
Weighted L^p -representations of Banach function spaces

A case study in improving theorem searchability
through a corollary generation protocol

by

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Plain Language Summary

To calculate the size of a whole, we often add up the sizes of its parts. However, this is not always possible as bulk discounts in a supermarket show. Buying several products together may cost less than buying those same products separately. Now suppose a supermarket had no discounts or extra charges. Then every basket full of products would cost exactly what those products would cost on their own. To find out how much you would have to pay, you would only have to take the amount of each product, multiply it by its price and add everything together. Mathematicians call this a weighted sum. This thesis starts by exploring an analogous statement for functions. Suppose that whenever positive functions are added, the size of the result is exactly what their separate sizes add up to. Then, just as with the basket, the way sizes are assigned to functions must be a weighted sum as well.

Here this result is called the seed theorem, from which the thesis grows outwards into larger mathematics. Along the way, we find and prove several more general theorems from which it can be recovered. These theorems are written in languages that a reader starting from the seed theorem would not yet know, making them hard to find. However, once a more general theorem is known recovering the seed theorem from it is often very easy.

We then try to make these general theorems easier to find by introducing a way for authors to automatically write out a theorem's simpler consequences and attach them as searchable corollary notes. This would help someone searching for the simpler consequence to find the more general theorem that contains it. The thesis ends with an experiment that treats the recoveries of the seed theorem as a case study and tests how well this protocol works, showing that the protocol could indeed help make general theorems more visible.

Abstract

This thesis studies which conditions on the norm of a Banach function space force a Banach function space to be a weighted L^p -space. The thesis starts out from a result we call the seed theorem: if X is a Banach function space over a σ -finite measure space whose norm satisfies $\|f + g\|_X = \|f\|_X + \|g\|_X$ for all $f, g \in X_+$, then $X = L^1(\Omega, w d\mu)$ isometrically for some weight $w > 0$. From this starting point, the thesis relates the seed theorem to three more general statements: an isometric $L^p(w d\mu)$ -representation under disjoint p -additivity, a theorem about weighted L^p -inequalities controlling a family of maps and finally an $L^p(w d\mu)$ -equivalence for (p, b) -convexity and (p, d) -concavity. Recovering the seed theorem from these general statements exposes an asymmetry: once the general theorem is known deriving the specific case is very easy. Finding the general theorem from the specific case is harder, since the general theorem may be formulated in a different mathematical language.

This thesis introduces a protocol that uses large language models to generate specific cases of general theorem statements and add them as corollary notes. Using the three statements above as general statements and the seed theorem as the specific case, we test experimentally whether the protocol generates corollary notes that recover the seed theorem. In the second iteration, the protocol produces corollary notes that come very close to the seed theorem for two of the three general statements. In a controlled search experiment, adding these corollary notes moves both corresponding general statements into the top ten results for every query about the seed theorem. Without these corollary notes, neither appears in the top ten.

Acknowledgements

I want to thank Emiel Lorist for encouraging me to combine the purely mathematical research with the experimental chapter on improving theorem searchability. The combination was a lot of fun and it taught me more than either part would have done on its own. Apparently, this project was not positively additive.

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Chapter 1

Introduction

In a weighted L^1 -space, the norm is positively additive:

$$\int_{\Omega} (f + g)w \, d\mu = \int_{\Omega} fw \, d\mu + \int_{\Omega} gw \, d\mu \quad (f, g \geq 0).$$

This thesis turns this observation around by asking whether positive additivity is enough to force a Banach function space to be a weighted L^1 -space. This turns out to be true and is formalized in what we call the seed theorem, as this thesis grows outward from it. The seed theorem raises the question whether this phenomenon is part of broader theory, leading us to the guiding problem of this thesis: how can one find surrounding theory when it is written in a different mathematical language or dialect?

After developing some intuition about the seed theorem and proving it, we build out the surrounding theory in steps. We first extend L^1 to L^p -representation under disjoint p -additivity. Then a theorem in which weighted L^p -inequalities control a whole family of maps gives another way to prove the seed theorem, while making us see exact additivity as a combination of two inequalities. These inequalities point us towards letting the exponents in these inequalities move independently. This allows us to explore which exponent region can still be expected to lead to a weighted L^p -equivalence. We end up at (p, b) -convexity and (p, d) -concavity, obtaining an $L^p(w \, d\mu)$ -equivalence. Along the way, we recover the seed theorem at each stage.

Following this path makes it clear that generalizing upward from a small theorem is hard, while reducing downward from a known general theorem is much easier. Moreover, once a general theorem has been found it reveals a bit of the larger language in which the smaller theorem sits. A shortcut to finding these theorems would thus also be a shortcut to these bits of larger languages, which would otherwise be unknown. The thesis asks whether we can create these shortcuts by exploiting the asymmetry of effort between generalizing and reducing.

For this we introduce the corollary generation protocol, in which large language models generate corollary notes for general theorems. In principle, these notes could contain all recoveries of a general theorem at lower levels of generality. Adding them as searchable metadata would then provide the shortcut as the general theorem becomes searchable in the language of the small theorem. Our exploration of weighted L^p -representations leaves us with a natural case study for testing this protocol experimentally and iterating on it.

The chapters follow this path. Chapter 2 introduces the seed language, after which Chapter 3 visualizes and proves the seed theorem itself. Chapter 4 then develops the larger languages from which the seed theorem can be recovered. What we learn there becomes the case study for Chapter 5, where we formulate, test and iterate the corollary generation protocol. Chapter 6 closes this thesis with an outlook.

Chapter 2

The seed language

To understand how the seed theorem reduces from larger theorems in other languages, we first introduce its own.

2.1 Measure theory and Banach spaces

Banach function spaces are, first of all, spaces of measurable functions. For measure theory we follow the Veraar–Lorist lecture notes [1] whenever possible. These notes state the material better than any summary here would, especially as they are very concise (which I really liked and longed for afterwards in later courses). However, a few definitions and theorems are stated, because they either adjust the source statement to our later needs or give a proof pattern for our main proofs. Absolute continuity and the Radon–Nikodym theorem are not in the notes and come from Rudin [2].

Definition 2.1 (The space $L^0(\Omega)$). *Let (Ω, Σ, μ) be a σ -finite measure space. Let $L^0(\Omega)$ be the set of measurable functions $f : \Omega \rightarrow \mathbb{C}$, modulo the equivalence relation of equality μ -a.e. An element of $L^0(\Omega)$ is thus an equivalence class of functions. We also write $L^0(\Omega, \mu)$ when we want to show which measure is used. For real-valued $f, g \in L^0(\Omega)$, we define $f \leq g$ if $f(\omega) \leq g(\omega)$ for μ -a.e. $\omega \in \Omega$. We call this the pointwise order. We write $L^0(\Omega)_+ = \{f \in L^0(\Omega) : f \geq 0\}$.*

Indicator functions are the start of a proof pattern that returns throughout this thesis: first prove a formula for indicator functions, then extend it to simple functions by finite linearity and finally take increasing limits to reach all non-negative measurable functions. The next two definitions and the two theorems after them set up this pattern.

Definition 2.2 (Indicator function). *Let $E \in \Sigma$. The indicator function of E is the function $\mathbf{1}_E : \Omega \rightarrow \mathbb{R}$ given by*

$$\mathbf{1}_E(\omega) = \begin{cases} 1, & \omega \in E, \\ 0, & \omega \notin E. \end{cases}$$

Finite linear combinations of indicator functions are the simple functions.

Definition 2.3 (Simple function). *A function $s : \Omega \rightarrow \mathbb{C}$ is called a simple function if it is measurable and takes only finitely many values. Equivalently, for some $n \in \mathbb{N}$, it can be written as*

$$s = \sum_{k=1}^n a_k \mathbf{1}_{E_k},$$

where $a_1, \dots, a_n \in \mathbb{C}$ and $E_1, \dots, E_n \in \Sigma$. When a simple function is written as a finite sum over pairwise disjoint measurable sets, we omit terms with zero coefficient, so that we can divide by the coefficients.

With these simple functions, we can approximate every non-negative measurable function from below.

Theorem 2.4 (Approximation by simple functions). *Let $f : \Omega \rightarrow [0, \infty]$ be measurable. Then there exists a sequence of simple functions $(f_n)_{n \geq 1}$ such that $0 \leq f_n \uparrow f$ pointwise.*

The real-analysis-scarred undergraduate reader is in luck. Of the convergence theorem package from the notes, our proofs only use the Monotone Convergence Theorem.

Theorem 2.5 (Monotone Convergence Theorem). *Let $f_n : \Omega \rightarrow [0, \infty]$ be measurable for $n \geq 1$ and suppose that*

$$0 \leq f_n \uparrow f \quad \text{pointwise.}$$

Then f is measurable and

$$\lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu = \int_{\Omega} f d\mu.$$

We state the L^p -definition, because spaces being equal to a weighted L^p -space is central in this thesis. A weight w gives a set E the new size $\int_E w d\mu$ instead of $\mu(E)$.

Definition 2.6 (L^p -spaces and weighted L^p -spaces). For $p \in [1, \infty)$, define

$$L^p(\Omega) := \left\{ f \in L^0(\Omega) : \int_{\Omega} |f|^p d\mu < \infty \right\}, \quad \|f\|_{L^p(\Omega)} := \left(\int_{\Omega} |f|^p d\mu \right)^{1/p}.$$

A weight on Ω is a measurable function $w : \Omega \rightarrow [0, \infty)$. The measure

$$E \mapsto \int_E w d\mu$$

is denoted by $w d\mu$, and, for a weight with $w > 0$ μ -a.e., the weighted L^p -space is the L^p -space of this measure:

$$L^p(\Omega, w d\mu) := \left\{ f \in L^0(\Omega) : \int_{\Omega} |f|^p w d\mu < \infty \right\}, \quad \|f\|_{L^p(\Omega, w d\mu)} = \left(\int_{\Omega} |f|^p w d\mu \right)^{1/p}.$$

Suppose that ν is a measure on the same measurable space (Ω, Σ) . We want to know whether it can be represented by integrating a density with respect to μ , i.e. whether $\nu(E) = \int_E w d\mu$ for some weight w . A necessary condition is that when $\mu(E) = 0$, then also $\nu(E) = 0$. This condition is called absolute continuity.

Definition 2.7 (Absolute continuity of measures). Let μ and ν be measures on (Ω, Σ) . We say that ν is absolutely continuous with respect to μ , and write

$$\nu \ll \mu,$$

if

$$\mu(E) = 0 \implies \nu(E) = 0 \quad (E \in \Sigma).$$

The Radon–Nikodym theorem turns a measure into a weighted integral. We will use this in the proof of the seed theorem to generate the weight.

Theorem 2.8 (Radon–Nikodym). Let μ and ν be σ -finite measures on (Ω, Σ) . If $\nu \ll \mu$, then there exists a measurable function

$$w : \Omega \rightarrow [0, \infty)$$

such that

$$\nu(E) = \int_E w d\mu \quad (E \in \Sigma).$$

The function w is unique up to equality μ -a.e. and is called the Radon–Nikodym derivative of ν with respect to μ . We write

$$w = \frac{d\nu}{d\mu}.$$

The following example shows where the Radon–Nikodym weight comes from.

Example 2.9 (Visualizing Radon–Nikodym). Suppose that A_1, A_2, A_3 are pairwise disjoint sets with $\Omega = A_1 \cup A_2 \cup A_3$, and take $\Sigma = \sigma(A_1, A_2, A_3)$. Suppose that we have two measures μ and ν , with $0 < \mu(A_k) < \infty$ and $\nu(A_k) < \infty$ for each k . If we want ν to be written as weighted μ -mass on each subset, then that weight is forced by

$$\nu(A_k) = w_k \mu(A_k).$$

Thus $w_k = \nu(A_k)/\mu(A_k)$: the density is the amount of ν -mass per unit of μ -mass.

If we define

$$w = \sum_{k=1}^3 w_k \mathbf{1}_{A_k},$$

then for every set E that is a union of these subsets,

$$\int_E w d\mu = \sum_{A_k \subseteq E} w_k \mu(A_k) = \sum_{A_k \subseteq E} \nu(A_k) = \nu(E).$$

	A_1	A_2	A_3
$\mu(A_k)$	2	3	2
$\nu(A_k)$	6	3	4
w_k	3	1	2

Table 2.1: Finite-subset intuition for the Radon–Nikodym density. On each subset A_k , the density is the ratio of $\nu(A_k)$ to $\mu(A_k)$.

The same ratios can be drawn as a step-function density. In Figure 2.1, each block A_k has horizontal width $\mu(A_k)$, while the vertical height is the density value w_k . The area over A_k is therefore

$$w_k \mu(A_k) = \nu(A_k).$$

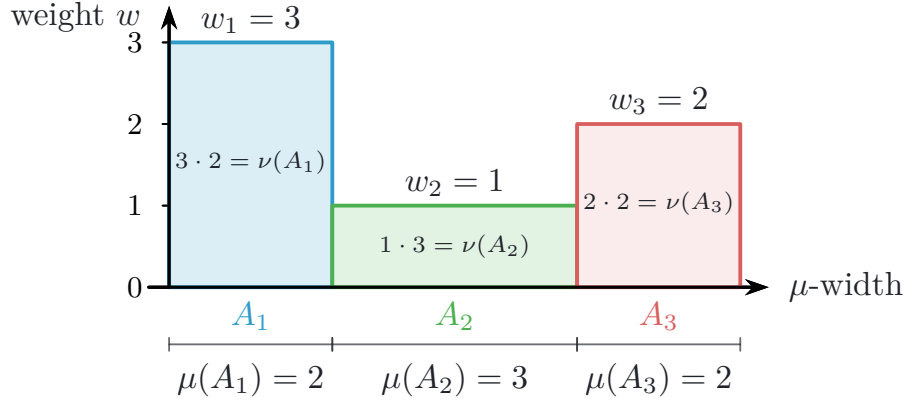


Figure 2.1: The subsetwise ratios drawn as a density. Each subset A_k is drawn with width $\mu(A_k)$, and the height is $w_k = \nu(A_k)/\mu(A_k)$. The area over A_k shows the corresponding $\nu(A_k)$.

We end with the definition of a Banach space.

Definition 2.10 (Cauchy sequences and Banach spaces). *Let $(X, \|\cdot\|_X)$ be a normed vector space. A sequence $(x_n)_{n \geq 1}$ in X converges to $x \in X$ if $\|x_n - x\|_X \rightarrow 0$, and it is a Cauchy sequence if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ with $\|x_n - x_m\|_X < \varepsilon$ for all $n, m \geq N$. The space X is called complete if every Cauchy sequence in X converges to an element of X . A complete normed vector space is called a Banach space.*

2.2 Banach function spaces

A Banach function space is a Banach space inside $L^0(\Omega)$. However, the additional properties one wants that space to have differ per definition. We follow the paper *Banach Function Spaces Done Right*, in which Lorst–Nieraeth show that the older Luxemburg–Zaanen definition includes more spaces without losing basic properties [3]. Their convention is the language this thesis works in. This section states what we use from the paper, and shows why this convention suits this thesis.

Definition 2.11 (Banach function space). *Let $X \subseteq L^0(\Omega)$ be a Banach space with norm $\|\cdot\|_X$. The space X is called a Banach function space over (Ω, μ) if it satisfies the following properties:*

(i) Ideal property: if $f \in X$ and $g \in L^0(\Omega)$ with

$$|g| \leq |f|,$$

then $g \in X$ and

$$\|g\|_X \leq \|f\|_X.$$

(ii) Saturation property: for every measurable $E \subseteq \Omega$ of positive measure, there exists a measurable $F \subseteq E$ of positive measure with

$$\mathbf{1}_F \in X.$$

A definition shows only what it asks, not what it does not.

Definition 2.12 (Fatou property). Let X be a Banach function space over (Ω, μ) . We say that X has the Fatou property if, whenever

$$0 \leq f_n \uparrow f \quad \text{pointwise } \mu\text{-a.e.}, \quad f_n \in X, \quad \sup_n \|f_n\|_X < \infty,$$

one has $f \in X$ and

$$\|f\|_X = \sup_n \|f_n\|_X.$$

Besides the ideal property, the Bennett–Sharpley convention requires the Fatou property and a stricter form of saturation: for every measurable $E \subseteq \Omega$ with $\mu(E) < \infty$, the indicator $\mathbf{1}_E$ must be in X . They also require every $f \in X$ to be integrable over such E [3, Definition 1.1]. The next example gives a weighted L^1 -space that satisfies saturation, but fails the Bennett–Sharpley indicator condition.

Example 2.13 (A weighted L^1 -space that Bennett–Sharpley excludes as a Banach function space). Suppose that we take $\Omega = (0, 1)$ with the Lebesgue measure and the weight $w(x) = \frac{1}{x}$:

$$X = L^1((0, 1), \frac{1}{x} dx), \quad \|f\|_X = \int_0^1 |f(x)| \frac{1}{x} dx.$$

The Bennett–Sharpley convention requires $\mathbf{1}_E \in X$ for every set E of finite measure, but for $E = (0, \frac{1}{2})$,

$$\|\mathbf{1}_E\|_X = \int_0^1 \mathbf{1}_E(x) \frac{1}{x} dx = \int_0^{1/2} \frac{1}{x} dx = \infty,$$

so $\mathbf{1}_E \notin X$, and thus X would not be a Banach function space. The saturation property in our convention only asks for a positive-measure subset of E whose indicator is in X . So we can always cut away the blow-up near 0: taking the subset $F = (\frac{1}{8}, \frac{1}{2}) \subseteq E$ gives

$$\|\mathbf{1}_F\|_X = \int_0^1 \mathbf{1}_F(x) \frac{1}{x} dx = \int_{1/8}^{1/2} \frac{1}{x} dx = \log 4 < \infty,$$

so $\mathbf{1}_F \in X$. We can take a subset in the same way for every positive-measure set $A \subseteq (0, 1)$: there is a $\delta > 0$ for which the subset $A \cap (\delta, 1) \subseteq A$ still has positive measure, and on it the weight is at most $\frac{1}{\delta}$, so $\mathbf{1}_{A \cap (\delta, 1)} \in X$. Thus X satisfies saturation.

One would hope that all weighted L^p -spaces with weight $0 < w \in L^0(\Omega)$ are Banach function spaces. The next proposition shows that in our convention they are, while Example 2.13 provides a counterexample for the Bennett–Sharpley convention.

Proposition 2.14 (Weighted L^p -spaces are Banach function spaces). Let (Ω, Σ, μ) be a σ -finite measure space, let $p \in [1, \infty)$ and let w be a weight with $w > 0$ μ -a.e. Then

$$L^p(\Omega, w d\mu)$$

is a Banach function space over (Ω, μ) .

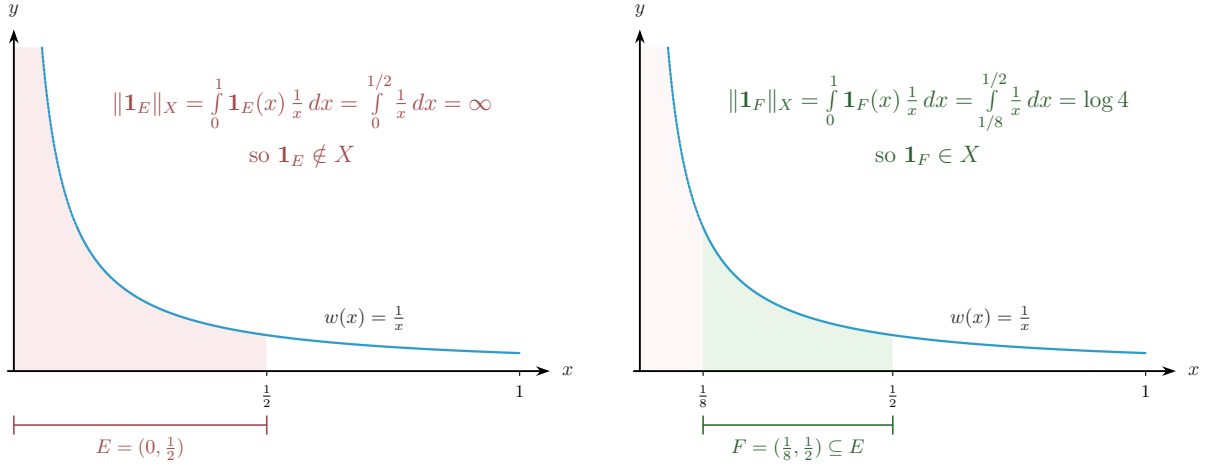


Figure 2.2: The weight $w(x) = \frac{1}{x}$ on $(0, 1)$. In the left panel the red area is $\|1_E\|_X$, infinite because of the blow-up near 0. In the right panel the green area is $\|1_F\|_X = \log 4$. The cut at $\frac{1}{8}$ removes the blow-up.

Remark 2.15. This is [3, Proposition 3.17] applied to $L^p(\Omega)$ with the weight $w^{1/p}$, because $\|f\|_{L^p(\Omega, w d\mu)} = \|fw^{1/p}\|_{L^p(\Omega)}$. The paper proves its results for quasi-Banach function spaces. The later results we take from it are stated in the Banach case. We do not restate the proofs, as the specialization is direct.

This makes the Lorist–Nieraeth convention suit this thesis well. We will show that we also do not need the Fatou property in the definition. For that we first need some tools.

f and $|f|$ belong to X together and have the same norm. It follows from the ideal property, but because we use it so often we state it as a lemma.

Lemma 2.16 (f and $|f|$). *Let X be a Banach function space over (Ω, μ) . For $f \in L^0(\Omega)$ one has*

$$f \in X \iff |f| \in X,$$

and in this case

$$\| |f| \|_X = \|f\|_X.$$

The next lemma says that a norm limit and a pointwise μ -a.e. limit of the same sequence are equal μ -a.e. We will use it often when approximating functions from below [3, Proposition 2.3].

Lemma 2.17 (Equality of norm limits and pointwise μ -a.e. limits). *Let X be a Banach function space over (Ω, μ) . If*

$$f_n \rightarrow f \quad \text{in } X \quad \text{and} \quad f_n \rightarrow g \quad \text{pointwise } \mu\text{-a.e.},$$

then $f = g$ μ -a.e.

The definition of saturation is not always the easiest form to use in proofs. The next theorem gives two equivalent formulations [3, Proposition 2.5].

Theorem 2.18 (Equivalent formulations of saturation). *Let (Ω, μ) be a σ -finite measure space and let $X \subseteq L^0(\Omega)$ be a Banach space satisfying the ideal property. Then the following are equivalent:*

- (i) X satisfies the saturation property;
- (ii) there is an increasing sequence of measurable sets $(F_n)_{n \geq 1}$ with

$$1_{F_n} \in X \quad (n \geq 1), \quad \bigcup_{n=1}^{\infty} F_n = \Omega;$$

- (iii) X has a weak order unit, i.e. there is a $u \in X$ with

$$u > 0 \quad \mu\text{-a.e.}$$

We now show that a positive additive Banach function space in our convention has the Fatou property. Here $X_+ := X \cap L^0(\Omega)_+$ stands for the non-negative elements of X .

Lemma 2.19 (Positive additivity gives the Fatou property). *Let X be a Banach function space over (Ω, μ) . Suppose that*

$$\|f + g\|_X = \|f\|_X + \|g\|_X \quad (f, g \in X_+).$$

Then X has the Fatou property.

Proof. Suppose that $f \in L^0(\Omega)_+$ and

$$0 \leq f_n \uparrow f \quad \text{pointwise } \mu\text{-a.e.}, \quad f_n \in X, \quad \sup_n \|f_n\|_X < \infty.$$

For $m \geq n$, $f_m - f_n \geq 0$, and since X is a vector space, $f_m - f_n \in X_+$. Thus

$$f_m = f_n + (f_m - f_n), \quad f_n \in X_+, \quad f_m - f_n \in X_+.$$

By positive additivity,

$$\|f_m\|_X = \|f_n\|_X + \|f_m - f_n\|_X, \quad \text{so} \quad \|f_m - f_n\|_X = \|f_m\|_X - \|f_n\|_X.$$

Since $(\|f_n\|_X)_{n \geq 1}$ is increasing and $\sup_n \|f_n\|_X < \infty$ by assumption, it converges, so

$$|\|f_m\|_X - \|f_n\|_X| \rightarrow 0 \quad (m, n \rightarrow \infty).$$

For the X -distance, take $m \geq n$. The case $n \geq m$ is the same by symmetry. Then $\|f_m\|_X - \|f_n\|_X \geq 0$, and

$$\|f_m - f_n\|_X = \|f_m\|_X - \|f_n\|_X = |\|f_m\|_X - \|f_n\|_X|.$$

Therefore $\|f_m - f_n\|_X \rightarrow 0$ as $m, n \rightarrow \infty$, and $(f_n)_{n \geq 1}$ is Cauchy in X . Since X is complete, there is a $g \in X$ such that $f_n \rightarrow g$ in X . By Lemma 2.17,

$$g = f \quad \mu\text{-a.e.}$$

Thus $f \in X$, and

$$|\|f_n\|_X - \|f\|_X| \leq \|f_n - f\|_X \rightarrow 0, \quad \text{so} \quad \|f\|_X = \lim_{n \rightarrow \infty} \|f_n\|_X = \sup_n \|f_n\|_X.$$

□

So for a positive additive space it does not even matter whether the Fatou property is part of the definition. We will also need it. For example, when proving that a positive additive Banach function space is a weighted L^1 -space, we will first show that $X \subseteq L^1(\Omega, w d\mu)$ and that the norms are equal. The next proposition [3, Proposition 3.10] shows that the Fatou property then forces equality.

Proposition 2.20 (Isometric inclusions become equalities under the Fatou property). *Let X and Y be Banach function spaces over the same measure space (Ω, μ) . Suppose that*

$$X \subseteq Y$$

and that

$$\|f\|_X = \|f\|_Y \quad (f \in X).$$

If X has the Fatou property, then

$$X = Y.$$

The Fatou property is about increasing limits. What about decreasing limits?

Definition 2.21 (Order continuity). *Let X be a Banach function space over (Ω, μ) . We say that X is order-continuous if, for sequences $(f_n)_{n \geq 1}$ in X with*

$$f_n \downarrow 0 \quad \text{pointwise } \mu\text{-a.e.},$$

we have

$$\|f_n\|_X \downarrow 0.$$

Positive additivity also forces order continuity.

Lemma 2.22 (Positive additivity gives order continuity). *Let X be a Banach function space over (Ω, μ) . Suppose that*

$$\|f + g\|_X = \|f\|_X + \|g\|_X \quad (f, g \in X_+).$$

Then X is order-continuous.

Proof. Suppose that $f_n \in X$ and $f_n \downarrow 0$ pointwise μ -a.e. Then $0 \leq f_1 - f_n \uparrow f_1$, and since X is a vector space, $f_1 - f_n \in X_+$. By positive additivity,

$$\|f_n\|_X = \|f_1\|_X - \|f_1 - f_n\|_X.$$

The ideal property gives that the norms $\|f_1 - f_n\|_X$ increase and are bounded by $\|f_1\|_X$. By Lemma 2.19, X has the Fatou property, so $\|f_1 - f_n\|_X \uparrow \|f_1\|_X$. Thus $\|f_n\|_X \downarrow 0$. $\square$

The previously mentioned real-analysis-scarred reader probably has a fishy feeling by now. And they would be right too. The Fatou property and order continuity quietly contain the whole convergence theorem package, but now for the X -norm instead of for integrals, while positive additivity gives both. Monotone convergence returns as the Fatou property, and dominated convergence returns as order continuity. Fatou's lemma follows by applying the Fatou property to $g_n = \inf_{k \geq n} f_k$.

Chapter 3

The seed theorem

Now that we know a few words and sentences in the Banach function space language, we can state the seed theorem in it.

Theorem 3.1 (A positive additive Banach function space is a weighted L^1 -space). *Let X be a Banach function space over a σ -finite measure space (Ω, Σ, μ) . If*

$$\|f + g\|_X = \|f\|_X + \|g\|_X \quad (f, g \in X_+),$$

then there exists a weight $w > 0$ μ -a.e. such that $X = L^1(\Omega, w d\mu)$ with

$$\|f\|_X = \int_{\Omega} |f| w d\mu \quad (f \in X).$$

The conclusion is an *isometric identification*. The two spaces are the same set inside $L^0(\Omega)$, with the same norm. This is stronger than the isomorphism we encounter later, which only maps X onto an L^p -space over another measure space, with equivalent instead of equal norms.

Before proving the theorem, we build some intuition for what it says.

3.1 Visualizing the seed theorem

The L^1 -norm is additive on non-negative functions because the integral is linear. If $f, g \geq 0$, then

$$\|f + g\|_{L^1} = \int_{\Omega} (f + g) d\mu = \int_{\Omega} f d\mu + \int_{\Omega} g d\mu = \|f\|_{L^1} + \|g\|_{L^1}.$$

The only extra thing we know about $\|\cdot\|_X$ is that it is positively additive. The theorem says that this already forces it to be a weighted L^1 -norm. We will visualize the equality in an example with three sets.

Example 3.2. Suppose that we take $\Omega = [0, 7]$ and split it into the pairwise disjoint pieces

$$A_1 = [0, 2), \quad A_2 = [2, 5), \quad A_3 = [5, 7].$$

Set $\Sigma = \sigma(\{A_1, A_2, A_3\})$, and let μ be the measure on (Ω, Σ) determined by

$$\mu(A_1) = 2, \quad \mu(A_2) = 3, \quad \mu(A_3) = 2.$$

Every measurable function is then of the form

$$u = \alpha \mathbf{1}_{A_1} + \beta \mathbf{1}_{A_2} + \gamma \mathbf{1}_{A_3}.$$

Let X be the space of all such u , with norm

$$\|u\|_X = 6|\alpha| + 3|\beta| + 4|\gamma|.$$

This norm makes X a Banach function space. If $u, v \in X_+$, all coefficients are non-negative and

$$\|u + v\|_X = \|u\|_X + \|v\|_X.$$

The norms on indicator sets are

$$\|\mathbf{1}_{A_1}\|_X = 6, \quad \|\mathbf{1}_{A_2}\|_X = 3, \quad \|\mathbf{1}_{A_3}\|_X = 4.$$

We define a set function on Σ by $\nu(E) = \|\mathbf{1}_E\|_X$. For the three pieces this gives

$$\nu(A_1) = \|\mathbf{1}_{A_1}\|_X, \quad \nu(A_2) = \|\mathbf{1}_{A_2}\|_X, \quad \nu(A_3) = \|\mathbf{1}_{A_3}\|_X.$$

Positive additivity also makes ν additive. For instance,

$$\nu(A_1 \cup A_3) = \|\mathbf{1}_{A_1 \cup A_3}\|_X = \|\mathbf{1}_{A_1} + \mathbf{1}_{A_3}\|_X = \nu(A_1) + \nu(A_3).$$

As in the Radon–Nikodym example from the last chapter, the weight on each block is the ratio of the new set value to the original mass:

$$w_1 = \frac{\nu(A_1)}{\mu(A_1)}, \quad w_2 = \frac{\nu(A_2)}{\mu(A_2)}, \quad w_3 = \frac{\nu(A_3)}{\mu(A_3)}.$$

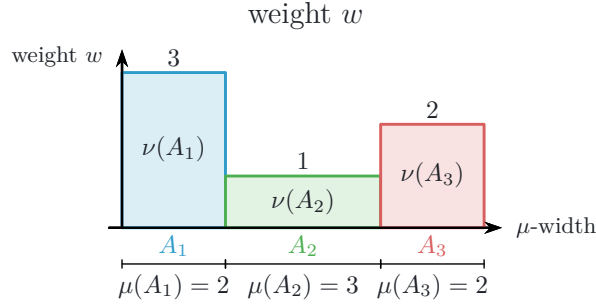


Figure 3.1: The forced weight w , drawn per block.

Take the step function

$$s_0 = \frac{1}{2} \mathbf{1}_{A_1} + 2 \mathbf{1}_{A_2} + \frac{3}{2} \mathbf{1}_{A_3}.$$

We now visualize the norm equality for this s_0 . First we show the weighted integrand by multiplying s_0 with w in Figure 3.2. Then we show the X -norm in Figure 3.3 and use $\nu(A_i) = w_i \mu(A_i)$, for $i = 1, 2, 3$, to cut it into the rectangles that appear in the graph of $s_0 w$.

$$s_0 \cdot w = s_0 w$$

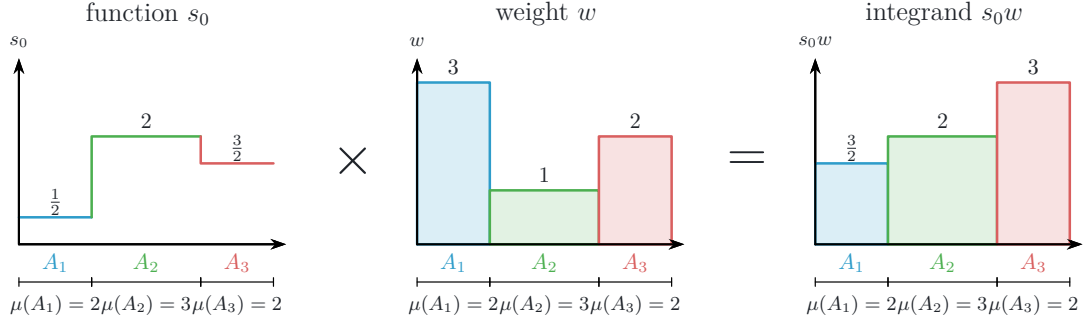


Figure 3.2: The weight scales the heights of s_0 into the integrand $s_0 w$, whose area is $\int s_0 w d\mu$.

$$\|s_0\|_X = \int s_0 w d\mu$$

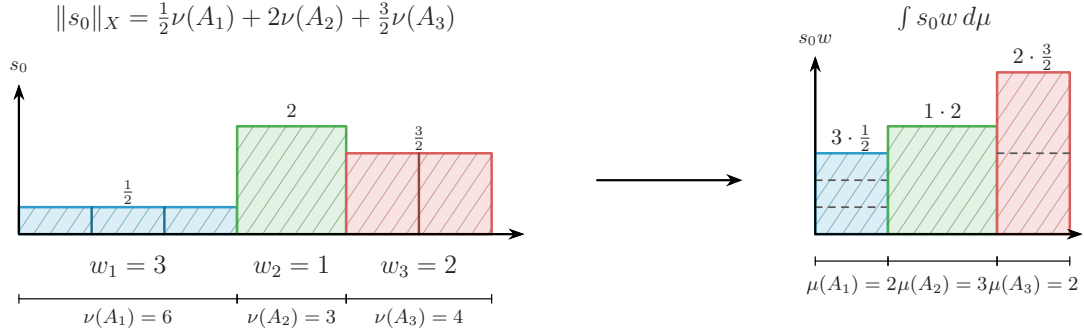


Figure 3.3: On the left, the area is $\frac{1}{2}\nu(A_1) + 2\nu(A_2) + \frac{3}{2}\nu(A_3) = \|s_0\|_X$. For $i = 1, 2, 3$, the relation $\nu(A_i) = w_i \mu(A_i)$ cuts the ν -width into w_i μ -widths. Stacking those pieces gives the graph of $s_0 w$ on the right.

We can also show the equality for a general non-negative step function algebraically. Let

$$s = a_1 \mathbf{1}_{A_1} + a_2 \mathbf{1}_{A_2} + a_3 \mathbf{1}_{A_3} \quad (a_1, a_2, a_3 \geq 0).$$

Homogeneity and additivity on non-negative functions give

$$\begin{aligned} \|s\|_X &= a_1 \|\mathbf{1}_{A_1}\|_X + a_2 \|\mathbf{1}_{A_2}\|_X + a_3 \|\mathbf{1}_{A_3}\|_X \\ &= a_1 \nu(A_1) + a_2 \nu(A_2) + a_3 \nu(A_3). \end{aligned}$$

Integrating the step function sw gives

$$\begin{aligned} \int_{\Omega} sw d\mu &= a_1 w_1 \mu(A_1) + a_2 w_2 \mu(A_2) + a_3 w_3 \mu(A_3) \\ &= a_1 \nu(A_1) + a_2 \nu(A_2) + a_3 \nu(A_3). \end{aligned}$$

Thus

$$\|s\|_X = \int_{\Omega} sw d\mu$$

for every non-negative measurable function on this finite σ -algebra. For a general function

$$u = \alpha \mathbf{1}_{A_1} + \beta \mathbf{1}_{A_2} + \gamma \mathbf{1}_{A_3},$$

by Lemma 2.16, $\|u\|_X = \| |u| \|_X$. Since

$$|u| = |\alpha| \mathbf{1}_{A_1} + |\beta| \mathbf{1}_{A_2} + |\gamma| \mathbf{1}_{A_3}$$

is non-negative, the previous calculation gives

$$\|u\|_X = \int_{\Omega} |u| w d\mu.$$

3.2 Proof of the seed theorem

To prove the seed theorem, we follow the steps of the example, now for a general X .

Proof of Theorem 3.1. Suppose that (Ω, Σ, μ) is a σ -finite measure space, that X is a Banach function space over (Ω, μ) and that

$$\|f + g\|_X = \|f\|_X + \|g\|_X \quad (f, g \in X_+).$$

Step 1: Building the norm into a measure. Define a set function $\nu : \Sigma \rightarrow [0, \infty]$ by

$$\nu(E) = \begin{cases} \|\mathbf{1}_E\|_X, & \mathbf{1}_E \in X, \\ +\infty, & \mathbf{1}_E \notin X. \end{cases}$$

Checking that ν is a measure. We first prove that ν is monotone. If $E \subseteq F$ and $\nu(F) = \infty$, then $\nu(E) \leq \nu(F)$. If $\nu(F) < \infty$, then $\mathbf{1}_F \in X$, and

$$0 \leq \mathbf{1}_E \leq \mathbf{1}_F \implies \mathbf{1}_E \in X, \quad \|\mathbf{1}_E\|_X \leq \|\mathbf{1}_F\|_X \implies \nu(E) \leq \nu(F).$$

- (1) We have $\nu(\emptyset) = 0$, since $\emptyset \in \Sigma$ and $\|\mathbf{1}_\emptyset\|_X = 0$.
- (2) ν is non-negative, since it is defined by a non-negative norm or by the value $+\infty$.
- (3) We prove countable additivity, beginning with finite additivity. Let $E, F \in \Sigma$ be disjoint. If $\nu(E) + \nu(F) < \infty$, then $\mathbf{1}_E, \mathbf{1}_F \in X_+$ and $\mathbf{1}_{E \cup F} = \mathbf{1}_E + \mathbf{1}_F$. By positive additivity,

$$\nu(E \cup F) = \|\mathbf{1}_{E \cup F}\|_X = \|\mathbf{1}_E + \mathbf{1}_F\|_X = \nu(E) + \nu(F).$$

If $\nu(E) + \nu(F) = \infty$, then at least one of $\nu(E)$ and $\nu(F)$ is infinite. Since both E and F are contained in $E \cup F$, monotonicity gives $\nu(E \cup F) = \infty$. Thus finite additivity holds.

Now let $(E_j)_{j \geq 1}$ be pairwise disjoint sets in Σ , and put $G_N := \bigcup_{j=1}^N E_j$ and $E := \bigcup_{j=1}^\infty E_j$. By finite additivity, $\nu(G_N) = \sum_{j=1}^N \nu(E_j)$. Since $G_N \subseteq E$, monotonicity gives

$$\sum_{j=1}^\infty \nu(E_j) = \sup_N \sum_{j=1}^N \nu(E_j) = \sup_N \nu(G_N) \leq \nu(E).$$

If $\sup_N \nu(G_N) = \infty$, then this forces $\nu(E) = \infty$. So suppose that $\sup_N \nu(G_N) < \infty$. Then $\nu(G_N) < \infty$ for all N , thus $\mathbf{1}_{G_N} \in X$, and

$$\sup_N \|\mathbf{1}_{G_N}\|_X = \sup_N \nu(G_N) < \infty.$$

Since $0 \leq \mathbf{1}_{G_N} \uparrow \mathbf{1}_E$ μ -a.e., Lemma 2.19 gives $\mathbf{1}_E \in X$ and

$$\nu(E) = \|\mathbf{1}_E\|_X = \sup_N \|\mathbf{1}_{G_N}\|_X = \sum_{j=1}^\infty \nu(E_j).$$

Thus ν is countably additive.

As (1)–(3) hold, ν is a measure on (Ω, Σ) .

Step 2: Applying Radon–Nikodym. We check that ν is σ -finite and that $\nu \ll \mu$.

- (1) By Theorem 2.18, choose measurable sets $F_n \uparrow \Omega$ such that $\mathbf{1}_{F_n} \in X$ for $n \geq 1$. Then $\nu(F_n) = \|\mathbf{1}_{F_n}\|_X < \infty$, so ν is σ -finite.
- (2) Suppose that $\mu(E) = 0$. Then $\mathbf{1}_E = 0$ in $L^0(\Omega)$, so $\mathbf{1}_E \in X$, $\nu(E) = 0$ and thus $\nu \ll \mu$.

Since μ is σ -finite by assumption, and we have shown that ν is σ -finite and $\nu \ll \mu$, we can apply the Radon–Nikodym theorem, Theorem 2.8, to obtain a measurable function $w : \Omega \rightarrow [0, \infty)$ such that

$$\nu(E) = \int_E w \, d\mu \quad (E \in \Sigma).$$

$$\|\mathbf{1}_E\|_X = \int_E w \, d\mu \quad (\mathbf{1}_E \in X).$$

Also $w > 0$ μ -a.e. Indeed, if $\mu(\{w = 0\}) > 0$, then saturation gives a measurable $G \subseteq \{w = 0\}$ with $\mu(G) > 0$ and $\mathbf{1}_G \in X$. But then

$$0 < \|\mathbf{1}_G\|_X = \nu(G) = \int_G w \, d\mu = 0,$$

which is impossible.

Step 3: The norm identity by approximation. We now prove the norm identity for arbitrary functions in X .

3.1. From indicator functions to non-negative simple functions in X . Let $s \in X$ be a non-negative simple function. If $s = 0$, both sides are zero. Otherwise, by the simple-function convention, write

$$s = \sum_{k=1}^r a_k \mathbf{1}_{E_k}, \quad a_k > 0,$$

where the E_k 's are pairwise disjoint. For every k we have

$$0 \leq \mathbf{1}_{E_k} \leq a_k^{-1} s \implies \mathbf{1}_{E_k} \in X$$

by the ideal property. By positive additivity, induction and the Radon–Nikodym formula for ν ,

$$\|s\|_X = \sum_{k=1}^r a_k \|\mathbf{1}_{E_k}\|_X = \sum_{k=1}^r a_k \nu(E_k) = \sum_{k=1}^r a_k \int_{E_k} w \, d\mu = \sum_{k=1}^r \int_{E_k} a_k w \, d\mu = \int_{\bigcup_{k=1}^r E_k} s w \, d\mu = \int_{\Omega} s w \, d\mu.$$

Here we use that the E_k 's are pairwise disjoint, that $s = a_k$ on E_k and that $s = 0$ outside $\bigcup_{k=1}^r E_k$.

3.2. From non-negative simple functions to non-negative elements of X . Let $f \in X_+$. Choose non-negative simple functions $(s_j)_{j \geq 1}$ by Theorem 2.4 such that $0 \leq s_j \uparrow f$ μ -a.e. and $s_j \leq f$. Then by the ideal property all the simple functions s_j are also in X_+ , and by Step 3.1,

$$\|s_j\|_X = \int_{\Omega} s_j w \, d\mu.$$

Since $0 \leq s_j \leq f$, the ideal property gives $\|s_j\|_X \leq \|f\|_X$, and therefore $\sup_j \|s_j\|_X < \infty$. Since $w > 0$, we have $0 \leq s_j w \uparrow f w$ μ -a.e. The monotone convergence theorem, Theorem 2.5, and Lemma 2.19 give

$$\begin{aligned} \|f\|_X &= \sup_j \|s_j\|_X = \sup_j \int_{\Omega} s_j w \, d\mu \\ &= \lim_{j \rightarrow \infty} \int_{\Omega} s_j w \, d\mu = \int_{\Omega} f w \, d\mu = \|f\|_{L^1(\Omega, w \, d\mu)}. \end{aligned}$$

3.3. From non-negative elements to arbitrary elements of X . Let $f \in X$. Then $|f| \geq 0$, and by Lemma 2.16 also $|f| \in X$ and $\|f\|_X = \||f|\|_X$. Applying Step 3.2 to $|f|$ gives

$$\|f\|_X = \||f|\|_X = \int_{\Omega} |f| w \, d\mu.$$

Since $f \in X$, we have $\|f\|_X < \infty$, so $\int_{\Omega} |f| w \, d\mu < \infty$, and therefore $f \in L^1(\Omega, w \, d\mu)$. This proves $X \subseteq L^1(\Omega, w \, d\mu)$ with equality of norms on X .

Step 4: Reverse inclusion. Since $w > 0$ μ -a.e., the measures μ and $w \, d\mu$ have the same null sets. Therefore Proposition 2.14 shows that $L^1(\Omega, w \, d\mu)$ is a Banach function space over (Ω, μ) . By Step 3, $X \subseteq L^1(\Omega, w \, d\mu)$ with equal norms on X , and by Lemma 2.19, X has the Fatou property. Applying Proposition 2.20 with $Y = L^1(\Omega, w \, d\mu)$, we conclude

$$X = L^1(\Omega, w \, d\mu),$$

with equality of norms. □

Looking outward from the seed theorem

As we have now uncovered the seed theorem in its native language, curiosity gives rise to the question whether this phenomenon is part of a broader theory. For now, the seed theorem is an illuminated shore in a larger sea of fog.



Figure 3.4: Caspar David Friedrich, *Der Mönch am Meer*, 1808–1810. Oil on canvas. Alte Nationalgalerie, Staatliche Museen zu Berlin, Berlin.

Chapter 4

The seed theorem recovered from larger languages

We first broaden the seed theorem in our own language. Whenever there is a constant in a theorem, no knowledge of further theory or great scientific creativity is needed to ask the question: “What if it becomes a variable?” Thus we can ask what conditions are needed to get an isometry with weighted L^p -spaces. Afterwards, we split the additivity assumption into an upper and a lower inequality, allowing us to use a theorem from a different dialect. Loosening these inequalities then leads us to a more general representation theorem through a theorem in Banach-lattice language. Along the way we recover the seed theorem from each more general theorem.

4.1 $L^p(w d\mu)$ -isometry under disjoint p -additivity

For $p > 1$ the L^p -norm is not positively additive. For $f, g \geq 0$ and $w > 0$ μ -a.e.,

$$\int_{\Omega} (f + g)^p w d\mu \geq \int_{\Omega} f^p w d\mu + \int_{\Omega} g^p w d\mu,$$

with equality only when $fg = 0$ μ -a.e. Therefore in this section we try to prove an isometry between X and a weighted L^p -space under the assumption of disjoint p -additivity. For $p = 1$, this would mean that the seed theorem is true under a weaker assumption.

Definition 4.1 (Disjoint p -additivity). *Let X be a Banach function space over (Ω, μ) and let $1 \leq p < \infty$. We say that X is disjointly p -additive if, for every finite pairwise disjoint family $f_1, \dots, f_n \in X_+$,*

$$\left\| \sum_{k=1}^n f_k \right\|_X^p = \sum_{k=1}^n \|f_k\|_X^p.$$

Here pairwise disjoint means that

$$f_i f_j = 0 \quad \mu\text{-a.e.} \quad (i \neq j).$$

For a disjointly p -additive Banach function space X over a σ -finite measure space, proving the isometric identification with a weighted L^p -space,

$$X = L^p(\Omega, w d\mu)$$

as sets with equal norms is very similar to the proof of Theorem 3.1. But the pairwise disjoint requirement causes one new problem.

In the seed case, the Fatou property was given by Lemma 2.19, whose proof uses positive additivity on f_n and $f_m - f_n$. In the disjoint p -additive case these two functions are not necessarily disjoint.

Lemma 4.2 (Splitting a simple function). *Let $1 \leq p < \infty$, and let X be disjointly p -additive. If $0 \leq f \leq g$ are simple functions in X , then*

$$\|f\|_X^p + \|g - f\|_X^p \leq \|g\|_X^p.$$

Proof. If $g = 0$, then also $f = 0$, and both sides are 0. Thus we may assume that $g \neq 0$. We split g into f and $g - f$. The two pieces are not necessarily disjoint, so we first put f and g on one common finite partition. After discarding the pieces on which $g = 0$, choose pairwise disjoint measurable sets $B_1, \dots, B_m$ on which both f and g are constant, and write

$$g = \sum_{k=1}^m d_k \mathbf{1}_{B_k}, \quad d_k > 0.$$

Let c_k be the value of f on B_k . Since $0 \leq f \leq g$, we have $0 \leq c_k \leq d_k$. Also $f = g = 0$ outside $\bigcup_k B_k$. For each k , $\mathbf{1}_{B_k} = d_k^{-1} g \mathbf{1}_{B_k} \in X$ by the ideal property, so

$$f = \sum_k c_k \mathbf{1}_{B_k}, \quad g - f = \sum_k (d_k - c_k) \mathbf{1}_{B_k}, \quad g = \sum_k d_k \mathbf{1}_{B_k}$$

are each sums over pairwise disjoint supports.

For $0 \leq c_k \leq d_k$, set $a = c_k/d_k \in [0, 1]$. Since $p \geq 1$, we have $t^p \leq t$ for $0 \leq t \leq 1$, so $a^p \leq a$ and $(1 - a)^p \leq 1 - a$. Adding,

$$a^p + (1 - a)^p \leq a + (1 - a) = 1,$$

and multiplying by d_k^p ,

$$c_k^p + (d_k - c_k)^p \leq d_k^p.$$

Therefore, by disjoint p -additivity and homogeneity of the norm,

$$\|f\|_X^p + \|g - f\|_X^p = \sum_k (c_k^p + (d_k - c_k)^p) \|\mathbf{1}_{B_k}\|_X^p \leq \sum_k d_k^p \|\mathbf{1}_{B_k}\|_X^p = \|g\|_X^p. \quad \square$$

Lemma 4.3 (Disjoint p -additivity gives the Fatou property). *Let $1 \leq p < \infty$, and let X be disjointly p -additive. Then X has the Fatou property.*

Proof. Suppose that $f \in L^0(\Omega)_+$ and

$$0 \leq f_n \uparrow f \quad \text{pointwise } \mu\text{-a.e.}, \quad f_n \in X, \quad \sup_n \|f_n\|_X < \infty.$$

We must show that $f \in X$ with $\|f\|_X = \sup_n \|f_n\|_X$. We approximate the f_n from below by simple functions $s_n \uparrow f$, use Lemma 4.2 to show that (s_n) is Cauchy in X , and identify its limit as f . For each i , Theorem 2.4 gives simple functions $\sigma_{i,k}$ with $0 \leq \sigma_{i,k} \uparrow f_i$ as $k \rightarrow \infty$. Put $s_n = \max_{i \leq n} \sigma_{i,n}$. Each s_n is simple, and

$$\sigma_{i,n} \leq s_n \leq f_n \quad (i \leq n),$$

the upper bound because $\sigma_{i,n} \leq f_i \leq f_n$. The bound $s_n \leq f_n$ and the ideal property give

$$s_n \in X, \quad \|s_n\|_X \leq \|f_n\|_X, \quad \text{so} \quad \sup_n \|s_n\|_X < \infty.$$

The s_n increase with n , as the index set $\{i \leq n\}$ and each $\sigma_{i,n}$ grows. Letting $n \rightarrow \infty$ in these bounds gives

$$f_i \leq \lim_n s_n \leq f \quad \text{for every } i,$$

and $f_i \uparrow f$ forces $\lim_n s_n = f$, so $s_n \uparrow f$ μ -a.e. The s_n are simple and increasing, so Lemma 4.2 applied to $s_n \leq s_m$ ($m \geq n$) gives

$$\|s_m - s_n\|_X^p \leq \|s_m\|_X^p - \|s_n\|_X^p.$$

The numbers $\|s_n\|_X^p$ increase and are bounded, so they converge, and the right-hand side tends to 0. Thus

$$\|s_m - s_n\|_X \rightarrow 0 \quad (m, n \rightarrow \infty),$$

and (s_n) is Cauchy in X . By completeness $s_n \rightarrow g$ in X for some $g \in X$, and Lemma 2.17 gives

$$g = f \quad \mu\text{-a.e.}$$

Thus $f \in X$, and

Since $s_n \rightarrow f$ in X ,

$$\|f\|_X = \lim_n \|s_n\|_X.$$

Also $0 \leq s_n \leq f_n \leq f$, so the ideal property gives

$$\lim_n \|s_n\|_X \leq \sup_n \|f_n\|_X \leq \|f\|_X.$$

Thus $\|f\|_X = \sup_n \|f_n\|_X$. □

Theorem 4.4 (Disjoint p -additivity gives an isometric $L^p(w d\mu)$ -representation). *Let X be a Banach function space over a σ -finite measure space (Ω, Σ, μ) , and let $1 \leq p < \infty$. Suppose that X is disjointly p -additive. Then there exists a weight $w > 0$ μ -a.e. such that $X = L^p(\Omega, w d\mu)$ with*

$$\|f\|_X = \left(\int_{\Omega} |f|^p w d\mu \right)^{1/p} \quad (f \in X).$$

Proof. The new part was handled by the two preceding lemmas. This proof follows the seed-theorem proof, replacing $\nu(E)$ with $\nu_p(E) := \|\mathbf{1}_E\|_X^p$, $\|f\|_X$ with $\|f\|_X^p$ and, for non-negative functions, f with f^p . The last step is then applied to $|f|$. We omit the arguments we can take from there.

Step 1: Building the norm into a measure. Define $\nu_p : \Sigma \rightarrow [0, \infty]$ by

$$\nu_p(E) = \begin{cases} \|\mathbf{1}_E\|_X^p, & \mathbf{1}_E \in X, \\ +\infty, & \mathbf{1}_E \notin X. \end{cases}$$

The monotonicity, empty-set, non-negativity and infinite-value cases are the same as in Step 1 of the seed-theorem proof. The finite additivity check for disjoint E, F with finite $\nu_p(E) + \nu_p(F)$ is now

$$\nu_p(E \cup F) = \|\mathbf{1}_E + \mathbf{1}_F\|_X^p = \|\mathbf{1}_E\|_X^p + \|\mathbf{1}_F\|_X^p = \nu_p(E) + \nu_p(F).$$

Countable additivity is Step 1 of the seed-theorem proof followed by one extra step: in the bounded case Lemma 4.3 gives $\|\mathbf{1}_E\|_X = \sup_N \|\mathbf{1}_{G_N}\|_X$ for $E = \bigcup_j E_j$ and $G_N = \bigcup_{j \leq N} E_j$, and taking p -th powers gives $\nu_p(E) = \sup_N \nu_p(G_N)$. Thus ν_p is a measure on (Ω, Σ) .

Step 2: Applying Radon–Nikodym.

Following Step 2 of the seed-theorem proof, saturation gives that ν_p is σ -finite, and the null-set argument gives $\nu_p \ll \mu$. The Radon–Nikodym theorem, Theorem 2.8, gives a measurable function $w : \Omega \rightarrow [0, \infty)$ such that

$$\nu_p(E) = \int_E w d\mu \quad (E \in \Sigma).$$

The same positivity argument gives $w > 0$ μ -a.e., with the norm now raised to the p -th power:

$$0 < \|\mathbf{1}_G\|_X^p = \nu_p(G) = \int_G w d\mu = 0,$$

which is impossible for a positive-measure $G \subseteq \{w = 0\}$ with $\mathbf{1}_G \in X$.

Step 3: The norm identity by approximation. The indicator formula is

$$\|\mathbf{1}_E\|_X^p = \int_E w d\mu \quad (\mathbf{1}_E \in X).$$

Following Step 3 of the seed-theorem proof, disjoint p -additivity gives, for non-negative simple $s \in X$,

$$\|s\|_X^p = \int_{\Omega} s^p w d\mu.$$

The same argument, now approximating from below and using Lemma 4.3 and $s_j^p \uparrow f^p$, gives for $f \in X_+$

$$\|f\|_X^p = \int_{\Omega} f^p w d\mu.$$

Applying this to $|f|$ gives

$$\|f\|_X^p = \||f|\|_X^p = \int_{\Omega} |f|^p w d\mu.$$

Since $f \in X$, we have $\|f\|_X < \infty$, so $\int_{\Omega} |f|^p w d\mu < \infty$, and therefore $f \in L^p(\Omega, w d\mu)$. This proves $X \subseteq L^p(\Omega, w d\mu)$ with equality of norms on X .

Step 4: Reverse inclusion.

This is Step 4 of the seed-theorem proof with L^1 replaced by L^p . Lemma 4.3 gives the Fatou property. Therefore Proposition 2.20 with $Y = L^p(\Omega, w d\mu)$ gives

$$X = L^p(\Omega, w d\mu)$$

with equality of norms. □

Thus we get the weaker L^p -version of the seed theorem, in which disjoint p -additivity on a Banach function space forces an isometry with a weighted $L^p(w d\mu)$ -space. This shows that the additivity condition in the seed theorem was stricter than necessary.

Corollary 4.5 (Recovery of the seed theorem from disjoint p -additivity). *Let X be a Banach function space over a σ -finite measure space (Ω, Σ, μ) . Suppose that*

$$\|f + g\|_X = \|f\|_X + \|g\|_X \quad (f, g \in X_+).$$

Then there exists a weight $w > 0$ μ -a.e. such that $X = L^1(\Omega, w d\mu)$ with

$$\|f\|_X = \int_{\Omega} |f| w d\mu \quad (f \in X).$$

Proof. Let $f_1, \dots, f_n \in X_+$ be pairwise disjoint. Positive additivity and induction give

$$\left\| \sum_{k=1}^n f_k \right\|_X = \sum_{k=1}^n \|f_k\|_X.$$

Thus X is disjointly 1-additive. Applying Theorem 4.4 with $p = 1$ gives the stated isometry. $\square$

4.2 $L^p(w d\mu)$ -bounds via vector-valued estimates

We now recover the seed theorem using a theorem from Loris's MSc thesis [4, Theorem 6.1.2, pp. 110–111], which asks when one weighted L^p -seminorm can control a whole family of maps. It is written in a dialect of our language, as it uses a slightly different definition of Banach function spaces and some new vocabulary. Just as one thinks in one's native language when reading foreign texts, we first translate it and then recover the seed theorem.

Theorem 4.6 (Weighted versus vector-valued inequalities). *Let $1 \leq p < \infty$. Let X be a p -convex order-continuous Banach function space on (Ω, Σ, μ) , let Y be a semi-normed space and let Γ be a family of operators from X to Y such that*

$$T(\lambda x) = |\lambda| T(x) \quad (T \in \Gamma, x \in X, \lambda \text{ a scalar}).$$

Then the estimate

$$\left(\sum_{k=1}^n \|T_k x_k\|_Y^p \right)^{1/p} \leq C_1 \left\| \left(\sum_{k=1}^n |x_k|^p \right)^{1/p} \right\|_X$$

for all $n \geq 1$, all $T_1, \dots, T_n \in \Gamma$ and all $x_1, \dots, x_n \in X$, holds if and only if there exists a weight $w \geq 0$ such that

$$\left(\int_{\Omega} |x|^p w d\mu \right)^{1/p} \leq \|x\|_X \quad (x \in X),$$

and

$$\|Tx\|_Y \leq C_2 \left(\int_{\Omega} |x|^p w d\mu \right)^{1/p} \quad (T \in \Gamma, x \in X).$$

Moreover, the constants satisfy

$$M^{(p)}(X)^{-1} \leq \frac{C_1}{C_2} \leq 1.$$

We translate the p -convexity condition into our own language separately, as it will return later.

Definition 4.7 (p -convexity). *Let X be a Banach function space over (Ω, μ) and let $1 \leq p < \infty$. We say that X is p -convex if there exists a constant $A \geq 1$ such that, for every finite family $f_1, \dots, f_n \in X$,*

$$\left\| \left(\sum_{k=1}^n |f_k|^p \right)^{1/p} \right\|_X \leq A \left(\sum_{k=1}^n \|f_k\|_X^p \right)^{1/p}.$$

The least admissible constant is denoted by $M^{(p)}(X)$.

We now translate the rest of the theorem.

Theorem 4.8 (Weighted versus vector-valued inequalities, in our language). *Let X be a Banach function space with the Fatou property over a σ -finite measure space (Ω, Σ, μ) , and let $1 \leq p < \infty$. Assume that X is order-continuous and p -convex with constant A . Let Y be a normed space and let Γ be a family of maps from X to Y such that*

$$T(\lambda f) = |\lambda|T(f) \quad (T \in \Gamma, f \in X, \lambda \text{ a scalar}).$$

Then the estimate

$$\left(\sum_{k=1}^n \|T_k f_k\|_Y^p \right)^{1/p} \leq C_1 \left\| \left(\sum_{k=1}^n |f_k|^p \right)^{1/p} \right\|_X$$

for all $n \geq 1$, all $T_1, \dots, T_n \in \Gamma$ and all $f_1, \dots, f_n \in X$, holds if and only if there exists a weight $w \geq 0$ such that

$$\left(\int_{\Omega} |f|^p w \, d\mu \right)^{1/p} \leq \|f\|_X \quad (f \in X),$$

and

$$\|Tf\|_Y \leq C_2 \left(\int_{\Omega} |f|^p w \, d\mu \right)^{1/p} \quad (T \in \Gamma, f \in X).$$

Moreover,

$$A^{-1} \leq \frac{C_1}{C_2} \leq 1.$$

Proof. We translate Lorist's theorem into our native language by specializing it. The source version of a Banach function space requires domination, a weak order unit, and the Fatou property. For us these are the ideal property, saturation (Theorem 2.18), and the assumed Fatou property. Our order continuity coincides with Lorist's (Definition 2.21, [4, Definition 2.4.1 and the following paragraph, p. 23]). Lorist's source convention is real at this point. We use the corresponding complexification described before that definition [4, complexification paragraph, p. 22; Section 2.4 opening, p. 23]. and assumed p -convexity with constant A gives $M^{(p)}(X) \leq A$. A normed space is semi-normed, and the homogeneity of Γ is unchanged. So Lorist's theorem applies and gives the equivalence, with $M^{(p)}(X)^{-1} \leq C_1/C_2 \leq 1$. Since $M^{(p)}(X) \leq A$, this gives $A^{-1} \leq C_1/C_2 \leq 1$. $\square$

This theorem looks quite different from the seed theorem. It does not give an isometry with an L^p -space, but only the two inequalities:

$$\left(\int_{\Omega} |f|^p w \, d\mu \right)^{1/p} \leq \|f\|_X, \quad \|Tf\|_Y \leq C_2 \left(\int_{\Omega} |f|^p w \, d\mu \right)^{1/p}.$$

If we take $Y = X$ and use the modulus map $T(f) = |f|$, then the two inequalities become a two-sided inequality for $\|f\|_X$. Thus to recover the seed theorem we check whether the seed assumptions give the needed estimate for the modulus map and $C_2 = 1$ for equality.

Corollary 4.9 (Recovery of the seed theorem from vector-valued estimates). *Let X be a Banach function space over a σ -finite measure space (Ω, Σ, μ) . Suppose that*

$$\|f + g\|_X = \|f\|_X + \|g\|_X \quad (f, g \in X_+).$$

Then there exists a weight $w > 0$ μ -a.e. such that $X = L^1(\Omega, w \, d\mu)$ with

$$\|f\|_X = \int_{\Omega} |f| w \, d\mu \quad (f \in X).$$

Proof. Set $p = 1$, $Y = X$, and

$$\Gamma = \{T\}, \quad T(f) = |f|.$$

Then Y is normed. By Lemma 2.16, T maps X into X , and $T(\lambda f) = |\lambda|T(f)$.

Step 1: Checking the hypotheses. By Lemma 2.19 and Lemma 2.22, positive additivity gives the Fatou property and order continuity.

For $f_1, \dots, f_n \in X$, the triangle inequality and Lemma 2.16 give

$$\left\| \sum_{k=1}^n |f_k| \right\|_X \leq \sum_{k=1}^n \| |f_k| \|_X = \sum_{k=1}^n \|f_k\|_X.$$

Thus X is 1-convex with constant $A = 1$.

We now check the vector-valued estimate. Since $\Gamma = \{T\}$, each T_k is T . For $f_1, \dots, f_n \in X$, Lemma 2.16 and positive additivity give

$$\sum_{k=1}^n \|Tf_k\|_X = \sum_{k=1}^n \| |f_k| \|_X = \left\| \sum_{k=1}^n |f_k| \right\|_X.$$

This is the $p = 1$ vector-valued estimate in Theorem 4.8, with $C_1 = 1$.

Step 2: Norm equality. With $A = 1$ and $C_1 = 1$, the theorem's bound $A^{-1} \leq C_1/C_2 \leq 1$ forces $C_2 = 1$. Thus Theorem 4.8 gives a weight $w \geq 0$ such that the two weighted inequalities become

$$\int_{\Omega} |f|w \, d\mu \leq \|f\|_X = \| |f| \|_X \leq \int_{\Omega} |f|w \, d\mu.$$

This gives the norm identity

$$\|f\|_X = \int_{\Omega} |f|w \, d\mu \quad (f \in X).$$

Step 3: Reverse inclusion. Since $f \in X$, we have $\|f\|_X < \infty$, so $\int_{\Omega} |f|w \, d\mu < \infty$, and therefore $f \in L^1(\Omega, w \, d\mu)$. This proves $X \subseteq L^1(\Omega, w \, d\mu)$ with equality of norms on X . As in Step 2 of the seed-theorem proof, saturation and the norm identity give $w > 0$ μ -a.e., and as in Step 4 the Fatou property and Proposition 2.20 give

$$X = L^1(\Omega, w \, d\mu).$$

□

What does this teach us? To fulfil the requirements of Lorist's theorem, we needed to reformulate additivity into two separate inequalities:

$$\|f + g\|_X \leq \|f\|_X + \|g\|_X, \quad \|f\|_X + \|g\|_X \leq \|f + g\|_X \quad (f, g \in X_+).$$

The first is the triangle inequality, which is always given for a Banach function space. Thus in this setting asking for positive additivity is the same as asking for the reverse inequality on the positive cone. This reverse inequality can be formalized as the concavity condition.

Definition 4.10 (p -concavity). *Let X be a Banach function space over (Ω, μ) and let $1 \leq p < \infty$. We say that X is p -concave with constant $B \geq 1$ if, for every finite family $f_1, \dots, f_n \in X$,*

$$\left(\sum_{k=1}^n \|f_k\|_X^p \right)^{1/p} \leq B \left\| \left(\sum_{k=1}^n |f_k|^p \right)^{1/p} \right\|_X.$$

In trying to reach a broader weighted L^p -representation, p -convexity and p -concavity point towards one more sea to traverse. What if we allow the exponents in these inequalities to vary?



Figure 4.1: Rembrandt van Rijn, *Christ in the Storm on the Sea of Galilee*, 1633. Oil on canvas. Isabella Stewart Gardner Museum, Boston. Missing since the 1990 theft.

4.3 $L^p(w d\mu)$ -equivalence under (p, b) -convexity and (p, d) -concavity

We now introduce mixed convexity and mixed concavity, in which the exponents on the convex and concave sides are allowed to move independently.

Definition 4.11 (Mixed convexity and mixed concavity). *Let X be a Banach function space over (Ω, μ) . For $1 \leq a, b \leq \infty$, we say that X is (a, b) -convex with constant $A \geq 1$ if for every finite family $f_1, \dots, f_n \in X$,*

$$\left\| \left(\sum_{k=1}^n |f_k|^b \right)^{1/b} \right\|_X \leq A \left(\sum_{k=1}^n \|f_k\|_X^a \right)^{1/a}.$$

For $1 \leq c, d \leq \infty$, we say that X is (c, d) -concave with constant $B \geq 1$ if for every finite family $f_1, \dots, f_n \in X$,

$$\left(\sum_{k=1}^n \|f_k\|_X^c \right)^{1/c} \leq B \left\| \left(\sum_{k=1}^n |f_k|^d \right)^{1/d} \right\|_X.$$

When one of the exponents is ∞ , the corresponding expression is the maximum over $k = 1, \dots, n$.

Even though we will prove the weighted L^p -representation only for (p, b) -convexity and (p, d) -concavity, we start with the four-exponent mixed definition because it allows us to show why we narrow the range of exponents.

We first check what restrictions the inequalities themselves force on the exponents. Choose $0 \neq f \in X$. Testing the mixed inequalities with $f_1 = \dots = f_n = f$ gives

$$\begin{aligned} n^{1/b} \|f\|_X &\leq A n^{1/a} \|f\|_X, \\ n^{1/c} \|f\|_X &\leq B n^{1/d} \|f\|_X \quad (n \geq 1). \end{aligned}$$

Since these inequalities hold for arbitrarily large n , we get $1/b \leq 1/a$ and $1/c \leq 1/d$, so $b \geq a$ and $d \leq c$. All four exponents lie in $[1, \infty]$. Figure 4.2 shows the available regions.

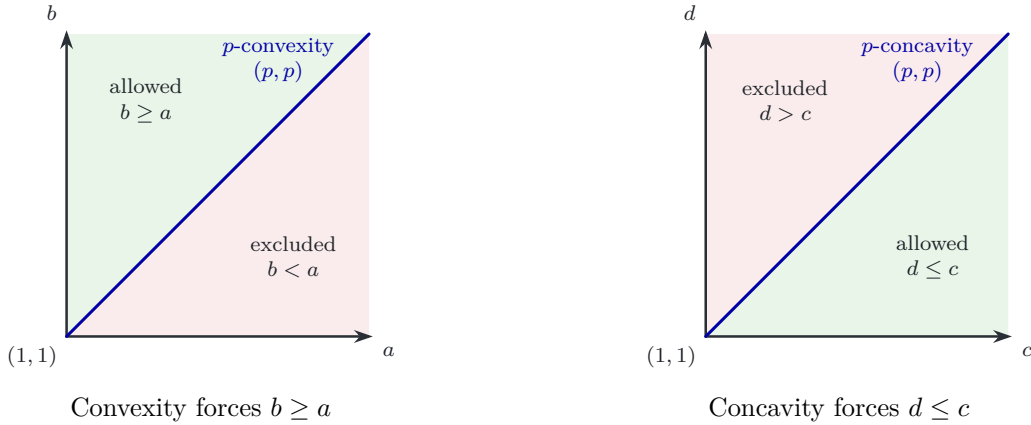


Figure 4.2: The repeated-vector test forces $b \geq a$ on the convex side and $d \leq c$ on the concave side. The blue diagonals mark the p -convexity and p -concavity (p, p) cases.

As explained in Section 4.1, a weighted L^p -space is disjointly p -additive. Therefore we narrow the exponent range to one in which the mixed inequalities estimate $(\sum_k \|f_k\|_X^p)^{1/p}$ from both sides by $\|\sum_k f_k\|_X$ on disjoint positive sums.

Let $f_1, \dots, f_n \in X_+$ be pairwise disjoint. Then $(\sum_k |f_k|^b)^{1/b} = \sum_k f_k = (\sum_k |f_k|^d)^{1/d}$, so the mixed

convexity and mixed concavity inequalities become

$$\begin{aligned} \left\| \sum_k f_k \right\|_X &= \left\| \left(\sum_k |f_k|^b \right)^{1/b} \right\|_X \leq A \left(\sum_k \|f_k\|_X^a \right)^{1/a}, \\ \left(\sum_k \|f_k\|_X^c \right)^{1/c} &\leq B \left\| \left(\sum_k |f_k|^d \right)^{1/d} \right\|_X = B \left\| \sum_k f_k \right\|_X. \end{aligned}$$

Now if we set $a = c = p$, we can combine the inequalities:

$$A^{-1} \left\| \sum_k f_k \right\|_X \leq \left(\sum_k \|f_k\|_X^a \right)^{1/a} = \left(\sum_k \|f_k\|_X^p \right)^{1/p} = \left(\sum_k \|f_k\|_X^c \right)^{1/c} \leq B \left\| \sum_k f_k \right\|_X$$

Together with $b \geq a$ and $d \leq c$, this leaves $1 \leq d \leq p \leq b \leq \infty$. Thus, for a (p, b) -convex and (p, d) -concave Banach function space in this range, $(\sum_k \|f_k\|_X^p)^{1/p}$ can be estimated from both sides by $\|\sum_k f_k\|_X$. This is where we now look for a weighted L^p -representation.

Now that we have a range we think should work, is there a theorem that gives the isometry? Not directly, but García-Sánchez's recent survey *Convexity and Concavity in Banach Lattices* [5] helps. The relevant part is the following corollary.

Corollary 4.12 (Upper and lower p -estimates give a lattice isomorphism onto $L_p(\mu)$, in García-Sánchez's language). *Let X be a Banach lattice that satisfies an upper p -estimate and a lower p -estimate for some $1 \leq p < \infty$. Then X is lattice isomorphic to $L_p(\mu)$ for some measure μ , with distortion $K^{(\uparrow p)}(X)K^{(\downarrow p)}(X)$.*

García-Sánchez states the corollary in Banach-lattice language. Since we are interested in what this result says for Banach function spaces, we take our dictionary and grammar book to translate the relevant part of García-Sánchez's corollary [5, Corollary 5.12, p. 34] into our native language. We do not explore Banach lattice language itself.

Theorem 4.13 (An $L^p(\nu)$ -representation under (p, b) -convexity and (p, d) -concavity, in our language). *Let X be a Banach function space over (Ω, μ) . If $1 \leq p < \infty$, $1 \leq d \leq p \leq b \leq \infty$ and X is (p, b) -convex with constant A and (p, d) -concave with constant B , then there exist a measure space (Ω', ν) and a bounded linear bijection $T : X \rightarrow L^p(\nu)$ such that*

$$T|f| = |Tf| \quad (f \in X), \quad T^{-1}|g| = |T^{-1}g| \quad (g \in L^p(\nu)),$$

with $\|T\| \|T^{-1}\| \leq AB$.

Proof. Let $X_{\mathbb{R}}$ be the real-valued part of X . The maps $f \mapsto \Re f$ and $f \mapsto \Im f$ are contractions by the ideal property, so $X_{\mathbb{R}}$ is closed and $X = X_{\mathbb{R}} + iX_{\mathbb{R}}$. With the pointwise order, $X_{\mathbb{R}}$ is a real Banach lattice. If $f_1, \dots, f_n \in X_+$ are pairwise disjoint, then

$$\left(\sum_{k=1}^n |f_k|^b \right)^{1/b} = \sum_{k=1}^n f_k = \left(\sum_{k=1}^n |f_k|^d \right)^{1/d}.$$

Thus (p, b) -convexity and (p, d) -concavity give the upper and lower p -estimates on $X_{\mathbb{R}}$ with constants A and B [5, Definition 3.2, p. 10]. Then Corollary 4.12 gives a real lattice isomorphism

$$T_{\mathbb{R}} : X_{\mathbb{R}} \rightarrow L_{\mathbb{R}}^p(\nu), \quad \|T_{\mathbb{R}}\| \|T_{\mathbb{R}}^{-1}\| \leq AB.$$

where $L_{\mathbb{R}}^p(\nu)$ denotes the real-valued part of $L^p(\nu)$. Define $T(u + iv) = T_{\mathbb{R}}u + iT_{\mathbb{R}}v$ for $u, v \in X_{\mathbb{R}}$. The lattice-isomorphism property from García-Sánchez [5, Section 2.1, p. 5] says that $T_{\mathbb{R}}$ commutes with the lattice functional calculus. Thus, for $f = u + iv$,

$$T|f| = T_{\mathbb{R}}((u^2 + v^2)^{1/2}) = ((T_{\mathbb{R}}u)^2 + (T_{\mathbb{R}}v)^2)^{1/2} = |Tf|.$$

The same calculation, applied to $T_{\mathbb{R}}^{-1}$, gives

$$T^{-1}|g| = |T^{-1}g|.$$

For $f \in X$ and $g \in L^p(\nu)$,

$$\|Tf\|_{L^p(\nu)} = \|T_{\mathbb{R}}|f|\|_{L^p(\nu)} \leq \|T_{\mathbb{R}}\| \|f\|_X, \quad \|T^{-1}g\|_X = \|T_{\mathbb{R}}^{-1}|g|\|_X \leq \|T_{\mathbb{R}}^{-1}\| \|g\|_{L^p(\nu)}.$$

Therefore $\|T\| \|T^{-1}\| \leq AB$. □

The theorem above does not yet give a weight on the original measure space (Ω, μ) . It gives a bijection $T : X \rightarrow L^p(\nu)$ for some measure space (Ω', ν) . Still, this is enough to make the work from Section 4.1 useful again. In the proof, we first normalize T and then use the $L^p(\nu)$ -norm to define a norm N on X . The proof shows that this norm is disjointly p -additive, and then Theorem 4.4 gives the weight on (Ω, μ) .

Theorem 4.14 (An $L^p(w d\mu)$ -equivalence under (p, b) -convexity and (p, d) -concavity). *Let X be a Banach function space over a σ -finite measure space (Ω, Σ, μ) . If $1 \leq p < \infty$, $1 \leq d \leq p \leq b \leq \infty$ and X is (p, b) -convex with constant A and (p, d) -concave with constant B , then there exists a weight $w > 0$ μ -a.e. such that*

$$X = L^p(\Omega, w d\mu)$$

as sets of functions, and

$$\|f\|_{L^p(\Omega, w d\mu)} \leq \|f\|_X \leq AB \|f\|_{L^p(\Omega, w d\mu)} \quad (f \in X).$$

Proof. Step 1: The $L^p(\nu)$ -representation. By Theorem 4.13, there is a measure space (Ω', ν) and a bounded linear bijection

$$T : X \longrightarrow L^p(\nu)$$

such that T and T^{-1} preserve absolute values and

$$\|T\| \|T^{-1}\| \leq AB.$$

Step 2: Normalizing the map and comparing norms. If $X = \{0\}$, saturation forces $\mu(\Omega) = 0$, and $w = 1$ gives the conclusion. Therefore assume $X \neq \{0\}$. Let $\alpha := \|T\|$ and $\beta := \|T^{-1}\|$. Since $X \neq \{0\}$ and T is injective, T is not the zero map, so $\alpha > 0$. The operator norm gives only $\|Tf\|_{L^p(\nu)} \leq \alpha \|f\|_X$, so we set $S := \alpha^{-1}T$, giving $\|Sf\|_{L^p(\nu)} \leq \|f\|_X$. Both S and $S^{-1} = \alpha T^{-1}$ are bounded and preserve absolute values. Put

$$N(f) := \|Sf\|_{L^p(\nu)} \quad (f \in X).$$

Then $N(f) = \alpha^{-1} \|Tf\|_{L^p(\nu)} \leq \|f\|_X$, while

$$\|f\|_X = \|S^{-1}Sf\|_X \leq \alpha\beta \|Sf\|_{L^p(\nu)} = \alpha\beta N(f).$$

Using $\alpha\beta = \|T\| \|T^{-1}\| \leq AB$, we obtain

$$N(f) \leq \|f\|_X \leq AB N(f) \quad (f \in X),$$

so N and $\|\cdot\|_X$ are equivalent norms on X .

Step 3: X_N is a disjointly p -additive Banach function space. Write X_N for X equipped with the norm N . We check the hypotheses of Theorem 4.4.

(1) X_N is a Banach space.

(a) *Norm.* For $f \in X$ we have $Sf \in L^p(\nu)$, so $N(f) = \|Sf\|_{L^p(\nu)} < \infty$. If $N(f) = 0$, then $Sf = 0$, so $f = 0$ by injectivity of S . For every scalar λ ,

$$N(\lambda f) = \|S(\lambda f)\|_{L^p(\nu)} = |\lambda| N(f),$$

and, using the linearity of S and the triangle inequality in $L^p(\nu)$,

$$N(f + g) = \|Sf + Sg\|_{L^p(\nu)} \leq \|Sf\|_{L^p(\nu)} + \|Sg\|_{L^p(\nu)} = N(f) + N(g).$$

(b) *Completeness.* Since S and S^{-1} are bounded, N and $\|\cdot\|_X$ are equivalent norms on X , so the identity map $(X, \|\cdot\|_X) \rightarrow (X_N, N)$ is a uniform homeomorphism. Uniform homeomorphisms preserve completeness, and $(X, \|\cdot\|_X)$ is complete, so X_N is complete.

(2) X_N satisfies the ideal property. The map S is positive: if $h \geq 0$, then $Sh = S|h| = |Sh| \geq 0$. So if $g \in L^0(\Omega)$ satisfies $|g| \leq |f|$ with $f \in X$, then $g \in X$ by the ideal property of X , and $|Sg| = S|g| \leq S|f| = |Sf|$, thus $N(g) \leq N(f)$.

(3) X_N is saturated, since X_N and X have the same elements and saturation only asks which indicators lie in the space.

Finally, X_N is disjointly p -additive. Let $f_1, \dots, f_n \in X_+$ be pairwise disjoint. Each $Sf_k = S|f_k| = |Sf_k| \geq 0$, and for $i \neq j$,

$$|Sf_i - Sf_j| = |S(f_i - f_j)| = S|f_i - f_j| = S(f_i + f_j) = Sf_i + Sf_j.$$

For non-negative functions, $|u - v| = u + v$ is equivalent to $uv = 0$, so $Sf_1, \dots, Sf_n$ are positive and pairwise disjoint in $L^p(\nu)$, and

$$N\left(\sum_{k=1}^n f_k\right)^p = \left\|\sum_{k=1}^n Sf_k\right\|_{L^p(\nu)}^p = \sum_{k=1}^n \|Sf_k\|_{L^p(\nu)}^p = \sum_{k=1}^n N(f_k)^p.$$

Step 4: The weight on (Ω, μ) . By Theorem 4.4 there is a weight $w > 0$ μ -a.e. with $X_N = L^p(\Omega, w d\mu)$ and

$$N(f) = \left(\int_{\Omega} |f|^p w d\mu\right)^{1/p} = \|f\|_{L^p(\Omega, w d\mu)} \quad (f \in X).$$

Since X_N and X have the same elements, $X = L^p(\Omega, w d\mu)$ as sets of functions. Substituting this norm identity into $N(f) \leq \|f\|_X \leq AB N(f)$ gives

$$\|f\|_{L^p(\Omega, w d\mu)} \leq \|f\|_X \leq AB \|f\|_{L^p(\Omega, w d\mu)} \quad (f \in X).$$

□

This gives a more general weighted L^p -equivalence theorem. Although it is not isometric in general, the seed theorem can be recovered.

Corollary 4.15 (Recovery of the seed theorem from (p, b) -convexity and (p, d) -concavity). *Let X be a Banach function space over a σ -finite measure space (Ω, Σ, μ) . Suppose that*

$$\|f + g\|_X = \|f\|_X + \|g\|_X \quad (f, g \in X_+).$$

Then there exists a weight $w > 0$ μ -a.e. such that $X = L^1(\Omega, w d\mu)$ with

$$\|f\|_X = \int_{\Omega} |f| w d\mu \quad (f \in X).$$

Proof. Let $f_1, \dots, f_n \in X$. By Lemma 2.16, $|f_k| \in X_+$ and $\||f_k|\|_X = \|f_k\|_X$. Positive additivity and induction give

$$\left\|\sum_{k=1}^n |f_k|\right\|_X = \sum_{k=1}^n \||f_k|\|_X = \sum_{k=1}^n \|f_k\|_X.$$

This gives both $(1, 1)$ -convexity and $(1, 1)$ -concavity with constants $A = B = 1$. Applying Theorem 4.14 gives the stated isometry. □

As this chapter has now demonstrated what it takes to generalize from positive additivity to the more general theorem for (p, b) -convexity and (p, d) -concavity, the point of this thesis becomes clear. Once a more general theorem is known, reducing it to a specific case is often very easy while finding that general theorem from the specific case is much harder. Figure 4.3 visualizes this asymmetry by showing the route we took next to the short recovery back to the seed theorem.

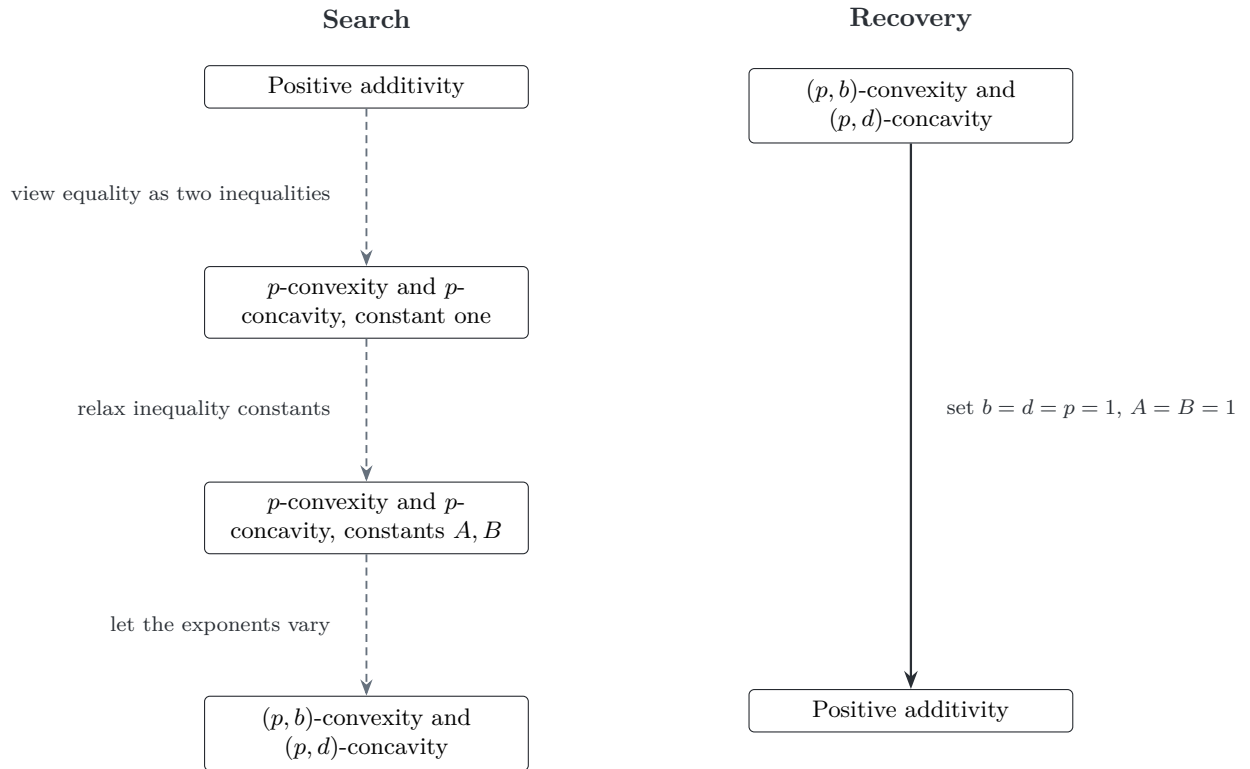


Figure 4.3: The route toward (p, b) -convexity and (p, d) -concavity is hard to guess from the seed theorem. The route back is short once the general theorem is known.

Chapter 5

Bridging the language barrier

Several of the theorems that appear in the previous chapter were suggested to me by my supervisor. A curious student working on their own might try to extend additivity to disjoint p -additivity, but would not search for weighted/vector-valued inequalities or for (p, b) -convexity and (p, d) -concavity, as those words would not yet be in their vocabulary.

The student might think: “why would the author not just provide as many special cases, examples and corollaries, in as many languages as reasonably possible so that I could find the general theorem by searching for my case in my language?” The author, on the other hand, wants to write statements that are general, compact and recognizable inside the author’s mathematical field. Furthermore, they see mountains of extra work in writing and checking extra statements.

The asymmetry from the previous chapter suggests that the student might be right. If the downward route is easier once the general theorem is known, then the reduction should indeed be made at the source. To exploit this asymmetry, we introduce a protocol for author-side corollary generation that produces searchable reductions downwards. Afterwards, we test experimentally whether these reductions make the relevant general theorems easier to find. We formalize the following research question.

To what extent do corollaries generated by a large language model make general theorem statements easier to find from specific cases?

We first explain the experimental setup shared by all runs. Each iteration then begins by stating the protocol used for that run.

5.1 Experimental setup

Unlike search repositories such as Google Scholar or arXiv, which return documents, it is more natural for us to make individual theorem statements the searchable objects. A recent project, TheoremSearch, takes a similar direction. The authors extract theorem statements and use a large language model to rephrase them for search [6]. We take a bigger step: instead of only rephrasing theorem statements, we try to generate nearby corollary statements.

The haystack to hide our needles in consists of *theorem chunks*: theorem statements, with a small window of surrounding text, extracted from arXiv. We enlarge the haystack until simple seed language keyword search no longer finds the targets trivially. This happens at roughly 31,000 theorem chunks, which is still modest compared to the number of theorems in the wider literature.

Target and control theorem chunks

The needles are the theorems from the previous chapter from which we recover the seed theorem. We label them T1–T3 in Table 5.1. Here T stands for theorem chunk. For a target theorem, the ideal protocol would give a corollary close to the seed theorem and formulate it in the kind of language a seeker of the seed theorem would use.

There is a reasonable chance that any effect the protocol would have could be explained by a simple heuristic: a theorem chunk with more text is easier to find, regardless of whether it reduces to the seed theorem. To control for this effect, we add three control theorem chunks, labeled T4–T6. They have nearby vocabulary, but do not recover the seed theorem, at least directly. Table 5.1 lists both groups. The same step of generating corollary notes is also applied to T4–T6. The full statements of T4–T6 are shown in Section A.1.

Label	Theorem	Title
<i>Target theorem chunks</i>		
T1	Theorem 4.4	Disjoint p -additivity gives an isometric $L^p(w d\mu)$ -representation.
T2	Theorem 4.6	Weighted versus vector-valued inequalities.
T3	Theorem 4.14	An $L^p(w d\mu)$ -equivalence under (p, b) -convexity and (p, d) -concavity.
<i>Control theorem chunks</i>		
T4	Theorem A.1	Hadamard means of positive kernel operators.
T5	Theorem A.2	Haar multipliers and branch variation.
T6	Theorem A.3	Parabolic fractional maximal functions and Sawyer testing.

Table 5.1: The theorem chunks used as targets and controls.

Corpora and search system

To test what effect the protocol would have in a realistic search setting, we need an A/B comparison under a modern search method. To avoid building an entire modern search system ourselves, the experiment uses the third-party system Vectara. Vectara allows us to upload our own searchable collections of theorem chunks, which it then calls corpora. We label the baseline corpus C0 and the test corpus C1. C0 contains the 30,738 theorem chunks without generated corollary notes. In each iteration, C1 represents the corpus in which the generated corollary notes are appended to the theorem chunks T1–T6.

On these corpora, the chosen Vectara search method is **hybrid_010_rerank**. This method combines traditional keyword-based search with embedding-based semantic search. This makes the test stricter than if we used pure keyword search. Intuitively, embedding-based semantic search represents text fragments as vectors in a high-dimensional space, where fragments with similar meanings are located closer together even when they do not share the same keywords. The 010 indicates that 10% of the retrieval weight is assigned to pure keyword search, while 90% is assigned to embedding-based semantic search.

Queries and metrics

For the comparison, we use five queries that assume a seeker already knows the seed language, but not the broader language of the general theorems. They are labeled Q1–Q5 and are listed in Table 5.2. To keep the experiment realistic, the queries vary in quality.

Query	Text
Q1	Banach function space positive additivity implies weighted L1
Q2	norm additive on nonnegative measurable functions weighted L1 representation
Q3	characterize function norms additive on positive functions
Q4	when does a Banach function space have an L1 norm with weight
Q5	positive cone norm additivity forces weighted integral norm

Table 5.2: The five queries on which Vectara ranks the corpora.

For each query on each corpus, Vectara’s **hybrid_010_rerank** returns a list of theorem chunks ordered by its estimated relevance to the query. As in a Google search, rank 1 is the top result, so a lower rank means that the theorem chunk appears higher in the list.

We also need to define clear metrics. For every query, we list the rank of each target theorem T1–T3 and each control theorem T4–T6 in both corpora. Since results outside the first ten are much less useful to a searcher, we also count how many targets and controls appear in the first ten results.

5.2 First iteration

Corollary generation

The prompt used for corollary generation has to satisfy two requirements. The output needs to be in a human-readable format, as the output is the entry point for a seeker. And more importantly, the prompt

has to remain blind to the seed theorem. Therefore the model receives only the theorem statement. The prompt asks for corollary reductions, without specifying a direction in which the theorem should be reduced.

To ensure mathematical correctness, each new statement has to come with a short argument explaining why it follows from the theorem. The model is also asked to rephrase the new statements in different mathematical languages. The exact prompt is included in Section A.2, so this can be checked directly. The generation uses OpenAI API calls with model `gpt-5.5` and reasoning effort set to `xhigh`, with one API call for each of the theorems T1–T6.

Results and interpretation

Generated corollary notes and interpretation. The first iteration of the protocol creates the generated corollary notes. Before proceeding to the quantitative rank comparison, we first look qualitatively at the generated corollary notes for T1–T3. We use the internal numbering of the generated output.

T1. *Generated corollary note.* The closest generated corollary note is Corollary 6, which gives the weighted L^1 -conclusion under disjoint 1-additivity. The closest search formulation is: “Every disjointly additive Banach function space is a weighted L^1 -space.”

Interpretation. Corollary 6 almost recovers the seed theorem. The only thing left is to conclude the statement for positive additivity, since positive additivity on X_+ implies disjoint 1-additivity.

T2. *Generated corollary notes.* The closest generated corollary notes are Corollary 7, which sets $p = 1$, and Corollary 4, which restricts the operator family to one map, $\Gamma = \{T\}$. The closest search formulation is: “At $p = 1$, the theorem characterizes an ℓ^1 -sum estimate by common $L^1(w)$ -domination.”

Interpretation. No generated corollary note chooses $Y = X$ and the modulus map $T(f) = |f|$ needed for the recovery and therefore no corollary comes close to the seed theorem.

T3. *Generated corollary notes.* The closest generated corollary notes are Corollary 6, which gives the weighted L^1 -conclusion under $(1, b)$ -convexity and $(1, 1)$ -concavity, and Corollary 7, which states that $AB = 1$ gives equality of norms. The closest search formulation is: “The endpoint $p = 1$ of Theorem 3: $(1, b)$ -convexity and $(1, 1)$ -concavity force a weighted L^1 -space.”

Interpretation. Since Corollary 6 and Corollary 7 are not combined, the recovery is not made.

The first iteration of the protocol has not yet completed the reduction to the seed theorem in all target cases. However, the generated corollary notes and their search formulations do move towards the seed language and theorem. Therefore, we can still run the search test, asking whether these partial reductions already improve retrieval.

Search results and interpretation. By adding the generated corollary notes to the corresponding theorem chunks, we create C1 from C0 and upload both corpora to Vectara. Using `hybrid_010_rerank`, we then compare the ranks on the five queries. Tables 5.3 and 5.4 show the target and control ranks.

Query	T1		T2		T3		Top-10 hits	
	C0	C1	C0	C1	C0	C1	C0	C1
Q1	54	1	1256	614	30	9	0	2
Q2	690	113	5166	3498	92	21	0	0
Q3	794	140	1358	1055	83	171	0	0
Q4	262	11	1214	405	37	7	0	1
Q5	3589	783	3782	2820	921	266	0	0

Table 5.3: Ranks of the target theorems T1–T3 in the baseline corpus C0 and the corpus C1 after adding generated corollary notes. Rank 1 is the top result, so a lower rank means that the theorem appears higher in the list. The last two columns count how many targets appear in the first ten results.

Query	T4		T5		T6		Top-10 hits	
	C0	C1	C0	C1	C0	C1	C0	C1
Q1	203	203	698	639	2549	2549	0	0
Q2	320	321	434	398	293	294	0	0
Q3	146	146	524	364	841	841	0	0
Q4	633	634	22	21	828	829	0	0
Q5	673	674	23291	19747	189	189	0	0

Table 5.4: Ranks of the control theorems T4–T6 in the baseline corpus C0 and the corpus C1 after adding generated corollary notes. Rank 1 is the top result, so a lower rank means that the theorem appears higher in the list. The last two columns count how many controls appear in the first ten results.

Although the generated corollary notes do not yet recover the seed theorem fully, the results are still positive. T1 and T2 improve for every query, and T3 improves for four of the five queries. For practical search, ranks below the first ten matter much less. For Q1 two target theorems and for Q4 one target theorem now appear in the top ten, with T1 even in first place for Q1. This means that these theorems would now actually be found, unlike in C0. The ranks of the control theorems do not change much, and no control theorem reaches the top ten. The improvement for T1–T3 therefore cannot be explained by a general heuristic that adding text helps every theorem chunk equally. The results thus suggest that the generated corollary notes make the target theorems easier to find.

5.3 Second iteration

The first iteration of the protocol is far from optimized, while the generated corollary notes give clear directions for improvement. Therefore we run a second iteration of the protocol.

Corollary generation

The generated corollary notes from the first iteration show three problems: not all of the needed specializations are made, the specializations needed to recover the seed theorem stay in separate corollaries and the corollary assumptions are not reformulated to positive additivity when they are mathematically equivalent.

Moving from single large language model calls to multiple stages of agentic loops lets us address these problems. In an agentic loop, a large language model repeatedly works on the output produced so far instead of answering the prompt only once. To keep the corollary generation protocol blind to the seed theorem, each run is done in an isolated Codex run that contains only the input intended for the protocol. Here Codex denotes the OpenAI agent interface used to run the prompts. The model is `gpt-5.5` with reasoning effort `xhigh`. For each stage, we give Codex the corresponding prompt. The prompts are shown in Section A.3. Once that prompt is completed, the next prompt is given.

For each theorem, once again the goal is to generate corollaries at lower levels of generality that future seekers are likely to search for. The second iteration of the protocol starts with the input S0 and then has five generation stages S1–S5. Below we explain each of the stages:

S0 *Input.* The input for the protocol is a single theorem statement. In contrast to the first iteration, we now also add the definitions used in that theorem statement. An author would provide the surrounding paper as input. For us adding this entire thesis is unfortunately not possible, as it would point Codex towards the seed theorem.

S1 *Enumerate corollaries.* Codex first gets the task to derive every corollary that follows from the theorem, with a short argument for each.

S2 *Combine corollaries.* Then Codex loops over the corollaries found so far, combining and specialising them further.

S3 *Curate the set.* As S1 and S2 lead to an explosion in the number of corollaries, we let Codex curate the set afterwards to avoid duplicates and weak entries. In the current setup this is also needed to reduce the number of corollaries to a size that can still be inspected.

S4 *Search formulations.* Each kept corollary gets its own Codex run with the goal of restating it in different mathematical languages.

S5 *Typeset to LaTeX.* Finally, we let Codex format all corollaries and search formulations into clean LaTeX.

Figure 5.1 makes this second iteration of the protocol a bit easier to follow.

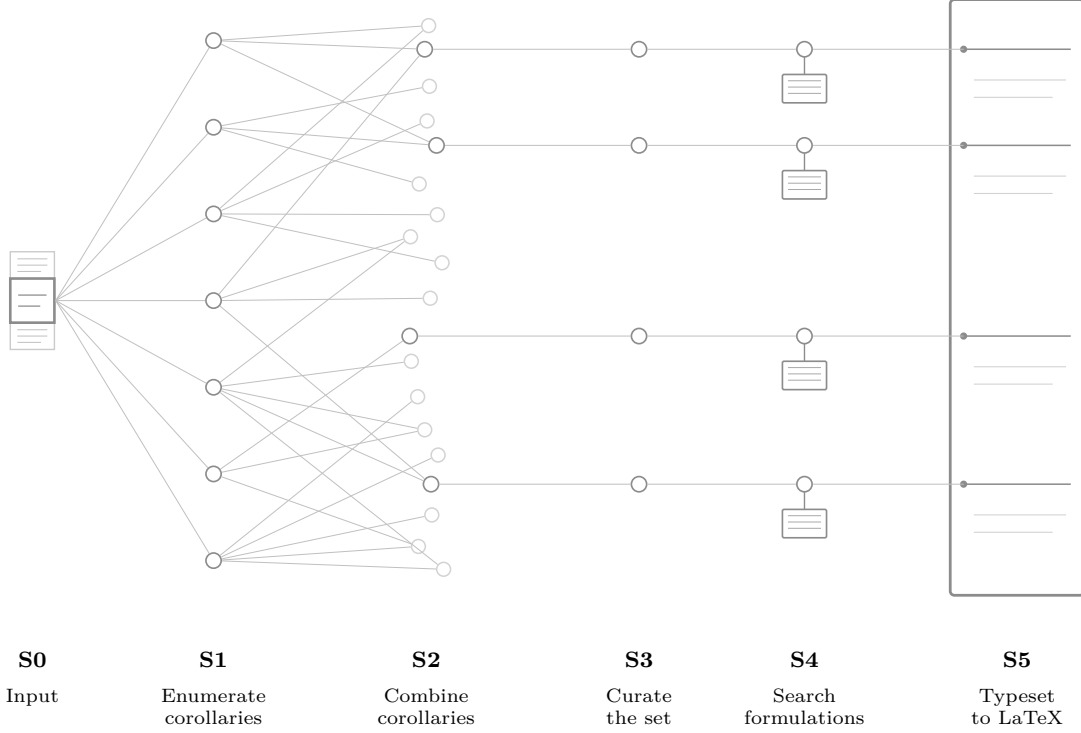


Figure 5.1: Schematic of the second iteration of the protocol for a single theorem, from the input S0 on the left through stages S1–S5 to the document of typeset corollaries on the right.

Results and interpretation

Generated corollary notes and interpretation. The second iteration of the protocol creates generated corollary notes. Before proceeding to the quantitative rank comparison, we first look qualitatively at the generated corollary notes for T1–T3. We use the internal numbering of the generated output. This time we also compare the notes to the first iteration.

T1. *Generated corollary note.* The closest generated corollary note is Corollary 7, which gives the weighted L^1 -conclusion under disjoint 1-additivity, with the Fatou property explicit in the generated corollary note. The closest search formulation is: “A Banach space of measurable functions with the ideal property, indicators on positive-measure subsets and the Fatou increasing-limit rule becomes $L^1(w d\mu)$ once disjoint positive pieces have additive norms.”

Interpretation. Compared with the first iteration, the corollary itself is very similar. However, the search formulations are much more descriptive and closer to the seed language.

T2. *Generated corollary note.* The closest generated corollary note is Corollary 23, which sets $p = 1$ and writes the weighted integral estimate with one constant C . The closest search formulation is: “A family of scalar-homogeneous maps satisfies the one-sum vector inequality precisely when it is uniformly bounded by the same weighted L^1 -seminorm sitting below the given function norm.”

Interpretation. As in the first iteration, the choices $Y = X$ and $T(f) = |f|$ are not made in any corollary, so the generated corollary notes still do not provide the useful recovery reduction. However, because the search formulations are again more descriptive, the generated corollary notes for T2 do move closer to the seed language as well.

T3. *Generated corollary note.* The closest generated corollary note is Corollary 8, which gives an isometric weighted L^1 -representation under $(1, 1)$ -convexity with constant 1 and $(1, 1)$ -concavity with constant 1. The closest search formulation is: “ $X = L^1(\Omega, w d\mu)$ isometrically when a complete ideal of measurable functions over a σ -finite measure space contains an indicator of a positive subset inside every positive-measure set and has additive norm on absolute-value sums for every finite family.”

Interpretation. For T3 there is a very large improvement compared with the first iteration. The first iteration had the $(1, b)$ -convexity and $(1, 1)$ -concavity conditions and the condition $AB = 1$ in separate corollaries. Since the second iteration also chooses $b = 1$ and combines this with $AB = 1$, T3 now attains an isometric weighted L^1 -representation.

It would be quite reasonable to conclude from this qualitative comparison alone that the second iteration of the protocol produces better results than the first iteration. The recovery through T2 remains open, which is not surprising, as the choices $Y = X$ and $T(f) = |f|$ ask for more creativity. For T1 and T3, the generated corollary notes get close, apart from the final translation back to positive additivity.

Search results and interpretation. As we still want to quantify the exact effect on search, we run Vectara in the same way as in the first iteration. We add the generated corollary notes to the corresponding theorem chunks, create C1 from C0 and upload both corpora to Vectara. Using `hybrid_010_rerank`, we then compare the ranks on the five queries. Tables 5.5 and 5.6 show the target and control ranks in this second iteration.

Query	T1		T2		T3		Top-10 hits	
	C0	C1	C0	C1	C0	C1	C0	C1
Q1	54	10	1256	769	30	1	0	2
Q2	690	1	5166	249	92	2	0	2
Q3	794	2	1358	58	83	1	0	2
Q4	262	7	1214	138	37	3	0	2
Q5	3589	2	3782	23	921	1	0	2

Table 5.5: Second iteration. Ranks of the target theorems T1–T3 in the baseline corpus C0 and the corpus C1 after adding the generated corollary notes from the second iteration. Rank 1 is the top result, so a lower rank means that the theorem appears higher in the list. The last two columns count how many targets appear in the first ten results.

Query	T4		T5		T6		Top-10 hits	
	C0	C1	C0	C1	C0	C1	C0	C1
Q1	203	203	698	537	2549	2499	0	0
Q2	320	113	434	450	293	296	0	0
Q3	146	148	524	522	841	842	0	0
Q4	633	634	22	22	828	829	0	0
Q5	673	676	23291	19869	189	188	0	0

Table 5.6: Second iteration. Ranks of the control theorems T4–T6 in the baseline corpus C0 and the corpus C1 after adding the generated corollary notes from the second iteration. The last two columns count how many controls appear in the first ten results.

The second iteration of the protocol gives a large improvement in retrieval of the target theorems. In the first iteration, the target theorems appeared in the top ten only three times across the five queries. Now T1 and T3 appear in the top ten for every query. Also, for every query, at least one of T1 and T3 appears in the top three. This means that a searcher would now immediately encounter a relevant target theorem for all five seed-level queries. As expected, T2 remains outside the top ten, since no recovery has been made. The equivalent search formulations do contain language that links it to the seed case, so T2 moves upward, but not far enough to be found in practice. The control theorems do not show the same behaviour. Therefore, the improvement for T1–T3 can again not be explained by a general heuristic that adding text helps every theorem chunk equally.

Chapter 6

Concluding remarks and outlook

This thesis started by introducing the language of Banach function spaces. Then we proved the seed theorem, showing that positive additivity of a Banach function space norm forces an isometric identification with a weighted L^1 -space. From there we explored this phenomenon through disjoint p -additivity, vector-valued estimates and (p, b) -convexity/ (p, d) -concavity, recovering the seed theorem in each case. These recoveries showed an asymmetry. Generalizing from the seed theorem is hard, as one has to search for theorems in an unknown language, while the recovery is very easy once the general theorem is known. We set out to exploit this asymmetry by creating a protocol that uses large language models to turn general theorem statements into searchable corollaries at lower levels of generality. As the second iteration of the protocol showed that some of the general theorems became much easier to find from the language of the seed theorem, we can conclude that this approach can improve theorem search. However, this experiment covered only a few target theorems, and the recoveries were not made when they required a genuinely creative step. A stronger claim would also require more independent runs because large language model outputs are stochastic.

Thus, as the protocol has not been tested more generally or iterated to perfection, it is not yet finished. However, I do not think such a protocol should ever be finished. Instead of recommending that other authors run this exact protocol, it would be more fruitful to treat it as an open source project and let them extend it. For example, other authors could improve the protocol by introducing prompts specific to their own mathematical fields. The same open structure would also let the protocol improve as better large language models become available. I would also advocate for something even broader: opening up a second layer of text inside paper repositories, dedicated to mathematics produced by large language models. Besides the now foreseen consequences of improving searchability, it is easy to imagine how this can further improve research. Furthermore, a strict separation between a layer of large language model text and human written papers could have the side benefit of keeping the human layer readable for humans.

By increasing theorem searchability, the perspective of an individual mathematician would change as well. Ridges and mountains would start showing in the sea of fog, pointing toward broader worlds.



Figure 6.1: Caspar David Friedrich, *Der Wanderer über dem Nebelmeer*, c. 1818. Oil on canvas. Hamburger Kunsthalle, Hamburg.

Image credits

- Figure 3.4** Caspar David Friedrich, *Der Mönch am Meer*, 1808–1810, oil on canvas, Alte Nationalgalerie, Staatliche Museen zu Berlin. Image source: Google Arts & Culture via Wikimedia Commons, Public Domain Mark 1.0. Accessed 3 June 2026. https://commons.wikimedia.org/wiki/File:Caspar_David_Friedrich_-_Der_M%C3%B6nch_am_Meer_-_Google_Art_Project.jpg
- Figure 4.1** Rembrandt van Rijn, *Christ in the Storm on the Sea of Galilee*, 1633, oil on canvas, Isabella Stewart Gardner Museum, Boston. Missing since the 1990 theft. Image source: Isabella Stewart Gardner Museum via Wikimedia Commons, public domain. Accessed 3 June 2026. https://commons.wikimedia.org/wiki/File:Rembrandt_Christ_in_the_Storm_on_the_Lake_of_Galilee.jpg
- Figure 6.1** Caspar David Friedrich, *Der Wanderer über dem Nebelmeer*, c. 1818, oil on canvas, Hamburger Kunsthalle, Hamburg. Image source: Wikimedia Commons, Public Domain Mark 1.0. Accessed 7 June 2026. https://commons.wikimedia.org/wiki/File:Caspar_David_Friedrich_-_Wanderer_above_the_Sea_of_Fog.jpeg

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Appendix A

Search experiment material

A.1 Control theorems

The following three theorems are the negative controls used in the experiment. They receive generated corollary notes together with T1–T3, but they are not expected to recover the seed theorem.

T4. Generation label: Theorem 4. Chunk `arxiv-1612_01767-013`.

Theorem A.1 (Hadamard means of positive kernel operators). *Let A and B be positive kernel operators on $L^2(X, \mu)$. Then*

$$\begin{aligned} \|A^{(1/2)} \circ B^{(1/2)}\| &\leq \rho((A^*A)^{(1/2)} \circ (B^*B)^{(1/2)})^{1/2} \\ &\leq \rho((A^*AB^*B)^{(1/2)} \circ (B^*BA^*A)^{(1/2)})^{1/4} \\ &\leq \rho(A^*AB^*B)^{1/4} = \|AB^*\|^{1/2} \leq \|A\|^{1/2}\|B\|^{1/2}, \end{aligned}$$

and, if $\alpha \in [0, 1]$,

$$\|A^{(\alpha)} \circ B^{(1-\alpha)}\| \leq \rho((A^*A)^{(\alpha)} \circ (B^*B)^{(1-\alpha)})^{1/2} \leq \|A\|^\alpha \|B\|^{1-\alpha}.$$

If the underlying space is $\ell^2(R)$, then

$$\|A^{(\alpha)} \circ B^{(\beta)}\| \leq \rho((A^*A)^{(\alpha)} \circ (B^*B)^{(\beta)})^{1/2} \leq \|A\|^\alpha \|B\|^\beta$$

whenever $\alpha, \beta > 0$ and $\alpha + \beta \geq 1$.

T5. Generation label: Theorem 5. Chunk `arxiv-2102_10088-004`.

Theorem A.2 (Haar multipliers and branch variation). *Let $(a_I)_{I \in \mathcal{D}^+}$ be a collection of scalars, and let D be the associated Haar multiplier. Define*

$$\text{trn}(D) = \sup \left(\sum_{k=1}^{\infty} |a_{I_k} - a_{I_{k-1}}| + \lim_k |a_{I_k}| \right),$$

where the supremum is taken over all $(I_k)_{k=0}^{\infty} \in [\mathcal{D}^+]$. Then D is bounded, and hence extends to a bounded linear operator on $L_1(X)$, if and only if $\text{trn}(D) < \infty$. More precisely,

$$\|D\| \leq \text{trn}(D) \leq 3\|D\|.$$

T6. Generation label: Theorem 6. Chunk `arxiv-2410_01012-002`.

Theorem A.3 (Parabolic fractional maximal functions and Sawyer testing). *Let $1 < q \leq r < \infty$, let $0 \leq \alpha < 1$, and let w, v be weights. Then*

$$[(w, v)]_{S_{q, r, \alpha}^+} = \sup_R \left(\int_{R^+} v^{1-q'} dx dt \right)^{-1/q} \left(\int_R (M_\alpha^+(v^{1-q'} \chi_{R^+}))^r w dx dt \right)^{1/r} < \infty$$

if and only if there is a constant C such that

$$\left(\int_{\mathbb{R}^{n+1}} (M_{\alpha, c}^+ f)^r w dx dt \right)^{1/r} \leq C [(w, v)]_{S_{q, r, \alpha}^+} \left(\int_{\mathbb{R}^{n+1}} |f|^q v dx dt \right)^{1/q}$$

for every $f \in L_{\text{loc}}^1(\mathbb{R}^{n+1})$.

A.2 Corollary generation protocol (first iteration)

Input	a. <code>theorem_label</code> b. <code>theorem_text</code>
Task	a. Formulate all useful corollaries supported by <code>theorem_text</code>. b. As a guideline, produce 6 to 15 corollaries, that will serve as reductions of the theorem to smaller cases. c. Use fewer if more cannot be genuinely supported by the theorem in combination with mathematical steps that take up at most one page. d. Use more only if they are mathematically distinct and useful for search. Equivalent language formulations will be done separately. e. Include reductions, special cases, endpoint cases, restricted-family cases, equivalent readings and short theorem-internal derivation etc. f. Exclude unsupported corollaries.
Allowed mathematical steps	a. Fix parameters, quantified objects, auxiliary data, or families already present in <code>theorem_text</code> to get to said reductions. b. For example, endpoint, singleton, one-object, diagonal and minimal nontrivial case paths are examples of how to reduce theorems. c. Rewrite an abstract hypothesis as a simpler condition on the remaining objects. d. Combine hypotheses and conclusions already present in <code>theorem_text</code>. e. Do not use external theorems, outside applications, target problems, or unstated completion arguments.
Equivalent search formulations	a. Give 2 to 6 short formulations in different mathematical language when possible. b. Each formulation must state the same mathematical claim as the corollary. c. Prefer genuinely different search language: conclusion language, hypothesis language, endpoint language, representation language, estimate language, or object-language already present in <code>theorem_text</code>. d. Do not add a new application, stronger conclusion, weaker hypothesis, or target problem. e. If no genuinely different equivalent formulation exists, return an empty list. f. The equivalent formulations are to help expand to what a searcher might ask.
Argument	a. State the specialization, restriction, fixed object, or reading of the theorem. b. Explain why the corollary follows from <code>theorem_text</code>. c. Keep the argument short but mathematically checkable.
Output	For each corollary write the following in human-readable LaTeX. a. Label. b. Statement in formal mathematical statement language. c. Equivalent search formulations. d. Argument.

Return the following valid JSON:

```
{
  "theorem_label": "Theorem N",
  "corollaries": [
    {
      "label": "Corollary 1",
      "statement": "...",
      "equivalent_search_formulations": [
        "..."
      ],
      "argument": "..."
    }
  ]
}
```

A.3 Corollary generation protocol (second iteration)

S1. Enumerate corollaries.

Run this as a Codex /goal. Keep going until no further reduction or corollary can be added.

Input: `./input/theorem.json`, a theorem with its definitions, lemmas, and results.

Reduce the theorem to every more special or more concrete true statement that follows from it together with this material. This includes fixing each arbitrary object to specific concrete cases. You may use any knowledge to propose candidate reductions, but every step of the argument must follow from the theorem and this material alone. Be exhaustive and complete rather than selective. There is no upper limit.

Give each record a statement and a short argument, and leave `equivalent_search_formulations` empty.

In `./output/pass-log.md`, note what you considered and rejected, and your stopping reason.

Output: valid JSON to `./output/metadata.json` of the form
`{"theorem_label":"...", "corollaries":[{"label":"...", "statement":"...", "equivalent_search_formulations":[], "argument":"..."}]}`.

S2. Combine corollaries.

Run this as a Codex /goal. Keep going until the set is a fixed point, with no further true record following from the existing ones.

Input: `./input`, `theorem.json` plus `corollaries.json`, a set of records for the theorem.

Add every further record that is true and follows from the theorem, the supplied material, and the existing records by a checkable argument. Carry all existing records through unchanged, and justify every step only from the theorem, this material, and the existing records.

Work in rounds. In each round, specialize and combine the existing records in every way that still follows, including re-specializing records that already fixed something. Then repeat on the records you add, until nothing new follows.

Keep a smaller case even when a more general one subsumes it, as long as its statement is simpler or more concrete.

Each record keeps a statement and a short argument, and leaves `equivalent_search_formulations` empty.

In `./output/pass-log.md`, note what you considered and rejected, and your stopping reason.

Output: valid JSON to `./output/metadata.json` with all original and new records, same schema.

S3. Curate the set.

Run this as a Codex /goal. Keep going until the kept set is stable and each kept record earns its place.

Input: `./input`, `theorem.json` plus `corollaries.json`, a JSON object with a "corollaries" array of records, each with a statement and an argument.

Keep a focused set most useful to a future searcher who does not yet know this theorem and works in elementary terms. A record earns its place when its conclusion is concrete and self-contained, stable in terms a non-expert could search for, and when it opens a genuinely different search from the records already kept.

Be strict about what counts as a genuinely different search. When several records are minor variations on the same idea, such as the same kind of object at different sizes or one statement specialized in routine ways, keep only the one or two clearest representatives a searcher would actually use and drop the rest. Also remove duplicates, purely technical intermediate bounds, and trivial or vacuous restatements.

Still keep a specific special case that opens its own distinct search, even when a more general record implies it. The sharpest and most elementary special cases are

exactly what to keep. A case a searcher would phrase very differently from the general statement stays, even when it is part of something larger. Across the kept set, span the different directions of reduction.

Keep each kept record and all its fields unchanged. Do not edit, merge, or rewrite them.

In `./output/pass-log.md`, say why for each kept and each removed record.

Output: valid JSON to `./output/metadata.json`, a JSON object with a "corollaries" array containing only the kept records, same schema.

S4. Search formulations.

Run this as a Codex /goal. Keep going until this one record carries every distinct concrete restatement a specific searcher would use.

Input: `./input`, `theorem.json` with the theorem and its definitions, plus `corollaries.json`, a JSON object whose "corollaries" array holds a single record. Its statement and argument are given, and its `equivalent_search_formulations` is empty.

Think carefully about this one record. Fill its `equivalent_search_formulations` with short restatements of the same claim, in the concrete working language of a searcher who faces one specific situation it covers but does not know this theorem. Lead with the concrete conclusion about the objects, in plain terms a non-expert would search. Use the definitions to state the situation directly. Strip the theorem's internal parameter names, labels, and machinery rather than leaning on them. Describe the concrete situation rather than naming an abstract class it falls under.

Give genuinely different angles rather than near-duplicates. Cover the assumed property and what it forces, the conclusion as a feature one would observe, a plain-word rendering of any formula, and concrete instances of the objects where natural.

Change the statement and argument in no way, add no new claim, and weaken no hypothesis. Use only what the theorem and packet license.

Output: valid JSON to `./output/metadata.json`, the same object with `equivalent_search_formulations` filled.

S5. Typeset to LaTeX.

Run this as a Codex /goal. Keep going until every record is in the output, verbatim, correctly typeset, with balanced braces. Do not stop early; if anything is missing, continue. Then stop: do not loop trying to verify the document beyond that.

Input: `./input`, `corollaries.json`, a JSON object with a "corollaries" array of records, each with a label, a statement, an argument, and `equivalent_search_formulations`.

Write a LaTeX body to `./output/appendix-body.tex` that presents the records in thesis style. Output the body only, with no preamble and no `\begin{document}`: it will be compiled inside a standard article with `amsthm` and a `\newtheorem{corollary}` already defined.

For each record, in the same order as the input, emit exactly this block:

```
% RECORD: <label>
\begin{corollary}
<statement>
\end{corollary}
\smallskip\noindent\textit{Search formulations.}
\begin{itemize}\setlength{itemsep}{0pt}
\item <formulation 1>
\item <formulation 2>
...
\end{itemize}
\smallskip\noindent\textit{Argument.} <argument>
\bigskip
```

The `% RECORD` line carries the record's exact label and lets the content be checked against the source. Keep the blocks in input order and keep the format identical for every record so the appendix is consistent.

Typeset the mathematics, do not change it. Wrap each expression in `\( \)` and convert

the notation: $||x||$ becomes `\lVert x\rVert`, $|x|$ becomes `\lvert x\rvert`, $\int_{\Omega}$ becomes `\int_{\Omega}`, $\sum_{k=1}^n$ becomes `\sum_{k=1}^n`, $\hat{p}$ and $_X$ become `\hat{p}` and `\_X`, $\leq$ becomes `\leq`, $\geq$ becomes `\geq`, infinity becomes `\infty`, $d\mu$ becomes `\,d\mu`, L^1 becomes `L^1`. Leave wording such as (1,1)-convex as prose. Outside math, escape `_` & `%` # as `\_` `\&` `\%` `\#`.

Do not add, drop, merge, reorder, shorten, summarize, or rephrase any record, statement, formulation, or argument. Only typeset what is already there. Every record in the input must appear once, with all of its fields, with the content unchanged.

You cannot compile here and do not need to: an external step compiles the body and reports any LaTeX error back to you to fix on a later round. Write valid LaTeX directly instead. Balance every `{ }` and every `\( \)`, wrap each mathematical expression in `\( \)`, and escape `_` & `%` # outside math. Stop as soon as the body holds all of the records verbatim, in order, with balanced braces. Do not loop trying to verify compilation yourself.

In `./output/pass-log.md`, note the record count you wrote.

Output: `./output/appendix-body.tex` (the LaTeX body described above).