SOME PROBLEMS RELATING TO
A TELEPHONE SYSTEM
EMPLOYING NON-HOMING SELECTORS

N. RODENBURG

Kiesysteme

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PROEFSCHRIFT TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAP AAN DE TECHNISCHE HOGESCHOOL TE DELFT, OP GEZAG VAN DE RECTOR MAGNIFICUS Dr O. BOTTEMA, HOOGLERAAR IN DE AFDELING DER ALGEMENE WETENSCHAPPEN, VOOR EEN COMMISSIE UIT DE SENAAT TE VERDEDIGEN OP WOENSDAG 6 MEI 1953, DES NAMIDDAGS TE 4.00 UUR

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NICOLAAS RODENBURG
ELECTROTECHNISCH INGENIEUR
GEBOREN TE GRONINGEN

DIT PROEFSCHRIFT IS GOEDGEKEURD DOOR DE PROMOTOR,

PROF. DR IR W. TH. BÄHLER

Aan de Directie van de N.V. Philips' Telecommunicatie Industrie te Hilversum betuig ik mijn hartelijke dank voor de toestemming, om dit gedeelte van mijn werk in deze vorm te publiceren.

Aan mijn Ouders

Aan mijn Vrouw

SOME PROBLEMS RELATING TO A TELEPHONE SYSTEM EMPLOYING NON-HOMING SELECTORS

SUMMARY

Various problems relating to a telephone system employing non-homing selectors have been studied. Every selector stage of this system is controlled by common equipments, each of which serves exclusively a number of selectors. When the dial pulses are received in the common-control circuit, a high-speed uniselector, that hunts for a free trunk in the desired group, is started. Testing must be done before the subscriber sends the next train of pulses. The speed of the selector must, therefore, be high enough to enable it to make one revolution in the inter-digital pause.

The following problems were examined:

In many cases the outlets of a selector are connected to the next stage with an interconnected multiple. Such multiples are well-known in conjunction with homing selectors, but for non-homing selectors new schemes will have to be contrived.

Starting from Erlang's "ideal arrangement", some multiples for non-homing selectors were set up and investigated for their traffic-carrying capacity by means of artificial traffic. From these measurements, various rules could be derived for the construction of these multiples. The conclusion, that interconnected multiples for non-homing selectors need not be inferior to the well-known gradings for homing selectors, seems to be justified.

A second problem of importance is met with when calculating the selector quantities, if common-controls are used for the set up. Although the formulae previously derived by Jacobeus can be used for all stages, it proved to be possible — especially in the case of the final selectors — to obtain a better approximation formula. For the cases that are of practical importance, Tables are given of the traffic-carrying capacity. By applying non-homing selectors, the average travel per call is considerably decreased compared with that for homing selectors. This average travel has been calculated for several cases, for full- as well as for limited availability groups.

The conclusion of these calculations is that a non-homing selector makes 7 to 10 times less revolutions than a homing selector in the same period.

Lastly, the required selector speed, which is closely related to the minimum length of the inter-digital pause, was further considered. Although the selector speed is based on the shortest interdigital pause which occurs with the "lost motion" dial, the effect of a lower speed or a shorter interdigital time must still be examined. The calling subscriber will receive busy tone if he sends a new dialling train before the selector has been positioned, so that too fast dialling or a too slow selector speed will reduce the grade of service. This reduction has been calculated for different cases. Results show the selector speed to be less critical than is often expected.

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Definition of symbols

m = number of contacts of a selector, giving access to the lines of one direction.
N = number of trunks in an interconnected multiple.
q = grade of service (= probability of calls being lost).
k = number of circuits, served by one common control.
s = average holding time of a common-control circuit.
h = average duration of call, including the interval s.
$\beta = \frac{s}{h}$.

$$E_m(a) = \frac{\frac{a^m}{m!}}{1 + a + \frac{a^2}{2!} + \dots \frac{a^m}{m!}}.$$

c = number of outlets of a selector.
n = number of directions which may be connected to a selector.
w = average number of contacts traversed.

CHAPTER I

1. Introduction

The Philips Telecommunication Industries have developed the *UR 49* telephone system, which employs some principles having found little or no description in literature. The object of this study is to give a theoretical explanation of these principles.

The *UR 49* system has been realized with a selector and a relay of novel design, details of which can be found elsewhere [1] [2]). The selector employed is a 100-point uniselector, which can be coupled to a common drive system, giving it a rotational speed of three revolutions per second. This high selector speed allows a simple telephone system to be designed.

Before a short description of this system is given, it may be stated with what means the selector is positioned on a free trunk of a desired direction. The selector has four brushes and contact arcs a, b, c and d (*fig. 1*), of which a and b form the speech path whilst arc c contains the test wires to the succeeding stage of selection, and arc d serves for the positioning of the selector. The dial impulses, determining the number of the group of trunks to which the selector must be directed, are sent, via the a and b-wires, to a control circuit, common to a number of selectors. This series of impulses is registered on a small stepping switch or on a relay counting chain, the final condition of which corresponds to the number of pulses of the series. The common-control circuit now marks, in the d-arc of the selector, those contact positions to which, in the c-arc of the selector, are connected the test wires of the desired group. Upon the termination of the impulse train the selector receives a start signal and will rotate until the high-speed test relay in the common-control circuit, which relay is connected in series with the d- and c-arcs, is energized. This is only possible when a free trunk is encountered in the marked group.

The principles of the *UR 49* system may now be described as follows:

Groups of 100 subscribers are served by 100-point 1st line finders, the number of which is determined by the traffic to be carried and by the grade of service desired. The wiper side of these 1st line finders must now be connected to the so-called connecting circuit, associated with the 1st group selector. This connection can be established in various ways:

a. the wiper sides of all 1st line finders are wired to the banks of groups of 2nd line finders, the wiper sides of which are directly connected to the connecting circuits.

b. the wiper sides of all 1st line finders are directly connected to the connecting circuits.

c. some 1st line finders are wired directly to connecting circuits, whilst others are wired to the banks of 2nd line finders.

The method to be preferred depends on the relative cost of the selectors and the connecting circuits, and on the price of the relay sets, necessary in the three different cases. An important part is played,

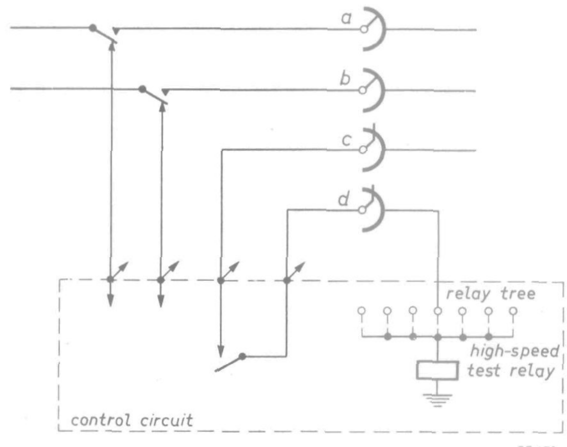

Fig. 1. Positioning of the selector. The dial impulses are sent into a common-control circuit via the a-and b-wires. The desired group is marked on the d-arc by means of a relay-counting chain. The selector, when started, tests the c-arc for a free trunk which belongs to the direction marked on the d-arc.

moreover, by the magnitude of the traffic and by the number of subscribers of the exchange. In general the third method will be preferred.

For a correct understanding, method *a* will be described first (*fig. 2*). The line finders of a number of groups of 100 lines are wired to the contact banks of a group of 2nd line finders. The wipers of the 2nd line finders are connected to a connecting circuit, the right-hand side of which gives access to a 1st group selector. The connecting circuits arranged in two ways. *Fig. 3* indicates a method, analogous to the one described above. This method has the disadvantage of associating the common-control circuits with one 100-group only. In order

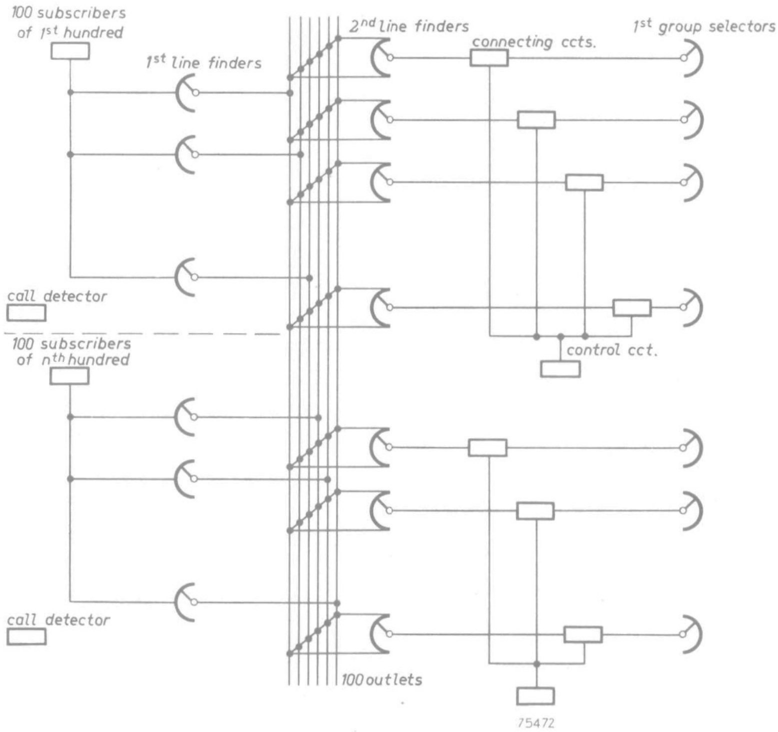

Fig. 2. Connecting scheme with 1st and 2nd line finders.

can be coupled, by means of a relay, to a common-control circuit which controls the positioning of the 2nd and 1st line finders and of the 1st group selector. One common-control circuit is provided for every 6 connecting circuits; the coupling circuit is so arranged, that only one connecting circuit at a time may be connected to the common control. When a subscriber originates a call, a call detector, one of which is provided per group of 100 subscribers, looks for a free common-control circuit belonging to a group of 2nd line finders which give access to the calling subscriber and of which at least one of the associated 6 connecting circuits is idle. At the same time, the call detector marks those contacts in the 2nd line-finder banks, to which idle 1st line finders of the calling 100-group are connected. The common-control circuit now starts the 2nd line finder, which will search for a contact marked by the call detector. The 1st line finder will then be started, and when the calling line has been found, the subscriber will obtain dialling tone from the common-control circuit.

If method *b* is applied, the system may again be

to keep the influence of the common-control circuits on the grade of service small, appreciably fewer than 6 selectors may be served by one common control, and the total number of common-control circuits thereby becomes fairly large. An improvement may be obtained by associating the

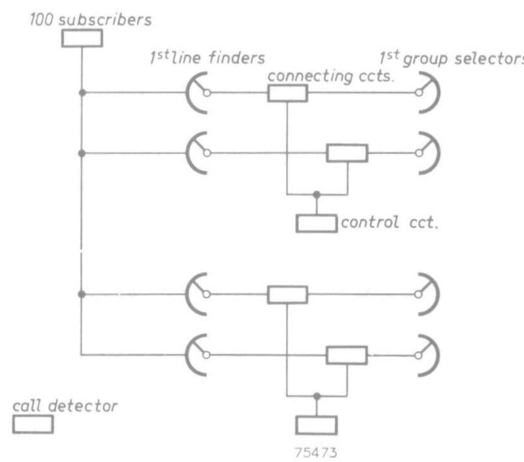

Fig. 3. Connecting scheme with 1st line finders exclusively. The common-control circuits only serve the line-finders and 1st group selectors of one 100-group.

common-control circuits with several 100-groups, as indicated by *fig. 4*. One common control is now associated with a number of connecting circuits, all belonging to different 100-groups. A call originating in a certain 100-group may now be served by as many common-control circuits as there are line finders in the group, provided these common controls are not occupied for the establishment of connections in other 100-groups. The task of the call detector is now somewhat more complicated, since it must not only look for a free common control having at least one free connecting circuit associated with it, but it must look for a free common control having a free connecting circuit serving the calling 100-group.

Method *c* is a combination of methods *a* and *b*, with the understanding that the caller is only connected to a 2nd line finder when no direct path is available. As soon as the caller has received dialling tone, he may start dialling the first digit. The dialling pulses are received by the common-control circuit, and approximately 120 ms after the last pulse of the train the 1st group selector is started. The 1st group selector searches for a free outlet, which may be, for instance, a 2nd group selector in the group marked by the common-control circuit. The 1st group selector will only be stopped if the 2nd group selector as well as its associated common-control circuit are free. The 2nd group selectors are connected to the 1st group selectors by means of an interconnected multiple, and ten 2nd group selectors are served by one common control. If every 1st group selector has access to, for instance, ten 2nd group selectors, the grading is so formed that these 10 selectors are served by as many different common-control circuits as possible. The circuit arrangements are such, that the 1st group selector tests, via a relay in the 2nd group selector, on a free common control. Unless both the 2nd group selector and the common control are idle, the 1st group selector will not test free. The 1st group selector must have found a free outlet before the subscriber sends the next train of pulses. If the minimum interdigital pause is assumed to be 450 ms *), it must be guaranteed that the selector can make a complete revolution within this interval. As indicated before, about 120 of these 450 ms are lost, leaving 330 ms for the selector to complete a revolution. This leads to the requirement that the speed of the selector must be high enough to enable it to traverse 300 contacts per second. The system is so arranged that the subscriber receives busy tone if, for some reason or other, a new train of impulses arrives before the selector has been positioned.

*) Measurements have shown that, with a "lost motion" dial, only approximately 1% of the subscribers produces interdigital pauses shorter than 450 ms.

With the method described there is no longer any necessity to give the selectors a home position, and the *UR 49* system is consequently arranged to use non-homing selectors, which do not return to zero position after a conversation. If now the outlets to a certain direction are spread evenly over the circumference of the contact bank, the average number of contacts which the selector

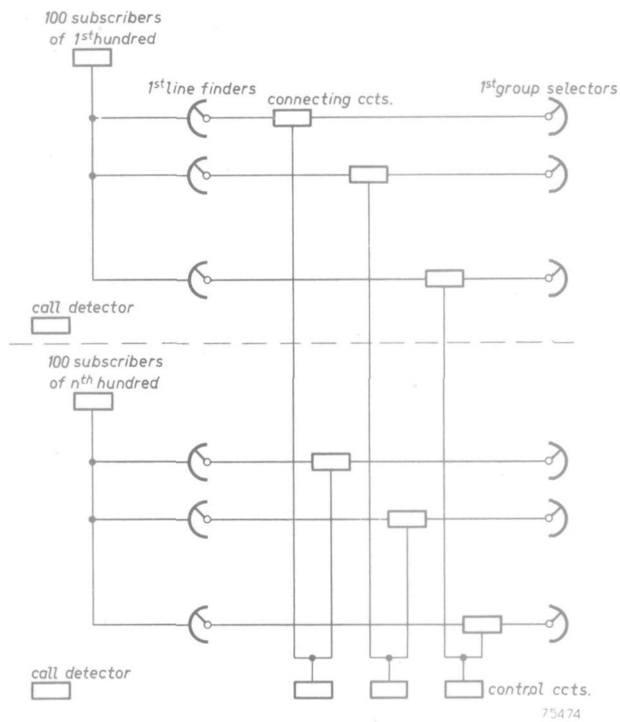

Fig. 4. Connecting scheme with 1st line finders exclusively. The common-control circuits serve cord circuits belonging to more than one 100-group.

must traverse per conversation becomes very small. The connection between the final selectors and their associated common-control circuits is made according to a diagram, analogous to one of the diagrams already described for 1st line finders directly connected to connecting circuits.

Fig. 5 outlines a method for the case of final selectors being accessible to the penultimate stage in a full-availability group. The method to be followed in case the final selectors are reached via an interconnected multiple will be treated later on in this study. The control circuits are common to a group of, for instance, 1000 subscribers. The first common-control circuit is connected with all 1st final selectors of each of the ten 100-groups, the second to all 2nd final selectors, and so on. The marking of the d-arc is given via a marking translator, common to all control circuits of the 1000-group, and the d-arcs are multipled over all final selectors of this group. The tens and units digits dialled by the subscriber are received by the common-control circuit. This then waits until the marking translator becomes free and then marks, via the marking translator, one of the 100 wires

1. The multiple for non-homing selectors, with the outlets to a certain direction spread over the arc, will deviate from the conventional type. The known multiples are always based on selectors with a home position and cannot be applied here. The manner in which the multiples for non-homing selectors will be executed and the efficiencies to be expected from them, will be discussed in Chapter II.

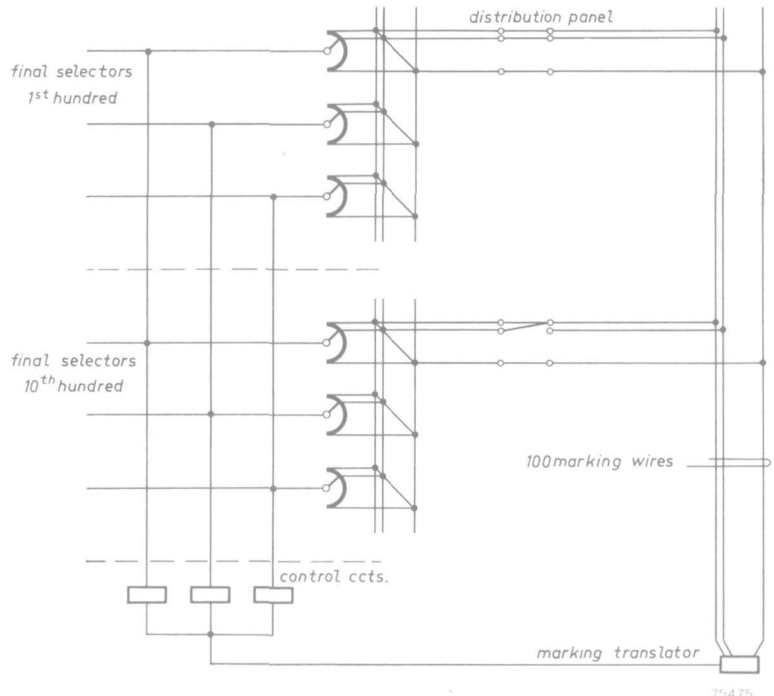

Fig. 5. Positioning of the final selector. The common-controls serve a group of 1000 subscribers. If the tens and units digits are received, the common-control circuit waits until the marking translator becomes free and then marks via this device one of the 100 wires of the marking multiple.

of the marking multiple. The final selector with which the common control is connected at that instant, is started simultaneously and positioned on the contact indicated by the marking wire. Since there is only one common marking multiple, only one final selector of the 1000-group can be served by the marking translator at a time. An extra delay is therefore encountered between the termination of the units digit and the moment the called subscriber is rung. Due to the high speed of the selector this delay is, however, negligible.

In the above, only those properties of the *UR 49* system have been brought forward which give rise to some questions of principle.

The application of the system to large cities and to trunk switching networks, where a route register will, as a rule, be called in, will be no further enlarged upon.

The following questions arise from the arrangement of the system as outlined above:

2. Due to the use of common-control circuits serving a number of selectors, the usual methods for the calculation of selector quantities are inapplicable. This is due to the fact that, during the time a common-control circuit is occupied in positioning a certain selector, the other selectors, associated with the same common control, are inaccessible. This leads to an increased probability of loss. Although the effect of the common-control circuits has already been treated in literature, some improvements may nevertheless be applied to the formulae which have been deduced. Chapter III is dedicated to the treatment of these formulae.
3. Although the reduction of the average number of contacts to be traversed per call, due to the use of non-homing selectors, appears evident, it is important to know the magnitude of this reduction. In Chapter IV the average selector travel per call is calculated for different cases.

4. It has been argued here that, if the interdigital pause is equal to or greater than 450 ms, a selector speed of 300 contacts per second is sufficient to reach all outlets within this time. The question arises at once as to what will be the consequences of a different type of telephone dial, and of a subscriber dialling very fast and producing interdigital pauses shorter than 450 ms. It might be possible, moreover, that the speed of the common-drive system of the selectors deviates somewhat from the nominal value. It has been stated already that the calling subscriber will receive busy tone if he sends a new dialling train before the selector has been positioned, so that too fast dialling or a too slow selector speed will reduce the grade of service. It is evident that the influence of the factors mentioned should be small. This influence is more closely investigated in Chapter V.

CHAPTER II

Interconnected multiples for non-homing selectors

2.0 General considerations

When the required number of trunks between two successive selector stages exceeds the number of selector-bank contacts available for a certain direction, it becomes necessary to resort to trunk groups with limited availability. The selectors are divided into a number of groups, each of which can no longer reach all trunks, but only a limited number of trunks. Although the grade of service of limited-availability groups for homing selectors can only be calculated with difficulty or not at all, so many data are known from practice, that the application of such groups causes no difficulty [3]. If, with non-homing selectors, the trunks for one direction were connected to one group of consecutive bank contacts, the result obtained would be practically the same as for homing selectors, since a selector would, in practically all cases, start from a position outside the wanted group. In the case, for instance, of the 1st group selector of a 10,000 line local exchange, the selector arc of 100 contacts will practically always be divided up into 10 groups of 10 outlets, and in 9 cases out of 10, the 10 contacts of a wanted group are searched in the same direction, starting from the same contact.

It has been mentioned in the introduction that in the *UR 49* system the trunks of a certain direction are not arranged consecutively, but are spread evenly over the circumference of the selector bank. A trunk to a certain direction is followed by trunks to each of all the other directions, before another trunk of the same direction is encountered. The trunks of each direction are spread evenly over the arc, and the group to one particular direction may be considered as belonging to a non-homing selector which can reach only the trunks of this direction. The opinion has long been prevalent, that for non-homing selectors the grade of service of interconnected multiples must be inferior to that foı homing selectors. This is indeed true, if the well-known grading of the British Post Office is used, which grading is, in principle, based on the fact that the selector has a home position. Erlang has already mentioned a kind of multiple, entirely different from this grading [4]) *), although it is impossible to apply this in practice. This multiple has, however, been used by O'Dell as a basis for the calculation of the grade of service of his gradings. Erlang divides the number of selectors into a number of groups such, that each group — if m contacts of the selector give access to the multiple of N trunks — searches a different permutation of N trunks, m at a time. This implies that the probability that m particular trunks out of N are busy, is the same for every permutation of N trunks, m at a time, and also, that all trunks carry the same amount of traffic. The number of groups which must be formed in this way is equal to $\frac{N!}{(N-m)!}$ for homing selectors. For non-homing selectors which start from an arbitrary m position, the number of groups may be reduced to $\frac{1}{m}\,\frac{N!}{(N-m)!}$. Erlang has expressed the grade of service of a multiple of this kind in a form which is valid for homing as well as for non-homing selectors. This expression will be more closely considered in Chapter IV. The grade of service according to this formula is better than for a grading of the same number of trunks, but it is evident that in practical cases the number of selectors is much too small to permit the formation of the required number of groups. Although the general form of the relevant equations can indeed be established for any arbitrary multiple, the solution of the system cannot be obtained, since the number of equations is 2^N.

It seems obvious that multiples should be found which are built in a manner similar to Erlang's, but

*) It is little known, that the same form of multiple had already been mentioned in 1921 by Molina [5]).

with a limited number of groups. Calculations for a very small number of trunks have shown, that it is quite possible to design multiples with a small number of groups, having a grade of service only slightly inferior to that of Erlang's multiple. As has been pointed out before, these calculations are impracticable in reality, whilst methods of approximation have until now not been found. To this may be added, that even approximation methods would add little to the solution of the problem, since for a given number of trunks the number of practicable multiples is still very large. All that seems possible as yet, is to design a number of multiples from theoretical considerations and to test these with the artificial traffic method [3]).

One is naturally led to look for multiples which, analogous to Erlang's, apply approximately the same amount of traffic to each trunk. To this end all trunks must occupy, as far as possible, equivalent positions and should therefore be connected to the same number of selector outlets (in different groups). The groups should be well mixed in order that peaks of traffic in one particular group shall not affect a few groups seriously, whilst others are hardly affected at all.

The investigation of these multiples will now be carried out as follows: First of all, for a number of N trunks and a traffic of a erlangs, some multiples of various design will be tested. The results will yield the conclusion that one particular multiple should be preferred. Based on this preferred multiple, some new multiples will be designed for different numbers of trunks. From these latter cases a few will have to be tested and, as will be shown, will satisfy the expectations. In these considerations the influence of the common-control circuits has been disregarded; this influence will be calculated in Chapter III. The artificial-traffic test method will, however, be discussed briefly beforehand, whilst, in addition, the principles will be given of a semi-automatic traffic machine, which allow the time-consuming investigations to be speeded up appreciably.

2.1 Investigation by means of the artificial-traffic method

The method used is that given by Kosten [6]) [7]), which is based on an exponential distribution of holding times, with the result that the holding time no longer plays a part in the process of investigation. The method may be illustrated with an example:

If it is desired to determine the grade of service of a full-availability group of 7 trunks, the traffic offered being 3 erlangs, the following significance is attributed to a set of 10 numbers:

1: a conversation (if in progress) on trunk 1 is terminated.
2: ,, ,, (,, ,, ,,) ,, ,, 2 ,, ,, .
3: ,, ,, (,, ,, ,,) ,, ,, 3 ,, ,, .
⋮
7: ,, ,, (,, ,, ,,) ,, ,, 7 ,, ,, .
8: a call appears.
9: ditto.
0: ditto.

Numbers are now thrown by means of a *10*-position roulette, for example, and each time the corresponding action is taken. If trunks *1* and *2* are occupied and number *9* is thrown, then, if homing selectors are being considered, trunk *3* will be occupied. A record is kept of the condition of all trunks and it can be seen at once when a call is being lost due to all trunks being occupied. To obtain sufficiently accurate results, very many numbers must be thrown. If non-homing selectors are being examined, a second number must be thrown each time, to indicate the contact from which the selector starts searching. For interconnected multiples a third number must be thrown to determine the sub-group in which the call is supposed to appear. The traffic intensity may be varied by means of the ratio of the numbers of calls appearing and disappearing. If N be the number of trunks and a the traffic intensity, then N "appearance" and a "disappearance" numbers may be taken. Instead of a roulette, Tables of random numbers may be used to advantage [8]). In order to keep an easy record of the condition of all trunks, a machine has been designed, arranged to analyse groups of a maximum of 33 trunks, distributed at random over a number of sub-groups not exceeding 12, whilst the number of outlets per sub-group could be either 10 or 20. The random numbers, taken from the Table, are given a significance, indicating:

1. Appearance or disappearance of a call.
2. In case of a disappearance, the number of the trunk on which the conversation must terminate.
3. In case of an appearance, the sub-group in which the call appears and the number of the contact from which the selector starts searching.

These data are registered in the machine by means of 3 rows of push-button keys. The machine then either determines the trunk which must be occupied, or whether the call must be considered as lost, because all trunks accessible to a group are busy. The numbers of successful and lost calls are registered on counters. Further particulars on the method of measurement[9]), on the machine and on the results of measurements, will be found in *appendix I*.

2.2 Results of investigated multiples

The following interconnected multiples have been tested successively, each multiple containing 25 trunks and having a traffic of 16 erlangs applied to it, whilst each sub-group has 10 outlets to the multiple:

a. Grading destined for homing selectors (*fig. 6*).

A number of interconnected multiples for non-homing selectors have been examined.

b. An interconnected multiple with 5 sub-groups, in which the trunks are distributed in an arbitrary manner, with the understanding that the trunks of each sub-group also appear in the other sub-groups. Each trunk is connected to 2 outlets *). The conception of the multiple is as follows:

Group	*Outlet No* 1	2	3	4	5	6	7	8	9	10
I	11	25	19	8	14	24	23	4	22	21
II	20	23	9	16	18	24	7	13	3	25
III	18	6	12	19	20	2	17	21	15	22
IV	10	11	12	13	5	14	15	16	17	1
V	1	2	3	4	5	6	7	8	9	10

Numbers are trunk numbers.

c. A multiple also containing 5 sub-groups, the trunks being, however, arranged more regularly (*fig. 7*).

d. A multiple with 6 sub-groups, in which some trunks are connected to 2 and others to 3 sub-groups (*fig. 8*).

e. A multiple of slightly different design, also with 6 subgroups (*fig. 9*).

f. A 6-group multiple in which not the trunks of a sub-group appear in all other sub-groups (*fig. 10*).

g. A multiple analogous to the preceding one (*fig. 11*).

The grades of service measured are reproduced in the following Table, which also gives the grade of service of E r l a n g's ideal multiple, and of the grading mentioned under *a*, according to the B.P.O.

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*) The diagram of the multiple has not been drawn, as this would not clarify the principle.

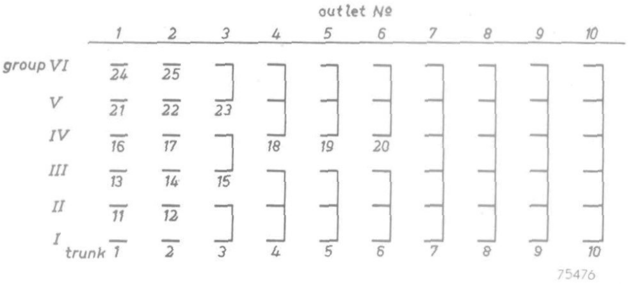

Fig. 6. Grading of 25 lines for homing selectors. $a = 16$ erlangs, $N = 25$, $q = 0.035 \pm 0.0059$.

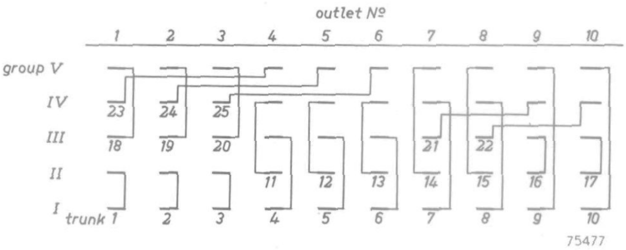

Fig. 7. Interconnected multiple for non-homing selectors. $a = 16$ erlangs, $N = 25$, $q = 0.033 \pm 0.0056$.

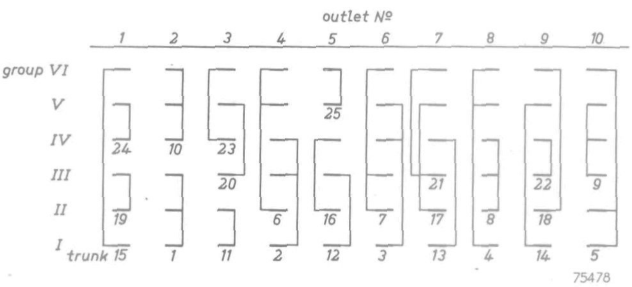

Fig. 8. Interconnected multiple for non-homing selectors. $a = 16$ erlangs, $N = 25$, $q = 0.031 \pm 0.004$.

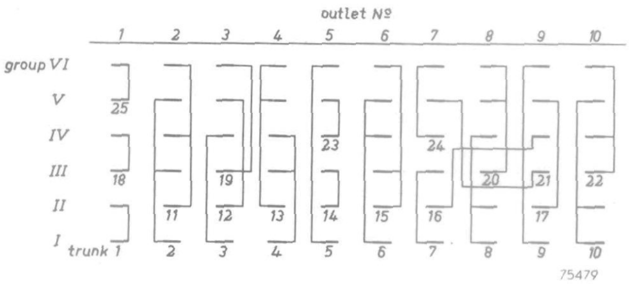

Fig. 9. Interconnected multiple for non-homing selectors. $a = 16$ erlangs, $N = 25$, $q = 0.032 \pm 0.0064$.

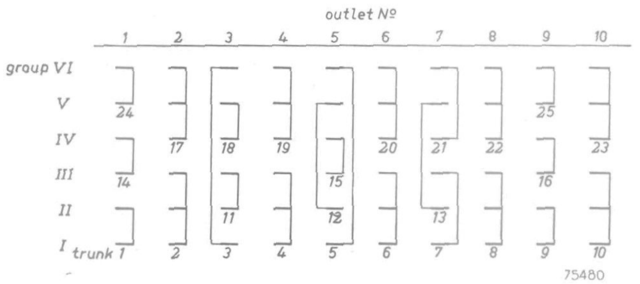

Fig. 10. Interconnected multiple for non-homing selectors. $a = 16$ erlangs, $N = 25$, $q = 0.042 \pm 0.0075$.

	Grade of service measured	Homing sel. BPO	Erlang
a. Homing selectors	0.035 ± 0.0059	0.033	0.027
b. 5 sub-groups, irregular	0.032 ± 0.0048	0.033	0.027
c. 5 s.grps, regular	0.033 ± 0.0056	0.033	0.027
d. 6 s.grps, fully mixed	0.031 ± 0.004	0.033	0.027
e. 6 s.grps, fully mixed	0.032 ± 0.0064	0.033	0.027
f. 6 s.grps, not completely mixed	0.042 ± 0.0075	0.033	0.027
g. 6 s.grps, not completely mixed	0.044 ± 0.0073	0.033	0.027

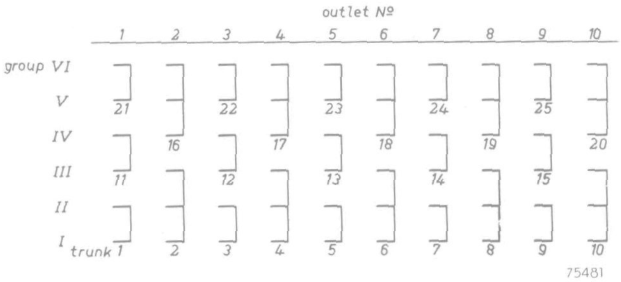

Fig. 11. Interconnected multiple for non-homing selectors. $a = 16$ erlangs, $N = 25$, $q = 0.044 \pm 0.0073$.

Finally, a multiple consisting of 20 trunks in 4 sub-groups, in which the trunks of each sub-group appeared in all other sub-groups, was tested on the traffic machine of the British Post Office *). This multiple had the following form:

Group	Outlet No 1	2	3	4	5	6	7	8	9	10
I	10	9	8	7	17	16	15	20	19	18
II	6	5	4	14	13	12	11	18	19	20
III	3	2	1	11	12	13	14	15	16	17
IV	1	2	3	4	5	6	7	8	9	10

The results obtained were:

Traffic offered	*Grade of service measured*	*Homing sel. BPO*	*Erlang*
8 erlangs	0.0031 ± 0.001	0.0023	0.001
10 „	0.0094 ± 0.0017	0.011	0.007
13.3 „	0.066 ± 0.006	0.06	0.041

A number of conclusions can be drawn from the above measurements:

1. A multiple for non-homing selectors need not be inferior to one for homing selectors (results *b*, *c*, *d*, *e*).
2. Interconnected multiples, in which the trunks of a sub-group do not appear in all other sub-groups, provide a lower grade of service than those in which they do appear in all others (results *f* and *g*).
3. When the trunks of a sub-group do appear in all other sub-groups, the form of the multiple is not of great importance.

*) My sincere thanks are due to *Messrs* Bell, Harmston and McGrath of the British Post Office for their cooperation in the investigation of these multiples.

2.3 Some newly designed multiples

Since the testing of multiples for various numbers of trunks and for various traffic values would demand too much time, the rules which have been established until now have been utilized in the construction of a number of multiples which may be expected to yield good results.

1. A multiple for 10—20 trunks, consisting of four sub-groups with 10 outlets per group. For 20 trunks the multiple appears as follows:

Group	*Outlet No* 1	2	3	4	5	6	7	8	9	10
I	1	2	3	4	5	6	7	8	9	10
II	1	2	3	11	12	13	14	15	16	10
III	17	18	19	4	5	6	14	15	16	20
IV	17	18	19	11	12	13	7	8	9	20

For each outlet of every sub-group the number of the trunk connected to it has been indicated. It can be seen from the Table that:

4 trunks of sub-group I are also accessible to sub-group II;

3 trunks of sub-group I are also accessible to sub-group III;

3 trunks of sub-group I are also accessible to sub-group IV.

Similar rules apply to the other sub-groups.

If less than 20 trunks are required in the multiple, four instead of two groups may be connected in parallel in some columns. This has been drawn in *fig. 12* where the dotted connections are taken away in the numbered sequence when the number of trunks grows from 10 to 20. The dotted connections are made in such a manner that the distribution of the trunks over the multiple is as even as possible.

2. A multiple of 20—40 trunks, with 8 sub-groups and 10 outlets per sub-group.

For 40 trunks, the multiple takes the following form:

Group	*Outlet No* 1	2	3	4	5	6	7	8	9	10
I	1	2	3	4	5	6	7	8	9	10
II	1	11	12	13	14	15	16	8	17	18
III	19	2	20	21	22	23	16	24	9	18
IV	19	25	26	4	14	27	28	24	17	10
V	29	11	20	30	31	6	28	32	33	34
VI	29	35	12	21	5	27	36	32	37	38
VII	39	25	3	30	22	15	36	40	33	38
VIII	39	35	26	13	31	23	7	40	37	34

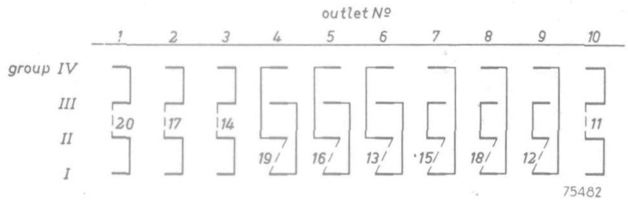

Fig. 12. Interconnected multiple for non-homing selectors for 10—20 trunks. The dotted connections are removed in numbered order as the trunks increase from 10 to 20.

The multiple has been so designed that:

2 trunks of group I can be reached from group II,
2 „ „ „ I „ „ „ „ „ III,
2 „ „ „ I „ „ „ „ „ IV,
1 trunk „ „ I „ „ „ „ „ V,
etc.

The same rules apply to other groups.

In the multiple drawn in *fig. 13* the dotted connections again indicate the manner in which the multiple is allowed to grow from 20 to 40 trunks.

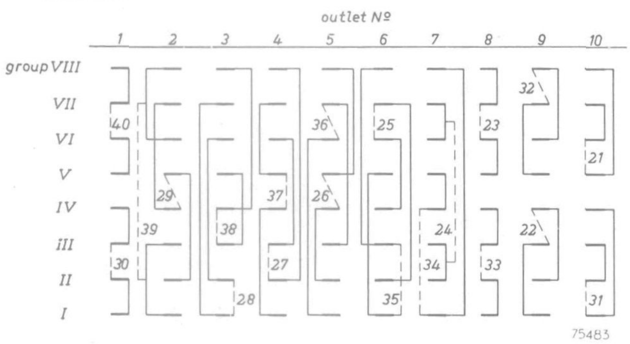

Fig. 13. Interconnected multiple for non-homing selectors for 20—40 trunks. The dotted connections have the same meaning as in fig. 12.

3. A multiple for 40—50 trunks, with 10 sub-groups and 10 outlets per sub-group. For 50 trunks the multiple assumes the following shape:

Group	Outlet No 1	2	3	4	5	6	7	8	9	10
I	1	2	3	4	5	6	7	8	9	10
II	1	11	12	13	14	15	16	17	18	19
III	20	2	12	21	22	23	24	25	26	27
IV	28	29	3	13	22	30	31	32	33	34
V	35	36	37	4	14	23	31	38	39	40
VI	41	42	43	44	5	15	24	32	39	40
VII	35	42	45	46	47	6	16	25	33	34
VIII	28	36	43	46	48	49	7	17	26	27
IX	20	29	37	44	47	49	50	8	18	19
X	41	11	45	21	48	30	50	38	9	10

In this multiple, group I contains one trunk accessible to group II, one accessible to group III, and so on, to one trunk accessible to group IX, and two trunks accessible to group X. This systematic procedure is also followed for the other groups, with the understanding that group II has two trunks in common with group IX, group III two trunks in common with group VIII, etc.

An outline of the multiple is given in *fig. 14*.

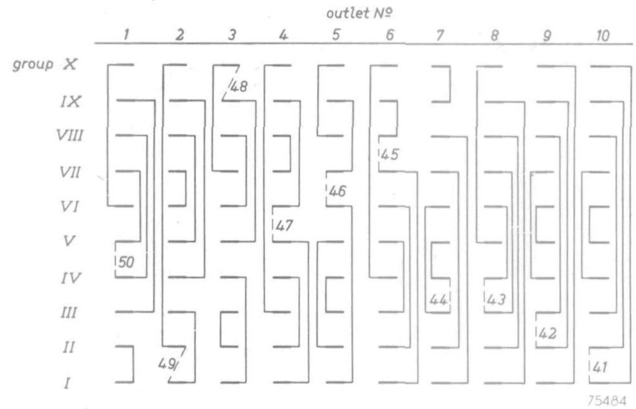

Fig. 14. Interconnected multiple for non-homing selectors for 40—50 trunks. The dotted connections have the same meaning as in fig. 12.

4. A multiple for 30 to 60 trunks, with 12 sub-groups and 10 outlets per sub-group. The form of the multiple is as follows for 60 trunks:

Group	Outlet No 1	2	3	4	5	6	7	8	9	10
I	1	2	3	4	5	6	7	8	9	10
II	1	11	12	13	14	15	16	17	18	19
III	20	2	12	21	22	23	24	25	26	27
IV	28	29	3	13	22	30	31	32	33	34
V	20	35	36	4	14	30	37	38	39	40
VI	28	11	36	21	5	41	42	43	44	45
VII	46	47	48	49	50	6	16	25	39	34
VIII	51	29	48	52	53	54	7	17	44	27
IX	46	35	55	56	57	54	24	8	18	45
X	51	58	59	49	57	15	31	43	9	40
XI	60	47	59	56	53	23	42	38	33	10
XII	60	58	55	52	50	41	37	32	36	19

5. A multiple for 40 to 80 trunks, with 16 sub-groups and 10 outlets per group. For the maximum size of this multiple the trunks of a certain group also appear in 10 other groups. The following Table shows the form of the multiple for 80 trunks.

Group	Outlet No 1	2	3	4	5	6	7	8	9	10
I	1	2	3	4	5	6	7	8	9	10
II	1	11	12	13	14	15	16	17	18	19
III	20	2	12	21	22	23	24	25	26	27
IV	28	29	3	13	22	30	31	32	33	34
V	35	36	37	4	14	23	31	38	39	40
VI	41	36	42	43	5	15	24	32	44	45
VII	46	29	37	47	48	6	16	25	49	50
VIII	51	11	42	21	52	30	7	53	54	55
IX	35	56	57	58	48	59	60	8	18	27
X	28	61	62	43	52	63	64	65	9	19
XI	20	66	57	47	67	68	69	70	39	10
XII	51	66	62	71	72	73	74	17	44	34
XIII	46	61	75	76	77	59	74	70	26	40
XIV	41	78	79	58	77	73	69	65	33	55
XV	80	56	79	76	72	68	64	38	54	50
XVI	80	78	75	71	67	63	60	53	49	45

6. A multiple for 20 to 40 trunks, with 4 sub-groups and 20 outlets per sub-group. This multiple can be obtained by doubling the scheme of category 1.
7. A multiple for 40 to 80 trunks, with 8 sub-groups and 20 outlets per sub-group. This multiple can be obtained by doubling the scheme of category 2.

2.4 **Measurements on some of the multiples of 2.3**

The grade of service of some of the multiples thus realized has been measured with the aid of the artificial-traffic method. The following multiples have been examined:

h. a 25-trunk multiple according to the scheme of category 2, with 17.23 erlangs.

i. a 33-trunk multiple according to the scheme of category 2, with 22.8 erlangs.

j. a 25-trunk multiple according to the scheme of category 6, with 19 erlangs.

The following results were obtained:

Measured	*Homing selectors BPO*	*Erlang's ideal grading*
h. 0.044 ± 0.0062	0.064	0.042
i. 0.055 ± 0.0081	0.061	0.036
j. 0.028 ± 0.006	0.042	0.04

From the many measurements taken, the conclusion may be drawn that the grade of service of multiples for non-homing selectors is not inferior to that given by the B.P.O. for homing selectors. There is, therefore, no objection to the application of the Tables drawn up by the B.P.O. for homing selectors, to multiples designed for non-homing selectors.

CHAPTER III

The influence of the common-control circuits on the grade of service

3.0 **General considerations**

Since an idle selector in a certain stage is only accessible if the associated common-control circuit is not occupied, the probability of loss of a selector stage will be higher than in those cases where each selector is individually controlled. It may thus occur in circuit arrangements employing common-control circuits per stage, that the wanted trunks are not all occupied, but that a connection may still not be established, because all the switching devices which are needed to gain access to a free trunk are occupied. The added probability of loss which originates in this manner is called *internal blocking*, whilst a circuit arrangement of this kind may be considered as a special case of a so-called *link* system. A few examples of known link systems are the connecting cords and the operator's positional circuit in a manual exchange, the line finder- and pre-selector arrangement of Siemens und Halske and the circuit arrangements of the Crossbar system. Although approximative solutions have been given in incidental cases, the first general investigation of link systems was carried out by Jacobeus [10]). Exact solutions are only obtainable for very simple cases which, as a rule, do not occur in practice. For more complicated arrangements exact solutions cannot be obtained because of the enormous amount of calculation required.

Jacobeus has earned great merit by deducing approximation formulae from which numerical results can be easily obtained. Although the approximation is not very close in many cases, the results yielded by the formulae appear to be in fairly good accordance with the results of measurements. The probability of loss of the selector stages in the *UR 49* system can in all cases be determined by means of Jacobeus' formulae; it is possible, however, in various cases to arrive at simple results by means of a better method of approximation. The following cases will be examined in this chapter:

1. The probability of loss of the final selector stage in a full-availability group and of the 1st group selector, when this is directly connected to a 1st line finder and when the common-control circuits are associated with several hundred-groups.
2. The probability of loss of the first group selector, when this is connected to a second line finder.
3. The probability of loss of a group-selector stage and of the final selector, both in an interconnected multiple.

To begin with, the method of Jacobeus will be briefly explained.

3.1 **Jacobeus' method**

Assume that m is a number of lines, to which access can only be obtained with the aid of m other devices, each of which has access to only one of the m lines. It is also assumed that the probability that p out of these m lines are occupied, is equal to $G(p)$. In this case, loss will be encountered if the $m - p$ devices, associated with those lines which are still free, are occupied. If the probability for the latter condition be called $H(m-p)$, the probability of loss becomes:

$$W = \sum_{p=0}^{m} H(m-p)\ G(p).$$

Since the two functions H and G are not entirely independent of each other — both being influenced by the same source of calls — the formula is not entirely exact.

For the H- and G-functions Jacobeus uses those of Bernoulli or of Erlang, according to whether the probability of a new call depends on the number of occupied lines or not. This is also inexact, since the circuit arrangements of the *link* system certainly influence the probability of a number of lines being busy. If $f(z)$ denotes the probability that z out of the m devices are busy, then an expression for $H(x)$ — i.e. the probability that x particular devices are busy — can easily be deduced if it is assumed that the busy devices are arbitrarily distributed over the total of m. It will be clear that

$$H(x) = \sum_{z=x}^{m} \frac{C_{z-x}^{m-x}}{C_z^m} f(z),$$

where $$C_p^q = \frac{q!}{p!\,(q-p)!}\,.$$

If the formulae of Erlang or Bernoulli are chosen for $f(z)$, one finds:

$$H(x) = \frac{E_m(a)}{E_{m-x}(a)} \quad \text{and} \quad H(x) = \frac{a^x}{m^x}\,,$$

respectively, where

$$E_k(a) = \frac{\dfrac{a^k}{k!}}{1 + a + \dfrac{a^2}{2!} + \ldots \dfrac{a^k}{k!}}.$$

It is rendered plausible by Jacobeus that the effect of the approximations which lie at the base of his formulae, is only fairly slight and that they yield a somewhat too high probability of loss.

3.2 The grade of service of the final selector

The grade of service (= probability of loss) of the final selector in a full-availability group and of the first group selector, when directly connected to a first line finder, will now be examined. Let the number of groups of final selectors or of group selectors served by the same common-control circuits, be k, whilst m denotes the number of trunks per group and consequently also the number of common-control circuits (*fig. 15*, see p. 25). If a call appears in one of the groups, this can only be successful if one or more of the m trunks are free, and also one or more of these free trunks can be connected to a free common-control circuit. The number of calls per unit of time, appearing in a group, is called n, the average holding time of a common-control circuit s, and the average duration of a call, including the holding time of the common control, h. Let the ratio $\dfrac{s}{h}$ be called β.

3.2.1 *Jacobeus' formula*

The traffic, offered to a group of m trunks, amounts to $nh = a$ erlangs; the total traffic, offered to the common-control circuits, to $kns = ka\beta = b$ erlangs. Blocking occurs if in a group, p out of m trunks are busy and if the remaining $m-p$ trunks are connected with busy common-control circuits. The probability that $m-p$ particular common-control circuits are busy, is equal to

$$\frac{E_m(b)}{E_p(b)}\,,$$

and the probability that p out of the m trunks in a group are busy, to:

$$\frac{\dfrac{a^p}{p!}}{1 + a + \dfrac{a^2}{2!} + \ldots + \dfrac{a^m}{m!}}\,.$$

The probability of loss amounts to:

$$q_1 = \sum_{p=0}^{m} \frac{\dfrac{a^p}{p!}}{1 + a + \dfrac{a^2}{2!} + \ldots + \dfrac{a^m}{m!}} \frac{E_m(b)}{E_p(b)}\,,$$

which, after rearrangement, reduces to:

$$q_1 = \frac{aE_m(a) - bE_m(b)}{a-b} \quad \text{if } a \neq b. \qquad (1)$$

In many cases $E_m(b) \ll E_m(a)$ and one finds:

$$q_1 = \frac{1}{1-\beta k} E_m(a).$$

3.2.2. *A new approximate formula*

In deriving his formula, Jacobeus has, among other things, not taken into account the possibility that a number of common-control circuits are already connected with trunks of the group under consideration. This becomes apparent if the value $k = 1$ is substituted in the expression for q_1. The result should then be: $q_1 = E_m(a)$. It proves to be possible to apply a correction to this expression, whilst continuing to describe the number of busy trunks per group and the number of busy common-control circuits by means of Erlang's formula.

If p trunks of a certain group are busy, x of these are connected with a common-control circuit, whilst, if r common-control circuits are busy in all, $r - x$ common-control circuits are connected with trunks in other groups. Blocking will now be encountered if the $m - p$ free trunks have to be served by $m - p$ of the $r - x$ common-control circuits, occupied in building up connections in other groups.

The $r - x$ common-control circuits can be distributed in C^{m-x}_{r-x} different ways over the $m - x$. Among these C^{m-x}_{r-x} combinations are a number which correspond exactly to the $m - p$ common-control circuits which are connected with the free trunks of the group, namely:

$$C^{m-x-(m-p)}_{r-x-(m-p)} = C^{p-x}_{r-x-m+p},$$

and the probability that exactly the $m - p$ free trunks are inaccessible, because they are connected with $m - p$ of the $r - x$ busy common-control circuits, therefore becomes:

$$\frac{C^{p-x}_{r-x-m+p}}{C^{m-x}_{r-x}}.$$

Let $W_p(x)$ be the probability that, if p trunks of a group are occupied, x of these are connected with a common-control circuit. Moreover, $\varphi_x(r)$ is defined as the probability that, when x common-control circuits are connected to a certain group, a total number of r are occupied. The probability of loss may now be written as:

$$q_2 = \sum_{p=0}^{m} \frac{\frac{a^p}{p!}}{1 + a + \ldots \frac{a^m}{m!}} \sum_{x=0}^{p} W_p(x) \times \sum_{r=x+m-p}^{m} \frac{C^{p-x}_{r-x-m+p}}{C^{m-x}_{r-x}} \varphi_x(r) \qquad (2)$$

An expression for $W_p(x)$ may easily be found, since the probability of a busy trunk being connected with a common-control circuit is equal to β, and consequently:

$$W_p(x) = C^p_x \beta^x (1 - \beta)^{p-x}.$$

The probability that r common-control circuits are occupied, being:

$$W(r) = \frac{\frac{b^r}{r!}}{1 + b + \ldots \frac{b^m}{m!}},$$

the probability that x out of these r are connected to a certain group, is:

$$C^r_x \left(\frac{1}{k}\right)^x \left(1 - \frac{1}{k}\right)^{r-x}.$$

Applying B a y e s' theorem, the expression for $\varphi_x(r)$ may be given as:

$$\varphi_x(r) = \frac{W(r)\, C_x \left(\frac{1}{k}\right)^x \left(1 - \frac{1}{k}\right)^{r-x}}{\sum_{r=x}^{m} W(r)\, C^r_x \left(\frac{1}{k}\right)^x \left(1 - \frac{1}{k}\right)^{r-x}}$$

If we put $b\left(1 - \frac{1}{k}\right) = \lambda$, the expression for $\varphi_x(r)$ may be rewritten as:

$$\varphi_x(r) = \frac{\frac{\lambda^{r-x}}{(r-x)!}}{1 + \lambda + \frac{\lambda^2}{2!} + \ldots \frac{\lambda^{m-x}}{(m-x)!}}.$$

Substituting these expressions in (2), and elaborating the last summation, leads to:

$$q_2 = \sum_{p=0}^{m} \frac{\frac{a^p}{p!}}{1 + a + \frac{a^2}{2!} + \ldots \frac{a^m}{m!}} \sum_{x=0}^{p} C^p_x \beta^x (1 - \beta)^{p-x} \times \frac{E_{m-x}(\lambda)}{E_{p-x}(\lambda)}.$$

When k becomes infinite, q_2 changes into Jacobeus' expression q_1, as only $\varphi_o(r)$ exists in that case, whilst for $k = 1$ it changes into $E_m(a)$.

The summations of q_2 may be interchanged:

$$q_2 = \frac{1}{1 + a + \ldots \frac{a^m}{m!}} \sum_{x=0}^{m} a^x \frac{\beta^x}{x!} E_{m-x}(\lambda) \times \sum_{p=x}^{m} \frac{[a(1-\beta)]^{p-x}}{(p-x)!} \cdot \frac{1}{E_{p-x}(\lambda)}. \qquad (3)$$

If the last summation be called S, it may easily be found by calculating $S - \frac{\alpha}{\lambda} S$ (if $\alpha \neq \lambda$), where $\alpha = a(1 - \beta)$. One finds:

$$S = \frac{\alpha^{m-x}}{(m-x)!} \frac{1}{\lambda - \alpha} \left[\frac{\lambda}{E_{m-x}(\alpha)} - \frac{\alpha}{E_{m-x}(\lambda)}\right].$$

Substitution in (3) leads to:

$$q_2 = \frac{E_m(a)}{\alpha - \lambda} \left[\alpha - \lambda \sum_{x=0}^{m} C^m_x \beta^x (1 - \beta)^{m-x} \frac{E_{m-x}(\lambda)}{E_{m-x}(\alpha)}\right].$$

The last summation cannot be realized and will be approximated. If we note, that:

$$\frac{E_{m-x}(\lambda)}{E_{m-x}(\alpha)} < \frac{(\lambda)^{m-x}}{(\alpha)^{m-x}} e^{\alpha-\lambda},$$

we shall find by substitution:

$$\lambda \sum_{x=0}^{m} C^m_x \beta^x (1 - \beta)^{m-x} \frac{E_{m-x}(\lambda)}{E_{m-x}(\alpha)} < \lambda (\beta k)^m e^{\alpha-\lambda}.$$

For usual values of k, β and a, the second member of the inequality is, however, negligibly small compared to α, so that q_2 may be written:

$$q_2 = \frac{a}{a - \lambda} E_m(a) = \frac{1 - \beta}{1 - \beta k} E_m(a).$$

For $a = \lambda$, the elaboration of equation (3) follows a somewhat different course.

The meaning of $a = \lambda$ is that

$$a(1 - \beta) = \beta a(k - 1),$$

or that $\beta k = 1$, and one finds for q_2:

$$q_2 = E_m(a) \Big[m(1 - \beta) + 1 - a +$$

$$+ a \sum_{x=0}^{m} C_x^m \beta^x (1 - \beta)^{m-x} E_{m-x}(a) \Big].$$

In practical cases the summation is $\ll 1$ and q_2 may then be written as:

$$q_2 = E_m(a) \Big[(m - a)(1 - \beta) + 1 \Big].$$

3.2.3 *Exact calculation for $m = 1$*

For $m = 1$ the probability of loss can be calculated by an exact method. For that purpose we assume $f(x)$ to be the probability that x out of the k trunks are occupied, whilst the (only) common-control circuit is free. We further assume that $g(x)$ be the probability that a conversation is in progress on x trunks, whilst the common-control circuit is also occupied. The call that is being handled by the common-control circuit is not yet considered to represent a conversation.

The holding time s of the common-control circuit, and the conversational time $h - s$ are assumed to have an *exponential* character. For the condition of statistical equilibrium the following equations may be established:

$$f(x) = g(x - 1) \frac{dt}{s} + f(x + 1)(x + 1) \frac{dt}{h - s}$$

$$+ f(x) \Big[1 - (k - x) \frac{a\,dt}{h} - \frac{x\,dt}{h - s} \Big],$$

or:

$$\Big[(k - x) a\beta + x \frac{\beta}{1 - \beta} \Big] f(x) - g(x - 1)$$

$$- (x + 1) \frac{\beta}{1 - \beta} f(x + 1) = 0 \qquad (4)$$

and

$$g(x) = f(x) \frac{(k - x)\,a\,dt}{h} + g(x + 1) \frac{(x + 1)\,dt}{h - s}$$

$$+ g(x) \Big[1 - \frac{x\,dt}{h - s} - \frac{dt}{s} \Big],$$

or

$$\Big[1 + \frac{x\beta}{1 - \beta} \Big] g(x) - f(x)(k - x) a\beta$$

$$- (x + 1) \frac{\beta}{1 - \beta} g(x + 1) = 0. \qquad (5)$$

For the boundary case $x = k$, the following equations obtain:

$$\frac{k\beta}{1 - \beta} f(k) = g(k - 1) \text{ and } g(k) = 0\,, \qquad (6)$$

$$\Big[1 + \frac{(k - 1)\beta}{1 - \beta} \Big] g(k - 1) = a\beta f(k - 1).$$

For the solution of this system of equations the method of the generating functions [6]) will be used. A brief exposition of this method may be given here. The functions $f(x)$ and $g(x)$ only have a meaning for entire, positive values of the argument, viz. $f(0)$, $f(1)$, $f(2)$ ...., respectively $g(0)$, $g(1)$, $g(2)$, etc.

One may now define the following power series in y:

$$F(y) = f(0) + f(1)\,y + f(2)\,y^2 + \dots .$$
$$G(y) = g(0) + g(1)\,y + g(2)\,y^2 + \dots ,$$

and it is said, that the functions $f(x)$ and $g(x)$ are *generated* by the series $F(y)$ and $G(y)$, which ,henceforth, are represented by the notations:

$$f(x) \triangleq F(y) \text{ and } g(x) \triangleq G(y), \text{ respectively.}$$

It will be evident that

$$a\,f(x) \triangleq a\,F(y) \text{ and}$$
$$f(x) + g(x) \triangleq F(y) + G(y).$$

Moreover:

$$F(0) = f(0) \text{ and } G(0) = g(0).$$

The following rules may now be deduced:

If $f(x) \triangleq F(y)$, then:

$$f(x + 1) \triangleq \frac{1}{y} [F(y) - F(0)],$$
$$f(x - 1) \triangleq y F(y),$$
$$x f(x) \triangleq y \frac{dF}{dy}\,.$$

Consequently:

$$(x + 1) f(x + 1) \triangleq \frac{dF}{dy}\,,$$

The rules given may now be used to determine the generating functions of the first members of equations (4) and (5). Since these first members are equal to zero, their generating functions must also be equal to zero. In this way it is found that:

$$(qy - 1) \frac{dF}{dy} + pF = ryG. \qquad (7)$$

$$(y - 1) \frac{dG}{dy} + uy \frac{dF}{dy} + rG - pF = 0, \qquad (8)$$

with

$$u = a(1 - \beta),\; q = 1 - u,\; p = uk,\; r = \frac{1 - \beta}{\beta}.$$

Addition of (7) and (8) produces the new equation:

$$\frac{dF}{dy} + \frac{dG}{dy} = rG. \tag{9}$$

Elimination of F from (7) and (9) yields an equation for G:

$$(qy - 1)\frac{d^2G}{dy^2} + \frac{dG}{dy}\left[yru + p + q + r\right]$$
$$- G\left[ru\,(k-1)\right] = 0.$$

By substituting $(qy - 1)\dfrac{ru}{q^2} = z$, this equation may be transformed into:

$$z\frac{d^2K}{dz^2} + \frac{dK}{dz}(z + \gamma) - \alpha K = 0, \tag{10}$$

wherein $\gamma = \dfrac{r}{q^2} + \dfrac{p+q}{q}$ and $\alpha = k - 1$.

The solution of equation (10) is the confluent hypergeometric series [11]):

$$K(z) = C\left[1 + \frac{\alpha}{1!\,\gamma}z + \frac{\alpha(\alpha-1)}{2!\,\gamma(\gamma+1)}z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!\,\gamma(\gamma+1)(\gamma+2)}z^3 + \ldots\right],$$

where C is a constant.

The second solution to (10) is not analytic for $z = 0$ and need not be considered.

One notes that, since $\alpha = k - 1$, the hypergeometric series changes into a polynomial with $k - 1$ as the highest power for z. This means that $g(k) = 0$ and that the boundary conditions (6) are satisfied.

For $G(y)$ is found:

$$G(y) = C\left[1 + \frac{\alpha}{1!\,\gamma}\frac{ru}{q^2}(qy-1) + \frac{\alpha(\alpha-1)}{2!\,\gamma(\gamma+1)}\left(\frac{ru}{q^2}\right)^2(qy-1)^2 + \ldots + \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+2)}{(k-1)!\,\gamma(\gamma+1)\ldots(\gamma+k-2)} \times \left(\frac{ru}{q^2}\right)^{k-1}(qy-1)^{k-1}\right] = C\left[1 + \psi(y)\right].$$

For the determination of $F(y)$ this result may be combined with relation (9), yielding:

$$\frac{dF}{dy} = C\left[r\left\{1 + \psi(y)\right\} - \frac{d\psi(y)}{dy}\right].$$

Since $\left(\dfrac{dF}{dy}\right)_{y=0} = f(1)$, one finds for $f(1)$:

$$f(1) = C\left[r\left\{1 + \psi(0)\right\} - \left(\frac{d\psi(y)}{dy}\right)_{y=0}\right].$$

Integration of $\dfrac{dF}{dy}$ yields the relation:

$$F(y) = C\left[r\int\left\{1 + \psi(y)\right\}dy - \psi(y) + C_1\right]$$

and for $f(0) = F(0)$:

$$f(0) = C\left[r\left[\int\left\{1 + \psi(y)\right\}dy\right]_{y=0} - \psi(0) + C_1\right].$$

The relation between $f(0)$ and $f(1)$ is, however, given by equation (4), viz.

$$f(1) = ka(1-\beta)f(0).$$

From this relation and the expressions which have been deduced for $f(0)$ and $f(1)$, the constant C_1 may be determined:

$$C_1 = \frac{1}{ka(1-\beta)}\left[r\left\{1 + \psi(0)\right\} - \left(\frac{d\psi(y)}{dy}\right)_{y=0}\right] + \psi(0) - r\left[\int\left\{1 + \psi(y)\right\}dy\right]_{y=0}.$$

The constant C may be determined from the condition $\sum_{o}^{k} f(x) + \sum_{o}^{k-1} g(x) = 1$, which corresponds to the relation: $F(1) + G(1) = 1$.

The following relation is thus found for C:

$$C\left[1 + r\int_0^1\left\{1 + \psi(y)\right\}dy + \psi(0) + \frac{1}{ka(1-\beta)}\left[r\left\{1+\psi(0)\right\} - \left(\frac{d\,\psi(y)}{dy}\right)_{y=0}\right]\right] = 1.$$

The traffic carried by the common-contro circuits, amounts to:

$$\sum_{x=0}^{k-1} g(x) = C\left[1 + \psi(1)\right].$$

The probability of loss is then equal to:

$$q_3 = \frac{a\beta k - C\left[1 + \psi(1)\right]}{a\beta k}.$$

Numerical results may be easily obtained in this way. For the simple case where $k = 2$, the general solution can be found without difficulty. In this case:

$$G(y) = 1 + \frac{ru}{\gamma q^2}(qy - 1)$$

and, therefore,

$$\psi(y) = \frac{ru}{\gamma q^2}(qy - 1).$$

The result is:

$$q_3 = 1 - \frac{r(r+1)(1+u)}{(1+u)(2u + r^2u + r^2 + 2ru + r) + ru}.$$

Substituting the values for u and r, and neglecting terms containing 2nd and higher powers of β, yields:

$$q_3 = E_1(a)\left[1 + \beta \frac{1}{(1+a)^2}\right].$$

3.2.4 *Comparison of results*

If, as has been done in 3.2.2, the possibility is taken into account that a number of common-control circuits is already connected with trunks of the existing group, the probability of loss found is a factor $1 - \beta$ smaller than that given by Jacobeus. For the case $m = 1$, $k = 2$, the method of paragraph 3.2.2 is entirely confirmed by the exact method of 3.2.3, since for these conditions one finds:

$$q_1 = E_1(a)(1 + 2\beta).$$
$$q_2 = E_1(a)(1 + \beta).$$
$$q_3 = E_1(a)\left(1 + \frac{\beta}{(1+a)^2}\right).$$

The value of q_3 is somewhat smaller still than q_2; since $a \ll 1$ for a normal probability of loss, it is evident that for this case, q_2 represents a considerably better approximation than q_1.

A second confirmation of the expression q_2 is provided by the results of measurements examined by Jacobeus [10]. These measurements were taken by the Swedish Telephone Administration in a telephone exchange at Stockholm, equipped according to the Ericsson system. In this system a number of groups of connecting circuits, equipped with 500-point line finders, is served by a common group of registers in the manner indicated by fig. 15. Although no busy tone is given when no connection can be obtained, and the subscriber may continue to wait until a line finder becomes free, the majority of subscribers, if dialling tone is not returned at once, make a new call some time later.

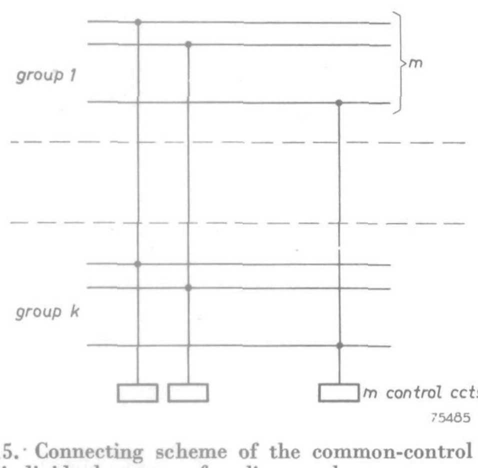

Fig. 15. Connecting scheme of the common-control circuits for k individual groups of m lines each.

Jacobeus finds that the values provided by his formula for q_1 must be multiplied by approximately 0.82 to be in accordance with the results of the measurements. This factor is partly due to the correlation between the traffic on the group of line finders under consideration and that on the registers. As for these measurements $\beta = 0.1$; the correction factor $1 - \beta$ of formula for q_2 becomes 0.9 which, to some extent, explains the factor 0.82. It now seems justified to accept the formula for q_2 as giving a good approximation of the problem considered. In *appendix II* have been included some results, calculated from this formula, which may be used in the calculation of the number of final selectors.

3.3 The grade of service of the first group selector, when 2nd line finders are employed

A common-control circuit again serves k selectors, whilst a total of m common-control circuits is available (*fig. 16*). Let the number of trunks in the group be mk and the traffic offered be a erlangs,

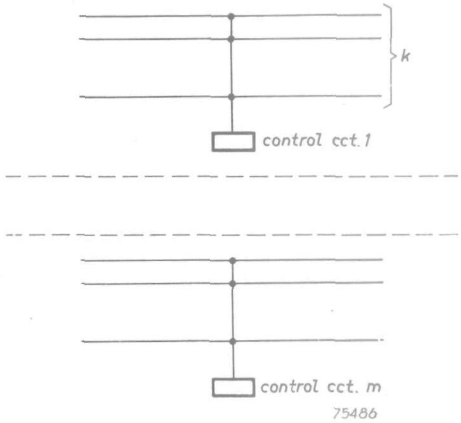

Fig. 16. Connecting scheme of the common-control circuits for one group of mk lines.

of which a proportion $a\beta$ is offered to the common-control circuits. It is evident, that for this problem a much greater correlation exists between the traffic on the common-control circuits and that on the trunk group, than for the case of 3.2. There, it may be recalled, the common-control circuits were only partly associated with the trunk group under consideration and were largely occupied by traffic in other groups. In this case, however, the traffic on the trunk group springs from the entire traffic on the common-control circuits.

For an originating call a choice can, as a rule, be made from several connecting possibilities, namely, from all free common-control circuits which still have a free trunk available in their part of the trunk group, and it is necessary to make an assumption as to how this choice is determined. The following reasonable assumptions can be made:

a. The call occupies that common-control circuit which has access to the greatest number of free trunks. This has the effect that the free trunks are spread as much as possible over the various sections of the group, reducing the internal blocking to its lowest proportion, since now, in the majority of cases, a free common-control circuit will have a free trunk associated with it.

b. The probability of being seized by a call is the same for all free common-control circuits having a free trunk available.

c. The probability that a free common-control circuit is seized, is proportional to the number of free trunks it has available.

The assumption sub *b* most nearly approaches the circuit arrangements of the *UR 49* system, in which a call, appearing in a certain hundred-group, tests all the common-control circuits in succession until a free one is found that also has a free trunk associated with it.

Since the calls of each hundred-group start testing for a free common-control circuit at a different point of the chain, one may expect a call, appearing in an arbitrary hundred-group, to seize an arbitrary common-control circuit.

On the whole, the three assumptions do not differ very much, however, since the probability of being occupied is greater for a common-control circuit associated with a sub-group having many busy trunks, than for one associated with a sub-group with only a few busy trunks. This will always produce the effect that the free trunks are spread more or less evenly over the sub-groups.

Besides by Jacobeus, much attention has also been given to this problem by Vaulot [12]) and Fortet [13]). Both these authors have given an exact solution of the problem for $m = 1$. Vaulot, for his solution, assumed "exponential" holding times both for the common-control circuits and for the trunks. Fortet, on the contrary, assumed constant holding times. Under Vaulot's assumption, the equations for the condition of statistical equilibrium can be easily established. If $f(x)$ be the probability that x out of the k trunks are occupied whilst the common-control circuit is free, and $g(x)$ the probability that the x trunks are occupied whilst the common-control circuit is equally busy, then these equations may be written as follows:

$$f(x)(u + x) = r\,g(x - 1) + (x + 1)\,f(x + 1),$$
$$g(x)(r + x) = (x + 1)\,g(x + 1) + u\,f(x).$$

In these equations:

$$u = a(1-\beta);\ r = \frac{1-\beta}{\beta} \text{ and } \sum_{o}^{k} f(x) + \sum_{o}^{k-1} g(x) = 1.$$

For the boundary case $x = k$, the equations are:

$$k\,f(k) = r\,g(k - 1);\ g(k) = 0;\ \text{and}$$
$$[r + (k + 1)]\ g(k - 1) = u\,f(k - 1).$$

For an arbitrary value of k, a solution can be found, but it is relatively complicated.

For $k = 2$, the probability of loss may be written:

$$q = 1 - \frac{1}{a\beta}\,[g(0) + g(1)]$$

which, after substitution of the values for $g(0)$ and $g(1)$, and after rearrangement, reduces to:

$$q = \frac{\frac{1}{2}a^2(1-\beta)(1+\beta^2) + a\beta}{1 + a + \frac{1}{2}a^2(1-\beta)(1+\beta^2)}.$$

For $m > 1$, an exact solution is difficult to obtain and an approximate formula must be established. First of all, the solutions of Jacobeus and Fortet will be examined.

3.3.1 *Jacobeus' formulae*

Jacobeus deduces 2 formulae, depending on the method employed to select a free common-control circuit. Under his first assumption, which corresponds to the one mentioned under *a* in paragraph 3.3, one trunk in each group remains free as long as possible and internal blocking only occurs if $m(k - 1)$ trunks are occupied. The trunk group is, therefore, inaccessible to an incoming call, if $m(k - 1)$ trunks or less are occupied whilst all common-control circuits are busy, or else, if $m(k - 1) + p$ trunks are occupied whilst the common-control circuits associated with the $m - p$ sub-groups which still have a free trunk available, are all busy.

In accordance with the exposition given in 3.1, Jacobeus puts the probability, that a certain $(m - p)$ common-control circuits are occupied, at:

$$\frac{E_m(\beta a)}{E_p(\beta a)}.$$

The probability of loss is established as:

$$q_1 = E_m(\beta a) \sum_{q=0}^{m(k-1)} \frac{\dfrac{a^q}{q!}}{1 + a + \dfrac{a^2}{2!} + \ldots + \dfrac{a^{mk}}{(mk)!}}$$
$$+ \sum_{p=1}^{m} \frac{\dfrac{a^{m(k-1)+p}}{[m(k-1)+p]!}}{1 + a + \dfrac{a^2}{2!} + \ldots + \dfrac{a^{mk}}{(mk)!}} \cdot \frac{E_m(\beta a)}{E_p(\beta a)}.$$

After some rearrangement and omitting some negligible terms, Jacobeus finds:

$$q_1 = \frac{E_{mk}(a)}{1 - k\beta} + E_m(a\beta).$$

The second formula is based on the assumption corresponding to c) in 3.3, i.e., that the busy trunks are distributed in an arbitrary manner over the groups, since the probability of being seized is the same for all free trunks of idle sub-groups. Loss is now encountered when p common-control circuits are occupied whilst all trunks of the $m - p$ sub-groups of which the common-control circuits are idle, are equally busy. One finds:

$$q_2 = \sum_{p=0}^{m} \frac{\dfrac{(a\beta)^p}{p!}}{1 + a\beta + \ldots \dfrac{(a\beta)^m}{m!}} \frac{E_{mk}(a)}{E_{pk}(a)}.$$

A good approximation of this formula cannot be given.

In deducing these formulae, no account has been taken of the fact that the traffic on the trunks has sprung from the traffic on the common-control circuits. This correlation has been taken into account by Fortet. Although Fortet gives several approximate formulae, only the most important one will be given here.

3.3.2 *Fortet's approximation method*

Although Fortet deduces his formulae for an arbitrary distribution function of the holding time of the common-control circuits, the *exponential* distribution will be used here for the sake of simplicity.

Let $p_x(\tau_1, \tau_2, \ldots \tau_x)\, d\tau_1\, d\tau_2 \ldots d\tau_x$ be the probability, that x particular common-control circuits are busy, of which the first one has been busy for a time τ_1, the second one for a time τ_2, etc. Let, moreover, $w_x(\tau_1 \ldots \tau_x)$ be the probability that, if x particular common-control circuits have been busy for the times τ_1, τ_2, etc., all trunks of the groups associated with the $m - x$ idle common-control circuits are occupied.

For the quantities p_x the following equations can be established:

$$p_x(\tau_1 + dt, \tau_2 + dt, \ldots \tau_x + dt)\, d\tau_1\, d\tau_2 \ldots d\tau_x$$
$$= \int_0^\infty (m-x)\, p_{x+1}(\tau_1, \tau_2, \ldots \tau_x, \tau) \frac{dt}{s}\, d\tau_1 d\tau_2 \ldots d\tau_x\, d\tau$$
$$+ [p_x(\tau_1, \tau_2, \ldots \tau_x)\, d\tau_1\, d\tau_2 \ldots d\tau_x]$$
$$\times \left[1 - \frac{a\, dt}{h}\left\{1 - w_x(\tau_1, \tau_2, \ldots \tau_x)\right\} - \frac{x\, dt}{s}\right].$$

Or otherwise:

$$s \sum_{\lambda=1}^{x} \frac{\partial p_x}{\partial \tau\lambda} + \left[a\beta\left\{1 - w_x(\tau_1, \tau_2, \ldots \tau_x)\right\} + x\right] p_x$$
$$= (m - x)\int_0^\infty p_{x+1}(\tau_1, \tau_2, \ldots \tau_x, \tau)\, d\tau. \qquad (11)$$

For $x = 0$ and $x = m$ the equations become:

$$a\beta(1 - w_0)\, p_0 = m \int_0^\infty p_1(\tau)\, d\tau \qquad \text{and}$$

$$s \sum_{\lambda=1}^{m} \frac{\partial p_m(\tau_1, \tau_2, \ldots \tau_m)}{\partial \tau\lambda} + m p_m(\tau_1, \tau_2, \ldots \tau_m) = 0 \quad (12)$$

The probability of x common-control circuits being occupied is:

$$f(x) = C_x^m \int_0^\infty \ldots \int_0^\infty p_x(\tau_1 \ldots \tau_x)\, d\tau_1 \ldots d\tau_x \quad (13)$$

whilst at the same time the relation

$$\sum_{x=0}^{m} f(x) = 1, \qquad \text{holds good.} \quad (14)$$

If now $w_x(\tau_1 \ldots \tau_x)$ is replaced by a constant value φ_x, the system of equations is satisfied by:

$$p_x(\tau_1, \ldots \tau_x) = \frac{1}{C_x^m} f(x) \frac{1}{s^x} e^{-\frac{\tau_1 + \ldots + \tau_x}{s}} \qquad (15)$$

where $f(x)$ must be solved from:

$$\left[a\beta(1 - \varphi_x) + x\right] f(x) = (x + 1) f(x + 1)$$

as will become apparent after substitution of (15) in (11). The two boundary conditions (12) are, moreover, satisfied by (15).

For $f(x)$ one finds at once the solution:

$$f(x) = \frac{\dfrac{(a\beta)^x}{x!} A_x}{\sum\limits_{n=0}^{m} \dfrac{(a\beta)^n}{n!} A_n},$$

where

$A_x = (1 - \varphi_0)(1 - \varphi_1) \ldots (1 - \varphi_{x-1})$ and $A_0 = 1$

For φ_x, Fortet takes the value, assumed by w_x when every single τ becomes equal to zero, which value is evidently also one of those that are independent of τ. In addition, $w_x(0, 0, \ldots 0)$ is calculated for the case where β approaches zero.

Let $q_x(\tau_1, \ldots \tau_x)\, d\tau_1 \ldots d\tau_x$ be the probability that x particular common-control circuits are busy and have been for times $\tau_1, \tau_2, \ldots \tau_x$, whilst at the same time all trunks in groups having idle associated common-control circuits, are busy. It follows from this definition that:

$$w_x(\tau_1 \ldots \tau_x) = \frac{q_x(\tau_1 \ldots \tau_x)}{p_x(\tau_1 \ldots \tau_x)}$$

and therefore:

$$\varphi_x = \lim_{s \to 0} \lim_{\tau \to 0} \frac{q_x(\tau_1 \ldots \tau_x)}{p_x(\tau_1 \ldots \tau_x)}.$$

Now the probability $q_x(0, 0, \ldots 0)$ holds for the situation q_x where all trunks of $m - x$ particular groups are occupied, the remaining x groups still having idle trunks, whilst at least x calls arrived during the interval dt. The probability of x calls arriving in an interval t is equal to:

$$\frac{1}{h^x} \frac{(at)^x}{x!} e^{-\frac{at}{h}},$$

and for an infinitesimal interval dt:

$$\frac{\left(\frac{a}{h}\right)^x}{x!} (dt)^x.$$

Therefore:

$$q_x(0, 0, \ldots 0) = \frac{\left(\frac{a}{h}\right)^x}{x!} (dt)^x q_x$$

(the probability of more than x calls arriving is less by an order of magnitude dt).

When s approaches 0, the probability of q_x can be calculated as if Erlang's formula were valid for the group. One finds:

$$\lim_{s \to 0} \lim_{\tau \to 0} p_x(\tau_1 \ldots \tau_x) = \frac{1}{C_x^m} \frac{\left(\frac{a}{h}\right)^x}{x!} B_x,$$

where B_x is the value assumed by A_x when the values of all τ's and of s approach zero. Following this line of reasoning, the value of φ_x may now be established as:

$$\varphi_x = \frac{C_x^m q_x}{B_x} = \frac{h_x}{B_x}.$$

One easily finds that:

$$B_x = \prod_{\alpha=0}^{x-1} \varphi_\alpha = 1 - \sum_{\alpha=0}^{x-1} h_\alpha.$$

The traffic carried by the common-control circuits amounts to $\sum_{x=1}^{m} x f(x)$, and the traffic that is offered amounts to $a\beta$. The probability of loss consequently becomes:

$$q_3 = 1 - \frac{1}{a\beta} \sum_{x=1}^{m} x f(x) = \frac{\sum_0^m \varphi_x B_x \frac{(a\beta)^x}{x!}}{\sum_0^m B_x \frac{(a\beta)^x}{x!}}$$

$$= \frac{\sum_0^m h_x \frac{(a\beta)^x}{x!}}{\sum_{x=0}^{m} \left(1 - \sum_{\alpha=0}^{x-1} h_\alpha\right) \frac{(a\beta)^x}{x!}}.$$

Assuming that the free trunks are spread in an arbitrary manner over the group which corresponds to c of para. 3.3, h_x can now be determined as follows:

The probability that $(m - x)$ particular groups are fully occupied is equal to:

$$\frac{E_{mk}(a)}{E_{xk}(a)}.$$

This probability may be expressed in terms of the quantities q_x as follows:

$$\frac{E_{mk}(a)}{E_{xk}(a)} = q_0 + q_1 C_1^x + q_2 C_2^x + \ldots + q_x,$$

which yields:

$$q_x = \frac{E_{mk}(a)}{E_{xk}(a)} - \sum_{\alpha=0}^{x-1} q_\alpha C_\alpha^x.$$

By introducing the quantity U_x defined by $q_x = U_x E_{mk}(a)$, h_x may be expressed as:

$$h_x = C_x^m U_x E_{mk}(a), \qquad \text{with}$$

$$U_x = \frac{1}{E_{xk}(a)} - \sum_{\alpha=0}^{x-1} U_\alpha C_\alpha^x \qquad \text{and } U_0 = 1.$$

One finds for the probability of loss:

$$q_3 = E_{mk}(a) \frac{\sum_{x=0}^{m} U_x C_x^m \frac{(a\beta)^x}{x!}}{1 + \sum_{x=1}^{m} \left[1 - \sum_{\alpha=0}^{x-1} U_\alpha C_\alpha^m E_{mk}(a)\right] \frac{(a\beta)^x}{x!}}.$$

3.3.3 *A new approximation method*

Jacobeus, in the approximation methods described in para. 3.3.1, did not take into account the correlation between the traffic on the trunks and that on the common-control circuits. It is possible, however, to introduce this correlation if one waives the assumption that the probability of $m - p$ common-control circuits being occupied, be represented by the expression $\frac{E_m(\beta a)}{E_p(\beta a)}$, but deduces this probability directly from the condition of the trunk group. It is assumed that the distribution of the busy trunks over the group is arbitrary and that the probability of z trunks being busy, be equal to:

$$f(z) = \frac{\dfrac{a^z}{z!}}{1 + a + \ldots + \dfrac{a^{mk}}{(mk)!}}.$$

Moreover, the assumption is made that, if x trunks of a particular sub-group are busy, the probability that the associated common-control circuit be occupied, is equal to $x\beta$. If one ignores the fact that the common-control circuits can only handle one call at a time, this probability may be stated as $1 - (1-\beta)^x \approx x\beta$. In reality this probability will be somewhat larger, because calls in one sub-group must be handled one after the other by the common-control circuit. For a constant conversational time, an impression may be gained of the fault made, by the following

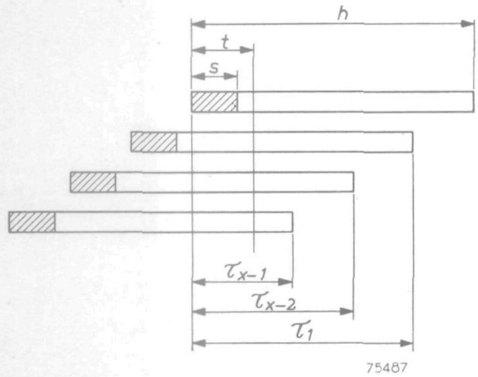

Fig. 17. Diagram of x simultaneous conversations on a group of trunks served by one common-control. The holding time of a call is h; that of a common-control is s.

reasoning: The x calls that are in progress simultaneously, are arranged in their order of arrival, and the moment of arrival of the last call is taken as the origin of time (*fig. 17*). The probability that at the moment of observation, a time t has elapsed since the beginning of the last call, whilst the other calls are still to continue for periods

$$\tau_1 - t,\ \tau_2 - t,\ \ldots\ \tau_{x-1} - t,$$

respectively, is equal to:

$$C\, dt\, d\tau_{x-1}\, d\tau_{x-2}\ \ldots\ d\tau_1.$$

The constant C can be determined from the fact that one of these conditions must be a *possible* condition and is given by:

$$C \int\limits_{(x-2)s}^{h-s} d\tau_1 \ldots \int\limits_{2s}^{\tau_{x-4}-s} d\tau_{x-3} \int\limits_{s}^{\tau_{x-3}-s} d\tau_{x-2} \int\limits_{0}^{\tau_{x-2}-s} d\tau_{x-1} \int\limits_{0}^{\tau_{x-1}} dt = 1,$$

or:

$$C\,\frac{1}{x!}\Big[h - (x-1)\,s\Big]^x = 1.$$

The probability that at the instant of observation no connection is being established by the common-control circuit, is equal to:

$$C \int\limits_{(x-1)s}^{h-s} d\tau_1 \ldots \int\limits_{2s}^{\tau_{x-3}-s} d\tau_{x-2} \int\limits_{s}^{\tau_{x-2}-s} d\tau_{x-1} \int\limits_{s}^{\tau_{x-1}} dt = C\,\frac{1}{x!}\,(h - xs)^x.$$

It may be inferred from the foregoing, that the probability that the common-control circuit be occupied, is:

$$1 - \left[\frac{h - xs}{h - (x-1)\,s}\right]^x \approx x\beta\,\Big\{1 + \tfrac{1}{2}\beta\,(x-1)\Big\}.$$

For the relevant values of x and β there is no objection against putting this probability equal to $x\beta$.

For the determination of the probability of loss, the method of Jacobeus can now again be used. If z trunks of the group are busy, the probability that α subgroups are completely occupied, whilst an arbitrary number of trunks in the remaining sub-groups are free, may be expressed as:

$$C_\alpha^m\,\frac{C^k_{x_{\alpha+1}}\,C^k_{x_{\alpha+2}}\ldots C^k_{x_m}}{C_z^{mk}},$$

where

$$z = \alpha k + x_{\alpha+1} + x_{\alpha+2} + \ldots + x_m\,.$$

Loss is now encountered if the common-control circuits, associated with the $m - \alpha$ incompletely occupied groups, are all busy, the probability of which is:

$$x_{\alpha+1}\,x_{\alpha+2}\ldots x_m\,\beta^{m-\alpha}\,.$$

The probability of loss, if z trunks are occupied in all, whilst α sub-groups are completely occupied, is then equal to:

$$f(z)\,\frac{C_\alpha^m\,C^k_{x_{\alpha+1}}\,C^k_{x_{\alpha+2}}\ldots C^k_{x_m}}{C_z^{mk}}\,x_{\alpha+1}\,x_{\alpha+2}\ldots x_m\,\beta^{m-\alpha}. \tag{16}$$

The total probability of loss is obtained by the summation of this expression over:

$$1 \leqslant x \leqslant k - 1;\ \alpha;\ z.$$

Expression (16) may be rewritten as:

$$f(z)\,\frac{C_\alpha^m}{C_z^{mk}}\,(k\beta)^{m-\alpha}\,C^{k-1}_{x_{\alpha+1}-1}\,C^{k-1}_{x_{\alpha+2}-1}\ldots C^{k-1}_{x_m-1}\,.$$

Summation over all values of x_α between 1 and $k - 1$ yields:

$$f(z)\,\frac{C_{\alpha}^{m}}{C_{z}^{mk}}\,(k\beta)^{m-\alpha}\Big[C_{z-\alpha\;k-(m-\alpha)}^{(m-\alpha)\,(k-1)} - C_{1}^{m-\alpha}\,C_{z-\alpha\;k-(m-\alpha)-(k-1)}^{(m-\alpha-1)\,(k-1)}$$

$$+ \ldots. + (-1)^{m-\alpha-1}\,C_{m-\alpha-1}^{m-\alpha}\,C_{z-\alpha\;k-(m-\alpha)-(m-\alpha-1)\,(k-1)}^{k-1}\Big].$$

Summation over α yields the result:

$$\frac{1}{C^{mk}}f(z)\Big[C_{km-z}^{m(k-1)}\,(k\beta)^{m} + C_{km-z}^{(m-1)\,(k-1)}\,C_{1}^{m}\,(k\beta)^{m-1}(1-k\beta)$$

$$+ C_{km-z}^{(m-2)\,(k-1)}\,C_{2}^{m}\,(k\beta)^{m-2}\,(1-k\beta)^{2} + \ldots.\Big].$$

The probability of loss is ultimately found by summation over z:

$$q_4 = E_{mk}\,(a)\sum_{p=0}^{m}\frac{1}{E_{m\,k-\{m+p\,(k-1)\}}\,(a)}\,C_{p}^{m}\,(k\beta)^{m-p}(1-k\beta)^{p}.$$

3.3.4 *Comparison of results*

First of all, the assumptions underlying the various formulae that have been deduced, will be stated once again. Jacobeus' formula for the probability of loss q_1 is based on the assumption, that the traffic on the trunks and on the common-control circuits can be described by means of the formula of Erlang, whilst no correlation is assumed to exist between the two classes of traffic. The system is so arranged, moreover, that the occupied trunks are spread evenly over all sub-groups. For small values of β the traffic on the trunks is likely to follow approximately Erlang's formula; the traffic on the common-control circuits will, however, satisfy this formula to a lesser degree, whilst the correlation between the two classes of traffic is very important. The distribution of the busy trunks over the sub-groups will tend to be more or less uniform, since the common-control circuits of heavily loaded sub-groups are more likely to be busy than those of the more lightly loaded sub-groups. This even distribution will to some extent be obtained for any arrangement of the system, although not to the same degree that has been assumed here.

The probability of loss, obtained under the assumption that no correlation exists between the traffic on the trunks and that on the common-control circuits, will be too high; the assumption of an even distribution of the busy trunks will, however, yield a too low value.

Jacobeus' second formula, for q_2, only differs from that for q_1 because the assumption is now made of an arbitrary distribution of the busy trunks over the entire group. One may now expect the value, yielded by q_2 for the probability of loss, to be too high.

Fortet's method proceeds from the exact equations for the system, but is ultimately led to approximate the quantity $w_x\,(\tau_1 \ldots. \tau_x)$, i.e., the probability that, when x common-control circuits are occupied, all trunks of the sub-groups associated with the $m - x$ idle common-control circuits, are busy. The approximation is found by calculating w_x for the case where all τ's approach zero and by equating β to zero at the same time. The value of w_x for $\tau = 0$ is higher than for any other value of τ, whilst, in addition, β is assumed to be zero and the distribution of the busy trunks over the group to be arbitrary. As a consequence, the probability of loss found will be too high.

The method followed for the determination of q_4 proceeds, just as Jacobeus' method, from the formula of Erlang for the group, but the correlation between the traffic on the trunks and that on the common-control circuits has been taken into account. The correlation as calculated is, however, somewhat too small, since it has been assumed that the probability for each trunk of a sub-group to be connected with the common-control circuit, is equal to β. The distribution of the busy trunks over the group has also been assumed to be arbitrary in this case. Although some factors are present that will cause the probability of loss found to be too low, this effect will, on the other hand, be compensated. For $k = 1$ this formula becomes exactly $E_m\,(a)$.

The following sets of values for the probability of loss have been calculated from the various formulae:

$m = 1;\ k = 2;\ a = 0.15;\ \beta = {}^{1}/_{20}$		$m = 2;\ k = 2;\ a = 0.5;\ \beta = {}^{1}/_{20}$	
Exact value:	0.0157		
q_1	0.0183	q_1	0.002059
q_2	0.017	q_2	0.00234
q_3	0.017	q_3	0.00278
q_4	0.0161	q_4	0.00234
$\beta = 0$	0.0097	$\beta = 0$	0.00158
$m = 2;\ k = 6;\ a = 6;\ \beta = {}^{1}/_{20}$		$m = 6;\ k = 6;\ a = 25;\ \beta = {}^{1}/_{20}$	
q_1	0.0497	q_1	0.01297
q_2	0.0515	q_2	0.0155
q_3	0.0535	q_3	0.0252
q_4	0.0424	q_4	0.0138
$\beta = 0$	0.01136	$\beta = 0$	0.008022

$m = 6;\ k = 10;\ a = 40;\ \beta = {}^{1}/_{40}$	
q_1	0.001416
q_2	0.00166
q_3	0.00268
q_4	0.00124
$\beta = 0$	0.000179

These results, as obtained from the various formulae, differ relatively little. In the majority of cases Fortet's formula yields the highest values. It seems entirely justified, therefore, to use, for practical applications, the formula which can be most easily calculated, and, therefore, to describe the problem as stated by means of Jacobeus' first formula:

$$q_1 = \frac{E_{mk}(a)}{1 - k\beta} + E_m(a\beta).$$

In *appendix III* some Tables are given where a is a variable and where q_1, m, k, β, are given quantities. The formula yields no results if one subgroup contains less than k trunks. This may happen, however, if the traffic is too high to be carried by mk trunks, but too low to justify $(m + 1) k$ trunks. In such a case one will equip one sub-group with less than k trunks. The value of a, belonging, for a determined probability of loss, to this number of trunks, can be found by interpolation from the values for the completely equipped sub-groups.

3.3.5 *The number of trunks, served by one common-control circuit*

If the formula for q_1 is considered, it is evident that for *small* values of m, (for example 1 or 2), and for higher values of k (for example 6), the probability of loss is for the largest part due to the term $E_m(a\beta)$. For low traffic values and small trunk groups, fewer trunks may, therefore, be served by one common-control circuit than for larger groups. For *larger* trunk groups, m is of course larger and the efficiency of the common-control circuits increases rapidly. For larger groups, the number of trunks per common-control circuit can, therefore, be more freely chosen and the probability of loss is no longer determined exclusively by the term $E_m(a\beta)$.

When a certain traffic must be carried with a given probability of loss, the number of trunks of the group, for a large number of common-control circuits, will approach that needed for $\beta = 0$. If, however, the number of common-control circuits is reduced, the number of trunks increases. The value of k to be chosen is, therefore, to a large extent determined by the cost of the trunk circuits and of the common-control circuits. For groups of costly long-distance trunks, for instance, it is economical to choose a low value of k. However, k is also determined by the design of the equipment racks. In the *UR 49* system, the values $k = 6$ or $k = 5$ are chosen for the combination 2nd line finder — cord circuit — 1st group selector. For the most usual traffic values a minimum of cost is hereby obtained, while the rack design remains, moreover, attractive.

3.4 **The connecting diagram for final selector common-control circuits in small central offices**

The circuit arrangements for the positioning of final selectors mentioned in 3.2, are too costly for small central offices, since too many common-control circuits are employed. The number of hundred-groups is limited in small offices, and the traffic on the common-control circuits is, therefore, small. It thus becomes possible to connect no longer *one*, but *two* final selectors per hundred-group to one common-control circuit (*fig. 18*).

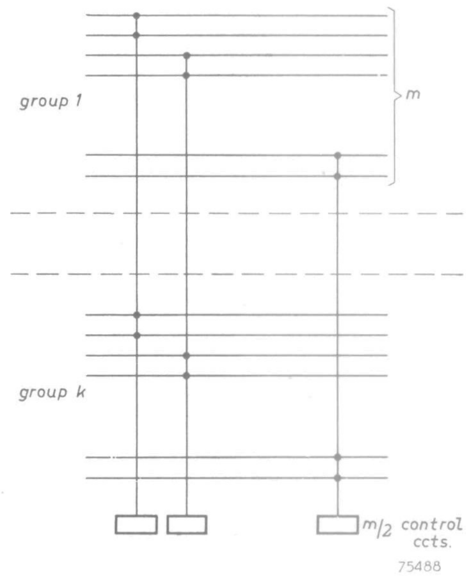

Fig. 18. The connecting diagram for final selector common-control circuits in small central offices.

If $k = 1$, the formula found in para. 3.3 can be directly applied and the probability of loss is given by:

$$q = \frac{E_m(a)}{1 - 2\beta} + E_{m/2}(a\beta).$$

If $k > 1$, a different solution must be found. Now it has become apparent in paragraphs 3.2 and 3.3, that Jacobeus' methods, whereby no correlation is assumed to exist between the traffic on the trunks and that on the common-control circuits, nevertheless yield a very good approximation of the problem. For the present, more complicated, case which is a combination of the problems treated in 3.2 and 3.3, a solution will, therefore, be given with the aid of Jacobeus' method.

Let the traffic on one group of m trunks be a erlangs, the traffic on the common-control circuits

$b = \beta ak$ erlangs. The trunks of one group are divided into $m/2$ sub-groups. Loss will now be encountered if both trunks of $m/2 - x$ sub-groups are occupied, while the common-control circuits associated with the remaining x sub-groups are occupied, although these sub-groups each contain at least one idle trunk. It is now assumed that the occupied trunks are distributed in an arbitrary manner over the group of m trunks. This is not entirely true, since the tendency is again present for the busy trunks to be spread evenly over the $m/2$ sub-groups. The probability of loss, obtained under the assumption of an arbitrary distribution, will, however, be slightly too high and the approximation is, therefore, on the safe side. Under this assumption, the probability that $m/2 - x$ sub-groups are completely occupied whilst the remaining sub-groups each contain at least one idle trunk, has already been deduced in para. 3.3.2 and amounts to:

$$h_x = C_x^{m/2}\, U_x E_m(a)$$

where

$$U_x = \frac{1}{E_{2x}(a)} - \sum_{\alpha=0}^{x-1} U_\alpha C^x, \quad \text{and } U_0 = 1.$$

The probability that x particular common-control circuits are occupied, is equal to:

$$\frac{E_{m/2}(b)}{E_{m/2-x}(b)}$$

and the probability of loss is found as:

$$q = \sum_{x=0}^{m/2} h_x \frac{E_{m/2}(b)}{E_{m/2-x}(b)}\,.$$

For various values of q, k, β and m, the corresponding values of a have been calculated. A survey is given in *appendix IV*.

3.5 The influence of the common-control circuits with interconnected multiples

In the problems that have been treated until now, the influence of the common-control circuits is usually determined by means of the known probability that the trunks of a group are busy, in the case where no common-control circuits are present. For interconnected multiples, on the other hand, this probability is only approximately known and the problem thereby assumes a much more complicated character. Since the conclusion has been drawn in para. 2.4 that interconnected multiples, conceived for non-homing selectors, have no worse efficiency than those for homing selectors, O'Dell's reasoning whereby the formulae for interconnected multiples on homing selectors are obtained, will be followed for the determination of the influence of the common-control circuits. The use of non-homing selectors, moreover, provides the advantage that the traffic load on all trunks of the group is now approximately the same, and it is, therefore, assumed that the probability of being occupied is the same for all combinations of z trunks of the group. The effect of the common-control circuits is considered in the following cases:

a. The common-control circuits serve exclusively for the positioning of the selectors included in the multiple under consideration. This applies, as will be evident from Chapter I, for example, to 2nd group selectors where racks are employed mounting 50 selectors and five common-control circuits, each common-control circuit serving a maximum of 10 selectors.

b. The common-control circuits also serve for the positioning of selectors forming part of other multiples. This occurs where final selectors form part of an interconnected multiple.

3.5.1 *O'Dell's method for the determination of the grade of service of graded multiples for homing selectors*

O'Dell proceeds from Erlang's formula for the ideally arranged multiple (see Chapter IV). If in this formula a and N (the number of trunks) are large compared to m (the number of available outlets in the selector bank), the probability of loss may be given approximately by:

$$q = \frac{(a_\infty)^m}{(N_\infty)^m}\,.$$

If $N \leqslant m$, no interconnecting is applied and the relation between probability of loss, m and a can be exactly calculated by means of Erlang's formula for a full-availability group. Let, for a given probability of loss, a_m be the traffic offered to m trunks in a full-availability group, then, for $N \geqslant m$ and again for a given probability of loss, the following formal relation may be written:

$$a - a_m = \lambda (N - m),$$

wherein λ is dependent on a, m, N and the probability of loss. The trunk efficiency a/N can be written as:

$$\frac{a}{N} = \frac{a_m}{m} + \left(1 - \frac{m}{N}\right)\left(\lambda - \frac{a_m}{m}\right).$$

It now appears from the formulae for the full-availability group, and from measurements, that the trunk efficiency increases with the number of trunks. For $N = m$ the trunk efficiency is equal to

$\frac{a_m}{m}$, and for $N > m$ the trunk efficiency will certainly not be inferior to this value. A value $\frac{a_m}{m}$ may therefore be chosen for λ wherewith the trunk efficiency will then always remain equal to $\frac{a_m}{m}$ for $N > m$. On the other hand, for large values of N in Erlang's ideally arranged multiple, the trunk efficiency will approach $\frac{a_\infty}{N_\infty} = (q)^{1/m}$. For a large value of N the above formula becomes $\frac{a}{N} = \lambda$, so that for large values of N, λ may be chosen as $\lambda_1 = \frac{a_\infty}{N_\infty} = q^{1/m}$. The real value of λ will thus lie between the two values $\frac{a_m}{m}$ and $q^{1/m}$.

O'Dell now considers two kinds of traffic, namely, smooth traffic and pure-chance traffic. The first kind of traffic is found in selector stages such as the 1st group selector, to which the 2nd group selectors are connected in an interconnected multiple. In the 1st group-selector stage the traffic from many groups of subscribers flows together, and as a result of the non-coincidence of the busy hours of these groups, the traffic in this stage loses its pronounced pure-chance character. For this reason O'Dell calls it "smooth" traffic. In accordance with the measurements made, for smooth traffic the value $\lambda_1 = \frac{a_\infty}{N_\infty} = q^{1/m}$ is taken for λ. For selector stages where the traffic cannot be called smooth but must, on the contrary, be described as pure-chance traffic, the value chosen for λ will be, in accordance with the measurements:

$$\lambda_2 = 0.53\, q^{1/m} + 0.47\, \frac{a_m}{m} \, .$$

For the interconnected multiples we have considered, which were conceived for selectors being positioned by means of common-control circuits, an analogous treatment is possible, the value of λ only being smaller as a result of the effect of the common-control circuits.

For the selector stages, mentioned under a in para. 3.5, the value of β is approximately $^1/_{40}$ to $^1/_{60}$. The traffic to be carried by the common-control circuits is then certainly small, but for small groups of trunks, if 10 selectors are controlled by one common-control circuit, the efficiency of the latter is so low, that a too high probability of loss would be obtained. As a result it becomes necessary for small trunk groups to put less than 10 selectors under the control of one common-control circuit. For 20 selectors for instance, the traffic on the common-control circuits amounts to 0.2 erlang. With 3 common-control circuits the traffic is already sufficiently large to create a probability of loss of 0.001, due to the presence of the control circuits alone.

The *UR 49* system can, therefore, be so arranged that for a group of up to 50 selectors, 5 common-control circuits are always provided. In the case where only 20 selectors are used, only 4 selectors will be served by one common-control circuit.

3.5.2 *The grade of service of groups with a maximum of 50 trunks* *)

For the calculation of $\lambda_1 = \frac{a_\infty}{N_\infty}$ one can now proceed as follows: Let v be the number of common-control circuits; $m = pv$ the total number of outlets appearing in the interconnected multiple; $N = vk$ the number of trunks of the group, where k is the number of selectors served by one common-control circuit. The traffic on the v common-control circuits is equal to βa, if a be the traffic that can be carried by N trunks for a given grade of service. The traffic on the common-control circuits and that on the trunks are assumed to be independent of each other, so that Jacobeus' method can be followed. For the design of the interconnecting scheme, the procedure described in Chapter II is used.

The selectors to whose outlets the multiple is connected, are divided into a number of groups, and a call appearing in one of these groups has a choice of m trunks. The common-control circuits are now connected to the trunks in such a manner that a call in each group, with the choice of m trunks, also has the choice of all the v common-control circuits. For groups of 50 selectors (where $v = 5$), the interconnected multiples and the connections to the common-control circuits take a form corresponding to the categories 3 and 7 of para. 2.3. These multiples have been drawn in *figs. 19* and *20* for $m = 10$ and $m = 20$ resp. The encircled numerals indicate the common-control circuit to which the outgoing trunks are connected. A call, appearing in a certain sub-group of the multiple, will now be lost if z out of the v common-control circuits are occupied (thereby disabling pz trunks) and if the remaining $m - pz$ trunks, otherwise accessible to the sub-group, are all busy. For the determination of λ_1, the number of trunks in the multiple is assumed to be very large, so that

*) Jacobeus appears to be following an analogous line of reasoning in designing multiples for the crossbar system.

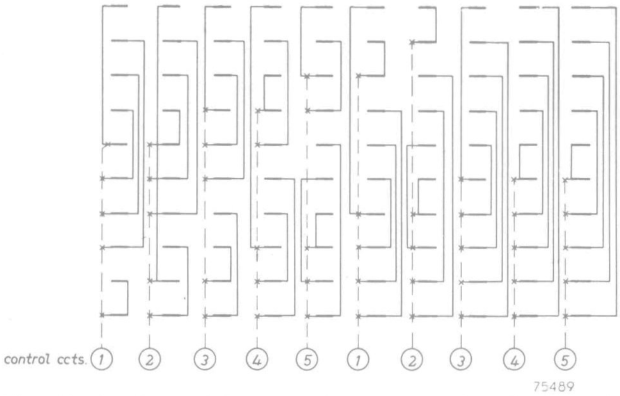

Fig. 19. 10 Group 10-contact interconnected multiple with 50 trunks and 5 common-control circuits. The encircled numerals represent the common-control circuit to which the outgoing trunks are connected.

the probability of $m - pz$ trunks of the group being busy may be expressed as:

$$\left(\frac{a_\infty}{N_\infty}\right)^{m-pz} = (\lambda_1)^{p(v-z)}.$$

For a given traffic of $\beta a = b$ erlangs on the common-control circuits, and assuming that Erlang's formula holds for the probability of the

one common circuit only, to which the formulae of para. 3.3 may be applied, viz:

$$q = \frac{E_m\,(a_m)}{1 - p\beta} + E_\nu\,(\beta a_m).$$

The curves for smooth traffic can now be determined, according to O'Dell, from:

$$a - a_m = \lambda_1\,(N - m);\quad q = \frac{E_\nu\,(\beta a)}{E_\nu\left(\dfrac{\beta a}{\lambda_1{}^p}\right)}\;;$$

$$q = \frac{E_m\,(a_m)}{(1 - p\beta)} + E_v\,(\beta a_m). \tag{17}$$

In the case of pure-chance traffic the first expression must be replaced by:

$$a - a_m = \lambda_2\,(N - m),\ \text{with}\ \lambda_2 = 0.53\,\lambda_1 + 0.47\frac{a_m}{m}\,.$$

3.5.3 *Interconnected multiples for more than 50 trunks*

When more than 50 selectors are needed in an interconnected multiple, a common-control circuit is provided for every 10 selectors. The influence of

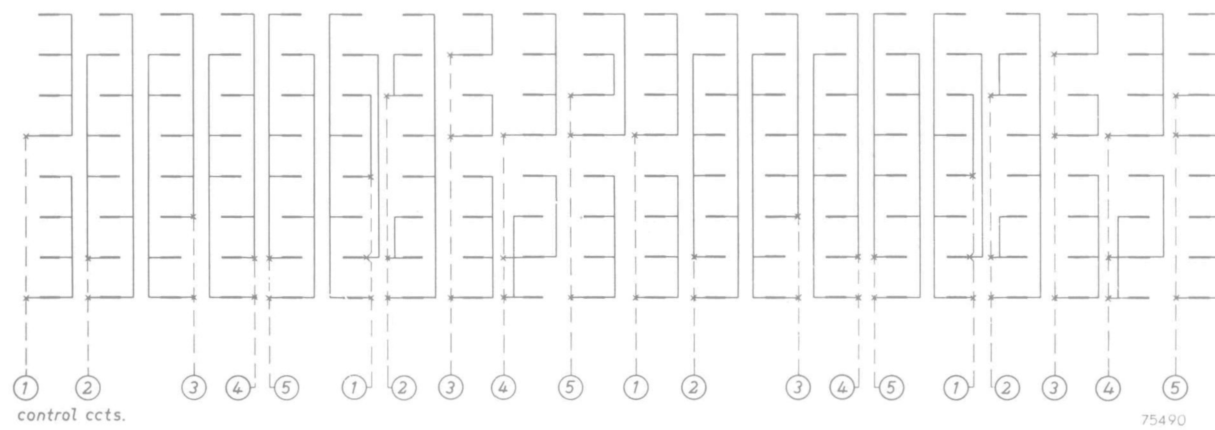

Fig. 20. 8 Group 20-contact interconnected multiple with 50 trunks and 5 common-control circuits.

common-control circuits being busy, the probability of loss is:

$$q = \sum_{z=0}^{v} (\lambda_1)^{p(v-z)} \frac{\dfrac{b^z}{z!}}{1 + b + \dfrac{b^2}{2!} + \ldots + \dfrac{b_v}{v!}}\,.$$

After rearrangement, this reduces to:

$$\frac{E_\nu\,(b)}{E_\nu\left(\dfrac{b}{\lambda_1{}^p}\right)}\,.$$

Subsequently, the value of a_m still has to be determined. There is no longer an interconnecting scheme in this case; p selectors are controlled by

the common-control circuits on the grade of service cannot now be calculated so easily, since in most cases the number of outlets of a sub-group is no longer a whole multiple of the number of common-control circuits. If, for instance, 70 selectors are used with 7 common-control circuits, and $m = 10$, then three common-control circuits will appear twice in one sub-group of the multiple and four of them only once. If the common-control circuits are connected to the multiple in such a manner that the probability that z of the k common-control circuits are busy, is equal for all combinations C_z^k, then the method followed in 3.5.2 can indeed be applied, but the expressions obtained will render the numerical determination of λ_1 fairly difficult.

A similar difficulty is encountered when, for instance, 130 selectors are used with 13 common-control circuits, and $m = 10$. Only 10 of the 13 common-control circuits will now appear in one sub-group of the multiple. For more than 50 trunks, however, the number of common-control circuits is already so large that the internal blocking becomes very small and the relation between a and N approaches that for a group without common-control circuits. It will, consequently, suffice to calculate a few points, whereupon the trend of the curve can be easily estimated for intermediate values. With the aid of 3.5.2 we have carried out these calculations for a few points of the curve, limiting ourselves to the more common values of m, viz. $m = 10$ and $m = 20$. Three cases have been examined, to each of which the system of equations (17) applies:

a. $m = 10$; $N = 100$; $v = 10$; $p = 1$.
b. $m = 20$; $N = 100$; $v = 10$; $p = 2$.
c. $m = 20$; $N = 200$; $v = 20$; $p = 1$.

For the case $m = 10$, $N = 200$, $v = 20$, the line of reasoning followed is somewhat different from that used in the foregoing ones. It may be expected that the way in which the common-control circuits are occupied, will now also be in accordance with the laws of the interconnected multiple. The traffic offered to the common-control circuits is equal to βa erlangs and the probability of z common-control circuits being busy in one sub-group of the multiple is approximately:

$$C_z^m \left(\frac{\beta a}{v}\right)^z \left(1 - \frac{\beta a}{v}\right)^{m-z}.$$

The equation for λ_1, therefore, now reads:

$$q_1 = \sum_{z=0}^{m} (\lambda_1)^{m-z}\, C_z^m \left(\frac{\beta a}{v}\right)^z \left(1 - \frac{\beta a}{v}\right)^{m-z},$$

or:

$$q_1 = \left[\frac{\beta a}{v} + \lambda_1 \left(1 - \frac{\beta a}{v}\right)\right]^m.$$

λ_1 can now be determined from this equation in conjunction with $a - a_m = \lambda_1 (N - m)$, whilst a_m follows from the equation $q = E_m(a_m)$ since, if the number of common-control circuits is kept constant, for $N = 10$, each trunk is provided with a common-control circuit. The occupancy of the common-control circuits has been assumed to be in accordance with Bernoulli's theorem. Under this assumption the probability of loss found will be too low.

Instead of Bernoulli's expression for the probability of z common-control circuits being busy, a different approximation, yielding a somewhat too high probability, may be used. This is done by assuming Erlang's formula to be valid for the common-control circuits associated with a sub-group of the multiple, the traffic offered to these circuits being $\dfrac{\beta a}{v} m = \dfrac{bm}{v}$.

The case of the common-control circuits may be considered as one where a traffic quantity b is offered to v trunks in an interconnected multiple. According to O'Dell's method, the relation between q and b is given by:

$$b - b_m = \lambda (v - m); \qquad q = E_m (b_m),$$

where
$$\lambda > \frac{b_m}{m}\ .$$

The first expression may be rearranged to yield:

$$\frac{mb}{v} = \frac{mb_m}{v} + \frac{m\lambda}{v}(v - m),$$

from which follows:

$$\frac{mb}{v} > b_m\ .$$

If, therefore, the probability of loss for the common-control circuits is assumed as $E_m\left(\dfrac{mb}{v}\right)$, then this value is higher than $E_m(b_m)$.

If Erlang's distribution is now assumed to apply to the occupancy of the common-control circuits, the probability of loss of the interconnected multiple is found as:

$$q_2 = \sum_{z=0}^{m} (\lambda_1)^{m-z} \frac{\dfrac{\left(\frac{mb}{v}\right)^z}{z!}}{1 + \dfrac{mb}{v} + \dots + \dfrac{\left(\frac{mb}{v}\right)^m}{m!}} = \frac{E_m\left(\dfrac{mb}{v}\right)}{E_m\left(\dfrac{mb}{v\lambda_1}\right)}$$

The difference between the two formulae is small, however, as will appear from the following values:

$\beta = 1/40$; $m = 10$; $N = 200$; $v = 20$;
$q_1 = q_2 = 0.01$; $a_1 = 112.83$ erlangs;
$a_2 = 112.2$ erlangs.

The Tables of section V of the *Appendix*, for the traffic-carrying capacities of the various multiples, have been composed by means of the expressions found in 3.5.2 and 3.5.3.

3.5.4 *The grade of service of final selectors in an interconnected multiple*

A multiple of category 1 of para. 2.3 will serve in this case (*fig. 21*). Since a common-control circuit is connected to every trunk, one can, when considering the traffic offered to the common-control circuits, also regard these as forming part of an interconnected multiple. The number of final selectors per hundredgroup is assumed to be N, which final selectors are accessible from the penultimate stage via an interconnected multiple

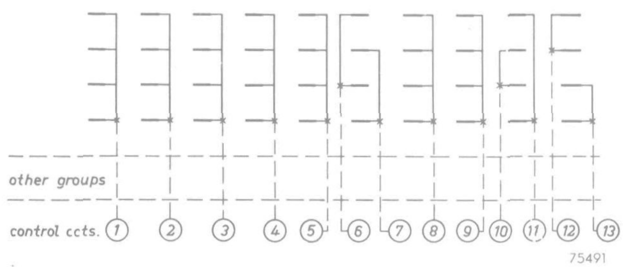

Fig. 21. 4 Group 10-contact interconnected multiple with 13 final selectors. The common-control circuits are connected to more than one multiple.

with m outlets per sub-group. The number of common-control circuits is also equal to N, whilst k final selector groups are served by the common-control circuits. The value a_m can now be determined by means of the formula of para. 3.2.2:

$$q = \frac{1-\beta}{1-\beta k} E_m(a_m).$$

Let the relation between a and N, if a be the traffic offered to a group of N final selectors, again be given by:

$$a - a_m = \lambda_2 (N - m),$$

where $$\lambda_2 = 0.53\lambda_1 + 0.47 \frac{a_m}{m}.$$

The traffic offered to the common-control circuits is $ka\beta$ erlangs and the efficiency of one of these circuits is $\frac{ka\beta}{N}$. The probability that z of the m common-control circuits in one sub-group of the multiple are occupied, may be fairly closely approached by:

$$C_z^m \left(\frac{ka\beta}{N}\right)^z \left(1 - \frac{ka\beta}{N}\right)^{m-z}.$$

By analogy with the procedure described in paragraphs 3.5.3 and 3.5.2, λ_1 may now be solved from:

$$q = \left[\frac{\beta ak}{N} + \lambda_1 \left(1 - \frac{\beta ak}{N}\right)\right]^m.$$

As in the previous paragraph, the probability of loss obtained is slightly too high, when the probability that z common-control circuits in a sub-group of the multiple are busy, is assumed as:

$$\frac{\dfrac{\left(\dfrac{\beta akm}{N}\right)^z}{z!}}{1 + \dfrac{\beta akm}{N} + \dots \dfrac{\left(\dfrac{\beta akm}{N}\right)^m}{m!}}.$$

λ_1 must then be determined from:

$$q = \frac{E_m\left(\dfrac{\beta akm}{N}\right)}{E_m\left(\dfrac{\beta akm}{N\lambda_1}\right)}.$$

The difference between the two formulae is, however, insignificant. The results of the first formula are given in section VI of the *Appendix*.

CHAPTER IV

The average travel per call of non-homing selectors

4.0 General considerations

It has already been explained in Chapter I that the high velocity of the selector renders the home position superfluous. As a result, the average number of contacts over which the selector will have to travel for each call will be very small. It needs no further explanation that this number will be 49.5 for the line finder and for the final selector, but for the various group-selector stages a thorough analysis is necessary. Traffic flows from the group selectors in various directions, either in full-availability trunk groups or to a subsequent selector stage via an interconnected multiple. In a 10,000 line exchange, for instance, the 2nd group selectors are connected to the 1st group-selector outlets in an interconnected multiple, whilst the final selectors are either connected to the 2nd group selectors in a straight (full availability) multiple or else in an interconnected multiple. When all groups of subscribers have the same traffic intensity, the outgoing trunk groups from the 1st and 2nd group selectors to the various 1000- and 100-groups will have to carry the same amount of traffic and will all be made up to the same pattern. In many cases this high degree of symmetry will not be found, however.

In this chapter, however, these symmetrical cases only will be considered, as it is but our intention to demonstrate the difference in the average travel per call of homing and non-homing selectors. If the selector has a total of c outlets, with n directions connected, each with m outlets available, then $c = nm$.

Homing selectors must evidently always travel the entire bank of c contacts for each call.

With non-homing selectors two cases must still be considered:

1. Starting from a certain contact, one encounters successively:
m contacts destined for direction 1,
m contacts destined for direction 2,
and so on to
m contacts destined for direction n.

It is obvious, that in $n - 1$ out of n cases the selector will start from a contact outside the selected group. It will, therefore, have to pass a large number of contacts before the wanted group is reached. Once this group is reached, however, only a small number of contacts need be searched to find a free trunk within it.

2. Starting from a certain contact, one finds successively:
1 contact of direction 1,
1 contact of direction 2,
and so on to
1 contact of direction n and subsequently again:
1 contact of direction 1,
1 contact of direction 2, etc.

The contacts associated with a certain direction are now spread uniformly over the selector bank. In any position the selector will now always be near a trunk of the wanted direction. If this trunk happens to be busy, $n - 1$ contacts must be skipped before another trunk of the wanted direction is found. For the calculation of the various cases it is necessary, in order to simplify the problem, to make a few assumptions. An exact treatment is almost out of the question and would not yield any new points of view, since the results are hardly influenced by the simplifications. Our object is, moreover, only a comparison between the various possibilities. The assumptions are:

a. The influence of the common-control circuits is neglected. It has been shown in Chapter III that the influence of the common-control circuits is not very great. For a given traffic and grade of service, only a few more trunks are required per group than without them. The trunk efficiency deteriorates only slightly as a result of the presence of the common-control circuits. This assumption, therefore, leads to a slightly too high value for the average travel.

b. When the average travel must be calculated for an interconnected multiple, this will be done for Erlang's ideally arranged multiple, since an exact expression can then be given for the trunk efficiency. Since the trunk efficiency of this multiple is superior to that of the schemes examined in Chapter II, this assumption will also yield a too high value.

The average selector travel will now be calculated successively for the following cases:

a. An interconnected multiple with the outlets to a certain direction grouped together in the bank.
b. A multiple with full availability under the same conditions.
c. An interconnected multiple, with the outlets to the various directions arranged as stated under *2* above.
d. A multiple with full availability under the same conditions.

A multiple with full availability can be considered as a special case of an interconnected multiple, and the formulae will, therefore, only be deduced for the cases *a* and *c*. It is assumed for these calculations that the selectors start from an arbitrary position. A traffic of a erlangs is offered to each direction and must be carried by an interconnected multiple of N trunks, with a choice of m trunks per selector. Erlang's ideal multiple will first be somewhat more closely considered.

4.1 Erlang's ideal interconnected multiple[14])

The multiple is divided into so many sub-groups, that each sub-group contains a different permutation of N trunks, m at a time. As a result, all trunks occupy the same relative position and the multiple is completely symmetrical. The number of sub-groups of which the multiple is composed, can be easily calculated. The number of combinations of N elements, m at a time, is C_m^N. The m elements of each combination can be arranged in $m!$ different ways. As the selector has no home position but starts searching from an arbitrary position, each permutation, through cyclic transposition, yields m further ones, and the number of independent permutations, therefore, is $(m - 1)!$ The number of sub-groups, mentioned before, is therefore $(m - 1)!C_m^N$. For practical values of m and N, the number of selectors is far too small to permit the subdivision into so many groups. When $m + p$ trunks of the multiple are busy, a number of sub-groups will be blocked and can no longer contribute to the traffic of the group. The number of combinations of $m + p$ trunks, m at a time, is equal to C_m^{m+p}, and each combination yields $(m - 1)!$ independent permutations. Only $(m - 1)!\,[C_m^N - C_m^{m+p}]$ sub-groups can, therefore, make a contribution to the traffic of the group. If now $m + p$ trunks of the group are busy, the total flow of traffic can only be:

$$\frac{(m-1)!\,[C_m^N - C_m^{m+p}]}{(m-1)!\,C_m^N}\,a = \left(1 - \frac{C_m^{m+p}}{C_m^N}\right) a$$
$$= W_{m+p}\, a \qquad (0 \leqslant p \leqslant N - m).$$

By means of the well-known method of the statistical equilibrium, the following equations for the probability $g(x)$ that x trunks of the group are occupied, can be established:

$$g(0) = g(0)\left(1 - \frac{a\,.\,dt}{h}\right) + g(1)\frac{dt}{h}\,,$$

where h represents the average duration of a call. For $x < m$:

$$g(x) = g(x-1)\frac{adt}{h} + (x+1)\frac{dt}{h}g(x+1)$$
$$+ g(x)\left(1 - x\frac{dt}{h} - a\frac{dt}{h}\right).$$

For $x \geqslant m + p$, with $0 \leqslant p \leqslant N - m$:

$$g(m+p) = g(m+p-1)\,W_{m+p-1}\,a\frac{dt}{h}$$
$$+ (m+p+1)\,g(m+p+1)\frac{dt}{h}$$
$$+ g(m+p)\left[1 - aW_{m+p}\frac{dt}{h} - (m+p)\frac{dt}{h}\right].$$

The solution to these equations can be written:

$$g(x) = \frac{a^x}{x!}g(0),$$
$$g(m+p) = \frac{a^{m+p}}{(m+p)!}\prod_{r=0}^{p-1} W_{m+r}\,g(0), \qquad (18)$$

or:

$$g(m+p) = \frac{a^{m+p}}{(m+p)!}K_{m+p}\,g(0),$$

with $\quad K_0 = K_1 = K_{m-1} = K_m = 1.$

Furthermore: $\quad \sum_0^N g(x) = 1.$

The traffic carried is $\sum_1^N x\,g(x)$, and the probability of loss can be determined from:

$$q = 1 - \frac{1}{a}\sum_1^N x\,g(x),$$

yielding for q:

$$q = \frac{\frac{a^m}{m!}(1 - K_{m+1}) + \frac{a^{m-1}}{(m+1)!}(K_{m+1} - K_{m+2}) + \ldots + \frac{a^{N-1}}{(N-1)!}(K_{N-1} - K_N) + \frac{a^N}{N!}K_N}{1 + a + \frac{a^2}{2!} + \ldots + \frac{a_m}{m!} + \frac{a_{m+1}}{(m+1)!}K_{m+1} + \ldots + \frac{a_N}{N!}K_N} \qquad (19)$$

Further:

$$\frac{a^{m+p}}{(m+p)!}(K_{m+p} - K_{m+p+1})$$
$$= \frac{a^{m+p}}{(m+p)!}\left[W_m \times W_{m+1} \ldots \times W_{m+p-1}\right](1 - W_{m+p}).$$

For $a \gg m$ and $N \gg m$, the product

$$W_m \times W_{m+1} \ldots \times W_{m+p-1}$$

becomes very nearly equal to one, whilst the term: $\frac{a^{m+p}}{(m+p)!}(1 - W_{m+p})$ becomes $\frac{a^{m+p}}{p!N!}(N-m)!$.

For $a \gg m$ and $N \gg m$, the expression for q therefore reduces to $\quad q \approx \left(\frac{a}{N}\right)^m$.

As has been stated before, the selectors of the different sub-groups all search different combinations of N trunks, m at a time. For the calculation of the number of contacts over which the selector has to travel before a free trunk of the desired direction is found, we shall have to establish the probability that p particular trunks out of the m are occupied. If z trunks of the group are busy, the following expression holds for the probability $\varphi(p)$ that p particular trunks out of the m of a sub-group are busy:

$$\varphi(p) = \frac{C_{z-p}^{N-p}}{C_z^N}\,. \qquad (20)$$

The probability that z trunks of the group are busy is given by expression (18), and one finds for the probability that p particular trunks of a sub-group are busy:

$$\psi(p) = \sum_{z=p}^{N}\frac{C_{z-p}^{N-p}}{C_z^N}g(z) = g(0)\sum_{z=p}^{N}\frac{C_{z-p}^{N-p}}{C_z^N}\frac{a^z}{z!}K_z\,. \qquad (21)$$

If we now define:

$$G_{N-p}(a) = \frac{\frac{a^{N-p}}{(N-p)!}}{K_p + aK_{p+1} + \frac{a^2}{2!}K_{p+2} + \ldots + \frac{a^{N-p}}{(N-p)!}K_N}\,, \qquad (22)$$

$\psi(p)$ may be written as:

$$\psi(p) = \frac{G_N(a)}{G_{N-p}(a)}\,. \qquad (23)$$

If no interconnecting is used (i.e., for $N = m$), all factors K are equal to one, and the already known expression:

$$\psi(p) = \frac{E_m(a)}{E_{m-p}(a)} \qquad \text{is obtained.}$$

For $p = m$, expression (23) must be identical to the probability of loss q that has already been calculated. With some rearrangement this can be shown to be true.

4.2 The average selector travel with a concentrated multiple

Let W_x be the probability that the selector must advance over exactly x contacts before a free trunk of the desired direction is found. The average travel then amounts to:

$$W = \sum_{1}^{c-1} x W_x \, .$$

It is possible that the hunting selector when starting, is already positioned within the group of contacts leading to the desired direction. As a rule, this will not be the case. This is, however, of no importance to the loading of the trunks, since the number of sub-groups is chosen such, that each combination of N trunks, m at a time, is searched. For the case where all trunks are occupied, the *UR 49* system is so arranged that the selector will make somewhat more than a full revolution. For the calculation it will be assumed that exactly one revolution is made and that, therefore, c contacts are passed. The probability W_x will be established for 3 domains of x, and within each of these for the cases where the selector starts from a position inside or outside the wanted group. It is also assumed that the start-position of the hunting selector is arbitrary. For the calculations is needed the probability that p particular trunks are busy, whilst the trunk having the number $(p + 1)$ is free. Let this probability be called $z(p)$.

It follows from (23):

$$\frac{G_N(a)}{G_{N-p}(a)} = z(p) + \frac{G_N(a)}{G_{N-p-1}(a)} \, .$$

$W(x)$ can now be deduced in the following way:

1. $0 < x < m$

a. The selector starts from a position *outside* the wanted group, $x - p$ contacts ahead of the first contact of this group. The first p trunks of the group must now be busy whilst trunk number $p + 1$ is free. One finds for $W_1(x)$:

$$W_1(x) = \frac{1}{c} \sum_{p=0}^{x-1} z(p),$$

or:

$$W_1(x) = \frac{1}{c}\left[1 - \frac{G_N(a)}{G_{N-x}(a)}\right].$$

b. The selector starts from a position *inside* the wanted group and encounters a continuous succession of x busy contacts before contact number $(x + 1)$ is tested free. The selector may start from $m - x$ different contacts and we may, therefore, put:

$$W_2(x) = \frac{m - x}{c} z(x)$$

$$= \frac{m - x}{c} G_N(a)\left[\frac{1}{G_{N-x}(a)} - \frac{1}{G_{N-x-1}(a)}\right].$$

For $0 < x < m$ one finds for $W(x)$:

$$W(x) = W_1(x) + W_2(x)$$

$$= \frac{1}{c} + \frac{1}{c} G_N(a)\left[(m - x - 1)\frac{1}{G_{N-x}(a)} - \frac{m - x}{G_{N-x-1}(a)}\right].$$

2. $m \leqslant x \leqslant c - m$

a. The selector starts from a position *outside* the wanted group, $x - p$ contacts ahead of the first contact of this group. The first p trunks of the group must be busy whilst the trunk number $(p + 1)$ is free.
One finds for $W_1(x)$:

$$W_1(x) = \frac{1}{c} \sum_{p=0}^{m-1} z(p) = \frac{1}{c}\left[1 - \frac{G_N(a)}{G_{N-m}(a)}\right].$$

b. If the selector starts from a position *inside* the group, at least $c - m + 1$ contacts will have to be traversed.

3. $c - m < x \leqslant c - 1$

a. The selector starts from a position *outside* the wanted group, $x - p$ contacts ahead of the first contact of this group. The first p trunks of the group must be busy whilst trunk number $(p + 1)$ is free. The extreme value of p is now determined, however, by $x - p = c - m$, and, consequently:

$$W_1(x) = \frac{1}{c} \sum_{p=x-c+m}^{m-1} z(p)$$

$$= \frac{1}{c} G_N(a)\left[\frac{1}{G_{N-x+c-m}(a)} - \frac{1}{G_{N-m}(a)}\right].$$

b. The selector starts from a contact *inside* the wanted group, p contacts ahead of the last contact of this group whilst these p contacts are all connected to busy trunks. The selector must now pass over $c - m + p$ contacts before the first contact of the group is reached again. As the total travel has been put at x, the first $x - c + m - p$ trunks must be tested busy before the next trunk is found free. The total number of busy trunks must, therefore, be $x - c + m$. The maximum value to be taken by p is $x - c + m$, and one finds:

$$W_2(x) = \frac{x - c + m}{c} z(x - c + m)$$
$$= \frac{x - c + m}{c} G_N(a) \left[\frac{1}{G_{N-x+c-m}(a)} - \frac{1}{G_{N-x+c-m-1}(a)}\right].$$

The expression for $W(x)$ may now be written as

$$W(x) = \frac{1}{c} G_N(a) \left[\frac{x-c+m+1}{G_{N-x+c-m}(a)} - \frac{x-c+m}{G_{N-x+c-m-1}(a)} - \frac{1}{G_{N-m}(a)}\right].$$

4. $x = m$
If all trunks are busy when a call arrives, the selector makes exactly one revolution and:

$$W(mn) = \frac{G_N(a)}{G_{N-m}(a)}.$$

From the various quantities that have been calculated, the average travel can now be determined as:

$$W = \sum_{1}^{c-1} W(x)\, x + mnW(m).$$

After substitution and evaluation, one finds:

$$W_c = \tfrac{1}{2} \frac{(n-1)\{(n-1)m+1\}}{n} + \left[1 + \frac{\frac{1}{2}(n-1)\{1+m(n-1)\}}{n}\right] \frac{G_N(a)}{G_{N-m}(a)} + \frac{2n-1}{n}\left[\frac{1}{G_{N-1}(a)} + \frac{1}{G_{N-2}(a)} + \dots + \frac{1}{G_{N-m+1}(a)}\right] G_N(a). \qquad (24)$$

wherein $mn = c$.

4.3 The average selector travel with a spread multiple

Of the m outlets to a certain direction, two consecutive ones will always be separated by $n - 1$ outlets to other directions. When the selector starts from a position, x contacts ahead of the first (not necessarily free) outlet to the wanted direction, where $0 \leqslant x \leqslant n - 1$, then the selector must pass $pn + x$ contacts before a free outlet to the wanted direction is met. This, evidently, assumes the first p outlets to be busy and outlet number $(p + 1)$ to be free. The probability that the selector starts from a position, x contacts ahead of the first wanted trunk, is $\frac{m}{c}$, and the probability that $pn + x$ contacts must be traversed, becomes:

$$W(pn + x) = \frac{m}{c} z(p) = \frac{m}{c}\left[\frac{G_N(a)}{G_{N-p}(a)} - \frac{G_N(a)}{G_{N-p-1}(a)}\right]. \qquad (25)$$

If it is again assumed that the selector makes exactly one revolution in case all trunks of the wanted direction are busy, the average selector travel per call may be calculated as:

$$W_s = \sum_{p=0}^{m-1} \sum_{x=0}^{n-1} (pn + x)\, W(pn + x) + c\frac{G_N(a)}{G_{N-m}(a)},$$

or, after evaluation:

$$W_s = \tfrac{1}{2}(n-1) + \tfrac{1}{2}(n+1)\frac{G_N(a)}{G_{N-m}(a)} + nG_N(a)\left[\frac{1}{G_{N-1}(a)} + \frac{1}{G_{N-2}(a)} + \dots + \frac{1}{G_{N-m+1}(a)}\right]. \qquad (26)$$

For a multiple with full availability, where $N = m$, the functions G of (24) and (26) become the functions E of Erlang for the full-availability group.

4.4 Comparison of obtained results

The following results have been obtained from a few examples:

a. Full-availability multiple ($N = m$).

$m = n = 10$				$n = 5;\ m = 20$			
a	$E_m(a)$	W_s	W_c	a	$E_m(a)$	W_s	W_c
3.4	0.0019	10.3	42.13	10	0.0019	8	34.63
4.44	0.01	14.3	43.23	12	0.01	11.7	36.22

b. Interconnected multiple with $N = 30$.

$m = n = 10$				$n = 5;\ m = 20$			
a	Prob. of loss	W_s	W_c	a	Prob. of loss	W_s	W_c
14	0.00194	13.73	42.84	17	0.00206	9.65	35.23
17	0.0104	18.4	44.87	20	0.0115	14.08	37.11

These results demonstrate that great advantages are gained when a spread multiple is used instead of a concentrated multiple. It must be pointed out, moreover, that the values given above, hold for a given value of a. During periods of slack traffic, a much lower average value is obtained. When the traffic approaches zero, one finds that:

W_s approaches $\frac{1}{2}(n-1)$ and

W_c approaches $\frac{1}{2}\frac{(n-1)\{(n-1)m+1\}}{n}$.

For $m = n = 10$, these values are 4.5 and 40.95, respectively. If the traffic is known as a function of time, it is of course possible to calculate the average selector travel over the day. For most exchanges this will be of the order of 10 to 15 contacts, which demonstrates that the average number of contacts passed per call is 7 to 10 times smaller for a non-homing selector with a uniformly spread multiple than for a homing selector. Finally, an approximate formula, applicable to the interconnected multiple, can be deduced for W_s and W_c. If the probability of loss is small, and when $N \gg m$, the second term can be neglected and $\left(\frac{a}{N}\right)^p$ may then be written for

$$\frac{G_N(a)}{G_{N-p}(a)}.$$

The expression

$$G_N(a)\left[\frac{1}{G_{N-1}(a)} + \frac{1}{G_{N-2}(a)} + \dots\right]$$

then changes into $\frac{a}{N-a}$, and one finds:

$$W_s = \tfrac{1}{2}(n-1) + \frac{na}{N-a},$$

$$W_c = \tfrac{1}{2}\frac{(n-1)\{(n-1)m+1\}}{n} + \frac{2n-1}{n}\frac{a}{N-a}.$$

CHAPTER V

The influence of the interdigital pause and of the selector speed on the grade of service

5.0 General considerations

The method of positioning the selectors has been discussed in Chapter I. As soon as the common-control circuit has received the digit dialled by the subscriber, the selector is started to hunt for a trunk to the desired direction. The time elapsing between the end of the last dialling pulse and the moment the selector starts to rotate, is approximately 120 milliseconds. If the next pulse train is sent t ms after the end of the one just received, the selector must have found a free outlet within $t - 120$ ms, as the pulse train would otherwise arrive at the next selector stage in mutilated form. To avoid this mutilation, busy tone is given in the *UR 49* system in case a new pulse train arrives before the selector has found a free outlet. It is possible, although not very probable, that the selector has to make nearly a full revolution before a free trunk is found. With a speed of 300 contacts per second, a complete revolution is made in approx. 330 ms. Therefore, if $t \geqslant 450$ ms, the selector can always be positioned within the time t. The interval t is largely dependent on the next digit dialled, on the type of telephone dial and on the disposition of the subscriber. If the digit 1 is dialled, the time needed to wind up the dial is very short. Most dials are, therefore, so designed that they have a lost-motion period of 200 ms, as a result of which the minimum interval between two successive pulse trains becomes 450 to 500 ms.

Various measurements have been made [15] [16], yielding an insight into the magnitude of the interdigital pause. From a number of measurements the distribution of the interdigital pauses has been found for a dial with a 200 ms lost-motion period. The measurements show that in only 2% of all cases the interdigital pause is smaller than 500 ms, and one may therefore conclude that with a selector speed of 300 contacts per second, the system will function properly.

It may happen to a few subscribers having telephone dials with a lost-motion period inferior to 200 ms, that the selector cannot make a complete revolution. The probability of loss is somewhat increased for these subscribers, since they not only receive busy tone when all trunks are busy, but also, in spite of free trunks still being available, when the selector cannot reach them in time. This added probability of loss is, however, only very small and is investigated in para. 5.1. Another extra probability of loss may be expected when the selector speed drops below 300 contacts per second due, for instance, to a variation of the supply voltage to the driving motors. This extra loss will then occur even with an interdigital pause of 450 ms. This influence of the selector speed is considered in para. 5.2 for the case where all interdigital pauses are 450 ms. In reality the effect will be smaller than calculated, since longer interdigital pauses will always be found as well. This is investigated in para. 5.3. It is evident that, the smaller the average number of contacts travelled is, the smaller these effects will be. The calculations have, consequently, only been made for the spread multiple as used in the *UR 49* system. The influence of the common-control circuits has been left out of account. Due to their presence, the trunk efficiency and, consequently, also the average selector travel decreases. The results of the calculations will thus be less favourable than may be expected in actual practice.

Table of interdigital pauses for a dial with a 200 ms lost-motion period.

milliseconds	*observations %*
$400 < t < 500$	2.0
$500 < t < 600$	6.4
$600 < t < 700$	8.7
$700 < t < 800$	9.8
$800 < t < 900$	10.5
$900 < t < 1000$	10.6
$1000 < t < 1100$	10.5
$1100 < t < 1200$	9.7
$1200 < t < 1300$	8.0
$1300 < t < 1400$	6.4
$1400 < t < 1500$	5.0
$1500 < t < 1600$	4.0
$1600 < t < 1700$	3.1
$1700 < t < 1800$	2.3
$1800 < t < 1900$	1.5
$1900 < t < 2000$	0.9
$2000 < t$	0.6

5.1 Grade of service in the case of a single subscriber, provided with a telephone dial permitting short interdigital pauses

Since in the case considered, only a few subscribers are assumed to be able to produce shorter interdigital pauses than 450 ms, the traffic on the trunk group is not influenced. Let t be the interdigital pause, b the time between the end of the last dial pulse and the start of the selector, and v the time necessary to cover a contact spacing. The number of contacts the selector may travel before the arrival of the next pulse train can be written as:

$$y = \frac{t - b}{v} = \alpha\, n + \beta.$$

If more than y contacts must be passed before a free trunk is found, the subscriber will receive busy tone. The probability of loss for this subscriber will be the probability that all trunks are busy, increased by the probability that more than y contacts must be covered if not all trunks are busy. With equation (25) of para. 4.3, this probability of loss may be expressed as:

$$q = \frac{G_N(a)}{G_{N-m}(a)} + \sum_{x=\beta+1}^{n-1} W(\alpha n + x) + \sum_{p=\alpha+1}^{m-1} \sum_{x=0}^{n-1} W(pn + x),$$

or:

$$q = \frac{(n - 1 - \beta)\, m}{c} G_N(a) \left[\frac{1}{G_{N-\alpha}(a)} - \frac{1}{G_{N-\alpha-1}(a)}\right] + \frac{G_N(a)}{G_{N-\alpha-1}(a)}.$$

A few results have been elaborated for $b = 120$ ms and $v = \frac{330}{99}$, the selector being assumed to travel exactly 99 contacts when $t = 450$ ms. The results are given for a full-availability group.

$N = m = n = 10$; $\beta = 9$			
t (ms)	α	a (erlangs) 4.5	3.4
		q	q
450	9	0.01	0.0019
416	8	0.012	0.0025
383	7	0.016	0.0033
350	6	0.020	0.0049
317	5	0.028	0.0076
283	4	0.041	0.0131
250	3	0.065	0.025
217	2	0.111	0.053
183	1	0.21	0.127
150	0	0.42	0.34

These calculations show clearly that, even if a subscriber succeeds in forcing a short interdigital pause, the probability of loss he will have to expect will only increase very little.

5.2 Closer consideration of the relation between interdigital pause, selector speed and grade of service

In the following considerations it is assumed that on the incidence of a call, the selector is in an arbitrary position. This is not entirely true for a high probability of loss, since the selector is directed to one particular contact each time busy tone is given. The influence of this detail will however be neglected.

Let $\psi(t)\,dt$ be the probability that the interdigital pause lies between t and $t + dt$, and b the time elapsing between the end of the last pulse of a train and the start of the selector. Let v, moreover, be the time needed to travel over a contact spacing. A call will not be established, either because all trunks are occupied or because the interdigital pause is too small to permit a free trunk being reached. If z trunks of the wanted direction are busy, the probability that exactly $pn + x$ contacts must be travelled to find a free trunk, may be calculated with the aid of equation (20) of para. 4.1, and on the lines of paragraphs 4.2 and 4.3, as:

$$W(pn + x) = \frac{m}{c}\,[\varphi(p) - \varphi(p+1)].$$

The time needed to travel these contacts may be determined from the relation:

$$\frac{t_{pn+x} - b}{v} = pn + x.$$

It follows from the above, that the probability that a call cannot be established when z trunks of the group are occupied, may be stated as:

$$\sum_{p=0}^{m-1}\sum_{x=x}^{n-1} W(pn+x)\int_0^{t_{pn+x}} \psi(t)\,dt.$$

Since the probability is $\varphi(m)$ that, with a total of z occupied trunks, the m outlets to which the selector has access, are all busy, the probability that the call will be successful, is:

$$R(z) = 1 - \varphi(m) - \sum_{p=0}^{m-1}\sum_{x=x}^{n-1} W(pn+x)\int_0^{t_{pn+x}} \psi(t)\,dt. \tag{27}$$

(If $z < m$, $\varphi(m)$ disappears from this expression and p ranges from 0 to z).

The calculation will now be effected under two different assumptions for the function $\psi(t)$, namely, for the case where all interdigital pauses are identical, and for a function analogous to the one mentioned in para. 5.0.

5.2.1 *The interdigital pause has the constant value t_0.*

In this case the integral of (27) is equal to 0 for all values of $t_{pn+x} < t_0$, and equal to 1 for $t_{pn+x} \geqslant t_0$.

If $\dfrac{t_0 - b}{v} = an + \beta$, (27) becomes:

$$1 - \varphi(m) - \sum_{x=\beta+1}^{n-1} W(an + x) \sum_{p=a+1}^{m-1}\sum_{x=0}^{n-1} W(pn + x)$$

$$= 1 - \frac{m}{c}\Big[(n - 1 - \beta)\,\varphi(a) + (\beta+1)\,\varphi(a+1)\Big].$$

Substituting the expression for $\varphi(p)$, this becomes:

$$R(z) = 1 - \frac{m}{c}\left[(n-1-\beta)\,\frac{C_{z-a}^{N-a}}{C_z^N} + (\beta+1)\,\frac{C_{z-a-1}^{N-a-1}}{C_z^N}\right] \tag{28}$$

The probability that a call can be established in the trunk group in the interval dt, if z trunks are busy, is now:

$$a\frac{dt}{h}\,V_z = a\frac{dt}{h}\left[1 - \frac{m}{c}\left\{(n - 1 - \beta)\,\frac{C_{z-a}^{N-a}}{C_z^N} + (\beta + 1)\,\frac{C_{z-a-1}^{N-a-1}}{C_z^N}\right\}\right]. \tag{29}$$

For $z < a$, the expression reduces to $a\dfrac{dt}{h}$; this is evident, since only so many trunks are occupied in the whole group in this case, that certainly no more than $an + \beta$ contacts need to be travelled. If the speed of the selector is so large that a complete revolution can always be made, β is equal to $n - 1$, and a to $m - 1$, which causes (29) to change to:

$$a\frac{dt}{h}\left[1 - \frac{C_{z-m}^{N-m}}{C_z^N}\right] = a\frac{dt}{h}\left[1 - \frac{C_m^{m+p}}{C_m^N}\right],$$

which corresponds to the expressions that have already been deduced in para. 4.1 for the ideal interconnected multiple. With the equations for the statistical equilibrium, the probability $h(z)$ that z trunks of the group are busy, is found as:

$$h(z) = h(z-1)\,a\frac{dt}{h}\,V_{z-1} + (z+1)\frac{dt}{h}\,h(z+1) + h(z)\left[1 - z\frac{dt}{h} - a\,V_z\,\frac{dt}{h}\right],$$

and also: $$\sum_{z=0}^{N} h(z) = 1.$$

The solutions to these equations read:

$$h(z) = \frac{a^z}{z!} K_z h(0);$$

$$K_z = \prod_{x=0}^{z-1} V_x \; ;$$

$$K_0 = 1;$$

$$h(0)\left[\sum_{z=0}^{N} \frac{a^z}{z!} K_z\right] = 1.$$

If we now define:

$$H_{N-p}(a) = \frac{\dfrac{a^{N-p}}{(N-p)!}}{K_p + aK_{p+1} + \dfrac{a^2}{2!}K_{p+2} + \ldots + \dfrac{a^{N-p}}{(N-p)!}K_N},$$

the probability of loss $q = 1 - \frac{1}{a}\sum_{z=1}^{N} z\, h(z)$ may be written as:

$$q = \frac{H_N(a)}{H_{N-m}(a)}.$$

We choose the example:

$N = 30$; $a = 14$; $m = n = 10$; $t_0 = 450$ ms; $b = 120$ ms			
Selector speed in contacts per sec	β	α	*Prob. of loss* q
300	9	9	0.00194
270	9	8	0.00322
240	9	7	0.00514
209	9	6	0.00845
179	9	5	0.0144
148	9	4	0.0256

These values are reproduced graphically in *fig. 22.*

The above Table shows that a decrease of the selector speed, due to variations of the driving motor, has but a slight effect on the probability of loss. A decrease of 10 to 20% of the speed will not be experienced as objectionable by the

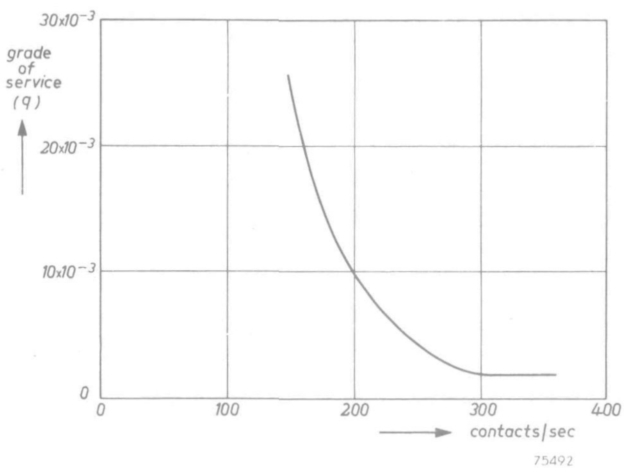

Fig. 22. Relation between grade of service and selector speed at a constant interdigital pause of 450 ms.

subscriber. Conversely, one may keep the selector speed constant and calculate the probability of loss as a function of the interdigital pause. This is graphically represented in *fig. 23.*

In the above, the extreme case has been considered where all interdigital pauses have the same value. The selector speed appears to be in no way

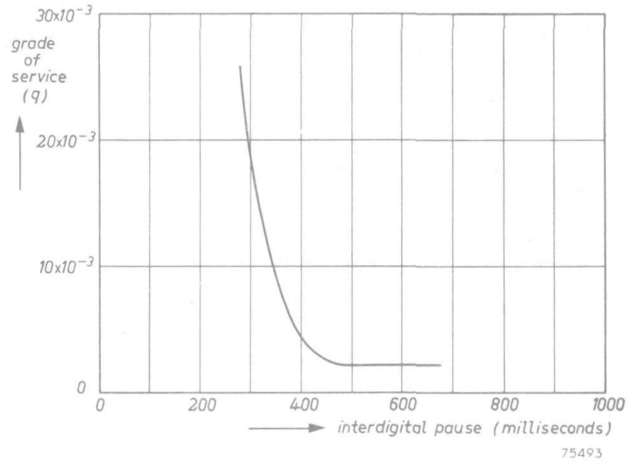

Fig. 23. Relation between grade of service and interdigital pause at a constant selector speed of 300 contacts/sec.

critical with regard to the proper operation of the system. The calculations just made are not, however, representative of actual practice. The real losses in the exchange due to a lowering of the selector speed, will be much lower still, as the interdigital pauses will mostly be longer than the minimum one. The system must of course be based on the minimum interdigital pause in order to give good service to each subscriber. It is remarkable, however, how little a lowering of the selector speed will be noticeable in the exchange. This will be demonstrated in the next paragraph.

5.2.2 *The interdigital pause follows the distribution law of 5.0*

The values of the Table of 5.0 have been used to make up the histogram of *fig. 24.* As the time is divided into class intervals, a choice must be made as to the points where the readings will be taken. This has been done at the mid-points of the intervals, namely, at 450, 550, etc. ms. In order to establish an equation for the distribution function $\psi(t)$, the first four moments of the measured values must be determined. The values 450 ms, 550 ms, etc, are taken as class marks. One finds for the first four moments:

$$m_1 = 1056 \text{ ms}; \; m_2 = 12{,}442 \times 10^4 \text{ ms}^2;$$
$$m_3 = 21{,}809 \times 10^6 \text{ ms}^3; \; m_4 = 412{,}760 \times 10^8 \text{ ms}^4.$$

From these, the usual statistical quantities may be obtained [7]):

The average value $m_1 = 1056$ ms.
The standard deviation $\sigma = \sqrt{m_2} = 353$ ms.
The asymmetry $\sqrt{\beta_1} = \frac{m_3}{m_2{}^{3/2}} = 0.496$.
The flatness $\beta_2 = \frac{m_4}{m_2^2} = 2.67$.

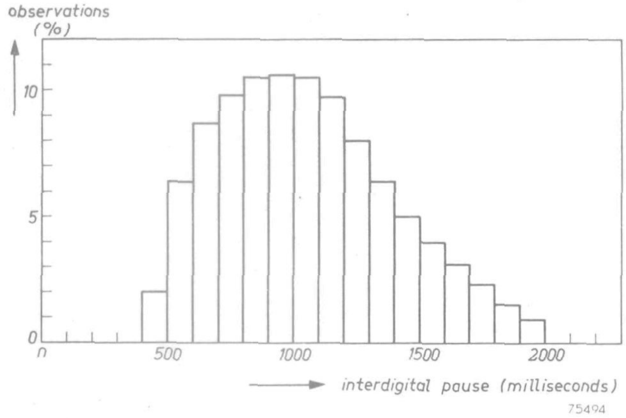

Fig. 24. Histogram of the distribution of the interdigital pause.

A curve must now be found having the same values of m_1, σ, $\sqrt{\beta_1}$ and β_2. The Gram-Charlier series can now be used to advantage [17]), viz:

$$\psi(t) = \frac{1}{\sigma}\left[\Phi_o\left(\frac{t-a}{\sigma}\right) + A_3\Phi_3\left(\frac{t-a}{\sigma}\right) + A_4\Phi_4\left(\frac{t-a}{\sigma}\right)\right]. \tag{30}$$

In this expression:

$$\Phi_o\left(\frac{t-a}{\sigma}\right) = e^{-\frac{1}{2}\left(\frac{t-a}{a}\right)^2}\frac{1}{\sqrt{2\pi}}$$

Φ_3 and Φ_4 are the 3rd and 4th derivatives of Φ_o with respect to the argument $\frac{t-a}{\sigma}$. The constants are so determined that:

$\sigma = \sqrt{m_2}$; $6A_3 = -\sqrt{\beta_1}$; $24\,A_4 = \beta_2{}^{-3}$; $a = m_1$.
After substitution of the constants one finds:

$$\psi(t) = \frac{1}{\sigma}\left[\Phi_o(y) - 0.083\,\Phi_3(y) - 0.0137\,\Phi_4(y)\right],$$

with $$y = \frac{t-1056}{353}.$$

The probability that the interdigital pause lies between t_a and t_b, may be written as:

$$\int_a^{t_b}\psi(t)\,dt = \int_{\frac{t_a-1056}{353}}^{\frac{t_b-1056}{353}}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy$$
$$-0.083\left[\Phi_2\left(\frac{t_b-1056}{353}\right) - \Phi_2\left(\frac{t_a-1056}{353}\right)\right]$$
$$-0.0137\left[\Phi_3\left(\frac{t_b-1056}{353}\right) - \Phi_3\left(\frac{t_a-1056}{353}\right)\right].$$

One finds:

Milliseconds	%
$-\infty$ — 300	0.23
300 — 400	1.67
400 — 500	2.5
500 — 600	4.9
600 — 700	7.0
700 — 800	9.0
800 — 900	10.7
900 — 1000	11.1
1000 — 1100	11.1
1100 — 1200	9.7
1200 — 1300	8.2
1300 — 1400	6.7
1400 — 1500	5.5
1500 — 1600	4.0
1600 — 1700	2.9
1700 — 1800	1.9
1800 — 1900	1.4
1900 — 2000	0.7
2000 — ∞	0.8

It is possible to calculate the quantities $R(z)$ of (27) with the aid of the now known quantity $\int_0^t \psi(t)\,dt$. (If (30) is used, the lower limit of the integral must be $-\infty$). This quantity corresponds very well with the measured values, as will be apparent from the following Table:

$\int_0^t \psi(t)$ is the probability that the interdigital pause is $< t$.

t (ms)	Calculated from (30) (%)	Measured (%)
400	1.9	
500	4.4	2.0
600	9.3	8.4
700	16.3	17.1
800	25.3	26.9
900	36.0	37.4
1000	47.1	48.0
1100	58.2	58.5
1200	67.9	68.2
1300	76.1	76.2
1400	82.8	82.6
1500	88.3	87.6
1600	92.3	91.6
1700	95.2	94.7
1800	97.1	97.0
1900	98.5	98.5
2000	99.2	99.4

The values are represented graphically in *fig. 25*.

The numerical calculation of $R(z)$ is difficult to carry out and the result will certainly deviate little from the true value, if $\int\psi(t)\,dt$ is approximated as follows:

$$0 \leqslant t \leqslant t_0 : \int_0^t \psi(t)\, dt = 0$$

$$t_0 \leqslant t \leqslant t_1 : \int_0^t \psi(t)\, dt = m\,(t - t_0)$$

$$t > t_1 : \int_0^t \psi(t)\, dt = 1.$$

Here:

$t_0 = 450$ ms ; $t_1 = 1550$ ms ; $m = \frac{1}{1100}$.

This approximation has also been drawn in fig. 25.

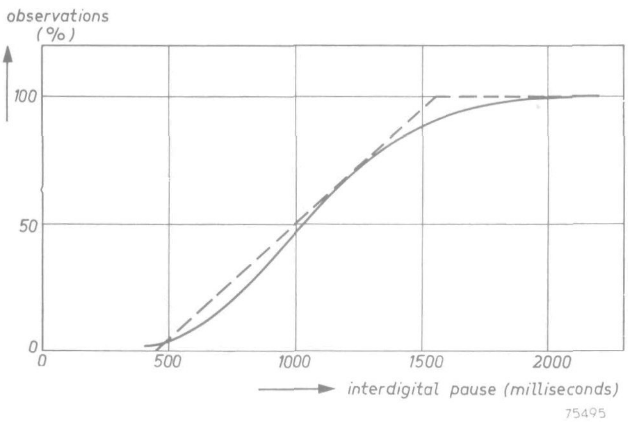

Fig. 25. Probability of the interdigital pause smaller than t ms. For calculating, the dotted line was used as an approximation.

Passing from t_{pn+x} to the number $pn + x$ of travelled contacts, one finds, with $\frac{t_o - b}{v} = an + \beta$:

$$pn + x \leqslant an + \beta; \int_0^t \psi(t)\, dt = 0;$$

$$an + \beta < pn + x \leqslant \frac{t_1 - b}{v};$$

$$\int_0^t \psi(t)\, dt = m\, \frac{t_0 - b}{a\,n + \beta} \left[pn + x - (an + \beta)\right]$$

$$= \gamma\,(pn + x - an - \beta);$$

$$pn + x > \frac{t_1 - b}{v}; \int_0^t \psi(t)\, dt = 1.$$

If the selector speed is greater than approximately 75 contacts per second, the last domain of $pn + x$ may be left unconsidered and $R(z)$ becomes:

$$R(z) = 1 - \varphi(m) - \sum_{x=\beta+1}^{n-1} W(an + x)\,\gamma\,(x - \beta)$$

$$- \sum_{p=a+1}^{m-1} \sum_{x=0}^{n-1} W(pn + x)\,\gamma\,(pn + x - an - \beta).$$

Introducing the known expression for $W(pn + x)$, one finds for $R(z)$:

$$R(z) = 1 - \varphi(m)$$

$$+ \varphi(m) \left[\gamma \left\{- \beta + \tfrac{1}{2}(n-1) + n(m-1-a)\right\}\right]$$

$$- \varphi(a)\,\gamma \left[\tfrac{1}{2} \frac{m}{c} (n - 1 - \beta)(n - \beta)\right]$$

$$+ \tfrac{1}{2}\gamma \frac{m\beta}{c} (1 + \beta)\,\varphi(a + 1)$$

$$- n\gamma \left[\varphi(a + 1) + \ldots . \varphi(m - 1)\right].$$

The probability of loss can now be found in the same way as in 5.2.1, for instance:

selector speed 200 contacts per second;
$N = m = n = 10$; $a = 3.4$ erlangs;
$t_0 = 450$ ms ; $b = 120$ ms;
Probability of loss: 0.00197.

This example shows, that the traffic loss increases only very slightly when the selector speed drops.

The examples treated in this chapter make it apparent that the lower limit of the selector speed is not very critical. The upper limit is fixed by the speed of operation of the test relay, and by the time needed by the stop pawl to engage the ratchet wheel. This limit is not very critical either, due to the dimensioning of these parts.

APPENDIX

I. The method of measurement of interconnected multiples

I.1. A series of six-digit numbers is taken from a Table of "random sample" numbers [8], each of which is given the following meaning:

a. The first three digits determine whether a call is initiated or terminated, and, in addition, if a call is initiated, in which group this occurs. When this three-digit number lies between 000 and x, it signifies the beginning; when it lies between x and 999, the end of a call. The interval 000 — x must be divided into as many sub-intervals as there are sub-groups in the multiple under investigation. The first three digits, therefore, indicate directly in which sub-group a call is initiated or whether a call must be terminated.
b. If a call is initiated, the fourth digit indicates the contact from which the selector starts hunting for a free trunk.
c. When a call must be terminated, the fifth and sixth digit indicate the number of the trunk to which this applies. Only interconnected multiples with 25 or 33 trunks have been investigated and a trunk number was, therefore,

represented by four different combinations of the hundred numbers in the first case, and by three in the last case, the combination 00 then not being used. The probability of the appearance of a number signifying the initiation of a call, is $\frac{x}{1000}$; that of a number signifying a termination is $\frac{1000-x}{1000}$. The probability that the latter case applies to one particular trunk, indicated by the 5th and 6th digits, is equal to $\frac{1}{N}$, where N is the number of trunks in the multiple. The traffic offered to the multiple is, therefore, equal to:

$$a = \frac{\frac{x}{1000}}{\frac{1000-x}{1000\,N}} = \frac{Nx}{1000-x}\,.$$

It is also possible to draw the number of the trunk on which the call must terminate in one step instead of in two steps, as has been done here. For that purpose another draw must then be made from c numbers, x of which signify the initiation, and pN the termination of a call on a particular trunk. These x numbers have, however, to be subdivided again into a number of classes of equal size to draw the number of the sub-group in which the call appears. Moreover, p must be a whole number. For the investigated multiples, where the number of groups and the offered traffic must vary, it is not always possible in that case to choose the value 1000 for c. Several of the combinations of three digits would then have to be left unused, and many more numbers would have to be drawn to obtain the same accuracy of measurement. The measurements would, consequently, demand a great deal more time.

I.2. 10,000 numbers were drawn for each multiple, after an initial condition of it had been assumed. After each series of 500 numbers, a reading was taken of the number of calls carried by each trunk and of the numbers of successful and lost calls in each group.

The accuracy of the measurements is difficult to determine, since the statistical distribution of the lost calls is not known. It is true that this distribution has been calculated for a few cases [9], but to this end, one must know the formulae describing the multiple. For the measurements as they have been carried out here, we have proceeded from a normal frequency distribution of lost calls. Let the average number of lost calls per series of 500 observations be $\overline{y}$, the standard error then being $\sigma = \frac{1}{20}\sqrt{\Sigma(y-\overline{y})^2}$, if y denotes the number of lost calls of a series. The probability that the number of lost calls lies between $\overline{y}+\sigma$ and $\overline{y}-\sigma$ is then 0.68, and the probability that it lies between $\overline{y}+2\sigma$ and $\overline{y}-2\sigma$, is 0.95. The expected number of calls per series is $V = 500 \times \frac{x}{1000}$ and the probability of loss has been assumed as

$$\frac{\overline{y} \pm \sigma}{V}\,.$$

I.3. In order to speed up the evaluation of the numbers as much as possible, and perform the selection of a free trunk and the registration of successful calls automatically, a machine was designed permitting the investigation of multiples

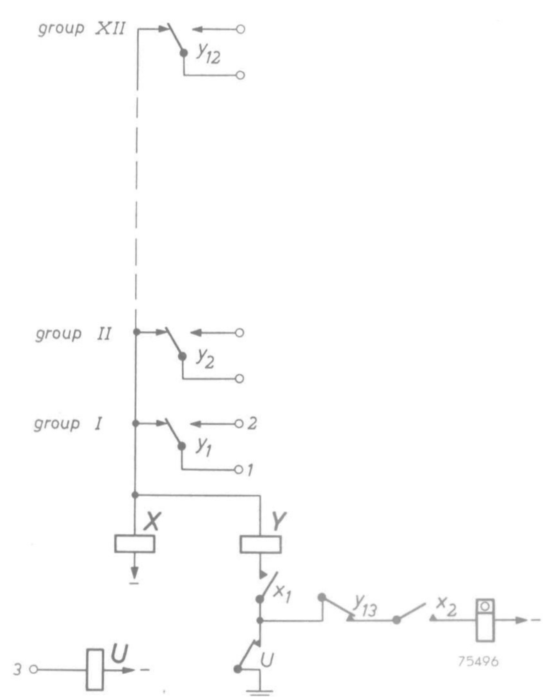

Fig. 26. Principle of a line device of a traffic machine.

with a maximum of 33 trunks, either in 12 subgroups with 10 outlets per sub-group, or in 6 subgroups with 20 outlets per sub-group. By means of patching cords, any desired multiple fulfilling the above-mentioned conditions, could be set up on a jack field. A short description of the principles of this machine may be given here: For each of the 33 trunks, the machine contains a line device comprising a few relays and a telephone subscriber's meter as a register (*fig. 26*). When the trunk is free and a call is placed in group 1, an earth potential pulse is applied to terminal 1, causing relay X to operate. Contacts x_1 and x_2 close, x_2 completing a circuit for the meter. Relay Y cannot operate while the pulse lasts, being kept short-circuited at terminal 1. As soon as the earth is disconnected from terminal 1, relay Y will be energized in series with X and, in operating, will transfer contacts y_1 to y_{12} incl. Contact y_{13} will interrupt the circuit for the meter, and the transfer of contacts y_1 to y_{12} incl. will cause the next earth potential pulse to be passed to terminal 2, which is in turn connected to terminal 1 of another line device. This pulse will thus by-pass all busy-line devices and cause the first free one to function. The line device can be brought back to the "free" condition by applying an earth pulse to terminal 3.

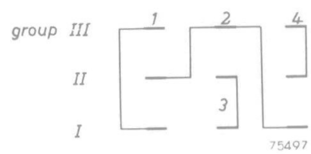

Fig. 27. Interconnected multiple used for the explanation of the traffic machine.

The multiple of *fig. 27* can obviously be realized by connecting the y contacts in the manner indicated by *fig. 28.* The terminals 1 and 2 of each line device were wired to jacks on the patching panel of the machine. The same thing was done with a number of sets of 10 pairs of contacts, the

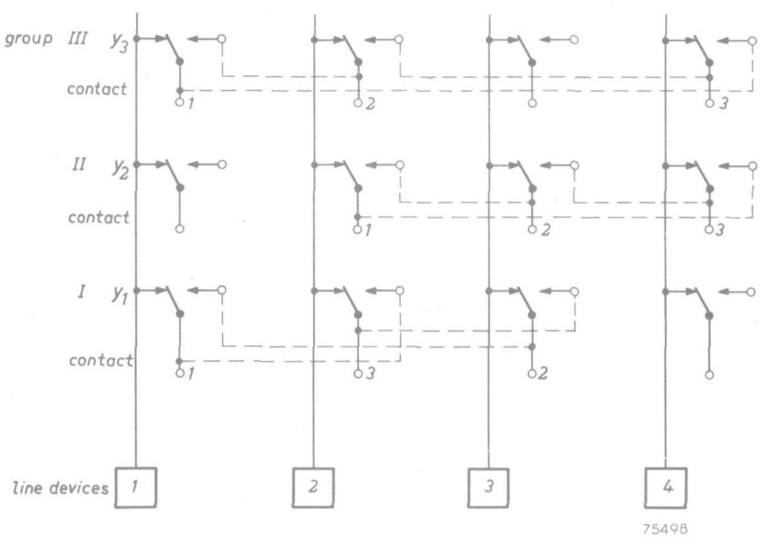

Fig. 28. Connection of the multiple of fig. 27 with the line-devices.

pairs of each set being interconnected as indicated in *fig. 29* while each pair corresponds to one of the 10 outlets of a sub-group. One has now only to read from the diagram of the multiple to what outlets the trunks should be connected and then to patch the "outlet" jacks to the "y-contact"

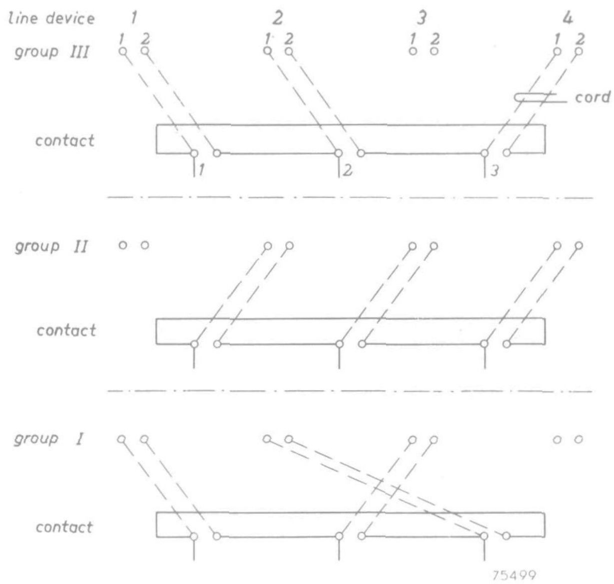

Fig. 29. Diagram of the patching-panel of the traffic machine.

jacks of the corresponding sub-group of the line device. This is drawn in fig. 29 for the multiple of fig. 27. For the investigation of sub-groups with 20 outlets, two sub-groups could be combined by means of a few switches on the patching panel.

Now, if a number must be put through the machine, indicating, for instance, that it must start searching for a free trunk from outlet 3 of sub-group 4, *sub-group button* number 4 must first be pressed, and then *outlet button* number 3. Via a combination of relay contacts a pulse is then passed to that line device of which the y-contact of sub-group 4 has been patched to outlet 3 in the same sub-group. When this line device is already busy, the pulse is passed to the line device connected to outlet 4, and so on. When all trunks of the subgroup are busy, the pulse is passed to a lost-calls register.

If the number indicates that a call must be terminated, a central *terminate*-button must be pressed first and then one of the 33 individual *terminate*-keys, causing a pulse to be applied to terminal 3 of the corresponding line device. As the numbers could be put through the machine in very rapid succession, a few interlocking and alarm circuits were provided in order to prevent errors, such as simultaneously pressing two buttons or pressing two outlet-buttons in succession, from having any effect.

1.4 Results of measurements.

For a number of multiples a to g, the numbers 000 to 389 were taken to indicate the initiation of a call. In the following Tables the number of successful calls per trunk, and the number of lost calls per series of 500 attempts, are given.

Multiple a: $a = 16$ erlangs $N = 25$				Multiple b: $a = 16$ erlangs $N = 25$			
Trunk No.	No. of calls	Series No.	Lost calls	Trunk No.	No. of calls	Series No.	Lost calls
1	174	1	9	1	160	1	5
2	150	2	0	2	153	2	7
3	161	3	2	3	156	3	5
4	169	4	4	4	145	4	5
5	124	5	8	5	162	5	10
6	122	6	10	6	161	6	4
7	140	7	3	7	154	7	7
8	115	8	0	8	135	8	1
9	89	9	4	9	146	9	1
10	76	10	10	10	150	10	7
11	172	11	7	11	143	11	6
12	143	12	5	12	142	12	5
13	182	13	16	13	156	13	8
14	147	14	3	14	154	14	5
15	170	15	7	15	166	15	7
16	173	16	20	16	144	16	21
17	163	17	0	17	153	17	1
18	178	18	7	18	141	18	3
19	145	19	13	19	145	19	8
20	117	20	8	20	146	20	9
21	184			21	155		
22	151			22	146		
23	160			23	137		
24	167			24	153		
25	151			25	148		
$\bar{y} = 6.8$; $\sigma = 1.15$; $V = 195$; $q = 0.035 \pm 0.0059$				$\bar{y} = 6.25$; $\sigma = 0.94$; $V = 195$; $q = 0.032 \pm 0.0048$			

Multiple c: $a = 16$ erlangs $N = 25$				Multiple d: $a = 16$ erlangs $N = 25$			
Trunk No.	No. of calls	Series No.	Lost calls	Trunk No.	No. of calls	Series No.	Lost calls
1	167	1	6	1	163	1	4
2	145	2	3	2	143	2	4
3	152	3	5	3	165	3	4
4	157	4	2	4	163	4	8
5	136	5	12	5	157	5	5
6	156	6	8	6	170	6	3
7	148	7	5	7	172	7	3
8	143	8	2	8	148	8	2
9	157	9	1	9	165	9	5
10	154	10	5	10	169	10	6
11	155	11	6	11	139	11	7
12	138	12	8	12	138	12	12
13	159	13	10	13	146	13	10
14	154	14	1	14	136	14	7
15	158	15	5	15	149	15	5
16	150	16	20	16	134	16	15
17	147	17	0	17	141	17	1
18	139	18	6	18	139	18	3
19	145	19	15	19	153	19	8
20	132	20	9	20	135	20	10
21	165			21	140		
22	156			22	148		
23	135			23	125		
24	141			24	137		
25	149			25	148		
$\bar{y} = 6.45$; $\sigma = 1.087$; $V = 195$; $q = 0.033 \pm 0.0056$				$\bar{y} = 6.1$; $\sigma = 0.77$; $V = 195$; $q = 0.031 \pm 0.004$			

Multiple e: $a = 16$ erlangs $N = 25$				Multiple f: $a = 16$ erlangs $N = 25$			
Trunk No.	No. of calls	Series No.	Lost calls	Trunk No.	No. of calls	Series No.	Lost calls
1	147	1	6	1	140	1	9
2	155	2	5	2	158	2	5
3	139	3	5	3	139	3	1
4	158	4	3	4	163	4	8
5	135	5	6	5	134	5	7
6	164	6	5	6	177	6	9
7	143	7	4	7	139	7	10
8	153	8	0	8	146	8	2
9	143	9	3	9	126	9	3
10	159	10	2	10	154	10	8
11	166	11	4	11	138	11	6
12	139	12	8	12	127	12	9
13	161	13	10	13	136	13	12
14	136	14	3	14	139	14	11
15	184	15	7	15	153	15	3
16	134	16	25	16	153	16	32
17	134	17	0	17	153	17	0
18	147	18	3	18	169	18	7
19	140	19	15	19	161	19	12
20	160	20	11	20	154	20	10
21	146			21	147		
22	157			22	138		
23	132			23	138		
24	144			24	144		
25	141			25	145		
$\bar{y} = 6.25$; $\sigma = 1.25$; $V = 195$; $q = 0.032 \pm 0.0064$				$\bar{y} = 8.2$; $\sigma = 1.45$; $V = 195$; $q = 0.042 \pm 0.0075$			

Multiple g: $a = 16$ erlangs $N = 25$			
Trunk No.	No. of calls	Series No.	Lost calls
1	138	1	9
2	157	2	5
3	137	3	2
4	161	4	9
5	131	5	7
6	174	6	8
7	131	7	12
8	145	8	2
9	129	9	3
10	152	10	8
11	139	11	11
12	126	12	11
13	136	13	11
14	142	14	3
15	153	15	31
16	153	16	2
17	153	17	4
18	168	18	12
19	160	19	9
20	154	20	8
21	146		
22	134		
23	133		
24	143		
25	141		
$\bar{y} = 8.35$; $\sigma = 1.39$; $V = 191$; $q = 0.044 \pm 0.0073$			

For the multiples h and i which follow, the numbers 000 to 407 were taken as "initiate" numbers.

Multiple h: $a = 17.23$ erlangs $N = 25$				Multiple i: $a = 22.8$ erlangs $N = 33$			
Trunk No.	No. of calls	Series No.	Lost calls	Trunk No.	No. of calls	Series No.	Lost calls
1	154	1	9	1	99	1	22
2	143	2	7	2	98	2	9
3	167	3	2	3	83	3	4
4	155	4	4	4	97	4	4
5	153	5	9	5	116	5	0
6	180	6	10	6	105	6	11
7	168	7	5	7	113	7	17
8	149	8	1	8	84	8	15
9	167	9	2	9	85	9	6
10	162	10	8	10	112	10	11
11	163	11	8	11	84	11	22
12	170	12	13	12	88	12	5
13	178	13	12	13	85	13	11
14	173	14	8	14	84	14	15
15	146	15	14	15	97	15	15
16	137	16	24	16	94		
17	173	17	1	17	81		
18	136	18	8	18	76		
19	119	19	16	19	75		
20	142	20	15	20	79		
21	131			21	92		
22	111			22	73		
23	115			23	79		
24	124			24	91		
25	131			25	69		
				26	81		
				27	82		
				28	79		
				29	80		
				30	80		
				31	80		
				32	65		
				33	86		
$\bar{y} = 8.8$; $\sigma = 1.25$; $V = 204$; $q = 0.044 \pm 0.0062$				$\bar{y} = 11.12$; $\sigma = 1.65$; $V = 204$; $q = 0.055 \pm 0.0081$			

For the last multiple, the numbers 000 to 431 were taken as "initiate" numbers.

Multiple j: $a = 19$ erlangs $N = 25$			
Trunk No.	No. of calls	Series No.	Lost calls
1	143	1	8
2	154	2	0
3	139	3	1
4	137	4	0
5	159	5	5
6	188	6	7
7	180	7	3
8	174	8	2
9	149	9	1
10	148	10	0
11	146	11	9
12	148	12	11
13	180	13	10
14	168	14	2
15	202	15	10
16	163	16	19
17	132	17	1
18	138	18	9
19	154	19	15
20	156	20	6
21	177		
22	168		
23	165		
24	172		
25	159		
$\bar{y} = 5.95$; $\sigma = 1.17$; $V = 216$; $q = 0.028 \pm 0.006$			

II. The grade of service of final selectors in a full—availability group

m = number of final selectors per 100-group. Also the number of common-control circuits associated with each hundred group.

k = number of final selector groups served by the same common-control circuits.

β = ratio of holding times of the common-control circuit and of the trunk (the latter including the common-control circuit holding time).

a = traffic in erlangs per 100-group.

q = probability of loss.

The Tables give the values of a for various values of m, k, β and q.

$q = 0.01$											
$k = 5$				$k = 8$				$k = 10$			
m	$\beta = {}^1/_{30}$	$\beta = {}^1/_{40}$	$\beta = {}^1/_{60}$	m	$\beta = {}^1/_{30}$	$\beta = {}^1/_{40}$	$\beta = {}^1/_{60}$	m	$\beta = {}^1/_{30}$	$\beta = {}^1/_{40}$	$\beta = {}^1/_{60}$
5	1.31	1.32	1.33	5	1.26	1.29	1.31	5	1.23	1.26	1.30
6	1.84	1.86	1.88	6	1.79	1.82	1.85	6	1.74	1.79	1.83
7	2.42	2.44	2.46	7	2.35	2.39	2.43	7	2.31	2.36	2.41
8	3.03	3.06	3.08	8	2.96	3.01	3.05	8	2.90	2.97	3.02
9	3.68	3.71	3.73	9	3.59	3.64	3.69	9	3.53	3.60	3.67
10	4.34	4.38	4.41	10	4.25	4.31	4.36	10	4.18	4.26	4.33

$q = 0.002$											
$k = 5$				$k = 8$				$k = 10$			
m	$\beta = {}^1/_{30}$	$\beta = {}^1/_{40}$	$\beta = {}^1/_{60}$	m	$\beta = {}^1/_{30}$	$\beta = {}^1/_{40}$	$\beta = {}^1/_{60}$	m	$\beta = {}^1/_{30}$	$\beta = {}^1/_{40}$	$\beta = {}^1/_{60}$
5	0.867	0.875	0.884	5	0.841	0.857	0.872	5	0.821	0.844	0.864
6	1.28	1.29	1.30	6	1.25	1.27	1.29	6	1.22	1.25	1.28
7	1.75	1.76	1.77	7	1.71	1.73	1.75	7	1.67	1.71	1.74
8	2.25	2.26	2.28	8	2.20	2.23	2.26	8	2.16	2.21	2.24
9	2.79	2.80	2.82	9	2.73	2.76	2.80	9	2.69	2.73	2.78
10	3.35	3.37	3.39	10	3.29	3.32	3.36	10	3.24	3.29	3.34

$q = 0.001$											
$k = 5$				$k = 8$				$k = 10$			
m	$\beta = {}^1/_{30}$	$\beta = {}^1/_{40}$	$\beta = {}^1/_{60}$	m	$\beta = {}^1/_{30}$	$\beta = {}^1/_{40}$	$\beta = {}^1/_{60}$	m	$\beta = {}^1/_{30}$	$\beta = {}^1/_{40}$	$\beta = {}^1/_{60}$
5	0.735	0.742	0.750	5	0.713	0.726	0.739	5	0.698	0.715	0.732
6	1.11	1.12	1.13	6	1.08	1.10	1.11	6	1.06	1.08	1.11
7	1.53	1.54	1.55	7	1.50	1.52	1.54	7	1.47	1.50	1.53
8	2.00	2.01	2.03	8	1.96	1.98	2.01	8	1.93	1.96	1.99
9	2.50	2.51	2.53	9	2.45	2.48	2.51	9	2.41	2.45	2.49
10	3.02	3.04	3.06	10	2.97	3.00	3.03	10	2.93	2.98	3.02

III. The grade of service of the first group selector

k = number of selectors served by one common-control circuit.

m = number of common-control circuits.

β = ratio of holding times of common-control circuit and of trunk (the latter including the common-control circuit holding time).

q = probability of loss.

The Tables give the values of a for a number of values of the other variables.

$k = 6$						
	$\beta = 1/20$			$\beta = 1/30$		
m	$q =$ 0.001	$q =$ 0.002	$q =$ 0.01	$q =$ 0.001	$q =$ 0.002	$q =$ 0.01
1	0.02	0.04	0.20	0.03	0.06	0.30
2	0.92	1.30	3.03	1.38	1.95	4.25
3	4.0	5.0	8.35	5.8	7.1	9.6
4	8.8	10.4	13.75	11.2	12.2	14.6
5	14.6	15.9	19.1	16.1	17.0	19.8
6	19.9	21.2	24.4	20.8	21.9	24.9
7	24.9	26.2	29.7	25.6	26.8	30.2
8	30.0	31.3	35.1	30.5	31.8	35.5
9	35.0	36.4	40.3	35.3	36.7	40.8
10	40.0	41.5	45.7	40.3	41.8	46.2
11	45.0	46.6	51.1	45.4	46.9	51.6
12	50.1	51.8	56.6	50.5	52.2	57.2
13	55.0	57.1	62.0	55.7	57.5	62.6
14	60.4	62.3	67.6	60.9	62.8	68.1
15	65.5	67.5	73.1	65.9	67.9	73.7

$k = 3$						
	$\beta = 1/20$			$\beta = 1/30$		
m	$q =$ 0.001	$q =$ 0.002	$q =$ 0.01	$q =$ 0.001	$q =$ 0.002	$q =$ 0.01
1	0.02	0.04	0.18	0.03	0.06	0.26
2	0.81	1.03	1.68	0.97	1.17	1.78
3	2.4	2.7	3.6	2.5	2.8	3.7
4	4.1	4.5	5.7	4.2	4.6	5.8
5	6.0	6.5	7.9	6.0	6.5	8.0
6	7.9	8.5	10.2	8.0	8.5	10.3
7	10.0	10.6	12.6	10.0	10.7	12.7
8	12.1	12.8	15.0	12.1	12.9	15.1
9	14.3	15.1	17.5	14.3	15.1	17.6
10	16.5	17.4	20.0	16.6	17.5	20.1
	$\beta = 1/40$			$\beta = 1/60$		
m	$q =$ 0.001	$q =$ 0.002	$q =$ 0.01	$q =$ 0.001	$q =$ 0.002	$q =$ 0.01
1	0.04	0.08	0.28	0.06	0.11	0.33
2	1.03	1.23	1.83	1.10	1.27	1.86
3	2.5	2.8	3.7	2.6	2.8	3.7

$k = 2$						
	$\beta = 1/20$			$\beta = 1/30$		
m	$q =$ 0.001	$q =$ 0.002	$q =$ 0.01	$q =$ 0.001	$q =$ 0.002	$q =$ 0.01
1	0.017	0.030	0.10	0.022	0.038	0.11
2	0.40	0.50	0.82	0.42	0.51	0.84
3	1.1	1.3	1.85	1.1	1.3	1.9
4	2.0	2.3	3.1	2.0	2.3	3.1
5	3.0	3.4	4.4			
6	4.2	4.6	5.8			
7	5.4	5.8	7.2			
8	6.6	7.2	8.7			
9	8.0	8.5	10.3			
10	9.3	10.0	11.9			
	$\beta = 1/40$			$\beta = 1/60$		
m	$q =$ 0.001	$q =$ 0.002	$q =$ 0.01	$q =$ 0.001	$q =$ 0.002	$q =$ 0.01
1	0.026	0.043	0.12	0.031	0.049	0.13
2	0.42	0.52	0.85	0.43	0.53	0.86
3	1.1	1.3	1.9	1.1	1.3	1.9
4	2.0	2.3	3.1	2.0	2.3	3.1

IV. The grade of service of final selectors in small central offices

m = number of final selectors per hundred-group = twice the number of common-control circuits.

k = the number of groups of final selectors served by a common-control circuit.

The Tables give the values of a for various values of the other variables.

$q = 0.01$										
	$k = 1$		$k = 2$		$k = 3$		$k = 4$		$k = 5$	
m	$\beta =$ 1/30	$\beta =$ 1/40	$\beta =$ 1/30	$\beta =$ 1/40	$\beta =$ 1/30	$\beta =$ 1/40	$\beta =$ 1/30	$\beta =$ 1/40	$\beta =$ 1/30	$\beta =$ 1/40
6	1.9	1.9	1.79	1.82	1.72	1.77	1.64	1.72	1.56	1.66
8	3.1	3.1	3.00	3.04	2.92	2.97	2.82	2.92	2.70	2.85
10	4.4	4.4	4.31	4.35	4.22	4.29	4.12	4.22	3.99	4.14

$q = 0.002$										
	$k = 1$		$k = 2$		$k = 3$		$k = 4$		$k = 5$	
m	$\beta =$ 1/30	$\beta =$ 1/40	$\beta =$ 1/30	$\beta =$ 1/40	$\beta =$ 1/30	$\beta =$ 1/40	$\beta =$ 1/30	$\beta =$ 1/40	$\beta =$ 1/30	$\beta =$ 1/40
6	1.3	1.3	1.23	1.26	1.17	1.22	1.10	1.17	1.02	1.12
8	2.3	2.3	2.22	2.25	2.15	2.20	2.07	2.15	1.96	2.09
10	3.4	3.4	3.32	3.35	3.25	3.30	3.17	3.25	3.05	3.19

V. The grade of service of interconnected multiples

N = number of trunks.
v = number of common-control circuits.
β = holding-time ratio of common-control circuit and of trunks.
q = probability of loss.

The Table gives the value of a for a given probability of loss as a function of the number of trunks.

10 outlets per sub-group

		Pure-chance traffic						Smooth traffic					
q		0.001		0.002		0.01		0.001		0.002		0.01	
β		$^1/_{40}$	$^1/_{60}$	$^1/_{40}$	$^1/_{60}$	$^1/_{40}$	$^1/_{60}$	$^1/_{40}$	$^1/_{60}$	$^1/_{40}$	$^1/_{60}$	$^1/_{40}$	$^1/_{60}$
N	v												
10	5	3.07	3.08	3.40	3.41	4.42	4.43	3.07	3.08	3.40	3.41	4.42	4.43
20	5	7.01	7.08	7.69	7.76	9.71	9.77	7.74	7.87	8.44	8.57	10.45	10.56
30	5	10.7	11.0	11.8	12.0	14.8	15.0	12.0	12.4	13.0	13.5	16.1	16.5
40	5	14.2	14.7	15.6	16.1	20.1	20.4	15.6	16.6	17.1	18.0	21.3	22.1
50	5	17.4	18.2	19.2	20.0	24.4	25.1	18.7	20.4	20.6	22.3	26.0	27.4
100	10	37.4	38.4	40.9	41.8	50.7	51.7	42.4	44.4	45.9	48.0	55.5	57.6
200	20	75.9	77.7	82.7	84.6	102.3	104.3	86.8	90.6	93.8	97.7	112.8	116.8

20 outlets per sub-group

		Pure-chance traffic						Smooth traffic					
q		0.001		0.002		0.01		0.001		0.002		0.01	
β		$^1/_{40}$	$^1/_{60}$	$^1/_{40}$	$^1/_{60}$	$^1/_{40}$	$^1/_{60}$	$^1/_{40}$	$^1/_{60}$	$^1/_{40}$	$^1/_{60}$	$^1/_{40}$	$^1/_{60}$
N	v												
20	5	9.32	9.35	9.96	10.00	11.90	11.93	9.32	9.35	9.96	10.00	11.90	11.93
30	5	15.0	15.1	15.9	16.1	18.7	18.8	15.8	16.1	16.7	17.0	19.4	19.6
40	5	20.2	20.7	21.5	22.0	25.3	25.6	21.3	22.4	22.8	23.7	26.6	27.1
50	5	24.7	26.0	26.6	27.7	31.7	32.3	25.5	28.1	27.7	29.8	33.0	34.3
100	10	54.2	55.4	57.4	58.4	66.1	66.9	60.0	62.3	63.0	65.2	71.0	72.8
200	20	112.0	113.7	118.0	119.6	134.5	136.1	126.4	130.2	131.9	135.5	146.3	149.5

VI. The grade of service of final selectors in an interconnected multiple

m = number of final selectors per 100-group = number of common-control circuits.
k = number of final selector groups served by a common group of control circuits.
β = ratio of holding times of common-control circuits and trunks.
q = probability of loss.

$q = 0.01$							
$k = 5$				$k = 10$			
m	$\beta = {}^1/_{30}$	$\beta = {}^1/_{40}$	$\beta = {}^1/_{60}$	m	$\beta = {}^1/_{30}$	$\beta = {}^1/_{40}$	$\beta = {}^1/_{60}$
10	4.34	4.38	4.41	10	4.18	4.26	4.33
15	6.95	7.02	7.08	15	6.66	6.81	6.94
20	9.55	9.66	9.74	20	9.14	9.35	9.54

$q = 0.002$							
$k = 5$				$k = 10$			
m	$\beta = {}^1/_{30}$	$\beta = {}^1/_{40}$	$\beta = {}^1/_{60}$	m	$\beta = {}^1/_{30}$	$\beta = {}^1/_{40}$	$\beta = {}^1/_{60}$
10	3.35	3.37	3.39	10	3.24	3.29	3.34
15	5.48	5.53	5.57	15	5.26	5.37	5.47
20	7.61	7.68	7.75	20	7.27	7.43	7.59

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STELLINGEN

I

In een telefoonsysteem, uitgerust met honderddelige draaikiezers zonder nulstand die toegang geven tot tien verkeersrichtingen, zal het gemiddelde aantal afgezochte contacten per gesprek ongeveer tien bedragen.

II

In een telefoonsysteem, ingericht met draaikiezers die geen nulstand bezitten en niet met registers worden bestuurd, zal een bepaald verband moeten bestaan tussen de snelheid van de kiezers en de duur van de kiespauze. Dit verband is minder critisch dan vaak wordt verondersteld.

III

De door Palm gegeven uitdrukking voor de kans op het bezet zijn van een bepaald aantal lijnen uit een groep is niet juist.

(C. Palm, Några följdsatser ur de Erlang'ska formlerna, Tekn. Medd. från Kungl. Telegrafstyrelsen, nr 1—3, 1943).

IV

De methoden waarmede de aantallen kiezers in telefooncentrales worden bepaald, kunnen uitkomsten geven die aanzienlijk verschillen. Meerdere klaarheid in deze methoden ware gewenst.

V

Bij de door Kruithof geconstrueerde verkeersmachine blijkt een verschil te bestaan tussen de werkelijke en de gemeten waarden van de formule van Erlang. Dit verschil kan worden verklaard uit het niet overeenstemmen van de principes van deze verkeersmachine met de veronderstellingen die aan de formule van Erlang ten grondslag liggen.

(J. Kruithof, Rotary traffic machine, Electr. Comm. **23**, p. 192, 1946).

VI

De door Hahn gegeven theorie over de grote verliezen bij kleine bundels is onjuist. Deze verliezen kunnen echter worden verklaard indien de veronderstelling wordt ingevoerd dat een abonné, die bezettoon ontvangt, zijn oproep herhaalt.

(F. H a h n, Eine Theorie der Verluste in Fernsprechanlagen, Veröff. aus dem Geb. der Nachr. technik **8**, p. 91, 1938).

VII

Bij de berekening van de z.g.n. schalm-systemen (*link*-systems) moet grote aandacht worden geschonken aan de overbelastbaarheid.

VIII

Voor het snel wijzigen van de frequentie van een radiozender wordt gebruik gemaakt van een automatische afsteminrichting. Bij een bepaalde uitvoering van deze inrichting worden bij een frequentiewijziging eerst alle afstemelementen naar een nulstand teruggedraaid en daarna in voorwaartse richting naar de nieuwe stand gebracht.

Indien men de elementen direct van de oude stand in de nieuwe brengt zonder de nulstand te passeren, wordt een twee tot drie maal grotere gemiddelde afstemsnelheid verkregen, al naar gelang zeer veel elementen voor de afstemming nodig zijn of slechts één daarvoor wordt gebruikt.

IX

In tegenstelling tot een modulator met niet-lineaire weerstanden, kan bij een modulator met niet-lineaire reactieve elementen het afgegeven vermogen van het gewenste modulatieproduct groter zijn dan het vermogen dat door de generator van het te moduleren signaal wordt geleverd.

X

Bij een modulator met niet-lineaire zelfinducties is de verhouding van het geleverde vermogen van het modulatieproduct tot het vermogen dat door de generator van het te moduleren signaal wordt geleverd, ten naaste bij gelijk aan de verhouding van de frequenties van de genoemde signalen.

XI

Electronische schakelingen in een systeem voor automatische telefonie zijn vooralsnog belangrijker voor de besturings- dan voor de verbindingsorganen.

XII

In vele gevallen is het twijfelachtig of een zeer gering onderhoud van een automatische telefooncentrale een hoge aanschaffingsprijs van de apparatuur rechtvaardigt.