 0.038 < x[i] < 0.052:
        return 1
    else:
        return 0


def g(i,j):
    if 0.038 < x[i] < 0.052:
        return 1
    else:
        return 0


def h(i,j):
    if 0.038 < x[i] < 0.052:
        return 1
    else:
        return 0


#Defining the functions for the pressure and the velocity inside the flute.
#The boundary conditions are incorporated in the function for the pressure inside the flut
def p(i, j, k):
    global dt, b, d
    rho0 = 1
#boundary conitions on p
    #check if for every hole there is at least one value of x inside the hole.
    if j == b and (0.038 < x[i] < 0.052 or 0.298 < x[i] < 0.306 or 0.581 < x[i] < 0.597 or
        return 0
    if i == d:
        return 0
    if j == -1:
        return p(i, 1, k)
    if j == b+1 and (x[i]<=0.038 or 0.052 < x[i] < 0.298 or 0.306 < x[i] < 0.581 or 0.597
        return p(i,b-1,k)
    if i == -1:
        return p(1,j,k)
    else:
        if k == 0:
            return f(i,j)
        if k == 1:
            return dt*g(i,j) + f(i,j)
        else:
            return dt*rho0* memuold[i,j] + mempold[i, j]

def u(i, j, k):
    global dt, dx, dy, b, d
    rho0 = 1
    if k == 0:
```

```
        return (p(i, j, 1) - p(i, j, 0))/(dt*rho0)
    if k == 1:
        return dt * h(i, j) + u(i,j,k-1)
    else:
        failsafejmax = j
        failsafeimin = i
        failsafejmin = j
        #When j is too large b+1 cannot be called and by the boundary conditions b-1
        if j == b:
            failsafejmax = failsafejmax - 2
        if i == 0:
            failsafeimin = failsafeimin + 2
        if j == 0:
            failsafejmin = failsafejmin + 2
        t1 = (mempold[i+1, j]-2*mempold[i, j]+ mempold[failsafeimin-1, j])/(dx*dx)
        t2 = (mempold[i, failsafejmax+1]-2*mempold[i, j]+ mempold[i, failsafejmin-1])
        return dt * (1/rho0) * (t1 + t2) + memuold[i, j]


# Creating a grid on which we can calculate the pressure inside the flute.
# It is not necessary for x and y to be of the same length.
x = np.linspace(0, 0.66, precx)
y = np.linspace(0, 0.02, precy)
X, Y = np.meshgrid(x, y, indexing="ij")

# Creating index grid to make calculations with the functions easier.
iax = np.arange(0, precx)
jax = np.arange(0, precy)

#Filling in mem-matrices with solutions
for k_global in range(timesteps):
    if k_global % int(timesteps/100) == 0:
        print("k : ", k_global)
    for i_global in iax:
        for j_global in jax:
            mempnew[i_global, j_global] = p(i_global, j_global, k_global)
            if i_global != precx-1:
                memunew[i_global, j_global] = u(i_global, j_global, k_global)
    mempold = mempnew
    memuold = memunew


#Choosing the last time step in mem-matrix for plot
Z = prep(X,Y)
for i_global in iax:
    for j_global in jax:
        Z[i_global, j_global] = mempnew[i_global, j_global]

# Plotting the first time step -> intital condition
ax = plt.axes(projection="3d", xlabel="x", ylabel="y", zlabel="p")

# Creating plot
ax.plot_surface(X, Y, Z)
# ax.set_zlim([-3,5])

# Showing plot
```

```
#print("count: ", count)
plt.show()
```

D
Comparing numerical and analytical method

To compare the numerical and analytic solution methods we plotted note c5 at the same time, with the same initial conditions. Figures D.1 and D.2 show that both methods result in approximately the same sound wave inside the flute. Both plots show the sound wave at time $t = 9.981684966311907 \cdot 10^{-3}$.

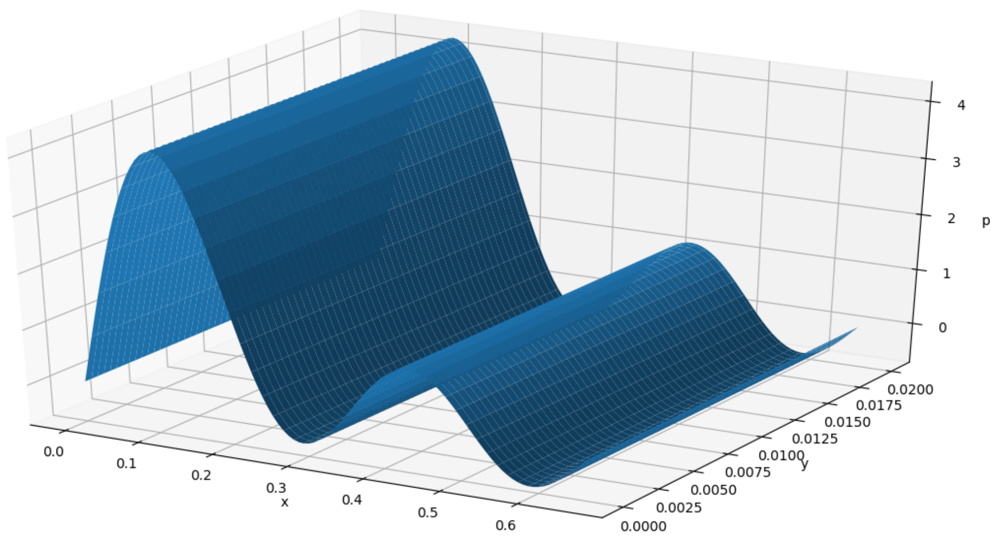


Figure D.1: Analytic solution of musical note c5

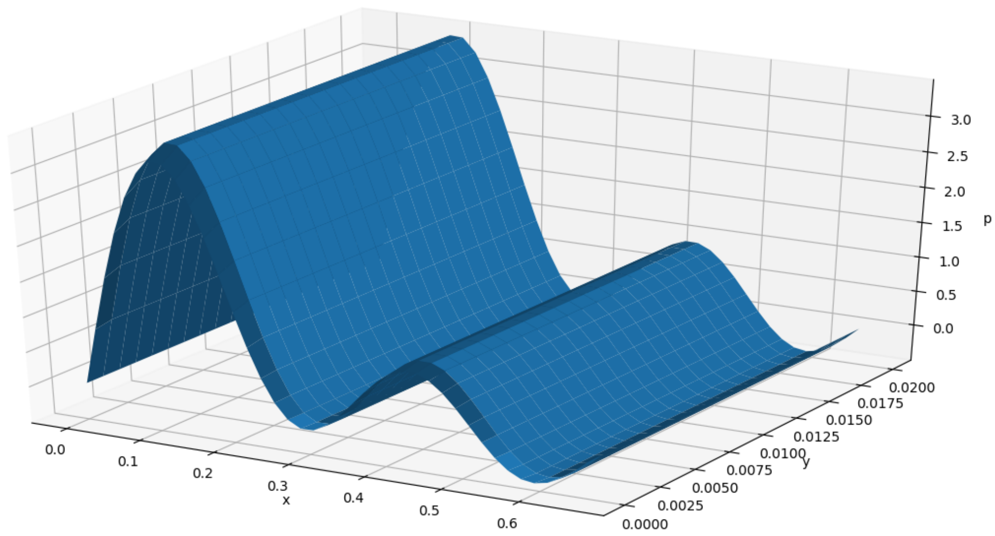


Figure D.2: Numeric solution of musical note c5