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The Clebsch-Gordan coefficients for a family of natural modules of the Modular Double of the quantum group $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$

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“The Clebsch-Gordan coefficients for a family of natural modules of the Modular Double of the quantum group $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$”

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Abstract

We will study the Clebsch-Gordan coefficients of the modular double of the quantum group $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$. This will be done by studying and taking a good look at how B. Ponsot and J. Teschner showed how to compute the Clebsch-Gordan coefficients [1]. Moreover, we will also take an introductory look at the concept of quantum groups by looking at some general theory on Hopf $*$-algebras and their representations.
The Clebsch-Gordan coefficients can roughly be described as a relation between a basis of a tensor product $U \otimes V$ of two simple $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-modules and a basis of the decomposition of $U \otimes V$ into simple modules. We will show that this relation can be explicitly described by an integral transformation.
Since this describes a relation between modules of a quantum group, the first part of this thesis will give the necessary information to introduce the reader to the concept of quantum groups and their modules. This will be done by introducing Hopf algebras and their modules and then look at their quantum deformations. This first part will also introduce several examples of algebras, Hopf algebras and quantum groups to make the reader get used to the concept of Hopf algebras and quantum groups.

Contents

Introduction

One goal of this thesis was to take a look at, and explain in detail, how B. Ponsot and J. Teschner computed the Clebsch-Gordan coefficients [1] of a certain natural module of the modular double of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ introduced by Faddeev [2]. Another goal was to give an introduction to the concept of quantum groups, as this theory is needed to understand the modular double and its representation.
The introduction to quantum groups has been approached in such a way that other master students with similar knowledge should be able to understand and be able to work with Hopf $*$-algebras and Hopf $*$-algebra representations. This is done by introducing the reader to the concept of quantum groups, but not only information on the Hopf $*$-algebra $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$. In the hope that the reader gets a good understanding of general concepts on Hopf $*$-algebras, a more general approach to the theory on quantum groups has been taken. However, there is still a big focus on $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$. Thus most of the examples on algebras and representations are chosen in a way that they have some similarities to $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ or its module.
Besides giving information that is needed to understand the structure of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and how to work with Hopf $*$-algebra modules, I have also added some extra details to the proofs and structures given by B. Ponsot and J. Teschner. All of this is added together in one thesis in the hope that all details on the Hopf $*$-algebra $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and the natural module of its modular double are easy to follow for other master students.

So, in this thesis we will be taking a look at [1], with a focus on the Clebsch-Gordan coefficients of the modular double $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})) \otimes U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ introduced by Faddeev [2]. This will be done in an introductory way, in the sense that the paper starts with background information that is needed to understand the subject of Quantum groups and their modules. After this background information, we will be following along [1] to construct the $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-module $\mathcal{P}_\alpha$ and calculate the Clebsch-Gordan coefficients. Thus most of the information around this specific module is given in this paper can also directly be found in [1].
However, in this paper we will immediately consider the module $\mathcal{P}_\alpha$ as a module of the modular double, instead of getting this fact as a consequence. This will not change any of the facts, but will more directly show why certain restrictions are chosen for the module $\mathcal{P}_\alpha$. Besides this, we will be giving a bit more details in some proofs and also add some more details of the module that are found in other papers.
Now, the Racah-Wigner coefficients of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, coefficients that depend on the Clebsch-Gordan coefficients, play an important role in Liouville theory. For example, the Racah-Wigner coefficients can be used to describe normalized fusion blocks of the fusion kernel. Surprisingly, another application of the Racah-Wigner coefficients is that they can be used to calculate the hyperbolic volume of a non-ideal tetrahedron. This was shown by J. Teschner and G.S. Vartanov in [3], where they also allude that there may be a three dimensional gauge theory that has the Racah-Wigner coefficients as its partition functions. So we see that the Racah-Wigner coefficients have several use cases in certain aspects of quantum field theory.
The paper will have the following structure: The first five sections will contain information on certain subjects that are necessary to understand what a $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-module is in general. All of these sections will take a more general approach to the theory to give the reader a good idea of how to work with the subjects. Now, in the first section we will be looking at algebras, coalgebras and Hopf algebras. In the second section we will be looking at the concept of modules of algebras and coalgebras. In the thirds section we will be looking at the concept of universal enveloping algebras of a Lie algebra. Now, since it is natural to introduce Lie algebras via Lie groups, Lie groups will also shortly be a subject of this paper, even though they will not play any role in the later parts. The fourth section will introduce quantum deformations of the real plane and of certain algebras and will also discuss what a Hopf $*$-algebra is.

The sixth section will be about a different $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-module: the Verma module. Most of the things that are discussed in this section can directly be translated to the infinite-dimensional module $\mathcal{P}_\alpha$. However, the action of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ on this module are a bit simpler, as they roughly send a basis element to another basis element. Therefore, this is a decent module to look at first to get used to the notation and how to work with $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-modules.
The final three sections will be about the main subject: the infinite-dimensional $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-module and its Clebsch-Gordan coefficients. Section seven will introduce the module and go over certain properties that it satisfies. The eighth section will be about the decomposition of the tensor product $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ into irreducible modules. This will in turn lead to the definition of the Clebsch-Gordan coefficients. The ninth section will end the thesis with a few small remarks on the Racah-Wigner coefficients. This part can be seen as a generalisation of section eight to the tensor product of more than just two modules.

1 Algebras, coalgebras and bialgebras

Our main goal is to study some properties of a certain module of the Hopf $*$-algebra $\mathcal{U}_q(\mathfrak{sl}(2))$. But before we will look at this Hopf $*$-algebra, we will first look at some general theory on Hopf algebras and their modules. After that, we will introduce the algebra $\mathcal{U}(\mathfrak{sl}(2))$ and finally look at its quantum deformation $\mathcal{U}_q(\mathfrak{sl}(2))$.
This first section will contain general information on algebra, coalgebras and bialgebras, together with some examples. Most of the information in this part is from [4], but certain information on algebras and algebra morphisms can be found in [5]. Do note that we will see some of the more concrete examples in later parts of this thesis. Since they will also be used as examples in later sections.
The later sections that will be about our main subject $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ will not use all of the information given here. The most important parts are the definition of a bialgebra, properties of morphisms and the coproduct and information on the structure of quotient algebras and the tensor algebra. The reason why the definition of a Hopf algebra is less important, is due to the fact that the antipode S, which defines this structure, is not used in the later sections. However, even though not all of the given information is used in later sections, it is still given to give the reader a good idea of how to work with bialgebras and Hopf algebras in general.

1.1 Algebras

Let's start with defining what an algebra is.

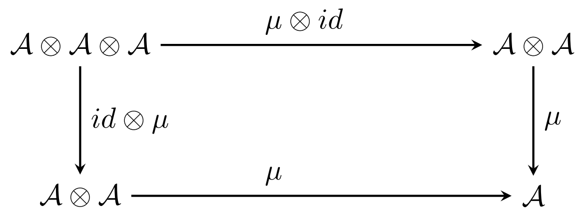
Definition 1.1. An algebra $\mathcal{A}$ over a ring $\mathcal{R}$ is a ring $\mathcal{A}$ with a ring map $\eta_{\mathcal{A}}\colon \mathcal{R} \to \mathcal{A}$, such that the image of $\eta_{\mathcal{A}}$ commutes with all of $\mathcal{A}$.

Now, a ring $\mathcal{R}$ is an additive group $\mathcal{R}$, with a multiplication $\mu_{\mathcal{R}}\colon \mathcal{R}\otimes\mathcal{R} \to \mathcal{R}$ that is associative, distributive and $\mathcal{R}$ contains a unit $1_{\mathcal{R}}$ such that $\mu_{\mathcal{R}}(1_{\mathcal{R}}, x) = x$ for all $x \in \mathcal{R}$. Thus we assume that any ring contains a unit. So, the ring map $\eta_{\mathcal{A}}$ allows us to define scalar multiplication on $\mathcal{A}$ via the map $\mathcal{R}\otimes\mathcal{A} \to \mathcal{A}$; $r\cdot a := \mu_{\mathcal{A}}(\eta_{\mathcal{A}}(r), a)$ for any $r \in \mathcal{R}$ and $a \in \mathcal{A}$. Furthermore, since any algebra $\mathcal{A}$ is defined in combination with some ring $\mathcal{R}$, it is often referred to as an $\mathcal{R}$-algebra. In the case of this thesis, every algebra will be a $\mathbf{k}$-algebra where $\mathbf{k}$ is some field. Note, that if $\mathcal{A}$ is a ring and $\mathbf{k}$ is a field, the map $\eta_{\mathcal{A}}$ of the above definition, induces a vector space structure on $\mathcal{A}$ over $\mathbf{k}$ via $k\cdot a$, $k \in \mathbf{k}$, $a \in \mathcal{A}$, as defined above. This then turns the multiplication map $\mu_{\mathcal{A}}\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ into a bilinear map.
Now, an algebra $\mathcal{A}$ is said to be *abelian*, or commutative, if $\mu_{\mathcal{A}}(a,b) = \mu_{\mathcal{A}}(b,a)$ $\forall a,b \in \mathcal{A}$. In other words, if the *center of* $\mathcal{A}$, $Z(\mathcal{A})$, is all of $\mathcal{A}$, where $Z(\mathcal{A}) = \{a \in \mathcal{A}\colon \mu_{\mathcal{A}}(a,b) = \mu_{\mathcal{A}}(b,a)\ \forall b \in \mathcal{A}\}$.

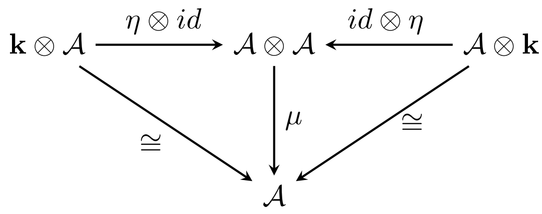
One can also define an algebra with the use of commuting diagrams. This is especially useful later on to quickly see that coalgebras are dual to algebras. However, it gives a less intuitive definition of what an algebra is, which is why the first definition is also given.

Definition 1.2. An algebra $\mathcal{A}$ over a field $\mathbf{k}$ is a triple $(\mathcal{A}, \mu, \eta)$ where $\mathcal{A}$ is a vector space and $\mu\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$, $\eta\colon \mathbf{k} \to \mathcal{A}$ are linear maps such that the following two graphs commute:

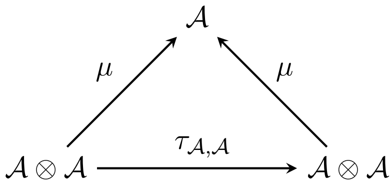
Associativity:



and unitality:



If, in addition, the following diagram commutes, then $\mathcal{A}$ is said to be an abelian algebra:



In the above definition $\tau_{\mathcal{A},\mathcal{B}}\colon \mathcal{A} \otimes \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ is the *flip function.* So, if $a \otimes b \in \mathcal{A} \otimes \mathcal{B}$, then $\tau_{\mathcal{A},\mathcal{B}}(a \otimes b) = b \otimes a$.
Note, if $\mathcal{A}$ is an algebra, not every $a \in \mathcal{A}$ needs to have a multiplicative inverse, i.e. there need not be a $b \in \mathcal{A}$ such that $\mu_{\mathcal{A}}(a, b) = 1 \ \forall a \in \mathcal{A}$. So, to make it easier to refer to invertible elements of an algebra, let $A^\times = \{a \in \mathcal{A}\colon \exists b \in \mathcal{A}\,(\mu_{\mathcal{A}}(a, b) = \mu_{\mathcal{A}}(b, a) = 1)\}$, be the set of all invertible elements of $\mathcal{A}$.

Some useful concepts regarding algebras are algebra morphisms, ideals and modules. These concepts will also be return quite often, as we will also introduce them with respect to, for example, bialgebras or Lie algebras.

Definition 1.3. Let $\mathcal{A}$, $\mathcal{B}$ be algebras, then an algebra morphism $f\colon \mathcal{A} \to \mathcal{B}$ is a ring map such that $f \circ \eta_{\mathcal{A}} = \eta_{\mathcal{B}}$.
In other words, f is a linear map and for $a, b \in \mathcal{A}$, $f(a+b) = f(a)+f(b)$, $f(\mu_{\mathcal{A}}(a, b)) = \mu_{\mathcal{B}}(f(a), f(b))$.

Definition 1.4. Let $\mathcal{A}$ be an algebra, then $\mathcal{I} \subset \mathcal{A}$ is a left-ideal (respectively a right ideal) if

$$\mu_{\mathcal{A}}(a, i) \in \mathcal{I} \quad \text{(respectively } \mu_{\mathcal{A}}(i, a) \in \mathcal{I}\text{)} \qquad \forall a \in \mathcal{A}, i \in \mathcal{I}$$

$\mathcal{I}$ is called a two-sided ideal, or an ideal, if $\mathcal{I}$ is both a left- and right-ideal.

Now, ideals of an algebra let us define what a quotient algebra is.

Theorem 1.5. *Let $\mathcal{A}$ be an algebra over a field* $\mathbf{k}$ *and let $\mathcal{I} \subset \mathcal{A}$ be an ideal. Then the bilinear map*

$$\mu_{\mathcal{A}/\mathcal{I}}\colon \mathcal{A}/\mathcal{I} \otimes \mathcal{A}/\mathcal{I} \to \mathcal{A}/\mathcal{I}; \qquad \mu_{\mathcal{A}/\mathcal{I}}((a+\mathcal{I}), (b+\mathcal{I})) = \mu_{\mathcal{A}}(a, b) + \mathcal{I} = ab + \mathcal{I},$$

defines an algebra structure on $\mathcal{A}/\mathcal{I}$ together with $\eta_{\mathcal{A}/\mathcal{I}} = \eta_{\mathcal{A}}$.
In particular, the natural map $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$; $a \mapsto a + \mathcal{I}$ becomes an algebra morphism with kernel $\mathcal{I}$ when $\mathcal{A}/\mathcal{I}$ is given this algebra structure.

Note that a quotient algebra $\mathcal{A}/\mathcal{I}$ contains classes $a + \mathcal{I}$ as its elements. This means that two differently written elements can be the same, as

$$a + \mathcal{I} = b + \mathcal{I} \iff a - b \in \mathcal{I}.$$

To show that $\mathcal{A}/\mathcal{I}$ is indeed an algebra with the above multiplication, it is sufficient to show that the above property holds for $\mu_{\mathcal{A}/\mathcal{I}}$. In other words, the only thing that needs to be shown is that if $a + \mathcal{I} = a' + \mathcal{I}$, $b + \mathcal{I} = b' + \mathcal{I}$, then $ab + \mathcal{I} = a'b' + \mathcal{I}$. Thus it suffices to show that $ab - a'b' \in \mathcal{I}$. Which follows from the fact that $\mathcal{I}$ is an ideal.

Now, let's give some relevant examples of algebras.
We will first give some general examples. Let $\mathcal{A}$ be some algebra. Then one can define an *opposite algebra* $\mathcal{A}^{op}$ as the same algebra $\mathcal{A}$, but with $\mu_{\mathcal{A}^{op}} = \mu_{\mathcal{A}} \circ \tau_{\mathcal{A},\mathcal{A}}$. Notice that if $\mathcal{A}$ is an abelian algebra, then $\mathcal{A} = \mathcal{A}^{op}$.

Another general example is the *polynomial algebra* $\mathcal{A}[x]$, where $\mathcal{A}$ is an algebra. This is the algebra of all polynomials of the form $\sum_{i=0}^{n} a_i x^i$, $a_i \in \mathcal{A}$. Note, $\mathcal{A}[x_1, \ldots, x_n]$ would then be the polynomial algebra with n variables.

A third and final general example is the *free algebra*. Let X be a set and let $\mathbf{k}$ be a field. Then, $\mathbf{k}\{X\}$ is the vector space with basis consisting of all elements $x_i \in X$ and the empty set .
Then, multiplication on $\mathbf{k}\{X\}$ can be defined by:

$$x_1 \cdot x_2 = x_1x_2, \text{ and } (x_1 \cdots x_n) \cdot (x_{n+1} \cdots x_{n+m}) = x_1x_2 \cdots x_{n+m} \text{ for } x_1, \ldots, x_{n+m} \in X.$$

This multiplication turns $\mathbf{k}\{X\}$ into an algebra, called the free algebra on X.
Now, if we take $X = \{x_1, \ldots, x_n\}$ and let $\mathcal{I} \subseteq \mathbf{k}\{X\}$ be the ideal generated by elements of the form $x_ix_j - x_jx_i$ with $i, j \in \{1, \ldots, n\}$. Then we see that $\mathbf{k}\{X\}/\mathcal{I}$ is an abelian algebra. Moreover, we get $\mathbf{k}\{X\}/\mathcal{I} \cong \mathbf{k}[x_1, \ldots, x_n]$.

We will end this part with two more explicit examples, which will return in later parts. Let $\mathcal{A}$ be an algebra. Then $M_n(\mathcal{A})$ is the algebra of all $n \times n$-matrices with entries in $\mathcal{A}$, where the multiplication is the standard matrix multiplication. This gives rise to two specific examples, $GL_2(\mathcal{A})$ and $SL_2(\mathcal{A})$.
Let $\mathcal{A}$ be an abelian algebra, then

$$GL_2(\mathcal{A}) = \left\{ m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A}) \colon ad - bc = det(m) \in \mathcal{A}^\times \right\},$$

$$SL_2(\mathcal{A}) = \{a \in GL_2(\mathcal{A}) \colon det(a) = 1\},$$

are the matrix algebra of invertible matrices and matrices with determinant equal to 1 respectively.
We can also define two polynomial algebras that are related to the above two algebras in a certain sense:

$$GL(2) = M(2)[t]/((ad - bc)t - 1),$$
$$SL(2) = GL(2)/(t - 1) = M(2)/(ab - bc - 1),$$

with $M(2)$ the polynomial algebra $\mathbf{k}[a, b, c, d]$. Notice that both $GL(2)$ and $SL(2)$ are also abelian algebras.
The names for these algebras make sense, as $Hom_{\text{Alg}}(M(2), \mathcal{A}) \cong M_2(\mathcal{A})$ for any abelian algebra $\mathcal{A}$, with $Hom_{Alg}(\mathcal{A}, \mathcal{B})$ the algebra of algebra morphisms from $\mathcal{A}$ to $\mathcal{B}$. The same is true when $M(2)$ is replaced with either $GL(2)$ or $SL(2)$, which follows from the fact that $Hom_{\text{Alg}}(\mathbf{k}[x_1, \ldots, x_n], \mathcal{A}) \cong \mathcal{A}^n$ and $M_2(\mathcal{A}) \cong \mathcal{A}^4 = \bigoplus_{i=1}^4 \mathcal{A}$. So the algebras $M(2)$, $GL(2)$ and $SL(2)$ are related to the matrix algebras $M_2(\mathcal{A})$, $GL_2(\mathcal{A})$ and $SL_2(\mathcal{A})$ for $\mathcal{A}$ abelian via an isomorphism between morphisms and matrices.
For example, the isomorphism $Hom_{Alg}(M(2), \mathcal{A}) \cong M_2(\mathcal{A})$, $\mathcal{A}$ an abelian algebra, $f \colon M(2) \to \mathcal{A}$ an algebra morphism, is given by:

$$f \mapsto \begin{pmatrix} f(a) & f(b) \\ f(c) & f(d) \end{pmatrix}.$$

The same function is used for $GL(2)$ and $SL(2)$. Note, we will use $Hom(\mathcal{A}, \mathcal{B})$ for the set of linear maps from $\mathcal{A} \to \mathcal{B}$. Also, we will be using $Hom_{text}(\mathcal{A}, \mathcal{B})$ in general for the set of morphisms of a certain structure. For example, $Hom_{Alg}(\mathcal{A}, \mathcal{B})$ is the set of algebra morphisms and $Hom_{Lie}(\mathcal{A}, \mathcal{B})$ the set of Lie algebra morphism, with Lie algebras a structure that will be introduced later on in section 3.2.
Eventually it will be shown that the vector spaces $GL(2)$ and $SL(2)$ can also be turned into coalgebras and even Hopf-algebras. Later on we will mostly be concerned with $SL(2)$, but $M(2)$ and $GL(2)$ also have nice properties and are two other explicit examples that are closely related to $SL(2)$.

1.2 Tensor products of vector spaces and algebras

This next part will be on tensor algebras, since $\mathcal{U}_q(sl(2))$ will be defined as a quotient algebra of a tensor algebra. Roughly speaking, a tensor algebra is just the tensor product of two algebras. However, unlike the tensor product of vector spaces, we do need to make sure that the unit and multiplication maps are well-defined.

We will end this part with looking at the tensor algebra. This is a tensor product of vector spaces that will be turned into an algebra. Do note that Appendix A: Tensor Products contains information on the tensor product between vector space that will be assumed to be known.

Theorem 1.6. *Let* $\mathcal{A}, \mathcal{B}$ *be algebras over* $\mathbf{k}$*, then the bilinear map*

$$\mu_{\mathcal{A}\otimes\mathcal{B}} \colon (\mathcal{A}\otimes\mathcal{B})\otimes(\mathcal{A}\otimes\mathcal{B}) \to \mathcal{A}\otimes\mathcal{B}; \qquad \mu_{\mathcal{A}\otimes\mathcal{B}}(a\otimes b, a'\otimes b') = \mu_{\mathcal{A}}(a,a')\otimes\mu_{\mathcal{B}}(b,b') = aa'\otimes bb'$$

defines an algebra structure on $\mathcal{A}\otimes\mathcal{B}$*, where* $\mathcal{A}\otimes\mathcal{B}$ *is called the tensor product of* $\mathcal{A}$ *and* $\mathcal{B}$*.*

Note that $1\otimes 1 \in \mathcal{A}\otimes\mathcal{B}$ is the unit of this tensor algebra. Also, by defining $i_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}\otimes\mathcal{B}$, $i_{\mathcal{A}}(a) = a\otimes 1$ (similar for $\mathcal{B}$), we get two algebra morphisms such that,

$$a\otimes b = i_{\mathcal{A}}(a)i_{\mathcal{B}}(b) = i_{\mathcal{B}}(b)i_{\mathcal{A}}(a); \qquad \forall a\in\mathcal{A},\, b\in\mathcal{B}.$$

We also have the following universal property of the tensor product of algebras:

Proposition 1.7. Let $\mathcal{A},\mathcal{B},\mathcal{C}$ be three algebras over $\mathbf{k}$ and $f\colon \mathcal{A}\to\mathcal{C}$ and $g\colon\mathcal{B}\to\mathcal{C}$ two algebra morphisms such that

$$\forall (a,b)\in\mathcal{A}\times\mathcal{B}\,(\ f(a)g(b) = g(b)f(a)\ )\,.$$

Then, $\exists! f\otimes g\colon \mathcal{A}\otimes\mathcal{B}\to\mathcal{C}$, algebra morphism, such that $(f\otimes g)\circ i_{\mathcal{A}} = f$ and $(f\otimes g)\circ i_{\mathcal{B}} = g$.
In particular, if $\mathcal{C}$ is abelian, we get $\mathrm{Hom}_{Alg}(\mathcal{A}\otimes\mathcal{B},\mathcal{C}) \cong \mathrm{Hom}_{Alg}(\mathcal{A},\mathcal{C})\times$ $\mathrm{Hom}_{Alg}(\mathcal{B},\mathcal{C})$.

In other words, we see that all algebra morphisms of $\mathcal{A}\otimes\mathcal{B}$ to an algebra $\mathcal{C}$ can be constructed precisely by the pairs (f,g) of algebra morphisms $f\colon\mathcal{A}\to\mathcal{C}$ and $g\colon\mathcal{B}\to\mathcal{C}$ such that $Im(f)$ and $Im(g)$ commute in $\mathcal{C}$.
This proposition can than be used to show the following:

Proposition 1.8. Let X be a set and let $I\subset\mathbf{k}\{X\}$ be an ideal of the free algebra. Let $\mathcal{A} = \mathbf{k}\{X\}/I$ be the quotient algebra and let X', X'' be copies of X with I', I'' ideals of $\mathbf{k}\{X'\}$ and $\mathbf{k}\{X''\}$ respectively such that they resemble the ideal $I\subset X$.
Then, $\mathcal{A}\otimes\mathcal{A}\cong A^{\otimes 2} = \mathbf{k}\{X'\sqcup X''\}/\,(I',\, I'',\, X'X''-X''X')$, where $X'\sqcup X''$ is the disjoint union of X' and X'' and $X'X''-X''X'$ is the two-sided ideal of $\mathbf{k}\{X'\sqcup X''\}$ generated by the elements $x'x''-x''x'$, $x'\in X'$, $x''\in X''$.

Proof. For any $x\in X$, let $x'\in X'$ and $x''\in X''$ be the corresponding copy of x. Let $\varphi'\colon\mathcal{A}\to\mathcal{A}^{\otimes 2}$ and $\varphi''\colon\mathcal{A}\to\mathcal{A}^{\otimes 2}$ be given by $\varphi'(x) = x'$, $\varphi''(x) = x''$.
As φ' and φ'' are algebra morphisms and $x'y'' = y''x'$, $x'\in X', y''\in Y''$, in $\mathcal{A}^{\otimes 2}$. It follows from proposition 1.7, that we have a unique algebra morphism $\varphi\colon\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}^{\otimes 2}$; $x\otimes y\mapsto x'y''$.
Conversely, let $\psi\colon\mathcal{A}^{\otimes 2}\to\mathcal{A}\otimes\mathcal{A}$, $\psi(x') = x\otimes 1$, $\psi(x'') = 1\otimes x$ for $x'\in X'$, $x''\in X''$. ψ is also an algebra morphism and also the inverse of φ, hence $\mathcal{A}\otimes\mathcal{A}\cong\mathcal{A}^{\otimes 2}$. ☐

Later on we'll use this proposition to show more facts about $M(2)$, $GL(2)$ and $SL(2)$. But for now we'll move on to shortly look at the tensor algebra.

Definition 1.9. Let V be a vector space and let $T^0(V) = \mathbf{k}$, $T^1(V) = V, T^2(V) = V\otimes V = V^{\otimes 2}$ and $T^n(V) = V^{\otimes n}$. As we have $T^n(V)\otimes T^m(V)\cong T^{n+m}(V)$, we can define an associative multiplication on $T(V) = \bigoplus_{n\in\mathbb{N}} T^n(V)$ given by,

$$(x_1\otimes\cdots\otimes x_n)(x_{n+1}\otimes\cdots\otimes x_{n+m}) = x_1\otimes\cdots\otimes x_{n+m}, \qquad x_1,\ldots,x_{n+m}\in V.$$

The vector space $T(V)$, together with the above multiplication, is called the *tensor algebra* of V

We can also define $i_V : V \to T(V)$ to embed the elements of V into $T(V)$, just like we did above for embedding $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{B}$. This embedding, together with the multiplication of $T(V)$, gives us the following simplification:

$$x_1 \otimes \cdots \otimes x_n = i_V(x_1) \cdots i_V(x_n).$$

So we can set $x_1 \cdots x_n = x_1 \otimes \cdots \otimes x_n \in T(V)$ for $x_1, \ldots, x_n \in V$. This mostly helps with keeping things short and clean when dealing with products of elements of V in $T(V)$, which is something that will come up a lot. Especially when we will look at $\mathcal{U}(sl(2))$, which we will later show to be a tensor algebra generated by elements of the "vector space" $sl(2)$.

A nice thing about the tensor algebra is that algebra morphisms from $T(V)$ to some algebra $\mathcal{A}$ are closely related to linear maps from V to $\mathcal{A}$. The tensor algebra is itself closely related to a certain free algebra. So, even though the tensor algebra looks really complicated, some properties regarding this algebra are closely related to simpler forms.

Proposition 1.10. Let $\mathcal{A}$ be any algebra and V some vector space.

(i) If $f : V \to \mathcal{A}$ is some linear map, then

$$\exists! \bar{f} : T(V) \to \mathcal{A} \ \left(\bar{f} \circ i_V = f\right), \text{ algebra morphism.}$$

In particular, the map $\bar{f} \mapsto \bar{f} \circ i_V$ is a bijection, so $\mathrm{Hom}_{Alg}(T(V), \mathcal{A}) \cong \mathrm{Hom}(V, \mathcal{A})$.

(ii) Let I be an indexing set for a basis of the vector space V. Then $T(V) \cong \mathbf{k}\{I\}$.

In general the tensor algebra $T(V)$ is not abelian. However, we can construct an abelian algebra by simply "dividing out" the non-commuting terms.

Definition 1.11. Let V be a vector space and let $I(V)$ be the ideal of $T(V)$ that's generated by all elements of the form $xy - yx$ for $x, y \in V$. Then $S(V) = T(V)/I(V)$ is an abelian algebra called the *symmetric algebra.*

The symmetric algebra $S(V)$ has some properties that are almost identical to the properties of the tensor algebra $T(V)$ of proposition 1.10.

Proposition 1.12. Let $\mathcal{A}$ be any algebra, V some vector space.

(i) If $f : V \to \mathcal{A}$ is a linear map such that $f(x)f(y) = f(y)f(x)$ for any $x, y \in V$. Then

$$\exists! \bar{f} : S(V) \to \mathcal{A} \ \left(\bar{f} \circ i_V = f\right), \text{ algebra morphism.}$$

In particular, if $\mathcal{A}$ is abelian, we again get that $\mathrm{Hom}_{Alg}(S(V), \mathcal{A}) \cong \mathrm{Hom}(V, \mathcal{A})$.

(ii) If I is an indexing set for a basis of V, then $S(V) \cong \mathbf{k}[I]$, the polynomial algebra on the set I.

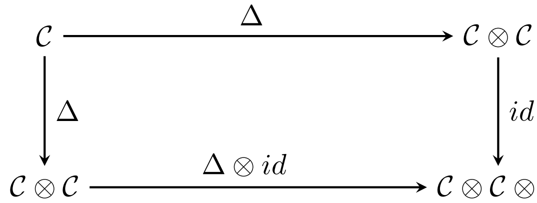
(iii) Let V' be another vector space. Then $S(V \oplus V') \cong S(V) \otimes S(V')$.

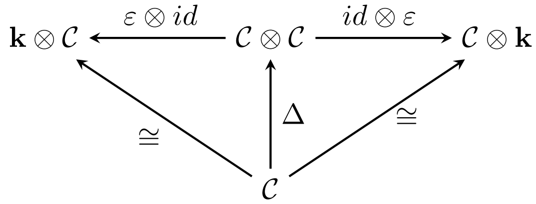
1.3 Coalgebras

Now that we have seen some examples of algebras, we will take a short look at coalgebras. After some short introduction, we will show that the algebras $M(2), GL(2)$ and $SL(2)$ we saw earlier, can also be turned into coalgebras.

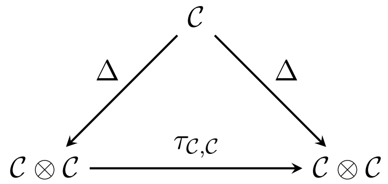
Coalgebras are, like algebras, vectors spaces with two operations defined on them. The definition of a coalgebra is also dual to that of an algebra.

Definition 1.13. A coalgebra is a triple $(\mathcal{C}, \Delta, \varepsilon)$ where $\mathcal{C}$ is a vector space and the *coproduct* $\Delta\colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and *counit* $\varepsilon\colon \mathcal{C} \to \mathbf{k}$ are linear maps such that the following two graphs commute:
Coassociativity; and counitality:





If in addition the following diagram, cocummunity, also commutes, then $\mathcal{C}$ is said to be a cocommutative coalgebra:



Notice that these graphs are exactly the same graphs as in the definition of an algebra 1.2, but the arrows are all going in the opposite direction. Hence, it is said that coalgebras are dual to algebras. Just like with algebras, we also have a coalgebra morphisms:

Definition 1.14. Let $(\mathcal{C}, \Delta, \varepsilon)$ and $\mathcal{C}', \Delta', \varepsilon')$ be two coalgebras. Then a linear map $f\colon \mathcal{C} \to \mathcal{C}'$ is a coalgebra morphism if

$$(f \otimes f) \circ \Delta = \Delta' \circ f \ \text{ and } \ \varepsilon = \varepsilon' \circ f.$$

Since $\Delta(x) \in \mathcal{C} \otimes \mathcal{C}$, we can in general only write it out as $\Delta(x) = \sum_i x_i' \otimes x_i''$ for $x_i', x_i'' \in \mathcal{C}$. But, sometimes the coproduct acts rather nicely on certain elements of the coalgebra.

Definition 1.15. Let $\mathcal{C}$ be a coalgebra and let $x \in \mathcal{C}\backslash\{0\}$. Then x is called a *grouplike* element of $\mathcal{C}$ if $\Delta(x) = x \otimes x$.

Later on we will see some useful cases for both coalgebra morphisms and grouplike elements. But we will first take a quick look at some generic examples of coalgebras, with three concrete examples given in 1.4. First, note that a field $\mathbf{k}$ can be turned into a coalgebra when taking $\Delta(1) = 1 \otimes 1$ and $\varepsilon(1) = 1$. So, any field is in particular also a coalgebra.

Similar to the opposite algebra, we can also define an *opposite coalgebra.* If $(\mathcal{C}, \Delta, \varepsilon)$ is a coalgebra, let $\Delta^{op} = \tau_{\mathcal{C},\mathcal{C}} \circ \Delta$, then $(\mathcal{C}, \Delta^{op}, \varepsilon)$ is also a coalgebra.

Also, the tensor product of two coalgebras $\mathcal{C}, \mathcal{C}'$ can also be given a coalgebra structure. For the counit one can take $\varepsilon \otimes \varepsilon'$, but, unlike with algebras, we cannot just use $\Delta \otimes \Delta'$ as the coproduct, since $\Delta \otimes \Delta'$ will then not be coassociative. No, for $\mathcal{C} \otimes \mathcal{C}'$ to be turned into a coalgebra, its coproduct needs to be defined as $\big(id \otimes \tau_{\mathcal{C},\mathcal{C}'} \otimes id\big) \circ (\Delta \otimes \Delta')$. So we see that $\big(\mathcal{C} \otimes \mathcal{C}', \big(id \otimes \tau_{\mathcal{C},\mathcal{C}'} \otimes id\big) \circ (\Delta \otimes \Delta'), \varepsilon \otimes \varepsilon'\big)$ is a coalgebra.

The tensor algebra $T(V)$ of a vector space V, can also be constructed as a coalgebra instead of an algebra, then called the tensor coalgebra $T'(V)$ of V. Note that the construction of $T(V)$ was done via the isomorphism $V^{\otimes n} \otimes V^{\otimes m} \cong V^{\otimes(n+m)}$. This isomorphism was used to construct a multiplication on the vector space $T(V)$. Similarly, we can use the isomorphism $V^{\otimes(n+m)} \cong V^{\otimes n} \otimes V^{\otimes m}$ to construct a coproduct on $T'(V) = (\oplus_{n\in\mathbb{N}} T^n(V), \Delta, \varepsilon)$ by $\Delta(x_1 \otimes \cdots \otimes x_n \otimes x_{n+1} \otimes \cdots \otimes x_{n+m}) = (x_1 \otimes \cdots \otimes x_n) \otimes (x_{n+1} \otimes \cdots \otimes x_{n+m})$. Together with $\varepsilon(v) = 1$, for $v \in V$, $T'(V)$ turns into a coalgebra. Do note that this coproduct just splits up the tensor product of vectors into two tensor products. So the n in the above explanation need not be fixed. This also means that we can just split the tensor product $x \otimes \cdots x_n \otimes x_{n+1} \otimes x_{n+m}$ at any spot. So $\Delta(x \otimes \cdots x_n \otimes x_{n+1} \otimes x_{n+m})$ is not really a "fixed" element, as we can place the brackets at any spot, but it is exactly one element of $T'(V)$.

A last example we will look at is similar to the free algebra. But before we will look at that, let us note a nice property of the dual spaces of a coalgebra and algebra.

Proposition 1.16. Let $\mathcal{C}$ be a coalgebra, let $\mathcal{A}$ be a finite dimensional algebra and let λ be as in A.5, then

1. the dual space $\mathcal{C}^*$ of $\mathcal{C}$ is an algebra with $\mu_{\mathcal{C}^*} = \Delta^* \circ \overline{\lambda}$ and $\eta_{\mathcal{C}^*} = \varepsilon^*$, and
2. the dual space $\mathcal{A}^*$ of $\mathcal{A}$ is a coalgebra with $\Delta_{\mathcal{A}^*} = \overline{\lambda}^{-1} \circ \mu^*$ and $\varepsilon = \eta^*$, where the superscript $*$ indicates the transpose of a linear map.

The above proposition can be proven by using the diagrams of definitions 1.2 and 1.13 and checking that they are true for the given $\Delta_{\mathcal{A}*}$, $\varepsilon_{\mathcal{A}*}$, $\mu_{\mathcal{C}*}$ and $\eta_{\mathcal{C}*}$. Do note that the above propositions has no condition for the coalgebra $\mathcal{C}$, but it does have one for the algebra $\mathcal{A}$. So, we see that only the dual $\mathcal{A}^*$ of an algebra need not be a coalgebra in general, but the dual $\mathcal{C}^*$ of a coalgebra can always be induced with an algebra structure.

Now, an example of a coalgebra that is similar to the free algebra is the *coalgebra of a set*. Let X be some set and set $\mathcal{C} = \mathbf{k}[X]$, the polynomial vector space with variables in X. Then $\mathcal{C}$ becomes a coalgebra with $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1 \; \forall x \in X$. Obviously, we can also construct a coalgebra of a set for a product set $X \times Y$, with X, Y sets. This gives rise to the coalgebra $\mathbf{k}[X \times Y]$. Now, the nice thing is that

$$\mathbf{k}[X \times Y] \simeq \mathbf{k}[X] \otimes \mathbf{k}[Y],$$

as one would expect. The isomorphism between these coalgebras is given by $\psi(x \otimes y) = (x, y)$, for all $x \in X$, $y \in Y$. So we can construct several examples of tensor products of coalgebras using the coalgebra of a set.
Besides helping us with constructing examples of tensor products of coalgebras, the coalgebra of a set can also be used to construct examples of algebras. If we use the above proposition with $\mathcal{C} = \mathbf{k}[X]$, for X a set, we get the function algebra $\mathcal{C}^*$ consisting of linear functionals $f\colon X \to \mathbf{k}$. This algebra has a unit given by $\eta = \varepsilon$ and multiplication defined as $(fg)(x) = \mu(f,g)(x) = f(x)g(x)$ for $x \in X$, $f, g \in \mathcal{C}^*$.

We finish this part by constructing the quotient coalgebra. This definition will look fairly similar to that of the quotient algebra, theorem 1.5.

Definition 1.17. Let $\mathcal{C}$ be a coalgebra, then $I \subset \mathcal{C}$ is a coideal if $\Delta(I) \subset I \otimes \mathcal{C} + \mathcal{C} \otimes I$ and $\varepsilon(I) = 0$.

Now, let $\mathcal{C}$ be a coalgebra and $I \subset \mathcal{C}$ a coideal of $\mathcal{C}$. Then $\Delta\colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ factors to

$$\overline{\Delta}\colon \mathcal{C}/I \to \mathcal{C} \otimes \mathcal{C}/(I \otimes \mathcal{C} + \mathcal{C} \otimes I) = \mathcal{C}/I \otimes \mathcal{C}/I$$

and $\varepsilon\colon \mathcal{C} \to \mathbf{k}$ factors to

$$\overline{\varepsilon}\colon \mathcal{C}/I \to \mathbf{k},$$

making $(\mathcal{C}/I, \overline{\Delta}, \overline{\varepsilon})$ a coalgebra, called the *quotient-coalgebra*. Later on we will see some examples of quotient-coalgebras. But for now, we will first look at Sweedler's notation for the coproduct. This notation will help us to write the coproduct in a nice and short fashion.

1.3.1 Sweedler's notation

Let $\mathcal{C}$ be a coalgebra. Up till now, we have not really looked at how $\Delta(x)$, $x \in \mathcal{C}$, looks in general. For some of the given examples its easy to write out $\Delta(x)$, e.g. for the coalgebra of a set we just get $\Delta(x) = x \otimes x$. But, in general we can only say that $\Delta(x) = \sum_i x_i' \otimes x_i''$ for $x_i', x_i'' \in \mathcal{C}$.
Due to the coassociativity of the coproduct, we get that $id \otimes \Delta \circ \Delta(x) = \Delta \otimes id \circ \Delta(x)$, which becomes

$$\sum_i x_i' \otimes \Delta(x_i'') = \sum_i \sum_j x_i' \otimes (x_i'')_j' \otimes (x_i'')_j'' = \sum_i \sum_j (x_i')_j' \otimes (x_i')_j'' \otimes x_i'' = \sum_i \Delta(x_i') \otimes x_i''.$$

This quickly becomes a mess of subscripts and upperscript, thus the coproduct will be written as

$$\Delta(x) = \sum_{(x)} x' \otimes x'',$$

and the coassoaciative property will be written as

$$\sum_{(x)} x' \otimes x'' \otimes x''' = \sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)} = \sum_{(x)} \left(\sum_{(x)} (x')' \otimes (x')'' \right) \otimes x'' = \sum_{(x)} x' \otimes \sum_{(x'')} (x'')' \otimes (x'')''.$$

This makes using the coproduct several times a lot more convenient, as we will just write $\sum_{(x)} x^{(1)} \otimes \cdots \otimes x^{(n)}$ for when the coproduct is applied n times to an element $x \in \mathcal{C}$.
With this, the condition for counitality can be rewritten as $\sum_{(x)} \varepsilon(x')x'' = x = \sum_{(x)} x'\varepsilon(x'')$. Here we use that $k \otimes \mathcal{C} \cong \mathcal{C}$ to just get x. This then lead to the following:

$$\sum_{(x)} x^{(1)} \otimes \varepsilon(x^{(2)}) \otimes x^{(3)} \otimes x^{(4)} \otimes x^{(5)} = \sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes x^{(4)},$$

via the same isomorphism $\mathbf{k} \otimes \mathcal{C} \cong \mathcal{C}$.

With this new notation, the condition for $\mathcal{C}$ to be cocommutative becomes

$$\sum_{(x)} x' \otimes x'' = \sum_{(x)} x'' \otimes x', \quad \forall x \in \mathcal{C}.$$

The left relation in definition 1.14 can in turn also be rewritten as,

$$f(\Delta(x)) = \sum_{(x)} f(x') \otimes f(x'') = \sum_{(f(x))} f(x)' \otimes f(x)'',$$

which closely resembles the condition for a linear map to be an algebra morphism.
Lastly, the coproduct of a tensor product of coalgebras $\mathcal{C} \otimes \mathcal{C}'$ can be written as,

$$\Delta(x \otimes y) = \sum_{(x\otimes y)} (x \otimes y)' \otimes (x \otimes y)'' = \sum_{(x)(y)} (x' \otimes y') \otimes (x'' \otimes y''),$$

which is a lot clearer than the original given coproduct of the tensor product of two coalgebras.

1.4 $M(2)$, $GL(2)$ and $SL(2)$ as coalgebras

We have seen some examples of coalgebras that looked like some of the earlier given examples of algebras. To end this section about coalgebras, we will look at the algebra examples $M(2)$, $GL(2)$, $SL(2)$ and show that they can also be turned into coalgebras. Coincidentally, this will also be a good introduction for the next section, which will be about *bialgebras*. As bialgebras will be shown to be vector spaces that are both algebras and coalgebras.

Recall that,

$$M(2) = \mathbf{k}[a,b,c,d],\ \ GL(2) = M(2)[t]/\left((ad-bc)\,t-1\right),\ \ SL(2) = GL(2)/(t-1) = M(2)/(ad-bc-1).$$

Then we get,

$$\begin{aligned} M(2)^{\otimes 2} &= \mathbf{k}[a',a'',b',b'',c',c'',d',d''],\\ GL(2)^{\otimes 2} &= M(2)^{\otimes 2}[t,t']/((a'd'-b'c')t'-1,(a''d''-b''c'')t''-1),\\ SL(2)^{\otimes 2} &= GL(2)^{\otimes 2}/(t'-1,t''-1) = M(2)^{\otimes 2}/(a'd'-b'c'-1,a''d''-b''c''-1)). \end{aligned}$$

Now, matrix multiplication is given by

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a'a''+b'c'' & a'b''+b'd'' \\ c'a''+d'c'' & c'b''+d'd'' \end{pmatrix}$$

and this gives rise to the following algebra morphisms: $\Delta_1\colon M(2)\to M(2)^{\otimes 2}$, $\Delta_2\colon GL(2)\to GL(2)^{\otimes 2}$ and $\Delta_3\colon SL(2)\to SL(2)^{\otimes 2}$, with

$$\begin{aligned} \Delta_i(a) = a'a''+b'c'',\ \Delta_i(b) = a'b''+b'd'',\\ \Delta_i(c) = c'a''+d'c'',\ \Delta_i(d) = c'b''+d'd'', \end{aligned}$$

for $i\in\{1,2,3,\}$ provided we set $\Delta_2(t) = t'\cdot t''$.
Let $\mathcal{A}$ be an abelian algebra, then

$$Hom_{Alg}(M(2),\mathcal{A}) \cong M_2(\mathcal{A}),\ Hom_{Alg}(GL(2),\mathcal{A}) \cong GL_2(\mathcal{A}) \text{ and } Hom_{Alg}(SL(2),\mathcal{A}) \cong SL_2(\mathcal{A}).$$

These isomorphisms let us identify Δ_i with the matrix multiplication in $M_2(\mathcal{A})$, $GL_2(\mathcal{A})$ and $SL_2(\mathcal{A})$. Thus its easier to rewrite these morphisms as:

$$\Delta_i\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \Delta_i(a) & \Delta_i(b) \\ \Delta_i(c) & \Delta_i(d) \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$$

We can also apply proposition 1.8 to note that $M(2)^{\otimes 2}\cong M(2)\otimes M(2)$, $GL(2)^{\otimes 2}\cong GL(2)\otimes GL(2)$ and $SL(2)^{\otimes 2}\cong SL(2)\otimes SL(2)$. Recall that this isomorphism maps $a'\mapsto a\otimes 1$ and $a''\mapsto 1\otimes a$. So, if we also apply this isomorphism, we can further rewrite Δ_i to, $\Delta_1\colon M(2)\to M(2)\otimes M(2)$, $\Delta_2\colon GL(2)\to GL(2)\otimes GL(2)$ and $\Delta_3\colon SL(2)\to SL(2)\otimes SL(2)$, which can be written in matrix form as:

$$\Delta_i\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\otimes\begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \text{and } \Delta_2(t) = t\otimes t.$$

Now, since Δ_i are compositions of two algebra morphisms, they are also algebra morphisms from $M(2)$, $GL(2)$ and $SL(2)$ to $M(2)\otimes M(2)$, $GL(2)\otimes GL(2)$ and $SL(2)\otimes SL(2)$ respectively.
For now we've only constructed an algebra morphism, but the Δ_i are at least possible candidates for a coproduct. But we also need a counit if we want to show that $M(2), GL(2)$ and $SL(2)$ are coalgebras. So, let

$$\varepsilon_1\colon M(2)\to\mathbf{k},\ \varepsilon_2\colon GL(2)\to\mathbf{k} \text{ and } \varepsilon_3\colon SL(2)\to\mathbf{k}$$

be algebra morphism defined by $\varepsilon_i(a) = \varepsilon_i(d) = 1$, $\varepsilon_i(b) = \varepsilon(c) = 0$ and $\varepsilon_2(t) = 1$. The claim is that Δ_i and ε_i turn the above vector spaces into coalgebras. To show this, it's sufficient to show that Δ_i is

coassociative and ε_i is counital for the generators a, b, c, d and also t in the case of $GL(2)$.
First, note that $t \in GL(2)$ is grouplike, so Δ_2 is coassociative for t and

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

due to how Δ_i is constructed. Hence Δ_i is coassociative for $i \in \{1, 2, 3\}$.
Secondly, $\varepsilon_2(t) = 1$ and note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \varepsilon_i \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \varepsilon_i \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

so the counitality graph of definition 1.13 commutes for the generators a, b, c, d and t. Hence ε_i, $i \in \{1, 2, 3\}$, is indeed a counit and $(M(2), \Delta_1, \varepsilon_1)$, $(GL(2), \Delta_2, \varepsilon_2)$ and $(SL(2), \Delta_3, \varepsilon_3)$ are coalgebras.

1.5 Bialgebras and Hopf algebras

We have seen that $M(2)$, $GL(2)$ and $SL(2)$ are both algebras and coalgebras. Moreover, the way we constructed a coalgebra structure on these spaces even turned their respective coproducts and counits into algebra morphisms. Thus, all three vector space have an algebra structure and a coalgebra structure, which in particular is defined via algebra morphisms. This is precisely the definition of a bialgebra.

Definition 1.18. A *bialgebra* is a quintuple $(\mathcal{H}, \mu, \eta, \Delta, \varepsilon)$ such that $(\mathcal{H}, \mu, \eta)$ is an algebra, $(\mathcal{H}, \Delta, \varepsilon)$ is a coalgebra and μ, η are coalgebra morphisms or Δ, ε are algebra morphism.

In this definition, the vector space $\mathcal{H} \otimes \mathcal{H}$ implicitly gets the structure of both a tensor product of algebras and a tensor product of coalgebras. Also, it is sufficient to know if either μ, η are coalgebra morphisms or that Δ, ε are algebra morphisms to know if $(\mathcal{H}, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra, since both conditions are equivalent:

Theorem 1.19. *Let $(\mathcal{H}, \mu, \eta, \Delta, \varepsilon)$ be a quintuple such that $(\mathcal{H}, \mu, \eta)$ is an algebra and $(\mathcal{H}, \Delta, \varepsilon)$ is a coalgebra. Then the following are equivalent:*

1. *The maps μ and η are coalgebra morphisms.*
2. *The maps Δ and ε are algebra morphisms.*

The proof of this theorem comes down to just checking that the graphs that express that Δ and ε are algebra morphisms are the same as the graphs that express that μ and η are coalgebra morphisms. Using Sweedler's notation, the conditions that Δ and ε are algebra morphisms becomes:

$$\Delta(xy) = \sum_{(xy)} (xy)' \otimes (xy)'' = \sum_{(x)(y)} x'y' \otimes x''y'' = \Delta(x)\Delta(y), \quad \Delta(1) = 1 \otimes 1,$$
$$\varepsilon(xy) = \varepsilon(x)\varepsilon(y), \quad \varepsilon(1) = 1 \; \forall x, y \in \mathcal{H}.$$

Definition 1.20. Let $\mathcal{H}, \mathcal{H}'$ be bialgebras and $f \colon \mathcal{H} \to \mathcal{H}'$ a linear map. Then f is called a *bialgebra morphism* if f is both an algebra and coalgebra morphism.

We have already seen some examples of bialgebras, for example $M(2)$, $GL(2)$ and $SL(2)$. We have also seen some indirect examples, for example proposition 1.16 shows that if $\mathcal{H}$ is a finite dimensional bialgebra, then the dual $\mathcal{H}^*$ is also a bialgebra. But, before we continue with some facts on bialgebras, we will give some more examples of bialgebras. These examples will look similar to earlier given examples of algebras and coalgebras.

Let $\mathcal{H}$ be a bialgebra, then we can construct three different bialgebras that looks similar to the opposite algebra or opposite coalgebra. These three bialgebras are given by:

$$\mathcal{H}^{op} = (\mathcal{H}, \mu^{op}, \eta, \Delta, \varepsilon), \quad \mathcal{H}^{cop}(\mathcal{H}, \mu, \eta, \Delta^{op}, \varepsilon) \quad \text{and} \quad \mathcal{H}^{op\,cop} = (\mathcal{H}, \mu^{op}, \eta, \Delta^{op}, \varepsilon).$$

The example $M(2)$ can also be generalised to construct a bialgebra structure on $M(n) = \mathbf{k}[x_{11}, x_{12}, \ldots, x_{nn}]$, the polynomial algebra in n^2 variables. Set

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj} \text{ and } \varepsilon(x_{ij}) = \delta_{ij},$$

then Δ and ε are algebra morphisms, with Δ still resembling matrix multiplication. Together with the algebra structure of the polynomial algebra, $(M(n), \mu, \eta, \Delta, \varepsilon)$ turns into a bialgebra. Note that $n = 2$ does indeed give us the same bialgebra as $M(2)$ defined in the previous section.

A final example of a bialgebra is once again a vector space which we know has both an algebra and coalgebra structure. Namely, the tensor algebra $T(V)$. However, to define a bialgebra structure, we do need to construct a different coproduct on $T(V)$.

Theorem 1.21. *Let V be a vector space, then $\exists!$ bialgebra structure on $T(V)$ such that*

$$\Delta(v) = 1 \otimes v + v \otimes 1 \quad \textit{and} \quad \varepsilon(v) = 0 \;\; \forall v \in V.$$

This is a cocommutative bialgebra and for any $v_1, v_2, \ldots, v_n \in V$ we have

$$\varepsilon(v_1 \cdots v_n) = 0,$$
$$\Delta(v_1 \cdots v_n) = 1 \otimes v_1 \cdots v_n + \sum_{p=1}^{n-1} \sum_{\sigma} v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)} + v_1 \cdots v_n \otimes 1,$$

where σ runs over all permutations of S_n such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \;\; \textit{and} \;\; \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(n),$$

the so called $(p, n-p)-$shuffles.

Notice that $v_1 \cdots v_n$ is again the same short hand notation as in definition 1.9. For the full proof of this theorem, see [4]. This proof mostly comes down to using induction on n to show that ε and $\Delta$are algebra morphisms, then checking that $(T(V), \Delta, \varepsilon)$ is indeed a cocommutative coalgebra with the given coproduct and counit.

Now that we have a general idea of what bialgebras are, we can finally talk about Hopf algebras. Roughly speaking, a Hopf algebra $\mathcal{H}$ is just a bialgebra, but with an extra endomorphism $S \colon \mathcal{H} \to \mathcal{H}$ added to the structure. Since the endomorphism S is what makes a Hopf algebra different from a bialgebra, we will first construct S. Later on we will also see that the endomorphism S is unique if it exists. So we need not be able to define a Hopf algebra structure on a bialgebra, but if we can, the Hopf algebra structure is unique.

Definition 1.22. Let $(\mathcal{A}, \mu, \eta)$ be an algebra and $(\mathcal{C}, \Delta, \varepsilon)$ a coalgebra. Let $\star \colon Hom(\mathcal{C}, \mathcal{A}) \otimes Hom(\mathcal{C}, \mathcal{A}) \to Hom(\mathcal{C}, \mathcal{A})$ be a bilinear map given by,

$$(f \star g)(x) = \mu \circ f \otimes g \circ \Delta(x), \qquad x \in \mathcal{C}.$$

Then $\star$ is called the *convolution* on $Hom(\mathcal{C}, \mathcal{A})$.

Recall that $Hom(\mathcal{C},\mathcal{A})$ is the set of linear maps from $\mathcal{C}$ to $\mathcal{A}$. The convolution on $Hom(\mathcal{C},\mathcal{A})$ can be rewritten as $(f \star g)(x) = \sum_{(x)} f(x')g(x'')$ for any $x \in \mathcal{C}$ using Sweedler's notation.
The convolution can be used as a multiplication on $Hom(\mathcal{C},\mathcal{A})$. Together with a unit, $\eta \circ \varepsilon$, it even becomes an algebra.

Proposition 1.23.

1. The triple $(Hom(\mathcal{C},\mathcal{A}), \star, \eta \circ \varepsilon)$ is an algebra.
2. The map $\lambda_{\mathcal{C},\mathcal{A}}\colon A \otimes C^* \to Hom(\mathcal{C},\mathcal{A})$, with $\lambda_{\mathcal{C},\mathcal{A}}$ as in corollary A.6, is an algebra morphism.

Proof. 1. Note, for $x \in \mathcal{C}$, $f, g, h \in Hom(\mathcal{C},\mathcal{A})$,

$$((f \star g) \star h)(x) = \sum_{(x)} f(x')g(x'')h(x''') = (f \star (g \star h))(x)$$

and

$$((\eta\circ\varepsilon)\star f)(x) = \sum_{(x)} \left(\eta(\varepsilon(x'))\right) f(x'') = \sum_{(x)} \varepsilon(x')f(x'') = \sum_{(x)} f(\varepsilon(x')x'') = f(\sum_{(x)} \varepsilon(x')x'') = f(x).$$

Showing $f \star (\eta \circ \varepsilon) = f$ goes similar. These show that $\star$ is associative and $\eta \circ \varepsilon$ is a unit, thus $Hom(\mathcal{C},\mathcal{A})$ has an algebra structure.

2. Let $a, b \in \mathcal{A}$, $\alpha, \beta \in \mathcal{C}^*$. Then $\forall x \in \mathcal{C}$,

$$(\lambda_{\mathcal{C},\mathcal{A}}(a \otimes \alpha) \star \lambda_{\mathcal{C},\mathcal{A}}(b \otimes \beta))(x) = \sum_{(x)} \alpha(x')\beta(x'')ab = (\alpha\beta)(x)ab = (\lambda_{\mathcal{C},\mathcal{A}}(ab \otimes \alpha\beta))(x),$$

so $\lambda_{\mathcal{C},\mathcal{A}}$ preserves the product and $(\lambda_{\mathcal{C},\mathcal{A}}(1 \otimes \varepsilon))(x) = \varepsilon(x)1 = (\eta \circ \varepsilon)(x)$. Hence, $\lambda_{\mathcal{C},\mathcal{A}}$ is indeed an algebra morphism.

□

Now, if $\mathcal{H}$ is a bialgebra, one can look at $Hom(\mathcal{H},\mathcal{H}) = End(\mathcal{H})$ and we get the following definition:

Definition 1.24. Let $\mathcal{H}$ be a bialgebra. Then $S \in End(\mathcal{H})$ is called an *antipode* of $\mathcal{H}$ if $S \star id_{\mathcal{H}} = id_{\mathcal{H}} \star S = \eta \circ \varepsilon$.
Also, if a bialgebra $\mathcal{H}$ has an antipode S, then $\mathcal{H}$ is called a Hopf algebra, which we will denote as $(\mathcal{H}, \mu, \eta, \Delta, \varepsilon, S)$.

In other words, an endomorphism S of $\mathcal{H}$ is an antipode if it is the inverse of the endomorphism $id_{\mathcal{H}}$ in the algebra $(End(\mathcal{H}), \star, \eta \circ \varepsilon)$. Using Sweedler's notation, we see that the antipode satisfies:

$$\sum_{(x)} x'S(x'') = \varepsilon(x)1 = \sum_{(x)} S(x')x''.$$

Now, if a bialgebra $\mathcal{H}$ has an antipode S, then it is unique. Since, if S and S' are both antipodes of $\mathcal{H}$, then

$$S = S \star (id_{\mathcal{H}} \star S') = (S \star id_{\mathcal{H}}) \star S' = S'.$$

So we see that a bialgebra has at most one Hopf algebra structure, thus having a Hopf algebra structure is a property of the bialgebra itself. Recall that we will be working with a Hopf algebra in the final few sections. Since the existence of the antipode is determined by the bialgebra structure, it is natural to look at the concepts of Hopf algebras. Even though we will not use the antipode S of our Hopf algebra explicitly in the later sections. However, we will later see that we can derive another structure on a Hopf algebra from its antipode S, a $*$-structure. Since we will use this $*$-structure, we do use the antipode implicitly in the later sections.

Similar to algebras, coalgebras and bialgebras, we also have the concept of Hopf algebra morphisms.

Definition 1.25. Let $\mathcal{H}$ and $\mathcal{H}'$ be two Hopf algebras with antipodes S and S' respectively. Let $f\colon \mathcal{H} \to \mathcal{H}'$ be a linear map. Then f is a *Hopf algebra morphism* if it is a bialgebra morphisms and $f \circ S = S' \circ f$.

Remark: the definition of a Hopf algebra morphism f is a bit redundant, as any bialgebra morphism $f\colon \mathcal{H} \to \mathcal{H}'$ between two Hopf algebras $\mathcal{H}$, $\mathcal{H}'$ is automatically a Hopf algebra morphism, in the sense that f will always satisfy $f \circ S = S' \circ f$. However, showing that it is sufficient for $f\colon \mathcal{H} \to \mathcal{H}'$ to just be bialgebra morphism for it to be a Hopf algebra morphism, is not trivial and it also makes it less obvious that Hopf algebra morphisms commute with the two antipodes. So, the property $f \circ S = S' \circ f$ is stated in the above definition to highlight this property of Hopf algebra morphisms.
Nonetheless, we will still prove that any bialgebra morphism $f\colon \mathcal{H} \to \mathcal{H}'$ between two Hopf algebras $\mathcal{H}$, $\mathcal{H}'$ is automatically a Hopf algebra morphism in the sense of the above definition.

Proof. Let $\mathcal{H}$, $\mathcal{H}'$ be two Hopf algebras with antipodes S and S' respectively. Let $f\colon H \to \mathcal{H}'$ be a bialgebra morphism. We want to show that $f \circ S = S' \circ f$. To do that, we will first need the following: Let $\mathcal{A}$ and $\mathcal{A}'$ be algebras and $\mathcal{C}$ and $\mathcal{C}'$ coalgebras. Let $g\colon \mathcal{A} \to \mathcal{A}'$ be an algebra morphism and $h\colon \mathcal{C}' \to \mathcal{C}$ a coalgebra morphism.
Claim: the map $\psi_{g,h}\colon Hom(\mathcal{C},\mathcal{A}) \to Hom(\mathcal{C}',\mathcal{A}');\ p \mapsto g \circ p \circ h$ is an algebra morphism.
First, recall from proposition 1.23 that $Hom(\mathcal{C},\mathcal{A})$ is an algebra with the convolution $\star$ as its product. Now, to prove this claim, let $p, q \in Hom(\mathcal{C},\mathcal{A})$ be arbitrary. Then,

$$\begin{aligned}\psi_{g,h}(p \star q) &= \psi_{g,h}(\mu_{\mathcal{A}} \circ p \otimes q \circ \Delta_{\mathcal{C}}) = g \circ (\mu_{\mathcal{A}} \circ p \otimes q \circ \Delta_{\mathcal{C}}) \circ h = (g \circ \mu_{\mathcal{A}}) \circ p \otimes q \circ (\Delta_{\mathcal{C}} \circ h)\\ &\overset{h \text{ coalg morph}}{=} (g \circ \mu_{\mathcal{A}}) \circ p \otimes q \circ (h \otimes h \circ \Delta_{\mathcal{C}'}) \overset{g \text{ alg morph}}{=} (\mu_{\mathcal{A}'} \circ g \otimes g) \circ p \otimes q \circ (h \otimes h \circ \Delta_{\mathcal{C}'})\\ &= \mu_{\mathcal{A}'} \circ (g \otimes g \circ p \otimes q \circ h \otimes h) \circ \Delta_{\mathcal{C}'} = \mu_{\mathcal{A}'} \circ (g \circ p \circ h \otimes g \circ q \circ h) \circ \Delta_{\mathcal{C}}\\ &= \mu_{\mathcal{A}'} \circ \psi_{g,h}(p) \otimes \psi_{g,h}(q) \circ \Delta_{\mathcal{C}} = \psi_{g,h}(p) \star \psi_{g,h}(q)\end{aligned}$$

Thus we have proven our claim.
Now, we will finally show that $f \circ S = S' \circ f$. This will be shown by proving that $f \circ S$ is a left-inverse and $S' \circ f$ is a right-inverse of f in $Hom(\mathcal{H},\mathcal{H}')$. This implies that $f \circ S = S' \circ f$ and also that they are both the unique inverse of f in $Hom(\mathcal{H},\mathcal{H}')$.

$$\begin{aligned}&f \circ S \star f = f \circ S \circ id_{\mathcal{H}} \star f \circ id_{\mathcal{H}} \circ id_{\mathcal{H}} = \psi_{f,id_{\mathcal{H}}}(S) \star \psi_{f,id_{\mathcal{H}}}(id_{\mathcal{H}}) \overset{\text{by claim}}{=} \psi_{f,id_{\mathcal{H}}}(S \star id_{\mathcal{H}}) = \eta_{\mathcal{H}'} \circ \varepsilon_{\mathcal{H}}\\ &f \star S' \circ f = \psi'_{id_{\mathcal{H}'},f}(id_{\mathcal{H}'}) \star \psi'_{id_{\mathcal{H}'},f}(S') = \psi'_{id_{\mathcal{H}'},f}(id_{\mathcal{H}'} \star S') = \eta_{\mathcal{H}'} \circ \varepsilon_{\mathcal{H}}\end{aligned}$$

The last step of both equations use the fact that both $\psi_{f,id_{\mathcal{H}}}$ and $\psi_{id_{\mathcal{H}'},f}$ are algebra morphisms, thus they send the unit of $Hom(\mathcal{H},\mathcal{H})$ and $Hom(\mathcal{H}',\mathcal{H}')$ respectively, to the unit of $Hom(\mathcal{H}',\mathcal{H}')$ (as that is the codomain of both morphisms), which is $\eta_{\mathcal{H}'} \circ \varepsilon_{\mathcal{H}}$. Hence, f has both a right- and left-inverse in $Hom(\mathcal{H},\mathcal{H}')$ and it follows that $f^{-1} = f \circ S = S' \circ f$. Concluding that any bialgebra morphism between two Hopf algebras is indeed a Hopf algebra morphism. ☐

We will now give some examples of Hopf algebras. This will be done by showing that some of the bialgebra examples that we saw before, have in particular a Hopf algebra structure.
We will start with the dual space of a Hopf algebra. Let $\mathcal{H}$ be a finite-dimensional Hopf algebra with antipode S. Recall that this implies that the dual $\mathcal{H}^*$ is a bialgebra. Moreover, $\mathcal{H}^*$ is a Hopf algebra with antipode S^*, where S^* is the transpose of S, so $S^*(\alpha)(x) = \alpha(S(x))$ for $\alpha \in \mathcal{H}^*, x \in \mathcal{H}$.

Let G be a group, then the bialgebra $\mathbf{k}[G]$ has an antipode S given by $S(x) = x^{-1}$ as $\Delta(x) = x \otimes x$. Indeed,

$$S \star id_G(x) = xS(x) = S(x)x = \varepsilon(x)1 = 1 \implies S(x) = x^{-1}.$$

We have seen that $GL(2)$ and $SL(2)$ are bialgebras, but they are also Hopf algebras. Define S by:

$$S(a) = (ad-bc)^{-1}d,\ S(b) = -(ad-bc)^{-1}b,$$
$$S(c) = -(ad-bc)^{-1}c,\ S(d) = (ad-bc)^{-1}a,\ S(t) = t^{-1},$$

in matrtix form this becomes $S\begin{pmatrix} a & b \\ c & d\end{pmatrix} = (ad-bc)^{-1}\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}$. Then S is an antipode for either $GL(2)$ or $SL(2)$. Showing that they are indeed Hopf algebras.
Notice that we assumed that it is sufficient to define the S for how it acts on the generators of the bialgebra and that this a;sp shows that it is an antipode. The following theorem shows that this is indeed the case.

Theorem 1.26. *Let $(\mathcal{H}, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra.*

i) $S\colon \mathcal{H} \to \mathcal{H}^{op\,cop}$ is a bialgebra morphism,

ii) The following statements are equivalent:

a) $S^2 = id_{\mathcal{H}}$,
b) $\forall x \in \mathcal{H}\ \sum_{(x)} S(x'')x' = \varepsilon(x)1$,
c) $\forall x \in \mathcal{H}\ \sum_{(x)} x''S(x') = \varepsilon(x)1$

iii) If $\mathcal{H}$ is abelian or cocommutative, then $S^2 = id_{\mathcal{H}}$

Proof. To show i), we need to show that $S(xy) = S(y)S(x)\ \forall x, y \in \mathcal{H}$ and $(S\otimes S)\Delta = \Delta^{op}S$, $\varepsilon \circ S = \varepsilon$. This can be rewritten into

$$\sum_{(S(x))} S(x)' \otimes S(x)'' = \sum_{(x)} S(x'') \otimes S(x').$$

We start with by showing that $S(xy) = S(y)S(x)$. We will do this by showing that $\rho\colon \mathcal{H}\otimes\mathcal{H} \to \mathcal{H};\ (x\otimes y) \mapsto S(xy)$ and $\nu\colon \mathcal{H}\otimes\mathcal{H};(x\otimes y) \mapsto S(y)S(x)$ are the left and right inverse respectively for $\mu\colon \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}$ in the algebra $Hom(\mathcal{H}\otimes\mathcal{H}, \mathcal{H})$.
Recall that $Hom(\mathcal{H}\otimes\mathcal{H}, \mathcal{H})$ is an algebra with $\star$ as its multiplication and $\eta\circ\varepsilon$ as its unit by proposition 1.23. So we want to show that $\rho\star\mu = \eta\circ\varepsilon = \mu\star\nu$. Then it follows that $\rho = \nu$, as it implies that they are the inverse of μ, which is unique.
Now, let $x, y \in \mathcal{H}$ be arbitrary, then

$$(\rho\star\mu)(x\otimes y) = \sum_{(x\otimes y)} \rho((x\otimes y)')\mu((x\otimes y)'') = \sum_{(x)(y)} \rho(x'\otimes y')x''y'' = \sum_{(xy)} S((xy)')(xy)'' = \eta\circ\varepsilon(xy)$$

and

$$(\mu\star\nu)(x\otimes y) = \sum_{(x\otimes y)} \mu((x\otimes y)')\nu((x\otimes y)'') = \sum_{(x)(y)} x'y'\nu(x''\otimes y'') = \sum_{(x)(y)} x'y'S(y'')S(x'')$$
$$= \sum_{(x)} x'\left(\sum_{(y)} y'S(y'')\right)S(x'') = \sum_{(x)} x'(\eta\varepsilon(y))S(x'') = \eta\varepsilon(x)\eta\varepsilon(y) = \eta\varepsilon(xy),$$

thus it follows that $\rho = \eta$. And $id\star S(x) = \eta\varepsilon(x) \implies id\star S(1) = S(1) = 1$.
Similarly, set $\rho = \Delta\circ S$ and $\nu = (S\otimes S)\circ\Delta^{op}$, then $\rho, \nu \in Hom(\mathcal{H}, \mathcal{H}\otimes\mathcal{H})$, so $\rho = \nu$ will follow from showing that $\rho\star\Delta = \Delta\star\nu = (\eta\otimes\eta)\circ\varepsilon$, i.e. showing that ρ and ν are a left- and right inverse of Δ

in $Hom(\mathcal{H}, \mathcal{H}\otimes\mathcal{H})$ respectively.
Let $x \in \mathcal{H}$ be arbitrary, then

$$(\rho\star\Delta)(x) = \sum_{(x)} \Delta(S(x'))\Delta(x'') = \Delta(\sum_{(x)} S(x')x'') = \Delta(\eta\varepsilon(x)) = ((\eta\otimes\eta)\varepsilon)(x)$$

and

$$\begin{aligned}(\Delta\star\nu)(x) &= \sum_{(x)} \Delta(x')((S\otimes S)(\Delta^{op}(x''))) = \sum_{(x)} (x'\otimes x'')(S(x^{(4)})\otimes S^{(3)}) = \sum_{(x)} x'S(x^{(4)})\otimes x''S(x^{(3)})\\ &= \sum_{(x)} x'S(x^{(3)})\otimes\eta\varepsilon(x'') = \sum_{(x)} x'\varepsilon(x'')S(x^{(3)})\otimes 1 = \sum_{(x)} x'S(x'')\otimes 1 = \varepsilon(x)1\otimes 1.\end{aligned}$$

Thus it follows that $\rho = \nu$.
Lastly,

$$\varepsilon(S(x)) = \varepsilon(S(\sum_{(x)}\varepsilon(x')x'')) = \varepsilon(\sum_{(x)}\varepsilon(x')S(x'')) = \varepsilon(\eta\varepsilon(x)) = \varepsilon(x).$$

Together with the previous calculations, it follows that $S\colon\mathcal{H}\to\mathcal{H}^{op\,cop}$ is indeed a bialgebra morphism.
For ii), we will show that a) is equivalent to b) and leave out the proof that a) is equivalent to c) as this can be shown in a similar fashion. Again, since $S, S^2 \in Hom(\mathcal{H}, \mathcal{H}^{op\,cop})$ and $id_{\mathcal{H}}\star S = id_{Hom(\mathcal{H},\mathcal{H}^{op\,cop})}$, it is enough to show that $S\star S^2 = id_{Hom(\mathcal{H},\mathcal{H}^{op\,cop})}$, since inverses are unique.
So, let $x\in\mathcal{H}$ and assume b). Then,

$$(S\star S^2)(x) = \sum_{(x)} S(x')S^2(x'') = S(\sum_{(x)} S(x'')x') = S(\varepsilon(x)1) = \varepsilon(x)S(1) = \varepsilon(x)1,$$

so $S^2 = id_{\mathcal{H}}$. And, if $S^2 = id_{\mathcal{H}}$,

$$\sum_{(x)} S(x'')x' = S^2(\sum_{(x)} S(x'')x') = S(\sum_{(x)} S(x')S^2(x'')) = S(\sum_{(x)} S(x')x'') = \varepsilon(x)1,$$

thus it follows that $S(x'')x' = \varepsilon(x)1$, showing that a) $\iff$ b).
For *iii*), note that if $\mathcal{H}$ is abelian, we have $\sum_{(x)} S(x'')x' = \eta\varepsilon(x) \implies S^2 = id_{\mathcal{H}}$ by *ii*). And if $\mathcal{H}$ is cocommutative, we get again that $\sum_{(x)} S(x'')x' = \eta\varepsilon(x)$. Showing that $S^2 = id_{\mathcal{H}} \iff \mathcal{H}$ is abelian or cocommutative. □

Corollary 1.27. Let $(\mathcal{H},\mu,\eta,\Delta,\varepsilon,S)$ be a Hopf algebra, then $H^{op\,cop}$ together with S is another Hopf algebra. Also, $S\colon\mathcal{H}\to\mathcal{H}^{op\,cop}$ is a Hopf algebra morphism.
Moreover, if S is an isomorphism with inverse S^{-1}, then $H^{op} = (\mathcal{H},\mu^{op},\eta,\Delta,\varepsilon,S^{-1})$ and $H^{cop} = (\mathcal{H},\mu,\eta,\Delta^{op},\varepsilon,S^{-1})$ are isomorphic Hopf algebras.

The following lemma will finally show that it is indeed enough to check that S is an antipode on the generators for it to be an antipode on the whole bialgebra.

Lemma 1.28. Let $\mathcal{H}$ be a bialgebra and $S\colon\mathcal{H}\to\mathcal{H}^{op}$ an algebra morphism. If $\mathcal{H}$ is generated as an algebra by a subset $X\subset\mathcal{H}$ such that $\sum_{(x)} x'S(x'') = \varepsilon(x)1 = \sum_{(x)} S(x')x''$, $\forall x\in X$, then S is an antipode for $\mathcal{H}$.

Proof. It is sufficient to show that if $\sum_{(x)} S(x')x'' = \varepsilon(x)1 = \sum_{(x)} x'S(x'')$ for $X, y\in X$, then this also holds for the product xy. Using theorem 1.26, we see that,

$$\begin{aligned}\sum_{(xy)} (xy)'S((xy)'') &= \sum_{(x)(y)} x'y'S(x''y'') = \sum_{(x)} x'\left(\sum_{(y)} y'S(y'')\right)S(x'')\\ &= \sum_{(x)} x'S(x'')\varepsilon(y) = \varepsilon(x)\varepsilon(y) = \varepsilon(xy)\end{aligned}$$

The proof that $\sum_{(xy)} S((xy)')(xy)'' = \varepsilon(xy)$ goes similar. □

Using this lemma, we can even show that the tensor algebra $T(V)$ and symmetric algebra $S(V)$ are also Hopf algebras. One can check that the tensor algebra $T(V)$, for a vector space V, turns into a Hopf algebra with antipode $S\colon T(V) \to T(V)$ given by $S(1) = 1$, $S(v_1 v_2 \cdots v_n) = (-1)^n v_n \cdots v_2 v_1$.
For $S(V) = T(V)/I$, with I the ideal generated by the elements of the form $xy - yx$, to be a Hopf algebra, we first need to show that it's a bialgebra. To do this, it suffices to show that I is also a coideal of $T(V)$. Since $T(V)$ is already a bialgebra, I being an ideal and coideal of $T(V)$ allows us to induce a quotient-bialgebra structure on $S(V)$.
We will show that $I \subseteq T(V)$ is also a coideal. Take $x, y \in T(V)$, $v, w \in V$, then any element of I is a sum of elements of the form $x(vw - wv)y$. Then

$$\Delta(x(vw - wv)y) = \sum_{(x)(y)} \big(x'(vw - wv)y' \otimes x''y'' + x'y' \otimes x''(vw - wv)y''\big) \in I \otimes T(V) + T(V) \otimes I$$

and $\varepsilon(x(vw - wv)y) = \varepsilon(x)(\varepsilon(vw - wv)\varepsilon(y) = 0$, thus I is indeed a coideal. So $S(V)$ can be turned into a Hopf algebra by inducing it with the quotient-bialgebra structure and taking the same antipode S as $T(V)$, but projected to $S(V)$.

2 Modules and comodules

In this section the concept of modules and comodules will be introduced. In short, modules are vector spaces for which the vectors can be multiplied with elements of an algebra. Since this multiplication will depend on the algebra, the module structure of a vector space will also depend on the algebra that acts on it. Later on we will even see that we sometimes know all the possible modules of certain algebras.
Now, in this section we will first look at some basic properties of algebra modules. After that we will look at coalgebra modules. A few examples of both type of modules will be given, however we will wait with giving explicit examples of modules until a later section.
Do note that the concept of modules will play a major part in this thesis. Since the Clebsch-Gordan coefficients can be seen as a relation between two different modules. So we will be talking a lot about modules in the later sections.

2.1 Algebra modules

Definition 2.1. Let $\mathcal{A}$ be an algebra. Then, an $\mathcal{A}$-*module* V is a vector space with a bilinear map $(a, v) \mapsto av$ from $\mathcal{A} \times V \to V$ such that $a'(av) = (a'a)v$ and $1v = v$ for $a', a \in \mathcal{A}$ and $v \in V$.

Note that an $\mathcal{A}$-module is only multiplied from the left by the algebra $\mathcal{A}$. Such a module can also be referred to as a left-module. We will often refer to the bilinear map as $\mathcal{A}$ acting on V.
We can define a right-module in a similar, but with the bilinear map given by $(v, a) \mapsto va$ from $V \times \mathcal{A} \to V$ and again $(va)a' = v(aa')$, $v1 = v$ for $a, a' \in \mathcal{A}$. But, notice that right-modules are just left-modules of $\mathcal{A}^{op}$, so its easier to just refer to them as $\mathcal{A}-$modules and only consider multiplication from the left.

Recall that algebras are themselves vector spaces. So, if we have an algebra $(\mathcal{A}, \mu, \eta)$, we can induce an $\mathcal{A}$-module structure on the vector space $\mathcal{A}$ with the use of the multiplication $\mu\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$.
This just gives us that $\mathcal{A}$ acts on $\mathcal{A}$ by $aa' = \mu(a, a')$, $a, a' \in \mathcal{A}$. So, this is not the most interesting example of an algebra module. But it is still a fun little fact.

Just like vector spaces have subspaces, modules also have subspaces called submodules. And similar to algebra morphisms preserving the multiplication of both algebras, modules also have functions that preserve the multiplication between the vector space and the algebra.

Definition 2.2. Let V, V' be $\mathcal{A}$-modules and let $f\colon V \to V'$ be a linear map. Then, f is called $\mathcal{A}$-*linear* or an $\mathcal{A}$-module morphism if $f(av) = af(v)$ for $a \in \mathcal{A}$, $v \in V$.

Definition 2.3. Let V be an $\mathcal{A}$-module, then a subspace $W \subset V$ is called an $\mathcal{A}$-*submodule* if the inclusion map from W into V is $\mathcal{A}$-linear.

Another small fact is, that if $\mathcal{A}$ is an algebra and V is an $\mathcal{A}$-module, then the action of $\mathcal{A}$ on V induces an algebra morphism $\rho\colon \mathcal{A} \to End(V)$. This algebra morphism is defined as

$$\rho(a)(v) = av, \;\; a \in \mathcal{A} \;, v \in V,$$

where av is the usual left action of $\mathcal{A}$ on V. The algebra morphism ρ is called a representation of $\mathcal{A}$ on V. This immediately shows that any algebra module V induces a representation of $\mathcal{A}$ on V. But, any algebra morphism from $\mathcal{A}$ to $End(V)$ also induces an $\mathcal{A}$-module structure on the vector space V. In other words, the language of representations and modules is equivalent and one could interchange the use of the words modules and representations. It also is not unheard of to only use the word representation, when technically working with both the representation and induced module structure.

But, to make it a bit easier to follow, we will try to make the distinction between representations and modules when talking about the morphism and vector space respectively.

We will continue with some facts regarding $\mathcal{A}$-modules. Let $V_1, \dots, V_n$ be $\mathcal{A}$-modules. Then $V_1 \oplus \cdots \oplus V_n$ is also an $\mathcal{A}$-module with $a(v_1, \dots, v_n) = (av_1, \dots, av_n)$ for $a \in \mathcal{A}$, $v_1 \in V_1, \dots, v_n \in V_n$. The concept of direct sums helps to categorize modules. Since, if $V' \subset V$ is a submodule, then one could look at the module $V = V' \oplus V \backslash V'$, given that $V \backslash V'$ is also an $\mathcal{A}$-module. This can then be used to study the module V by looking at all the "smaller" parts that make up V.

Definition 2.4. Let $\mathcal{A}$ be an algebra and V an $\mathcal{A}$-module. Then V is called *simple* if the only submodules of V are $\{0\}$ and V itself.
V is called *semisimple* if V is isomorphic to a direct sum of simple $\mathcal{A}$-modules.
V is called indecomposable if it is **not** isomorphic to a direct sum of two non-zero $\mathcal{A}$-modules.

This gives rise to the following proposition:

Proposition 2.5. Let $\mathcal{A}$ be an algebra and let V be any finite-dimensional $\mathcal{A}$-module with $V' \subset V$ any $\mathcal{A}$-submodule. The following statements are equivalent:

i) $\exists V''$, $\mathcal{A}$-module, such that $V \cong V' \oplus V''$.

ii) If V' is simple, $\exists V''$, $\mathcal{A}$-module, such that $V \cong V' \oplus V''$.

iii) $\exists p\colon V \to V'$, $\mathcal{A}$-linear map, such that $p^2 = p$.

iv) If V' is simple, $\exists p\colon V \to V'$, $\mathcal{A}$-linear, such that $p^2 = p$.

v) Any finite-dimensional $\mathcal{A}$-module is semisimple.

We will now look at one example of an $\mathcal{A}$-module of an algebra $\mathcal{A}$. First, recall that an algebra $\mathcal{A}$ has multiplication $\mu\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$; $(a', a) \mapsto a'a$. So, let $W = \mathcal{A} \otimes V$ be the vector space that is the tensor product of $\mathcal{A}$ with some vector space V. Then W is an $\mathcal{A}$-module as $\mathcal{A}$ acts on W by

$$a'w = a'(a \otimes v) = \mu(a', a) \otimes v, \quad a', a \in \mathcal{A},\ v \in V,\ w \in W.$$

Such an $\mathcal{A}$-module is called a *free module* over an algebra $\mathcal{A}$.
Since an $\mathcal{A}$-module V is a vector space, it also has a basis. A basis $M \subset V$ is a subset $\{v_i\}_{i \in I}$ such that $(a_i)_{i \in I} \mapsto \sum_{i \in I} a_i v_i$ from $\oplus_{i \in I} \mathcal{A}$ to V is an isomorphism. Moreover, by proposition A.3 and the remarks above that proposition, we get that $\oplus_{i \in I} \mathcal{A} \cong \oplus_{i \in I} (\mathcal{A} \otimes \mathbf{k}) \cong \mathcal{A} \otimes W$, $W = \oplus_{i \in I} \mathbf{k}$. Thus it follows that an $\mathcal{A}$-module has a basis if and only if it is a free module.

Now, let $\mathcal{A}$ be an algebra and let U, V be $\mathcal{A}$-modules. Then $U \otimes V$ is a $\mathcal{A} \otimes \mathcal{A}$-module with $(a \otimes a')(u \otimes v) = au \otimes a'v$, $a, a' \in \mathcal{A}$, $u \in U$ and $v \in V$. Say $\mathcal{A}$ is also a bialgebra, then the coproduct Δ allows us to equip the $\mathcal{A} \otimes \mathcal{A}$-module $U \otimes V$ with an $\mathcal{A}$-module structure via

$$a(u \otimes v) = \Delta(a)(u \otimes v) = \sum_{(a)} a'u \otimes a''v, \quad a \in \mathcal{A},\ u \in U \text{ and } v \in V,$$

as Δ is an algebra morphism.
When $\mathcal{A}$ is a bialgebra, we can also equip any vector space V with a trivial $\mathcal{A}$-module structure with the use of the counit ε. Since, $av = \varepsilon(a)v$ defines a bilinear map from $\mathcal{A} \times V \to V$.

Proposition 2.6. Let $\mathcal{A}$ be a bialgebra. Let U, V and W be $\mathcal{A}$-modules and give $\mathbf{k}$ the trivial $\mathcal{A}$-module structure obtained via the counit. Then $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$ and $\mathbf{k} \otimes V \cong V \cong V \otimes \mathbf{k}$ are isomorphic as $\mathcal{A}$-modules.
Furthermore, if $\mathcal{A}$ is cocommutative, then $\tau_{V,W}\colon V \otimes W \tilde{\to} W \otimes V$ is an $\mathcal{A}$-module isomorphism.

Now, if $\mathcal{A}$ is also a Hopf algebra, we can even define an $\mathcal{A}$-module structure on the vector space $Hom(V,W)$. Let V, W be two $\mathcal{A}$-modules and let $((a\otimes a')f)(v) = af(a'v)$, $a, a' \in \mathcal{A}$, $f \in Hom(V,W)$, $v \in V$. This induces an $\mathcal{A}\otimes\mathcal{A}^{op}$- module on $Hom(V,W)$, since

$$\big((a\otimes a')(b\otimes b)f\big)(v) = \big((ab\otimes b'a')f\big)(v) = abf(b'a'v) = a\big((b\otimes b')f\big)(a'v) = \big((a\otimes a')((b\otimes b')f)\big)(v),$$

with $a, a', b, b' \in \mathcal{A}$, $f \in Hom(V,W)$ and $v \in V$.
Now, since $\mathcal{A}$ is a Hopf algebra, $(id\otimes S)\circ\Delta\colon \mathcal{A}\to\mathcal{A}\otimes\mathcal{A}^{op}$ is an algebra morphism. We can then induce a $\mathcal{A}$-module structure on $Hom(V,W)$ by having $\mathcal{A}$ act on $Hom(V,W)$ via $af = (id\otimes S)\circ\Delta(a)f$ for $a\in\mathcal{A}$, $f\in Hom(V,W)$. This action is explicitly given by:

$$(af)(v) = \sum_{(a)} a'f(S(a'')v).$$

The above also allows us to construct an $\mathcal{A}$-module structure on the dual space V^* of a vector space V. Take $W = \mathbf{k}$ and give it the trivial $\mathcal{A}$-module structure. Then for $f\in V^*$, $v\in V$ and $a\in\mathcal{A}$ we get $(af)(v) = f(S(a)v)$.

Proposition 2.7. Let $\mathcal{A}$ be a Hopf algebra and U, U', V and V' be $\mathcal{A}$-modules such that either U or U' and V or V' are finite-dimensional vector spaces. Then, if $\tau_{U^*,V'}$ is $\mathcal{A}$-linear, the linear map

$$\lambda\colon Hom(U,U')\otimes Hom(V,V')\to Hom(V\otimes U, U'\otimes V')$$

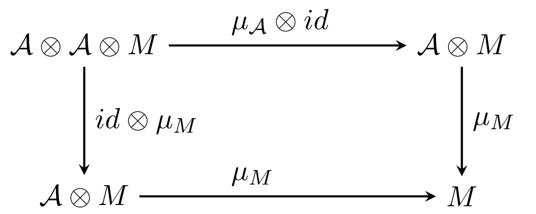
of corollary A.5 is $\mathcal{A}$-linear.
In particular, the maps $\lambda\colon U^*\otimes V^*\to(V\otimes U)^*$ and $\lambda_{U,V}\colon V\otimes U^*\to Hom(U,V)$ are $\mathcal{A}$-linear.

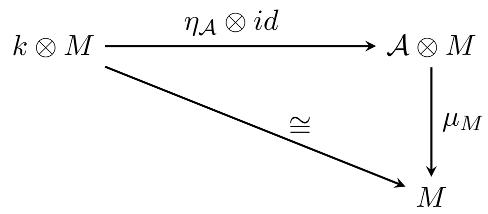
The proof of the first part of this proposition is another good exercise for rewriting equations using Sweedler's notation. We have already seen a few proofs that do the same, so this proof will be left to the reader. Do note that $\tau_{U^*,V'}$ being $\mathcal{A}$-linear is not too useful for proving the first part, but the fact that it implies that $\mathcal{A}$ is cocommutative is helpful.

2.2 Comodules

Comodules will not be looked at in this thesis much beyond this section, but it is nice to at least introduce them. Especially since they are dual to modules, similarly as coalgebras are dual to algebras. We have seen that algebra modules are vector spaces on which algebras or bialgebras can act with multiplication. Now, coalgebras have a comultiplication. Thus we will see that comodules will be vector spaces on which a coalgebra coacts with comultiplication.
Since this definition will be dual to the definition of an algebra module, we will use diagrams to define what a coalgebra is. Similar to definition 1.13, it will then immediately be clear why the concept of comodules is dual to that of modules.

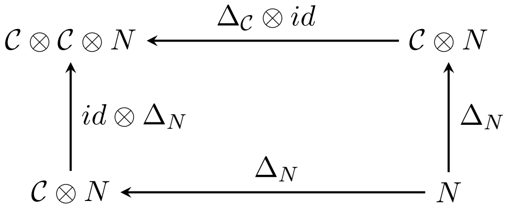
Definition 2.8. Let $(\mathcal{A},\mu_{\mathcal{A}},\eta_{\mathcal{A}})$ be an algebra. Then an $\mathcal{A}$-*module* is a pair (M,μ_M), with M a vector space and $\mu_M\colon\mathcal{A}\times M\to M$ a linear map such that the following graphs commute:
Associativty: and Unitality:



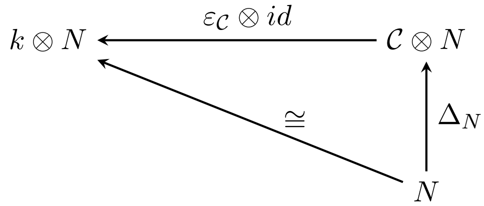


Then a comodule is defined by reversing the arrows of the above diagrams:

Definition 2.9. Let $(\mathcal{C}, \Delta_\mathcal{C}, \varepsilon_\mathcal{C})$ be a coalgebra. Then a $\mathcal{C}$-*comodule* is a pair (N, Δ_N), with N a vector space and $\Delta_N \colon N \to \mathcal{C} \otimes \mathcal{N}$ a linear map such that the following graphs commute:
Associativity:



and Unitality:



We also have similar definitions for comodule morphisms and subcomodules:

Definition 2.10. Let $\mathcal{C}$ be a coalgebra and N, N' be $\mathcal{C}$-comodules. A linear map $f \colon N \to N'$ is a $\mathcal{C}$-*comodule morphism* if $(id \otimes f) \circ \Delta_N = \Delta_{N'} \circ f$.
If in addition $N' \subset N$ is a subspace, then N' is a *subcomodule* of N if $\Delta_N(N') \subset \mathcal{C} \otimes N'$.

Similar to modules, the above definitions technically defines left-comodules. But again, since right-comodules of a comodule $\mathcal{C}$ are just left-comodules of $\mathcal{C}^{op}$, we will just refer to left-comodules as modules.

We will end this short section with some examples that are fairly similar to the examples given of modules and also one final property of tensors of comodules. This last property will look fairly similar to proposition 2.6.
Let $\mathcal{C}$ be a comodule. Then, similar to the fact that algebras are also modules, $(\mathcal{C}, \Delta)$ is a $\mathcal{C}$-comodule. The coaction of $\mathcal{C}$ on $\mathcal{C}$ is simply given by $\Delta(c)$, $c \in \mathcal{C}$.
Let $\mathcal{C}^*$ be the dual vector space of $\mathcal{C}$ and let (N, Δ_N) be a $\mathcal{C}$-comodule. Then we know that $\mathcal{C}^*$ is an algebra by proposition 1.16 and the dual N^* of N is a right-$\mathcal{C}^*$-module. The right action of $\mathcal{C}^*$ on N^* is given by, for $f \in N^*$, $g \in \mathcal{C}^*$,

$$fg = \Delta_N^* \circ \lambda(f, g), \quad \text{with } \lambda \colon N^* \otimes C^* \to (C \otimes N)^* \text{ of theorem A.5.}$$

One could place a coalgebra structure on the dual M^* of an algebra $\mathcal{A}$-module M in a similar fashion. However, it needs to be assumed that $\mathcal{A}$ is a finite-dimensional algebra, else A^* need not be a coalgebra.
Now, let $\mathcal{H}$ be a bialgebra and N, M $\mathcal{H}$-comodules. Then we can also define a $\mathcal{H}$-comodule structure on the tensor product $N \otimes M$, just like we did before with modules. In this case, we do still need $\mathcal{H}$ to be a bialgebra, else it will at most just be a $\mathcal{H} \otimes \mathcal{H}$-comodule. If we define

$$\Delta_{N \otimes M} = (\mu \otimes id_{N \otimes M})(id_\mathcal{H} \otimes \tau_{N,\mathcal{H}} \otimes id_M)(\Delta_N \otimes \Delta_M),$$

then $(N \otimes M, \Delta_{N \otimes M})$ turns into a $\mathcal{H}$-comodule.
Lastly, we can also define a trivial comodule, but this time using the unit η of a bialgebra $\mathcal{H}$ instead of the counit ε. So, let V be a vector space. Then $V \cong \mathbf{k} \otimes V \xrightarrow{\eta \otimes id_V} \mathcal{H} \otimes V$ induces V with a $\mathcal{H}$-comodule structure, called the *trivial comodule*. Thus $\Delta_V(v) = (\eta(1) \otimes v)$.

To end this section, lets note a proposition that is similar to proposition 2.6:

Proposition 2.11. Let $\mathcal{H}$ be a bialgebra. Let M, N and P be $\mathcal{H}$-comodules and give $\mathbf{k}$ the trivial $\mathcal{H}$-comodule structure. Then $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$ and $\mathbf{k} \otimes M \cong M \cong M \otimes \mathbf{k}$ are isomorphic as $\mathcal{H}$-comodules.
Furthermore, if $\mathcal{H}$ is abelian, then $\tau_{M,N} \colon M \otimes N \cong N \otimes M$ is an $\mathcal{H}$-comodule isomorphism.

3 Lie algebras and the enveloping algebra

Since the Hopf algebra $\mathcal{U}_q(sl(2))$ is the quantum deformation of $\mathcal{U}(\mathfrak{sl}(2))$, as in, if one takes $q = 1$, then $U_1(\mathfrak{sl}(2)) = \mathcal{U}(sl(2))$. It is natural to first look at the classical object before we will discuss the quantum deformed one. However, since $\mathcal{U}(\mathfrak{sl}(2))$ is the so called universal enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$. The first step is to understand the properties of the Lie algebra $\mathfrak{sl}(2)$, as $\mathcal{U}(\mathfrak{sl}(2))$ will be generated by the basis elements of $\mathfrak{sl}(2)$.
In this section we will study the concept of Lie algebras. We will do this by first looking at Lie groups, even though we will not use them further on. The reason for this, is that Lie algebras are tangent spaces of Lie groups and it feels natural to me to introduce Lie algebras in this way. However, since we do not necessarily need to know anything on Lie groups, we will also define Lie algebras without the use of Lie groups.
After we have introduced Lie algebras and the enveloping algebra. We will end this section by studying $\mathcal{U}(\mathfrak{sl}(2))$ and certain specific modules of this algebra.

3.1 Lie groups and the exponential map

Now, even though Lie algebras can be introduced fully algebraically as is done in [6], they are still closely connected to Lie groups. And as I find it natural to first speak of Lie groups and then about Lie algebras, the concept of Lie groups will be shortly introduced in this first section. We mostly use information from [7] for the part on Lie groups.
Do note that these concepts will not be of use for our subject, so this section is fully optional. Lie algebras will in turn also be reintroduced algebraically in a later part in definition 3.22. Due to this, we will also not give any proofs on any of the facts on Lie groups. However, the proofs can all be fined in [7].

Definition 3.1. A Lie group G is a smooth manifold G equipped with smooth maps:

$$\mu\colon G \times G \to G;\ (x, y) \mapsto xy,$$
$$i\colon G \to G;\ x \mapsto x^{-1},$$

turning G into a group.
The *unit* of G will be denoted by $e \in G$, thus we get that $\mu(x, i(x)) = e$, $x \in G$

Similar to algebras and coalgebras, we also have Lie group homomorphisms.

Definition 3.2. Let G and H be Lie groups and let $\phi\colon G \to H$ be a smooth map. Then, if ϕ is a group homomorphism from G to H, ϕ is called a *Lie group homomorphism.*
If, in addition, ϕ is a bijection and its inverse ϕ^{-1} is also a group homomorphism, then ϕ is called a *Lie group isomorphism.* In this case, if $G = H$, ϕ is called an *automorphism.*

We will give some short and simple examples of Lie groups.
First, let G be a finite group. Then G itself is a Lie group of dimension 0.
Now, let G be a vector space over the field $\mathbb{R}$. Then define $\mu(x, y) = x + y$ and $i(x) = -x$. Then, G turns into a Lie group with the functions μ and i. This is also an example of an abelian Lie group, i.e. a Lie group G such that $\mu(x, y) = \mu(y, x)\ \forall x, y \in G$.
For a third example, take $G = \mathbb{R}^\times$, the multiplicative group of real numbers. This is itself immediately a Lie group. Moreover, $G = \mathbb{R}_{>0} \cup \mathbb{R}_{<0}$ and the subgroup of positive real elements $\mathbb{R}_{>0} \subset G$ is also a Lie group, hence called a *Lie subgroup* similar to subalgebras.

Definition 3.3. Let G be Lie group and let $H \subseteq G$ be a subgroup. Then, if H is equipped with a Lie group structure, it is a *Lie subgroup* of G if the inclusion map $\iota\colon H \to G$ is a Lie group homomorphism.

We will give two more examples, but this time in the form of propositions. These will be a product of two Lie groups and a submanifold of a Lie group.

Proposition 3.4. Let G_1 and G_2 be two Lie groups with multiplication μ_1 and μ_2 and inverses i_1 and i_2 respectively. Define $\mu_{12}\colon (G_1 \times G_2) \times (G_1 \times G_2) \to G_1 \times G_2$ by $\mu_{12}((x_1, x_2), (y_1, y_2)) = (\mu_1(x_1, y_1), \mu_2(x_2, y_2))$ and $i_{12}\colon G_1 \times G_2 \to G_1 \times G_2$ by $i_{12}(x_1, x_2) = (i_1(x_1), i_2(x_2))$ for $x_1, y_1 \in G_1$, $x_2, y_2 \in G_2$.
Then $G_1 \times G_2$, together with μ_{12} and i_{12}, is a Lie group.

Proposition 3.5. Let G be a Lie group and let $H \subset G$ be an embedded submanifold of G and also a subgroup of G. Then H is a Lie group with multiplication $\mu|_{H\times H}$ and inverse $i|_H$.

Theorem 3.6. *Let G be a Lie group and let $H \subset G$ be a subgroup. Then the following are equivalent:*

1. *H is closed as a subset of G,*
2. *H is an embedded submanifold in G.*

Thus, the assumptions of the previous proposition 3.5 can also be reworded to $H \subset G$ being a subgroup of G and closed as a subset in G. Such a Lie subgroup is also referred to as a *closed Lie subgroup*. Do note that not all Lie subgroups are closed, so this theorem does not show how one could find all Lie subgroups.

Lastly we will see some examples that will look familiar, the Lie groups $GL(n, \mathbf{k})$ and $SL(n, \mathbf{k})$ for $n \in \mathbb{N}_{>0}$. First, let $Mat(n, \mathbf{k})$ be the set of $n \times n$-matrices with values in $\mathbf{k}$. Then $Mat(n, \mathbf{k})$, together with addition and scalar multiplication, is a linear space.
For $M \in Mat(n, \mathbf{k})$, let M_{ij} be the entry of M in the i-th row and j-th column and let $\zeta_{ij}\colon Mat(n, \mathbf{k}) \to \mathbf{k}$; $\zeta_{ij}(M) = M_{ij}$. Then, one can define the determinant function $det\colon Mat(n, \mathbf{k}) \to \mathbf{k}$ by:

$$det(M) = \sum_{\sigma \in S_n} sign(\sigma)\zeta_{1\sigma(1)}(M) \cdots \zeta_{n\sigma(n)}(M),$$

which shows that det is a smooth function. More specifically, det is a Lie group homomorphism.
Let $GL(n, \mathbf{k}) = \{M \in Mat(n, \mathbf{k})\colon det(M) \neq 0\}$ be the set of invertible matrices equipped with matrix multiplication. Then $GL(n, \mathbf{k})$ is the pre-image of $\mathbf{k}\{0\}$ under det and as det is smooth, $GL(n, \mathbf{k}) \subset Mat(n, \mathbf{k})$ is open and thus a smooth manifold.
Now, using the maps ζ_{ij}, matrix multiplication $\mu\colon GL(n, \mathbf{k}) \times GL(n, \mathbf{k}) \to GL(n, \mathbf{k})$ for matrices $M, N \in GL(n, \mathbf{k})$ is coordinate wise given by $\zeta_{ij}(\mu(M, N)) = \sum_{k=1}^{n} \zeta_{ik}(M)\zeta_{kj}(M)$. Thus, matrix multiplication μ is also a smooth map.
Lastly, to show that $GL(n, \mathbf{k})$ is a Lie group, we need to show that taking the inverse of a matrix is also a smooth map. This follows from the fact that $i(M) = det(M)^{-1} \cdot M^{co}$, where M^{co} is the co-matrix of M. Now, the co-matrix of M is defined coordinate wise as $\zeta_{ij}(M^{co}) = (-1)^{i+j} \cdot Mi_{ij}(M^T))$, with $Mi_{ij}(M)$ the $(i, j)-th$ minor of M, the determinant of the matrix obtained from deleting the i-th row and j-th column of M.
So we see that $GL(n, \mathbf{k})$ is indeed a Lie group, which is closely related to $GL_n(\mathcal{A})$ and even the same for $\mathcal{A} = \mathbf{k}$. Furthermore, we claim that $SL(n, \mathbf{k})$ is also a Lie group. To show this, we will use theorem 3.6. First, note that $SL(n, \mathbf{k}) = \{M \in Mat(n, \mathbf{k})\colon det(M) = 1\} \subset GL(n, \mathbf{k})$ is a subgroup of $GL(n, \mathbf{k})$. $SL(n, \mathbf{k})$ is also a closed subset of $GL(n, \mathbf{k})$, since its the kernel of det. Thus, from theorem 3.6 it follows that $SL(n, \mathbf{k})$ is indeed a Lie group. And we also have that $SL(n, \mathbf{k}) = SL_n(\mathcal{A})$ if $\mathcal{A} = \mathbf{k}$.
We will give one final remark about these last two examples, as the Lie groups noted in the remark below will also be used later on. Let V be a n-dimensional vector space over $\mathbf{k}$, then one could show that the set of linear endomorphisms from V to V, $End(V)$, is isomorphic to $Mat(n, \mathbf{k})$. This can be shown by fixing a basis $(e_1, \ldots, e_n)$ of V, then constructing matrices that act the same way as

endomorphisms on this basis of V.
Since the composition of functions in $End(V)$ corresponds to matrix multiplication in $Mat(n, \mathbf{k})$. And the determinant function on functions $f \in End(V)$ is independent of any chosen basis of V. It follows that $GL(V)$ is group and that $GL(V) \simeq GL(n, \mathbf{k})$, where $GL(V)$ is the set of invertible endomorphisms. Then, since the map sending an endomorphism f on V to its matrix form is a diffeomorphism. And $GL(V) \simeq GL(n, \mathbf{k})$ as groups, it follows that $GL(V)$ is even a Lie group, which is isomorphic to $GL(n, \mathbf{k})$
One could then define $SL(V) = \{M \in GL(V) \colon det(M) = 1\}$ in a similar manner, where det is again the function defined on endomorphisms of V via the isomorphism $End(V) \simeq Mat(n, \mathbf{k})$.
The Lie groups $GL(V)$ and $SL(V)$ are, in a sense, the generalised versions of $GL(n, \mathbf{k})$ and $SL(n, \mathbf{k})$, $n = dim(V)$. As it is defined for any vector space V and not just for $\mathbf{k}^n$.

To go from Lie groups to Lie algebras, we still need a few more tools. One of them is the exponential map, but we need to do some work to define it for Lie groups. First, let M be a smooth manifold and let $\mathcal{V}(M)$ be the linear space of smooth vector fields on M. Now, a vector field $v \in \mathcal{V}(G)$ is called *left invariant* if $\forall x, y \in G$ $T_y(l_x)v(y) = v(xy)$, where G is a Lie group, $l_x(y) = \mu(x, y)$ and $T_y(l_x)$ is the tangent map of l_x at y.
From this equation, it follows that the subspace of left invariant vector fields, $\mathcal{V}_L(G)$, is completely determined by its value at the identity e. As in, if $v \in \mathcal{V}_L(G)$ and $x \in G$, then $v(x) = T_e(l_x)(v(e))$. Moreover, there even exists an isomorphism between $\mathcal{V}_L(G)$ and $T_e(G)$, the tangent space of G at e;

Lemma 3.7. Let G be a Lie group, let $X \in T_eG$ and define $v_X(x) = T_e(l_x)(X)$. Then $v_X(x) \in \mathcal{V}_L(G)$ for $x \in G$ and the map sending $X \mapsto v_X$ defines a linear isomorphism from $T_e(G) \to \mathcal{V}_L(G)$. The inverse of this map is the map that sends $v \in \mathcal{V}_L(G)$ to $v(e) \in T_e(G)$.

Definition 3.8. Let G be a Lie group and let $X \in T_eG$. Then α_X is the *maximal integral curve* of v_X with initial point e. In other words, $\alpha_X(0) = e$, α_X is a smooth map and $\frac{d}{dt}\alpha_X(t) = v_X(\alpha_X(t))$

Definition 3.9. Let G be a Lie group and let $\mathbb{R}^+$ be the group of real numbers under addition. Then, a smooth group homomorphism $\phi \colon \mathbb{R}^+ \to G$ is called a *1-parameter subgroup* of G.

Lemma 3.10. The maximal integral curve α_X has all of $\mathbb{R}$ as its domain and it is a 1-parameter subgroup of G. Moreover, $\mathbb{R} \times T_e(G) \to G$; $(t, X) \mapsto \alpha_X(t)$ is a smooth map.

Using the above definition, we see that α_X being a 1-parameter subgroup means that $\alpha_X(s+t) = \alpha_X(s)\alpha_X(t)$ $\forall s, t \in \mathbb{R}$. With this, we can finally define the exponential map.

Definition 3.11. Let G be a Lie group. The *exponential map* $exp \colon T_eG \to G$ is given by $exp(X) = \alpha_X(1)$.

Note that if we look at the Lie group $GL(V)$ for some finite dimensional vector space V, the exponential map $exp \colon T_eGL(V) = End(V) \to GL(V)$ is given by $exp(M) = e^M$. Here e^M is the exponent of an endomorphism which can be defined by $e^M = \sum_{n=0}^{\infty} \frac{1}{n!}M^n$. Which shows that the exponential map does not need to be different to the usual known exponential function.
This fact can be deduced by showing that the maximal integral curve α_X satisfies $\frac{d}{dt}\alpha(t) = \alpha(t)X$. Since the map $t \mapsto e^tX$ is a solution to this equation, it follows that the exponential map $exp \colon End(V) \to GL(V)$ is given by the usual exponent of endomorphisms.

Lemma 3.12. $\forall s, t \in \mathbb{R}, X \in T_eG$, we have that

1. $exp(sX) = \alpha_X(s)$,
2. $exp((s+t)X) = exp(sX)exp(tX)$,
3. $exp \colon T_eG \to G$ is smooth and $T_0exp = id_{T_eG}$,

Moreover, $\exists U \subset T_eG$ open, $0 \in U$, and $\exists V \subset G$ open, $e \in V$, such that $exp|_U : U \xrightarrow{\simeq} V$ is a diffeomorphism.

Lemma 3.13. Let G be a Lie group and let $X \in T_eG$. Then $\phi \colon \mathbb{R}^+ \to G; t \mapsto exp(tX)$ is a 1-parameter subgroup. Moreover, any 1-parameter subgroup of G is of this form for some unique $X \in T_eG$.

This then leads to a really nice application of the exponential map:

Lemma 3.14. Let G and H be a Lie groups and let $\phi \colon G \to H$ be a Lie group homomorphism. Then the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ exp_G \uparrow & & \uparrow exp_H \\ T_eG & \xrightarrow{T_e\phi} & T_eH \end{array}$$

Before, we saw the left translation map $l_x \colon G \to G$ for a Lie group G, $x \in G$, but we can also define a right translation map $r_x \colon G \to G$; $r_x(y) = \mu(y, x)$ for $x, y \in G$. With both maps, we can define the *conjugation map* $C_x = l_x \circ r_x^{-1} \colon G \to G$; $C_x(y) = xyx^{-1}$ for $x, y \in G$. Note that $C_x(e) = e$, thus $T_e(C_x) \in GL(T_eG)$.

Definition 3.15. Let G be a Lie group and let $x \in X$. Then define $Ad(x) = T_eC_x$, so $Ad \colon G \to GL(T_eG)$ and its called the *adjoint representation* of G in T_eG.

The following lemma shows why calling Ad a representation is logical:

Lemma 3.16. Let $x \in G$ with G a Lie group, then $\forall X \in T_eG$
$C_x(exp(X)) = x\, exp(X)\, x^{-1} = exp(Ad(x)X)$.
Furthermore, $Ad \colon G \to GL(T_eG)$ is a Lie group homomorphism.

This shows that $Ad(x)$ acts on T_eG from the left and is also a Lie group morphism from G to $End(T_eG)$. Thus, it is a representation in the same sense as a representation that describes an algebra module. Only, this time it is a Lie group morphism instead of an algebra morphism. But, one could regard to T_eG as a G-module, where the left action of G on T_eG is constructed via the action of Ad.

3.2 Lie algebras

Note, we will first continue with defining what a Lie algebra is with the use of the concepts introduced in the sections about Lie groups. This will be done by constructing a bilinear form $[\,\cdot\,,\,\cdot\,]$ with the use of the adjoint-representation Ad. Then showing that the tangent space T_eG has some nice algebraic structure with respect to this bilinear form.
Now, since the first part of this section uses information on Lie groups, it is also optional and can be skipped. The definition of a Lie algebra will be given fully algebraic and this short dive into the concept of Lie groups will end at definition 3.22, where we will go back to the main topic of this thesis.

To start, we will first give the relation between Lie groups and Lie algebras.

Lemma 3.17. Let G be a Lie group. Then T_eG together with the bilinear map $[\cdot, \cdot] \colon T_eG \times T_eG \to T_eG$ given by $[X, Y] = T_eAd(X)Y$, $X, Y \in T_eG$, is a Lie algebra.

Due to this lemma, its natural to denote the Lie algebra T_eG associated with the Lie group G with the lower case Gothic letter $\mathfrak{g}$. Due to this, we will in general use lower case Gothic letters for Lie algebras. Also, if $\phi \colon G \to H$ is a Lie group morphism, one often denotes the associated tangent map $T_e\phi$ by ϕ_*.

Lemma 3.18. Let G, H be Lie groups and let $\phi\colon G \to H$ be a Lie group morphism. Then the associated tangent map $\phi_*\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra morphism. Moreover, the following diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ {\scriptstyle exp_G}\uparrow & & \uparrow{\scriptstyle exp_H} \\ \mathfrak{g} & \xrightarrow{\phi_*} & \mathfrak{h} \end{array}$$

commutes:

We will quickly sketch how to prove the above lemma. First, let G be a Lie group, then note that $Ad(e) = id_{T_eG}$ and $T_{id_{T_eG}}GL(T_eG) = End(T_eG)$. So we see that the tangent map at e of the adjoint representation Ad is linear.

Definition 3.19. Let the linear map $ad\colon T_eG \to End(T_eG)$ be given by $ad = T_eAd$.

Note that $\forall X \in T_eG$, $ad(X) = \frac{d}{dt}|_{t=0}Ad(exp(tX))$ and we even have that $Ad(exp(X)) = e^{adX}$. Now we finally arrived at the definition of the Lie bracket:

Definition 3.20. Let G be a Lie group and let $X, Y \in T_eG$. Then, the *Lie bracket* is given by $[X, Y] = ad(X)(Y)$.

Proposition 3.21. Let G be a Lie group and let $X, Y, Z \in T_eG$ be arbitrary. Then, the Lie bracket $[\cdot,\cdot]$ of G satisfies the following properties:

1. it is bilinear: $[X + Z, Y] = [X, Y] + [Z, Y]$ and $[X, Z + Y] = [X, Z] + [X, Y]$,
2. it is antisymmetric: $[X, Y] = -[Y, X]$,
3. it satisfies the Jacobi-identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

The last identity follows from the fact that if $\phi\colon G \to H$ is a Lie group morphism, then $T_e\phi[X, Y]_G = [T_e\phi X, T_e\phi Y]_H$. In other words, $\phi_* = T_e\phi$ is a Lie algebra morphism if ϕ is a Lie group morphism. Using this identity with the Lie group morphism $Ad\colon G \to GL(T_eG)$, together with the fact that

$$ad(X)(Y) = \frac{d}{dt}|_{t=0}Ad(exp(tX))(Y) = \frac{d}{dt}|_{t=0}(e^{tX}Ye^{-tX}) = XY - YX$$

for $X, Y \in GL(V)$ for some finite dimensional vector space V, will lead to the Jacobi identity, when one applies $ad[X, Y]$ to Z for $X, Y, Z \in G$.

Now, we finally have all the information that is needed to define what a Lie algebra is if we did not want to skip on the theory on Lie groups. As that theory is not needed for this thesis, and this was mostly a fun little detour, the definition of a Lie algebra will now also be given fully algebraic.

Definition 3.22. Let $\mathbf{k}$ be a field. A *Lie algebra* $\mathfrak{l}$ over $\mathbf{k}$ is a vector space equipped with an anti-symmetric bilinear form $[\cdot,\cdot]\colon \mathfrak{l} \times \mathfrak{l} \to \mathfrak{l}$ satisfying the Jacobi-identity, i.e. $\forall X, Y, Z \in \mathfrak{l}$

$$[X[Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

$\mathfrak{l}$ is called abelian if $[X, Y] = 0 \ \forall X, Y \in \mathfrak{l}$.

Since Lie algebras are almost always denoted by lower Gothic letters. We will also use them to denote Lie algebras. For example, we will often denote Lie algebras by $\mathfrak{g}$ or $\mathfrak{l}$.

Now, similar to previous structures, we again can define subalgebras, ideals, morphisms, the direct sum of two Lie algebras and also quotient algebras. These definitions are again fairly similar to the

versions of the previous discussed structures, so we will quickly go over these definitions.
First, if $\mathfrak{l}$ is a Lie algebra, $\mathfrak{l}' \subset \mathfrak{l}$ a subspace of $\mathfrak{l}$ such that $\forall X, Y \in \mathfrak{l}$ ($[X, Y] \in \mathfrak{l}'$), then $\mathfrak{l}'$ is called a *Lie subalgebra*.
An *ideal* of a Lie algebra $\mathfrak{l}$ is a subspace $\mathfrak{i} \subset \mathfrak{l}$ such that $\forall X \in \mathfrak{l}, Y \in \mathfrak{i}$ we have $[X, Y] \in \mathfrak{i}$. Note, due to the anti-symmetric property of the Lie bracket, we do not have left- and right-ideals this time. We can only have ideals, since $[\mathfrak{l}, \mathfrak{i}] = [\mathfrak{i}, \mathfrak{l}]$.
Let $\phi\colon \mathfrak{l} \to \mathfrak{k}$ be a linear map between two Lie algebras $\mathfrak{l}, \mathfrak{k}$. Then ϕ is called a *Lie algebra morphism* if $\phi([X, Y]_{\mathfrak{l}}) = [\phi(X), \phi(Y)]_{\mathfrak{k}}\ \forall X, Y \in \mathfrak{l}$.
Now, let $\mathfrak{l}, \mathfrak{k}$ be two Lie algebras. Then we can turn $\mathfrak{l} \oplus \mathfrak{k}$ into a Lie algebra by equipping it with a bracket $[(X, A), (Y, B)] = ([X, Y], [A, B])$ for $X, Y \in \mathfrak{l}$ and $A, B \in \mathfrak{k}$. Note that the injections from $\mathfrak{l}$ and from $\mathfrak{k}$ into $\mathfrak{l} \oplus \mathfrak{k}$ are both Lie algebra morphisms.
Lastly, let $\mathfrak{i} \subset \mathfrak{l}$ be an ideal. Then we can define a Lie algebra structure on the quotient space $\mathfrak{l}/\mathfrak{i}$. Let $X + \mathfrak{i}, Y + \mathfrak{i} \in \mathfrak{l}/\mathfrak{i}$. Then we can define a Lie bracket by taking $[X + \mathfrak{i}, Y + \mathfrak{i}] = [X, Y] + \mathfrak{i}$. This is indeed a well-defined Lie bracket, due to the fact that $\mathfrak{i}$ is an ideal of $\mathfrak{l}$. The vector space $\mathfrak{l}/\mathfrak{i}$ together with this Lie bracket is called the *quotient Lie algebra*.

We will give two short examples of a Lie algebras. Let $\mathcal{A}$ be an algebra and define $[\cdot, \cdot]\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by $[X, Y] = XY - YX$ for $X, Y \in \mathcal{A}$. Then $[\,\cdot\,,\,\cdot\,]$ is an antisymmetric bilinear form and for $X, Y, Z \in \mathcal{A}$, we see that

$$\begin{aligned}&[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]\\&= X(YZ - ZY) - (YZ - ZY)X + Y(ZX - XZ) - (ZX - XZ)Y + Z(XY - YX) - (XY - YX)Z = 0.\end{aligned}$$

Thus this bracket also satisfies the Jacobi-identity and induces a Lie algebra structure on the algebra $\mathcal{A}$. The Lie algebra induced by the algebra $\mathcal{A}$ will be denoted by $L(\mathcal{A})$.
This specific bracket is also referred to as the *commutator* of X and Y. Since, if it is just defined on any algebra $\mathcal{A}$, then the relation $XY - YX = 0$ for $X, Y \in \mathcal{A}$ is true if and only if X and Y commute with each other.
Another example is the *opposite Lie algebra*. Let $\mathfrak{l}$ be a Lie algebra with Lie bracket $[\,\cdot\,,\,\cdot\,]$. Then we can construct a new Lie algebra structure on $\mathfrak{l}$ by defining another bracket as

$$[X, Y]_{op} = -[X, Y] = [Y, X],$$

for $X, Y \in \mathfrak{l}$. Then, the Lie algebra $\mathfrak{l}$ with Lie bracket $[\,\cdot\,,\,\cdot\,]_{op}$ is denoted by $\mathfrak{l}^{op}$ and is called the opposite Lie algebra.

Another similar concept is that of representations. Recall that a representation of an algebra $\mathcal{A}$ on a vector space V was an algebra morphism $\rho\colon A \to End(V)$.

Definition 3.23. Let $\mathfrak{l}$ be a Lie algebra and let V be a vector space over $\mathbf{k}$. Then, a *representation* of $\mathfrak{l}$ in V is a Lie algebra morphism $\pi\colon \mathfrak{l} \to End(V)$.

This definition looks similar to that of a representation of an algebra. But this is not weird, since we even have the same concept of equivalence with modules.

Definition 3.24. Let $\mathfrak{l}$ be a Lie algebra and V a vector space over $\mathbf{k}$. Let V be endowed with a bilinear operation $(X, v) \mapsto Xv$.
Then, V is an *$\mathfrak{l}$-module* if

$$[X, Y]v = X(Yv) - Y(Xv),$$

for $X, Y \in \mathfrak{l}$, $v \in V$.

Now, the same concept of equivalence between modules and representations arises again due to the fact that the existence of a representation $\pi\colon \mathfrak{l} \to End(V)$ implies the existence of a $\mathfrak{l}$-module structure on V and vice versa. In other words, $\pi\colon \mathfrak{l} \to End(V)$ is a representation if and only if V is a $\mathfrak{l}$-module with the bilinear operation $Xv = \pi(X)(v)$ for $X \in \mathfrak{l}$, $v \in V$.
So the concept of modules of Lie algebras is similar to that of algebras. We will later see that there is another connection to algebra modules. That connection will even show that Lie algebra modules are basically algebra modules in a certain sense. But for now we will look at one more useful remark regarding modules and then at some more examples of Lie algebras.
Recall definition 2.4 on simple, semisimple and indecomposable algebra modules. These definitions are the same for Lie algebra modules. However, in the case of representations, the first two definitions are often called irreducible and completely reducible. One could just swap either term, but to make it easy to see if we are talking about a representation or a module, we will stick to this notion.

The next two examples will be related to the previously constructed bialgebras $GL(2)$ and $SL(2)$, but we will first start with the general cases. Let V be a vector space with $dim(V) = n$. Then, we see that $L(End(V))$ is a Lie algebra. More precisely, its the Lie algebra $End(V) = T_e(GL(V)) = \mathfrak{gl}(V)$ of the Lie group $GL(V)$, containing all endomorphisms of V.
Now, this was already stated in the part about Lie groups, but if V is a vector space over $\mathbf{k}$ with $dim(V) = n$, then $End(V) \simeq Mat_n(\mathbf{k})$ as Lie groups. Thus, it follows that the Lie algebra $\mathfrak{gl}(V) \simeq \mathfrak{gl}(n) = L(Mat_n(\mathbf{k}))$. This isomorphism of Lie algebras can also be shown algebraically.
Note, both $\mathfrak{gl}(V)$ and $\mathfrak{gl}(n)$ have their commutator as their bracket. By fixing a basis of V, one can define a map that sends $f \in End(V)$ to a matrix in $Mat_n(\mathbf{k})$. This can then be shown to be an algebra isomorphism from $End(V)$ onto $Mat_n(\mathbf{k})$, since taking compositions of functions will correspond to matrix multiplication under this map. Furthermore, since this map is linear, it will send the commutator of $End(V)$ to the commutator of $Mat_n(\mathbf{k})$. Hence $\mathfrak{gl}(V) \simeq \mathfrak{gl}(n)$.
Now, note that if $X, Y \in \mathfrak{gl}(n)$ have trace 0, then $[\,X\,,\,Y\,]$ also has trace 0. To be precise, $trace([\,X\,,\,Y\,]) = 0\ \forall X, Y \in \mathfrak{gl}(n)$. It follows that the subalgebra $\mathfrak{sl}(n) = \{X \in \mathfrak{gl}(n)\colon trace(X) = 0\}$ is also a Lie algebra, as $trace$ is a linear map and the bracket preserves the traceless property of matrices. It even follows that $\mathfrak{sl}(n) \subseteq \mathfrak{gl}(n)$ as an ideal, since the trace is a Lie algebra morphism $\mathfrak{gl}(n) \to \mathbf{k}$ and $\mathfrak{sl}(n)$ is the kernel of this map.
To quickly look back at Lie groups for the last time. Note that $SL(n, \mathbf{k}) = \{x \in GL(n, \mathbf{k})\colon det(x) = 1\}$. In other words, $x \in SL(n, \mathbf{k})$ if it is in the kernel of $det\colon GL(n, \mathbf{k}) \to \mathbf{k}$. Then, it follows that $\mathfrak{sl}(n) = T_e SL(n, \mathbf{k}) = \{X \in T_e GL(n, \mathbf{k})\colon T_e(det)(X) = 0\} = \{X \in \mathfrak{gl}(n)\colon trace(X) = 0\}$. So, by using the theory on Lie groups that we have seen, we do indeed get the same Lie algebra.

Now that we have seen the general cases, we will take a more detailed look at $\mathfrak{gl}(2)$ and $\mathfrak{sl}(2)$. To be precise, since $\mathfrak{gl}(2)$ is just the matrix Lie algebra of 2×2-matrices in the most general sense. It is more interesting to just focus on the Lie algebra $\mathfrak{sl}(2)$

3.2.1 The Lie algebra of $SL(2)$

First, for this part and the rest of this section, we will only work with the field $\mathbf{k} = \mathbb{C}$. In other words, we will only look at complex Lie algebras and the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of the Lie group $SL(2, \mathbb{C})$. From the general example, it follows that the definition of $\mathfrak{sl}(2)$ is given as

$$\mathfrak{sl}(2) = \{X \in \mathfrak{gl}(2)\colon trace(X) = 0\}.$$

We can construct a basis for this Lie algebra consisting of 3 elements, namely

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since any element of $\mathfrak{sl}(2)$ is a linear combination of X, Y and H and $[\cdot\,,\,\cdot]$ is a bilinear operator, the value of $[A\,,\,B]$, for any $A, B \in \mathfrak{sl}(2)$, can be deduced by knowing $[X\,,\,Y]$, $[H\,,\,X]$ and $[H\,,\,Y]$. Luckily, the values of $[X\,,\,Y]$, $[H\,,\,X]$ and $[H\,,\,Y]$ are rather nice. As we have that

$$[X\,,\,Y] = H,\;\; [H\,,\,X] = 2X,\;\; [H\,,\,Y] = -2Y.$$

Now, note that $\mathfrak{sl}(2) \subset \mathfrak{gl}(2)$ and, as noted before, it is even an ideal. Furthermore, note that $\mathbb{C}I = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} : \alpha \in \mathbb{C} \right\}$ is also a Lie subalgebra of $\mathfrak{gl}(2)$. So, it follows that $\mathfrak{gl}(2) \simeq \mathfrak{sl}(2) \oplus \mathbb{C}I$, which also holds if we used $k = \mathbb{R}$. This also shows that study of the Lie algebra $\mathfrak{gl}(2)$ can be reduced to that of $\mathfrak{sl}(2)$. Giving another reason why it is more interesting to just look at $\mathfrak{sl}(2)$.
To add to this, $\mathfrak{sl}(2)$ has no non-trivial ideals. This is an easy check, since if an ideal of $\mathfrak{sl}(2)$ contains H, then it is $\mathfrak{sl}(2)$. And when it contains either X or Y, it contains H. Since kernels of Lie algebra morphisms are ideals, it follows that any Lie algebra morphism $f\colon \mathfrak{sl}(2) \to \mathfrak{g}$, with $\mathfrak{g}$ a Lie algebra, is either injective or 0. Further showing that we can study properties of $\mathfrak{gl}(2)$ by just looking at $\mathfrak{sl}(2)$.

As a final remark on Lie algebras, a non-abelian non-zero Lie algebra with no non-trivial ideals is also called a *simple* Lie algebra. A Lie algebra that can be written as a direct sum of simple Lie algebras is called *semisimple*. These types of Lie algebras have been studied in detail and all types of semisimple Lie algebras are even classified [6]. So $\mathfrak{sl}(2)$ is a well-known and thoroughly studied Lie algebra, the same is true for its enveloping algebra, which we will look at next.

3.3 The enveloping algebra and the Verma modules

We will start this part with defining what the universal enveloping algebra is in an intuitive sense. Then show how to concretely define it. This will be done with the help of [6].
After that, we will look at the Verma module of the universal enveloping algebra of $\mathfrak{sl}(2)$. Do note that we will not look at what the Verma module will be in general, as that is a whole other topic on its own. The Verma module of $\mathcal{U}(\mathfrak{sl}(2))$ will mainly be used as a great example for something similar to the Clebsch-Gordan coefficients of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, namely the Clebsch-Gordan coefficients of $\mathcal{U}(\mathfrak{sl}(2))$. Which is why we will look at this specific Verma module.

Let $\mathfrak{l}$ be a Lie algebra. Intuitively, the *universal enveloping algebra* of $\mathfrak{l}$, $\mathcal{U}(\mathfrak{l})$, is an associative algebra such that:

1. $[X\,,\,Y] = XY - YX$ for $X, Y \in \mathfrak{l}$,
2. $\mathcal{U}(\mathfrak{l})$ is generated by the elements of $\mathfrak{l}$,
3. the algebra $\mathcal{U}(\mathfrak{l})$ is the maximal algebra that satisfies the previous properties.

Note, the multiplication in the first property is the multiplication of the algebra $\mathcal{U}(\mathfrak{l})$. Before giving the formal definition, we will look at two short examples.

Example 1: Let $\mathfrak{l}$ be a Lie algebra with basis given by a single element $X \neq 0$. Then, $\mathcal{U}(\mathfrak{l})$ needs to be generated by X and since $[X\,,\,X] = 0$, it follows that any algebra generated by X satisfies our first two properties.
To satisfy the last property, the algebra needs to not satisfy any other relation between the X^n, $n \in \mathbb{N}$, so the X^n have to be linearly independent. In other words, $\mathcal{U}(\mathfrak{l}) = \mathbb{C}[X]$.

Example 2: Let $\mathfrak{l}$ be an abelian Lie algebra, so $[X\,,\,Y] = 0$ for all $X, Y \in \mathfrak{l}$. Then, the first two properties are satisfied if $\mathcal{U}(\mathfrak{l})$ is any abelian algebra generated by $\mathfrak{l}$.
Now, as $S(\mathfrak{l})$ is an abelian algebra generated by $\mathfrak{l}$ and has no other relations regarding its elements. It can be shown that $S(\mathfrak{l})$ also satisfies the third property. In other words, the symmetric algebra $S(\mathfrak{l})$ is a universal enveloping algebra of an abelian Lie algebra $\mathfrak{l}$.

Now, the formal definition of the universal enveloping algebra is a bit similar to that of the tensor product of vector spaces. So, it is formally defined with the use of a theorem.

Theorem 3.25. *Let $\mathfrak{l}$ be any Lie algebra. Then, $\exists\mathcal{U}(\mathfrak{l})$, an algebra, together with a linear map $\iota\colon \mathfrak{l} \to \mathcal{U}(\mathfrak{l})$ such that:*

i) $\forall X, Y \in \mathfrak{l}$, $\iota([\,X\,,\,Y\,]) = \iota(X)\iota(Y) - \iota(Y)\iota(X)$,

ii) the algebra $\mathcal{U}(\mathfrak{l})$ is generated by the elements $\iota(X)$, $X \in \mathfrak{l}$,

iii) if $\mathcal{A}$ is an algebra such that $\exists j\colon \mathfrak{l} \to \mathcal{A}$, linear, with $j([\,X\,,\,Y\,]) = j(X)j(Y)-j(Y)j(X)$ $\forall X, Y \in \mathfrak{l}$, then $\exists!\varphi\colon \mathcal{U}(\mathfrak{l}) \to \mathcal{A}$, algebra morphism, such that $\varphi(1) = 1$ and $\varphi(\iota(X)) = j(X)$ $\forall X \in \mathfrak{l}$.

A pair $(\mathcal{U}(\mathfrak{l}), \iota)$ satisfying the above theorem is called a *universal enveloping algebra* for $\mathfrak{l}$. The proof of this theorem shows that we can define the universal enveloping algebra in a more intuitive manner with the use of the tensor algebra.

Definition 3.26. Let $\mathfrak{l}$ be a Lie algebra and let $\mathcal{U}(\mathfrak{l})$ be an algebra. Let $\iota_{\mathfrak{l}}\colon \mathfrak{l} \to L(\mathcal{U}(\mathfrak{l}))$ be a Lie algebra morphism that is also an embedding, thus $\iota_{\mathfrak{l}}(X) = X$ and $\iota_{\mathfrak{l}}([\,X\,,\,Y\,]) = XY - YX$ for $X, Y \in \mathfrak{l}$.
Let $I_{\mathfrak{l}} \subset T(\mathfrak{l})$ be the ideal of the tensor algebra generated by the elements $X \otimes Y - Y \otimes X - [\,X\,,\,Y\,]$.
Then, the *universal enveloping algebra* of $\mathfrak{l}$ is defined as $\mathcal{U}(\mathfrak{l}) = T(\mathfrak{l})/I_{\mathfrak{l}}$.

Note, the Lie algebra morphism $\iota_{\mathfrak{l}}$ can be defined as $\iota_{\mathfrak{l}} = \pi_{T(\mathfrak{l})} \circ \iota_{T(\mathfrak{l})}$, where $\iota_{T(\mathfrak{l})}$ is the canonical embedding of $\mathfrak{l}$ into the tensor algebra $T(\mathfrak{l})$ and $\pi_{T(\mathfrak{l})}$ the canonical surjection of $T(\mathfrak{l})$ onto $T(\mathfrak{l})/I_{\mathfrak{l}} = \mathcal{U}(\mathfrak{l})$. Now, the reason why its called the universal enveloping algebra, instead of just the enveloping algebra, is the fact that any two enveloping algebras of the same Lie algebra $\mathfrak{l}$ will be isomorphic. In other words, the universal enveloping algebra is unique up to isomorphisms for any fixed Lie algebra $\mathfrak{l}$.

Before we move on, we will look back at two examples that we gave earlier. We can now check that they are indeed universal enveloping algebras. But, this new definition will also make it a lot easier to see that the given algebras are indeed the enveloping algebras.

Example 1: Again, let $\mathfrak{l}$ be an algebra with basis given by a single element $X \neq 0$. Then, $I_{\mathfrak{l}} = \{0\}$, thus $\mathcal{U}(\mathfrak{l}) = T(\mathfrak{l}) \simeq \mathbb{C}[X]$, as the direct sum of $T(\mathfrak{l})$ implies that it only contains finite linear combinations of X^n, $n \in \mathbb{N}$.

Example 2: Let $\mathfrak{l}$ be an abelian Lie algebra. Then $I_{\mathfrak{l}}$ is the ideal generated by the elements of the form $XY - YX$, in other words, $\mathcal{U}(\mathfrak{l}) = S(\mathfrak{l}) = T(\mathfrak{l})/I_{\mathfrak{l}}$.

WE will now look at some properties of the enveloping algebra.

Theorem 3.27. *Let $\mathfrak{l}$ be a Lie algebra and let $\mathcal{A}$ be any algebra and $f\colon \mathfrak{l} \to L(\mathcal{A})$ any Lie algebra morphism.*
Then, $\exists!\varphi\colon \mathcal{U}(\mathfrak{l}) \to \mathcal{A}$, algebra morphism, such that $\varphi \circ \iota_{\mathfrak{l}} = f$.

This theorem can be rewritten into the following expression:

$$Hom_{Lie}(\mathfrak{l}, L(\mathcal{A})) \simeq Hom_{Alg}(\mathcal{U}(\mathfrak{l}), \mathcal{A}),$$

where $Hom_{Lie}(\mathfrak{l}, \mathfrak{l}')$ denotes the set of Lie algebra morphisms from $\mathfrak{l}$ into $\mathfrak{l}'$. It easily follows from the fact that one can extend $f\colon \mathfrak{l} \to L(\mathcal{A})$ to a function $\bar{f}\colon T(\mathfrak{l}) \to \mathcal{A}$.
There are two useful corollaries of this theorem. Which will help us show that the universal enveloping algebra is also a Hopf algebra.

Corollary 3.28. Let $f\colon \mathfrak{l} \to \mathfrak{l}'$ be any Lie algebra morphism. Then, $\exists!\mathcal{U}(f)\colon \mathcal{U}(\mathfrak{l}) \to \mathcal{U}(\mathfrak{l}')$, algebra morphism, such that $\mathcal{U}(f)\colon \iota_{\mathfrak{l}} = \iota_{\mathfrak{l}'} \circ f$.
Also, if $f'\colon \mathfrak{l} \to \mathfrak{l}''$ is another Lie algebra morphism, then $\mathcal{U}(f' \circ f) = \mathcal{U}(f') \circ \mathcal{U}(f)$.

The first part directly follows from theorem 3.27 and the second part follows from a direct calculation of $\mathcal{U}(f' \circ f) \circ \iota_{\mathfrak{l}} = \mathcal{U}(f') \circ \mathcal{U}(f) \circ \iota_{\mathfrak{l}}$.

Corollary 3.29. Let $\mathfrak{l}, \mathfrak{l}'$ be two Lie algebras and let $\mathfrak{l} \oplus \mathfrak{l}'$ be their direct sum. Then $\mathcal{U}(\mathfrak{l} \oplus \mathfrak{l}') \simeq \mathcal{U}(\mathfrak{l}) \otimes \mathcal{U}(\mathfrak{l}')$.

Proof. Let $f\colon \mathfrak{l} \oplus \mathfrak{l}' \to L(\mathcal{U}(\mathfrak{l}) \otimes \mathcal{U}(\mathfrak{l}'))$ given by $f(X, X') = \iota_{\mathfrak{l}}(X) \otimes 1 + 1 \otimes \iota_{\mathfrak{l}'}(X')$. Then f is a linear map and even a Lie algebra morphism. Then, by theorem 3.27, we have an algebra morphism $\varphi\colon \mathcal{U}(\mathfrak{l} \oplus \mathfrak{l}') \to \mathcal{U}(\mathfrak{l}) \otimes \mathcal{U}(\mathfrak{l}')$.
The converse goes by constructing an algebra morphism $\psi\colon \mathcal{U}(\mathfrak{l}) \otimes \mathcal{U}(\mathfrak{l}') \to \mathcal{U}(\mathfrak{l} \oplus \mathfrak{l}')$. This can also be done with the use of theorem 3.27. As it implies the existence of two algebra morphisms $\psi_1\colon \mathcal{U}(\mathfrak{l}) \to \mathcal{U}(\mathfrak{l} \oplus \mathfrak{l}')$ and $\psi_2\colon \mathcal{U}(\mathfrak{l}') \to \mathcal{U}(\mathfrak{l} \oplus \mathfrak{l}')$ with $\psi_1(X) = \iota_{\mathfrak{l} \oplus \mathfrak{l}'}(X, 0)$ and $\psi_2(X') = \iota_{\mathfrak{l} \oplus \mathfrak{l}'}(0, X')$. Then, by proposition 1.8, $\psi(X \otimes X') = \psi_1(X)\psi_2(X')$ is an algebra morphism as $\psi_1(X)\psi_2(X') = \psi_2(X')\psi_1(X)$ for $X \in \mathfrak{l}$, $X' \in \mathfrak{l}'$.
Then, since ψ and φ are inverses of each other. It follows that $\mathcal{U}(\mathfrak{l} \oplus \mathfrak{l}') \simeq \mathcal{U}(\mathfrak{l}) \otimes \mathcal{U}(\mathfrak{l}')$. □

Now, as was noted before, these two corollaries help us to endow a Hopf algebra structure on the enveloping algebra. One can define a coproduct $\Delta\colon \mathcal{U}(\mathfrak{l}) \to \mathcal{U}(\mathfrak{l}) \otimes \mathcal{U}(\mathfrak{l})$ by $\Delta = \varphi \circ \mathcal{U}(\delta)$, with $\varphi\colon \mathcal{U}(\mathfrak{l} \oplus \mathfrak{l}) \to \mathcal{U}(\mathfrak{l}) \otimes \mathcal{U}(\mathfrak{l})$ as in the proof above and $\delta(X) = (X, X)$ for $X \in \mathfrak{l}$.
The counit of $\mathcal{U}(\mathfrak{l})$ can be given by $\varepsilon = \mathcal{U}(0)$. With $0\colon \mathfrak{l} \to \{0\}$ the Lie algebra morphism that sends all elements of $\mathfrak{l}$ to 0. And the antipode can be defined as $S = \mathcal{U}(op)$, with the Lie algebra morphism $op\colon \mathfrak{l} \to \mathfrak{l}^{op}$, $op(X) = -X$ and $\mathfrak{l}^{op}$ the opposite Lie algebra.

Proposition 3.30. The enveloping algebra $\mathcal{U}(\mathfrak{l})$ of a Lie algebra $\mathfrak{l}$ is a cocommutative Hopf algebra with $\Delta = \varphi \circ \mathcal{U}(\delta)$, $\varepsilon = \mathcal{U}(0)$ and $S = \mathcal{U}(op)$.
Then, for $X_1, \ldots, X_n \in \mathfrak{l}$, we have $\Delta(X) = 1 \otimes X + X \otimes X$ and

$$\Delta(X_1 \cdots X_n) = 1 \otimes X_1 \cdots X_n + \sum_{p=1}^{n-1} \sum_{\sigma} X_{\sigma(1)} \cdots X_{\sigma(p)} \otimes X_{\sigma(p+1)} \cdots X_{\sigma(n)} + X_1 \cdots X_n \otimes 1,$$

with σ running over all (p, q)-shuffels of Sn. And $S(X_1 \cdots X_n) = (-1)^n X_n \cdots X_1$.

The proof for this boils down to just checking that the diagrams used in the Hopf algebra definition commute. In other words, it is sufficient to show that Δ is a well-defined algebra morphism and a coproduct. From this, it follows that if we have two $\mathcal{U}(\mathfrak{l})$-modules V, W. Then $V \otimes W$ turns into a $\mathcal{U}(\mathfrak{l})$-module via the action

$$X(v \otimes w) = \Delta(X)(v \otimes w) = Xv \otimes w + v \otimes Xw,$$

which is the same fact we saw earlier for any Hopf algebra $\mathcal{H}$.

One of the most important theorems about the universal enveloping algebra is the Poincaré-Birkhoff-Witt theorem, in short the PBW theorem. The reason for this, is that this theorem shows a lot of the structure of any enveloping algebra, which makes it a lot easier to work with, compared to only seeing it as a quotient algebra of the tensor algebra.

Theorem 3.31 (**Poincaré-Birkhoff-Witt**). *Let $\mathfrak{l}$ be a finite-dimensional Lie algebra with basis $X_1, \ldots, X_n$, then the elements of the form*

$$X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n},$$

for $k_i \in \mathbb{N}$, span all of $\mathcal{U}(\mathfrak{l})$ and are linearly independent.
Furthermore, the elements $X_1, \ldots, X_n$ are linearly independent.

The last remark in this theorem is why we can just write X, instead of $\iota_{\mathfrak{l}}(X)$ as it shows that it is indeed injective. Even though the theorem does not look too difficult to prove, showing that all $X_1^{k_1}X_2^{k_2}\cdots X_n^{k_n}$ are linearly independent is not easy. Due to this, the length of the proof is several pages long. So, the PBW-theorem will be taken as a fact in this paper and anyone that is interested in the details can read the full proof in [8]. Do note that we used the same finite-dimensional version of this theorem, but the theorem is still true for infinite dimensional Lie algebras.
As a consequence of the PBW-theorem, we have the following nice corollary about the enveloping algebra of a Lie subalgebra.

Corollary 3.32. Let $\mathfrak{l}' \subset \mathfrak{l}$ be a Lie subalgebra. Then there exists a natural injection $\iota\colon \mathcal{U}(\mathfrak{l}') \to \mathcal{U}(\mathfrak{l})$ given by $\iota(X_1\cdots X_n) = X_1\cdots X_n$ for $X_1,\ldots,X_n \in \mathfrak{l}'$.

Now, before we go to the Verma module of $\mathcal{U}(\mathfrak{sl}(2))$, we will return to the remark about Lie algebra representations and how they are related to algebra modules.

Proposition 3.33. let $\pi\colon \mathfrak{l} \to \mathfrak{gl}(V)$ be a Lie algebra representation. Then, $\exists!\bar{\pi}\colon \mathcal{U}(\mathfrak{l}) \to End(V)$ such that $\bar{\pi}(1) = I$ and $\bar{\pi}(X) = \pi(X)\ \forall X \in \mathfrak{l}$.

This follows immediately from applying theorem 3.25 with $\mathcal{A} = End(V)$ and $j(X) = \pi(X)$. Since the converse is also true, we have a equivalence relation between $\mathcal{U}(\mathfrak{l})$-modules and $\mathfrak{l}$-modules. As a $\mathcal{U}(\mathfrak{l})$-module is the same as an algebra representation $\bar{\pi}\colon \mathcal{U}(\mathfrak{l}) \to End(V)$. Which is equivalent to a Lie algebra representation $\pi\colon \mathfrak{l} \to \mathfrak{gl}(V)$, which in turn is the same as an $\mathfrak{l}$-module. So, we can relate any Lie algebra module to an algebra module.

3.3.1 The enveloping algebra $\mathcal{U}(\mathfrak{sl}(2)$ and its Verma modules

As was noted before, we do not want to fully delve into the theory on Verma modules, since it is mostly used as a good example for what we want to do later. It will also show how nice the finite dimensional modules of $\mathcal{U}(\mathfrak{sl}(2))$ are. But before we can look at these Verma modules, we first need to define $\mathcal{U}(\mathfrak{sl}(2))$. Luckily, this is an enveloping algebra that we can easily describe with the help of three generators.

Definition 3.34. The enveloping algebra $\mathcal{U}(\mathfrak{sl}(2))$ is isomorphic to the algebra generated by $X, Y, H \in \mathfrak{sl}(2)$ such that

$$XY - YX = H, \quad HX - XH = 2X, \quad HY - YH = -2Y.$$

For convenience, we will use the $[\,\cdot\,,\,\cdot\,]$ to denote the commutator, as we will only work with this bracket from this point onwards. Then the above relations can be rewritten into

$$[\,X\,,\,Y\,] = H, \quad [\,H\,,\,X\,] = 2X, \quad [\,H\,,\,Y\,] = -2Y.$$

Also, by the PBW-theorem, it follows that $\mathcal{U}(\mathfrak{sl}(2))$ has a basis given by $\{X^iY^jH^k\}_{i,j,k\in\mathbb{N}}$. Due to this, it is useful to look at certain relations between elements of $\mathcal{U}(\mathfrak{sl}(2))$. These relations are useful if one wants to calculate products of basis elements of $\mathcal{U}(\mathfrak{sl}(2))$ and, in our case, they are also useful in the construction of the Verma modules.

Lemma 3.35. For any $p, q \in \mathbb{N}$, $X, Y, H \in \mathcal{U}(\mathfrak{sl}(2))$, we have

$$\begin{gathered} X^pH^q = (H-2p)^qX^p, \qquad Y^pH^q = (H+2p)^qY^p, \\ [\,X\,,\,Y^p\,] = pY^{p-1}(H-p+1) = p(H+p-1)Y^{p-1}, \\ [\,X^p\,,\,Y\,] = pX^{p-1}(H+p-1) = p(H-p+1)X^{p-1} \end{gathered}$$

Another nice thing about the enveloping algebra $\mathcal{U}(\mathfrak{sl}(2))$ is that its center is generated by a single element.

Lemma 3.36. Let $C = XY + YX + \frac{H^2}{2} \in \mathcal{U}(\mathfrak{sl}(2))$, called the *Casimir element*, then $C \in Z(\mathcal{U}(\mathfrak{sl}(2)))$. Furthermore, $Z(\mathcal{U}(\mathfrak{sl}(2)))$ is generated by C.

We would love to show that the center is indeed generated by the Casimir element, but that is a lot easier with the help of certain modules. So, we will first at look some special modules of $\mathcal{U}(\mathfrak{sl}(2))$. To be precise, we want to look at the finite-dimensional simple modules.
Recall that a simple module is just a module with only trivial submodules. In other words, we want to look at finite-dimensional modules that we cannot make "smaller". We will prove that every finite-dimensional simple module of $\mathcal{U}(\mathfrak{sl}(2))$ is of the same type. All of these modules will be generated by a single vector with a special property.

Definition 3.37. Let V be a $\mathcal{U}(\mathfrak{sl}(2))$-module and let $\lambda \in \mathbb{C}$. If $v \in V\backslash\{0\}$ such that $Hv = \lambda v$, then v is said to be of weight λ.
Furthermore, if $Xv = 0$, then v is called a *highest weight vector* of weight λ.

Proposition 3.38. Any non-zero finite-dimensional $\mathcal{U}(\mathfrak{sl}(2))$-module has a highest weight vector.

Proof. $dim(V) < \infty$ and $\mathbb{C}$ is algebraically closed, so $\exists \lambda \, (Hv = \lambda v, v \neq 0)$. Assume that $Xv \neq 0$, else v would be of highest weight. Then,

$$H(X^n v) = (\lambda + 2n)(X^n v),$$

by lemma 3.36 and $(X^n v)_{n \geq 0}$ is a sequence of eigenvectors of H with distinct eigenvalues.
Hence $\exists n \in \mathbb{N} \left(X^n v \neq 0 \wedge X^{n+1} v = 0\right)$, as H can only have $dim(V)$ distinct eigenvalues. Thus $X^n v$ is of highest weight. ☐

Lemma 3.39. Let v be a highest weight vector of weight λ. Let $n \in \mathbb{N}$ and define $v_n = \frac{1}{n!} Y^n v$, then

$$Hv_n = (\lambda - 2n)v_n, \;\; Xv_n = (\lambda - n + 1)v_{n-1}, \;\; Yv_n = (n+1)v_{n+1}$$

So we see that if v is of highest weight λ and we order the v_n by their subscript, then v_n is an eigenvector of H, X sends v_n to the previous eigenvector v_{n-1} of H and Y sends v_n to the next eigenvector v_{n+1} of H. Thus, we can cycle through all the eigenvectors of H with the actions of X and Y on v.
Furthermore, we have seen that any non-zero finite-dimensional module has at least one highest weight vector v. Since we can define actions of $\mathcal{U}(\mathfrak{sl}(2))$ on v in such a way that we get new vectors v_k, we can construct a vector space that is, in a sense, generated by this vector v. The following theorem will show that every non-zero finite-dimensional simple $\mathcal{U}(\mathfrak{sl}(2))$-module is generated by a highest weight vector v in the sense of lemma 3.39.

Theorem 3.40. *Let V be a finite-dimensional $\mathcal{U}(\mathfrak{sl}(2))$-module generated by a highest weight vector v of weight λ. Then*

i) $\lambda = dim(V) - 1$,

ii) if $v_n = \frac{1}{n!} Y^n v$, *then* $v_n = 0$ *if* $n > \lambda$ *and* $\{v_0, \ldots, v_\lambda\}$ *is a basis for* V,

iii) the operator H *acting on* V *is diagonalizable with eigenvalues* $\{\lambda, \lambda - 2, \ldots, \lambda - 2\lambda = -\lambda\}$,

iv) any $w \in V$ *that is a highest weight vector is of weight* λ *and* $w = \alpha v$, $\alpha \in \mathbb{C}$,

v) V *is simple.*

Furthermore, any simple $\mathcal{U}(\mathfrak{sl}(2))$-module is generated by a highest weight vector. And, if W, W' are two $\mathcal{U}(\mathfrak{sl}(2))$-modules generated by highest weight vectors of weight λ, then $W \simeq W'$.

Proof. Note, $i), ii), iii)$ and $iv)$ are direct consequences of the formulas given in lemma 3.39 in combination with V being finite-dimensional. Since, again, from the fact that all v_n are eigenvectors of H with distinct eigenvalues, it follows that $v_n \neq 0$ and $v_{n+1} = 0$ for some $n = dim(V) - 1 \in \mathbb{N}$.
Lemma 3.39 shows then that $v_m = 0$ for all $m > n$ and $v_m \neq 0$ if $m < n$. It follows that $n = \lambda$ due to the fact that $0 = X0 = Xv_{n+1} = (\lambda - n)v_n$. From the formulas of this lemma, it also follows that $\{v_0, v_1, \dots, v_\lambda\}$ is a basis for V and that H acts diagonal on this basis.
For $iv)$, note that any highest weight vector is an eigenvector of H. Thus it is a linear combination of the v_i, but lemma 3.39 implies that $Xv_i = 0 \iff i = 0$. Hence, any highest weight vector v has to be of the form $v = \alpha v_0$.
For $v)$, let $0 \neq W \subset V$ be a $\mathcal{U}(\mathfrak{sl}(2))$-submodule. Then it has a highest weight vector $w \in W$ by proposition 3.38, hence $w = \alpha v$ for some $\alpha \in \mathbb{C}$ by $iv)$. Thus $v \in W \implies V \subset W \implies W = V$. Showing that V is indeed simple.
For the last remarks, note that $v)$ shows that if V is any simple finite-dimensional module, it will be generated by a highest weight vector v. Since the subspace generated by v would else be a submodule.
Lastly, if V, W are $\mathcal{U}(\mathfrak{sl}(2))$-modules generated by vectors v, w of highest weight λ, then $\phi\colon v \mapsto w$ is a $\mathcal{U}(\mathfrak{sl}(2))$-module isomorphism. □

These finite dimensional $\mathcal{U}(\mathfrak{sl}(2))$-modules are often denoted as $V(n)$, where $n \in \mathbb{N}$ is the weight of the vector that generates $V(n)$. So, $V(n)$ is of dimension $n+1$. This is a useful classification, since these modules are unique up to isomorphism by the last remark of the above theorem. These modules, or even any highest weight $\mathcal{U}(\mathfrak{sl}(2))$-module, are a quotient of the Verma module given in lemma 3.41. To show this, we first need to look at infinite-dimensional $\mathcal{U}(\mathfrak{sl}(2))$-modules.

Notice that we restricted our modules to be finite-dimensional. But, we can easily construct an infinite dimensional $\mathcal{U}(\mathfrak{sl}(2))$-module of highest weight λ with the formulas we saw before. Since the actions of X, Y and H defined in lemma 3.39 will induce a $\mathcal{U}(\mathfrak{sl}(2))$-module structure, even if they act on a non-finite-dimensional vector space.

Lemma 3.41. Let $V(\lambda)$ be a vector space with basis $\{v_i\}_{i\in\mathbb{N}}$, $v_i = \frac{1}{i!}Y^i v_0$, such that $X, Y, H \in \mathcal{U}(\mathfrak{sl}(2))$ act on the v_i as in lemma 3.39. Then, $V(\lambda)$ is a $\mathcal{U}(\mathfrak{sl}(2))$-module and it's generated by the highest weight vector v_0 of weight λ.

Proof. Note that the formulas in lemma 3.39 directly imply that $V(\lambda)$ is a $\mathcal{U}(\mathfrak{sl}(2))$-module, since, for $n \geq 0$,

$$\begin{aligned}
[X, Y]v_n &= (n+1)(\lambda - n)v_n - (\lambda - n + 1)nv_n = (\lambda - 2n)v_n, \\
[H, X]v_n &= ((\lambda - n + 1)(\lambda - 2(n-1)) - (\lambda - 2n)(\lambda - n + 1))v_{n-1} = 2Xv_n
\end{aligned}$$

Similar calculation shows that $[H, Y]v_n = -2Yv_n$.
Now, $Hv_0 = \lambda v_0$ and $Xv_0 = 0$, so v_0 is indeed a highest weight vector. Lastly, $Yv_n = (n+1)v_{n+1}$ shows that $v_{n+1} = \frac{1}{n!}Y^n v_n$, so $V(\lambda)$ is indeed generated by v_0. □

This lemma shows that we also have infinite dimensional $\mathcal{U}(\mathfrak{sl}(2))$-modules. This infinite dimensional $\mathcal{U}(\mathfrak{sl}(2))$-module of highest weight λ is called the *Verma module*. To avoid confusion, we will denote the Verma modules by $V(\lambda)$, or other Greek letters, and the finite-dimensional simple modules by $V(n)$, or other Roman letters.
Do note that the Verma modules $V(\lambda)$ do not need to be simple modules. Since, if we take $\lambda \in \mathbb{N}$, then $Xv_{\lambda+1} = (\lambda - \lambda)v_\lambda = 0$. So, the infinite dimensional module generated by a highest weight vector v_0 of weight λ, has a submodule generated by $v_{\lambda+1}$. Hence it is not simple.
A nice thing about the modules $V(n)$ is that the Casimir element and any element of the center acts on them as a scalar.

Lemma 3.42. Let $A \in Z(\mathcal{U}(\mathfrak{sl}(2)))$, then $Zv = \alpha v$ for any $v \in V(n)$, $\alpha \in \mathbb{C}$, $n \in \mathbb{N}$.
In particular, if $v \in V(n)$, then $Cv = \frac{n(n+2)}{2}v$.

Proof. Let $A \in Z(\mathcal{U}(\mathfrak{sl}(2)))$, then the action of A on $V(n)$ induces an endomorphism $\pi\colon \mathcal{U}(\mathfrak{sl}(2)) \to \mathcal{U}(\mathfrak{sl}(2))$; $v \mapsto Av$. Since A is central, π is $V(n)$-linear and it has an eigenvalue $\alpha \in \mathbb{C}$ due to $dim(V(n)) = n + 1 < \infty$.
Then, $\varphi = \pi - \alpha id_{V(n)}$ is once again a $\mathcal{U}(\mathfrak{sl}(2))$-linear endomorphism and $ker(\varphi) \subset V(n)$ is a $\mathcal{U}(\mathfrak{sl}(2))$-submodule. Since $V(n)$ is simple, it follows that $ker(\varphi) = V(n) \implies \pi(v) = \alpha v \,\forall v \in V(n)$. The proof that $Cv = \frac{n(n+2)}{2}v$ follows from a simple calculation on $v = v_0$. □

Note, since $HAv_i = AHv_i = (n-2i)Av_i$, for any $A \in Z(\mathcal{U}(\mathfrak{sl}(2)))$, it follows that the eigenvectors of A are the same as those of H, the vectors $v_i = \frac{1}{i!}Y^i v_0$.
Furthermore, we have the following universal property for all highest weight modules.

Proposition 3.43. Any highest weight $\mathcal{U}(\mathfrak{sl}(2))$-module V of highest weight λ is a quotient of the Verma module $V(\lambda)$.

Proof. Let v be a highest weight vector that generates V. Then $f\colon V(\lambda) \to V$; $v_i \mapsto \frac{1}{i!}Y^i v$ is a linear map and lemma 3.39 implies that f is $\mathcal{U}(\mathfrak{sl}(2))$-linear. Since $f(v_0)$ generates V, it follows that f is a surjective map, hence $\overline{f}\colon V(\lambda) \xrightarrow{\sim} V/ker(f)$ is a $\mathcal{U}(\mathfrak{sl}(2))$-module isomorphism. □

Note that this proposition shows again that the Verma modules $V(\lambda)$ need not be simple. Since the simple modules $V(n)$ are in particular the quotient of $V(\lambda)$ with $\lambda = n$. Hence, $V(\lambda)$ cannot be simple when $\lambda \in \mathbb{N}$, as $V(n)$ will always be a non-zero submodule that is not $V(\lambda)$ itself.

Lemma 3.44. $V(\lambda)$ is simple $\iff \lambda \notin \mathbb{N}$.

Proof. Note that the above remark already shows that $V(\lambda)$ is not simple if $\lambda \in \mathbb{N}$, thus $\lambda \notin \mathbb{N} \implies V(\lambda)$ is simple.
To show the converse, assume that $V(\lambda)$ is not simple. So there is a non-trivial submodule $V \subset V(\lambda)$. We can assume without loss of generality that V is simple, else we could take a submodule of V that is simple.
Now, $dim(V)$ is either finite or infinite. If $dim(V) < \infty$, then theorem 3.40 implies that $\lambda \in \mathbb{N}$.
So, let V be an infinite dimensional submodule of $V(\lambda)$. Then, $v \in V$ is a linear combination of the v_i, since $v \in V(\lambda)$. But, then $Y^n v = 0$ for some $n \in \mathbb{N}$. Hence $0 \neq Y^{n-1}v \in V \implies \alpha v_0 \in V$ for some $0 \neq \alpha \in \mathbb{C}$. Thus $V = V(\lambda)$ if $dim(V) = \infty$, so $V(\lambda)$ cannot have non-trivial infinite dimensional submodules.
Hence, $V(\lambda)$ is simple $\iff \lambda \notin \mathbb{N}$. □

We can use this, together with the theory on the Verma modules $V(n)$, to finally prove that C generates $Z(\mathcal{U}(\mathfrak{sl}(2)))$. So let's prove lemma 3.36.

Proof. Let $U^H \subset \mathcal{U}(\mathfrak{sl}(2))$ be given by $U^H = \{A \in \mathcal{U}(\mathfrak{sl}(2)) | AH = HA\}$. We want to construct an algebra morphism $\pi\colon U^H \to \mathbf{k}[H]$ and use it to make an isomorphism from $Z(\mathcal{U}(\mathfrak{sl}(2)))$ into $\mathbf{k}[H^2]$.
First, note that $A \in U^H \iff A = \sum_{n\in N} Y^n P_n X^n$, with $P_n \in \mathbf{k}[H]$. Since,

$$(Y^i H^j X^k)H = Y^i H^j (H-2k)X^k = Y^i (H-2k)H^j X^k = ((H+2i)-2k)Y^i H^j X^k,$$

so by the PBW-theorem, it follows that $A \in U^H \iff A = \sum_{n\in N} Y^n P_n X^n$.
Next, we claim that $I = Y\mathcal{U}(\mathfrak{sl}(2)) \cap U^H = \mathcal{U}(\mathfrak{sl}(2))X \cap U^H$ is an ideal. Note $Y\mathcal{U}(\mathfrak{sl}(2)) \cap U^H$ is obviously a left ideal and $\mathcal{U}(\mathfrak{sl}(2))X \cap U^H$ is obviously a right ideal. So if they are equal, it immediately follows that I is an ideal. Now, $A \in Y\mathcal{U}(\mathfrak{sl}(2)) \cap U^H \iff P_0 = 0 \iff A \in \mathcal{U}(\mathfrak{sl}(2))X \cap U^H$, so I is indeed an ideal.
Since I is an ideal, it follows that we have an algebra morphism $\pi\colon U^H \to U^H/I$; $A \mapsto A + I$. Note

that $\pi(A) = \pi(\sum_{n\in N} Y^n P_n X^n) = P_0 + I$ and $\mathbf{k}[H] \cap I = \{0\}$. So it follows that $U^H = \mathbf{k}[H] \oplus I$ and we can rewrite π to be the algebra morphism $\pi\colon U^H \to \mathbf{k}[H]; \sum_{n\in\mathbb{N}} Y^n P_n X^n \mapsto P_0$.
Then π is called the Harish-Chandra morphism. By lemma 3.42 we know that if $Z \in Z(\mathcal{U}(\mathfrak{sl}(2)))$, then Z acts as a scalar on any $v \in V(n)$. Moreover, we even have that $Zv = \pi(Z)(\lambda)v$ for any $v \in V$ with V a highest weight module of weight λ. As, if v_0 generates the highest weight module V of weight λ, then

$$Zv_0 = \sum_{n\in\mathbb{N}} Y^n P_n X^n v_0 = P_0 v_0 = P_0(\lambda)v_0$$

and since v_0 generates V, it follows that $Zv = \pi(Z)(\lambda)v$.
Furthermore, $\pi|_{Z(\mathcal{U}(\mathfrak{sl}(2)))}$ is **injective.** Since, if $0 \neq Z \in Z(\mathcal{U}(\mathfrak{sl}(2)))$ and $\pi(Z) = 0$, then $Z = \sum_{n=k}^{\infty} Y^n P_n X^n$ for some non-zero polynomial $P_k \in \mathbf{k}[K]$. And we get

$$\pi(Z)(\lambda)v_k = Zv_k = Y^k P_k X^k v_k = 0 \iff k = 0 \text{ for } v_k \in V(\lambda),\ \lambda \notin \mathbb{N}.$$

Thus it follows that $P_k = 0$, but we can repeat this for any $k \in \mathbb{N}$, thus $Z = 0$. So we see that the map π is injective.
Now, let $\delta\colon \mathbf{k}[H] \to \mathbf{k}[H]; f(H) \mapsto f(H-1)$. Then $\varphi = \delta \circ \pi\colon \mathcal{U}(\mathfrak{sl}(2))^H \to \mathbf{k}[H]$ is the normalized Harish-Chandra morphism and we see that $\varphi(Z)(\lambda) = \varphi(Z)(\lambda^{-1})$ if $Z \in Z(\mathcal{U}(\mathfrak{sl}(2)))$.
Since, if $v_n \in V(n-1)$ and $Z \in Z(\mathcal{U}(\mathfrak{sl}(2)))$, then $Xv_n = (n-1-n+1)v_{n-1} = 0$, so v_n is of weight $-n-1$. Thus $Zv_n = \pi(Z)(-n-1)v_n$, but $v_n \in V(n-1)$, thus $Zv_n = \pi(Z)(n-1)v_n$. So we see that

$$\varphi(Z)(n)v_n = \pi(Z)(n-1)v_n = \pi(Z)(-n-1)v_n = \varphi(Z)(-n)v_n.$$

Since both δ and π are injective, we again have that φ is injective. By the above, we have that $\varphi(Z(\mathcal{U}_q(\mathfrak{sl}(2)))) \subset \mathbf{k}[H^2]$. Also,

$$\varphi(C) = \varphi(XY + YX + \frac{H^2}{2}) = \varphi(2YX + H + \frac{H^2}{2}) = \delta(H + \frac{H^2}{2}) = \frac{H^2}{2} - \frac{1}{2}.$$

It follows that $\varphi(Z(\mathcal{U}(\mathfrak{sl}(2)))) = \mathbf{k}[H^2]$, since it contains the polynomial H^2. Hence, π is injective and surjective when restricted to $Z(\mathcal{U}(\mathfrak{sl}(2)))$, thus $Z(\mathcal{U}(\mathfrak{sl}(2))) \simeq \mathbf{k}[H^2]$ via $\varphi|_{Z(\mathcal{U}(\mathfrak{sl}(2)))}$.
Now, since $\varphi(C)$ generates $\mathbf{k}[H^2]$, it follows that C generates $Z(\mathcal{U}(\mathfrak{sl}(2)))$. □

As a remark on this proof, it is the specific case of theorem 7.4.5 of [9]. Since one can show that the Weyl group acts by $H \mapsto -H$ on $\mathbf{k}[H]$. So, the Harish-Chandra morphism indeed becomes an isomorphism from $Z(\mathcal{U}(\mathfrak{sl}(2)))$ to $\mathbf{k}[H^2]$, as those are the polynomials that are invariant for the change $H \mapsto -H$. For more details regarding this specific case, see [10]. This uses more theory on Lie algebras than we introduced, but rigorously goes over the case for $\mathcal{U}(\mathfrak{sl}(2))$ unlike [9], which only gives the general theorem. [10] also introduces all of the extra details that are necessary to understand the Harish Chandra morphism in full detail. So, all theory on Lie algebras given in this section is enough to understand what is done in this paper.

We started this part by saying that the finite-dimensional simple $\mathcal{U}(\mathfrak{sl}(2))$-modules $V(n)$ are a good example of what we'll be looking at later on. The reason for this, is the existence of the Clebsch-Gordan coefficients. Which, kinda like the 6j-symbols, show the relation between $V(n) \otimes V(m)$ and their decomposition into a direct sum of simple modules. These coefficients exist for any direct product of two finite-dimensional modules due to the following fact.

Theorem 3.45. *Any finite-dimensional $\mathcal{U}(\mathfrak{sl}(2))$-module is semisimple.*

Proof. It suffices to show that if V, V' are arbitrary finite-dimensional $\mathcal{U}(\mathfrak{sl}(2))$-modules, $V' \subset V$, then $\exists V''$, $\mathcal{U}(\mathfrak{sl}(2))$-module, such that $V = V' \oplus V''$.
This will be proven in two steps. We will first show that such a V'' exists in the case that $V' \subset V$ is

of codimension 1. Then we will proof that it holds in the case that $V' \subset V$ is of any codimension.
So, let V, V' be two $\mathcal{U}(\mathfrak{sl}(2))$-modules such that $V' \subset V$ is of codimension 1. We will show $V = V' \oplus V''$ via induction on $dim(V')$.
Let $dim(V') = 0$, then $V = V' \oplus V$. If $dim(V') = 1$, V' and V/V' are of dimension 1 and there is a basis $\{v_1 \in V', v_2 \notin V'\}$ such that $\mathfrak{sl}(2)v_1 = 0$ and $\mathfrak{sl}(2)v_2 \subset V'$. Then $[\,\mathfrak{sl}(2)\,,\,\mathfrak{sl}(2)\,]v_i = 0$ for $i = 1, 2$ and it follows that $\mathfrak{sl}(2)$ acts trivially on V. So we can take any subspace $V'' \subset V$ with $V'' \cap V' = \{0\}$ and get $V = V' \oplus V''$.
Assume $dim(V') = n > 1$ and that $\exists V''$, $\mathcal{U}(\mathfrak{sl}(2))$-module, such that $V = V' \oplus V''$ when $dim(V') < n$. Then, V' is either simple or it isn't.
Say V' is not simple. Then, $\exists V_1 \subset V'$, submodule, such that $0 < dim(V_1) < n$. Let $\pi \colon V \to V/V_1$ be the canonical projection, then $\pi(V') \subset \pi(V)$ is a submodule of codimension 1 and by the induction hypothesis $\exists W \subset \pi(V)$, submodule, such that $\pi(V) = \pi(V') \oplus W$. Which gives us $V = V' + \pi^{-1}(W)$. $dim(W) = 1$, thus $V_1 \subset \pi^{-1}(W)$ is a submodule of codimension 1 and the induction hypothesis again implies $\exists V'' \subset \pi^{-1}(W)$ such that $\pi^{-1}(W) = V_1 \oplus V''$. Now, by construction, $V = V' + V_1 + V'' = V' + V''$ and since $dim(V) = dim(V') + dim(V'')$, we get $V = V' \oplus V''$.
Now, if V' is simple, then C acts on V' as a non-zero scalar α. Thus, $\frac{C}{\alpha}$ acts as the identity operator on V'. Since $dim(V/V') = 1$, V/V' is a trivial $\mathcal{U}(\mathfrak{sl}(2))$-module, thus C/α is a projection of V onto V'. Furthermore, it follows that $\frac{C}{\alpha}$ is an $\mathcal{U}(\mathfrak{sl}(2))$-module morphism and $V = V' \oplus ker(\frac{C}{\alpha})$.
Finally, let $V' \subset V$ be $\mathcal{U}(\mathfrak{sl}(2))$-modules and define $W' \subset W$, vector spaces, with $W = \{f \colon V \to V' \,|\, f(v') = \alpha v',\ v' \in V',\ \alpha \in \mathbf{k}\}$ and $f \in W'$ if $f(V') = \{0\}$.
Then $W, W' \subset Hom(V, V')$, so for $f \in W$, $f(v') = \alpha v'\ \forall v' \in V'$, we have

$$(Xf)(v') = Xf(v') - f(Xv') = X(\alpha v') - \alpha(Xv') = 0.$$

Similarly, if $f \in W'$

$$(Xf)(v') = Xf(v') - f(Xv') = 0 - 0 = 0,$$

so both W and W' are $\mathcal{U}(\mathfrak{sl}(2))$-modules with the module structure induced by $Hom(V, V')$.
Since $W' \subset W$ is of codimension 1, by the previous part, we have $W = W' \oplus W''$, with W'' a $\mathcal{U}(\mathfrak{sl}(2))$-submodule of dimension 1. Then, W'' is generated by some $f \colon V \to V'$ such that $f(v') = \alpha v' \neq 0$ for $v' \in V$, hence $\frac{f}{\alpha}$ is a projection from V onto V'.
Lastly, since $dim(W'') = 1$, it is a trivial $\mathcal{U}(\mathfrak{sl}(2))$-modules, thus $Xf = 0$ for $X \in \mathfrak{sl}(2)$ and $f \in W''$. Hence, $Xf(v) - f(Xv) = 0\ \forall v \in V$ and it follows that $\frac{f}{\alpha}$ is a $\mathcal{U}(\mathfrak{sl}(2))$-module morphism. Thus $V = V' \oplus ker(\frac{f}{\alpha})$.
So, it follows that there always exists a $\mathcal{U}(\mathfrak{sl}(2))$-module V'', such that $V = V' \oplus V''$ when $V' \subset V$ are $\mathcal{U}(\mathfrak{sl}(2))$-modules. Hence, all finite-dimensional $\mathcal{U}(\mathfrak{sl}(2))$-modules are semisimple. $\square$

From this theorem, it follows that $V(n) \otimes V(m)$ is semisimple, since it is a $\mathcal{U}(\mathfrak{sl}(2))$-module as $\mathcal{U}(\mathfrak{sl}(2))$ is a Hopf-algebra. So it can be decomposed into a direct sum of simple modules.

Proposition 3.46. Let $n \geq m \in \mathbb{N}\backslash\{0\}$, then

$$V(n) \otimes V(m) \simeq \bigoplus_{k=0}^{m} V(n + m - 2k)$$

Proof. One can construct an embedding $\pi \colon V(n+m-2k) \to V(n) \otimes V(m)$, that is also $\mathcal{U}(\mathfrak{sl}(2))$-linear, by showing that $V(n) \otimes V(m)$ has a highest weight vector of weight $n + m - 2k$ for $0 \leq k \leq m$. π will be an embedding due to the fact that $V(n + m - 2k)$ is simple and $ker(\pi)$ is a submodule of $V(n+m-2k)$. The next lemma shows how to construct highest weight vectors of weight $n+m-2k$, so we will assume that we can embed any $V(n + m - 2k)$ into $V(n) \otimes V(m)$.
Now, since all $V(n + m - 2k)$ are simple and of distinct weight, their sum in $V(n) \otimes V(m)$ is direct. And, since $dim(\sum_{k=0}^{m} V(n+m-2k) = \sum_{k=0}^{m} n+m-2k+1 = (n+1)(m+1) = dim(V(n))dim(V(m))$, it follows that $V(n) \otimes V(m) \simeq \bigoplus_{k=0}^{m} V(n + m - 2k)$. $\square$

Lemma 3.47. Let $n \geq m \in \mathbb{N}\backslash\{0\}$ and let $v^{(n)} \in V(n)$, $v^{(m)} \in V(m)$ be highest weight vectors. Set $v_k^{(n)} = \frac{1}{k!}Y^k v^{(n)}$ and $v_k^{(m)} = \frac{1}{k!}Y^k w^{(m)}$ for $k \geq 0$. Then

$$v^{(n+m-2k)} = \sum_{i=0}^{k}(-1)^i \frac{(m-k+i)!(n-i)!}{(m-k)!n!} v_i^{(n)} \otimes v_{k-i}^{(m)}$$

is a highest weight vector of $V(n) \otimes V(m)$ of weight $n+m-2k$.

So we get two different bases for $V(n) \otimes V(m)$, namely $\{v_i^{(n)} \otimes w_j^{(m)}\}_{0 \leq i \leq n,\, 0 \leq j \leq m}$ and a basis given by all vectors $v_j^{(n+m-2k)} = \frac{1}{j!}Y^j v^{(n+m-2k)}$, for $0 \leq k \leq m$ and $0 \leq j \leq n+m-2k$. This last basis is easier to work with as a $\mathcal{U}(\mathfrak{sl}(2))$-module, as we can just use the known actions of X, Y and H on the Verma module $V(n+m-2k)$.
We can compare both of these basis and even go from one to the other. In other words, a relationship between a $v_j^{(n+m-2k)}$ and all $v_i^{(n)} \otimes w_j^{(m)}$ can be constructed. The coefficients describing this relation are called the *Clebsch-Gordan coefficients*. They are given by

$$v_p^{(n+m-2k)} = \sum_{0 \leq i \leq n,\, 0 \leq j \leq m} C_{i,j,p}^{n,m,n+m-2k} v_i^{(n)} \otimes v_j^{(m)}.$$

Now, one can show that the terms $v_p^{(n+m-2k)}$ are linear combinations of the vectors $v_i^{(n)} \otimes v_{k-i+p}^{(m)}$ by induction on p. Since,

$$\begin{aligned} v_1^{(n+m-2k)} &= Fv_0^{(n+m-2k)} = F\left(\sum_i \alpha_i v_i^{(n)} \otimes v_{k-i}\right) = \sum_i \alpha_i((i+1)v_{i+1} \otimes v_{k-i} + v_i \otimes (k-i+1)v_{k-i+1}) \\ &= \sum_i \alpha_i(i \cdot v_i \otimes v_{k-i+1} + (k-i+1)v_i \otimes v_{k-i+1}). \end{aligned}$$

And

$$\begin{aligned} v_{p+1}^{(n+m-2k)} &= Fv_p^{(n+m-2k)} = F\left(\sum_i \alpha_i v_i^{(n)} \otimes v_{k-i+p}\right) \\ &= \sum_i \alpha_i((i+1)v_{i+1} \otimes v_{k-i+p} + v_i \otimes (k-i+p+1)v_{k-i+p+1}) \\ &= \sum_i \alpha_i(k+p+1)v_i \otimes v_{k-i+p+1} \end{aligned}$$

From this, one can deduce that $C_{i,j,p}^{n,m,n+m-2k} = 0$ if $i+j \neq p+k$. Furthermore, recall that $A \in Z(\mathcal{U}(\mathfrak{sl}(2)))$ acts as a scalar on simple non-zero finite-dimensional modules. So, $Av_p^{n+m-2k} = \lambda v_p^{(n+m2k)}$. And since C generates the center of $\mathcal{U}(\mathfrak{sl}(2))$, we have, in particular, $Cv_p^{n+m-2k} = \lambda_C v_p^{(n+m2k)}$. This, together with the fact that $v_p^{(n+m-2k)} = \sum_i \alpha_i v_i^{(n)} \otimes v_{k-i+p}^{(m)}$, shows that $\sum_i \alpha_i v_i^{(n)} \otimes v_{k-i+p}^{(m)}$ has to be an eigenfunction of C for any $0 \leq p \leq n+m-2k$ and $0 \leq k \leq m$, given that $n \geq m$. These eigenfunctions can be described by the so called Hahn polynomials [11].
The fact that these are eigenfunctions for the Clebsch-Gordan coefficients is something that we will also see in the next case, the quantum case, and also in the part about the modular double. Here we just ended with these eigenfunctions as a last fun fact. But, we will use the eigenfunctions in a later part to fully compute the Clebsch-Gordan coefficients.

Since this is all fairly similar to that of the quantum case, we will stop here with looking at these coefficients. In the quantum case we will dive a bit deeper into some of the properties of these

coefficients, but most of that could be done similarly for this case. The biggest reason to focus more on the quantum case is the fact that our main goal is to look at certain modules of $\mathcal{U}_q(\mathfrak{sl}(2))$, which itself is a quantum deformation of $\mathcal{U}(\mathfrak{sl}(2))$.
This will also be where we stop looking at the classical case, and we will finally move on the the quantum analogues of most of the things that we have discussed up till now.

4 The quantum plane and quantum algebras

Now that we know what the universal enveloping Lie algebra $\mathcal{U}(\mathfrak{sl}(2))$ is, we can build towards explaining what the quantum enveloping algebra $\mathcal{U}_q(\mathfrak{sl}(2))$ is. We will do this by first introducing the quantum plane, which is a quantum deformation of the affine plane. In short, the quantum plane is nothing more than the affine plane, $\mathbf{k}[x, y]$, with a small deformation. This deformation will change the commutation property to be $yx = qxy$ instead.
After we studied the quantum plane, just like how we first looked at $SL(2)$ before we studied $\mathcal{U}(\mathfrak{sl}(2))$, we will first introduce $SL_q(2)$, the quantum analogue of $SL(2)$. This is, similar to the quantum plane, a one-parameter deformation of $SL(2)$. $SL_q(2)$ will be one of our first examples of quantum groups, which is why it's referred to as the quantum analogue of $SL(2)$.
We will end this chapter with a new type of Hopf algebra structure, the Hopf *-algebra. A Hopf $*$-algebra will be shown to be nothing more than a Hopf algebra with an extra involution $*$ added to the structure. Which is similar to the difference between bialgebras and Hopf algebras.
In the end we will show that the quantum algebras $SL_q(2)$ and $GL_q(2)$ are Hopf $*$-algebras and even show that the Hopf algebra $\mathcal{U}(\mathfrak{sl}(2))$ can also be turned into a Hopf $*$-algebra. Now, the $*$-structure of the Hopf $*$-algebra will not be used directly in later parts. But, it will still play an important part for when we construct a module for the Hopf $*$-algebra $\mathcal{U}_q(\mathfrak{sl}(2))$.

4.1 The quantum plane and the q-binomial formula

Let x, y be two non-zero elements, then the regular affine plane is the abelian algebra generated by x, y such that $yx = xy$. In other words, the affine plane is the algebra $\mathbf{k}[x, y]$. Now, the quantum plane is obtained by modifying the commutator relation $yx = xy$ to $yx = qxy$.

Definition 4.1. Let $q \in \mathbf{k}$ be an invertible element. Let $I_q \subset \mathbf{k}\{x, y\}$ be the ideal generated by the elements $yx - qxy$. Then the *quantum plane* is the quotient-algebra $\mathbf{k}_q[x, y] = \mathbf{k}\{x, y\}/I_q$.

So, one can just look at the quantum plane as the algebra generated by two non-zero elements x, y such that $yx = qxy$. Now, the quantum plane $\mathbf{k}_q[x, y]$ is abelian if and only if $q = 1$. Recall that if $\mathcal{I} \subseteq \mathbf{k}\{x, y\}$ is the ideal generated by elements of the form $xy - yx$, then $\mathbf{k}\{x, y\}/\mathcal{I} \cong \mathbf{k}[x, y]$. Thus, it follows that $\mathbf{k}_1[x, y] = \mathbf{k}[x, y]$, which is what one would expect. As it shows that the quantum deformation is just the classical case when the commutation relation is not deformed.

Proposition 4.2. Fix $q \in \mathbf{k}$ to be invertible and let $i, j > 0$. Then, $y^j x^i = q^{ij} x^i y^j$ for $x, y \in \mathbf{k}_q[x, y]$.

Eventually we want to be able to calculate powers of $x + y$ in the quantum plane, or similar, powers of linear combinations of elements of $\mathcal{U}_q(\mathfrak{sl}(2))$. To make this easier, we will introduce the Gauss polynomials.
For this part, fix $q \in \mathbf{k}$ to be an invertible element. Now, as a first remark, all of the below formulas will be equal to their classical formulas if $q = 1$.
First, set $(n)_q = 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1}$, for any $n \in \mathbb{N}_{>0}$. Then we can define the quantum analogue of the factorial, the *q-factorial*, by: $(0)!_q = 1$,

$$(n)!_q = (1)_q (2)_q \cdots (n)_q = \frac{(q-1)(q^2-1)\cdots(q^n-1)}{(q-1)^n}, \quad n \in \mathbb{N}_{>0}.$$

This is a polynomial in q with integral coefficients. Now, we can define the *Gauss polynomials*, the quantum analogue of the binomial coefficients, by

$$\binom{n}{k}_q = \frac{(n)!_q}{(k!)_q (n-k)!_q}, \quad n, k \in \mathbb{N}, \; k \leq n.$$

Similar to the q-factorial, this is also a polynomial in q with integral coefficients.

Proposition 4.3. Let $n, k \in \mathbb{N}_{>0}$ such that $k \leq n$. Then,

i) $\binom{n}{k}_q = \binom{n}{n-k}_q$,

ii) (q-Pascal identity)

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q = \binom{n-1}{k}_q + q^{n-k} \binom{n-1}{k-1}_q$$

The proof of this proposition is similar to that of the classical case. Now, this then leads to the following formula for $(x+y)^n$.

Proposition 4.4. Let x, y be variables such that $yx = qxy$, then for all $n \in \mathbb{N}_{>0}$ we have

$$(x+y)^n = \sum_{0 \leq k \leq n} \binom{n}{k}_q x^k y^{n-k}.$$

Another property of the binomial coefficients is the Chu-Vandermonde formula, which has the following quantum analogue.

Proposition 4.5. For $n, m, p \in \mathbb{N}$, $m \geq p \leq n$,

$$\binom{m+n}{p}_q = \sum_{0 \leq k \leq p} q^{(m-k)(p-k)} \binom{m}{k}_q \binom{n}{p-k}_q.$$

4.2 The algebras $M_q(2)$, $GL_q(2)$ and $SL_q(2)$

In this part we will be looking at the quantum deformation of the Hopf algebras $M(2)$, $SL(2)$ and $GL(2)$. We will then show that, similar to the classical case, that they are all bialgebras and that $SL_q(2)$ and $GL_q(2)$ are Hopf algebras in particular.
Just like in the classical case, we will be introducing the algebra $SL_2(q)$ as a quotient algebra of $M_q(2)$. For this part, fix a $q \in \mathbf{k}$ to be non-zero and such that $q^2 \neq -1$.

Theorem 4.6. *Let x, y be variables such that $yx = qxy$ and let a, b, c, d be four variables that commute with x and y. Let x', y', x'', y'' be four variables such that*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then we have an equivalence relation between

i) $y'x' = qx'y'$ *and* $y''x'' = qx''y''$,

ii) $ba = qab$, $db = qbd$, $ca = qac$, $dc = qcd$, $bc = cd$, $ad - da = (q^{-1} - q)bc$

This theorem follows from a simple computation, but it does show us how to define the quantum analogue of $M(2)$.

Definition 4.7. The algebra $M_q(2) = \mathbf{k}\{a, b, c, d\}/I_q$, with I_q the ideal generated by the six relations given in theorem 4.6 *ii*).

Just like other q-analogue things, $M_q(2)$ is also isomorphic, as Hopf algebras, to its classical form $M(2)$ when $q = 1$. For now we only know that $M_q(2)$ is an algebra, but we will soon see that it is indeed a Hopf algebra.

Since we associated the classical case with matrix algebras and also used the determinant to define $GL(2)$ and $SL(2)$, it is useful to introduce a quantum analogue of the determinant.

Proposition 4.8. The element $det_q = ad - q^{-1}bc = da - qbc \in M_q(2)$ is the *quantum determinant* and $det_q \in Z(M_q(2))$.

It is an easy check to see that det_q commutes with a, b, c and d.
We will see that the basis of $M_q(2)$ is exactly the same as that of $M(2) = \mathbf{k}[a, b, c, d]$. But since the proof uses Ore extensions, and we do not want to introduce those, the following theorem will be stated as a fact. The proof can be read in full details in [4].

Theorem 4.9. *The algebra $M_q(2)$ has no non-trivial zero-divisors and has a basis given by $\{a^i b^j c^k d^l\}_{i,j,k,l \geq 0}$.*

Thus we see that $M_q(2)$ has some of the same properties as $M(2)$.
Another similarity between $M_q(2)$ and $M(2)$ is that they are both bialgebras.

Theorem 4.10. *Let $\Delta\colon M_q(2) \to M_q(2) \otimes M_q(2)$ be the algebra morphism defined by:*

$$\Delta(a) = a \otimes a + b \otimes c,\ \Delta(b) = a \otimes b + b \otimes d,\ \Delta(c) = c \otimes a + d \otimes c,\ \Delta(d) = c \otimes b + d \otimes d,$$

and $\varepsilon\colon M_q(2) \to \mathbf{k}$ the algebra morphism defined by:

$$\varepsilon(a) = \varepsilon(d) = 1, \qquad \varepsilon(b) = \varepsilon(c) = 0.$$

The above algebra morphisms induce a bialgebra structure on $M_q(2)$. Furthermore, $\Delta(det_q) = det_q \otimes det_q$ and $\varepsilon(det_q) = 1$.

Similar to the classical case, we can rewrite Δ and ε into matrix forms:

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which are actually the same matrix forms as that of the classical case.
Showing that Δ and ε are indeed algebra morphisms is another simple computation. As it suffices to show that they are well-defined algebra morphisms for the generators a, b, c and d. Similarly, showing that Δ and ε are coassociative and counital respectively, is also done via an easy computation. Then, remember that $M_q(2)$ being both an algebra and coalgebra and Δ, ε being algebra morphisms, implies that it is indeed a bialgebra.

Now that we know what $M_q(2)$ is, we can follow the same steps as the classical case to introduce $GL_q(2)$ and $SL_q(2)$. They will be defined in the exact same way, but this time we will use the quantum determinant det_q. So, define $GL_q(2) = M_q(2)[t]/(t \cdot det_q - 1)$ and $SL_q(2) = M_q(2)/(det_q - 1) = GL_q(2)/(t - 1)$.
Also, identical to the classical case, we can turn $GL_q(2)$ and $SL_q(2)$ into Hopf algebras by using the same coproduct and counit as $M_q(2)$ and, again, set $\Delta(t) = t \otimes t$ and $\varepsilon(t) = 1$ for $t \in GL_q(2)$. This then leads to the following theorem.

Theorem 4.11. *The coproduct Δ and counit ε of $M_q(2)$ are well-defined on $GL_q(2)$ and $SL_q(2)$ with the extension $\Delta(t) = t \otimes t$ and $\varepsilon(t) = 1$ for $t \in GL_q(2)$. Furthermore, they induce a bialgebra structure on $GL_q(2)$ and $SL_q(2)$.*
In addition, if $S(a) = \frac{d}{det_q}$, $S(b) = \frac{-qb}{det_q}$, $S(c) = \frac{-c}{q det_q}$, $S(d) = \frac{a}{det_q}$ and $S(t) = t$. Then S is an antipode for $GL_q(2)$ and $SL_q(2)$ and induces a Hopf algebra structure on $GL_q(2)$ and $SL_q(2)$.

Since this is also a proof that boils down to simple computations, as one just needs to check the denoted conditions, we will once again skip writing it down. Do note that S can be rewritten in matrix form to

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = det_q^{-1}\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix},$$

which can be helpful to show that it is indeed an antipode.

As a final remark on these quantum groups, we will look at some details of the antipode S of $GL_q(2)$ and $SL_q(2)$. First, note that we get the classical algebras $GL(2)$ and $SL(2)$ if we take $q = 1$. But, contrary to the classical case, the antipode S defined above, need not be an involution.
Recall that the antipode S' of the classical case is defined as $S'\begin{pmatrix} a & b \\ c & d \end{pmatrix} = det^{-1}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, in other words, $S'(A) = A^{-1}$ for $A \in GL(2)$ or $SL(2)$. So, $S'^2 = id$.
Now, in the quantum case we get

$$S^{2n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & q^{2n}b \\ q^{-2n}c & d \end{pmatrix} = \begin{pmatrix} q^n & 0 \\ 0 & q^{-n} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} q^{-n} & 0 \\ 0 & q^n \end{pmatrix}.$$

So, if $q \in \mathbb{C}$ is a root of unity of order n, i.e. $q^n = 1$, then we see that $S^{2n} = id$, but $S^2 \neq id$. In other words, S^2 is of order n instead of order 1. We will see another example soon where q being a root of unity will give different results. So we see that one needs to be careful of the allowed choices for $q \in \mathbb{C}$, as the choice does impact some properties of the Hopf algebras $SL_q(2)$, $GL_q(2)$ and later on we will see that the choice of q also impacts properties of $\mathcal{U}_q(\mathfrak{sl}(2))$.

4.3 Hopf ∗-algebras

As was stated before, we can add another structure on $GL_q(2)$ and $SL_q(2)$. This will be the similar to the difference of a bialgebra and a Hopf algebra, where only one function. the antipode S, was added to the structure. In this case, to go from a Hopf algebra to a Hopf ∗-algebra, an antilinear involution ∗ is added to the structure.

Definition 4.12. Let $(\mathcal{H}, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra over $\mathbb{C}$. Then, $\mathcal{H}$ is a *Hopf ∗-algebra* if $\exists * \colon \mathcal{H} \to \mathcal{H}$ such that ∗ is a antilinear involution and

i) $* \colon \mathcal{H} \to \mathcal{H}^{op}$ is a real algebra morphism, i.e. ∗ is an antimorphism of real algebras,

ii) $* \colon \mathcal{H} \to \mathcal{H}$ is a real coalgebra morphism,

iii) $\forall x \in \mathcal{H}\,(\; S(S(x)^*)^* = x)$.

Two Hopf ∗-algebra structures $*_1, *_2$ are equivalent on $\mathcal{H}$ if $\exists \phi \colon \mathcal{H} \to \mathcal{H}$, Hopf algebra automorphism, such that $\phi(x^{*_1}) = \phi(x)^{*_2}\ \forall x \in \mathcal{H}$.

The following lemma will help with finding ∗-structures on Hopf algebras. Since it changes the requirement to finding an algebra morphism γ that is also an antimorphism of coalgebras. And for our cases, it is a lot easier to find algebra automorphisms instead of coalgebra automorphisms.

Lemma 4.13. Let $(\mathcal{H}, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra over $\mathbb{C}$. $\mathcal{H}$ has a Hopf ∗-structure if and only if $\exists \gamma \colon \mathcal{H} \to \mathcal{H}$, antilinear automorphism, such that

i) $\gamma \colon \mathcal{H} \to \mathcal{H}$ is a real algebra morphism,

ii) $\gamma \colon \mathcal{H} \to \mathcal{H}^{cop}$ is a real coalgebra morphism, i.e. γ is an antimorphism of real coalgebras,

iii) $\forall x \in \mathcal{H}$ we have $\gamma^2(x) = (S\gamma)^2(x) = x$.

Proof. Let $*\colon \mathcal{H} \to \mathcal{H}$ be an involution as in definition 4.12 and set $\gamma = S^{-1} \circ *$. Then γ is an antilinear algebra automorphism, as both $*$ and S^{-1} are antimorphisms. It is a coalgebra antimorphism, since $*$ is a coalgebra morphism and S^{-1} is a coalgebra antimorphism. As $S\gamma = *$, it follows that $S\gamma$ is an involution and

$$\gamma^2 = (S^{-1} \circ *)^2 = (* \circ S)^2)^{-1} = id_{\mathcal{H}}^{-1} = id_{\mathcal{H}},$$

where the second equality follows from $*^2 = id_{\mathcal{H}}$. Thus if $\mathcal{H}$ is a Hopf $*$-algebra, we indeed have a $\gamma = S^{-1} \circ *$ that satisfies the desired properties.
Now, let $\gamma\colon \mathcal{H} \to \mathcal{H}$ satisfy the above properties and define $* = S \circ \gamma$. Then we claim that $*$ satisfies the properties given in definition 4.12.
It follows directly that $*^2 = id_{\mathcal{H}}$, $*$ is an algebra antimorphism and a coalgebra morphism by similar reasons as noted before, and

$$(* \circ S)^2 = (S \circ \gamma \circ S)^2 = (S \circ \gamma)^2 \circ \gamma^{-1} \circ (S \circ \gamma)^2 \circ \gamma^{-1} = \gamma^{-2} = id_{\mathcal{H}}.$$

Thus $* = S \circ \gamma$ does indeed induce a Hopf $*$-algebra structure on $\mathcal{H}$. □

We will end this section by showing that three Hopf algebras that we saw before, can be turned into Hopf $*$-algebras.

Theorem 4.14. $\exists!$ *Hopf $*$-structure on the Hopf algebras $GL_q(2)$ and $SL_q(2)$ such that*

$$a^* = td, \quad b^* = -qtc, \quad c^* = -q^{-1}tb, \quad d^* = ta, \quad t^* = t^{-1}.$$

Proof. Let $\gamma\colon SL_q(2) \to SL_q(2)$ be antilinear and defined by $\gamma = T$, where T denotes the transpose map. So, $T(a) = a$, $T(b) = c$, $T(d) = d$ and $T^2 = id$. Then, γ is an involution and an algebra morphism. Also, note that it is a coalgebra antimorphism, since the transposition of matrices reverses the product.
Now, γ can be extended to $GL_q(2)$ by taking $\gamma(t) = t$. Since $det_q t = 1$ in $GL_q(2)$ and $\gamma(det_q t - 1) = det_q t - 1$, it is indeed also an algebra morphism on $GL_q(2)$.
Lastly, note that $(S\gamma)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = t\begin{pmatrix} d & -qc \\ -q^{-1}b & a \end{pmatrix}$, so

$$(S\gamma)^2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = S\gamma \begin{pmatrix} d & -qc \\ -q^{-1}b & a \end{pmatrix} S\gamma(t) = t \begin{pmatrix} a & b \\ c & d \end{pmatrix} t^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $(S\gamma)(t) = t$, so $S\gamma$ is indeed an involution. Thus, $* = S\gamma$ does induce a $*$-structure on both $GL_q(2)$ and $SL_q(2)$. □

The third and final Hopf algebra that we will discuss here is $\mathcal{U}(\mathfrak{sl}(2))$. Recall that we know that $\mathcal{U}(\mathfrak{sl}(2))$ is a cocommutative Hopf algebra by proposition 3.30. Thus, by lemma 4.13, it follows that we can define a $*$-structure on $\mathcal{U}(\mathfrak{sl}(2))$ if we can construct an antilinear automorphism on $\mathfrak{sl}(2)$, as the generators of $\mathfrak{sl}(2)$ also generate $\mathcal{U}(\mathfrak{sl}(2))$.
Note that we can ignore that γ needs to be an antimorphism of coalgebras. Since $\mathcal{U}(\mathfrak{sl}(2))$ is cocommutative, any coalgebra antimorphism is just a coalgebra morphism.

Proposition 4.15. The complex Hopf algebra $\mathcal{U}(\mathfrak{sl}(2))$ has 3 unique $*$-structures up to equivalence. These $*$-structures are given by:

1: $X^* = -X$, $Y^* = -Y$, $H^* = -H$,

2: $X^* = X$, $Y^* = Y$, $H^* = -H$,

3: $X^* = -Y$, $Y^* = -X$, $H^* = H$.

Note, the above fact can easily be shown with the use of the fact that any Lie algebra automorphism of $\mathfrak{sl}(2)$ is of the form $f(A) = uAu^{-1}$, with $A \in \mathfrak{sl}(2)$, $u \in SL(2, \mathbb{C})$. Since this fact relies on some more knowledge on Lie algebras, we will simply assume this as a fact.

Now, let $f\colon \mathfrak{sl}(2) \to \mathfrak{sl}(2)$ be an automorphism such that $f^2 = id$. It then follows that $f(A)$ is of one of the following forms for $A \in \mathfrak{sl}(2)$:

1: $f(A) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} A \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$,

2: $f(A) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{pmatrix} A \begin{pmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{pmatrix}$,

3: $f(A) = \begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix} A \begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix}$, with $b \neq 0$, $a \in \mathbb{R}$.

Now, we can construct the γ of lemma 4.13 by calculating $f(X)$, $f(Y)$ and $f(H)$ and letting $\gamma(U) = f(U)$ for $U \in \{X, Y, H\}$, then extend γ to be an antilinear automorphism. We then get the above $*$-structures by using $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for the matrix u that defines $f(A) = uAu^{-1}$. Lastly, one can use the fact that two $*$-structures will only be equivalent in this case when $\exists u' \in SL(2, \mathbb{C})$ such that $u'u = wu'$, which holds if and only if $u'uu'^{-1} = w$, where $u, w \in SL(2, \mathbb{R})$ are the two matrices that generate the two different $*$-structures. From this it will follow that the above three forms of $f(A)$ will indeed generate three non-equivalent $*$-structures.

5 The Hopf $*$-algebra $\mathcal{U}_q(\mathfrak{sl}(2))$

We will finally introduce the algebra that will be the main subject of this thesis, the algebra $\mathcal{U}_q(\mathfrak{sl}(2))$. Similar to the quantum analogues of $M(2), GL(2)$ and $SL(2)$, $\mathcal{U}_q(\mathfrak{sl}(2))$ is the quantum analogue of the Hopf $*$-algebra $\mathcal{U}(\mathfrak{sl}(2))$. In this chapter we will be introducing $\mathcal{U}_q(\mathfrak{sl}(2))$ and looking at some properties of it.
We will start with defining $\mathcal{U}_q(\mathfrak{sl}(2))$ for when $q \neq 1$. This will give a simpler definition, but we will obviously lose the property of $\mathcal{U}_q(\mathfrak{sl}(2))$ being isomorphic to the classical case $\mathcal{U}(\mathfrak{sl}(2))$ when $q = 1$. Because we still want $\mathcal{U}_q(\mathfrak{sl}(2))$ to be isomorphic to $\mathcal{U}(\mathfrak{sl}(2))$ when $q = 1$, a second definition will be given, for which $q = 1$ is a possible value.
After we know what $\mathcal{U}_q(\mathfrak{sl}(2))$ is, we will see that certain properties of $\mathcal{U}_q(\mathfrak{sl}(2))$ will be similar to that of $\mathcal{U}(\mathfrak{sl}(2))$, given that q is not a root of unity. To further illustrate the fact that one needs to keep in mind which values for q we allow, we will take a detailed look at modules of $\mathcal{U}_q(\mathfrak{sl}(2))$ for when $q^n = 1$ for some $n \in \mathbb{N}_{>0}$.
We end this section by showing that $\mathcal{U}_q(\mathfrak{sl}(2))$ is also a Hopf $*$-algebra. This will be done by showing all possible $*$-structures up to isomorphisms.

5.1 $\mathcal{U}_q(\mathfrak{sl}(2))$ as a Hopf algebra

Before we introduce the Hopf algebra $\mathcal{U}_q(\mathfrak{sl}(2))$, we will first look at some notation. This will be related to the q-binomial coefficients that we have seen in section 4.1.
First, fix $q \in \mathbb{C}$ such that $q \notin \{-1, 0, 1\}$, then we know that $\frac{1}{q-q^{-1}}$ is well-defined and $q^2 \neq 1$. Let $n \in \mathbb{N}$ and define $[n]_q = \frac{q^n - q^{-n}}{q-q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+1}$. Unlike $(n)_q = \frac{q^n-1}{q-1}$, $[n]_q$ is symmetric, as $[-n]_q = -[n]_q$ and $[m+n]_q = q^n[m]_q + q^{-m}[n]_q$.
Note, if $q^d = 1$, then we see that $[d]_q = 0$. This is not the only case that $[n]_q = 0$. Let $d \in \mathbb{N}_{>2}$ such that $q^d = 1$ and let

$$e = \begin{cases} d & \text{if } d \text{ is odd,} \\ d/2 & \text{if } d \text{ is even.} \end{cases}$$

Note that the restriction of d is due to our choice of q. Now, set $e = \infty$ if q is not a root of unity. Then,

$$[n]_q = 0 \iff n \equiv 0 \text{mod } e.$$

We can also take factorials of $[n]$ and use it to define another type of binomial coefficient. Let $k \in \mathbb{N}$ such that $k \leq n$ and set $[0]_q! = 1$, then $[k]_q! = [1]_q[2]_q \cdots [k]_q$ and, if $k > 0$, $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$. Now, the remark on $(n)_q$ was not fully out of place. Since we have

$$[n]_q = q^{-(n-1)}(n)_{q^2}, \quad [n]_q! = q^{-(n-1)/2}(n)!_{q^2}, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = q^{-k(n-k)} \begin{pmatrix} n \\ k \end{pmatrix}_{q^2},$$

so $[n]_q$ is fully related the the q-analogues of the factorial and binomial function. This also gives us a new formula for $(x+y)^n$, given that x, y are variables such that $yx = q^2xy$:

$$(x+y)^n = \sum_{k=0}^{n} q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}.$$

We will not be using this formula ourselves, but it is rather useful for calculating elements in $\mathcal{U}_q(\mathfrak{sl}(2))$. As we will see that there are elements in $\mathcal{U}_q(\mathfrak{sl}(2))$ that do satisfy the relation $yx = q^2xy$, hence the q^2 instead of just q like in section 4.1.
Now that we have seen some useful formulas to do calculations in $\mathcal{U}_q(\mathfrak{sl}(2))$, we will finally define the Hopf algebra $\mathcal{U}_q(\mathfrak{sl}(2))$ for $q \in \mathbb{C}\backslash\{-1, 0, 1\}$.

Definition 5.1. $\mathcal{U}_q(\mathfrak{sl}(2))$ is the complex algebra generated by the four elements E, F, K, K^{-1} such that

$$KK^{-1} = K^{-1}K = 1, \qquad KE = q^2EK, \qquad KF = q^{-2}FK, \qquad [\,E\,,\,F\,] = \frac{K - K^{-1}}{q - q^{-1}}.$$

Note that the bracket $[\,\cdot\,,\,\cdot\,]$ is the commutator bracket. Also, since $\mathcal{U}_1(\mathfrak{sl}(2))$ is undefined, we do not have the property that the quantum analogue case with $q = 1$ is equal or isomorphic to the classical case $\mathcal{U}(\mathfrak{sl}(2))$. In a bit we will see another definition of $\mathcal{U}_q(\mathfrak{sl}(2))$ that will allow every $q \neq 0$.
Obviously, we could have just used that as our definition immediately. But, to study the Clebsch-Gordan coefficients, we will only look at the case that $|q| = 1$ and q is not a root of unity. The definition of $\mathcal{U}_q(\mathfrak{sl}(2))$ that allows $q = 1$ is also a bit more complicated. So, since we are not that interested in the case $q = 1$, it is easier to use definition 5.1. However, we will still give the broader definition and look at the case when $q^d = 1$ for some $d \in \mathbb{N}$ for completeness sake.

We will now give some properties of $\mathcal{U}_q(\mathfrak{sl}(2))$.

Lemma 5.2. Let $\omega\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_q(\mathfrak{sl}(2))$ be given by

$$\omega(E) = F, \qquad \omega(F) = E, \qquad \omega(K) = K^{-1}.$$

Then ω extends uniquely to an algebra automorphism on $\mathcal{U}_q(\mathfrak{sl}(2))$.
Also, ω is often referred to as the *Cartan automorphism.*

Proof. To show that ω is indeed an algebra automorphism, we only need to check that it satisfies the 4 relations in definition 5.1. Now,

$$\begin{aligned}\omega(KE) &= \omega(K)\omega(E) = K^{-1}F = q^2FK^{-1} = q^2\omega(EK),\\ \omega([\,E\,,\,F\,]) &= FE - EF = -[\,E\,,\,F\,] = \frac{K^{-1} - K}{q - q^{-1}} = \omega\frac{K - K^{-1}}{q - q^{-1}}.\end{aligned}$$

The other relations go similar. Thus ω is indeed an algebra automorphism, as the image contains all 4 generators.
The uniqueness of ω follows due to how it is constructed. $\square$

Lemma 5.3. Let $m \in \mathbb{N}$ and $n \in \mathbb{Z}$, then

$$\begin{aligned}E^mK^n &= q^{-2mn}K^nE^m, \qquad F^mK^n = q^{2mn}K^nF^m,\\ [\,E\,,\,F^m\,] &= [m]F^{m-1}\frac{q^{-(m-1)}K - q^{m-1}K^{-1}}{q - q^{-1}} = [m]\frac{q^{m-1}K - q^{-(m-1)}K^{-1}}{q - q^{-1}}F^{m-1},\\ [\,E^m\,,\,F\,] &= [m]\frac{q^{-(m-1)}K - q^{m-1}K^{-1}}{q - q^{-1}}E^{m-1} = [m]E^{m-1}\frac{q^{m-1}K - q^{-(m-1)}K^{-1}}{q - q^{-1}}.\end{aligned}$$

Proof. The first two equations follow directly from $EK = q^{-2}KE$ and $FK = q^2KF$.
The third equation is proven by induction on m together with

$$[\,E\,,\,F^m\,] = [\,E\,,\,F^{m-1}\,]F + F^{m-1}[\,E\,,\,F\,] = [\,E\,,\,F^{m-1}\,]F + F^{m-1}\frac{K - K^{-1}}{q - q^{-1}}.$$

Since it is true trivially for $m = 1$, assume it holds for all $n < m$. Then,

$$[\,E\,,\,F^m\,] = [\,E\,,\,F^{m-1}\,]F + F^{m-1}\frac{K - K^{-1}}{q - q^{-1}} = [m-1]\frac{q^{m-2}K - q^{-(m-2)}K^{-1}}{q - q^{-1}}F^m + F^{m-1}\frac{K - K^{-1}}{q - q^{-1}}.$$

One concludes this equation by pulling F^{m-1} to the right using the second equation. Lastly, the final equation follows directly from applying the Cartan automorphism ω on the term $[\,E\,,\,F^m\,]$. $\square$

The next proposition will show that the basis of $\mathcal{U}_q(\mathfrak{sl}(2))$ is also nice, just like that of $\mathcal{U}(\mathfrak{sl}(2))$. But, since the proof once again uses Ore extensions, we will just state this as a fact. Do note that this is a really useful proposition, as it shows that we will only ever need to proof things for the terms $E^iF^kK^l$, given that what we need to show depends linearly on $E^iF^kK^l$.

Proposition 5.4. The set $\{E^iF^kK^l\}_{i,j\in\mathbb{N},l\in\mathbb{Z}}$ is a basis for $\mathcal{U}_q(\mathfrak{sl}(2))$. Furthermore, $\mathcal{U}_q(\mathfrak{sl}(2))$ has no zero-divisors.

Remember, for the quantum analogues, we normally had the nice, and intuitive, property that if $q = 1$, then we end up back in the classical case. For example, $SL_1(2) \simeq SL(2)$ and $(n)_1! = n!$. But, due to how $\mathcal{U}_q(\mathfrak{sl}(2))$ is constructed, we cannot define $\mathcal{U}_1(\mathfrak{sl}(2))$, as $\frac{1}{q-q^{-1}}$ is not defined in this case. Since we still want to relate $\mathcal{U}_q(\mathfrak{sl}(2))$ to $\mathcal{U}(\mathfrak{sl}(2))$ in such a way that $\mathcal{U}_1(\mathfrak{sl}(2)) \simeq \mathcal{U}(\mathfrak{sl}(2))$, we have the following proposition:

Proposition 5.5. Let $\mathcal{U}_q(\mathfrak{sl}(2))'$ be the algebra generated by the 5 elements E, F, K, K^{-1}, L such that

$$KK^{-1} = K^{-1}K = 1, \qquad KE = q^2EK, \qquad KF = q^{-2}FK, \qquad [E\,,\,F] = L,$$
$$(q-q^{-1})L = K - K^{-1}, \qquad [L\,,\,E] = q(EK + K^{-1}E), \qquad [L\,,\,F] = -q^{-1}(FK + K^{-1}F).$$

Then $\mathcal{U}_q(\mathfrak{sl}(2)) \simeq \mathcal{U}_q(\mathfrak{sl}(2))'$ for $q \notin \{-1, 0, 1\}$.

The bracket $[\,\cdot\,,\,\cdot\,]$ is again the commutator bracket.
Notice that this definition is a bit more complex than the simple $\mathcal{U}_q(\mathfrak{sl}(2))$ defined in definition 5.1. Just having one less generator and three less equation to worry about, will make the next parts a lot easier. But, everything will off course still be true when one uses $\mathcal{U}_q(\mathfrak{sl}(2))'$ instead of $\mathcal{U}_q(\mathfrak{sl}(2))$ everywhere, given that $q \notin \{-1, 0, 1\}$.

Proof. To show that $\mathcal{U}_q(\mathfrak{sl}(2))' \simeq \mathcal{U}_q(\mathfrak{sl}(2))$, we will construct an algebra automorphism. Now, set $q \in \mathbb{C}\backslash\{-1, 0, 1\}$ and let $\phi\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_q(\mathfrak{sl}(2))'$ be a linear map such that

$$\phi(E) = E, \qquad \phi(F) = F, \qquad \phi(K) = K.$$

Then ϕ is obviously a well-defined algebra morphism. And, since $\phi([E\,,\,F]) = L$, it follows that ϕ is surjective.
To show that ϕ is a bijection, we will construct a new algebra morphism $\psi\colon \mathcal{U}_q(\mathfrak{sl}(2))' \to \mathcal{U}_q(\mathfrak{sl}(2))$,

$$\psi(E) = E, \qquad \psi(F) = F, \qquad \psi(K) = K, \qquad \psi(L) = [E\,,\,F].$$

Note, obviously $\psi \circ \phi = id_{\mathcal{U}_q(\mathfrak{sl}(2))}$ and $\phi \circ \psi = id_{\mathcal{U}_q(\mathfrak{sl}(2))'}$, so it is only left to show that ψ is indeed a well-defined algebra morphism.
First, the first five equations of $\mathcal{U}_q(\mathfrak{sl}(2))'$ are obviously satisfied, so we only need to show that $\psi([L\,,\,E]) = q\psi(EK + K^{-1}E)$ and $\psi([L\,,\,F]) = -q^{-1}\psi(FK + K^{-1}F)$. Then it follows that ψ is an algebra morphism, as it is well-defined for all of its generators. Now,

$$\begin{aligned}\psi([L\,,\,E]) = [\psi(L)\,,\,\psi(E)] = [[E\,,\,F]\,,\,E] &= \frac{[K - K^{-1}\,,\,E]}{q-q^{-1}} \\ &= \frac{(q^2-1)EK + (q^2-1)K^{-1}E}{q-q^{-1}} = q(EK + K^{-1}E)\end{aligned}$$

$$\begin{aligned}\psi[L\,,\,F] = [[E\,,\,F]\,,\,F] &= \frac{[K - K^{-1}\,,\,F]}{q-q^{-1}} \\ &= \frac{(q^{-2}-1)FK + (q^{-2}-1)K^{-1}F}{q-q^{-1}} = -q^{-1}(FK + K^{-1}F.\end{aligned}$$

So, $\psi\colon \mathcal{U}_q(\mathfrak{sl}(2))' \to \mathcal{U}_q(\mathfrak{sl}(2))$ is indeed an algebra morphism. Thus, $\phi\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_q(\mathfrak{sl}(2))'$ is an algebra automorphism and $\mathcal{U}_q(\mathfrak{sl}(2)) \simeq \mathcal{U}_q(\mathfrak{sl}(2))'$. ☐

We will now show how $\mathcal{U}_1(\mathfrak{sl}(2))'$ is related to the classical case $\mathcal{U}(\mathfrak{sl}(2))$. Since the quantum case is generated by 5 elements and the classical case by only 3 elements, we do not get an equality.

Proposition 5.6. We have $\mathcal{U}_1(\mathfrak{sl}(2))' \simeq \mathcal{U}(\mathfrak{sl}(2))[K]/(K^2-1)$ and $\mathcal{U}(\mathfrak{sl}(2)) \simeq \mathcal{U}_1(\mathfrak{sl}(2))'/(K-1)$.

Proof. Note, $\mathcal{U}_1(\mathfrak{sl}(2))'$ is generated by E, F, K, K^{-1}, L such that

$$KK^{-1} = K^{-1}K = 1, \qquad KE = EK, \qquad KF = FK, \qquad [E\,,\,F] = L,$$
$$K - K^{-1} = 0, \qquad [L\,,\,E] = EK + K^{-1}E, \qquad [L\,,\,F] = -(FK + K^{-1}F).$$

The first three equations imply that $K \in Z(\mathcal{U}_1(\mathfrak{sl}(2))')$ and $K - K^{-1} = 0$ gives us that $K^2 = 1$, so $[L\,,\,E] = 2EK$, $[L\,,\,F] = -2FK$. Then, $\phi\colon \mathcal{U}_1(\mathfrak{sl}(2))' \to \mathcal{U}(\mathfrak{sl}(2))[K]$ with

$$\phi(E) = XK, \qquad \phi(F) = Y, \qquad \phi(K) = K, \qquad \phi(L) = HK,$$

is a well-defined algebra morphism with $\ker\phi$ being the ideal generated by K^2-1. In other words, $\mathcal{U}_1(\mathfrak{sl}(2))' \simeq \mathcal{U}(\mathfrak{sl}(2))/(K^2-1)$.
The other isomorphism $\mathcal{U}(\mathfrak{sl}(2)) \simeq \mathcal{U}_1(\mathfrak{sl}(2))'/(K-1)$ is obtained via $E \to X$, $F \to Y, L \to H, K \to 1$. This defines an algebra morphism from $\mathcal{U}_1(\mathfrak{sl}(2))'$ onto $\mathcal{U}(\mathfrak{sl}(2))$ with kernel generated by $K-1$. Hence, it induces an isomorphism $\mathcal{U}_1(\mathfrak{sl}(2))'/(K-1) \simeq \mathcal{U}(\mathfrak{sl}(2))$. □

For now we have only seen that $\mathcal{U}_q(\mathfrak{sl}(2))$ is an algebra. But it is indeed a Hopf algebra and even a Hopf $*$-algebra. To make it easier for ourselves, and since we are not fully interested in the case that q is a root of unity, we will fix $q \in \mathbb{C}\backslash\{0\}$ such that it is not a root of unity unless specified differently. So, from this point on $q \in \mathbb{C}\backslash\{0\}$ and $q^d \neq 1\ \forall d \in \mathbb{N}$, unless we define q otherwise.
This assumption will mostly avoid us having to do extra work due to powers of q sometimes being equal to 1. But we will not fully ignore the case that q is a root of unity. For example, we will also look at the Verma modules of $\mathcal{U}_q(\mathfrak{sl}(2))$ when q is a root of unity, but we will only look at this case as if $\mathcal{U}_q(\mathfrak{sl}(2))$ was an algebra.
We will use the Hopf algebra properties of $\mathcal{U}_q(\mathfrak{sl}(2))$ when q is not a root of unity.

Proposition 5.7. $\Delta\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_q(\mathfrak{sl}(2)) \otimes \mathcal{U}_q(\mathfrak{sl}(2))$ and $\varepsilon\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathbf{k}$ are algebra morphisms defined by:

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$
$$\varepsilon(E) = \varepsilon(F) = 0, \qquad \varepsilon(K) = \varepsilon(K^{-1}) = 1.$$

These algebra morphisms induce a bialgebra structure on $\mathcal{U}_q(\mathfrak{sl}(2))$.
Furthermore, $S\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_q(\mathfrak{sl}(2))$ given by,

$$S(E) = -EK^{-1}, \qquad S(F) = -KF, \qquad S(K) = K^{-1}, \qquad S(K^{-1}) = K,$$

defines an antipode for $\mathcal{U}_q(\mathfrak{sl}(2))$, i.e. $\mathcal{U}_q(\mathfrak{sl}(2))$ is a Hopf algebra.

Proof. Note,

$$\begin{aligned}
\Delta(K)\Delta(K^{-1}) &= \Delta(K^{-1})`\Delta K = 1,\\
\Delta(K)\Delta(E) &= (K \otimes K)(1 \otimes E + E \otimes K) = K \otimes KE + KE \otimes K^2 = q^2\Delta(E)\Delta(K)\\
\Delta(K)\Delta(F) &= (K \otimes K)(K^{-1} \otimes F + F \otimes 1) = F \otimes KF + KF \otimes K = q^{-2}\Delta(F)\Delta(K),\\
\Delta([E\,,\,F]) &= (1 \otimes E + E \otimes K)(K^{-1} \otimes F + F \otimes 1) - (K^{-1} \otimes F + F \otimes 1)(1 \otimes E + E \otimes K)\\
&= K^{-1} \otimes [E\,,\,F] + [E\,,\,F] \otimes K = \frac{\Delta(K) - \Delta(K^{-1})}{q - q^{-1}}.
\end{aligned}$$

So Δ is an algebra morphism. To check that Δ is coassociative, it suffices to show it for the generators of $\mathcal{U}_q(\mathfrak{sl}(2))$. For E,

$$(\Delta \otimes id)\Delta(E) = 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K = (id \otimes \Delta)\Delta(E).$$

This goes similar for F and K.
Now, ε is obviously an algebra morphism. It also satisfies the counit axiom, hence, $\mathcal{U}_q(\mathfrak{sl}(2))$ is at least a Hopf algebra.
Lastly, we want to show that S is an antipode. First,

$$\begin{aligned} S(K^{-1})S(K) = S(K)S(K^{-1}) = 1,& \\ S(KE) = S(E)S(K) = -EK^{-1}K^{-1} = -q^2K^{-1}EK^{-1} = q^2S(K)S(E) = S(q^2EK),& \\ S(KF) = S(F)S(K) = -KFK^{-1} = -q^{-2}K^{-1}KF = q^{-2}S(K)S(F) = S(q^{-2}FK),& \\ S([E\,,\,F]) = KFEK^{-1} - EK^{-1}KF = [F\,,\,E] = \frac{K^{-1}-K}{q-q^{-1}} = \frac{S(K)-S(K^{-1})}{q-q^{-1}},& \end{aligned}$$

so $S\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_q(\mathfrak{sl}(2))^{op}$ is an algebra morphism. Now, by lemma 1.28, it suffices to show that $\sum_{(x)} x'S(x'') = \sum_{(x)} S(x')x'' = \varepsilon(x)1$ for $x \in \{E, F, K, K^{-1}\}$. Now,

$$\begin{aligned} 1 \cdot S(E) + E \cdot S(K) = -EK^{-1} + EK^{-1} = 0 = \varepsilon(E),& \\ K^{-1} \cdot S(F) + F = K^{-1}(-KF) + F = 0 = \varepsilon(F),& \\ K \cdot S(K) = KK^{-1} = 1 = \varepsilon(K).& \end{aligned}$$

Applying S to the left side gives the same results, thus it follows from lemma 1.28 that S is indeed an antipode and $(\mathcal{U}_q(\mathfrak{sl}(2)), \mu, \eta, \Delta, \varepsilon, S)$ is indeed a Hopf algebra. □

Note, this is our first example of a non-abelian, non-cocommutative Hopf algebra. This is also clear from the fact that $S^2 \neq id$. Even though $S^2 \neq id$, it is a nice function.

Proposition 5.8. $S^2(X) = KXK^{-1}$ for any $X \in \mathcal{U}_q(\mathfrak{sl}(2))$.

Proof. $S^2(E) = q^2E = KEK^{-1}$, $\quad S^2(F) = q^{-2}F = KFK^{-1}$, $\quad S^2(K) = K$. □

This now shows that we have another example of a Hopf algebra with an antipode of finite order $2n$. As, if q is a $2n$-th root of unity, it follows that $S^{2n} = id$.

Since we know that the basis of $\mathcal{U}_q(\mathfrak{sl}(2))$ is given by $\{E^iF^kK^l\}_{i,j\in\mathbb{N},l\in\mathbb{Z}}$, if we can construct a general expression for $\Delta(E^iF^kK^l)$, it becomes a lot easier to calculate any coproduct in $\mathcal{U}_q(\mathfrak{sl}(2))$.

Proposition 5.9. Let $i, j \in \mathbb{N}$ and $k \in \mathbb{Z}$, then

$$\Delta(E^iF^jK^l) = \sum_{r=0}^{i}\sum_{s=0}^{j} q^{r(i-r)+s(j-s)-2(i-r)(j-s)} \begin{bmatrix} i \\ r \end{bmatrix}_q \begin{bmatrix} j \\ s \end{bmatrix}_q E^{i-r}F^sK^{l-(j-s)} \otimes E^rF^{j-s}K^{l+(i-r)}.$$

Proof. First,

$$\Delta(E^iF^jK^l) = \Delta(E)^i\Delta(F)^j\Delta(K)^l = (1 \otimes E + E \otimes K)^i(K^{-1} \otimes F + F \otimes 1)^j)(K^l \otimes K^l),$$

and

$$\begin{aligned} (E \otimes K)(1 \otimes E) = q^2(1 \otimes E)(E \otimes K),& \\ (K^{-1} \otimes F)(F \otimes 1) = q^2(F \otimes 1)(K^{-1} \otimes F).& \end{aligned}$$

So, we get that

$$\Delta(E)^i = \sum_{r=0}^{i} q^{r(i-r)} \begin{bmatrix} i \\ r \end{bmatrix}_q E^{i-r} \otimes E^r K^{i-r},$$
$$\Delta(F)^j = \sum_{s=0}^{j} q^s(j-s) \begin{bmatrix} j \\ s \end{bmatrix}_q F^s K^{-(j-s)} \otimes F^{j-s}.$$

□

We would love to also show that the center $Z(\mathcal{U}_q(\mathfrak{sl}(2)))$ is generated by the quantum Casimir element $C_q = EF + \frac{q^{-1}K+qK^{-1}}{(q-q^{-1})^2}$, since this also shows a nice property of $\mathcal{U}_q(\mathfrak{sl}(2))$, which is similar to that of $\mathcal{U}(\mathfrak{sl}(2))$. But, since this proof will use the concept of Verma modules of $\mathcal{U}_q(\mathfrak{sl}(2))$, this will have to wait until we have discussed $\mathcal{U}_q(\mathfrak{sl}(2))$-modules. This will be done in the next section. The last thing that we will do in this section, will be showing that $\mathcal{U}_q(\mathfrak{sl}(2))$ is a Hopf $*$-algebra.

5.2 The $*$-structures of $\mathcal{U}_q(\mathfrak{sl}(2))$

Now that we have shown that $\mathcal{U}_q(\mathfrak{sl}(2))$ is a Hopf algebra, we want to finally conclude the remark made at the start, that $\mathcal{U}_q(\mathfrak{sl}(2))$ is a Hopf $*$-algebra.
Recall that the Hopf algebra structure is unique if it exists, but that is not the case for the $*$-structures. We saw that this was the case for $\mathcal{U}(\mathfrak{sl}(2))$, as it had 3 unique $*$-structures up to isomorphism. Here we will see that $\mathcal{U}_q(\mathfrak{sl}(2))$ can have up to three $*$-structures for certain fixed $q \in \mathbb{C}\backslash\{0\}$, with q not a root of unity.

Theorem 5.10. *Let $q \in \mathbb{C}$ such that $q^n \neq 1$ for $n \in \mathbb{N}_{>1}$. Then, a Hopf $*$-algebra structure can be induced on $\mathcal{U}_q(\mathfrak{sl}(2))$ if and only if $q^2 \in \mathbb{R}$ or $|q| = 1$.*
Moreover, any Hopf $*$-algebra structure is equivalent to one of the following three $*$-structures:

i) $E^* = E, \quad F^* = F, \quad K^* = K$, *if* $|q| = 1$,

ii) $E^* = KF, \quad F^* = EK^{-1}, \quad K^* = K$, *if* $q \in \mathbb{R}\backslash\{0\}$,

iii) $E^* = iKF, \quad F^* = iEK^{-1}, \quad K^* = K$, *if* $q = \lambda i$, $\lambda \in \mathbb{R}\backslash\{0\}$.

Proof. We will prove that the above three $*$-structures are indeed the only three up to equivalence in the following manner. First, we will show some properties of the coproduct of $\mathcal{U}_q(\mathfrak{sl}(2))$. Then we will prove that all Hopf algebra morphisms $f\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_{q'}(\mathfrak{sl}(2))$ are of the same form. Lastly, we will proof that the above three $*$-structures are indeed the only three up to equivalence. This will be done by constructing 5 $*$-structures, then ending with the fact that any $*$-structure will be equivalent to one of those five and that two pairs of those $*$-structures are equivalent with each other.
So our first step is to prove certain properties of $\Delta\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_q(\mathfrak{sl}(2)) \otimes \mathcal{U}_q(\mathfrak{sl}(2))$. Let $x \in \mathcal{U}_q(\mathfrak{sl}(2))$, then we will show that:

i) $\Delta(x) = x \otimes x \iff x = K^l$,

ii) $\Delta(x) = 1 \otimes x + x \otimes K \iff x$ is a linear combination of E and KF,

iii) $\Delta(x) = K^{-1} \otimes x + x \otimes 1 \iff x$ is a linear combination of F and EK^{-1},

iv) $\Delta(x) = 1 \otimes x + x \otimes K^{-1} \iff x = 0$.

Recall that $\Delta(E^iF^jK^l) = \sum_{r=0}^{i}\sum_{s=0}^{j} q^{r(i-r)+s(j-s)-2(i-r)(j-s)} \begin{bmatrix} i \\ r \end{bmatrix} \begin{bmatrix} j \\ s \end{bmatrix} E^{i-r}F^sK^{l-(j-s)} \otimes E^rF^{j-s}K^{l+(i-r)}$. So, if x is grouplike, then $r = 0$ and $s = 0$. Thus, $i = 0$ and $j = 0$, hence $x = K^l$, which proves $i)$ as K^l is obviously grouplike itself.
Now, note that $\Delta(E)$ and $\Delta(F)$ are of the second and third form respectively. Thus, $\Delta(S(F))$ and $\Delta(S(E))$ are also of the second and third form respectively, as $S\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_q(\mathfrak{sl}(2))^{cop}$ is an algebra morphism. This at least shows the if implications of $ii), iii)$ and equation $iv)$.
For the “only if” implications, note that Δ is linear, so we only need to check $\Delta(E^iF^jK^l)$. Note that $\Delta(E^{i_1}F^{j_1}K^{l_1})$ and $\Delta(E^{i_2}F^{j_2}K^{l_2})$ only have at least one similar term if they are equal. In other words, $\Delta(x+y)$ will only be of the desired form if both $\Delta(x)$ and $\Delta(y)$ are of the desired form. Now, an easy check shows that if $\Delta(x) = 1 \otimes x + x \otimes K$, then either $i = 0$ and $j = k = 1$ or $i = 1$ and $j = k = 0$. Thus $x = \alpha E + \beta KF$. A similar check for $iii)$ and $iv)$ also shows that the "only if“ implications are indeed true.
We want to use this to show that the following is true for Hopf algebra isomorphisms between $\mathcal{U}_q(\mathfrak{sl}(2))$ and $\mathcal{U}_{q'}(\mathfrak{sl}(2))$.
Claim: $\exists f\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_{q'}(\mathfrak{sl}(2))$, Hopf algebra isomorphism $\iff q' = \pm q^{\pm 1}$ and if $q' = q$, then

$$f(E) = \alpha E, \quad f(F) = \alpha^{-1}F, \quad f(K) = K, \text{ for } \alpha \in \mathbb{C}\backslash\{0\}.$$

Let $f\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_{q'}(\mathfrak{sl}(2))$ be a Hopf algebra isomorphism. Then, if $x \in \mathcal{U}_q(\mathfrak{sl}(2))$ is grouplike, it follows that $f(x)$ is also grouplike. Since $f\otimes f(\Delta(x)) = f(x)\otimes f(x) = \Delta(f(x))$. Thus, $f(K) = K^{\pm 1}$ as f has to be bijective. Since $\Delta(E) = 1\otimes E+E\otimes K$, it follows that $\Delta(f(E)) = 1\otimes f(E)+f(E)\otimes f(K)$, thus $f(K) = K$. Else, $\Delta(f(E)) = 0$ by $iv)$, which implies that $f(E) = 0$, contradicting the fact that f is bijective.
So we at least have that $f(K) = K$. Also, by similar reasoning with $ii)$ and $iii)$, $f(E) = \alpha E + \beta KF$ and $f(F) = \alpha' F + \beta' EK^{-1}$. Now,

$$\alpha KE + \beta K^2F = f(KE) = q^2 f(EK) = q^2(\alpha EK + \beta KFK) = \alpha q^2 q'^{-2} + \beta q^2 q'^2 K^2 F,$$

so $q' = \pm q$ if $\alpha \neq 0$ and $q' = \pm q^{-1}$ if $\beta \neq 0$ and $\alpha \neq 0$ if and only if $\beta = 0$ as $q \neq 0$.
Similarly,

$$\alpha' KF + \beta' KEK^{-1} = f(KF) = q^{-2} f(FK) = q^{-2}(\alpha' FK + \beta' E) = \alpha' q^{-2} q'^2 + \beta' q^{-2} q'^{-2} KEK^{-1},$$

so $q' = \pm q$ if $\alpha' \neq 0$ and $q' = \pm q^{-1}$ if $\beta' \neq 0$ and $\alpha' \neq 0$ if and only if $\beta' = 0$ as $q \neq 0$. Thus it follows that f can only be a Hopf algebra isomorphism if and only if $q = \pm q^{\pm 1}$.
Now, set $q' = q$, so $f\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_q(\mathfrak{sl}(2))$. Then $f(K) = K$, $f(E) = \alpha E$ and $f(F) = \alpha' F$. Since

$$[\,E\,,\,F\,] = [\,f(E)\,,\,f(F)\,] = \alpha\alpha'[\,E\,,\,F\,],$$

it follows that $\alpha' = \alpha^{-1}$ and f is indeed of the above form.
As an extra remark, this last equation can also be used to show that $\alpha' = -\alpha^{-1}$ if $q' = -q$.
Since $*$ is a coalgebra morphism, by the same reasoning as above, $K^* = K$, $E^* = \alpha E$ or $E^* = \alpha KF$ and $F^* = \alpha F$ or $F^* = \alpha EK^{-1}$. As changing the algebra morphism property to an algebra antimorphism property in the above proof, does not impact the conclusion that at most one of α and β can be non-zero.
Since $*$ has to be an involution, it also follows that we have two options for $*$, $E^* = \alpha E$ and $F^* = \alpha^{-1}F$ or $E^* = \alpha KF$ and $F^* = \alpha^{-1}EK^{-1}$. Set $*$ to be given by $K^* = K$, $E^* = \alpha E$ and $F^* = \alpha^{-1}F$. Then

$$\alpha EK = (KE)^* = (q^2EK)^* = \alpha\overline{q}^2 KE = \alpha\overline{q}^2 q^2 EK,$$

so $\overline{q}^2q^2 = 1$ implies $|q| = 1$. When $*$ is given by $K^* = K$, $E^* = \alpha KF$ and $F^* = \alpha^{-1}EK^{-1}$, we see that

$$\alpha^{-1}E = (KF)^* = (q^{-2}FK)^* = \alpha^{-1}\overline{q}^{-2}KEK^{-1} = \alpha^{-1}\overline{q}^{-2}q^2E,$$

so $\overline{q}^{-2}q^2 = 1$ implies $q^2 \in \mathbb{R}$. So it follows that $\mathcal{U}_q(\mathfrak{sl}(2))$ has a $*$-structure if and only if $|q| = 1$ or if $q^2 \in \mathbb{R}$.

Claim: we have the following 5 $*$-structures for $\mathcal{U}_q(\mathfrak{sl}(2))$ and they are unique up to equivalence for their given q,

1. $E^* = E, \quad F^* = F, \quad K^* = K$, if $|q| = 1$,
2. $E^* = KF, \quad F^* = EK^{-1}, \quad K^* = K$, if $q \in \mathbb{R}_{>0}$,
3. $E^* = -KF, \quad F^* = -EK^{-1}, \quad K^* = K$, if $q \in \mathbb{R}_{<0}$,
4. $E^* = iKF, \quad F^* = iEK^{-1}, \quad L^* = K$, if $q = \lambda i$, $\lambda \in \mathbb{R}_{>0}$
5. $E^* = -iKF, \quad F^* = -iEK^{-1}, \quad L^* = K$, if $q = \lambda i$, $\lambda \in \mathbb{R}_{<0}$.

It is easy to see that any of these $*$ are at least involutions and that $* \circ S \circ * \circ S = id$. Also, from the above it immediately follows that these are our only options up to the choice of α and that they are at least coalgebra morphisms. But, we have also seen that they are algebra antimorphisms, as that property implies what choice of q is possible and vice versa.

So, by construction, we immediately see that these 5 options are indeed $*$ structures and also our only options up to the constant α. Fix $q \in \mathbb{C}$ and let $*_1$ be the corresponding $*$-structure given above, so $\alpha = 1$, and let $*_2$ be the corresponding $*$-structure but with $\alpha \in \mathbb{C}\backslash\{0\}$ arbitrary.

Since $f\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_q(\mathfrak{sl}(2))$, $f(K) = K$, $f(E) = \alpha(E)$, $f(F) = \alpha^{-1}F$ is a Hopf algebra automorphism, with the property that $f(x^{*_2}) = f(x)^{*_1}$ with $*_2$. It immediately follows that $*_1 \simeq *_2$, in other words, any $*$-structure on $\mathcal{U}_q(\mathfrak{sl}(2))$ is equivalent to one of the above 5.

Now, we claimed at the start that there are only 3 $*$-structures on $\mathcal{U}_q(\mathfrak{sl}(2))$ up to equivalence. Note that if $q \in \mathbb{R}_{>0}$, then $-q \in \mathbb{R}_{<0}$ and if $q = \lambda i$, $\lambda \in \mathbb{R}_{>0}$ then $-q = \lambda' i$, $\lambda' \in \mathbb{R}_{<0}$. And earlier it was shown that there exists a Hopf algebra automorphism $f\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_{q'}(\mathfrak{sl}(2))$ if and only if $q' = \pm q^{\pm 1}$, so options 2 and 3 and options 4 and 5 have the possibility to be equivalent $*$-structures. Note that option 1 is in general unique, due to the fact that if $|q| = 1$, it need not be true that $\pm q^{\pm 1}$ is of the correct form for any of the other $*$-structures.

Now, fix $q \in \mathbb{R}_{>0}$, then we have already seen that $f\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_{-q}(\mathfrak{sl}(2))$ given by

$$f(K) = K, \qquad f(E) = E, \qquad f(F) = -F,$$

is a Hopf algebra automorphism. Furthermore,

$$f(E^*) = f(KF) = -KF = E^* = f(E)^*, \qquad f(F^*) = f(EK^{-1}) = EK^{-1} = -F^* = f(F)^*,$$

so it follows that the $*$ of $\mathcal{U}_q(\mathfrak{sl}(2))$ is equivalent to that of $\mathcal{U}_{-q}(\mathfrak{sl}(2))$. In other words, since q was chosen arbitrarily, $*$-structures 2 and 3 are equivalent. A similar calculation shows that $*$-structures 4 and 5 are equivalent.

Since we also showed that $\alpha \in \mathbb{C}\backslash\{0\}$ can be chosen arbitrarily and the $*$-structure will be equivalent to any of the 5 structures given with $\alpha = 1$. It indeed follows that $\mathcal{U}_q(\mathfrak{sl}(2))$ has only 3 possible $*$-structures up to equivalence. □

6 The Verma modules of $\mathcal{U}_q(\mathfrak{sl}(2))$

In this section we will look at the Verma modules of $\mathcal{U}_q(\mathfrak{sl}(2))$. All of this will look fairly similar to the classical case, if we assume that q is not a root of unity. In the case that $q^d = 1$ for some $d \in \mathbb{N}_{>0}$, we will still see that some simple finite-dimensional modules will be highest weight modules, but not all of them.
Since this part will look fairly similar to the classical case. It will be structured in a similar fashion. We will first start with looking at highest weight modules. Then show that any finite-dimensional simple $\mathcal{U}_q(\mathfrak{sl}(2))$-module is a highest weight module when q is not a root of unity. We will then define the Verma module of $\mathcal{U}_q(\mathfrak{sl}(2))$ and use that to show that the quantum Casimir element generates the center $Z(\mathcal{U}_q(\mathfrak{sl}(2)))$. After that, we will show that any finite-dimensional $\mathcal{U}_q(\mathfrak{sl}(2))$-module is semisimple and define the quantum Clebsch-Gordan coefficients.
We will end this section with some theory on the $\mathcal{U}_q(\mathfrak{sl}(2))$-modules for when $q^d = 1$ for some $d \in \mathbb{N}_{>0}$, but $q \neq \pm 1$. That part will be structured similarly to any of the previous parts on finite-dimensional simple modules, but we will see that it becomes a bit more complicated when q is a root of unity.

6.1 The Verma modules of $\mathcal{U}_q(\mathfrak{sl}(2))$ when q is not a root of unity

As indicated by the title, we will assume that $q \in \mathbb{C}$ is not a root of unity in this part. To be precise, we will always assume that q is not a root of unity, unless it is specified to be a root of unity. Now, the following definitions are similar to that of the classical case.

Definition 6.1. Let V be a $\mathcal{U}_q(\mathfrak{sl}(2))$-module and let $\lambda \in \mathbb{C} \setminus \{0\}$. Then if $v \in V$, $v \neq 0$ and $Kv = \lambda v$, then λ is called a *weight* of v. Furthermore, v is a *highest weight vector* of weight λ if $Ev = 0$ and $Kv = \lambda v$. And a $\mathcal{U}_q(\mathfrak{sl}(2))$-module generated by a highest weight vector of weight λ is called a highest weight module of weight λ.

Lemma 6.2. Let V be a $\mathcal{U}_q(\mathfrak{sl}(2))$-module and let $\lambda \in \mathbb{C} \setminus \{0\}$ be a weight of $v \in V$. Then $q^2\lambda$ is a weight of Ev and $q^{-2}\lambda$ is a weight of Fv.

Proof. $K(Ev) = q^2 EKv = q^2\lambda Ev$ and $K(Fv) = q^{-2}FKv = q^{-2}\lambda Fv$, showing that $q^2\lambda$ and $q^{-2}\lambda$ are indeed weights of Ev and Fv respectively. □

Proposition 6.3. Any non-zero finite-dimensional $\mathcal{U}_q(\mathfrak{sl}(2))$-module V has a highest weight vector. Furthermore, the actions of E and F on V are nilpotent.

Proof. Let V be a non-zero finite-dimensional $\mathcal{U}_q(\mathfrak{sl}(2))$-module. Showing that it has a highest weight vector is similar to that of proposition 3.38.
Now, to show that the actions of E and F are nilpotent, it suffices to show that they can only have 0 as their eigenvalue. Say $\exists v \in V\{0\}$ such that $Ev = \lambda v$ with $\lambda \in \mathbb{C}\{0\}$. Then, $EKv = q^{-2}KEv = q^{-2}\lambda Kv$, so $\{K^n v\}_{n\in\mathbb{N}}$ is a sequence of eigenvectors with distinct eigenvalues $q^{-2n}\lambda$. But, since $dim(V) < \infty$, this is impossible. So E is nilpotent.
The proof goes similar for F, but then $\{K^n v\}_{n\in\mathbb{N}}$ is a sequence of eigenvectors with distinct eigenvalues $q^{2n}\lambda$, again showing that F is nilpotent. □

To construct certain highest weight modules, we will follow the same setup as in the classical case.

Lemma 6.4. Let v be a highest weight vector of weight λ. Let $v_0 = v$ and $v_n = \frac{1}{[n]!}F^n v$, $n \in \mathbb{N}_{>0}$. Then

$$Kv_n = \lambda q^{-2p} v_n, \qquad Ev_n = \frac{q^{-(n-1)}\lambda - q^{n-1}\lambda^{-1}}{q - q^{-1}} v_{n-1}, \qquad Fv_n = [n+1]v_{n+1}.$$

This directly follows from lemma 5.3. It also shows that we have a similar cyclic behaviour for highest weight vectors as in the classical case. In the sense that they are eigenvectors of H and E sends it to the "previous" eigenvector and F sends it to the "next" eigenvector. This behaviour allows us, once again, to determine all simple finite-dimensional modules.

Theorem 6.5. *Let V be a finite-dimensional $\mathcal{U}_q(\mathfrak{sl}(2))$-module generated by a vector v of highest weight λ, then*

i) $\lambda = \varepsilon q^d$, *with* $\varepsilon = \pm 1$ *and* $dim(V) = d + 1$,

ii) if $v_n = \frac{1}{[n]!}F^n v$, *then* $v_n = 0$ *if* $n > d$ *and* $\{v, v_1, \ldots, v_d\}$ *is a basis of* V,

iii) the operator K *acting on* V *is diagonalizable with eigenvalues* $\{\varepsilon q^d, \varepsilon q^{d-2}, \ldots, \varepsilon q^{-d+2}, \varepsilon q^{-d}\}$,

iv) any $w \in V$ *that is a highest weight vector is of weight* λ *and of the form* αv, $\alpha \in \mathbb{C}$,

v) V *is simple.*

Furthermore, any simple finite-dimensional $\mathcal{U}_q(\mathfrak{sl}(2))$-module is generated by a highest weight vector. And if W, W' are two $\mathcal{U}_q(\mathfrak{sl}(2))$-modules generated by highest weight vectors of weight λ, then $W \simeq W'$.

Proof. *i*), *ii*), *iii*) and *iv*) are a direct consequence of the previous lemma in combination with V being finite-dimensional. By lemma 6.4, we get that $\{v_i\}_{i\geq 0}$ is a sequence of eigenvectors of K with distinct eigenvalues. Thus $\exists d = dim(V) - 1 \in \mathbb{N}$ such that $v_d \neq 0$, but $v_{d+1} = 0$. Then,

$$0 = Ev_{d+1} = \frac{q^{-d}\lambda - q^d\lambda^{-1}}{q - q^{-1}} v_d,$$

shows that $q^{-d}\lambda = q^d\lambda^{-1}$ implies $\lambda = \pm q^d$. The other details are similar to the classical case, see theorem 3.40. □

Similar to the classical case, we again have unique, up to isomorphisms, simple $\mathcal{U}_q(\mathfrak{sl}(2))$-modules of dimension $d + 1$ generated by highest weight vectors of weight εq^n. Since they are generated by highest weight vectors of weight εq^d, we will denote these modules by $V_{\varepsilon,d}$. Note, the formulas given in lemma 6.4 can be rewritten for $V_{\varepsilon,d}$ to:

$$Kv_n = \varepsilon q^{d-2n} v_n, \qquad Ev_n = \varepsilon[d - n + 1]v_{n-1}, \qquad Fv_n = [n + 1]v_{n+1}.$$

Now, similar to the classical case, the formulas given in lemma 6.4 also define an infinite dimensional $\mathcal{U}_q(\mathfrak{sl}(2))$-module.

Lemma 6.6. Let $V(\lambda)$ be a vector space with basis $\{v_i\}_{i\in\mathbb{N}}$, then the formulas in lemma 6.4 with $Ev_0 = 0$ induce a $\mathcal{U}_q(\mathfrak{sl}(2))$-module structure on $V(\lambda)$ and it is generated by the highest weight vector v_0.

Proof. That the formulas induce a $\mathcal{U}_q(\mathfrak{sl}(2))$-module structure immediately follow from simple computations.
Now, from these formulas, it follows that $Kv_0 = \lambda v_0$ and $Ev_0 = 0$, so v_0 is indeed a highest weight vector. Furthermore, $Fv_n = [n + 1]v_{n+1}$ shows that $v_n = \frac{1}{[n]!}F^n v_0$ $\forall n \in \mathbb{N}$, thus $V(\lambda)$ is indeed generated by v_0. □

The module $V(\lambda)$ that is generated by a highest weight vector of weight λ is called the Verma module, just like in the classical case. Just like in the classical case, these Verma modules also need not be simple. We could again show this by constructing a submodule, but, just like in the classical case, we have another way to show this fact.

Proposition 6.7. Any highest weight $\mathcal{U}_q(\mathfrak{sl}(2))$-module V of highest weight λ is a quotient of the Verma module $V(\lambda)$.

Proof. The proof is similar to that of the classical case, proposition 3.43, but the linear map is now given by $f: V(\lambda) \to V;\ v_i \mapsto \frac{1}{[i]!}F^i v_0$. □

This proposition is the quantum analogue of proposition 3.43 and shows that the Verma modules $V(\lambda)$ need not be simple in the quantum case. Since the simple modules V_{ε,q^n} are in particular a quotient of the Verma module $V(\varepsilon q^n)$, it follows that $V(\varepsilon q^n)$ is not simple for $n \in \mathbb{N}_{>1}$. Thus it follows that $V(\lambda)$ is at least never simple when $\lambda = \pm q^n$ for $n \in \mathbb{N}_{>1}$, since it has V_{ε,q^n} as a submodule. Furthermore, we even have a similar fact in the quantum case for when $V(\lambda)$ can only be simple.

Lemma 6.8. Let $0 \neq \lambda \in \mathbb{C}$, then $V(\lambda)$ is simple if and only if $\lambda \neq \pm q^n$ for some $n \in \mathbb{N}$.

Proof. The above remark already shows that $V(\lambda)$ is not simple if $\lambda = \pm q^n$ for some $n \in \mathbb{N}$. Thus, if $V(\lambda)$ is simple, then $\lambda \neq \pm q^n$ for some $n \in \mathbb{N}$.
For the converse, assume that $V(\lambda)$ is not simple. So, there is a submodule $V \subset V(\lambda)$ that is not trivial. Then, we can assume without loss of generality that V is a simple module, since if V is not simple, we can take a submodule of V that is simple. Now, $dim(V)$ is either finite or infinite. If it is finite, it immediately follows from theorem 6.5 that $\lambda = \pm q^n$ for some $n \in \mathbb{N}$.
So let $dim(V) = \infty$. Then, for $v \in V$, since $v \in V(\lambda)$, it is a linear combination of the v_i. Hence, $E^n v = 0$ for some $n \in \mathbb{N}$. But, V is a submodule, so $E^{n-1}v \in V$ and $E^n v = 0$ hence $E^{n-1}v = \alpha v_0$ for some $0 \neq \alpha \in \mathbb{C}$, as $\lambda^{-1} \neq \lambda$. Thus $V = V(\lambda)$ and it follows that $V(\lambda)$ cannot contain a non-trivial infinite dimensional submodule.
Hence $V(\lambda)$ is simple if and only if $\lambda \neq \pm q^n$ for some $n \in \mathbb{N}$. □

Something else that is also similar, is the fact that central elements act like scalars on non-zero finite-dimensional simple $\mathcal{U}_q(\mathfrak{sl}(2))$-modules.

Lemma 6.9. Let $A \in Z(\mathcal{U}_q(\mathfrak{sl}(2)))$, then $Av = \alpha v$ for any $v \in V_{\varepsilon,\lambda}$, $\alpha \in \mathbb{C}$.

Since the proof of lemma 3.42 did not use any property of the $\mathcal{U}(\mathfrak{sl}(2))$-module $V(n)$, except that it was non-zero, finite-dimensional and simple. A similar proof can be used to show that this also holds in the quantum case. So, it directly follows that any $X \in Z(\mathcal{U}_q(\mathfrak{sl}(2)))$ acts as a scalar on any non-zero simple finite-dimensional $\mathcal{U}_q(\mathfrak{sl}(2))$-module V.
This is still true when q is a root of unity. Only, when q is a root of unity, it will hold for different modules. Since we will see that the finite-dimensional simple modules will not necessarily have the same form when $q^d = 1$ for some $d \in \mathbb{N}_{>1}$.

Another property that is similar to the classical case is the following theorem:

Theorem 6.10. *Any finite-dimensional $\mathcal{U}_q(\mathfrak{sl}(2))$-module is semisimple, if q is not a root of unity.*

To proof this, we will closely follow the proof of the theorem 3.45, which is the classical version of this theorem. For this we will need the following:

Proposition 6.11. The element $C_q = EF + \frac{q^{-1}K+qK^{-1}}{(q-q^{-1})^2} = FE + \frac{qK+q^{-1}K^{-1}}{(q+q^{-1})^2} \in Z(\mathcal{U}_q(\mathfrak{sl}(2)))$.

The proof of this proposition is a simple check that C_q indeed commutes with E, F and K. Note that the equality follows from the fact that $[\,E\,,\,F\,] = EF - FE = \frac{(q-q^{-1})(K-K^{-1})}{(q-q^{-1})^2}$.
The element C_q is called the *quantum Casimir element* and in a bit we will see that it generates the center $Z(\mathcal{U}_q(\mathfrak{sl}(2)))$, just like in the classical case.

Lemma 6.12. $\exists C \in Z(\mathcal{U}_q(\mathfrak{sl}(2)))$ such that C acts on $V_{\varepsilon,0}$ as 0 and on $V_{\varepsilon',d}$ as a non-zero scalar when $d > 0$.

Proof. Let $C = C_q - \varepsilon\frac{q+q^{-1}}{(q-q^{-1})^2}$. Then it acts on $V_{\varepsilon,0}$ as

$$\varepsilon\frac{q+q^{-1}}{(q-q^{-1})^2} - \varepsilon\frac{q+q^{-1}}{(q-q^{-1})^2} = 0$$

and on $V_{\varepsilon',d}$ as
$$\varepsilon' \frac{q^{d+1} + q^{-(d+1)}}{(q - q^{-1})^2} - \varepsilon \frac{q + q^{-1}}{(q - q^{-1})^2}.$$
This is not 0, since $\varepsilon' \frac{q^{d+1}+q^{-(d+1)}}{(q-q^{-1})^2} - \varepsilon \frac{q+q^{-1}}{(q-q^{-1})^2} = 0 \implies q^{2n+2} - \varepsilon\varepsilon' q^{n+2} - \varepsilon\varepsilon' q^n + 1 = 0 \iff (q^{n+2} - \varepsilon\varepsilon')(q^n - \varepsilon\varepsilon') = 0 \iff q$ is a root of unity. So C doesn't act as 0 on $V_{\varepsilon',d}$ for $d > 0$. □

We will now prove theorem 6.10. Since this will be fairly similar to the proof of the classical case, theorem 3.45, some of the details will be omitted.

Proof. We will follow the proof of the classical case step by step. Let V, V' be finite-dimensional $\mathcal{U}_q(\mathfrak{sl}(2))$-modules with $V \subset V'$. We first show $V = V' \oplus V''$, V'' a $\mathcal{U}_q(\mathfrak{sl}(2))$-module, for when $V' \subset V$ is of codimension 1.
It is trivial if $dim(V') = 0$. If $dim(V') = 1$, V' and V/V' are 1-dimensional modules with weights ε_1 and ε_2. If $\varepsilon_1 \neq \varepsilon_2$, there is a basis $\{v_1, v_2\}$ of V with $V' = \mathbf{k}v_1$ such that $Kv_i = v_i$, $Ev_i = 0 = Fv_i$ for $i = 1, 2$, hence $V = \mathbf{k}v_1 \oplus \mathbf{k}v_2$.
If $\varepsilon_1 = \varepsilon_2$, there is a basis $\{v_1, v_2\}$ of V with $V' = \mathbf{k}v_1$ and $Kv_1 = \varepsilon_1 v_1$, $Kv_2 = \varepsilon_1 v_2 + \alpha v_1$, Then $Ev_1 = 0$ and $Ev_2 = \lambda v_1 + \mu v_2$, so
$$\varepsilon_1 \lambda v_1 + \mu(\varepsilon_1 v_2 + \alpha v_1) = KEv_2 = q^2 EKv_2 = \varepsilon_1 q^2(\lambda v_1 + \mu v_2),$$
so $Ev_2 = 0$. From a similar computation, it also follows that $Fv_i = 0$ for $i = 1, 2$. Hence K acts as K^{-1} on V and it follows that $\alpha = -\alpha$, so K acts diagonal on V. In other words, we have the same case as when $\varepsilon_1 \neq \varepsilon_2$.
Now, assume the assertion holds for $dim(V') < n$ and let $dim(V) = n$. Then V' is simple or not simple. When V is not simple, the same arguments can be used as in theorem 3.45. So assume V is simple.
Then V/V' is a module of weight $\varepsilon = \pm 1$ and a similar argument shows that $\frac{C}{\alpha}$ is a projection of V onto V'. Thus, $V = V' \oplus ker(\frac{C}{\alpha})$, with C as in the previous lemma.
For the general case, one can use similar arguments as in the proof of theorem 3.45 with the same spaces W and W' to show that the generator f of W'', $W = W' \oplus W''$ acts as a non-zero scalar α on V. Hence $\frac{f}{\alpha}$ is a projection of V onto V'. Once again, showing that f is $\mathcal{U}_q(\mathfrak{sl}(2))$-linear, completes the proof. □

6.2 The center of $\mathcal{U}_q(\mathfrak{sl}(2))$

Similar to the classical case, we can use the theory on Verma modules to show that the center of $\mathcal{U}_q(\mathfrak{sl}(2))$ is generated by a single element. We will show that $Z(\mathcal{U}_q(\mathfrak{sl}(2)))$ is generated by the quantum analogue of the Casimir element. But, we still need to be careful with our choice of q.

Lemma 6.13. Let $q \in \mathbb{C}\backslash\{0\}$ such that $q^n = 1$, $n \in \mathbb{N}_{>2}$, so q is a root of unity.
Let $e = \begin{cases} n & \text{if n is odd,} \\ n/2 & \text{if n is even.} \end{cases}$ Then, $K^e, E^e, F^e \in Z(\mathcal{U}_q(\mathfrak{sl}(2)))$.

Proof. Note, that $q^{2e} = 1$ and $[e] = 0$, so it follows from lemma 5.3 that K^e, E^e and F^e indeed commute with every element of $\mathcal{U}_q(\mathfrak{sl}(2))$, as they commute with all of its generators. □

This small lemma immediately shows why we need to be careful with our choice of q, since the elements K^e, E^e and F^e are obviously not in $Z(\mathcal{U}_q(\mathfrak{sl}(2)))$ if q is not a root of unity.
The goal for this part is to describe to center $Z(\mathcal{U}_q(\mathfrak{sl}(2)))$. We will do this with the *Harish-Chandra homomorphism*, since we will see that this is an isomorphism from $Z(\mathcal{U}_q(\mathfrak{sl}(2)))$ to a subalgebra of $\mathbf{k}[K, K^{-1}]$.

Definition 6.14. Let $\mathcal{U}_q(\mathfrak{sl}(2))^K = \{X \in \mathcal{U}_q(\mathfrak{sl}(2)) | XK = KX\}$ and let $\varphi\colon \mathcal{U}_q(\mathfrak{sl}(2))^K \to \mathbf{k}[K, K^{-1}]$ be a projection. Then φ is the so called *Harish-Chandra homomorphism.*

It is not immediately clear why this is an algebra morphism or even a projection, so we will show this. First, note that $\mathcal{U}_q(\mathfrak{sl}(2))^K$ is a subalgebra of $\mathcal{U}_q(\mathfrak{sl}(2))$ and $\mathbf{k}[K, K^{-1}]$ is also an algebra. So, φ is at least a map between two algebras.

Lemma 6.15. $X \in \mathcal{U}_q(\mathfrak{sl}(2))^K \subset \mathcal{U}_q(\mathfrak{sl}(2))$ if and only if $X = \sum_{i\geq 0} F^i P_i E^i$ with $P_i \in \mathbf{k}[K, K^{-1}]$ for $i \geq 0$.

Proof. Since $K(F^i K^l E^j)K^{-1} = q^{2(j-i)} F^i K^l E^j$ and $\{F^i K^l E^j\}_{i,j\in\mathbb{N},l\in\mathbb{Z}}$ was a basis of $\mathcal{U}_q(\mathfrak{sl}(2))$. It follows that $KXK^{-1} = X \iff X = \sum_{i\geq 0} F^i P_i E^i$. □

Lemma 6.16. Let $I = F\mathcal{U}_q(\mathfrak{sl}(2)) \cap \mathcal{U}_q(\mathfrak{sl}(2))^K = \mathcal{U}_q(\mathfrak{sl}(2))E \cap \mathcal{U}_q(\mathfrak{sl}(2))^K$, then it follows that I is an ideal of $\mathcal{U}_q(\mathfrak{sl}(2))^K$ and $\mathcal{U}_q(\mathfrak{sl}(2))^K = \mathbf{k}[K, K^{-1}] \oplus I$.

Proof. Let $X = \sum_{i\geq 0} F^i P_i E^i \in \mathcal{U}_q(\mathfrak{sl}(2))^K$. Then, if $X \in \mathcal{U}_q(\mathfrak{sl}(2))E$, it follows that $P_0 = 0$, hence $X \in F\mathcal{U}_q(\mathfrak{sl}(2))$. Conversely, if $X \in F\mathcal{U}_q(\mathfrak{sl}(2))$ it follows that $P_0 = 0$, hence $X \in \mathcal{U}_q(\mathfrak{sl}(2))E$ and $F\mathcal{U}_q(\mathfrak{sl}(2)) \cap \mathcal{U}_q(\mathfrak{sl}(2))^K = \mathcal{U}_q(\mathfrak{sl}(2))E \cap \mathcal{U}_q(\mathfrak{sl}(2))^K$.
Note, $F\mathcal{U}_q(\mathfrak{sl}(2)) \cap \mathcal{U}_q(\mathfrak{sl}(2))^K$ is a right ideal and $\mathcal{U}_q(\mathfrak{sl}(2))E \cap \mathcal{U}_q(\mathfrak{sl}(2))^K$ a left ideal, as $\mathcal{U}_q(\mathfrak{sl}(2))^K$ is an algebra. Thus it follows from the above that I is indeed an ideal.
Hence, we have an algebra morphism $\pi\colon \mathcal{U}_q(\mathfrak{sl}(2))^K \to \mathcal{U}_q(\mathfrak{sl}(2))^K/I$; $a \mapsto a + I$. Since $a + I = b + I$ if and only if $P_0^a = P_0^b$, with $a = \sum_i F^i P_i^a E^i, b = \sum_i F^i P_i^b E^i$, and the form given in lemma 6.15 is unique, it follows that we can turn this into an algebra morphism $\pi\colon \mathcal{U}_q(\mathfrak{sl}(2))^K \to \mathbf{k}[K, K^{-1}] \oplus I$, $a \mapsto (P_0^a, I)$. □

This shows that the map $\varphi\colon \mathcal{U}_q(\mathfrak{sl}(2))^K \to \mathbf{k}[K, K^{-1}]$ is is indeed an algebra morphism and it is the projection of $\mathcal{U}_q(\mathfrak{sl}(2))^K$ onto $\mathbf{k}[K, K^{-1}]$ via π.
We want to use φ to describe $Z(\mathcal{U}_q(\mathfrak{sl}(2)))$. Since $Z(\mathcal{U}_q(\mathfrak{sl}(2))) \subset \mathcal{U}_q(\mathfrak{sl}(2))^K$ and it is also a subalgebra, we will do this by looking at the Harish-Chandra morphism restricted to $Z(\mathcal{U}_q(\mathfrak{sl}(2)))$.

Proposition 6.17. Let V be a highest weight module of $\mathcal{U}_q(\mathfrak{sl}(2))$ with highest weight λ. Then, for any $Z \in Z(\mathcal{U}_q(\mathfrak{sl}(2))), v \in V$, we have $Zv = \varphi(Z)(\lambda)v$.

Proof. Let v_0 be of highest weight λ, the vector that generates V, and let $Z \in Z(\mathcal{U}_q(\mathfrak{sl}(2)))$. Then, $Z = \varphi(Z) + \sum_{i>0} F^i P_i E^i$. As $Ev_0 = 0$, $Kv_0 = \lambda v_0 \implies Zv_0 = \varphi(Z)(\lambda)v_0$.
Now, if $v \in V$ arbitrary, then $v = Xv_0$ for $X \in \mathcal{U}_q(\mathfrak{sl}(2))$, thus $Zv = ZXv_0 = XZv_0 = \varphi(Z)(\lambda)Xv_0 = \varphi(Z)(\lambda)v$. □

Lemma 6.18. Let $Z \in Z(\mathcal{U}_q(\mathfrak{sl}(2)))$, then $\varphi(Z) = 0$ if and only if $Z = 0$.

Proof. Let $Z \in Z(\mathcal{U}_q(\mathfrak{sl}(2)))$, $Z \neq 0$, with $\varphi(Z) = 0$. Then $Z = \sum_{i=k}^{l} F^i P_i E^i$ for $0 < k \leq l, k, l \in \mathbb{N}$ and some P_i non-zero, $P_k \neq 0$. Let $V(\lambda)$ be a Verma module of highest weight $\lambda \neq \varepsilon q^n$, $n \in \mathbb{N}$.
Then, $Ev_p = 0$ if and only if $p = 0$ and $Zv_k = \varphi(Z)(\lambda)v_k = 0$. Also, $Zv_k = F^k P_k E^k v_k = cP_k(\lambda)v_k$, $c \in \mathbb{C}\backslash\{0\}$, thus $P_k(\lambda) = 0$. From this it follows that P_k is a polynomial with infinitely many roots, but $P_k \neq 0$, hence $Z = 0$. □

We want to end this part with concluding that the center $Z(\mathcal{U}_q(\mathfrak{sl}(2)))$ is generated by the quantum Casimir element $C_q = EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$. We will do this in a similar fashion as in the classical case. So, we will show that the $\varphi(Z)$ have a special properties for $Z \in Z(\mathcal{U}_q(\mathfrak{sl}(2)))$, then normalise the Harish-Chandra morphism with the use of this. Similar to the classical case, this normalised morphism will be an isomorphism when restricted to $Z(\mathcal{U}_q(\mathfrak{sl}(2)))$
So, to start, we have the following property for central elements of $\mathcal{U}_q(\mathfrak{sl}(2))$:

Lemma 6.19. Let $Z \in Z(\mathcal{U}_q(\mathfrak{sl}(2)))$, then $\varphi(Z)(q^{-1}\lambda) = \varphi(Z)(q^{-1}\lambda^{-1})$.

Proof. Let $n > 0$ and consider the Verma module $V(q^{n-1})$. Then,

$$Ev_n = \frac{q^{-(n-1)}q^{n-1} - q^{n-1}q^{-(n-1)}}{q - q^{-1}} v_n = 0.$$

So, v_n is of highest weight $q^{n-1-2n} = q^{-n-1}$. By the previous proposition, $Zv_n = \varphi(Z)(q^{-n-1})v_n$, but $v_n \in V(q^{n-1})$, so $Zv_n = \varphi(Z)(q^{n-1})v_n$. In other words,

$$\varphi(Z)(q^{-n-1}) = \varphi(Z)(q^{n-1}).$$

It follows that $\varphi(Z)(q^{-n}) = \varphi(Z)(q^n)$ and hence $\varphi(Z)(q^{-1}\lambda) = \varphi(Z)(q^{-1}\lambda^{-1})$. □

Let $\delta_q \colon \mathbf{k}[K, K^{-1}] \to \mathbf{k}[K, K^{-1}]$ be the map given by $\delta_q(P(\lambda)) = P(q^{-1}\lambda)$. Then $\delta_q \circ \varphi$ is the normalized Harish Chandra morphism and we will see that it is an isomorphism from $Z(\mathcal{U}_q(\mathfrak{sl}(2)))$ into $\mathbf{k}[K + K^{-1}]$.
To show this, we will use the following lemma in combination with the remark that $\delta_q(\varphi(Z))$ is still a Laurent polynomial for $Z \in Z(\mathcal{U}_q(\mathfrak{sl}(2)))$.

Lemma 6.20. Any element of $P \in \mathbf{k}[K, K^{-1}]$ satisfying $P(\lambda) = P(\lambda^{-1})$ is a polynomial in $K + K^{-1}$.

This follows from a simple induction argument. So, for $Z \in Z(\mathcal{U}_q(\mathfrak{sl}(2)))$, $\delta_q(\varphi(Z))$ is a Laurent polynomial with the property that $\delta_q(\varphi(Z))(\lambda) = \delta_q(\phi(Z))(\lambda^{-1})$. Hence, $\delta_q(\varphi(Z))$ is a polynomial in $K + K^{-1}$. The next theorem will conclude that the center $Z(\mathcal{U}_q(\mathfrak{sl}(2)))$ is generated by the quantum Casimir element C_q.

Theorem 6.21. *Let q not be a root of unity, then $Z(\mathcal{U}_q(\mathfrak{sl}(2)))$ is generated by C_q. Furthermore, $\delta_q \circ \varphi|_{Z(\mathcal{U}_q(\mathfrak{sl}(2)))} \colon Z(\mathcal{U}_q(\mathfrak{sl}(2))) \xrightarrow{\sim} \mathbf{k}[K + K^{-1}]$.*

Proof. We know that $\varphi|_{Z(\mathcal{U}_q(\mathfrak{sl}(2)))}$ is injective, so it is only left to show that it is surjective. By the remark, we know that $Im(\delta_q \circ \varphi|_{Z(\mathcal{U}_q(\mathfrak{sl}(2)))}) \subset \mathbf{k}[K + K^{-1}]$.
Now, $\varphi(C_q) = \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$, thus $\delta_q(\varphi(C_q)) = \frac{K+K^{-1}}{(q-q^{-1})^2}$. Thus $Im(\delta_q \circ \varphi|_{Z(\mathcal{U}_q(\mathfrak{sl}(2)))})$ contains the generator $K + K^{-1}$ of $\mathbf{k}[K + K^{-1}]$, hence $\delta \circ \varphi(Z(\mathcal{U}_q(\mathfrak{sl}(2)))) = \mathbf{k}[K + K^{-1}]$ and $Z(\mathcal{U}_q(\mathfrak{sl}(2)))$ is indeed generated by C_q. □

6.3 The quantum 3j-symbols of $\mathcal{U}_q(\mathfrak{sl}(2))$

Now that we know that finite-dimensional $\mathcal{U}_q(\mathfrak{sl}(2))$-modules are semisimple, we can discuss a property of $\mathcal{U}_q(\mathfrak{sl}(2))$ that is similar to the Clebsch-Gordan coefficients in the classical case. But this time we want to go one step further. Besides only showing a relation between two bases of $V_{\varepsilon,n} \otimes V_{\varepsilon',m}$, we also want to show that the obtained basis is orthogonal.
Since we want to look at an orthogonality property, our first step is to construct a bilinear form on $\mathcal{U}_q(\mathfrak{sl}(2))$.

Proposition 6.22. $\exists!$ algebra antiautomorphism $T \colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_q(\mathfrak{sl}(2))$ such that $T(E) = KF$, $T(F) = EK^{-1}$, $T(K) = K$ and T is a coalgebra morphism.

Proof. First, assume that a linear map T exists as described above, such that $T(AB) = T(B)T(A)$ for $A, B \in \mathcal{U}_q(\mathfrak{sl}(2))$. Then,

$$T(KEK^{-1}) = K^{-1}(KF)K = FK = q^2KF = T(q^2E),$$
$$T([E, F]) = -[T(E), T(F)] = -[KF, EK^{-1}] = [E, K] = \frac{K - K^{-1}}{q - q^{-1}}.$$

A similar calculation for $KFK = q^{-2}F$ shows that it is indeed an algebra antimorphism of $\mathcal{U}_q(\mathfrak{sl}(2))$. It obviously is an isomorphism, hence an automorphism. Showing that T is a coalgebra morphism also follows from similar calculations.
Lastly, the existence and uniqueness follow directly from the properties of T. □

Theorem 6.23. *On the simple $\mathcal{U}_q(\mathfrak{sl}(2))$-modules $V_{\varepsilon,d}$ there exists a unique non-degenerate symmetric bilinear form such that $(v_0, v_0) = 1$ and $(Xv, v') = (v, T(X)v')$ for $X \in \mathcal{U}_q(\mathfrak{sl}(2))$, $v, v' \in V_{\varepsilon,d}$ and T defined as in proposition 6.22.*
Furthermore, the basis vectors $v_n = \frac{1}{[n]_q!}F^n v_0$, $n \geq 0$, are pairwise orthogonal and $(v_i, v_i) = q^{-(d-n-1)n}\begin{bmatrix} d \\ i \end{bmatrix}_q$

Proof. Again, assume that a bilinear form as described above exists on $V_{\varepsilon,d}$. Then we want to show that it is unique, so we want to show that the v_i are pairwise orthogonal and that the last formula holds. Now,

$$(v_i, v_j) = \frac{1}{[i]_q!}(Fv_0, v_j) = \frac{1}{[i]_q!}(v_0, T(F)^i v_j) = \frac{1}{[i]_q!}(v_0, (EK^{-1})^i v_j).$$

Since $(EK^{-1})^i = q^{i(i+1)}K^{-i}E^i$ for $i > 0$, it follows that $T(F)^i v_j = \alpha E^i v_j$, $0 \neq \alpha \in \mathbf{k}$, hence $(v_i, v_j) = 0$ if $i > j$. As $(\cdot, \cdot)$ is symmetric, $(v_i, v_j) = 0$ if $i < j$. Lastly,

$$(v_i, v_i) = \frac{1}{[i]_q!}q^{i(i+1)}(v_0, K^{-i}E^i v_i) = \varepsilon^i q^{i(i+1)}\frac{[d]_q!}{[i]_q![d-1]_q!}(v_0, K^{-i}v_0) = q^{i(i+1)-di}\begin{bmatrix} d \\ i \end{bmatrix}_q (v_0, v_0).$$

Hence, this bilinear form is unique. For existence, note that $(v_i, v_j) = q^{-(d-i-1)i}\begin{bmatrix} d \\ i \end{bmatrix}_q \delta_{ij}$ clearly is a non-degenerate bilinear form. A quick computation shows that this satisfies the last relation given in the theorem, showing that a bilinear form with the properties given above truly exists and is unique. □

We can use this bilinear form to show that the two different bases of $V_{1,n} \otimes V_{1,m}$ that we will find, are orthogonal. Before that, we want to rewrite $V_{1,n} \otimes V_{1,m}$ as a direct sum of simple modules. That will give us our two bases and the relation between these basis will be described by the so called *3j-symbols*, also known as the Clebsch-Gordan coefficients.
Note that $V_{\varepsilon,d} \simeq V_{\varepsilon,0} \otimes V_{1,d} \simeq V_{1,d} \otimes V_{\varepsilon,0}$. Since we have that $V_{\varepsilon,0} = \mathbf{k}$, it suffices to only look at the modules $V_{1,n} \otimes V_{1,m}$, instead of $V_{\varepsilon,n} \otimes V_{\varepsilon',m}$.

Theorem 6.24. *Let $n, m \in \mathbb{N}$, $n \geq m$, then $V_{1,n} \otimes V_{1,m} \simeq V_{1,n+m} \oplus V_{1,n+m-2} \oplus \cdots \oplus V_{1,n-m}$.*

The proof of this theorem is exactly the same as in the classical case, proposition 3.46. The following lemma shows that we will also have all the highest weight vectors that we want in the quantum case.

Lemma 6.25. Let $n, m \in \mathbb{N}$, $n \geq m$ and let $v^{(n)} \in V_{1,n}$, $v^{(m)} \in V_{1,m}$ be highest weight vectors of weight q^n and q^m respectively. Set $v_k^{(n)} = \frac{1}{[k]!}F^k v^{(n)}$ and $w_k^{(m)} = \frac{1}{[k]!}F^k w^{(m)}$ for $p \geq 0$. Then

$$v^{(n+m-2k)} = \sum_{i=0}^{k}(-1)^i \frac{[m-k+i]![n-i]!}{[m-k]![n]!}q^{-i(m-2k+i+1)}v_i^{(n)} \otimes v_{k-i}^{(m)}$$

is a highest weight vector of $V(n) \otimes V(m)$ of weight q^{n+m-2k}.

One can prove that $v^{(n+m-2k)}$ is indeed a highest weight vector by simply calculating how K and E act on it.
Just like in the classical case, we now have two different bases for $V_{1,n} \otimes V_{1,m}$, namely $\{v_i^{(n)} \otimes v_j^{(m)}\}_{0 \leq i \leq n, 0 \leq j \leq m}$ and $\{v_p^{(n+m-2k)}\}_{0 \leq k \leq m, 0 \leq p \leq n+m-2k}$, with $v_p^{(n+m-2k)} = \frac{1}{[p]!}F^p v^{n+m-2k}$. This first basis directly comes from the tensor product and the second one is more suited to work with when regarding $V_{1,n} \otimes V_{1,m}$ as a $\mathcal{U}_q(\mathfrak{sl}(2))$-module.
Since they are two bases of the same vector spaces, we can compare them with each other. In this case, it is also a rather nice comparison, since

$$v_p^{(n+m-2k)} = \sum_{0 \leq i \leq n, 0 \leq j \leq m} \begin{bmatrix} n & m & n+m-2k \\ i & j & p \end{bmatrix} v_i^{(n)} \otimes v_j^{(m)},$$

for certain coefficients $\begin{bmatrix} n & m & n+m-2k \\ i & j & p \end{bmatrix}$, which are referred to as the *quantum $3j$-symbols* and are defined for $0 \leq k \leq m$ and $0 \leq p \leq n+m-2k$.

Lemma 6.26. For a fixed k and p, v_p^{n+m-2k} is a linear combination of vectors of the form $v_i^{(n)} \otimes v_{p-i+k}^{(m)}$, so $\begin{bmatrix} n & m & n+m-2k \\ i & j & p \end{bmatrix} = 0$ if $i+j \neq p+k$.
Furthermore,

$$\begin{bmatrix} n & m & n+m-2k \\ i & j+1 & p+1 \end{bmatrix} = \frac{[j+1]q^{-(n-2i)} + [i]}{[p+1]} \begin{bmatrix} n & m & n+m-2k \\ i & j & p \end{bmatrix}.$$

Proof. We will proof this via induction on p. First, the case $k=0$ follows directly from lemma 6.25. Suppose we have $v_p^{n+m-2k} = \sum_i \alpha_i v_i^{(n)} \otimes v_{p-i+k}^{(m)}$, then

$$[p+1]v_{p+1}^{(n+m-2k)} = Fv_p^{(n+m-2k)} = \sum_i \alpha_i \left(K^{-1}v_i^{(n)} \otimes Fv_{k-i+p}^{(m)} + Fv_i^{(n)} \otimes v_{k-i+p}^{(m)} \right).$$

From this, it follows that v_p^{n+m-2k} is a linear combination of vectors of the form $v_i^{(n)} \otimes v_{p-i+k}^{(m)}$. The last equation also follows directly from the above. □

As stated before, the two bases we have given here, are orthogonal bases. Furthermore, there are also orthogonality relations for the $3j$-symbols. With these we can even express the vectors $v_i^{(n)} \otimes v_j^{(m)}$ in terms of the vectors v_p^{n+m-2k}. So we don't just have 1 original bases and 1 basis that is expressed in terms of the old one. We can indeed express both bases in terms of the other.
To do this, we will use the bilinear form of theorem 6.23. Note that if we equip both $V_{1,n}$ and $V_{1,m}$ with this bilinear form, then we can also define another symmetric bilinear form on $V_{1,n} \otimes V_{1,m}$ given by $(v_1 \otimes v_1', v_2 \otimes v_2') = (v_1, v_2)(v_1', v_2')$ for $v_1, v_2 \in V_{1,n}$, $v_1', v_2' \in V_{1,m}$.

Lemma 6.27. The above defined symmetric bilinear form is non-degenerate and the basis $\{v_i^{(n)} \otimes v_j^{(m)}\}_{0 \leq i \leq n, 0 \leq j \leq m}$ is orthogonal with respect to this bilinear form.
Furthermore, $\forall X \in \mathcal{U}_q(\mathfrak{sl}(2))$, $w_1, w_2 \in V_{1,n} \otimes V_{1,m}$, $(Xw_1, w_2) = (w_1, T(X)w_2)$.

Proposition 6.28. i) The basis $\{v_p^{(n+m-2k)}\}_{0 \leq k \leq m,\, 0 \leq p \leq n+m-2k}$ is orthogonal with respect to the bilinear of lemma 6.27.

ii) For fixed p, q, k, l, we have

$$0 = \sum_{i,j} q^{-i(n-i-1)-j(m-j-1)} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} n & m & n+m-2k \\ i & j & p \end{bmatrix} \begin{bmatrix} n & m & n+m-2q \\ i & j & l \end{bmatrix},$$

when $p \neq l$ or $k \neq q$ and

$$\sum_{i,j} q^{-i(n-i-1)-j(m-j-1)} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} n & m & n+m-2k \\ i & j & p \end{bmatrix}^2 = q^{-p(n+m-2k-p-1)} \begin{bmatrix} n+m-2k \\ p \end{bmatrix}.$$

iii) Given i and j, we have

$$v_i^{(n)} \otimes v_j^{(m)} = q^{-i(n-i-1)-j(m-j-1)} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \sum_{k=0}^{m} \sum_{p=0}^{n+m-2k} q^{p(n+m-2k-p-1)} \frac{\begin{bmatrix} n & m & n+m-2k \\ i & j & p \end{bmatrix}}{\begin{bmatrix} n+m-2k \\ p \end{bmatrix}} v_k^{(n+m-2k)}.$$

The proofs of these two statements comes down to doing some calculations with the use of both the $3j$-symbols and the defined bilinear form. Since we are not too interested in how to do to use the $3j$-symbols for these kind of calculations, the two proofs of these statements will be omitted, but can

be found in [4].
Now, similar to the classical case, since the central elements of $\mathcal{U}_q(\mathfrak{sl}(2))$ act like scalars on the modules $V_{\varepsilon,d}$, it also follows that the $\sum_{0\leq i\leq n, 0\leq j\leq m} \begin{bmatrix} n & m & n+m-2k \\ i & j & p \end{bmatrix} v_i^{(n)} \otimes v_j^{(m)}$ are eigenfunctions of the quantum Casimir element C_q, in the sense that

$$\begin{aligned}\lambda_{C_q} v_p^{n+m-2k} = C_q v_p^{n+m-2k} &= \sum_{0\leq i\leq n, 0\leq j\leq m} C_q \begin{bmatrix} n & m & n+m-2k \\ i & j & p \end{bmatrix} v_i^{(n)} \otimes v_j^{(m)} \\ &= \sum_{0\leq i\leq n, 0\leq j\leq m} \lambda_{C_q} \begin{bmatrix} n & m & n+m-2k \\ i & j & p \end{bmatrix} v_i^{(n)} \otimes v_j^{(m)}.\end{aligned}$$

This can then be used to get an explicit formula for the Clebsch-Gordan coefficients. In this case, similar to the classical case, the Clebsch-Gordan coefficients can also be described by certain polynomials, the q-Hahn polynomials [12].

Before we move on to the $\mathcal{U}_q(\mathfrak{sl}(2))$-module that will be our main subject for the rest of this thesis. We will take a short detour to look at the case when q is a root of unity.

6.4 The Verma modules of $\mathcal{U}_q(\mathfrak{sl}(2))$ when q is a root of unity

We have often said that the choice of q matters a lot for when we looked at certain properties of $\mathcal{U}_q(\mathfrak{sl}(2))$. Moreover, we even choose $q \in \mathbb{C}\backslash\{0\}$ to specifically not be a root of unity for most of the last two sections. To really indicate why we did this, we will look at the simple finite-dimensional modules of $\mathcal{U}_q(\mathfrak{sl}(2))$ for when $q^d = 1$ for some $d \in \mathbb{N}_{>2}$. This will then be a second example of why we need to be careful with how we choose q, with lemma 6.13 being our first example.
So, for this part, let $q \in \mathbb{C}\backslash\{0\}$ such that $q^d = 1$ for some $d \in \mathbb{N}_{>2}$. So $q \neq \pm 1$. Recall that $e = \begin{cases} d & \text{if } d \text{ is odd,} \\ d/2 & \text{if } d \text{ is even.} \end{cases}$ and that $[e] = 0$.
The next proposition shows that in certain cases, there is no difference between the Verma modules $V_{\varepsilon,n}$ when q is or is not a root of unity.

Proposition 6.29. Let $n \in \mathbb{N}$ such that $0 \leq n < e-1$ and let V be a non-zero simple $\mathcal{U}_q(\mathfrak{sl}(2))$-module with $dim(V) < e$. Then $V \simeq V_{\varepsilon,n}$, $\varepsilon = \pm 1$.

This proof is exactly the same as when q is not a root of unity, theorem 6.5, since $1, q^2, \ldots, q^n$ are all distinct scalars when $n < e$. The biggest difference when q is a root of unity are when we look at modules of dimension $\geq e$.

Proposition 6.30. There are no simple finite-dimensional $\mathcal{U}_q(\mathfrak{sl}(2))$-modules of dimension $> e$.

Proof. We will show that if V is a $\mathcal{U}_q(\mathfrak{sl}(2))$-module, $dim(V) > e$, then we can find a non-zero submodule $V' \subset V$. Thus showing that V cannot be simple.
Suppose there is an eigenvector $0 \neq v \in V$ of K such that $Fv = 0$, then the vector space generated by v is a submodule. So, let V' be generated by the vectors $v, Ev, \ldots, E^{e-1}v$, then V' is stable under the action of K, $E(E^i v) = E^{i+1}v \in V$ if $i+1 < e-1$ and $E(E^{e-1}v) = E^e v = \alpha v$ for some $\alpha \in \mathbb{C}$ as $E^e \in Z(\mathcal{U}_q(\mathfrak{sl}(2)))$ by lemma 6.13. And since $Fv = 0$, and $F(E^i v) = E^i(Fv) - [i]\frac{q^{-(i-1)}K - q^{i-1}K^{-1}}{q-q^{-1}} E^{m-1}v$, V' is also stable under F, hence $V' \subset V$ is a non-zero submodule that is not V itself.
Now suppose that there is no non-zero eigenvector v of K such that $Fv = 0$, so $Fv \neq 0$. Then we claim that V' generated by $v, Fv, \ldots, F^{e-1}v$ is a non-zero submodule of V. It again suffices to show that V' is stable under E, F and K. First, V' is again obviously stable under K and since $F(F^i v) = F^{i+1}v$ and $F^e v = \alpha v$ for some $0 \neq \alpha \in \mathbb{C}$, V' is also stable under F.

This time $\alpha \neq 0$, else $F(F^{e-1}v) = 0$ with $F^{e-1}v$ an eigenvector of K, contradicting our assumption. Lastly, for $i > 0$,

$$E(F^i v) = EF(F^{i-1}v) = (C_q - \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2}(F^{i-1}v) = \beta F^{i-1}v - \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2}(F^{i-1}v),$$

thus $E(F^i v) \in V'$ for $i > 0$, with C_q the q-Casimir element and $\beta \in \mathbb{C}$. Since $Ev = \alpha^{-1}EF^e v$, it follows that $Ev \in V'$. Hence $V' \subset V$ is a non-zero submodule that is not V itself. So, for both cases we could construct a non-zero submodule that is not V, hence V cannot be a simple module when $dim(V) > e$. □

The last part of this section will be dedicated to show that we can still describe all simple $\mathcal{U}_q(\mathfrak{sl}(2))$-modules of dimension e. Immediately notice that proposition 6.3 does not hold in this case. Since the proof relies on the sequence $(X^n v)_{n\in\mathbb{N}}$ to give distinct eigenvalues for K for all $n \in \mathbb{N}$. But, by lemma 6.13, $E^e \in Z(\mathcal{U}_q(\mathfrak{sl}(2)))$, thus $E^e v$ has the same eigenvalue for K as v. Thus such a sequence need not lead to a highest weight vector.
So, to find all possible simple $\mathcal{U}_q(\mathfrak{sl}(2))$-modules of dimension e, we will first construct two e-dimensional vector spaces, then show that any simple $\mathcal{U}_q(\mathfrak{sl}(2))$-module of dimension e is isomorphic to one of those two vector spaces.
First, set $a, b, \lambda \in \mathbb{C}$, $\lambda \neq 0$ and let $V(\lambda, a, b)$ be a vectors space of dimension e with basis $\{v_0, \dots, v_{e-1}\}$. For $0 \leq p < e-1$, set

$$Kv_p = \lambda q^{-2p} v_p, \qquad Ev_{p+1} = \left(\frac{q^{-p}\lambda - q^p\lambda^{-1}}{q-q^{-1}}[p+1] + ab\right)v_p, \qquad Fv_p = v_{p+1},$$

and $Ev_0 = av_{e-1}$, $Fv_{e-1} = bv_0$ and $Kv_{e-1} = \lambda q^{-2(e-1)}v_{e-1}$. These formulas induce a $\mathcal{U}_q(\mathfrak{sl}(2))$-module structure on $V(\lambda, a, b)$.
Secondly, let $\mu, c \in \mathbb{C}$, $\mu \neq 0$ and let $\tilde{V}(\mu, c)$ be another vector space of dimension e with basis $\{v_0, \dots, v_{e-1}\}$. For $0 \leq p < e-1$, set

$$Kv_p = \mu q^{2p} v_p, \qquad Fv_{p+1} = \frac{q^{-p}\mu^{-1} - q^p\mu}{q-q^{-1}}[p+1]v_p, \qquad Ev_p = v_{p+1},$$

and $Fv_0 = 0$, $Ev_{e-1} = cv_0$ and $Kv_{e-1} = \mu q^{-2}v_{e-1}$. These formulas induce a $\mathcal{U}_q(\mathfrak{sl}(2))$-module structure on $\tilde{V}(\mu, c)$.

Theorem 6.31. *Any simple $\mathcal{U}_q(\mathfrak{sl}(2))$-module of dimension e is isomorphic to one of the following modules:*

1. *$V(\lambda, a, b)$ with $b \neq 0$,*
2. *$V(\lambda, a, 0)$ with $\lambda \neq \pm q^{j-1}$, $1 \leq j \leq e-1$.*

Proof. Simple calculations will show that the actions of E, F, K on the vector spaces $V(\lambda, a, b)$ and $\tilde{V}(\mu, c)$ induce a $\mathcal{U}_q(\mathfrak{sl}(2))$-module structure. So, we at least know that the given vector spaces are modules. More precisely, lemma 5.3 shows that any vector space with basis given by $\{F^n v\}_{0\leq n<e}$ and $\{E^n v\}_{0\leq n<e}$ have a $\mathcal{U}_q(\mathfrak{sl}(2))$-module with the actions of E, F, K given by the formulas of $V(\lambda, a, b)$ and $\tilde{V}(\mu, c)$ respectively. So, to show that any simple $\mathcal{U}_q(\mathfrak{sl}(2))$-module of dimension e is isomorphic to one of the two given above, it suffices to show that can always generate a basis $\{v, Ev, \dots, E^{e-1}v\}$ for some vector v of a module V.
Let V be an e-dimensional simple $\mathcal{U}_q(\mathfrak{sl}(2))$-module. Then, by an earlier remark, we cannot assume that it has a highest weight vector. But, since $\mathbb{C}$ is algebraically closed, we still have that
$\exists 0 \neq v \in V \quad (Kv = \lambda v)$. And $KFv = q^{-2}FKv = q^{-2}\lambda Fv$, so Fv and Ev are also eigenvectors of K. Since $0, q^{-2}, \dots, q^{-e+1}$ are distinct numbers and $F^e, E^e \in Z(\mathcal{U}_q(\mathfrak{sl}(2)))$, it follows that the sequence

$(F^n v)_{n\in\mathbb{N}}$ gives us e distinct eigenvectors of K, if $F^i v \neq 0$ when $i < e$. If $F^i v = 0$ for some $0 \leq i < e$, then we still have e distinct eigenvectors from the sequence $(E^n v)_{n\in\mathbb{N}}$ if, again, $E^j v \neq 0$ if $j < e$.
Note, if $F^i v = 0$ and $E^j v = 0$ for some $0 \leq i < e$, $0 \leq j < e$, then $\{v, Ev, \dots, E^j v, Fv, \dots, F^i v\}$ is a basis for V. Since, if these are not e linear independent vectors, it is a submodule of V. As $KEFv = EFKv = \lambda EFv \implies EFv = \alpha v$, $\alpha \in \mathbb{C}$.
Similarly, $FEv = \alpha' v$, $\alpha' \in \mathbb{C}$. So it follows that the vector space with basis $\{v, Ev, \dots, E^i v, Fv, \dots, F^j v\}$ is closed under the actions of F and E, hence it has to be equal to V.
Thus, we get three options.

1. We have a basis of V given by $\{v, Fv, \dots, F^{e-1}v\}$,
2. we have a basis of V given by $\{v, Ev, \dots, E^{e-1}v\}$,
3. we have a basis of V given by $\{v, Fv, \dots, F^i v, Ev, \dots, E^j v\}$.

This means that in the case of 1, we can change the basis to $\{v_0, \dots, v_{e-1}\}$ with $Fv_p = v_{p+1}$ and $Fv_{e-1} = F^e v_0 = bv_0$ for some $b \in \mathbb{C}$. In other words, we get that $V \simeq V(\lambda, a, b)$.
For the second and third case, note that we assume that $F^i v = 0$ for some $0 < i < e$. So, in both cases, we can look at the sequence $(E^n F^{i-1} v)_{n\in\mathbb{N}}$. This sequence contains e different eigenvectors of K, else V would not be simple by a previous remark. Thus, in both cases, we can construct a different basis, given by $\{F^{i-1}v, Fv^{i-2}v, \dots, Fv, v, Ev, \dots, E^{e-i}v\}$. This can be rewritten into $\{v_0, \dots, v_e\}$ with $Ev_p = v_{p+1}$ and $Fv_0 = 0$, $Ev_{e-1} = E^e v_0 = cv_0$ for some $c \in \mathbb{C}$. In other words, both cases are isomorphic to $\tilde{V}(\mu, c)$.
We claim that $\tilde{V}(\mu, c) \simeq V(\mu^{-1}, 0, c)$. This directly follows from the fact that the Cartan automorphism ω of lemma 5.2 is $\mathcal{U}_q(\mathfrak{sl}(2))$-linear with the actions defined on $\tilde{V}(\mu, c)$ and $V(\mu^{-1}, 0, c)$. So, this only leaves us with one possible option for a simple $\mathcal{U}_q(\mathfrak{sl}(2))$-module up to isomorphisms.
First, say $b \neq 0$. Then we see that $(F^n v_i)_{n\in\mathbb{N}}$ is a sequence of e different vectors for all $0 \leq i \leq e-1$, as $Fv_{e-1} = bv_0 \neq 0$. So, $V(\lambda, a, b)$ cannot have a non-trivial submodule in this case.
So, let $b = 0$, then $Fv_{e-1} = 0$. Then, if $Ev_i = 0$, it follows that $\{v_{i+1}, v_{i+2}, \dots, v_{e-1}\}$ generates a submodule of $V(\lambda, a, 0)$. Thus, $V(\lambda, a, 0)$ can only be a simple module when $Ev_{i+1} \neq 0$ for $0 \leq i < e-1$.
Now, since $b = 0$,

$$Ev_{i+1} = \frac{q^{-i}\lambda - q^i\lambda^{-1}}{q - q^{-1}}[p+1]v_i = 0 \iff q^{-i}\lambda - q^i\lambda^{-1} = 0 \iff \lambda = \pm q^i.$$

Thus $V(\lambda, a, 0)$ is a simple $\mathcal{U}_q(\mathfrak{sl}(2))$-module $\iff \lambda \neq \pm q^{j-1}$ for $1 \leq j \leq e-1$. □

7 A natural module for the modular double of $\mathcal{U}_q(\mathfrak{sl}(2))$

In this section we will finally introduce the modular double of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, $\mathcal{Q}$, together with a vector space $\mathcal{P}_\alpha$ on which we can define a natural $\mathcal{Q}$-module structure.
However, before we will introduce the modular double and its natural module space, we will reintroduce the Hopf algebra $\mathcal{U}_q(\mathfrak{sl}(2))$. This will be done to get the same form for $\mathcal{U}_q(\mathfrak{sl}(2))$ as is used in [1]. After reintroducing $\mathcal{U}_q(\mathfrak{sl}(2))$, we will show that our first definition will be a Hopf subalgebra of the new one. After reintroducing $\mathcal{U}_q(\mathfrak{sl}(2))$ and defining the $\mathcal{Q}$-module $\mathcal{P}_\alpha$. We will look at some properties of this module. So this section will mainly focus on the module $\mathcal{P}_\alpha$ itself. In the next section we will look at the other main subject of this thesis, the Clebsch-Gordan coefficients of $\mathcal{P}_\alpha$.

7.1 The Hopf $*$-algebra $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$

Before we are going to define our new module, we will look at another way to construct $\mathcal{U}_q(\mathfrak{sl}(2))$, as both [1] and [3] use a different construction for this Hopf algebra. After constructing this Hopf algebra, we will show that we can project our earlier definition of $\mathcal{U}_q(\mathfrak{sl}(2))$, definition 5.1, into the new one. In other words, we will show that the Hopf algebra constructed in [1] and [3] is a bigger Hopf algebra, but contains our earlier definition of $\mathcal{U}_q(\mathfrak{sl}(2))$ as a subalgebra.

Definition 7.1. The Hopf $*$-algebra $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ is the complex vector space generated by E, F, K and K^{-1} such that

$$KE = qEK, \qquad KF = q^{-1}FK, \qquad [\,E\,,\,F\,] = \frac{K^2 - K^{-2}}{q - q^{-1}},$$

with coproduct $\Delta\colon \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})) \to \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})) \otimes \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ given by

$$\Delta(K) = K \otimes K, \qquad \Delta(E) = E \otimes K + K^{-1} \otimes E, \qquad \Delta(F) = F \otimes K + K^{-1} \otimes F,$$

counit given by

$$\varepsilon(K) = \varepsilon(K^{-1}) = 1, \qquad \varepsilon(E) = \varepsilon(F) = 0,$$

antipode $S\colon \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})) \to \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ given by

$$S(K) = K^{-1}, \qquad S(E) = KEK^{-1}, \qquad S(F) = KFK^{-1}$$

and $*$-structure induced by the relations

$$K^* = K, \qquad E^* = E, \qquad F^* = F.$$

Notice that there are a lot of small things that are different when compared to definition 5.1 of $\mathcal{U}_q(\mathfrak{sl}(2))$. For example, the use of q instead of q^2 in the commuting relation with K, the bracket $[\,E\,,\,F\,]$ uses K^2 instead of K and the coproduct does not have $1 \otimes X$ terms for $X \in \mathfrak{sl}(2)$. However, the differences with the bracket $[\,E\,,\,F\,]$ will help us to show why $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ contains "more" elements than $\mathcal{U}_q(\mathfrak{sl}(2))$.

Proposition 7.2. Let $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ be as above and let $\mathcal{U}_q(\mathfrak{sl}(2))$ be generated by E', F', K' and K'^{-1} as in definition 5.1. Then, $\pi\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, given by

$$\pi(K') = K^2, \qquad \pi(E') = EK, \qquad \pi(F') = K^{-1}F,$$

is an injective Hopf $*$-algebra morphism. In other words, we can identify $\mathcal{U}_q(\mathfrak{sl}(2))$ with a subalgebra of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ via π.

Proof. It suffices to show that π is a well-defined Hopf algebra morphism for just E', F', K and K'^{-1} as those generate $\mathcal{U}_q(\mathfrak{sl}(2))$. First, $\pi(K'K'^{-1}) = K^2K^{-2} = 1$ and

$$\pi(K'E') = \pi(K')\pi(E') = K^2EK = q^2EKK^2 = \pi(q^2E'K'),$$
$$\pi(K'F') = K^2(K^{-1}F) = K^{-1}K^2F = q^{-2}K^{-1}FK^2 = \pi(q^{-2}F'K').$$

Also,

$$\pi([E', F']) = \pi\left(\frac{K' - K'^{-1}}{q - q^{-1}}\right) = \frac{K^2 - K^{-2}}{q - q^{-1}} = [E, F] = [EK, K^{-1}F] = [\pi(E'), \pi(F')].$$

So we see that π is at least an algebra morphism. We also have $\Delta(\pi(K')) = K^2 \otimes K^2 = \pi(K') \otimes \pi(K')$ and

$$\Delta(\pi(E')) = \Delta(E)\Delta(K) = EK \otimes K^2 + 1 \otimes EK = (\pi \otimes \pi) \circ \Delta(E')$$
$$\Delta(\pi(F')) = \Delta(K^{-1})\Delta(F) = K^{-1}F \otimes 1 + K^{-2} \otimes K^{-1}F = (\pi \otimes \pi) \circ \Delta(F),$$

so π is a bialgebra morphism as ε acts the same on $\mathcal{U}_q(\mathfrak{sl}(2))$ and $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$.
For the antipode S, notice that $\pi(E'K'^{-1}) = EK^{-1}$ and $\pi(K'F') = KF$, thus

$$\pi(S'(E')) = EK^{-1} = K^{-1}KEK^{-1} = S(EK) = S(\pi(E')),$$
$$\pi(S'(F')) = KF = KFK^{-1}K = S(K^{-1}F) = S(\pi(F')),$$

and $\pi(S'(K')) = S(K^2)$, with S' the antipode of $\mathcal{U}_q(\mathfrak{sl}(2))$. Thus we see that $\pi \circ S' = S \circ \pi$, showing that π is also a Hopf algebra morphism.
Recall that $\mathcal{U}_q(\mathfrak{sl}(2))$ also has the $*$-structure that sends the generators E, F, K, K^{-1} to itself. So, if we induce $\mathcal{U}_q(\mathfrak{sl}(2))$ with this $*$-structure, π turns into a Hopf $*$-algebra morphism.
The injectivity of π is obvious. And we can identify $\mathcal{U}_q(\mathfrak{sl}(2))$ with the subalgebra of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ generated by $K^2, K^{-2}, EK, K^{-1}F$, given that we induce $\mathcal{U}_q(\mathfrak{sl}(2))$ with the same $*$-structure as $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$. □

As a remark, all the theory we discussed on the Verma modules of $\mathcal{U}_q(\mathfrak{sl}(2))$ also works for $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, it will only differ on some constants. As $[E, F] = \frac{K^2-K^{-2}}{q-q^{-1}}$ and $EK = qKE$, it follows that we get the same formulas as in lemma 5.3, but with K^2 instead. Then we get the same formulas as lemma 6.4, but with λ^2 in the terms of E. So we can indeed redo all the proofs the same way, but with different values for λ.
Now, since the relations between K, E, F are different in $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ when compared to the relations of $\mathcal{U}_q(\mathfrak{sl}(2))$. It also follows that the quantum Casimir element C_q of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ is different than the one we defined in proposition 6.11 for $\mathcal{U}_q(\mathfrak{sl}(2))$.
Similar to proposition 6.11, note that

$$EF - FE - \frac{qK^2 + q^{-1}K^{-2}}{(q - q^{-1})^2} = -\frac{q^{-1}K^2 + qK^{-2}}{(q - q^{-1})^2}.$$

Definition 7.3. Let $C_q = -FE - \frac{qK^2+q^{-1}K^{-2}}{(q-q^{-1})^2}$, then $C_q = -EF - \frac{q^{-1}K^2+qK^{-2}}{(q-q^{-1})^2}$ and $C_q \in Z(\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ is called the *quantum Casimir element* of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$.

That C_q is central, follows from a similar proof as in proposition 6.11. Since we are already talking about the quantum Casimir element, we will sketch how we will use this element in the next section. This will also show why we care about reintroducing this specific element.
Recall that we said that one can use the Casimir element to compute the Clebsch-Gordan coefficients of the Verma modules of $\mathcal{U}(\mathfrak{sl}(2))$ and $\mathcal{U}_q(\mathfrak{sl}(2))$. This could be done via eigenfunction equations of the

Casimir element and the same reasoning will be used in the next section. In other words, we will see that the Clebsch-Gordan coefficients that will be constructed in the next section, can be computed via eigenfunctions equations for C_q in a similar sense as in section 6.3.
But, before we will talk about the Clebsch-Gordan coefficients, we will first look at a natural module for the modular double of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and some of its properties.

7.2 A short remark on unbounded operators

Since the actions of the generators E, F, K of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ will be given by unbounded operators on the module that we will construct, we will quickly look at some definitions and properties of unbounded operators, as the unboundedness of these operators will be one of the reasons why some of the statements or proofs will look a lot more complex when compared to previous statements on modules or tensor products of modules (e.g. section 3.3 or chapter 6).

Definition 7.4. Let X, Y be Banach spaces, then an *unbounded operator* from X to Y is a pair $(A, D(A))$ with $D(A) \subseteq X$ and $A\colon D(A) \to Y$ a linear operator.
The subspace $D(A) \subseteq X$ is known as the domain of A.

So we see that an unbounded operator on a Hilbert space $\mathcal{H}$, only differs from a bounded operator on $\mathcal{H}$ by the fact that it need not be able to act on all of $\mathcal{H}$. However, this also means that we need to be careful when we use an unbounded operator A or have A act multiple times on a single element. Since we need to be certain that every element on which we let A act, is an element of its domain $D(A)$.
Now, the operators that we will be using later on, will not just be unbounded operators. To be precise, they will be unbounded positive self-adjoint operators. So, we will also note a few facts on these types of operators.
Let A be an unbounded operator on some Hilbert space $\mathcal{H}$. Then, A is called *densely defined* if $D(A) \subseteq \mathcal{H}$ dense.

Definition 7.5. Let A be a densely defined operator on a Hilbert space $\mathcal{H}$. Then, the *adjoint* of A is defined as the operator $(A^*, D(A^*))$ with

$$D(A^*) = \{y \in \mathcal{H} \,|\, \exists g \in \mathcal{H} \ (\forall x \in D(A) \ (x, g) = (Ax, y))\}$$

and A^* is defined as $A^*y = g$ for $y \in D(A^*)$.

So we see that the adjoint operator A^* is similar to the adjoint operator of a bounded operator. But we once again need to be careful of the domain for which A^* is defined.
Now, A is called *self-adjoint* if we have that $A^* = A$ and $D(A) = D(A^*)$. Note that we do need that A is densely defined, else the adjoint is not defined.

We ends this short remark with a remark on when an operator is positive and with a statement on when an operator has a self-adjoint extension. This last statement will be given due to the fact that the representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ that we will define will have self-adjoint extensions.

Definition 7.6. The spectrum $\sigma(A)$ of A is defined as

$$\sigma(A) = \{\lambda \in \mathbb{C} \,|\, A - \lambda I \text{ has no inverse}\}$$

Do note that if A is an unbounded operator, we have that the inverse B of $A - \lambda I$ need not be unbounded. However, we do need that $Bx \in D(A)$ for any $x \in \mathcal{H}$, as we have $x = (A - \lambda I)Bx = A(Bx) - \lambda I(Bx)$.
Now, if A is a self-adjoint operator such that $\sigma(A) \subseteq [0, \infty)$, then we call A a *positive self-adjoint operator*. Note that it is unnecessary to keep calling a positive operator self-adjoint, as it is self-adjoint

by definition. However, we call them self-adjoint positive operators to make sure that it is obvious that the operators are self-adjoint.
Lastly, if A is an unbounded densely defined self-adjoint positive operator, we can construct a positive self-adjoint extension B that will be defined on the closure $\overline{D(A)} = \mathcal{H}$. Note, if $x \in D(A)$, then $Bx = Ax$. Since we will not use these extensions ourselves, but are sometimes interested in the existence of these extensions, just knowing when they exist suffices for our case.

7.3 A short remark on the Fourier-transform

Besides the use of unbounded operators, the module that we will study will also make use of the Fourier transform. So we will also shortly talk about the Fourier-transform before talking about the $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-module.
We will give the definition of the Fourier-transform together with some small facts and useful theorems for our setting.

Definition 7.7. Let $f \in L^1(\mathbb{R})$, then the *Fourier transform* of f is the function $\tilde{f}\colon \mathbb{R} \to \mathbb{C}$ given by

$$\tilde{f}(\omega) = \int_{\mathbb{R}^d} e^{-2\pi i\omega x} f(x)dx, \quad \omega \in \mathbb{R}^d.$$

The Fourier transform $\tilde{f}$ is again a continuous function and it even vanishes at infinity. So, $\tilde{f} \in C_0(\mathbb{R}^d)$. Moreover, we have the following inversion formula:

Theorem 7.8. *Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, such that $\tilde{f} \in L^1(\mathbb{R})$, then*

$$f(x) = \int_{\mathbb{R}} e^{2\pi i\omega x}\tilde{f}(\omega)d\omega,$$

for almost all $x \in \mathbb{R}$.

Note, the inversion theorem also holds when $f \in S(\mathbb{R})$. And if $f \in S(\mathbb{R})$, we even get that $f(x) = \int_{\mathbb{R}} e^{2\pi i\omega x}\tilde{f}(\omega)d\omega$. Now, since $S(\mathbb{R}) \subset L^2(\mathbb{R})$ is dense, one can extend the Fourier transform restricted to $S(\mathbb{R})$ to an isometry to all of $L^2(\mathbb{R})$. Due to this fact, we will consider the given Fourier-transform as a bijective unitary operator on $L^2(\mathbb{R})$.
Lastly, we have the following useful theorem and lemma that show when the Fourier transform of a function is analytic.

Theorem 7.9 (Payley-Wiener)**.** *Let $f \in L^2(\mathbb{R})$, $a_\pm > 0$, then $(e^{2\pi x a_+} + e^{-2\pi x a_-})f \in L^2(\mathbb{R}) \iff \tilde{f}$ has an analytic continuation to $\{\omega \in \mathbb{C} \,|\, Im(\omega) \in (a_-, a_+)\}$ such that $\forall \omega_2 \in (-a_-, a_+)$*

$$\tilde{f}(\cdot + i\omega_2) \in L^2(\mathbb{R}) \quad and \quad sup_{\omega_2 \leq b} \int_{\mathbb{R}} |\tilde{f}(\omega_1 + i\omega_2)|^2 d\omega_1 < \infty \quad \forall b \in (-a_-, a_+).$$

Lemma 7.10. Let $f \in S(\mathbb{R}) = \{f \in C^\infty(\mathbb{R},\mathbb{C}) \,|\, \forall \alpha,\beta \in \mathbb{N} \left(||f||_{\alpha,\beta} = \sup_{x\in\mathbb{R}} |x^\alpha(\partial^\beta f)(x)| < \infty\right)$. The following are equivalent:

i) $f = F|_{\mathbb{R}}$ of some function $F\colon D \to \mathbb{C}$, $D \subseteq \mathbb{C}$, that is meromorphic in $\{z \in \mathbb{C} \,|\, Im(z) \in (-a_-, a_+)\}, a_\pm > 0$, with finitely many poles in the upper and lower half plane $I_\pm = \{z \in \mathbb{C} \,|\, \pm Im(z) > 0\}$ and every map $F_y(x) = F(x+iy)$, $y \in (-a_-, a_+)$, is of rapid decrease.

ii) $\tilde{f}(\omega)$ has the two following asymptotic behaviours if $\omega \to \pm\infty$:

$$\tilde{f}(\omega) = -2\pi i \sum_{z\in I_-} e^{-2\pi i z\omega} Res(F(z)) + \tilde{f}_{a_+}(\omega) \qquad \omega \to \infty,$$

$$\tilde{f}(\omega) = +2\pi i \sum_{j\in I_+} e^{-2\pi i z\omega} Res(F(z)) + \tilde{f}_{a_-}(\omega) \qquad \omega \to -\infty,$$

with $\tilde{f}_{a_\pm}$ decaying faster than $e^{-2\pi a|\omega|}$ $\forall a \in (-a_-, a_+)$ as $\omega \to \pm\infty$.

7.4 A family of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-modules

In this part we will finally introduce the modular double of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and a natural module of this algebra. But, before we will define the modular double of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and construct our desired module, we will first fix some variables. These variables will be fixed for the rest of this paper, unless stated otherwise.
Let $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ be as above and fix

$$q = e^{\pi i b^2} \text{ for } b \in (0,1) \cap \mathbb{R}\backslash\mathbb{Q}, \qquad Q = b + b^{-1}, \qquad \alpha = Q/2 + is,\ s \in \mathbb{R}.$$

Then we see that q is not a root of unity and $|q| = 1$. Thus our choice of $*$ is valid according to theorem 5.10. Furthermore, set $\tilde{q} = e^{\pi i b^{-2}}$ and let $\tilde{K}^{\pm 1}$, $\tilde{E}$ and $\tilde{F}$ be the generators of $U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$. Then $U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ is another quantum enveloping algebra and we get the following definition of the modular double by Faddeev [2]:

Definition 7.11. Let $\mathcal{Q} = \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})) \otimes U_{\tilde{q}}(\mathfrak{sl}(2),\mathbb{R})$ be a tensor algebra. Then $\mathcal{Q}$ is called *the modular double* of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$.

Now, the elements of $\mathcal{Q}$ are given by $U \otimes \tilde{U}$ for $U \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and $\tilde{U} \in U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$. Then $U \otimes 1 \in \mathcal{Q}$ is the projection of $U \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ into $\mathcal{Q}$. To make notation cleaner from this point on, we will just write $U \in \mathcal{Q}$ instead of $U \otimes 1$ for the projection into $\mathcal{Q}$. This will be done as we will only work with $\mathcal{Q}$-modules from this point onwards.
Moreover, note that if $U \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, $\tilde{U} \in U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$, then we get $U\tilde{U} = \tilde{U}U$ in $\mathcal{Q}$. So the projections of the algebras $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and $U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ commute with each other in $\mathcal{Q}$. However, they are both not contained in $Z(\mathcal{Q})$, since one could easily find an element $V \otimes 1$ or $1 \otimes \tilde{V}$ in $\mathcal{Q}$ that would not commute with some U or $\tilde{U}$ respectively. Thus the commutative property between $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and $U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ is a special property of these two Hopf $*$-algebras.
Now, since U,$\tilde{U} \in \mathcal{Q}$ commute, it also follows that if V is a $\mathcal{Q}$-module, the actions of U and $\tilde{U}$ commute on V. In other words, $U\tilde{U}v = \tilde{U}Uv$ for any $v \in V$.
As we can also choose to only act with elements of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ or only elements of $U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$, it even follows that any $\mathcal{Q}$-module will be both a $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and $U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$-module. So we can simplify our notation even more, and just talk about elements of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ or $U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ instead of elements of $\mathcal{Q}$.
Furthermore, due to how our actions will be defined. It suffices to only talk about the action of $U_(\mathfrak{sl}(2,\mathbb{R}))$, since every proof will also be true for the actions of $U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$. This will partly be due to the fact that the only change between the two Hopf $*$-algebras is $b \mapsto b^{-1}$ and the only difference between the actions will also be $b \mapsto b^{-1}$. Thus one can basically replace b with b^{-1} in every proof or statement made for elements of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ to get the proof or statement for the dual $U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$.
This is also precisely what we will do. So we will almost always only talk about the action of $U \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and not talk about the action of $\tilde{U} \in U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$. Due to the fact that almost all proofs and statements will be symmetric for the change $b \mapsto b^{-1}$.

Before we will start with the construction of the $\mathcal{Q}$-modules, we will give one last remark on the modular double. The module that we will look at, was originally a reason for Faddeev to unify the algebras $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and $U_{\tilde{q}}(\mathfrak{sl}(2),\mathbb{R})$ into one Hopf algebra. The reason for this is that the modules $\mathcal{P}_\alpha$ that we will construct below have a nice self-duality property. It is both a $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and $U_{\tilde{q}}(\mathfrak{sl}(2),\mathbb{R})$-module by construction. Furthermore, the module $\mathcal{P}_\alpha$ will not change when we replace b with b^{-1} in its definition. Thus the module $\mathcal{P}_\alpha$ will be self dual in the sense of replacing b with b^{-1}.
However, we will immediately consider the modules $\mathcal{P}_\alpha$ as $\mathcal{Q}$-modules, instead of having them also be $\mathcal{Q}$-modules as a consequence of this self-duality. The reason for this is that the construction of the module $\mathcal{P}_\alpha$ will feel more natural this way in a mathematical sense, since we will see that $\mathcal{P}_\alpha$ will be the largest subspace of $L^2(\mathbb{R})$ on which the actions of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and $U_{\tilde{q}}(\mathfrak{sl}(2),\mathbb{R})$ are well-defined.

But to show this, we will need to use the fact that both U and $\tilde{U}$, for $U \in \{E, F, K\}$, $\tilde{U} \in \{\tilde{E}, \tilde{F}, \tilde{K}\}$, have to be able to act on $\mathcal{P}_\alpha$ at the same time.

Now, we will finally be looking at a certain "well-behaved" family of $\mathcal{Q}$-modules. In the sense that, the representations π coming from these modules, gives us self-adjoint integrable operators $\pi(E), \pi(F)$ and $\pi(K)$. Similarly, the actions of $\tilde{E}, \tilde{F}$ and $\tilde{K}$ will also be given by self-adjoint integrable operators. So the actions of $\mathcal{Q}$ will be generated by self-adjoint operators.
As was said above, to not constantly refer to both triples of generators of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and $U_{\tilde{q}}(\mathfrak{sl}(2),\mathbb{R})$ separately, we will in general only refer to the elements of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ from this point onwards. Again, do note that anything that will be said about the action of $U \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, will also hold for $\tilde{U}$, but one just needs to replace every q with $\tilde{q}$ or, equivalently, every b with b^{-1}.

Since we have $U^* = U$ for $U \in \{K, E, F\}$, we need $\pi(E), \pi(F)$ and $\pi(K)$ to be self-adjoint operators to have π to be a $*$-representation. Furthermore, there is no representation on $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ that is generated by bounded self-adjoint operators. So, the actions of K, E, F that we will construct, will be given by unbounded operators. Due to this, some care is needed to end up with an interesting $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-module.
In [13] a notion of "well-behavedness" for $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-modules generated by self-adjoint operators was defined. Here they define when a representation is called integrable and it leads to a natural notion for well-behavedness for $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-modules. We will not be too concerned with all of the details of this definition, but we will try to give enough details to understand what it means to be an irreducible integrable representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$.

The $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-module that we will construct, will be dependent on one parameter, α, and it will be a well-defined module for all allowed α. In other words, the module can be described as a one-parameter family. Thus, we will not just construct one module, but we will construct a whole family of modules, that are all similar, but differ due to the variable α.
Also, the actions of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ on this module will be given via finite difference operators T_x^{ia}, the *shift operator*, and the operator $\mathbf{x}$. These are unbounded self-adjoint positive operators that act on functions $f \in L^2(\mathbb{R})$, with

$$\mathbf{x}f(x) = xf(x) \qquad \text{and} \qquad T_x^{ia} f(x) = f(x + ia).$$

Note, $f(x) \in Dom(T_x^{ia})$, i.e. T_x^{ia} can only act on $f(x) \in L^2(\mathbb{R})$, if f has an analytic continuation to $\{z \in \mathbb{C} \,|\, im(z) \in [a_-, a_+]\}$, $a_- \leq 0, a_+ \geq 0$.

Now, our one-parameter class of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-modules will be denoted by $\mathcal{P}_\alpha$ and be given by

$$\begin{aligned} \mathcal{P}_\alpha &= \{f : \mathbb{C} \to \mathbb{C} \,|\, f \text{ is entire and } \tilde{f} \text{ is meromorphic on } \mathbb{C} \text{ with possible poles } \omega \in \Omega_\pm\}, \\ \Omega_\pm &= \{\pm s \pm i\left(\frac{Q}{2} + nb + mb^{-1}\right),\ n, m \in \mathbb{N}\}. \end{aligned}$$

So, if $\omega_\pm \in \Omega_\pm$, then we get that

$$\begin{aligned} \omega_+ &= s + i\left(\frac{Q}{2} + nb + mb^{-1}\right) = i(-\alpha + Q + nb + mb^{-1}) = i(-\alpha + (n+1)b + (m+1)b^{-1}), \\ \omega_- &= -s - i\left(\frac{Q}{2} + nb + mb^{-1}\right) = i(\alpha - Q - nb - mb^{-1}) = i(\alpha - (n+1)b - (m+1)b^{-1}), \end{aligned}$$

for $n, m \in \mathbb{N}$. The last form will be useful for us to show that $\mathcal{P}_\alpha$ is a module. For some details on entire and meromorphic functions, see Appendix B: Complex Functions. In short, if $f \in \mathcal{P}_\alpha$, then f is analytic on all of $\mathbb{C}$ and its Fourier transform $\tilde{f}$ is analytic on $\mathbb{C} \setminus (\Omega_+ \cup \Omega_-)$.

At the moment $\mathcal{P}_\alpha$ is just a vector space of functions. We can induce a module structure on this space by having E, F, K act on functions $f \in \mathcal{P}_\alpha$, $x \in \mathbb{R}$, by

$$\begin{aligned}
Kf(x) &= T_x^{\frac{ib}{2}} f(x) = f(x + \frac{ib}{2}),\\
Ef(x) &= e^{2\pi bx} \frac{e^{\pi ib(Q-\alpha)} T_x^{\frac{ib}{2}} - e^{-\pi ib(Q-\alpha)} T_x^{\frac{-ib}{2}}}{e^{\pi ib^2} - e^{-\pi ib^2}} f(x) = e^{2\pi bx} \frac{e^{\pi ib(Q-\alpha)} f(x + \frac{ib}{2}) - e^{-\pi ib(Q-\alpha)} f(x - \frac{ib}{2})}{e^{\pi ib^2} - e^{-\pi ib^2}}\\
Ff(x) &= e^{-2\pi bx} \frac{e^{\pi ib(Q-\alpha)} T_x^{-\frac{ib}{2}} - e^{-\pi ib(Q-\alpha)} T_x^{\frac{ib}{2}}}{e^{\pi ib^2} - e^{-\pi ib^2}} f(x) = e^{-2\pi bx} \frac{e^{\pi ib(Q-\alpha)} f(x - \frac{ib}{2}) - e^{-\pi ib(Q-\alpha)} f(x + \frac{ib}{2})}{e^{\pi ib^2} - e^{-\pi ib^2}}.
\end{aligned}$$

To make all of this easier to write, denote

$$[x]_b = \frac{\sin(\pi bx)}{\sin(\pi b^2)}, \qquad d_x = \frac{1}{2\pi} \frac{d}{dx}, \qquad [d_x + a]_b = \frac{e^{\pi iba} T_x^{\frac{ib}{2}} - e^{-\pi iba} T_x^{\frac{-ib}{2}}}{e^{\pi ib^2} - e^{-\pi ib^2}},$$

then we can simplify the actions of E, F, K to

$$Kf(x) = T_x^{\frac{ib}{2}} f(x), \qquad Ef(x) = e^{2\pi bx}[d_x + Q - \alpha]_b f(x), \qquad Ff(x) = e^{-2\pi bx}[-d_x + Q - \alpha]_b f(x),$$

which in turn gives rise to a representation $\pi_\alpha \colon \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})) \to End(\mathcal{P}_\alpha)$ by

$$\pi_\alpha(K) = T_x^{\frac{ib}{2}}, \qquad \pi_\alpha(E) = e^{2\pi bx}[d_x + Q - \alpha]_b, \qquad \pi_\alpha(F) = e^{-2\pi bx}[-d_x + Q - \alpha]_b.$$

Recall that we get the actions of $U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ by replacing b with b^{-1} in the above given actions. To summarise:

Definition 7.12. Let $q = e^{\pi ib^2}$, $b \in (0,1) \cap \mathbb{R} \setminus \mathbb{Q}$, $Q = b + b^{-1}$ and $\alpha = Q/2 + is$ for $s \in \mathbb{R}$. Then, we have a natural family of $\mathcal{Q}$-modules given by

$$\begin{aligned}
\mathcal{P}_\alpha &= \{f : \mathbb{C} \to \mathbb{C} \,|\, f \text{ is entire and } \tilde{f} \text{ is meromorphic on } \mathbb{C} \text{ with possible poles } \omega \in \Omega_\pm\},\\
\Omega_\pm &= \{\pm s \pm i\left(\frac{Q}{2} + nb + mb^{-1}\right),\ n, m \in \mathbb{N}\},
\end{aligned}$$

with the actions of $E, F, K \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and $\tilde{E}, \tilde{F}, \tilde{K} \in U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ given by

$$\begin{aligned}
\pi_\alpha(K) &= T_x^{\frac{ib}{2}}, & \pi_\alpha(E) &= e^{2\pi bx}[d_x + Q - \alpha]_b, & \pi_\alpha(F) &= e^{-2\pi bx}[-d_x + Q - \alpha]_b.\\
\pi_\alpha(\tilde{K}) &= T_x^{\frac{ib^{-1}}{2}}, & \pi_\alpha(\tilde{E}) &= e^{2\pi b^{-1}x}[d_x + Q - \alpha]_{b^{-1}}, & \pi_\alpha(\tilde{F}) &= e^{-2\pi b^{-1}x}[-d_x + Q - \alpha]_{b^{-1}}.
\end{aligned}$$

To show that π_α is a well-defined representation, equivalently that $\mathcal{P}_\alpha$ is a well-defined $\mathcal{Q}$-module, we need to check two things.

Lemma 7.13. i) if $f \in \mathcal{P}_\alpha$, then Ef, Ff, $Kf \in \mathcal{P}_\alpha$, and

ii) E, F, K, K^{-1} generate a $\mathcal{Q}$-module structure on $\mathcal{P}_\alpha$ with the above defined actions.

In other words, $\mathcal{P}_\alpha$ has a well-defined module structure and $\pi_\alpha \colon \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})) \to End(\mathcal{P}_\alpha)$ is a well-defined representation.

Proof. Note, the operator T_x^{ia} gets mapped to the operator $e^{-a\omega}$ by the Fourier transform in the sense that

$$\widetilde{T_x^{ia} f}(\omega) = \int_{\mathbb{R}} e^{-2\pi i\omega x} f(x + ia)dx = \int_{\mathbb{R}} e^{-2\pi i\omega(x-ia)} f(x)dx = e^{-2\pi\omega a}\tilde{f}(\omega),$$

so $\pi_\alpha(K)$ gets mapped to the operator $e^{-\pi b\omega}$. Similarly, $\pi_\alpha(E)$ and $\pi_\alpha(F)$ get mapped to the operators $[-i\omega+\alpha]_b T_\omega^{ib}$ and $[i\omega+\alpha]_b T_\omega^{-ib}$ respectively. As,

$$\begin{aligned}(e^{\pi ib^2}-e^{-\pi ib^2})\widetilde{\pi_\alpha}(E)\tilde{f}(\omega) &= \int_{\mathbb{R}} e^{-2\pi i\omega x}e^{2\pi bx}\left(e^{\pi ib(Q-\alpha)}f(x+\frac{ib}{2})-e^{-\pi ib(Q-\alpha)}f(x-\frac{ib}{2})\right)dx\\ &= \int_{\mathbb{R}} e^{-2\pi i(\omega+ib)(x-\frac{ib}{2})}e^{\pi ib(Q-\alpha)}f(x)dx-\int_{\mathbb{R}} e^{-2\pi i(\omega+ib)(x+\frac{ib}{2})}e^{-\pi ib(Q-\alpha)}f(x)dx\\ &= \left(e^{-\pi b(\omega+ib)}e^{\pi ib(Q-\alpha)}-e^{\pi b(\omega+ib)}e^{-\pi ib(Q-\alpha)}\right)\int_{\mathbb{R}} e^{-2\pi i(\omega+ib)x}f(x)dx\\ &= \left(-e^{-\pi ib(-i\omega+\alpha)}+e^{\pi ib(-i\omega+\alpha}\right)\int_{\mathbb{R}} e^{-2\pi i(\omega+ib)x}f(x)dx,\end{aligned}$$

with $\widetilde{\pi_\alpha}(U)$ denoting the Fourier-transformed action of $U\in\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ on $\tilde{f}(\omega)$. A similar calculation shows that $\pi_\alpha(F)$ gets send to the operator $[i\omega+\alpha]_b T_\omega^{-ib}$ by the Fourier-transform.
Now, $f\in\mathcal{P}_\alpha$, so f is an entire function, so $T_x^{ia}f(x)=f(x+ia)$ will still be an entire function. Similarly, since e^z, $z\in\mathbb{C}$, is an entire function, $e^x f(x)$ will be an entire function. So it follows that Kf, Ef, Ff are at least entire functions.
Secondly, multiplying a meromorphic function g with an entire function h, gives us a meromorphic function hg with the same poles as g. By the above, we see that $\pi_\alpha(K),\pi_\alpha(E),\pi_\alpha(F)$ get send to operators that multiply $\tilde{f}(\omega)$ with entire functions, as $\sin(z)$ is also an entire function, and shift ω to $\omega\pm ib$.
If $x\in\Omega_\pm$, so x is a pole of $\tilde{f}$, then $x\mp ib$ is a pole of $T_x^{\pm ib}\tilde{f}$. To show that the Fourier transform of Ef, Ff, Kf still only have poles in $\Omega_\pm$, we need to check that if $x\notin\Omega_\pm$, then $\pi_\alpha(\tilde{U})f(x)$ is well-defined in $\mathbb{C}$, for $U\in\{E,F,K\}$.
Since K gets send to the operator $e^{-\pi b\omega}$. It follows that $\widetilde{\pi_\alpha}(K)\tilde{f}(\omega)$ won't have poles outside of $\Omega_\pm$. It even has the same poles as $\tilde{f}(\omega)$. Note, every possible pole is of the form $i(-\alpha\pm(nb+mb^{-1}))$, $n,m\in\mathbb{N}_{>0}$ and E,F both get send to a multiple of the shift operator $T_\omega^{\pm ib}$ by the Fourier transform. Since, $i(-\alpha+mb^{-1})+ib\in\Omega_+$, $m\in\mathbb{N}_{>0}$ and $\omega_m=i(-\alpha+mb^{-1})\notin\Omega_\pm$. We see that ω_m is a possible pole of $\widetilde{\pi_\alpha}(E)\tilde{f}(\omega)$, but should not be a possible pole if $Ef\in\mathcal{P}_\alpha$. Also, ω_m is the only point such that $\omega_m\notin\Omega_\pm$, but $\omega_m+ib\in\Omega_+$. In other words, ω_m is the only point that could be a pole of $\widetilde{\pi_\alpha}(E)\tilde{f}(\omega)$, which should not be a pole if π_α is a representation.
Similarly, we get that $\omega'_m=i(\alpha-mb^{-1})\notin\Omega_\pm$, $m\in\mathbb{N}_{>0}$, but $\omega'_m-ib\in\Omega_-$. So, ω'_m is a possible pole of $\widetilde{\pi_\alpha}(F)\tilde{f}(\omega)$, but should not be a pole if π_α is a representation. This is again the only point such that $\omega'_m\notin\Omega_\pm$, but $\omega'_m-ib\in\Omega_-$. Thus there are two cases that we are concerned with and we want to know what the values of $\widetilde{\pi_\alpha}(E)\tilde{f}(\omega_m)$ and $\widetilde{\pi_\alpha}(F)\tilde{f}(\omega'_m)$ are. As these values could be ∞, but they should have values in $\mathbb{C}$.
Recall, $[x]_b=\frac{\sin(\pi bx)}{\sin(\pi b^2)}$, thus $[x]_b=0\iff x=nb^{-1}$, $n\in\mathbb{Z}$. It follows that $[-i\omega+\alpha]_b T_\omega^{ib}$ is the zero-operator in $\omega=\omega_m$ and $[i\omega+\alpha]_b T_\omega^{-ib}$ is the zero-operator in $\omega=\omega'_m$. So, $\widetilde{\pi_\alpha}(E)\tilde{f}(\omega_m)=0$, $\widetilde{\pi_\alpha}(F)\tilde{f}(\omega'_m)=0\ \forall f\in\mathcal{P}_\alpha$. Thus, ω_m is not a possible pole of $\widetilde{\pi_\alpha}(E)\tilde{f}$ and ω'_m is not a possible pole of $\widetilde{\pi_\alpha}\tilde{f}$. Hence, $Ef, Ff\in\mathcal{P}_\alpha$.
It follows that the Fourier transform sends Ef, Ff, Kf to meromorphic functions with possible poles in $\Omega_\pm$. So, $\pi_\alpha(U)\colon\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))\to End(\mathcal{P}_\alpha)$ for $U\in\{E,F,K\}$.
By $i)$ we know that the actions of E,F,K,K^{-1} are well-defined on $\mathcal{P}_\alpha$, so the only thing left to check is that the relations between E,F,K,K^{-1} are still satisfied when considered as operators on $\mathcal{P}_\alpha$.
Let $f\in\mathcal{P}_\alpha$, then

$$\pi_\alpha(KE)f(x)=e^{2\pi b(x+\frac{ib}{2})}\frac{e^{\pi ib(Q-\alpha)}f(x+ib)-e^{-\pi ib(Q-\alpha)}f(x)}{e^{\pi ib^2}-e^{-\pi ib^2}}=e^{\pi ib^2}\pi_\alpha E\pi_\alpha Kf(x)=q\pi_\alpha(EK)f(x),$$

a similar calculation shows that $\pi_\alpha(KF) = \pi_\alpha(q^{-1}FK)$. Lastly, note

$$\begin{aligned}\pi_\alpha(E)\pi_\alpha(F) &= \frac{-e^{-\pi ib^2}T_x^{ib} - e^{\pi ib^2}T_x^{-ib} + e^{2\pi ib(Q-\alpha)-\pi ib^2} + e^{-2\pi ib(Q-\alpha)+\pi ib^2}}{(q-q^{-1})^2} \\ &= \frac{-q^{-1}T_x^{ib} - qT_x^{-ib} + e^{2\pi ib(Q-\alpha)}q^{-1} + e^{-2\pi ib(Q-\alpha)}q}{(q-q^{-1})^2}, \\ \pi_\alpha(F)\pi_\alpha(E) &= \frac{-e^{\pi ib^2}T_x^{ib} - e^{-\pi ib^2}T_x^{-ib} + e^{2\pi ib(Q-\alpha)-\pi ib^2} + e^{-2\pi ib(Q-\alpha)+\pi ib^2}}{(q-q^{-1})^2} \\ &= \frac{-qT_x^{ib} - q^{-1}T_x^{-ib} + e^{2\pi ib(Q-\alpha)}q^{-1} + e^{-2\pi ib(Q-\alpha)}q}{(q-q^{-1})^2},\end{aligned}$$

so it follows that $\pi_\alpha([\,E\,,\,F\,]) = \pi_\alpha(\frac{K^2-K^{-2}}{q-q^{-1}})$.
Thus, the defined actions of E, F, K on $\mathcal{P}_\alpha$ induce a $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-module structure and $\pi_\alpha\colon \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})) \to End(\mathcal{P}_\alpha)$ is an algebra morphism, i.e. π_α is a representation.
Now, it follows that $\mathcal{P}_\alpha$ is also a $U_{\tilde{q}}(\mathfrak{sl}(2),\mathbb{R})$-module, due to the fact that the above proof still holds when we replace b with b^{-1}. Thus, since the actions of E, F, K commute with the actions of $\tilde{E}, \tilde{F}, \tilde{K}$ on $\mathcal{P}_\alpha$, it follows that $\mathcal{P}_\alpha$ is indeed a $\mathcal{Q}$-module. □

Note, in the above proof we did use the fact that $f \in \mathcal{P}_\alpha$ was an entire function to show that $\pi_\alpha(K)f \in \mathcal{P}_\alpha$. However, a much weaker condition for $f(x)$ is sufficient if we are only concerned with $\mathcal{P}_\alpha$ as a $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-module, instead of a $\mathcal{Q}$-module. Since $\pi_\alpha(K)f(x) = f(x + \frac{ib}{2})$, it follows that we only need that $f(x)$ has an analytic extension to $\{z \in \mathbb{C}\,|\, im(z) \in [\frac{inb}{2} - \varepsilon, \frac{inb}{2} + \varepsilon]\}$ for some $\varepsilon > 0$ and $n \in \mathbb{Z}$. However, since we defined $\mathcal{P}_\alpha$ as a $\mathcal{Q}$-module, we also need to be able to act with $\pi_\alpha(\tilde{K})$ on $f \in \mathcal{P}_\alpha$. Hence, $f(x)$ needs to also have an analytic extension to $\{z \in \mathbb{C}\,|\, im(z) \in [\frac{imb^{-1}}{2} - \varepsilon, \frac{imb^{-1}}{2} + \varepsilon]\}$ for some $\varepsilon > 0$ and $m \in \mathbb{Z}$.
In particular, $f(x)$ needs to have an analytic extension to a strip around $\frac{i}{2}(nb + mb^{-1})$ for any $n, m \in \mathbb{Z}$ when we allow $\pi_\alpha(K)^n\pi_\alpha(\tilde{K})^m$ to act on $f(x) \in \mathcal{P}_\alpha$. And since $\{nb + mb^{-1}\,|\, n, m \in \mathbb{Z}\} \subseteq \mathbb{R}$ dense due to $b \in \mathbb{R}\backslash\mathbb{Q}$, it indeed follows that $f \in \mathcal{P}_\alpha$ needs to be entire.

The module $\mathcal{P}_\alpha$ has the following intertwining property:

Lemma 7.14. The $\mathcal{Q}$-modules $\mathcal{P}_\alpha$ and $\mathcal{P}_{Q-\alpha}$ are unitarily equivalent. I.e. $\exists\tilde{\mathcal{I}}_\alpha\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$, unitary operator, such that $\tilde{\mathcal{I}}_\alpha(\mathcal{P}_\alpha) = \mathcal{P}_{Q-\alpha}$ and $\widetilde{\pi_{Q-\alpha}}(U)(\tilde{\mathcal{I}}_\alpha\tilde{f}) = \tilde{\mathcal{I}}_\alpha(\widetilde{\pi_\alpha}(U)\tilde{f})$ $\forall U \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and $f \in \mathcal{P}_\alpha$.

Proof. Let $\tilde{\mathcal{I}}_\alpha\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be given by $(\tilde{\mathcal{I}}_\alpha\tilde{f})(\omega) = \frac{S_b(\alpha-i\omega)}{S_b(Q-\alpha-i\omega)}\tilde{f}(\omega)$, with $S_b(x)$ defined as in Appendix D: Special Functions, and let $f \in \mathcal{P}_\alpha$. Then $|\frac{S_b(\alpha-i\omega)}{S_b(Q-\alpha-i\omega)}| = 1$ and it follows that $\tilde{\mathcal{I}}_\alpha$ is a unitary operator.
Furthermore, the poles of $S_b(x)$ are given by $-nb - mb^{-1}$ and the zeros of $S_b(x)$ are given by $Q + nb + mb^{-1}$, $n, m \in \mathbb{N}$. So, it follows that $\frac{S_b(\alpha-i\omega)}{S_b(Q-\alpha-i\omega)}$ has poles in $\omega = i(-\alpha - nb - mb^{-1})$ and $\omega = i(\alpha + nb + mb^{-1})$. So, $\tilde{\mathcal{I}}_\alpha\tilde{f}$ does have possible poles of the desired form.
Note, $\frac{S_b(\alpha-i\omega)}{S_b(Q-\alpha-i\omega)}$ is zero at $\omega = i(Q - \alpha + nb + mb^{-1})$ and $\omega = i(\alpha - Q - nb - mb^{-1})$. So, $\tilde{\mathcal{I}}_\alpha\tilde{f}$ is zero if ω is a possible pole of $\tilde{f}$. In other words, $\tilde{\mathcal{I}}_\alpha\tilde{f}$ is meromorphic with possible poles in $\Omega_\pm$ of $\mathcal{P}_{Q-\alpha}$.
Lastly, the asymptotic behaviour of $\frac{S_b(\alpha-i\omega)}{S_b(Q-\alpha-i\omega)}$ as $\omega \to \pm\infty$, shows that $\tilde{\mathcal{I}}_\alpha\tilde{f} \in L^2(\mathbb{R})$. As $\frac{S_b(\alpha-i\omega)}{S_b(Q-\alpha-i\omega)}$ acts as $e^{2\pi\alpha|\omega|}$ as $\omega \to \pm\infty$ and lemma 7.10 shows us that $\tilde{f}$ decays faster. Thus it follows that $\tilde{\mathcal{I}}_\alpha$ indeed maps $\mathcal{P}_\alpha$ to $\mathcal{P}_{Q-\alpha}$.
Now it is only left to show that $\widetilde{\pi_{Q-\alpha}}(U)(\tilde{\mathcal{I}}_\alpha\tilde{f}) = \tilde{\mathcal{I}}_\alpha(\widetilde{\pi_\alpha}(U)\tilde{f})$ $\forall U \in \{E, F, K\}$. First, the Fourier transformed action of K is given by $e^{-\pi b\omega}$ for any α. So, the Fourier transformed action of K commutes with the action of $\tilde{\mathcal{I}}_\alpha$.

For the action of E, note that

$$T_\omega^{ib}\tilde{\mathcal{I}}_\alpha(\omega) = \frac{S_b(\alpha - i\omega + b)}{S_b(Q-\alpha-i\omega+b)} = \frac{\sin(\pi b(\alpha - i\omega))}{\sin(\pi b(Q-\alpha-i\omega))}\tilde{\mathcal{I}}_\alpha.$$

It follows that $\widetilde{\pi_{Q-\alpha}}(E)\tilde{\mathcal{I}}_\alpha\tilde{f}(\omega) = [-i\omega + Q - \alpha]_b T_\omega^{ib}\tilde{\mathcal{I}}_\alpha\tilde{f}(\omega) = \tilde{\mathcal{I}}_\alpha[-i\omega+\alpha]_b T_\omega^{ib}\tilde{f}(\omega) = \tilde{\mathcal{I}}_\alpha\widetilde{\pi_\alpha}(E)\tilde{f}(\omega)$. Similarly, note that $S_b(x) = 2\sin(\pi b(x-b))S_b(x-b) = -i(q^{-1}e^{\pi ibx} - qe^{-\pi ibx})S_b(x-b)$. So,

$$T_\omega^{-ib}\tilde{\mathcal{I}}_\alpha(\omega) = \frac{S_b(\alpha - iw - b)}{S_b(Q-\alpha-i\omega-b)} = \frac{q^{-1}e^{\pi ib(Q-\alpha-i\omega)} - e^{-\pi ib(Q-\alpha-i\omega-b)}}{q^{-1}e^{\pi ib(\alpha-i\omega)} - qe^{-\pi ib(\alpha-i\omega)}} = \frac{\sin(\pi b(i\omega+\alpha))}{\sin(\pi b(i\omega+Q-\alpha))}\tilde{\mathcal{I}}_\alpha,$$

and it follows that $\widetilde{\pi_{Q-\alpha}}(F)\tilde{\mathcal{I}}_\alpha\tilde{f}(\omega) = [i\omega+Q-\alpha]_b T_\omega^{-ib}\tilde{\mathcal{I}}_\alpha\tilde{f}(\omega) = \tilde{\mathcal{I}}_\alpha[i\omega+\alpha]_b T_\omega^{-ib}\tilde{f}(\omega) = \tilde{\mathcal{I}}_\alpha\widetilde{\pi_\alpha}(F)\tilde{f}(\omega)$. Thus, $\widetilde{\pi_{Q-\alpha}}(U)\tilde{\mathcal{I}}_\alpha = \tilde{\mathcal{I}}_\alpha\widetilde{\pi_\alpha}(U)$ for $U \in \{E,F,K\}$ and it follows that $\widetilde{\pi_{Q-\alpha}}(U)\tilde{\mathcal{I}}_\alpha = \tilde{\mathcal{I}}_\alpha\widetilde{\pi_\alpha}(U)$ $\forall U \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$.
Since the proof is symmetric under the transformation $b \mapsto b^{-1}$, it follows that $\tilde{\mathcal{I}}_\alpha$ also commutes with $\tilde{U} \in \{\tilde{E},\tilde{F},\tilde{K}\}$. Hence, $\mathcal{P}_\alpha \simeq \mathcal{P}_{Q-\alpha}$.
As a last remark, we get that $\mathcal{I}_\alpha f(x) = \int_{\mathbb{R}} B_\alpha(x-x')f(x)dx'$ with $B_\alpha(x-x') = S_b(2\alpha)\frac{S_b(\frac{Q}{2}+i(x-x')-\alpha)}{S_b(\frac{Q}{2}+i(x-x')+\alpha)}$ by the inverse Fourier transformation. We can see $\mathcal{I}_\alpha$ as the unitary operator that acts on the non-Fourier transformed functions $f(x) \in \mathcal{P}_\alpha$. □

Thus we have an equivalence relation between two different modules of the same family. So we see that not all modules of our family are necessarily unique.
Furthermore, this relation will later on be used to show that we can define the Clebsch-Gordan coefficients of $\mathcal{P}_\alpha$ in more than one way. One definition will have nice analytic properties and the other choice will be more natural, as it will make the Clebsch-Gordan coefficients invariant under the above equivalence relation. In other words, it is possible to construct the Clebsch-Gordan coefficients such that, if $\tilde{\mathcal{I}}_\alpha$ is used to go from $\mathcal{P}_\alpha$ to $\mathcal{P}_{Q-\alpha}$, the only change in the Clebsch-Gordan coefficients of $\mathcal{P}_\alpha$ and $\mathcal{P}_{Q-\alpha}$ will be that α changes to $Q-\alpha$.

We now know that our vector space $\mathcal{P}_\alpha$ is indeed a $\mathcal{Q}$-module with our defined actions of E, F, K and $\tilde{E},\tilde{F},\tilde{K}$. But, at the moment this class of modules seems a bit random and artificial. However, we will soon see that $\mathcal{P}_\alpha$ is a natural vector space to induce a module structure of the modular double of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ on. This is the case due to two facts. The first is that $\pi_\alpha(E), \pi_\alpha(F)$ and $\pi_\alpha(K)$ are positive self-adjoint operators on $L^2(\mathbb{R})$, the second fact is that $\mathcal{P}_\alpha$ is the largest space for which our given actions of E, F, K and $\tilde{E},\tilde{F},\tilde{K}$ are well-defined.
More precisely, the first fact should be that π_α is an integrable representation. Hence, we will show that π_α is integrable, as this will show several nice properties of $\mathcal{P}_\alpha$. Also, the fact that our operators are self-adjoint and unbounded is also a consequence of the fact that $\mathcal{P}_\alpha$ was build as an integrable representation.
However, it is best to keep in mind that we will care most about the fact that $\pi_\alpha(U)$, $U \in \{E,F,K\}$, will be a positive self-adjoint operator. As that is in turn used to show that $\mathcal{P}_\alpha$ is the largest subspace of $L^2(\mathbb{R})$ such that π_α is a well-defined representation. This property is also closely connected to the fact that $\mathcal{P}_\alpha$ is a natural choice for a module of the modular double of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$. Since it allows us to define the action of $\tilde{E},\tilde{F},\tilde{K} \in U_{\tilde{q}}(\mathfrak{sl}(2),\mathbb{R})$ as certain powers of $\pi_\alpha(E), \pi_\alpha(F)$ and $\pi_\alpha(K)$ respectively.

7.4.1 π_α is an integrable representation

The notion of an integrable representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ has been defined in [13]. We will go over some definitions and conclusions to give a broad idea of what it means to be an integrable representation in this sense.
This definition makes use of the definition of an integrable representation of $\mathbb{R}_q[x,y]$ given in [14], so

we will start with given a short explanation on this definition. In other words, we give all the details that are needed to understand what an integrable representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ is in this part. This will be done by summarising some main points of both [14] and [13]. However, we will omit the proofs and some of the technical details, since they use results on operator theory that we do not want to discus here.
Before we go on to the definitions and conclusion, let's quickly sketch why it is called an integrable representation. This will also sort of show why it is defined this way.

The notion of an integrable representation is used in representation theory of Lie algebras, $*$-representations and operator relations. For example, in the case of Lie algebras, a representation would be called integrable if the representation would come from the associated Lie group. Recall that the associated Lie algebra $\mathfrak{g}$ of a Lie group G is defined as T_eG. So, in a sense, calling a Lie algebra representation of $\mathfrak{g}$ integrable if it originated from a Lie group representation of G sounds logical. As we can sort of go back to the Lie group via integration.
Now, in the case of representation theory of Lie algebras and mathematical physics, one sometimes deals with self-adjoint operators that also satisfy a certain algebraic relation. For example, if one wants to define a representation of the relation $ab - ba = -i$, one could require that the self-adjoint operators a, b satisfy the Weyl relation $e^{ita}e^{isb} = e^{its}e^{isb}e^{ita}$, $s, t \in \mathbb{R}$.
As another example, if a is self-adjoint, b is normal and $F\colon \mathbb{R} \to \mathbb{R}$, then one could define the integrable representations such that $ab = F(a)b$, by requiring $f(a)b \subseteq bf(F(a))$ for $f \in L^\infty(\mathbb{R})$. So, in a way, being integrable is a notion of well-behavedness.

Since the definition of an integrable representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ is defined via integrable representations of $\mathbb{R}_q[x,y]$, the real quantum plane. We will first look at the latter case.
For this part, let X, Y be self-adjoint operators on some Hilbert space $\mathcal{H}$, let $q = e^{i\varphi}$ with $|\varphi| < \pi$ and $q^2 \neq 1$. Suppose $Y > 0$ or $Y < 0$ and that $YX = qXY$. If $f \in \mathbb{C}[\lambda]$, then it follows that $f(Y)X = Xf(qY)$. So, it is natural to define integrability by having $f(Y)X = Xf(qY)$ be satisfied for some nice functions f.

Definition 7.15. Let $Y > 0$ or $Y < 0$ and X be a self adjoint operator. Then, a pair $\{Y, X\}$ of self-adjoint operators is called an *integrable representation* of $\mathbb{R}_q[x,y]$ if $\exists k \in \mathbb{Z}$ such that

$$|Y|^{it}X = e^{(-\varphi-2\pi k)t}X|Y|^{it}, \quad t \in \mathbb{R}.$$

If $Y \geq 0$ or $Y \leq 0$, let $\mathcal{H}_0 = ker(Y)$ and $Y_1 = Y|_{\mathcal{H}_0^\perp}$. Then, $\{Y, X\}$ is called an *integrable representation* of $\mathbb{R}_q[x,y]$ if $\exists X_0, X_1$, self-adjoint operators, X_0 acting on $\mathcal{H}_0$ and X_1 on $\mathcal{H}_0^\perp$ such that $X = X_0 \oplus X_1$ and $\{Y_1, X_1\}$ is an integrable representation in the previous sense, as $Y_1 > 0$ or $Y_1 < 0$.

Note, if $\{Y, X\}$ is an integrable representation, then we see that $Y^2X = q^2XY^2$. Then, if $ker(Y) = \{0\}$, it follows that $Y^2 > 0$ and that integrability for $\{Y^2, X\}$ is defined as usual. This reasoning can then be used to define integrability for the pair $\{Y, X\}$, Y, X self-adjoint operators, using the polar decomposition $Y = U_Y|Y|$. So, integrability can be defined for any pair $\{Y, X\}$ of self-adjoint operators. Thus, $Y \geq 0$ or $Y \leq 0$ can be shown to be an optional requirement.
Since the operators we use in the representation of $\mathcal{Q}$ are all positive, we will not be looking at this more general definition. However, one can find more details about integrability in [14]. This goes over more details on general integrability and also looks at integrable representations of two other quantum groups.
Now, [14] concludes with the following theorem on irreducible integrable representations $\{Y, X\}$ of $\mathbb{R}_q[x,y]$:

Theorem 7.16. *Each irreducible integrable representation $\{Y, X\}$ of $\mathbb{R}_q[x,y]$ is unitarily equivalent to one of the following models:*

$(I)_{\varepsilon_1,\varepsilon_2,k}$: *$Y = \varepsilon_1 e^{Q}$, $X = \varepsilon_2 e^{(-\varphi-2\pi k)P}$ on $\mathcal{H} = L^2(\mathbb{R})$, with $P = -i\frac{d}{dt}$, Q the multiplication operator by t and $\varepsilon_1, \varepsilon_2 \in \{-1,1\}$, $k \in \mathbb{Z}$,*

$(II)_k$: *$Y = \begin{pmatrix} e^{Q} & 0 \\ 0 & e^{-Q} \end{pmatrix}$, $X = \begin{pmatrix} 0 & e^{(-\varphi-2\pi(k+1))P} \\ e^{(-\varphi-2\pi(k+1))P} & 0 \end{pmatrix}$ on $\mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, with $k \in \mathbb{Z}$,*

$(III)_{\alpha,0}$: *$Y = \alpha$, $X = 0$ on $\mathcal{H} = \mathbb{C}$, with $\alpha \in \mathbb{R}$,*

$(III)_{0,\alpha}$: *$Y = 0$, $X = \alpha$ on $\mathcal{H} = \mathbb{C}$, with $\alpha \in \mathbb{R}$.*

Immediately note that this is about irreducible representations. So any representation that is unitarily equivalent to one in the above list, will automatically be irreducible in addition to being integrable. Also, this list of integrable representations $\{Y, X\}$ will be used to construct a full list of possible integrable representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$.

Now, if $\{Y, X\}$ is a pair of self-adjoint operators with trivial kernels, i.e. $ker(Y) = ker(X) = \{0\}$. Then it follows that if $\{Y, X\}$ is an irreducible integrable representation of $\mathbb{R}_q[x,y]$ if and only if it is unitarily equivalent to either:

(M_1): $Y = e^{\alpha \mathbf{x}} \otimes u$, $X = e^{2\beta P} \otimes v$ on $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathcal{K}$, with $\alpha, \beta \in \mathbb{R}$ such that $2\alpha\beta = \pi b^2 + 2k\pi$, $k \in \mathbb{Z}$ and u, v are commuting self-adjoint unitary operators on the Hilbert space $\mathcal{K}$,

(M_{-1}): $\mathrm{Y} = e^{\alpha x} \otimes \sigma_0 \otimes I$, $X = e^{2\beta P} \otimes \sigma_1 \otimes I$ on $L^2(\mathbb{R}) \otimes \mathbb{C}^2 \otimes \mathcal{K}$, with $\alpha, \beta \in \mathbb{R}$ such that $2\alpha\beta = \pi b^2 + (2k+1)\pi$, $k \in \mathbb{Z}$, $\mathcal{K}$ a Hilbert space and

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since we will be dealing with operators that are self-adjoint with trivial kernels, these two models are the only ones that we will be dealing with. More precisely, for the next part it is assumed that K, E, F are self-adjoint operators and $ker(E) = ker(F) = \{0\}$. This is almost the same setting as what we had in the construction of the module $\mathcal{P}_\alpha$, so these assumptions make sense for this specific case. Do note that the operators K, E, F are unbounded as a consequence of E and F having trivial kernels. So, trying to construct "well-behaved" representations is also partly where some of the difficulties arise from.

Definition 7.17. Let $\{K, E, F\}$ be a triple of self-adjoint operators on a Hilbert space $\mathcal{H}$. Then $\{K, E, F\}$ is an *integrable representation* of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ if $\{K, E\}$ and $\{F, K\}$ are integrable representations of $\mathbb{R}_q[x,y]$ and if the closure of $C_q = -FE - \frac{qK^2 - q^{-1}K^{-2}}{(q-q^{-1})^2}$ is a self-adjoint operator that strongly commutes with K, E, F.

Note, if T is a bounded operator and S is a possibly unbounded operator on a Hilbert space $\mathcal{H}$, then T and S commute if $TS \subseteq ST$. So T and S commute when $Dom(S) = Dom(TS) \subseteq Dom(ST)$ and $TSv = STv$ for $v \in Dom(S)$.
Now, two self-adjoint operators A, B on $\mathcal{H}$ *strongly commute* if $(A-z)^{-1}$ and B commute for $z \in \mathbb{C}\backslash\mathbb{R}$. $(A-z)^{-1}$ is well-defined for $z \in \mathbb{C}\backslash\mathbb{R}$ as $\sigma(A) \subset \mathbb{R}$, with $\sigma(A)$ the spectrum of A (defined in [15]).

Since the integrability of $\{K, E, F\}$ is fully dependent on the integrability on $\mathbb{R}_q[x,y]$, the above two models can be used to describe integrable representations on $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$. Let $\mathcal{K}$ be a Hilbert space and let c be a self-adjoint operator on $\mathcal{K}$. Then we get the following two models that describe the structure of integrable representations on $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$:

(M_1): $K = e^{\alpha \mathbf{x}} \otimes u$, $E = e^{2\beta P} \otimes v$, $F = (q-q^{-1})^{-2} e^{-\beta P}(e^{2\alpha \mathbf{x}} + e^{-2\alpha \mathbf{x}})e^{-\beta P} \otimes v + e^{-2\beta P} \otimes vc$ on $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathcal{K}$,

(M_{-1}): $K = e^{\alpha\mathbf{x}} \otimes \sigma_0 \otimes I$, $E = e^{2\beta P} \otimes \sigma_1 \otimes I$, $F = -(q-q^{-1})^{-2} e^{-\beta P}(e^{2\alpha\mathbf{x}} + e^{-2\alpha\mathbf{x}})e^{-\beta P} \otimes \sigma_1 \otimes I + e^{-2\beta P} \otimes \sigma_1 \otimes c$ on $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2 \otimes \mathcal{K}$,

where $\alpha, \beta, k, \sigma_0, \sigma_1$ are as before.
These two models indeed satisfy the definition of an integrable representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$. However, F need not be self-adjoint necessarily. As one can show that it is sufficient to have that K, E are self-adjoint and F is symmetric. In [13] it is checked what condition needs to be satisfied for F to be a self-adjoint operator, but we will skip over this proposition to immediately look at the three conclusions that follow this proposition.
These conclusions precisely show us when $\{K, E, F\}$ is an irreducible integrable representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ with E, F, K self-adjoint. Furthermore, it also follows that the models we obtain will be $*$-representations in the sense that U^* acts as the adjoint of U on the given Hilbert spaces.

Theorem 7.18. *Let $\{K, E, F\}$ be a triple of self-adjoint operators on a Hilbert space $\mathcal{H}$ with $ker(E) = ker(F) = \{0\}$, then the following are equivalent:*

i) *$\{K, E, F\}$ is an integrable representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$,*

ii) *$\{K, A\}$ is an integrable representation of $\mathbb{R}_q[x, y]$ and C_q is a self-adjoint operator that strongly commutes with K and E,*

iii) *$\{K, E, F\}$ is unitarily equivalent to either model M_1 with $n = 0$, $e_+(c) = 0$ or M_{-1} with $2n+1+sign(\varphi) = 0$, $e_-(c) = 0$ and $e_+(c)$, $e_-(c)$ are the spectral projections of the self-adjoint operator C discussed in the above models corresponding to the intervals $(-\infty, (q+\overline{q})(q-q^{-1})^{-2})$ and $((q+\overline{q})(q-q^{-1})^{-2}, \infty)$ respectively.*

Note, in our specific case of $\mathcal{P}_\alpha$, we have $\varphi = \pi b^2$. Thus we already see that we only have one possible representation up to equivalence.
The models can be simplified with the use of the next corollary, which also shows that $\{K, E, F\}$ is a $*$-representation.

Corollary 7.19. Let $\{K, E, F\}$ be an integrable representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, then $\exists D \subset D(K) \cap D(K^{-1}) \cap D(E) \cap D(F)$ such that

i) $KD = D$, $ED = D$, $FD \subseteq D$, $|K|^{it}D = D$, $|E|^{it}D = D$ for $t \in \mathbb{R}$,

ii) D is a core for K, K^{-1}, E, F, i.e. $\overline{U|_D} = \overline{U}$ for $U \in \{K, K^{-1}, E, F\}$ with $\overline{U}$ the closure of U,

iii) the relations between elements of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ also hold for vectors in D.

As noted before, we get the following corollary, which will conclude two possible models of irreducible integrable representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})$, with the requirement of K, E, F being self-adjoint operators.

Corollary 7.20. Any irreducible integrable representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ is unitarily equivalent to one of the following:

$(I)_{\varepsilon_1,\varepsilon_2,c}$: $K = \varepsilon_1 e^{\alpha\mathbf{x}}$, $E = \varepsilon_2 e^{2\beta P}$, $F = \varepsilon_2 (q-q^{-1})^{-2} e^{-\beta P}(c(q-q^{-1})^2 + e^{2\alpha\mathbf{x}} + e^{-2\alpha\mathbf{x}})$ on $\mathcal{H} = L^2(\mathbb{R})$ with $2\alpha\beta = \varphi$, $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$, $c \leq (q+\overline{q})(q-q^{-1})^{-2}$,

$(II)_c$: $K = e^{\alpha\mathbf{x}} \otimes \sigma_0$, $E = e^{2\beta P} \otimes \sigma_1$, $F = (q-q^{-1})^{-2} e^{-\beta P}(c(q-q^{-1})^2 - e^{2\alpha\mathbf{x}} - e^{-2\alpha\mathbf{x}})e^{-\beta P} \otimes \sigma_1$ on $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$ with $2\alpha\beta = \varphi - sign(\varphi)\pi$, $c \geq (q+\overline{q})(q-q^{-1})^{-2}$.

In other words, we see that there is only one possible integrable representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ that acts on $L^2(\mathbb{R})$ with $q = e^{\pi i b^2}$, $b \in (0,1)\backslash(\mathbb{R} \cap \mathbb{Q})$. This is also a $*$-representation due to the previous corollary. Also, it is a reasonable definition, as it is natural to want that the operators K, E, F are

self-adjoint with our $*$-structure. Besides that, this definition makes sure that both the real powers and complex powers of K, E, F satisfy the relations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$.
Now, do note that we only have $\mathcal{P}_\alpha \subset L^2(\mathbb{R})$ as our vector space. Thus our representation π_α doesn't act on all on $L^2(\mathbb{R})$, even though the above constructions do act on all of $L^2(\mathbb{R})$. The reason for not having $\pi_\alpha(U)$, $U \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, act on all of $L^2(\mathbb{R})$ is due to the fact that the self-adjoint extensions of $\pi_\alpha(U)$ and $\pi_\alpha(\tilde{U})$ need not commute on all of $L^2(\mathbb{R})$. But, since we want $\mathcal{P}_\alpha$ to be a $\mathcal{Q}$-module, we do need that these two operators commute on the whole module. However, we do get the following fact, that shows that we can at least extend our representations $\pi_\alpha(U)$ to all of $L^2(\mathbb{R})$.

Proposition 7.21. The operators $\pi_\alpha(K), \pi_\alpha(E)$ and $\pi_\alpha(F)$ generate an integrable representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$. So,

i) $\pi_\alpha(K), \pi_\alpha(E), \pi_\alpha(F)$ have self-adjoint extensions to all of $L^2(\mathbb{R})$,

ii) the unitary operators $\pi_\alpha(E)^{it}, \pi_\alpha(F)^{it}$ and $\pi_\alpha(K)^{it}$ satisfy the commutation relations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, i.e.
$$\pi_\alpha(K)^{is}\pi_\alpha(E)^{it} = q^{-ts}\pi_\alpha(E)^{it}\pi_\alpha(K)^{is}, \qquad \pi_\alpha(K)^{is}\pi_\alpha(F)^{it} = q^{ts}\pi_\alpha(F)^{it}\pi_\alpha(K)^{is}, \quad s,t \in \mathbb{R}$$

iii) the action of the q-Casimir element C_q strongly commutes with $\pi_\alpha(K)$, $\pi_\alpha(E)$ and $\pi_\alpha(F)$.

So we see that the extension of $\mathcal{P}_\alpha$ to $L^2(\mathbb{R})$ is our only choice up to equivalence for a representation with the above nice properties. Also, note that we immediately get that $\mathcal{P}_\alpha$ is a simple module as a consequence of corollary 7.20 giving us only irreducible representations. Note, [1] proofs the above proposition by showing that the actions of E, F, K on $\mathcal{P}_\alpha$ are given by
$$E = J_\alpha^{-1}\tilde{E}_\alpha J_\alpha = T_\omega^{ib}, \quad J_\alpha^{-1}\tilde{F}_\alpha J_\alpha = [\alpha + i\omega]_b T_\omega^{-ib}[\alpha - i\omega]_b, \quad J_\alpha^{-1}\tilde{K}_\alpha J_\alpha = e^{\pi b\omega},$$
with the unitary operator J_α given by $J_\alpha\tilde{f}(\omega) = S_b(\alpha - i\omega)\tilde{f}(\omega)$. The above E, F, K are easily recognized as the E, F, K in model $(I)_{1,-1,c}$ if one replaces the x with ω in this model.
Next we will see that $\mathcal{P}_\alpha$ is also a maximal subspace of $L^2(\mathbb{R})$ with the property that $\pi_\alpha(U)$, $U \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ is well-defined.

7.4.2 $\mathcal{P}_\alpha$ is a maximal subspace of $L^2(\mathbb{R})$

From the previous part we know that the operators $\pi_\alpha(U)$, $U = E, F, K$, are self-adjoint operators. Furthermore, they are also positive operators by construction. So the above definition of integrability is completely defined via the first definition, definition 7.15.
Now, to show that $\mathcal{P}_\alpha$ is a maximal subspace, we will first construct a basis for $\mathcal{P}_\alpha$. Since $\pi_\alpha(E), \pi_\alpha(F)$, $\pi_\alpha(K)$ are positive operators, we see that
$$\pi_\alpha(C_q)^l\pi_\alpha(K)^{\frac{n}{2}}\pi_\alpha(E)^m\pi_\alpha(K)^{\frac{n}{2}} \quad \text{and} \quad \pi_\alpha(C_q)^l\pi_\alpha(K)^{\frac{n}{2}}\pi_\alpha(F)^m\pi_\alpha(K)^{\frac{n}{2}},\ l,m \in \mathbb{N},\ n \in \mathbb{Z}$$
are also positive operators. The linear span of the elements $(C_q)^l(K)^{\frac{n}{2}}(E)^m(K)^{\frac{n}{2}}$ and $(C_q)^l(K)^{\frac{n}{2}}(F)^m(K)^{\frac{n}{2}}$ forms a basis B_q for $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$. So we get that $\pi_\alpha(b)$, $b \in B_q$ is a positive operator.
As noted before, the actions of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ on $L^2(\mathbb{R})$ are given by unbounded operators. Which is why our module is given by $\mathcal{P}_\alpha$ and not all of $L^2(\mathbb{R})$. But, from the next lemma, it will follow that $\mathcal{P}_\alpha$ is the largest subspace of $L^2(\mathbb{R})$ on which all of the operators $\pi_\alpha(U)$, $U \in \mathcal{Q}$ are well-defined.

Lemma 7.22. $\mathcal{P}_\alpha$ is the largest space on which all $\pi_\alpha(U \otimes \tilde{U}) = \pi_\alpha(U) \otimes \pi_\alpha(\tilde{U})$, $U \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})), \tilde{U} \in U_{\tilde{q}}(\mathfrak{sl}(2),\mathbb{R})$ are well defined. In other words,
$$\mathcal{P}_\alpha = \bigcap_{b\otimes\tilde{b}\in B_q\otimes B_{\tilde{q}}} Dom(\pi_\alpha(b\otimes\tilde{b})).$$
Furthermore, $\mathcal{P}_\alpha$ is a Fréchet space for all possible α with topology induced by the seminorms
$$||f||_b = \sup_{k\in\mathbb{R}} |(\pi_\alpha(b\otimes\tilde{b})f)(k)|, \quad b\otimes\tilde{b} \in B_q \otimes B_{\tilde{q}}.$$

This lemma is proven in [16]. The biggest difference with our definition of $\mathcal{P}_\alpha$, is that this proof shows that $e^{a|\omega|}f(\omega) \in L^2(\mathbb{R})$, $a > 0$, is a necessary condition for $\pi_\alpha(K)^n f$ to be well defined for $n \in \mathbb{Z}$. But, this is equivalent with $f \in \mathcal{P}_\alpha$ being an entire function, since the proof uses the actions of K, E, F under the Fourier transform.
Note, both the condition that $f(x)$ needs to be entire and that $e^{a|\omega|}\tilde{f}(\omega) \in \mathcal{P}_\alpha$ for all $a \in \mathbb{R}$ follow due to the fact that $\{nb + mb^{-1} \,|\, n, m \in \mathbb{Z}\} \subset \mathbb{R}$ is a dense subset and that we want to act with both U and $\tilde{U}$ on $\mathcal{P}_\alpha$.
In other words, this condition follows due to the fact that $\mathcal{P}_\alpha$ is a $\mathcal{Q}$-module. As the actions of K and $\tilde{K}$ imply that we either need to be able to analytically extend $f(x)$ to a small strip around $nb + mb^{-1}$ or $e^{-\pi(nb+mb^{-1})k}\tilde{f}(k) \in \mathcal{P}_\alpha$ for all $n, m \in \mathbb{Z}$.
Recall that $\{nb + mb^{-1} \,|\, n, m \in \mathbb{Z}\}$ is only dense in $\mathbb{R}$ due to the fact that $b \in \mathbb{R}\backslash\mathbb{Q}$. So this specific choice for b is necessary to get that $\mathcal{P}_\alpha$ is a maximal subspace.

Note that the basis B_q of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ consists of only positive operators. So, the positivity of the operators $\pi_\alpha(U)$, $U \in \{E, F, K\}$, helps with showing that $\mathcal{P}_\alpha$ is the largest possible subspace of $L^2(\mathbb{R})$ that we can turn into a $\mathcal{Q}$-module, which is given by an integrable representation.
Furthermore, with the positivity of the operators $\pi_\alpha(E), \pi_\alpha(F), \pi_\alpha(K)$, we can directly define the operators $\pi_\alpha(\tilde{E}), \pi_\alpha(\tilde{F}), \pi_\alpha(\tilde{K})$. Since we have

$$\pi_\alpha(\tilde{E}) = \pi_\alpha(E)^{\frac{1}{b^2}}, \qquad \pi_\alpha(\tilde{F}) = \pi_\alpha(F)^{\frac{1}{b^2}}, \qquad \pi_\alpha(\tilde{K}) = \pi_\alpha(K)^{\frac{1}{b^2}},$$

which we can define via functional calculus due to the operators being positive. Combined with the fact that the actions of $\pi_\alpha(U)$ and $\pi_\alpha(\tilde{U})$, $U \in \{E, F, K\}$, $\tilde{U} \in \{\tilde{E}, \tilde{F}, \tilde{K}\}$, commute on $\mathcal{P}_\alpha$ by construction. We see that it is not a weird assumption to want the actions of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ on $\mathcal{P}_\alpha$ to be given by positive operators.
Furthermore, we saw that the chosen actions lead to the fact that $\mathcal{P}_\alpha$ is an integrable module. This was also shown to be a unique module up to unitary equivalence. Together with the maximality of $\mathcal{P}_\alpha$, we see why $\mathcal{P}_\alpha$ is a natural choice to define a $\mathcal{Q}$-module structure on. As $\mathcal{P}_\alpha$ also naturally has the same self-duality property for $b \to b^{-1}$ as $\mathcal{Q}$.

Besides being maximal, $\mathcal{P}_\alpha$ has another nice property. It is dense in $L^2(\mathbb{R})$. This is due to the fact that π_α is an integrable representation and $\pi_\alpha(E), \pi_\alpha(F), \pi_\alpha(K)$ can be extended to self-adjoint operators defined on all of $L^2(\mathbb{R})$, which is also true for the actions of $\tilde{E}, \tilde{F}$ and $\tilde{K}$.
Due to this, one could ask why we even bother with $\mathcal{P}_\alpha$ and not just use the self-adjoint extensions. As we could define the action of any element of $\mathcal{Q}$ on any $f \in L^2(\mathbb{R})$ via these self-adjoint extensions. However, the issue is that it is not clear whether or not the self adjoint extensions of $\pi_\alpha(U)$ would commute with the self-adjoint extensions of $\pi_\alpha(\tilde{U})$ for $U \in \{E, F, K\}$, $\tilde{U} \in \{\tilde{E}, \tilde{F}, \tilde{K}\}$ on all of $L^2(\mathbb{R})$. And in general, they do not need to commute. So, for any arbitrary $f \in L^2(\mathbb{R})$, it need not be true that $U\tilde{U}f = \tilde{U}Uf$, with $Uf, \tilde{U}f$ representing the action of the self-adjoint extension of $\pi_\alpha(U)$ and $\pi_\alpha(\tilde{U})$ respectively.
Which is why we work with $\mathcal{P}_\alpha$ as our $\mathcal{Q}$-module. Do note that we could turn all of $L^2(\mathbb{R})$ into a $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-module via these self-adjoint extensions. The above issue only arises if we are talking about modules of the modular double $\mathcal{Q}$, not when we consider $\mathcal{P}_\alpha$ as a $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-module.

Now that we have a basic idea of what our module $\mathcal{P}_\alpha$ is, why it is defined in this way and what some properties are. We can finally move on to describing the Clebsch-Gordan coefficients of $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$.

8 The Clebsch-Gordan coefficients of the module $\mathcal{P}_\alpha$

In this section we will be looking at the Clebsch-Gordan coefficients of the tensor product $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$. We will do this by studying how [1] calculated these coefficients and adding some extra details to the proofs that are given in that paper.
We will see that, similar to the classical and quantum case, we can decompose the tensor product of two simple modules $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$. However, unlike the finite dimensional case, we cannot decompose $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ into a direct sum of simple modules. However, we can decompose it into a direct integral of simple modules, which is a generalisation of the direct sum, as shown in Appendix C: The Direct Integral.

8.1 Small remark on distributions

Since the Clebsch-Gordan coefficients will be defined with the use of distributions, we will shortly talk about them. In short, a distribution Φ is a linear functional that acts on Schwartz functions. This is done via integration, so if $f \in S(\mathbb{R})$, then $\Phi f = \int_{\mathbb{R}} \Phi(x) f(x) dx$. So, in a sense, one can think of a distribution Φ as a function that acts on $f \in L^2(R)$ by integrating the product $(\Phi \cdot f)(x)$.

Recall that the Schwartz-space $S(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ and the Fourier-transformation is even a unitary operator on $S(\mathbb{R})$ and can be extended to an isometry to all of $L^2(\mathbb{R})$. Due to this, and since we are working with functions in $L^2(\mathbb{R})$, it is useful to consider the space $S'(\mathbb{R})$ of tempered distributions. Then, if $\Phi \in S'(\mathbb{R})$, it acts on any $f \in S(\mathbb{R})$ such that

$$\Phi f = \int_{\mathbb{R}} \Phi(x) f(x) dx.$$

Thus $\Phi \in S'(\mathbb{R})$ is a linear functional $\Phi\colon S(\mathbb{R}) \to \mathbb{C}$. We can then also consider the Fourier transform of a distribution $\Phi \in S'(\mathbb{R})$ by setting $\tilde{\Phi}\tilde{f} = \Phi f$. It then follows that we have

$$\Phi(x) = \int_{\mathbb{R}} e^{-2\pi i \omega x} \tilde{\Phi}(\omega) d\omega$$

as the inverse Fourier-transform for a distribution Φ. Notice that we have $e^{-2\pi i \omega x}$ instead of $e^{2\pi i \omega x}$ in this case. This is due to the fact that

$$\int_{\mathbb{R}} \tilde{\Phi}(\omega) \tilde{f}(\omega) d\omega = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-2\pi i \omega x} \tilde{\Phi}(\omega) f(x) dx d\omega = \int_{\mathbb{R}} \Phi(x) f(x) dx.$$

Now, for our case, we will be defining distributions with the use of limits of the index of certain families of meromorphic functions. Let $\{\Phi_\varepsilon\}$, $\varepsilon > 0$, be a family of meromorphic functions $\Phi_\varepsilon(x)$ that contain a certain strip around $\mathbb{R}$ in their domain. Furthermore, assume that $\forall \varepsilon > 0$, $\Phi_\varepsilon(x)$ decreases rapidly as $x \to \infty$ and that $\Phi_\varepsilon(x)$ has finitely many poles with residues that do not depend on ε and have distance ε from $\mathbb{R}$.
Then it follows that $\Phi := \lim_{\varepsilon \downarrow 0} \Phi_\varepsilon$ is a distribution that acts on $S(\mathbb{R})$, so $\Phi \in S'(\mathbb{R})$.
It follows that we can generalise lemma 7.10 to the space $S'(\mathbb{R})$.

Lemma 8.1. Let $\Phi \in S'(\mathbb{R})$, then the following are equivalent:

i) $\Phi = \lim_{\varepsilon \downarrow 0} \Phi_\varepsilon$, with Φ_ε, $\varepsilon > 0$, representing a restriction to $\mathbb{R}$ of a function $\Phi_\varepsilon(x)$ that is meromorphic in $\{z \in \mathbb{C} \,|\, Im(z) \in (-a_-, a_+)\}$, $a_\pm > 0$, with finitely many poles in the upper and lower half plane $I_\pm = \{z \in \mathbb{C} \,|\, \pm Im(z) > 0\}$ at $J_\pm^\varepsilon = \{z_j \pm i\varepsilon \,|\, z_j \in \pm I_\pm\}$ and every function $\Phi_{\varepsilon,y}(x) = \Phi_\varepsilon(x + iy)$, $x, y \in \mathbb{R}$, $y \in (-a_-, a_+)$ are of rapid decrease.

ii) $\tilde{\Phi}$ is represented by a function $\tilde{\Phi}(\omega) \in C^\infty(\mathbb{R})$ that has the following asymptotic behaviours as $\omega \to \pm\infty$:

$$\tilde{\Phi}(\omega) = +2\pi i \sum_{z_j \in J_+} e^{2\pi i z_j \omega} Res(\Phi(z_j)) + \tilde{\Phi}_{a_+}(\omega) \qquad \omega \to \infty,$$
$$\tilde{\Phi}(\omega) = -2\pi i \sum_{z_j \in J_-} e^{2\pi i z_j \omega} Res(\Phi(z_j)) + \tilde{\Phi}_{a_-}(\omega) \qquad \omega \to -\infty,$$

with $\tilde{\Phi}_{a_\pm}$ decaying faster than $e^{-2\pi a|\omega|}$ $\forall a \in (-a_-, a_+)$ as $\omega \to \pm\infty$.

The biggest difference with the above lemma and lemma 7.10 is the change of signs. This is due to the fact that the inverse Fourier-transform has a swapped sign for distributions.
As a quick remark, note that we have already seen some uses of distributions before. The non-Fourier-transformed version of the unitary operator $\tilde{\mathcal{I}_\alpha}$ of lemma 7.14 is defined as

$$\mathcal{I}_\alpha f(x) = \int_{\mathbb{R}} B_\alpha(x - x') f(x) dx',$$

hence $\mathcal{I}_\alpha$ is a distribution. Another example of a distribution is the Dirac δ. As we get that

$$\int_{\mathbb{R}} \delta(x' - x) f(x) dx = f(x'),$$

hence $\delta(x' - x)$ is a linear functional on $C(\mathbb{R})$.
We can mostly think about distributions as functions, but it is good to keep in mind why we can integrate $\Phi(x) \cdot f(x)$ when $f(x) \in L^2(\mathbb{R})$ if $\Phi \in S'(\mathbb{R})$. Also, we will use the changed Fourier-transform of distributions later on too.

8.2 The decomposition of $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$

Since $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ is a Hopf $*$-algebra, we can turn $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ into a $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-module, with the use of the coproduct. This gives us:

$$U(f \otimes g) = \Delta(U)(f \otimes g), \qquad U \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})),\ f \in \mathcal{P}_{\alpha_2},\ g \in \mathcal{P}_{\alpha_1}.$$

Now, let $\pi_{21}(U) = \pi_{\alpha_2} \otimes \pi_{\alpha_1}(\Delta(U))$, then π_{21} is the representation belonging to the module $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ as

$$\pi_{21}(U)(f \otimes g) = \Delta(U)(f \otimes g) = \sum_{(U)} U' f \otimes U'' g = \sum_{(U)} \pi_{\alpha_2}(U') f \otimes \pi_{\alpha_1}(U'') g = \pi_{\alpha_2} \otimes \pi_{\alpha_1}(\Delta(U))(f \otimes g),$$

for $U \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, $f \in \mathcal{P}_{\alpha_2}$, $g \in \mathcal{P}_{\alpha_1}$.
As before with the classical case and quantum case, we want to decompose $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ into a "direct sum" of simple modules. However, we will soon see that we cannot do this with the use of a direct sum, but we will need to use the direct integral. For some details on the definition of the direct integral, see Appendix C: The Direct Integral. In short, the direct integral is a generalisation of the direct sum. So, one could think of $\int_{\mathcal{S}}^{\oplus} \mathcal{P}_\alpha d\mu(\alpha)$ as if it is $\bigoplus_{\alpha \in \mathcal{S}} \mathcal{P}_\alpha$, but with a different requirement than $F \in \oplus_{\alpha \in \mathcal{S}} \mathcal{P}_\alpha$ if and only if $0 \neq F(\alpha) = f_\alpha \in \mathcal{P}_\alpha$ for finitely many $\alpha \in \mathcal{S}$. The precise requirement would be that

$$F \in \int_{\mathcal{S}}^{\oplus} \mathcal{P}_\alpha d\mu(\alpha) \iff \int_{\mathcal{S}} ||F(\alpha)||^2_{L^2(\mathbb{R})} d\mu(\alpha) < \infty,$$

so every $F(\alpha) = f_\alpha \in \mathcal{P}_\alpha$ needs to be square-integrable.

Before moving on to the Clebsch-Gordan coefficients, we will note the following fact:

Lemma 8.2. $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1} \subset L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ is dense.

So, the fact that $\mathcal{P}_\alpha \subset L^2(\mathbb{R})$ is dense, still holds in the case of the tensor product. Now, the main part of this section will be proving the following theorem:

Theorem 8.3. *Let* $\pi_{21} : \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})) \to End(\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1})$ *be given by* $\pi_{21}(U)(f \otimes g) = \pi_{\alpha_2} \otimes \pi_{\alpha_1}(\Delta(U))(f \otimes g)$. *Then,*

$$\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1} \simeq \int_{\mathcal{S}}^{\oplus} \mathcal{P}_\alpha d\mu(\alpha), \qquad S = Q/2 + i\mathbb{R}.$$

The isomorphism can explicitly be described in terms of $C_{\alpha_2,\alpha_1} : L^2(\mathbb{R} \times \mathbb{R}) \to L^2(\mathcal{S} \times \mathbb{R}, d\mu(\alpha_3)dx_3)$,

$$C_{\alpha_2,\alpha_1}(f(x_2,x_1)) = F_f(\alpha_3,x_3) = \int_{\mathbb{R}} \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix} f(x_2,x_1)dx_2dx_1,$$

with $d\mu(\alpha) = |S_b(2\alpha)|^2$ *and the Clebsch-Gordan coefficients* $\begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}$ *are given by:*

$$\begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix} = e^{-\frac{\pi i}{2}(\Delta_{\alpha_3} - \Delta_{\alpha_2} - \Delta_{\alpha_1})} D_b(\beta_{32}; y_{32}) D_b(\beta_{31}; y_{31}) D_b(\beta_{21}; y_{21}),$$

with $\Delta_\alpha = \alpha(Q-\alpha)$, $D_b(\alpha; y) = \frac{S_b(y)}{S_b(y+\alpha)}$ *and*

$$\begin{aligned} y_{32} &= i(x_3 - x_2) + \frac{1}{2}(\alpha_3 - \alpha_2), & \beta_{32} &= -\alpha_3 + \alpha_2 - \alpha_1 + Q, \\ y_{31} &= -i(x_3 - x_1) + \frac{1}{2}(\alpha_3 - \alpha_1), & \beta_{31} &= -\alpha_3 + \alpha_1 - \alpha_1 + Q, \\ y_{21} &= -i(x_2 - x_1) - \frac{1}{2}(\alpha_2 + \alpha_1) - Q + \alpha_3, & \beta_{21} &= \alpha_3 + \alpha_2 + \alpha_1 - Q. \end{aligned}$$

Furthermore, the isomorphism is $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$*-linear, so we have*

$$C^{\alpha_3}_{\alpha_2,\alpha_1}(\pi_{21}(U)f(x_2,x_1)) = \pi_{\alpha_3}(U)C^{\alpha_3}_{\alpha_2,\alpha_1} f(x_2,x_1), \qquad U \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})),$$

where $C^{\alpha_3}_{\alpha_2,\alpha_1} f(x_2,x_1) = F_f(\alpha_3,x_3) \in \mathcal{P}_{\alpha_3}$ *is the projection of* $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ *into* $\mathcal{P}_{\alpha_3}$.

Note, the last remark in the above theorem shows that the actions of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ intertwine with our decomposition coordinate wise. But since it is true for any $\alpha_3 \in \mathcal{S}$, it follows that the action of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ intertwines with $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ and $\int_{\mathcal{S}}^{\oplus} \mathcal{P}_\alpha d\mu(\alpha)$. Also, we could write the representation belonging to $\int_{\mathcal{S}}^{\oplus} \mathcal{P}_\alpha d\mu(\alpha)$ as $\int_{\mathcal{S}}^{\oplus} \pi_\alpha d\mu(\alpha)$, which again shows a similarity with direct sums. As we get $\int_{\mathcal{S}}^{\oplus} \pi_\alpha d\mu(\alpha)(U)F = \int_{\mathcal{S}}^{\oplus} \pi_\alpha(U)F_\alpha d\mu(\alpha)$, where $F \in \int_{\mathcal{S}}^{\oplus} \mathcal{P}_\alpha d\mu(\alpha)$ and F_α is the projection of F on $\mathcal{P}_\alpha$. In other words, the representation $\int_{\mathcal{S}}^{\oplus} \pi_\alpha d\mu(\alpha)$ acts coordinate wise on the elements of $\int_{\mathcal{S}}^{\oplus} \mathcal{P}_\alpha d\mu(\alpha)$. Furthermore, recall that the representations π_{α_3} were integrable. Assuming that we indeed have that the map C_{α_2,α_1} is a unitary isomorphism, it follows that $\pi_{\alpha_2} \otimes \pi_{\alpha_1}$ is also an integrable representation. This follows from the fact that the properties of proposition 7.21 hold for $\pi_{\alpha_2} \otimes \pi_{\alpha_1}$ by constructing the self-adjoint extension via $C^{-1}_{\alpha_2,\alpha_1}\left(\int_{\mathcal{S}}^{\oplus} \pi_{\alpha_3} d\mu(\alpha_3)\right)$. The other two properties also directly follow by using this inverse mapping.

8.3 Some properties of the Clebsch-Gordan coefficients

Since the proof of theorem 8.3 is rather long, we will do that in the next part. For now, we will focus on some small facts about the Clebsch-Gordan coefficients.

Corollary 8.4. The Clebsch-Gordan coefficients $\begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}$ have the following orthogonality and completeness relations:

$$\int_{\mathbb{R}} \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}^* \begin{bmatrix} \beta_3 & \alpha_2 & \alpha_1 \\ y_3 & x_2 & x_1 \end{bmatrix} dx_2 dx_1 = |S_b(2\alpha_3)|^{-2} \delta(\alpha_3 - \beta_3)\delta(x_3 - y_3),$$
$$\int_{\mathcal{S}} |S_b(2\alpha_3)|^2 d\alpha_3 \int_{\mathbb{R}} \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}^* \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & y_2 & y_1 \end{bmatrix} dx_3 = \delta(x_2 - y_2)\delta(x_1 - y_1).$$

Here $\begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}^*$ denotes the complex conjugate of $\begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}$.

This corollary is similar to proposition 6.28, where we saw an orthogonality relation of the 3j-symbols for the Verma modules of the quantum case.

Recall that we have an equivalence between the modules $\mathcal{P}_\alpha$ and $\mathcal{P}_{Q-\alpha}$. At the moment the Clebsch-Gordan coefficients $\begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}$ have nice analytic properties, but it is not true that $C^{\alpha_3}_{\alpha_2,\alpha_1}$ intertwines with the unitary operator $\mathcal{I}_\alpha$ of lemma 7.14. But, since $\mathcal{P}_\alpha$ and $\mathcal{P}_{Q-\alpha}$ are equivalent, it could be natural to define $C^{\alpha_3}_{\alpha_2,\alpha_1}$ in such a way that it acts similar if we change α_i with $Q - \alpha_i$.

Lemma 8.5. Let $\mathcal{I}_\alpha$ be defined as in the proof of lemma 7.14 and set
$\begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}^{\text{int}} = N(\alpha_3, \alpha_2, \alpha_1) \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}$. Then

$$C^{\alpha_3}_{\alpha_2,\alpha_1}(1 \otimes \mathcal{I}_{Q-\alpha_1}) = C^{\alpha_3}_{\alpha_2,Q-\alpha_1}, \quad C^{\alpha_3}_{\alpha_2,\alpha_1}(\mathcal{I}_{Q-\alpha_2} \otimes 1) = C^{\alpha_3}_{Q-\alpha_2,\alpha_1}, \quad \mathcal{I}_{\alpha_3} C^{\alpha_3}_{\alpha_2,\alpha_1} = C^{Q-\alpha_3}_{\alpha_2,\alpha_1}$$

is true if $N(\alpha_3, \alpha_2, \alpha_1) = (S_b(2Q - \alpha_1 - \alpha_2 - \alpha_3) S_b(Q - \alpha_1 - \alpha_2 + \alpha_3) S_b(\alpha_1 - \alpha_2 + \alpha_3) S_b(-\alpha_1 + \alpha_2 + \alpha_3))^{-\frac{1}{2}}$, with $C^{\alpha_3}_{\alpha_2,\alpha_1}(f(x_2, x_1)) = \int_{\mathbb{R}} \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}^{\text{int}} f(x_2, x_1) dx_2 dx_1$.

This lemma is proven in [3]. Due to this lemma, we see that first definition of the Clebsch-Gordan coefficients, theorem 8.3, is not the only possible one. To be precise, we could let $N(\alpha_3, \alpha_2, \alpha_1)$ be any function and the intertwining property $C^{\alpha_3}_{\alpha_2,\alpha_1} \pi_{21}(U) = \pi_{\alpha_3}(U) C^{\alpha_3}_{\alpha_2,\alpha_1}$, $U \in \mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ would still be satisfied if we use the above definition for the Clebsch-Gordan coefficients. It follows that there is no canonical way to define the Clebsch-Gordan coefficients for our module $\mathcal{P}_\alpha$.

8.3.1 Proof of theorem 8.3

We will now finally show that the decomposition given in theorem 8.3 is well-defined and correct. This will be done in a similar way as in the classical and quantum case, subsection 3.3.1 and section 6.3. In those cases, both sides of $V(n) \otimes V(m) \simeq \bigoplus_k V(n + m - 2k)$ where constructed in such a way that they were generated by eigenvectors of H or K respectively. It was then only noted that one could get an explicit value for the Clebsch-Gordan coefficients by using the fact that the Casimir and the quantum Casimir element acted as a constant on the spaces $V(n + m - 2k)$. This time, we will use this fact to show that the Clebsch-Gordan coefficients $\begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}$ are of the desired form.

Thus, we will first get a decomposition of the space $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ into eigenspaces of the operator $\pi_{21}(K)$. Then, since C_q acts as a scalar on the space $\mathcal{P}_\alpha$, thus also on all of the $\mathcal{P}_{\alpha_3}$, we will find eigenfunctions of $\pi_{21}(C_q)$. These can then be used to show that the Clebsch-Gordan coefficients are as defined in theorem 8.3. This last step will be done by constructing a unitary operator from $\int_{\mathcal{S}}^{\oplus} \mathcal{P}_\alpha d\mu(\alpha)$ to $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$, using the eigenfunctions of $\pi_{21}(C_q)$. In other words, we will show that

C_{α_2,α_1} is an isomorphism by constructing an inverse map.
Now, since we do not just have a direct sum in this case, we have to do some more work. Recall that the direct integral $\int_{\mathcal{S}}^{\oplus} \mathcal{P}_\alpha d\mu(\alpha)$ contains elements $F(\alpha, x)$ in such a way that $F(\alpha, x) \in \mathcal{P}_\alpha$ and

$$\int_{\mathcal{S}} ||F(\alpha, x)||^2 d\mu(\alpha) < \infty.$$

So, we need to show that the measure μ satisfies this definition. However, note that

$$\int_{\mathcal{S}} ||F_f(\alpha_3, x_3)||^2_{\mathcal{P}_{\alpha_3}} d\mu(\alpha_3) = \int_{\mathcal{S}}\int_{\mathbb{R}} F_f(\alpha_3, x_3)F_f(\alpha_3, x_3)^* dx_3 d\mu(\alpha_3) = \int_{\mathcal{S}}\int_{\mathbb{R}} |F_f(\alpha_3, x_3)|^2 dx_3 d\mu(\alpha_3),$$

thus the extra condition of the direct integral will be satisfied by the fact that $F_f(\alpha_3, x_3) \in L^2(\mathcal{S} \times \mathbb{R}, d\mu(\alpha_3)dx_3)$. As the condition $\int_{\mathcal{S}} ||F_f(\alpha_3, x_3)||^2_{\mathcal{P}_{\alpha_3}} d\mu(\alpha_3) < \infty$ is equivalent to $F_f(\alpha_3, x_3)$ being square-integrable.
Lastly, we need to show that the projection of $F \in \int_{\mathcal{S}}^{\oplus} \mathcal{P}_\alpha d\mu(\alpha)$ onto $\mathcal{P}_\alpha$, intertwines with the action of U, $U \in \mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$. In other words, that we indeed have

$$C^{\alpha_3}_{\alpha_2,\alpha_1}(\pi_{21}(U)f(x_2, x_1)) = \pi_{\alpha_3}(U)C^{\alpha_3}_{\alpha_2,\alpha_1} f(x_2, x_1), \qquad U \in \mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R})).$$

The first step is to decompose $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ into eigenspaces of $\pi_{21}(K)$. We can do this by decomposing all of $L^2(\mathbb{R} \times \mathbb{R})$ into eigenspaces. Similar to showing that $\mathcal{P}_\alpha$ is a maximal subset and $\mathcal{P}_\alpha \simeq \mathcal{P}_{Q-\alpha}$, we will also use a Fourier-transformation to construct eigenspaces of $\pi_{21}(K)$.
Let $\mathcal{F}\colon L^2(\mathbb{R} \times \mathbb{R}) \to L^2(\mathbb{R} \times \mathbb{R});\ f(x_2, x_1) \mapsto F(\kappa_3, \alpha_3) = \int_{\mathbb{R}} e^{-\pi i\kappa_3 x_+} f(\frac{x_+ + x_-}{2}, \frac{x_+ - x_-}{2})dx_+$, with $x_\pm = x_2 \pm x_1$. Then,

$$\begin{aligned} \mathcal{F}(\pi_{21}(K)f(x_2, x_1)) &= \int_{\mathbb{R}} e^{-\pi i\kappa_3 x_+} f\left(\frac{x_+ + x_- + ib}{2}, \frac{x_+ - x_- + ib}{2}\right) dx_+ \\ &= \int_{\mathbb{R}} e^{-\pi i\kappa_3 (x_+ - ib)} f\left(\frac{x_+ + x_-}{2}, \frac{x_+ - x_-}{2}\right) dx_+ = e^{-\pi b\kappa_3} F(\kappa_3, x_-). \end{aligned}$$

So we get an eigenspace decomposition of $L^2(\mathbb{R} \times \mathbb{R})$ into spaces $L_{\kappa_3} = \{F(\kappa_3, x_-) \in L^2(\mathbb{R})\}$, with $\kappa_3 \in \mathbb{R}$, of functions with a fixed $\kappa_3 \in \mathbb{R}$. Thus we get the decomposition $L^2(\mathbb{R} \times \mathbb{R}) = \int_{\mathbb{R}}^{\oplus} L_{\kappa_3} d\kappa_3 \simeq \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}) d\kappa_3$.
Now, let $C_q' = C_q - \frac{2}{(q-q^{-1})^2}$. Then C_q' is still a central elements of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ and also acts as a scalar on $\mathcal{P}_\alpha$. The action of $\pi_{21}(C_q')$ on $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ is given by

$$\begin{aligned} &-\pi_2(FE) \otimes \pi_1(K)^2 - \pi_2(FK^{-1}) \otimes \pi_1(KE) - \pi_2(K^{-1}E) \otimes \pi_1(FK) - \pi_2(K^{-1})^2 \otimes \pi(FE) \\ &- \frac{1}{(q-q^{-1})^2}\left(q\pi_2(K)^2 \otimes \pi_1(K)^2 + q^{-1}\pi_2(K^{-1})^2 \otimes \pi_1(K^{-1})^2 + 2\right) \end{aligned}$$

Thus it follows that the transformed action of C_q' under the Fourier-transform $\mathcal{F}$ is given by

$$\begin{aligned} C_{21}(\kappa_3) - [\alpha_3 - \tfrac{Q}{2}]_b^2 = [&-ix_- - \tfrac{1}{2}(\alpha_1 + \alpha_2 - Q) + (\alpha_3 - \tfrac{Q}{2})]_b[-ix - \tfrac{1}{2}(\alpha_1 + \alpha_2 - Q) - (\alpha_3 - \tfrac{Q}{2})]_b \\ &- [-ix + \tfrac{1}{2}(\alpha_1 + \alpha_2) - Q]_b \left(e^{i\pi b(-ix - \frac{1}{2}(\alpha_1+\alpha_2))}\{\alpha_- \alpha_2 + i\kappa_3\}_b\right. \\ &\qquad \left. - e^{-i\pi b(-ix - \frac{1}{2}(\alpha_1+\alpha_2))}\{\alpha_1 - \alpha_2 - i\kappa_3\}_b\right) T_{x_-}^{-ib} \\ &+ [-ix + \tfrac{1}{2}(\alpha_1 + \alpha_2) - Q]_b[-ix + \tfrac{1}{2}(\alpha_1 + \alpha_2) - 2Q]_b T_{x_-}^{-2ib}, \end{aligned}$$

where $C_{21}(\kappa_3)$ denotes the operator that acts as $C_{21}(\kappa_3)\mathcal{F}(f(x_2, x_1)) = \mathcal{F}(C_q' f(x_2, x_1))$, i.e. it acts as the transformed action of C_q', with $[x]_b = \frac{\sin(\pi bx)}{\sin(\pi b^2)}$ and $\{x\}_b = \frac{\cos(\pi bx)}{i\sin(\pi b^2)}$. The full calculations of the

action of $C_{21}(\kappa_3)$ on $\mathcal{F}(f(x_2,x_1))$ can be found in Appendix E.
Notice that the action of $C_{21}(\kappa_3)$ only shifts $F(\kappa_3,x_-)$ in the x_- term. Thus, this action leaves the eigenspaces L_{κ_3} of $\pi_{21}(K)$ invariant. Furthermore, it follows that

$$\begin{aligned}
&\Phi_{\alpha_3}(\alpha_2,\alpha_1|\kappa_3|x) = M^{\alpha_3;\kappa_3}_{\alpha_2,\alpha_1} e^{\pi x(2Q-2\alpha_3-2\alpha_2+i\kappa_3)}\Theta_b(T,y_-)\Psi_b(U,V,W;y_+),\\
&y_+ = -ix-\tfrac{1}{2}(\alpha_1+\alpha_2)+\alpha_3, \quad y_- = -ix-\tfrac{1}{2}(\alpha_1+\alpha_2)+Q-\alpha_3,\\
&T=\alpha_3+\alpha_2+\alpha_1-Q, \quad U=Q-\alpha_3-\alpha_2+\alpha_1, \quad V=Q-i\kappa_3-\alpha_3, \quad W=Q-i\kappa_3-\alpha_2+\alpha_1,
\end{aligned}$$

is a solution to the eigenfunction equation $\left(C_{21}(\kappa_3)-[\alpha_3-\frac{Q}{2}]_b^2\right)^t\Phi=0$ and it is proven in [1] that $\{(\Phi_{\alpha_3})^* \,|\, \alpha_3\in\mathcal{S}\}$ is a set of generalised eigenfunctions for the operator $C_{21}(\kappa_3)$. In other words, the functions $\Phi^*_{\alpha_3}$ are eigenfunctions of $C_{21}(\kappa_3)$ in an algebraic sense, but do note that they are not necessarily elements of $\mathcal{P}_{\alpha_2}\otimes\mathcal{P}_{\alpha_1}$.

Lemma 8.6.

$$\begin{bmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ \kappa_3 & x_2 & x_1\end{bmatrix} = \int_{\mathbb{R}} e^{2\pi i\kappa_3 x_3}\begin{bmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{bmatrix} dx_3 = e^{-\pi i\kappa_3 x_+}\Phi_{\alpha_3}(\alpha_2,\alpha_1|\kappa_3|x_-),$$

given that $M^{\alpha_3;\kappa_3}_{\alpha_2,\alpha_1} = e^{\pi i\alpha_2(\alpha_2-Q+\alpha_3)}e^{-\pi i(Q-\alpha_3-i\kappa_3)(\alpha_2-\alpha_3)}$.

We then get the following corollary:

Corollary 8.7.
Let $\tilde{F}_f(\alpha_3,\kappa_3)=\int_{\mathbb{R}} e^{-\pi i\kappa_3 x_+}\Phi_{\alpha_3}(\alpha_2,\alpha_1|\kappa_3|x_-)f(x_2,x_1)dx_-dx_+ = \int_{\mathbb{R}}\begin{bmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ \kappa_3 & x_2 & x_1\end{bmatrix} f(x_2,x_1)dx_3dx_2dx_1$, then we get

$$f(x_2,x_1)=\int_{\mathbb{R}} e^{\pi i\kappa_3 x_+}\int_{\mathcal{S}}(\Phi_{\alpha_3}(\alpha_2,\alpha_1|\kappa_3|x_-)^*\tilde{F}_f(\alpha_3,\kappa_3)d\mu(\alpha_3)d\kappa_3$$

Note, the above corollary gives us two maps that are both defined as a composition of a Fourier transform with another operator. The function $\tilde{F}_f(\alpha_3,\kappa_3)$ is obtained via first sending $f(x_2,x_1)\to\int_{\mathbb{R}}\Phi_{\alpha_3}(\alpha_2,\alpha_1|\kappa_3|x_-)f(x_2,x_1)dx_-$, then applying the Fourier transform. Then $f(x_2,x_1)$ is finally obtained by sending $\tilde{F}_f(\alpha_3,\kappa_3)\to\int_{\mathcal{S}}(\Phi_{\alpha_3}(\alpha_2,\alpha_1|\kappa_3|x_-)^*\tilde{F}_f(\alpha_3,\kappa_3)d\mu(\alpha_3)$, then applying the inversion of the Fourier transform.
We already know that the Fourier transform is a unitary operator on $\mathcal{P}_{\alpha_2}\otimes\mathcal{P}_{\alpha_1}$, even an isometry, or at least the extended Fourier transform. So, it now suffices to show that sending $f(x_2,x_1)\to\int_{\mathbb{R}}\Phi_{\alpha_3}(\alpha_2,\alpha_1|\kappa_3|x_-)f(x_2,x_1)dx_-$ is also a unitary operator, to show that C_{α_2,α_1} is an isometry.

Lemma 8.8. The Fourier transformation of Φ_{α_3}, $\tilde{\Phi}_{\alpha_3}(\omega)=\tilde{\Phi}_{\alpha-3}(\alpha_2,\alpha_1|\kappa_3|\omega)$, decays exponentially as $\omega\to\infty$. Furthermore, it has the following asymptotic behaviour as $\omega\to-\infty$:

$$\tilde{\Phi}_{\alpha_3}(\omega)=N_+(\alpha_3)e^{2\pi i\omega x_+}+N_-(\alpha_3)e^{2\pi i\omega x_-}+R_-(\omega),$$

with $R_-(\omega)$ decaying exponentially as $\omega\to-\infty$, $x_\pm=\frac{i}{2}(\alpha_1+\alpha_2-Q)\pm i(\alpha_3-\frac{Q}{2})$ and $|N_\pm(\alpha_3)|^2=|S_b(2\alpha_3)|^{-2}$.

Proposition 8.9. Due to the previous lemma, we can define an "inner product" $(\Phi_{\alpha_3},\Phi_{\alpha_3'})$ as a bi-distribution given by

$$(\Phi_{\alpha_3},\Phi_{\alpha_3'})=|N_+(\alpha_3)|^2\delta(\alpha_3-\alpha_3')=|S_b(2\alpha_3)|^{-2}\delta(\alpha_3-\alpha_3').$$

Proof. To show that $(\Phi_{\alpha_3},\Phi_{\alpha_3'})=|S_b(2\alpha_3)|^{-2}\delta(\alpha_3-\alpha_3')$, we will consider $(C_{21}(\kappa_3)\Phi_{\alpha_3},\Phi_{\alpha_3'})-(\Phi_{\alpha_3},C_{21}(\kappa_3)\Phi_{\alpha_3'})$, with $(f,g)=\int_{\mathbb{R}}f(x)^*g(x)dx$ and $f(x)^*=\overline{f(\overline{x})}$, and show that it is of the desired form.
Let $\Phi_{\alpha_3}(x)=\Phi_{\alpha_3}(\alpha_2,\alpha_1|\kappa_3|x)$ and note that

$$C_{21}(\kappa_3)-[\alpha_3-\tfrac{Q}{2}]_b^2=\delta_+(x)e^{\pi ibQ}e^{2\pi bx}-\delta_0(x)+\delta_-(x)e^{-\pi ibQ}e^{-2\pi bx},$$

with

$$\begin{aligned}
\delta_+(x) &= T_x^{-ib}[d_x - \alpha_2 + \tfrac{i}{2}\kappa_3]_b[d_x - \alpha_1 - \tfrac{i}{2}\kappa_3]_b,\\
\delta_0(x) &= \frac{1}{2}\{0\}_b\left(\{Q\}_b T_x^{-2ib} - \left(e^{\pi b\kappa_3}\{2\alpha_2 - Q\}_b + e^{-\pi b\kappa_3}\{2\alpha_1 - Q\}_b\right) T_x^{-ib} + \{2\alpha_3 - Q\}_b\right)\\
\delta_-(x) &= T_x^{-ib}[d_x + \alpha_2 + \tfrac{i}{2}\kappa_3]_b[d_x + \alpha_1 - \tfrac{i}{2}\kappa_3]_b.
\end{aligned}$$

Since $[\alpha_3 - \frac{Q}{2}]_b^2$ and $\delta_0(x)$ are self-adjoint, it follows that

$$\begin{aligned}
((C_{21}(\kappa_3) - [\alpha_3 - \tfrac{Q}{2}]_b^2)\Phi_{\alpha_3}, \Phi_{\alpha_3'}) - (\Phi_{\alpha_3}, (C_{21}(\kappa_3)[\alpha_3 - \tfrac{Q}{2}]_b^2)\Phi_{\alpha_3'}) &= (C_{21}(\kappa_3)\Phi_{\alpha_3}, \Phi_{\alpha_3'}) - (\Phi_{\alpha_3}, C_{21}\Phi_{\alpha_3'})\\
&= \sum_{j=\pm}(\delta_j(x)e^{\pi ibjQ}e^{2\pi bjx}\Phi_{\alpha_3}(x), \Phi_{\alpha_3'}(x)) - (\Phi_{\alpha_3}(x), \delta_j(x)e^{\pi ibjQ}e^{2\pi bjx}\Phi_{\alpha_3'}(x)).
\end{aligned}$$

Since lemma 8.8 shows the behaviour of $\tilde{\Phi}_{\alpha_3}(\omega)$, we want to know the Fourier transformed action of $\delta_j(x)e^{\pi ibjQ}e^{2\pi bjx}$ as $(f(x), g(x)) = (\tilde{f}(\omega), \tilde{g}(\omega))$. Do note that in lemma 8.8 the distributional Fourier-transform was used, but in this case we will be using the standard Fourier-transform, similar to [1]. This will imply that the behaviour of $\tilde{\Phi}_{\alpha_3}(\omega)$ as $\omega \to \infty$ is the same, but we get

$$\tilde{\Phi}_{\alpha_3}(\omega) \to N_+(\alpha_3)e^{-2\pi i\omega x_+} + N_-(\alpha_3)e^{-2\pi i\omega x_-} + R_-(\omega), \qquad \text{as } \omega \to -\infty.$$

Let $f \in \mathcal{P}_\alpha$, then

$$\begin{aligned}
&\int_{\mathbb{R}} e^{-2\pi i\omega x}\delta_+(x)e^{\pi ibQ}e^{2\pi bx}\ f(x)dx\\
&= \frac{e^{\pi ibQ}}{(q-q^{-1})^2}\int_{\mathbb{R}} e^{-2\pi i\omega x}\quad \Big(e^{-\pi ib(\alpha_2+\alpha_1)}e^{2\pi bx}f(x) + e^{\pi ib(\alpha_2+\alpha_1)}e^{2\pi b(x-2ib)}f(x-2ib)\\
&\qquad - \left(e^{\pi ib(-\alpha_2+\alpha_1+i\kappa_3)} + e^{-\pi ib(-\alpha_2+\alpha_1+i\kappa_3)}\right)e^{2\pi b(x-ib)}f(x-ib)\Big)\,dx\\
&= \frac{e^{\pi ibQ}}{(q-q^{-1})^2}\left(e^{-\pi ib(\alpha_2+\alpha_1)} + e^{\pi ib(\alpha_2+\alpha_1)}e^{4\pi b\omega} - \left((e^{\pi ib(-\alpha_2+\alpha_1+i\kappa_3)}) + e^{-\pi ib(-\alpha_2+\alpha_1+i\kappa_3)}\right)e^{2\pi b\omega}\right)T_\omega^{ib}\tilde{f}(\omega)\\
&= \frac{1}{(q-q^{-1})^2}\left(e^{\pi b(s_2+s_1)} + e^{-\pi b(s_2+s_1)}e^{4\pi b\omega}q^2 + \left((e^{\pi ib(s_2-s_1-\kappa_3}) + e^{-\pi b(s_2-s_1-\kappa_3)}\right)e^{2\pi b\omega}q\right)T_\omega^{ib}\tilde{f}(\omega)\\
&= \tilde{\delta_+}(\omega)\tilde{f}(\omega+ib)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}} e^{-2\pi i\omega x}\delta_-(x)e^{-\pi ibQ}e^{-2\quad\pi bx}f(x)dx\\
&= \frac{e^{-\pi ibQ}}{(q-q^{-1})^2}\int_{\mathbb{R}} e^{-2\pi i\omega x}\quad \Big(e^{\pi ib(\alpha_2+\alpha_1)}e^{-2\pi bx}f(x) + e^{-\pi ib(\alpha_2+\alpha_1)}e^{-2\pi b(x-2ib)}f(x-2ib)\\
&\qquad - \left(e^{\pi ib(\alpha_2-\alpha_1+i\kappa_3)} + e^{-\pi ib(\alpha_2-\alpha_1+i\kappa_3)}\right)e^{-2\pi b(x-ib)}f(x-ib)\Big)\,dx\\
&= \frac{e^{-\pi ibQ}}{(q-q^{-1})^2}\left(e^{\pi ib(\alpha_2+\alpha_1)} \quad + e^{-\pi ib(\alpha_2+\alpha_1)}e^{4\pi b\omega} - \left((e^{\pi ib(\alpha_2-\alpha_1+i\kappa_3}) + e^{-\pi ib(\alpha_2-\alpha_1+i\kappa_3)}\right)e^{2\pi b\omega}\right)T_\omega^{-ib}\tilde{f}(\omega)\\
&= \frac{1}{(q-q^{-1})^2}\left(e^{\pi b(-s_2-s_1)} + e^{-\pi b(-s_2-s_1)}e^{4\pi b\omega}q^{-2} + \left((e^{\pi b(-s_2+s_1-\kappa_3}) + e^{-\pi b(-s_2+s_1-\kappa_3)}\right)e^{2\pi b\omega}q^{-1}\right)T_\omega^{-ib}\tilde{f}(\omega)\\
&= \tilde{\delta_-}(\omega)\tilde{f}(\omega-ib),
\end{aligned}$$

with $\alpha_k = \frac{Q}{2} + is_k$, $k \in \{1,2\}$, where $\tilde{\delta_j}(\omega)$, $j \in \{+,-\}$, is used for a shorthand notation for the Fourier transformed action of δ_j. It follows that

$$\begin{aligned}
&(C_{21}(\kappa_3)\Phi_{\alpha_3}, \Phi_{\alpha'_3}) - (\Phi_{\alpha_3}, C_{21}(\kappa_3)\Phi_{\alpha'_3}) \\
&= \lim_{W\to\infty} \sum_{j=\pm} \int_{-W}^{W} \Big((\tilde{\delta_j}(\omega)\Phi_{\alpha_3}(\omega + jib))^* \Phi_{\alpha'_3}(\omega) - (\tilde{\Phi_{\alpha_3}}(\omega))^* \tilde{\delta_j}(\omega)\Phi_{\alpha'_3}(\omega + jib) \Big) d\omega \\
&= \lim_{W\to\infty} \sum_{j=\pm} - \int_{-W}^{-W-jib} (\tilde{\Phi_{\alpha_3}}(\omega))^* \tilde{\delta_j}(\omega)\Phi_{\alpha'_3}(\omega + jib) d\omega - \int_{W-jib}^{W} (\tilde{\Phi_{\alpha_3}}(\omega))^* \tilde{\delta_j}(\omega)\Phi_{\alpha'_3}(\omega + jib) d\omega,
\end{aligned}$$

as $\int_{-W}^{W} (\tilde{\Phi_{\alpha_3}}(\omega))^* \tilde{\delta_j}(\omega)\Phi_{\alpha'_3}(\omega + jib) d\omega = \int_{-W}^{-W-jib} (\tilde{\Phi_{\alpha_3}}(\omega))^* \tilde{\delta_j}(\omega)\Phi_{\alpha'_3}(\omega + jib) d\omega$ $+ \int_{-W-jib}^{W-jib} (\tilde{\Phi_{\alpha_3}}(\omega))^* \tilde{\delta_j}(\omega)\Phi_{\alpha'_3}(\omega + jib) d\omega + \int_{W-jib}^{W} (\tilde{\Phi_{\alpha_3}}(\omega))^* \tilde{\delta_j}(\omega)\Phi_{\alpha'_3}(\omega + jib) d\omega$, since neither $\tilde{\delta_j}(\omega)$ nor $\tilde{\Phi_{\alpha_3}}(\omega)$ have singularities inside this contour. Furthermore,

$$\begin{aligned}
&\int_{-W-jib}^{W-jib} (\tilde{\Phi_{\alpha_3}}(\omega))^* \tilde{\delta_j}(\omega)\Phi_{\alpha'_3}(\omega + jib) d\omega = \int_{-W}^{W} \overline{\tilde{\Phi_{\alpha_3}}}(\omega - jib)\tilde{\delta_j}(\omega - jib)\Phi_{\alpha'_3}(\omega) d\omega \\
&= \int_{-W}^{W} \overline{\tilde{\Phi_{\alpha_3}}}(\omega - jib)\overline{\tilde{\delta_j}}(\omega)\Phi_{\alpha'_3}(\omega) d\omega = \int_{-W}^{W} (\tilde{\delta_j}(\omega)\tilde{\Phi_{\alpha_3}}(\omega + jib))^* \Phi_{\alpha'_3}(\omega) d\omega,
\end{aligned}$$

since $e^{4\pi b(\omega - jib)} q^{j2} = e^{4\pi b\omega} q^{-j2}$ and $e^{2\pi b(\omega - jib)} q^{j1} = e^{2\pi b\omega} q^{-j1}$. Note that every other term of $\tilde{\delta_\pm}(\omega)$ is either real or invariant under complex conjugation. Thus this is indeed enough to prove the above equalities.

Now, by lemma 8.8, $\tilde{\Phi_{\alpha_3}}(\omega)$ decays exponentially as $\omega \to \infty$, hence

$$\lim_{W\to\infty} \int_{W}^{W-jib} (\tilde{\Phi_{\alpha_3}}(\omega))^* \tilde{\delta_j}(\omega)\tilde{\Phi_{\alpha'_3}}(\omega + jib) d\omega = 0,$$

and it follows that

$$(C_{21}(\kappa_3)\Phi_{\alpha_3}, \Phi_{\alpha'_3}) - (\Phi_{\alpha_3}, C_{21}(\kappa_3)\Phi_{\alpha'_3}) = \lim_{W\to\infty} \sum_{j=\pm} \int_{-W-jib}^{-W} (\tilde{\Phi_{\alpha_3}}(\omega))^* \tilde{\delta_j}(\omega)\tilde{\Phi_{\alpha'_3}}(\omega + jib) d\omega.$$

Now, $\tilde{\delta_j}(\omega) \to \frac{e^{\pi jb(s_2+s_1)}}{(q-q^{-1})^2}$ as $\omega \to -\infty$, in combination with the previous noted asymptotic behaviour of $\tilde{\Phi}_{\alpha_3}(\omega)$, we get

$$\begin{aligned}
&(C_{21}(\kappa_3)\Phi_{\alpha_3}, \Phi_{\alpha'_3}) - (\Phi_{\alpha_3}, C_{21}(\kappa_3)\Phi_{\alpha'_3}) \to \\
&\frac{1}{(q-q^{-1})^2} \lim_{W\to\infty} \sum_{j=\pm} \int_{-W-jib}^{-W} N_+(\alpha_3)^* N_+(\alpha'_3) e^{-2\pi i\omega(s_3 - s'_3)} e^{2\pi jbs'_3} + N_-(\alpha_3)^* N_-(\alpha'_3) e^{2\pi i\omega(s_3 - s'_3)} e^{-2\pi jbs'_3} \\
&\qquad + N_+(\alpha_3)^* N_-(\alpha'_3) e^{-2\pi i\omega(s_3 + s'_3)} e^{-2\pi jbs'_3} + N_-(\alpha_3)^* N_+(\alpha'_3) e^{2\pi i\omega(s_3 + s'_3)} e^{2\pi jbs'_3} d\omega \\
&= \frac{1}{(q-q^{-1})^2} \lim_{W\to\infty} \sum_{j=\pm} \int_{W-jib}^{W} N_+(\alpha_3)^* N_+(\alpha'_3) e^{2\pi i\omega(s_3 - s'_3)} e^{2\pi jbs'_3} + N_-(\alpha_3)^* N_-(\alpha'_3) e^{2\pi i\omega(-s_3 + s'_3)} e^{-2\pi jbs'_3} \\
&\qquad + N_+(\alpha_3)^* N_-(\alpha'_3) e^{2\pi i\omega(s_3 + s'_3)} e^{-2\pi jbs'_3} + N_-(\alpha_3)^* N_+(\alpha'_3) e^{-2\pi i\omega(s_3 + s'_3)} e^{2\pi jbs'_3} d\omega \\
&= \frac{1}{(q-q^{-1})^2} \lim_{W\to\infty} \sum_{j=\pm} \sum_{\varepsilon_1,\varepsilon_2=\pm} \frac{N_{\varepsilon_1}(\alpha_3)^* N_{\varepsilon_2}(\alpha'_3)}{2\pi i(\varepsilon_1 s_3 - \varepsilon_2 s'_3)} e^{2\pi iW(\varepsilon_1 s_3 - \varepsilon_2 s'_3)} e^{2\pi j\varepsilon_2 bs'_3} (1 - e^{2\pi jb(\varepsilon_1 s_3 - \varepsilon_2 s'_3)}).
\end{aligned}$$

According to the proof in [1], the above term simplifies to

$$\big([ip'_3]_b^2 - [ip_3]_b^2\big) \lim_{W\to\infty} |N_+(\alpha_3)|^2 \frac{e^{2\pi iW(p_3 - p'_3)} - e^{-2\pi iW(p_3 - p'_3)}}{2\pi i(p_3 - p'_3)},$$

due to the Riemann-Lebesgue Lemma and the fact that

$$\frac{e^{2\pi bp_3'}+e^{-2\pi bp_3'}-e^{2\pi bp_3}-e^{-2\pi bp_3}}{(q-q^{-1})^2}=[ip_3']_b^2-[ip_3]_b^2.$$

From this, it follows that

$$(\Phi_{\alpha_3},\Phi_{\alpha_3'})=\lim_{W\to\infty}|N_+(\alpha_3)|^2\frac{e^{2\pi iW(p_3-p_3')}-e^{-2\pi iW(p_3-p_3')}}{2\pi i(p_3-p_3')},$$

and we get

$$\int_{\mathcal{S}}\lim_{W\to\infty}\frac{e^{2\pi iW(p_3-p_3')}-e^{-2\pi iW(p_3-p_3')}}{2\pi i(p_3-p_3')}F_f(\alpha_3,\kappa_3)d\mu(\alpha_3)=$$
$$\int_{\mathcal{S}}\lim_{W\to\infty}\frac{e^{2\pi iW(p_3)}-e^{-2\pi iW(p_3)}}{2\pi i(p_3)}F_f(\alpha_3+\alpha_3',\kappa_3)d\mu(\alpha_3)=F_f(\alpha_3',\kappa_3)\cdot\mu(\alpha_3'),$$

due to the Dirichlet kernel. It then follows that

$$\lim_{a\to 0}\int_{\mathbb{R}}\frac{\sin(\pi x)}{\pi x}f(x)dx=f(0)$$

for any $f\in S(\mathbb{R})$. From this, it follows that, if we set $\mu(\alpha_3)=|N_+(\alpha_3)|^{-2}=|S_b(2\alpha_3)|^2$, then

$$\int_{\mathcal{S}}(\Phi_{\alpha_3},\Phi_{\alpha_3'})\tilde{F}_f(\alpha_3,\kappa_3)d\mu(\alpha_3)=|N_+(\alpha_3')|^2\mu(\alpha_3')\tilde{F}_f(\alpha_3',\kappa_3)=\tilde{F}_f(\alpha_3',\kappa_3).$$

In other words, we see that $(\Phi_{\alpha_3},\Phi_{\alpha_3'})$ acts as $|N_+(\alpha_3)|^2\delta(\alpha_3-\alpha_3')$ if we set $\mu(\alpha_3)=|S_b(2\alpha_3)|^2$. □

From this, it follows that the operator sending $f(x_2,x_1)\to\int_{\mathbb{R}}\Phi_{\alpha_3}(\alpha_2,\alpha_1|\kappa_3|x_-)f(x_2,x_1)dx_-=\tilde{F}_f(\alpha_3,x_+)$ is a unitary operator if we set $\mu(\alpha)=|S_b(2\alpha)|^2$. Its inverse is then given by $\tilde{F}_f(\alpha_3,x_+)\to\int_{\mathcal{S}}(\Phi_{\alpha_3}(\alpha_2,\alpha_1|\kappa_3|x_-)^*\tilde{F}_f(\alpha_3,x_+)d\mu(\alpha_3)$. Since the composition of two unitary operators is still unitary, it follows that both $\tilde{F}_f(\alpha_3,\kappa_3)$ and $f(x_2,x_1)$ in corollary 8.7 are defined via unitary operators. Thus it follows that the map C_{α_2,α_1} is indeed a unitary isomorphism.
To conclude the proof of theorem 8.3, it is only left to show that $C^{\alpha_3}_{\alpha_2,\alpha_1}$ satisfies the intertwining property and also that $C^{\alpha_3}_{\alpha_2,\alpha_1}f(x_2,x_1)\in\mathcal{P}_{\alpha_3}$ for all $\alpha_3\in\mathcal{S}$.

Proposition 8.10. The projection $C^{\alpha_3}_{\alpha_2,\alpha_1}$ maps $\mathcal{P}_{\alpha_2}\otimes\mathcal{P}_{\alpha_1}$ into $\mathcal{P}_{\alpha_3}$. Furthermore, we have

$$C^{\alpha_3}_{\alpha_2,\alpha_1}(\pi_{21}(U)f(x_2,x_1))=\pi_{\alpha_3}(U)C^{\alpha_3}_{\alpha_2,\alpha_1}f(x_2,x_1),\qquad\forall U\in\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})).$$

Proof. First, due to the fact that $\begin{bmatrix}\alpha_3&\alpha_2&\alpha_1\\\kappa_3&x_2&x_1\end{bmatrix}$ has poles at $\kappa_3=\pm(Q-\alpha_3+nb+mn^{-1})$, $n,m\in\mathbb{N}$, and $f(x_2,x_1)$ is entire, it follows that the Fourier-transform of $F_f(\alpha_3,x_3)$,

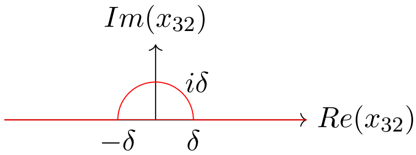
$$\tilde{F}_f(\alpha_3,\kappa_3)=\int_{\mathbb{R}^2}\begin{bmatrix}\alpha_3&\alpha_2&\alpha_1\\\kappa_3&x_2&x_1\end{bmatrix}f(x_2,x_1)dx_2dx_1,$$

has poles of the same form as elements of $\mathcal{P}_{\alpha_3}$. To show that $F_f(\alpha_3,x_3)$ is entire, we will give an explicit analytic continuation to $x_3\in\{x_3\in\mathbb{C}\,|\,im(x_3)\in[0,\frac{b}{2}]\}$ and $x_3\in\{x_3\in\mathbb{C}\,|\,im(x_3)\in[-\frac{b}{2},0]\}$. This can then be generalized to any $\{x_3\in\mathbb{R}\,|\,im(x_3)\in[0,\frac{nb}{2}]\}$, $n\in\mathbb{Z}$, and also holds for b^{-1}, due to the invariance of the Clebsch-Gordan coefficients for the change $b\to b^{-1}$ and also due to the fact that the proof is symmetric for the change $b\to b^{-1}$, since the poles we are dealing with are of the form $x_{3j}=\pm i(\varepsilon+nb+mb^{-1})$, $n,m\in\mathbb{N}$, $j\in\{2,1\}$, $\varepsilon>0$ and small.

Now, one can construct an explicit analytic continuation to $\{x_3 \in \mathbb{C}\,|\,im(x_3) \in [0, \frac{b}{2}]\}$ by first integrating over x_1, then by integrating $x_{32} = x_3 - x_2 - \frac{1}{2}(s_3 + s_2)$ along the contour C_1 which is the union of $(-\infty, -\delta]$, $[\delta, \infty)$ and the half-circle in the upper-half plane at $x_{32} = 0$ with radius δ, $b > \delta > \frac{b}{2}$. In other words, one gets

$$F_f(\alpha_3, x_3) = \int_{C_1}\int_{\mathbb{R}} \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix} f(x_2, x_1) dx_1 d_{x_3}2,$$

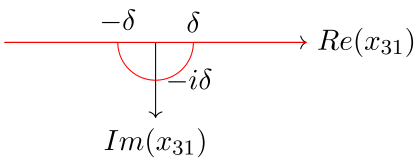
for an explicit analytic continuation, with C_1 visualised by:



To construct an explicit continuation to $\{x_3 \in \mathbb{C}\,|\,im(x_3) \in [-\frac{b}{2}, 0]\}$, one first integrates along x_2. Then integrates $x_{31} = x_3 - x_1 + \frac{1}{2}(s_3 + s_1)$ along the contour C_2, which is the union of $(-\infty, -\delta]$, $[\delta, \infty)$ and the half-circle in the lower-half plane at $x_3 1 = 0$ with radius δ, $b > \delta > \frac{b}{2}$. In other words, one gets

$$F_f(\alpha_3, x_3) = \int_{C_2}\int_{\mathbb{R}} \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix} f(x_2, x_1) dx_2 d_{x_3}1,$$

for an explicit analytic continuation, with C_2 visualised by:



Note, in both cases the half-circle is needed to avoid the pole in $\pm i(\varepsilon + nb + mb^{-1})$. Since this construction can be extended to $\{x_3 \in \mathbb{C}\,|\,im(x_3) \in [0, \frac{nb}{2}]\}$ for $n \in \mathbb{Z}$ and also works when b is replaced by b^{-1}, it follows that $F_f(\alpha_3, x_3)$ is entire. In other words, $C^{\alpha_3}_{\alpha_2,\alpha_1} f(x_2, x_1) \in \mathcal{P}_{\alpha_3}$ for any $f(x_2, x_1) \in \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$.
The last thing to prove is that the projection $C^{\alpha_3}_{\alpha_2,\alpha_1}$ is also $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$-linear. To do this, we will rewrite $C_{\alpha_2,\alpha_1}\pi_{21}(U) f(x_2, x_1)$, $U \in \{E, F, K\}$, then use the above analytic continuations to get to $\pi_3(U) F_f(\alpha_3, x_3)$. The analytic continuation is needed due to the fact that $\pi_3(U)$ contains the shift operator $T_{x_3}^{\frac{\pm ib}{2}}$ as $U \in \{E, F, K\}$.
First, note that $\pi_{21}(U)$ consists of a linear combination of the shift operators $T_{x_2}^{\frac{ib}{2}} T_{x_1}^{\frac{ib}{2}}$, $T_{x_2}^{-\frac{ib}{2}} T_{x_1}^{\frac{ib}{2}}$ and $T_{x_2}^{-\frac{ib}{2}} T_{x_1}^{-\frac{ib}{2}}$. Due to this, we are mostly concerned with how one would calculate $\int_{\mathbb{R}^2} \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix} T_{x_i}^{\pm\frac{ib}{2}} f(x_2, x_1) dx_2 dx_1$. Now,

$$\int_{\mathbb{R}^2} \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix} T_{x_2}^{\pm\frac{ib}{2}} f(x_2, x_1) dx_2 dx_1 = \int_{\mathbb{R}^2} \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix} f(x_2 \pm \tfrac{ib}{2}, x_1) dx_2 dx_1$$
$$= \int_{\mathbb{R}\mp\frac{ib}{2}}\int_{\mathbb{R}} \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix} f(x_2 \pm \tfrac{ib}{2}, x_1) dx_1 dx_2 + Res = \int_{\mathbb{R}}\int_{\mathbb{R}} \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 \mp \frac{ib}{2} & x_1 \end{bmatrix} f(x_2, x_1) dx_1 dx_2 + Res,$$

where we shifted the contour of x_2 to $\mathbb{R} \mp \frac{ib}{2}$ in the second step and Res is the contribution of the residue due to this shift. This would be the contribution of the pole at $x_2 = x_1 - \frac{i}{2}(2\alpha_3 - \alpha_1 - \alpha_2) -$

$i(\varepsilon+nb+mb^{-1})$ or $x_1 = x_2 - \frac{i}{2}(\alpha_1+\alpha_2-2\alpha_3)+i(\varepsilon+nb+mb^{-1})$, $n,m\in\mathbb{N}$, in the case of a shift in x_1. Since $\pi_{21}(U)$, $U\in\{E,F,K\}$, only contains shift operators as described above, it follows that we get

$$\int_{\mathbb{R}^2}\begin{bmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{bmatrix}\pi_{21}(U)f(x_2,x_1)dx_1dx_2 = \int_{\mathbb{R}^2}\left(\pi_{21}(U)^t\begin{bmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{bmatrix}\right)f(x_2,x_1)dx_2dx_1$$
$$=\int_{C_2}\int_{C_1}\left(\pi_{21}(U)^t\begin{bmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{bmatrix}\right)f(x_2,x_1)dx_2dx_1,$$

where $\pi_{21}(U)^t$ denotes the operator that acts on $\begin{bmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{bmatrix}$ via the above described shifts in the integral. For example, $(T_{x_1}^{\frac{ib}{2}})^t\begin{bmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{bmatrix} = \begin{bmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1-\frac{ib}{2}\end{bmatrix}$.
Since we have that $\pi_{21}(U)^t\begin{bmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{bmatrix} = \pi_3(U)\begin{bmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{bmatrix}$ for $U\in\{E,F,K\}$ [1], this follows from the functional equation of $S_b(x)$ as seen in Appendix D: Special Functions, It follows that we get

$$\int_{\mathbb{R}^2}\begin{bmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{bmatrix}\pi_{21}(U)f(x_2,x_1)dx_1dx_2 = \int_{\mathbb{R}^2}\pi_3(U)\begin{bmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{bmatrix}f(x_2,x_1)dx_1dx_2,$$

for $U\in\{E,F,K\}$. Lastly, since E,F,K generate $U_q(\mathfrak{sl}(2),\mathbb{R})$. It follows that the projection onto $\mathcal{P}_{\alpha_3}$, $C^{\alpha_3}_{\alpha_2,\alpha_1}$, is indeed $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-linear for any $\alpha_3\in\frac{Q}{2}+i\mathbb{R}$. □

We now see that the map $C_{\alpha_2,\alpha_1}\colon L^2(\mathbb{R}^2)\to L^2(\mathcal{S}\times\mathbb{R}, d\mu(\alpha_3)dx_3)$ is indeed a unitary isomorphism that explicitly describes how to decompose

$$\mathcal{P}_{\alpha_2}\otimes\mathcal{P}_{\alpha_1}\simeq\int_{\mathcal{S}}^{\oplus}\mathcal{P}_{\alpha_3}d\mu(\alpha_3).$$

Again, notice that the condition of $C^{\alpha_3}_{\alpha_2,\alpha_1}$ being $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-linear is necessary and sufficient for C_{α_2,α_1} to be a $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-module morphism. Without this fact, we would only have an isomorphism between the tensor product $\mathcal{P}_{\alpha_2}\otimes\mathcal{P}_{\alpha_1}$ and $\int_{\mathcal{S}}^{\oplus}\mathcal{P}_{\alpha_3}d\mu(\alpha_3)$ as vector spaces.

In the last two sections we showed how one could find the values of the Clebsch-Gordan coefficients and a similar strategy could also be used in sections 3 and 6 to calculate the Clebsch-Gordan coefficients of the Verma modules for the classical and the quantum case. However, since the operators in those two cases aren't unbounded operators, finding eigenfunctions does become a lot simpler. But the idea is similar.
In the next section we will that we can generalise the strategy to calculate the Clebsch-Gordan coefficients. Since we will be looking at the Racah-Wigner coefficients. In short, these coefficients describe a similar relation as the Clebsch-Gordan coefficients, but this time for the triple tensor $\mathcal{P}_{\alpha_3}\otimes\mathcal{P}_{\alpha_2}\otimes\mathcal{P}_{\alpha_1}$ instead of the regular tensor $\mathcal{P}_{\alpha_2}\otimes\mathcal{P}_{\alpha_1}$. However, we will not give an explicit calculation, but only describe how one could find an explicit formula, similar to sections 3 and 6.
Even though we will not do any explicit calculations, it is still useful to shortly look at these Racah-Wigner coefficients. As that will show how we could decompose any number of tensors of $\mathcal{P}_\alpha$ into a direct integral of simple modules. Concluding that we are not restricted to only the tensor product of two irreducible representations.

9 The Racah-Wigner coefficients for the module $\mathcal{P}_\alpha$

We have seen that we can decompose the tensor product $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ into a direct integral of simple modules. One could wonder if this process could be generalised to the tensor product of more than two modules. And this is indeed possible by decomposing "smaller" tensor products with known methods, until one gets a full decomposition.
In this section, we will look at how to decompose a triple tensor product of modules $\mathcal{P}_\alpha$ and we will also look at what problems one will find in that case. Most notable, similar to the case of $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$, we will find self-adjoint extensions for the operators $(\pi_{\alpha_3} \otimes \pi_{\alpha_2} \otimes \pi_{\alpha_1} \circ (\Delta \otimes id) \circ \Delta)(U)$, $U \in \{E, F, K\}$. But, since we will soon see that we can construct such an extension in two ways, it is not immediately clear that the self-adjoint extensions of these operators are unique.
Do note that this question arises due to the fact that the action of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ is integrable and unbounded. Since we are only talking about self-adjoint extensions due to the unboundedness. So this is mainly a problem we get due to our specific setting.

9.1 The decomposition of the triple tensor product

Let $\alpha_3, \alpha_2, \alpha_1 \in \mathcal{S} = \frac{Q}{2} + i\mathbb{R}$. Then, we want to decompose the triple tensor product $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ into a direct integral of irreducible representations. Now, the representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ on $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ is given by

$$\pi_{321} = (\pi_{\alpha_3} \otimes \pi_{\alpha_2} \otimes \pi_{\alpha_1}) \circ (\Delta \otimes id) \circ \Delta = (\pi_{\alpha_3} \otimes \pi_{\alpha_2} \otimes \pi_{\alpha_1}) \circ (id \otimes \Delta) \circ \Delta.$$

To decompose the module $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$, we will use what we know of the previous sections and we will also decompose it in two canonical ways. Note, $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1} = (\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2}) \otimes \mathcal{P}_{\alpha_1} = \mathcal{P}_{\alpha_3} \otimes (\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1})$, so we can see the triple tensor as the tensor product of $(\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2})$ and $\mathcal{P}_{\alpha_1}$ or as a the tensor product of $(\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1})$ and $\mathcal{P}_{\alpha_3}$. Since we already know how to fully decompose the tensor product of $\mathcal{P}_\alpha \otimes \mathcal{P}_\beta$ for arbitrary $\alpha, \beta \in \mathcal{S}$ with the isomorphism $C_{\alpha,\beta}$, we can first decompose $(\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2})$ into $\int_{\mathcal{S}}^{\oplus} \mathcal{P}_{\alpha_t} d\mu(\alpha_t)$, then decompose $\mathcal{P}_{\alpha_t} \otimes \mathcal{P}_{\alpha_1}$, $\forall \alpha_t \in \mathcal{S}$, to fully decompose the triple tensor. This decomposition is often referred to as the *t-channel*.
Similarly, we can decompose $\mathcal{P}_{\alpha_3} \otimes (\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1})$ by first applying $id \otimes C_{\alpha_2,\alpha_1}$ and then decomposing $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_s}$, $\forall \alpha_s \in \mathcal{S}$, which is often referred to as the *s-channel*. We will first describe the t-channel in more detail.
Define a unitary map $C_{(32)1} \colon L^2(\mathbb{R}^3) \to L^2(\mathcal{S}^2 \times \mathbb{R}, d\mu(\alpha_4) d\mu(\alpha_t) dx_4)$, $f(x_3, x_2, x_1) \mapsto F_f^t(\alpha_3, \alpha_t, x_4)$, then F_f^t is given by

$$F_f^t(\alpha_4, \alpha_t, x_4) = \lim_{\varepsilon_2 \downarrow 0} \lim_{\varepsilon_1 \downarrow 0} \int_{\mathbb{R}^2} \begin{bmatrix} \alpha_4 & \alpha_t & \alpha_1 \\ x_4 & x_t & x_1 \end{bmatrix}_{\varepsilon_2} \int_{\mathbb{R}^2} \begin{bmatrix} \alpha_t & \alpha_3 & \alpha_2 \\ x_t & x_3 & x_2 \end{bmatrix}_{\varepsilon_1} f(x_3, x_2, x_1) dx_3 dx_2 dx_1 dx_t,$$

$x_4 \in \mathbb{R}$, $\alpha_4, \alpha_t \in \mathcal{S}$. Note that this does indeed show that we first decompose $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2}$, then apply the unitary map of theorem 8.3 once more to fully decompose the triple tensor. To simplify notation, set $\mathbf{x} = (x_3, x_2, x_1)$ and $d\mathbf{x} = dx_3 dx_2 dx_1$, then we get

$$F_f^t(\alpha_4, \alpha_t, x_4) = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^3} \Phi_{\alpha_t}^t \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} (x_4, \mathbf{x}) f(\mathbf{x}) d\mathbf{x},$$

with

$$\Phi_{\alpha_t}^t \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} (x_4, \mathbf{x}) = \int_{\mathbb{R}} \begin{bmatrix} \alpha_4 & \alpha_t & \alpha_1 \\ x_4 & x_t & x_1 \end{bmatrix}_{\varepsilon} \begin{bmatrix} \alpha_t & \alpha_3 & \alpha_2 \\ x_t & x_3 & x_2 \end{bmatrix}_{\varepsilon} dx_t.$$

Since the map $C_{(32)1}$ is constructed as a composition of C_{α_3,α_2} and maps C_{α_t,α_1}, it follows that the action of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ on $\mathcal{P}_{\alpha_4}$ intertwines with the action of π_{321} on $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$. In other words, if $C_{(32)1}^{\alpha_4,t} f(\mathbf{x})$ is the projection of $F_f^t(\alpha_4, \alpha_t, x_4)$ onto $\mathcal{P}_{\alpha_4}^t$, then

$$\pi_{\alpha_4}(U) \left(C_{(32)1}^{\alpha_4,\alpha_t} f(\mathbf{x}) \right) = C_{(32)1} \left(\pi_{321}(U) f(\mathbf{x}) \right), \qquad \forall \alpha_4, \alpha_t \in \mathcal{S},\, U \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})),$$

similar to the intertwining property described in proposition 8.10.
Note, we need both α_t and α_4 as we have a double direct integral in this case, so there are two variables that act similar to coordinates. For example, only projecting onto α_4 would still give us a $\mu(\alpha_t)$-measurable vector field that depends on α_t. So we only get $L^2(\mathbb{R})$ functions after fixing both α_4 and $\alpha_t \in \mathcal{S}$.
From this, it also follows that

$$\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1} \simeq \int_{\mathcal{S}}^{\oplus} \int_{\mathcal{S}}^{\oplus} \mathcal{P}^t_{\alpha_4} d\mu(\alpha_4)\mu(\alpha_t).$$

As noted before, we have a second canonical way to decompose the triple tensor $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ by using the s-channel. Similar as above, define a unitary map $C_{3(21)} \colon L^2(\mathbb{R}^3) \to L^2(\mathcal{S}^2 \times \mathbb{R}, d\mu(\alpha_4)d\mu(\alpha_s)dx_4)$, $f(x_3, x_2, x_1) \mapsto F^s_f(\alpha_4, \alpha_s, x_4)$, with $F^s_f(\alpha_4, \alpha_s, x_4)$ given by

$$F^s_f(\alpha_4, \alpha_s, x_4) = \lim_{\varepsilon_2 \downarrow 0} \lim_{\varepsilon_1 \downarrow 0} \int_{\mathbb{R}^2} \begin{bmatrix} \alpha_4 & \alpha_3 & \alpha_s \\ x_4 & x_3 & x_s \end{bmatrix}_{\varepsilon_2} \int_{\mathbb{R}^2} \begin{bmatrix} \alpha_s & \alpha_2 & \alpha_1 \\ x_s & x_2 & x_1 \end{bmatrix}_{\varepsilon_1} f(x_3, x_2, x_1) dx_2 dx_1 dx_3 dx_s,$$

$x_4 \in \mathbb{R}$, $\alpha_4, \alpha_s \in \mathcal{S}$. This can then be simplified to

$$F^s_f(\alpha_4, \alpha_s, x_4) = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^3} \Phi^s_{\alpha_s} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} (x_4, \mathbf{x}) f(\mathbf{x}) d\mathbf{x},$$

with

$$\Phi^s_{\alpha_s} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} (x_4, \mathbf{x}) = \int_{\mathbb{R}} \begin{bmatrix} \alpha_4 & \alpha_3 & \alpha_s \\ x_4 & x_3 & x_s \end{bmatrix}_{\varepsilon} \begin{bmatrix} \alpha_s & \alpha_2 & \alpha_1 \\ x_s & x_2 & x_1 \end{bmatrix}_{\varepsilon} dx_s.$$

This then leads to the decomposition

$$\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1} \simeq \int_{\mathcal{S}}^{\oplus} \int_{\mathcal{S}}^{\oplus} \mathcal{P}^s_{\alpha_4} d\mu(\alpha_4) d\mu(\alpha_s)$$

and we also have a similar intertwining property given by

$$\pi_{\alpha_4}(U) \left(C^{\alpha_4, \alpha_s}_{3(21)} f(\mathbf{x}) \right) = C_{3(21)}(\pi_{321}(U) f(\mathbf{x})), \qquad \forall \alpha_4, \alpha_s \in \mathcal{S},\, U \in \mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R})).$$

Now, just like in the previous section, due to the fact that the maps $C_{(32)1}$ and $C_{3(21)}$ are unitary and the representations $\pi^{\alpha_t}_{\alpha_4}$ and $\pi^{\alpha_s}_{\alpha_4}$ are integrable, it follows that the representation π_{321} is also integrable. We can construct self-adjoint extensions of $\pi_{321}(U)$, $U \in \{E, F, K, C_q\}$, via the inverse maps $C^{-1}_{(32)1}$ and $C^{-1}_{3(21)}$. Call these self-adjoint extensions $\pi_{(32)1}(U) = C^{-1}_{(32)1} \int^{\oplus}_{\mathcal{S}^2} \pi^{\alpha_t}_{\alpha_4}(U) d\mu(\alpha_4) d\mu(\alpha_t)$ and $\pi_{3(21)}(U) = C^{-1}_{3(21)} \int^{\oplus}_{\mathcal{S}^2} \pi^{\alpha_s}_{\alpha_4}(U) d\mu(\alpha_4) d\mu(\alpha_s)$ respectively.
Now, since these two extensions are obtained via different unitary maps. It is not immediately clear that $\pi_{(32)1}(U)$ and $\pi_{3(21)}(U)$ act similar on all of $L^2(\mathbb{R}^3)$. In other words, the self-adjoint extensions, $\pi_{(32)1}(U)$ and $\pi_{3(21)}(U)$, of $\pi_{321}(U)$ to all of $L^2(\mathbb{R}^3)$ need not be the same or even equivalent. But, that would imply that a self-adjoint extension of $\pi_{321}(U)$ to all of $L^2(\mathbb{R}^3)$ need not be unique.
Luckily enough, we will soon see that the self-adjoint extensions will be the same and it will follow that the self-adjoint extension of $\pi_{321}(U)$ will be unique for $U \in \{E, F, K, C_q\}$.

To show that the self-adjoint extension of $\pi_{321}(U)$ is unique for $U \in \{E, F, K, C_q\}$, we will be looking at the relation between the distributions $\Phi^s_{\alpha_s} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}_{\varepsilon} (x_4, \mathbf{x})$ and $\Phi^t_{\alpha_t} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}_{\varepsilon} (x_4, \mathbf{x})$. Now, the Fourier-transform of these distributions is given by

$$\widetilde{\Phi}^i_{\alpha_i} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}_{\varepsilon} (\kappa_4, \mathbf{x}) = \int_{\mathbb{R}} e^{2\pi i \kappa_4 x_4} \Phi^i_{\alpha_i} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}_{\varepsilon} (x_4, \mathbf{x}) dx_4.$$

Furthermore, since $C_{(32)1}$ and $C_{3(21)}$ are unitary maps, we get the following relation between the functions $F_f^s(\alpha_4,\alpha_s,\kappa_4)$ and $F_f^t(\alpha_4,\alpha_t,\kappa_4)$:

$$\tilde{F_f}^s(\alpha_4,\alpha_s,\kappa_4)=\int_{\mathcal{S}^2}\int_{\mathbb{R}}\mathcal{K}\begin{bmatrix}\alpha_4 & \alpha_s & \kappa_4\\ \alpha_4' & \alpha_t & \kappa_4'\end{bmatrix}\tilde{F_f}^t(\alpha_4',\alpha_t,\kappa_4')d\mu(\kappa_4')d\mu(\alpha_4')d\mu(\alpha_t),$$

with the distribution $\mathcal{K}\begin{bmatrix}\alpha_4 & \alpha_s & \kappa_4\\ \alpha_4' & \alpha_t & \kappa_4'\end{bmatrix}$ given by:

$$\mathcal{K}\begin{bmatrix}\alpha_4 & \alpha_s & \kappa_4\\ \alpha_4' & \alpha_t & \kappa_4'\end{bmatrix}=\lim_{\rho\to\infty}\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}}\int_{-\rho}^{\rho}\int_{\mathbb{R}}\int_{-\rho}^{\rho}\left(\widetilde{\Phi}^t_{\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4' & \alpha_1\end{bmatrix}(\kappa_4',\mathbf{x})\right)^*\widetilde{\Phi}^s_{\alpha_s}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}(\kappa_4,\mathbf{x})dx_3dx_1dx_2.$$

Now, similar to the classical case, we will see that this relation will be 0 for a lot of values of α_4' and κ_4'. To be precise, we will see that $\mathcal{K}\begin{bmatrix}\alpha_4 & \alpha_s & \kappa_4\\ \alpha_4' & \alpha_t & \kappa_4'\end{bmatrix}$ will be 0 whenever $\alpha_4'\neq\alpha_4$ and $\kappa_4'\neq\kappa_4$. From this, it will follow that the self-adjoint extension of $\pi_{321}(U)$ will be unique for $U\in\{E,F,K,C_q\}$.

Proposition 9.1. The distribution $\mathcal{K}$ is of the form

$$\mathcal{K}\begin{bmatrix}\alpha_4 & \alpha_s & \kappa_4\\ \alpha_4' & \alpha_t & \kappa_4'\end{bmatrix}=\delta(\alpha_4-\alpha_4')\delta(\kappa_4-\kappa_4')K\begin{bmatrix}\alpha_4 & \alpha_s\\ \kappa_4 & \alpha_t\end{bmatrix},$$

with $K\begin{bmatrix}\alpha_4 & \alpha_s\\ \kappa_4 & \alpha_t\end{bmatrix}$ a distribution that only depends on $\alpha_4,\alpha_s,\alpha_t$ and κ_4.

Proof. To prove this proposition, we will show that the following holds:

$$\left(\left[\alpha_4-\frac{Q}{2}\right]_b^2-\left[\alpha_4'-\frac{Q}{2}\right]_b^2\right)\mathcal{K}\begin{bmatrix}\alpha_4 & \alpha_s & \kappa_4\\ \alpha_4' & \alpha_t & \kappa_4'\end{bmatrix}=0$$

$$\left(\kappa_4-\kappa_4'\right)\mathcal{K}\begin{bmatrix}\alpha_4 & \alpha_s & \kappa_4\\ \alpha_4' & \alpha_t & \kappa_4'\end{bmatrix}=0.$$

Then, as stated in the proof in [1], if the above claim is true, the statement in our proposition will follow due to a certain theorem. This known fact implies the following:
Let $f\in S(\mathbb{R})$ be such that $f(x)\neq 0$ if $x\neq x_0$ and $f(x_0)=0$, $x,x_0\in\mathbb{R}$ and $fg(x)\in S(\mathbb{R})$ $\forall g(x)\in S(\mathbb{R})$. Let $T\in S'(\mathbb{R})$ be a tempered distribution such that $Tf(x)=0$ $\forall x\in\mathbb{R}$, then $T=a_0(x_0)\delta(x-x_0)$. In other words, it suffices to show that the first claim of this proof is true. The proposition then follows from the above fact.

In our case, if we indeed have that $\left(\left[\alpha_4-\frac{Q}{2}\right]_b^2-\left[\alpha_4'-\frac{Q}{2}\right]_b^2\right)\mathcal{K}\begin{bmatrix}\alpha_4 & \alpha_s & \kappa_4\\ \alpha_4' & \alpha_t & \kappa_4'\end{bmatrix}=0$ and $(\kappa_4-\kappa_4')\mathcal{K}\begin{bmatrix}\alpha_4 & \alpha_s & \kappa_4\\ \alpha_4' & \alpha_t & \kappa_4'\end{bmatrix}=0$, it then follows from the above fact that $\mathcal{K}\begin{bmatrix}\alpha_4 & \alpha_s & \kappa_4\\ \alpha_4' & \alpha_t & \kappa_4'\end{bmatrix}=\delta(\kappa_4-\kappa_4')\delta_(\alpha_4-\alpha_4')a_0(\alpha_4,\kappa_4;\alpha_s;\alpha_t)$. As $\mathcal{K}\begin{bmatrix}\alpha_4 & \alpha_s & \kappa_4\\ \alpha_4' & \alpha_t & \kappa_4'\end{bmatrix}$ satisfies the conditions of the above fact if we take $f\in S(\mathbb{R}^3)$ with $f(\alpha_4,\alpha_s,\kappa_4)=0$ if $\alpha_4=\alpha_4'$ or $\kappa_4=\kappa_4'$.

Recall that $\pi_{21}(C_q')$ and $\pi_{21}(K)$ decomposed $L^2(\mathbb{R}^2)$ into eigenspaces with eigenvalues $e^{-\pi b\kappa_3}$ and $\left[\alpha_3-\frac{Q}{2}\right]_b^2$. As $\widetilde{\Phi}^i_{\alpha_i}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}(\kappa_4;\mathbf{x})=\lim_{\varepsilon\to 0}\widetilde{\Phi}^{\,i}_{\alpha_i}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_\varepsilon(\kappa_4;\mathbf{x})$, which consists of a multiplication of two Clebsch-Gordan coefficients, it follows that we also have

$$\pi_{\alpha_4}(K)\tilde{F_f^i}(\alpha_4,\alpha_i,\kappa_4)=e^{-\pi b\kappa_4}\tilde{F_f^i}(\alpha_3,\alpha_i,\kappa_4),\qquad\pi_{\alpha_4}(C_q')F_f^i(\alpha_4,\alpha_i,\kappa_4)=\left[\alpha_4-\tfrac{Q}{2}\right]_b^2\tilde{F_f^i}(\alpha_4,\alpha_i,\kappa_4),$$

with $i \in \{s,t\}$.
Now, it follows that we get

$$\left(\left[\alpha_4-\tfrac{Q}{2}\right]_b^2-\left[\alpha_4'-\tfrac{Q}{2}\right]_b^2\right)\mathcal{K}\begin{bmatrix}\alpha_4 & \alpha_s & \kappa_4\\ \alpha+4' & \alpha_t & \kappa_4'\end{bmatrix}=$$

$$\lim_{\varepsilon_1,\varepsilon_2\downarrow 0}\lim\rho\to\infty\int_{\mathbb{R}}\int_{-\rho}^{\rho}\int_{-\rho}^{\rho}\left(\left(\widetilde{\Phi}^t_{\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4' & \alpha_1\end{bmatrix}_{\varepsilon_1}(\kappa_4';\mathbf{x})\right)^*\pi_{321}(C_q')\widetilde{\Phi}^s_{\alpha_s}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_{\varepsilon_2}(\kappa_4;\mathbf{x})\right.$$
$$\left.-\left(\pi_{321}(C_q')\widetilde{\Phi}^t_{\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4' & \alpha_1\end{bmatrix}_{\varepsilon_1}(\kappa_4';\mathbf{x})\right)^*\widetilde{\Phi}^s_{\alpha_s}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_{\varepsilon_2}(\kappa_4;\mathbf{x})\right)dx_3dx_1dx_2$$

for the action of C_q', with the above regarded as a distribution.
To show that this indeed vanishes, one can do something similar as what was done in the proof of proposition 8.10. First, we have that $\pi_{321}(C_q')$ acts on $f(x_3,x_2,x_1)$ by shifts of the form $T_{x_1}^{is_1b}T_{x_2}^{is_2b}T_{x_3}^{is_3b}$, $s_i=\pm$, $i\in\{1,2,3\}$. Now, we can rewrite how $\pi_{321}(C_q')$ acts into shift operators of the form $T_{123}=T_{x_1}T_{x_2}T_{x_3}$, $T_{21}=T_{x_2}T_{x_1}^{-1}$ and $T_{32}=T_{x_3}T_{x_2}^{-1}$ to get

$$\pi_{321}(C_q')=\sum_{n_1=-3}^{3}\sum_{n_2=0}^{3}\sum_{n_3=0}^{3}P_{n_1,n_2,n_3}(x_3,x_2,x_1)T_{123}^{n_1ib}T_{21}^{\frac{2n_2}{3}ib}T_{32}^{\frac{2n_3}{3}ib},$$

with $P_{n_1,n_2,n_3}(x_3,x_2,x_1)$ some polynomial that depends on n_1,n_2,n_3 and x_3,x_2,x_1 and $T_{123}^a=T_{x_1}^aT_{x_2}^aT_{x_3}^a$, $T_{21}^a=T_{x_2}^aT_{x_1}^{-a}$, $T_{32}^a=T_{x_3}^aT_{x_2}^{-a}$. It follows that $\pi_{321}(C_q)$ contains positive imaginary shifts up to $2ib$ of the variables x_{21}, x_{32} and x_{31}, with x_{32} and x_{31} as in proposition 8.10 and $x_{21}=x_2-x_1+\frac{1}{2}(s_2+s_1-2s_3)$. Furthermore, the shifts of the form $T_{x_3}T_{x_2}T_{x_1}$ can be replaced by $e^{-2\pi i\kappa_4}$ in the above integral. So we indeed do end up with shifts in only x_{32}, x_{31} and x_{21}.
Now, by shifting the contours of integration, similarly to proposition 8.10, we get

$$\int_{\mathbb{R}}\int_{-\rho}^{\rho}\int_{-\rho}^{\rho}\left(\widetilde{\Phi}^t_{\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4' & \alpha_1\end{bmatrix}_{\varepsilon_1}(\kappa_4';\mathbf{x})\right)^*\pi_{321}(C_q')\widetilde{\Phi}^s_{\alpha_s}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_{\varepsilon_2}(\kappa_4;\mathbf{x})dx_3dx_1dx_2=$$
$$\int_{\mathbb{R}}\int_{-\rho}^{\rho}\int_{-\rho}^{\rho}\pi_{321}(C_q')^t\left(\widetilde{\Phi}^t_{\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4' & \alpha_1\end{bmatrix}_{\varepsilon_1}(\kappa_4';\mathbf{x})\right)^*\widetilde{\Phi}^s_{\alpha_s}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_{\varepsilon_2}(\kappa_4;\mathbf{x})dx_3dx_1dx_2=$$
$$\int_{\mathbb{R}}\int_{-\rho}^{\rho}\int_{-\rho}^{\rho}\left(\pi_{321}(C_q')\widetilde{\Phi}^t_{\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4' & \alpha_1\end{bmatrix}_{\varepsilon_1}(\kappa_4';\mathbf{x})\right)^*\widetilde{\Phi}^s_{\alpha_s}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_{\varepsilon_2}(\kappa_4;\mathbf{x})dx_3dx_1dx_2,$$

with $\pi_{321}(C_q)^t$ the transposed operator $\pi_{321}(C_q)$. It follows that we indeed have that

$$\left(\left[\alpha_4-\frac{Q}{2}\right]_b^2-\left[\alpha_4'-\frac{Q}{2}\right]_b^2\right)\mathcal{K}\begin{bmatrix}\alpha_4 & \alpha_s & \kappa_4\\ \alpha_4' & \alpha_t & \kappa_4'\end{bmatrix}=0.$$

A similar reasoning also shows that

$$(\kappa_4-\kappa_4')\,\mathcal{K}\begin{bmatrix}\alpha_4 & \alpha_s & \kappa_4\\ \alpha_4' & \alpha_t & \kappa_4'\end{bmatrix}=0,$$

when one uses the action of K instead of that of C_q. Hence, it follows that

$$\mathcal{K}\begin{bmatrix}\alpha_4 & \alpha_s & \kappa_4\\ \alpha_4' & \alpha_t & \kappa_4'\end{bmatrix}=\delta(\alpha_4-\alpha_4')\delta(\kappa_4-\kappa_4')K\begin{bmatrix}\alpha_4 & \alpha_s\\ \kappa_4 & \alpha_t\end{bmatrix},$$

by the first remark of the proof. □

From this proposition it now follows that the self-adjoint extension of $\pi_{321}(U)$ obtained via the inverse mappings $C^{-1}_{(32)1}$ and $C^{-1}_{3(21)}$ are the same for $U \in \{K, C_q\}$. Via similar reasoning as in the above proof, one can also show that we have $\pi_{(32)1}(U) = \pi_{3(21)}(U)$ for $U \in \{E, F\}$. In other words, the two self-adjoint extensions of $\pi_{321}(U)$ obtained via the inverse mappings $C^{-1}_{(32)1}$ and $C^{-1}_{3(21)}$ are the same for $U \in \{E, F, K, C_q\}$. So these self-adjoint extensions are indeed unique.

As a final remark on the triple tensor, we would like to quickly give a definition of the *Racah-Wigner coefficients*, also known as the *b-6j* symbols. Similar to the Clebsch-Gordan coefficients, the Racah-Wigner coefficients also gives us a relation via a linear transformation. But, instead of showing us how the decomposition is related to the tensor product, the Racah-Wigner coefficients give us a relation between the two differently obtained decompositions.

Proposition 9.2. We have the following relation between $\Phi^s_{\alpha_s}(x_4, \mathbf{x})$ and $\Phi^t_{\alpha_t}(x_4, \mathbf{x})$:

$$\Phi^s_{\alpha_s}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}(x_4, \mathbf{x}) = \int_{\mathcal{S}} \left\{\begin{matrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{matrix}\middle|\begin{matrix}\alpha_s\\ \alpha_t\end{matrix}\right\} \Phi^t_{\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}(x_4, \mathbf{x}) d\alpha_t,$$

where this relation is either considered as

i) a relation between two analytic functions in

$$\mathcal{A}^{(4)} = \{(x_1, x_2, x_3, x_4) \in \mathbb{C} \,|\, im(x_1) < im(x_2) < im(x_3) \wedge im(x_1) < im(x_4) < im(x_3) \wedge im(x_3 - x_1) < Q\}$$

ii) a relation between meromorphic functions on $\mathbb{C}^4$,

iii) a relation between distributions defined as boundary values of $\Phi^\delta_{\alpha_\delta}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}(x_4, \mathbf{x})$ for $\delta \in \{s, t\}$ and $(x_4, \mathbf{x}) \in \mathbb{R}^4$.

Note, the above can also be written as a relation between the functions $F^s_f(\alpha_4, \alpha_s, x_4)$ and $F^t_f(\alpha_4, \alpha_t, x_4)$. This gives a fairly similar looking relation,

$$F^s_f(\alpha_4, \alpha_s, x_4) = \int_{\mathcal{S}} \left\{\begin{matrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{matrix}\middle|\begin{matrix}\alpha_s\\ \alpha_t\end{matrix}\right\} F^t_f(\alpha'_4, \alpha_t, x_4) d\alpha_t,$$

which is obtained with the use of the transformation between $\tilde{F}^s_f(\alpha_4, \alpha_s, \kappa_4)$ and $\tilde{F}^t_f(\alpha_4, \alpha_s, \kappa_4)$ via the distribution $\mathcal{K}\begin{bmatrix}\alpha_4 & \alpha_s & \kappa_4\\ \alpha'_4 & \alpha_t & \kappa'_4\end{bmatrix}$.
Finally, one can get the following explicit value of the Racah-Wigner coefficients [1]:

Proposition 9.3.

$$\left\{\begin{matrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{matrix}\middle|\begin{matrix}\alpha_s\\ \alpha_t\end{matrix}\right\} = N \frac{S_b(\alpha_2 + \alpha_s - \alpha_1) S_b(\alpha_t + \alpha_1 - \alpha_4)}{S_b(\alpha_2 + \alpha_t - \alpha_3) S_b(\alpha_s + \alpha_3 - \alpha_4)}$$
$$\cdot |S_b(2\alpha_t)|^2 \int_{-i\infty}^{i\infty} \frac{S_b(U_1 + s) S_b(U_2 + s) S_b(U_3 + s) S_b(U_4 + s)}{S_b(V_1 + s) S_b(V_2 + s) S_b(V_3 + s) S_b(V_4 + s)} ds,$$

with N a certain constant and

$$\begin{aligned} U_1 &= \alpha_s + \alpha_1 - \alpha_2, & V_1 &= 2Q + \alpha_s - \alpha_t - \alpha_2 - \alpha_4,\\ U_2 &= Q + \alpha_2 - \alpha_2 - \alpha_1, & V_2 &= Q + \alpha_s + \alpha_t - \alpha_4 - \alpha_2,\\ U_3 &+ \alpha_s + \alpha_3 - \alpha_4, & V_3 &= 2\alpha_s,\\ U_4 &= Q + \alpha_s - \alpha_3 - \alpha_4, & V_4 &= Q. \end{aligned}$$

Do note that because the Racah-Wigner coefficients give us a relation between $F_f^s(\alpha_4, \alpha_s, x_4)$ and $F_f^t(\alpha_4', \alpha_t, x_4)$, any obtained explicit value for the Racah-Wigner coefficients depend on the normalization of the Clebsch-Gordan coefficients. In other words, the choice that is made for $N(\alpha_3, \alpha_2, \alpha_1)$ in lemma 8.5 will influence the obtained value for the Racah-Wigner coefficients.
Lastly, one can calculate the constant N of the Racah-Wigner coefficients either explicitly, or one could use a similar approach as what was done earlier to get an explicit value for the Clebsch-Gordan coefficients. In other words, one can describe the Racah-Wigner coefficients with the use of eigenfunctions of certain finite difference operators.
In[1] this is done by fixing three of the four variables of $x_1, \dots, x_4$ in proposition 9.2. This then gives a linear transformation that depends on only one variable. One can then solve an eigenfunction problem similarly to the eigenfunction problem of the Clebsch-Gordan coefficients to get a fully explicit value for the Racah-Wigner coefficients.

10 Summary

To give the reader a good idea how to work with Hopf $*$-algebras, we started with some general theory on quantum groups and their modules. This started with general information Hopf algebras, where we also looked at $M(2)$, $GL(2)$ and $SL(2)$ as they are a great explicit example on Hopf algebras. Especially since their quantum deformations $M_q(2)$, $GL_q(2)$ and $SL_q(2)$ have fairly similar structures. Since we also wanted to introduce $\mathcal{U}_q(\mathfrak{sl}(2))$ as a quantum deformation of $\mathcal{U}(\mathfrak{sl}(2))$, we also looked at some general theory on Lie algebras. This contained some useful facts, like the Poincaré-Birkhoff-Witt theorem 3.31 and the general structure of a universal enveloping algebra. This part also contained our first explicit example of a module, the Verma modules. We showed the form of every finite dimensional simple $\mathcal{U}(\mathfrak{sl}(2))$-module and that the same construction could be used to get the infinite-dimensional Verma modules.
The general theory ended with the study of the Hopf $*$-algebra $\mathcal{U}_q(\mathfrak{sl}(2))$. Here we showed all possible $*$-structures of $\mathcal{U}_q(\mathfrak{sl}(2))$, which was also done for $GL_q(2)$, $SL_q(2)$ and $\mathcal{U}(\mathfrak{sl}(2))$. We then proceeded to look at the Verma modules of $\mathcal{U}_q(\mathfrak{sl}(2))$. The study of these modules and all finite dimensional simple modules of $\mathcal{U}_q(\mathfrak{sl}(2))$ was shown to be fairly similar to that of $\mathcal{U}(\mathfrak{sl}(2))$, except for the fact that some care was needed for the choice of the parameter q. To further show why care is needed, we ended the general theory by looking at what happens when q is a root of unity. This changed several facts and possible options on the finite dimensional simple modules of $\mathcal{U}_q(\mathfrak{sl}(2))$.
We also looked at the tensor product of two Verma modules of $\mathcal{U}(\mathfrak{sl}(2))$ and $\mathcal{U}_q(\mathfrak{sl}(2))$ as an introduction the the part about the modular double $\mathcal{Q}$ introduced by Faddeev. There we looked at a simpler version of a decomposition into simple modules and also discussed how one could get an explicit formula for the Clebsch-Gordan coefficients. Even though the setting was simpler, most of the strategies could still be used in a similar way in the later sections on $\mathcal{Q}$.

Our next main subject was a certain module of $\mathcal{Q}$. Now, by defining $\mathcal{P}_\alpha$ to be a $\mathcal{Q}$-module by definition, we saw that this space was a natural choice to induce a $\mathcal{Q}$-module structure on. This was because of two facts. The first was that $\mathcal{P}_\alpha \subseteq L^2(\mathbb{R})$ was a maximal subspace on which $\pi_\alpha(E), \pi_\alpha(F)$ and $\pi_\alpha(K)$ where well-defined. Where we needed both actions of $\pi_\alpha(K)$ and $\pi_\alpha(\tilde{K})$ to deduce that $f \in \mathcal{P}_\alpha$ needs to be an entire function and the actions of $\pi_\alpha(E), \pi_\alpha(F), \pi_\alpha(\tilde{E}), \pi_\alpha(\tilde{F})$ showed that the Fourier-transform of f, $\tilde{f}$, has to be meromorphic with possible poles at

$$i(\alpha - Q - nb - mb^{-1}) \qquad \text{and} \qquad i(Q - \alpha + nb + mb^{-1}), \quad n, m \in \mathbb{Z}.$$

The other reason was due to the fact that the actions of E, F, K were defined via positive self-adjoint operators. Due to this fact, we can define the operators $\pi_\alpha(U)^{\frac{1}{b^2}}$ for $U \in \{E, F, K\}$. The action of $\pi_\alpha(U)^{\frac{1}{b^2}}$ is the same as that of $\pi_\alpha(\tilde{U})$. Combined with the fact that the actions of $\pi_\alpha(U)$ and $\pi_\alpha(\tilde{U})$ commute by construction, $\mathcal{P}_\alpha$ would automatically turn into a $\mathcal{Q}$-module if one wants $\pi_\alpha(U)^{\frac{1}{b^2}}$ to also act on the module.
Besides seeing that $\mathcal{P}_\alpha$ is a natural space to define a $\mathcal{Q}$-module structure on, we also saw that the representations π_α was unique up to isomorphism. This followed from the fact that π_α was an integrable representation and there was only one, up to isomorphisms, irreducible integrable $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ representation that acts on $L^2(\mathbb{R})$. So, in a way, $\mathcal{P}_\alpha$ is both a natural choice for a $\mathcal{Q}$-module and also the only choice up to isomorphism.

Later on we saw that the tensor product $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ was decomposable into irreducible representations, similar to the classical case. It followed that

$$\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1} \simeq \int_{\mathcal{S}}^{\oplus} \mathcal{P}_{\alpha_3} d\mu(\alpha_3), \text{ with } \mathcal{S} = \tfrac{Q}{2} + i\mathbb{R},$$

and this isomorphism was given via

$$L^2(\mathbb{R}^2) \ni f(x_2, x_1) \mapsto F_f(\alpha_3, x_3) = \int_{\mathbb{R}}^2 \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix} f(x_2, x_1) dx_2 dx_1.$$

We then discussed how to decompose $\int_{\mathcal{S}}^{\oplus} \mathcal{P}_{\alpha_3} d\mu(\alpha_3)$ into eigenspaces for the Fourier-transformed actions of $\pi_{\alpha_{21}}(K)$ and $\pi_{\alpha_{21}}(C_q)$. This then gave us the following equation for the Fourier-transformed Clebsch-Gordan coefficients in terms of the eigenfunctions Φ_{α_3}:

$$\begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ \kappa_3 & x_2 & x_1 \end{bmatrix} = \int_{\mathbb{R}} e^{2\pi i \kappa_3 x_3} \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix} dx_3 = e^{-\pi i \kappa_3 (x_1 + x_2)} \Phi_{\alpha_3}(\alpha_2, \alpha_1 | \kappa_3 | x_2 - x_1).$$

In the end we saw that we were not limited to only decomposing the tensor product of two modules into a direct integral of simple modules. For example, we showed that we got

$$\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1} \simeq \int_{\mathcal{S}^2} \mathcal{P}_{\alpha_4}^{\alpha_s} d\mu\alpha_4 d\mu(\alpha_s),$$

by first decomposing the $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$, then decomposing the obtained tensor products $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_s}$.

10.1 Outlook

We will end this by noting some possible options that could be studied in more detail or what one could study to add to this thesis. First, there are still some details of [1] that were not fully worked out. For example, the proof of proposition 8.9 was for a part assumed to be correct, as it was not possible to check some details in time. So, the details of this proof could be worked out in detail, or one could look at other papers or use other methods to verify some of the facts on the direct integral $\int_{\mathcal{S}}^{\oplus} \mathcal{P}_{\alpha} d\mu(\alpha)$.
Furthermore, we did not look at how the eigenfunctions Φ_{α_3} were calculated or look at the spectral analysis of the operator $\pi_{21}(C_q')$. This spectral analysis could be studied in more detail. The method used to find the eigenfunctions Φ_{α_3} could then also be used to give a detailed proof for a set of generalised eigenfunctions in the case of the Racah-Wigner coefficients.

Something that could be studied next is obviously the Racah-Wigner coefficients of $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$, as we only briefly went over a fact on the self-adjoint extension of $\pi_{321}(U)$ and gave the definition of the Racah-Wigner coefficients as a fact. This could then also be combined with looking at applications of the Racah-Wigner coefficients with respect to Liouville theory.
A more theoretical option could be to study universal R-matrices. These are, for example, used by Faddeev in [2] to construct the modular double and also to point out some of the difficulties. So this could then also be used to take a more detailed look at the exact construction of the modular double of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$.

Appendix A: Tensor Products

In this appendix we will look at some small facts on the tensor product of vector spaces. We will also look at how the tensor product of two linear maps $f\colon U \to U'$ and $g\colon V \to V'$ relate to linear maps $h\colon V \otimes U \to U' \otimes V'$.

Theorem A.1. *Let U, V be vector spaces over a field $\mathbf{k}$. Then $\exists\ U \otimes V$, vector space, and a bilinear map $\phi_0\colon U \times V \to U \otimes V$ such that $\forall\ W$, vector space,*

$$Hom(U \otimes V, W) \xrightarrow{\cong} Hom(U \times V, W); \qquad f \mapsto f \circ \phi_0,$$

is a linear isomorphism.

The vector space $U \otimes V$ in the above theorem is called the *tensor product* of U and V. This leads to the following definition,

Definition A.2. Let U, V be vector spaces over a field $\mathbf{k}$. The *tensor product* $U \otimes V$ of U and V is the vector space consisting of elements of the form $u \otimes v$, $u \in U$, $v \in V$, where $u \otimes v = \phi_0(u, v)$.

Since ϕ_0 is bilinear, we get the following relations in $U \otimes V$:
Let $u, u' \in U$, $v, v' \in V, \lambda \in \mathbf{k}$

$$\begin{aligned}(u + u') \otimes v &= u \otimes v + u' \otimes v,\\ u \otimes (v + v') &= u \otimes v + u \otimes v'\\ \lambda \cdot (u \otimes v) &= \lambda u \otimes v = u \otimes \lambda v,\end{aligned}$$

also, every element $w \in U \otimes V$ is of the form

$$w = \sum_{i=0}^{n} u_i \otimes v_i, \qquad u_0, \ldots, u_n \in U,\ v_0, \ldots, v_n \in V,\ n \in \mathbb{N}.$$

Now, a nice thing about the tensor product with vector spaces, is that it sort of acts distributively and commutatively. $\mathbf{k}$ also acts like the identity for the tensor product in the following sense:

$$\begin{aligned}(U \otimes V) \otimes W \cong \mathcal{U} \otimes (V \otimes W)&; \text{ via } (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w),\\ \mathbf{k} \otimes U \cong U \cong U \otimes \mathbf{k}&; \text{ via } \lambda \otimes u \mapsto \lambda \cdot u \text{ and } u \mapsto u \otimes 1,\\ U \otimes V \cong V \otimes U&; \text{ via } u \otimes w \mapsto w \otimes v.\end{aligned}$$

Besides acting commutatively on vector spaces, the tensor product also acts commutatively on the direct sum of vector spaces:

Proposition A.3. Let U be a vector space and let $(V_i)_{i \in I}$ be a family of vector spaces with I being its index set. Then,

$$U \otimes \bigoplus_{i \in I} V_i \cong \bigoplus_{i \in I} (U \otimes V_i)$$

Corollary A.4. Let U,V be vector spaces with bases $\{u_i\}_{i \in I}$ and $\{v_j\}_{j \in J}$ respectively. Then, $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$ forms a basis of $U \otimes V$ and $dim(U \otimes V) = dim(U)dim(V)$.

This corollary follows due to the fact that we can write $U = \bigoplus_{i \in I} \mathbf{k} u_i$ and $V = \bigoplus_{j \in J} \mathbf{k} v_j$.

The previous corollary allows us to define the tensor product of linear functions on vector spaces in an intuitive way. Let U, U' be vector spaces over a field $\mathbf{k}$ and V, V' be vector spaces over a field $\mathbf{k}'$. Since linear functions can be fully determined by how they act on the basis of their domains. If we have two linear functions $f\colon U \to U'$, $g\colon V \to V'$, we can define the tensor product $f \otimes g\colon U \otimes V \to U' \otimes V'$ by

$$f \otimes g(u \otimes v) = f(u) \otimes f(v); \quad u \in U,\, v \in V,$$

This then gives rise to the following map:

$$\lambda\colon \mathrm{Hom}(U, U') \otimes \mathrm{Hom}(V, V') \to \mathrm{Hom}(V \otimes U, U' \otimes V'); \quad f \otimes g \mapsto (f \otimes g) \circ \tau_{V,U}.$$

Theorem A.5. *The map λ is an isomorphism if at least one of the following pairs consist of finite dimensional vector spaces: (U, U'), (V, V') or (U, V).*

Corollary A.6. The map $\lambda_{U,V}\colon V \otimes U^* \to \mathrm{Hom}(U, V);\; v \otimes \alpha \mapsto \alpha(\cdot)v$ is an isomorphism if U or V is finite-dimensional.
In particular, if $dim(U) < \infty$, $\lambda_{U,U}\colon U \otimes U^* \to \mathrm{Hom}(U, U) = \mathrm{End(U)}$ is an isomorphism.

Here $V^* = \mathrm{Hom}(V, \mathbf{k})$. This new, less generic, $\lambda_{U,V}$ can also be used to express the more generic λ of Theorem A.5.

Lemma A.7. Let U, V be vector spaces over a field $\mathbf{k}$ and let U', V' be vector spaces over a field $\mathbf{k}'$, then the following diagram commutes:

$$\begin{array}{ccc}
U' \otimes U^* \otimes V' \otimes V^* & \xrightarrow{\lambda_{U,U'} \otimes \lambda_{V,V'}} & \mathrm{Hom}(U, U') \otimes \mathrm{Hom}(V, V') \\
\big\downarrow {\scriptstyle id \otimes \tau_{U^*,V'} \otimes id} & & \\
U' \otimes V' \otimes U^* \otimes V^* & & \big\downarrow {\scriptstyle \lambda} \\
\big\downarrow {\scriptstyle id \otimes id \otimes \lambda} & & \\
U' \otimes V' \otimes (V \otimes U)^* & \xrightarrow{\lambda_{V \otimes U, U' \otimes V'}} & \mathrm{Hom}(V \otimes U, U' \otimes V')
\end{array}$$

Also, note that $\lambda_{U,U'} \otimes \lambda_{V,V'}$ is invertible whenever U or U' is finite-dimensional, but both V and V' are not finite-dimensional vector spaces.

Appendix B: Complex Functions

This appendix summarises a few small facts about complex functions that are used in the definition of the module $\mathcal{P}_\alpha$. Proofs and more details can be found in [17].

Definition B.8. Let $f\colon D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ be a function. Then f is called *complex differentiable* at $a \in D$ if

$$lim_{z\to a}\frac{f(z)-f(a)}{z-a} \text{ exists.}$$

We have a few statements that are equivalent to the above definition of a function f being complex differentiable at a point a.

Proposition B.9. Let $f\colon D \to \mathbb{C}$, $D \subseteq \mathbb{C}$, then the following are equivalent:

i) f is complex differentiable at $a \in D$ with $l = f'(a)$,

ii) $\exists \phi\colon D \to \mathbb{C}$, continuous at $a \in D$, with $f(z) = f(a) + \phi(z)(z-a)$, $\phi(a) = l$, $z \in D$,

iii) $\exists \rho\colon D \to \mathbb{C}$, continuous at $a \in D$, with $f(z) = f(a) + l \cdot (z-a) + \rho(z)\cdot(z-a)$, $\rho(a) = 0$, $z \in D$,

iv) $\exists r\colon D \to \mathbb{C}$, continuous at $a \in D$, with $f(z) = f(a) + l\cdot(z-a) + r(z) \implies lim_{z\to a}\frac{r(z)}{z-a} = 0 \iff lim_{z\to a}\frac{r(z)}{|z-a|} = 0$.

Now that we know what it means for a function to be complex differentiable at some point a, we can finally talk about what it means to be *meromorphic* or *entire*. These two terms are used in the definition of $\mathcal{P}_\alpha$, thus we are mostly interested in what these terms mean.

Definition B.10. Let $f\colon D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ open, then f is called *analytic* or *holomorphic* in D if f is complex differentiable in all $a \in D$.
If $f\colon \mathbb{C} \to \mathbb{C}$, then f is called *entire* if f is analytic on all of $\mathbb{C}$.

Now, before being able to introduce the notion of being meromorphic, we first need to know what singularities of an analytic function $f\colon D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ open, are.

Definition B.11. Let $D \subseteq \mathbb{C}$ and let $a \in \mathbb{C}$. Then a is called an *accumulation point* of D if $\forall \varepsilon > 0 \exists d \in D$ such that $0 < |d-a| < \varepsilon$.

A subset $D \subseteq \mathbb{C}$ is called *discrete* if $a \in \mathbb{C}$ is an accumulation point of D, then $a \notin D$.
Now, let $\dot{U_r}(a) = \{z \in \mathbb{C} \,|\, 0 < |z-a| < r\}$, so $\dot{U_r}(a)$ is the open ball around a with radius r, with a removed.

Definition B.12. Let $f\colon D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ open and f analytic in D. Let $a \notin D$ such that $\exists r > 0$ with $\dot{U_r}(a) \subseteq D$. Then a is called a *singularity* of f.

Note that $D \cup \{a\} = D \cup U_r(a)$ is open.

Definition B.13. Let $a \notin D$ be a singularity of an analytic function $f\colon D \to \mathbb{C}$. Then a is called *removable* if $\exists \hat{f}\colon D \cup \{a\} \to \mathbb{C}$, analytic, with $\hat{f}|_D = f$. In other words, a is called a removable singularity if f can be analytically extended to $D \cup \{a\}$.
a is called a *non-essential* singularity if $\exists m \in \mathbb{Z}$ such that $g(z) = (z-a)^m f(z)$ has a removable singularity at a.
a is called a *pole* if a is a non-removable non-essential singularity.
Lastly, a is called essential if a is not an non-essential singularity.

We get the following facts on singularities of an analytic function $f\colon D \to \mathbb{C}$:

Proposition B.14. Let $f: D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ open, be an analytic function. Then a singularity $a \notin D$ is called:

i) removable $\iff$ f is bounded in a neighbourhood around a,

ii) a pole $\iff$ $lim_{z \to a}|f(z)| = \infty$,

iii) essential $\iff$ f comes arbitrary close to any value in any neighbourhood around a.

Now, meromorphic functions are precisely analytic functions on an open subset $D \subseteq \mathbb{C}$, which are extended to take on the value ∞ in every $a \in \mathbb{C}$ that is a pole.

Definition B.15. Let $f: D \to \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $D \subseteq \mathbb{C}$ open, is called *meromorphic* if

i) $S(f) = f^{-1}(\{\infty\}) = \{\omega \in \mathbb{C} \,|\, f(\omega) = \infty\}$ is discrete in D,

ii) $f_0 = f|_{D \setminus S(f)}$ is analytic,

iii) if $\omega \in S(f)$, then ω is a pole of f_0.

Lastly, lemma 7.10 uses the residue of $\tilde{f}$ to determine its asymptotic behaviour when $\omega \to \pm\infty$. The residue of an analytic function f can be determined by using the Laurent decomposition. This leads to the following lemma:

Lemma B.16. Let $f: U_r(a) \mapsto \mathbb{C}$ be analytic with a singularity in $a \in \mathbb{C}$. Set $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - a)^n$, then $Res(f(a)) = a_{-1}$ is the *residue* of f at a.
Furthermore,

i) a is removable $\iff$ $a_n = 0 \ \forall n < 0$,

ii) a is a pole $\iff$ $a_{-k} \neq 0$ and $a_n = 0 \ \forall n < -k$ for some $k \geq 0$,

iii) a is essential $\iff$ $a_n \neq 0$ for infinitely many $n < 0$.

For more details on the Laurent decomposition of a function, see [17], this also has more details on how to calculate the residues of a function using integrals along a contour.

Appendix C: The Direct Integral

This appendix will be dedicated to some details regarding direct integrals, as defined in [18]. This appendix is mostly here for completeness sake and to give concrete definitions. Around the end of this appendix we will once again see that the direct integral can be regarded as a generalisation of the direct sum.
For this appendix, it is assumed that the reader knows what Hilbert spaces are and what it means to integrate a function or operator with respect to a measure.

Definition C.17. A *Borel space* $(E, \mathcal{B})$ is a set E together with a set $\mathcal{B}$ of subsets of E such that

i) $\emptyset \in \mathcal{B}$,

ii) if $B_i \in \mathcal{B}$ for $i \in \mathbb{N} \implies \bigcup_{i\in\mathbb{N}} B_i \in \mathcal{B}$,

iii) if $B_i \in \mathcal{B}$ for $i \in \mathbb{N} \implies \bigcap_{i\in\mathbb{N}} B_i \in \mathcal{B}$.

The subset $B \in \mathcal{B}$ of E are called the *Borel sets of E*.

The third property for Borel sets is equivalent to $\mathcal{B}$ being closed under taking complements. Also, one can think of the Borel sets of a Borel space E as all of its open, or closed, sets.

Definition C.18. Let $(E, \mathcal{B}_E), (F, \mathcal{B}_F)$ be Borel spaces and let $f : E \to F$ be a surjective map. Then f is called a *Borel map* if the inverse map $f^{-1}(X) \in \mathcal{B}_E \; \forall X \in \mathcal{B}_F$.

Two examples of Borel structures are the *induced* Borel structure and the *discrete* one. If $(E, \mathcal{B})$ is a Borel space, $E' \subset E$ is a subset of E and $X \subset \mathcal{B}$ is a Borel space of E. Then, the set $X \cap E'$ define a Borel space structure on E', called the *induced* Borel structure. So $\mathcal{B}_{E'} = \{X \cap E' | X \in \mathcal{B}\}$.
A Borel space $(E, \mathcal{B})$ is called *discrete* if $\mathcal{B} = \{X | X \subset E\}$, so if every subset of E is a Borel set of E. This example will later on be used to show why one can think of a direct integral as a generalisation of direct sums. For now, we will move on to measures.

Definition C.19. Let $(Z, \mathcal{B})$ be a Borel space, a *positive measure* $\nu : \mathcal{B} \to [0, \infty]$ is a map such that

i) if $X_1, X_2, \ldots \in \mathcal{B}$ such that $X_i \cap X_j \neq \emptyset$ for all $i \neq j$, then $\nu(\bigcup_{i=0}^{\infty} X_i) = \sum_{i=1}^{\infty} \nu(X_i)$,

ii) $Z = \bigcup_{i=0}^{\infty} Y_i$ with $\nu(Y_i) < \infty \; \forall i \in \mathbb{N}$.

Now, a subset $Z' \subset Z$ is called *ν-negligible* if $Z' \subset X \in \mathcal{B}$, with $\nu(X) = 0$. And $Z' \subset Z$ is called *ν-measurable* if $Z' = X \cup N$, with $X \in \mathcal{B}$ and N ν-negligible.

Now, the set $M = \{Y \subset Z | Y \text{ is } \nu\text{-measurable}\}$ is closed under taking countable unions and countable intersections. So we can extend the measure ν to also be defined on M. Set $\nu(X \cup N) = \nu(X)$ for $X \in \mathcal{B}$ and N ν-negligible. Then ν is defined on M and is still a positive measure.

To define a direct integral, we will also need the concept of fields of Hilbert spaces. As the direct integral will be a field of Hilbert spaces together with a positive measure ν.

Definition C.20. Let $(Z, \mathcal{B})$ be a Borel space. A *field of complex Hilbert spaces* over Z is a map $\zeta \mapsto H(\zeta)$, $\zeta \in Z$, such that $H(\zeta)$ is a complex Hilbert space for all $\zeta \in Z$.

In other words, one could think of a field of Hilbert spaces as a collection of several Hilbert spaces that are indexed by the elements of a Borel space $(Z, \mathcal{B})$. From this definition, we see that the direct product $F = \Pi_{\zeta\in Z} H(\zeta)$ is a complex vector space. F contains vector fields x that are in a sense maps on Z, with $x(\zeta) \in H(\zeta)$. The vector fields x are called *vector fields over Z*.
Now, if $Y \subset Z$, then $y \in \Pi_{\zeta\in Y} H(\zeta)$ is called a vector field over Y. Notice that $\Pi_{\zeta\in Y} H(\zeta)$ is another vector space.

Lemma C.21. Let $(Z, \mathcal{B})$ be a Borel space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of vector fields such that the functions $\zeta \mapsto (x_m(\zeta), x_n(\zeta))$ are measurable.
For $\zeta \in Z$, let $\mathcal{X}(\zeta)$ be the complex vector space algebraically generated by the vector fields $x_n(\zeta)$ and set $d(\zeta) = dim(\mathcal{X}(\zeta))$. Then

$$z_p := \{\zeta \in Z | d(\zeta) = p\}, \quad p \in \mathbb{N},$$

is measurable and $\exists (y_n)_{n\in\mathbb{N}}$, a sequence of vector fields, such that

i) $\forall \zeta \in Z$ (the $y_n(\zeta)$ algebraically generate $X(\zeta)$),

ii) if $d(\zeta) = \aleph_0$, then the $y_n(\zeta)$, $n \in \mathbb{N}$, for an orthonormal system, if $d(\zeta) < \aleph_0$, the $y_0(\zeta), y_1(\zeta), \ldots, y_{d(\zeta)}(\zeta)$ form an orthonormal system and $y_n(\zeta) = 0$ for $n > d(\zeta)$,

iii) $\forall y_n \exists$ a cover of Z of disjoint measurable sets $Z_1, Z_2, \ldots$ such that

$$\forall Z_k \, y_n(\zeta) = \sum_i f_i(\zeta) x_i(\zeta),$$

with f_i measurable complex-valued functions which are identically 0 for sufficiently large i.

Now, if one looks at the direct integral, it is given as $\int_Z^{\oplus} H(\zeta) d(\zeta)$. So, we are kind of trying to integrate Hilbert spaces. So, to be able to define this, we need more than just functions to be measurable.

Definition C.22. Let $(Z, \mathcal{B})$ be a Borel space, $\zeta \to H(\zeta)$ a field of complex Hilbert spaces and $F = \Pi_{\zeta\in Z} H(\zeta)$. Then, the $H(\zeta)$ for a *ν-measurable field of complex Hilbert spaces* if $\exists S \subset F$, linear subspace, such that

i) $\forall y \in S$ ($\zeta \mapsto ||y(\zeta)||_{H(\zeta)}$ is ν-measurable),

ii) if $y \in F$ ($\forall s \in S(\zeta \mapsto (s(\zeta), y(\zeta))$ is ν-measurable)) $\implies y \in S$,

iii) $\exists (x_n)_{n\in\mathbb{N}}$, $x_n \in S$ ($\forall \zeta \in Z$ $(x_n(\zeta))_{n\in\mathbb{N}}$ is a total sequence in $H(\zeta)$).

The last point is similar to $(x_n(\zeta))_n \in \mathbb{N}$ being a basis for $H(\zeta)$, but not the same. However, for our purposes, it is good enough to think of the $(x_n(\zeta))_{n\in\mathbb{N}}$ as a basis for the Hilbert spaces $H(\zeta)$. As we will soon see that this is the case for when we are working with direct integrals.
Now, the vector fields $s \in S$ are called *ν-measurable vector fields.* And, the sequence $(x_n)_{n\in\mathbb{N}}$ of *iii*) is called a *fundamental sequence* of ν-measurable vector fields. Also, property *iii*) implies that the Hilbert spaces $H(\zeta)$ are separable.
Also, if x, y are measurable vector fields, then $(x(\zeta), y(\zeta))$ is a linear combination of $||(x+y)(\zeta)||^2_{H(\zeta)}$, $||(x-y)(\zeta)||^2_{H(\zeta)}$, $||(x+iy)(\zeta)||^2_{H(\zeta)}$ and $||(x-iy)(\zeta)||^2_{H(\zeta)}$ and it is a measurable function in ζ. We also see that if we multiply x with a complex-valued measurable function f, then $f \cdot x$ is again a measurable vector field.

To get a bit more used to measurable fields of Hilbert spaces, we will give two examples.
First, let $(Z, \mathcal{B})$ be a discrete Borel space. Then, every map $f : Z \to \mathbb{C}$ is measurable. It follows that the only option for $S \subset F$ such that $\zeta \to H(\zeta)$ is a measurable field of Hilbert spaces, is $S = F$.
Next, let H_0 be a separable complex Hilbert space and let $(Z, \mathcal{B})$ be a Borel space. Then we can define a ν-measurable field called the *constant field* corresponding to H_0 over Z in the following way:

i) set $H(\zeta) = H_0 \; \forall \zeta \in Z$,

ii) the ν-measurable vector fields $s \in S$ are precisely the ν-measurable maps from Z into H_0

This is indeed a ν-measurable field of Hilbert spaces. Property $i)$ of the definition is satisfied by construction.
Note, the constant map $s : \zeta \mapsto h$, $\zeta \in Z$, $h \in H_0$, is measurable, thus $s \in S$. Let $x \in F$ such that $\zeta \mapsto (x(\zeta), y(\zeta))$ is measurable $\forall y \in S$. Then $\zeta \mapsto (x(\zeta), s(\zeta)) = (x(\zeta), h)$, is measurable. Since this is true for all $h \in H_0$ and all $\zeta \in Z$, it follows that this map is only measurable when x is measurable itself. Hence $x \in S$. So property $ii)$ is also satisfied.
Lastly, if $(x_n)_{n\in\mathbb{N}}$ would be a total sequence in H_0, then $\zeta \mapsto a_n$ forms a fundamental sequence of measurable vector fields. Showing that this construction satisfies property $iii)$. Hence, this *constant field* corresponding to H_0 is indeed a measurable field of Hilbert spaces.

We now know when we have a measurable field for a given Borel space $(Z, \mathcal{B})$. However, we would also love to define a subspace for a measurable field in a certain sense.
So, let $Y \subset Z$ be a measurable subset. Then if y is a vector field over Y, it is called measurable if it can be extended to a measurable vector field over Z. Now, y can be extended to all of Z if $\zeta \mapsto (y(\zeta), x(\zeta))$, $\zeta \in Y$ is measurable for any $x \in F$.
Since, if the above holds for y, we can extend y to all of Z by setting $y(\zeta) = 0$ for $\zeta \in Z \backslash Y$. Then y is indeed a measurable vector field over Z.
Now, the measurable vector fields over Y also induce a measurable field of Hilbert space structure on $\zeta \mapsto H(\zeta)$ for $\zeta \in Y$. This measurable field is called the *induced* field by Y. And one can sort of think of this field as a subspace of $\zeta \mapsto H(\zeta)$ for $\zeta \in Z$.

As a final remark on measurable fields of Hilbert spaces, let ν_1 be a positive measure on Z that is equivalent to another positive measure ν_2. Then, if the $H(\zeta)$ form a ν_1 measurable field of complex Hilbert spaces with the subspace $S \subset F$, then it is also a measurable field of complex Hilbert spaces with respect to ν_2. In other words, the notation of ν-measurable Hilbert spaces involves the class of positive measures ν.

For the next part, let $(Z, \mathcal{B})$ be a Borel space, ν a positive measure on Z and $\zeta \to H(\zeta)$ a ν-measurable field of complex Hilbert spaces.

Proposition C.23. The set $z_p := \{\zeta \in Z | \ d(\zeta) = p\}$ is measurable.
Moreover, $\exists (y_n)_{n\in\mathbb{N}_{>0}}$, y_i ν-measurable vector fields, such that

i) if $d(\zeta) = \aleph_0$, then $(y_1(\zeta), y_2(\zeta), \ldots)$ is an orthonormal basis of $H(\zeta)$,

ii) if $d(\zeta) = p$, then $(y_1(\zeta), \ldots, y_p(\zeta))$ is an orthonormal basis of $H(\zeta)$ and $y_i(\zeta) = 0$ for $i > p$.

So we see that we can find a sequence of vector fields such that the first n vector fields in ζ form a basis for $H(\zeta)$. A sequence $(y_n)_{n\in\mathbb{N}_{>0}}$ satisfying the property of the above proposition is called a *measurable field of orthonormal bases.*

Proposition C.24. Let $(x_n)_{n\in\mathbb{N}_{>0}}$ be a fundamental sequence of measurable fields. $x \in F$ is measurable, if the functions $\lambda \mapsto (x(\zeta), x_i(\zeta))$ are measurable.

Proposition C.25. Let $p \in \{1, 2, \ldots, \aleph_0\}$ and set $z_p = \{\zeta \in Z | \ d(\zeta) = p\}$ and let H_p be a complex Hilbert space with $dim(H_p) = p$. Then, the field induced by $(H(\zeta))_{\zeta\in z_p}$ is isomorphic to the constant field corresponding to H_p.

Proposition C.26. Let $(x_n)_{n\in\mathbb{N}_{>0}}$ be a sequence of vector fields such that

i) the functions $\zeta \mapsto (x_i(\zeta), x_j(\zeta))$ are measurable for $i, j \in \mathbb{N}_{>0}$,

ii) $\forall \zeta \in Z$, the $(x_n(\zeta))_{n\in\mathbb{N}_{>0}}$ form a total sequence in $H(\zeta)$.

Then, $\exists!$ measurable field structure on the $H(\zeta)$'s such that all of the $x_i \in F$ are measurable.

We somehow want to turn a ν-measurable field of Hilbert spaces into a new Hilbert space. This new Hilbert space will then be defined as the direct integral. But, we cannot do that with just ν-measurability, we need something stronger.

Definition C.27. Let $\zeta \mapsto H(\zeta)$ be a ν-measurable field of complex Hilbert spaces. A vector field $x : \zeta \mapsto H(\zeta)$ is called *square-integrable* if x is measurable, so $x \in S$, and if $\int_Z ||x(\zeta)||^2_{H(\zeta)} d\nu(\zeta) < \infty$.

So, again, let $(Z, \mathcal{B})$ be a Borel space, $\nu\colon Z \to [0, \infty]$ a positive measure and $S \subset F = \Pi_{\zeta \in Z} H(\zeta)$ the subspace of measurable vector fields. Let $K := \{s \in S \mid \int ||s(\zeta)||^2 d\nu(\zeta) < \infty\}$, so K is the set of square-integrable vector fields. Then K is a complex vector space.
Now, for $x, y \in K$, $\zeta \mapsto (x(\zeta), y(\zeta))$ is a square integrable map. Define a scalar product $(\cdot, \cdot)\colon K \times K \to \mathbb{C}$ by

$$(x, y) = \int_Z (x(\zeta), y(\zeta)) d\nu(\zeta).$$

Then we can turn K into a pre-Hilbert space with respect to the norm $||x||^2 = (x, x) = \int_Z ||x(\zeta)||^2_{H(\zeta)} d\nu(\zeta)$. We see that if $x \in K$ and $||x|| = 0$, then x is almost zero everywhere. So, we can construct a Hausdorff pre-Hilbert space $\mathcal{H}$ associated with K by setting $x \in \mathcal{H}$ to be the class of $y \in K$ that are almost equal everywhere to $x \in K$.
We can consider the elements $x \in \mathcal{H}$ as vector spaces, but note that the values $x(\zeta)$ can only be determined up to within negligible sets. Since we are now dealing with classes of vector spaces that are equal almost everywhere, instead of regular vector spaces. Nonetheless, we get the following proposition:

Proposition C.28. The above defined Hausdorff pre-Hilbert space $\mathcal{H}$ is in fact a Hilbert space.
Furthermore, if a sequence $(x_n)_{n \in \mathbb{N}}$, $x_n \in H$, converges to some $x \in \mathcal{H}$ with respect to the norm of $\mathcal{H}$. Then, $\exists (x_{n_k})_{k \in \mathbb{N}}$, a subsequence of $(x_n)_{n \in \mathbb{N}}$, that converges to x almost everywhere.

Now, the above defined Hilbert space $\mathcal{H}$ is called the *direct integral* of the Hilbert spaces $H(\zeta)$, $\zeta \in Z$ and is denoted by $\mathcal{H} = \int_Z^{\oplus} H(\zeta) d\nu(\zeta)$. And if $x \in \mathcal{H}$, one can write $x = \int_Z^{\oplus} x(\zeta) d\nu(\zeta)$.
Do note that $\mathcal{H}$ depends on the choice of S, as K, the set of square-integrable vector fields, is a subspace of S. So, sometimes the direct integral is denoted as ${}^S\!\int_Z^{\oplus} H(\zeta) d\nu(\zeta)$, to indicate the specific choice of measurable vector fields.

Just like Hilbert spaces have subspaces, we can also identify subspaces of a direct integral. Let $Y \subset Z$ be a measurable subset, then the square-integrable vector fields x that vanish on Y form a closed linear subspace of $\mathcal{H}$. In other words, we have a subspace $\int_Y^{\oplus} H(\zeta) d\nu(\zeta)$, which we can identify as the direct integral of the induced measurable fields $\zeta \mapsto H(\zeta)$, $\zeta \in Y$.

We will end this appendix with two examples of direct integrals.
First, let $(Z, \mathcal{B})$ be a discrete Borel space and take $\nu(\zeta) = 1$ for $\zeta \in Z$, so ν is the counting measure. Recall that every map $Z \to \mathbb{C}$ is measurable, and thus, every vector field x is measurable.
It follows that $\int_Z ||x(\zeta)||^2 d\nu(\zeta) < \infty \iff \sum_{\zeta \in Z} ||x(\zeta)||^2 < \infty$. In other words, $\mathcal{H} = \bigoplus_{\zeta \in Z} H(\zeta)$. So, in this special case, the direct sum is the same as the direct sum. Which is why one can think of the direct integral as a generalisation of the direct sum. Furthermore, notice that we can identify the subspaces of $\mathcal{H}$ with the use of the Hilbert spaces $H(\zeta)$. This is also not always the case.

For a second example, let H_0 again be a separable Hilbert space and let $\zeta \mapsto H(\zeta)$ be the constant field over Z. The square-integrable vector fields are in this case the square-integrable functions from Z into H_0. So, $\mathcal{H} = \int_Z^{\oplus} H(\zeta) d\nu(\zeta) = L^2(H_0, \nu)$.

Appendix D: Special Functions

In this appendix we will describe a few facts and properties on special functions that are used to describe the Clebsch-Gordan coefficients. We will first introduce the so called *double sine function*, then build the other used special functions with the use of this one. To construct the double sine function, we will use two approaches. One is via the *quantum dilogarithm*, which is used to define the double sine function in [3], and the other approach is via the *double Gamma function* [1]. We will only be giving definitions and properties here, for proofs of some of the identities see [1].
Note, the quantum dilogarithm is also related to the double Gamma function. So both approaches do give the same function. We only describe both approaches to give more intuition on the double Sine function. Regardless, it is best to keep in mind that the properties of these functions is what we are most concerned about.

Since the quantum dilogarithm is also related to the double Gamma function. We will first start with describing the latter one. The *double Gamma function* is a special function introduced by Barnes and is defined by

$$\log \Gamma_2(s|\omega_1, \omega_2) = \left(\frac{\partial}{\partial t} \sum_{n_1,n_2=0}^{\infty} (s + n_1\omega_1 + n_2\omega_2)^{-t} \right)_{t=0} .$$

Now, the double Sine function is defined via the $\Gamma_b(x)$ function, which is given by

$$\Gamma_b(x) = \Gamma_2(x|b, b^{-1}).$$

So we see that this function is invariant under the change $b \to b^{-1}$. Furthermore, we can also define $\Gamma_b(x)$ with the following integral representation:

$$\log \Gamma_b(x) = \int_0^\infty \frac{1}{t} \left(\frac{e^{-xt} - e^{-\frac{Qt}{2}}}{(1 - e^{-bt})(1 - e^{-b^{-1}t})} - \frac{(Q - 2x)^2}{8e^t} - \frac{(Q - 2x)}{t} \right) dt.$$

Properties of $\Gamma_b(x)$ are:

functional equation: $\Gamma_b(x + b) = \sqrt{2\pi}\, b^{bx-\frac{1}{2}} \Gamma_b^{-1}(bx)\Gamma_b(x)$

analyticity: $\Gamma_b(x)$ is a meromorphic function with poles at $x = -nb - mb^{-1}$, $n, m \in \mathbb{N}$.

Now, the quantum dilogarithm is related to the double Gamma function, but was also eventually introduced independently. Let $x \in \mathbb{C}$, $im(x) < \frac{Q}{2}$, then we get the following integral representation of for the quantum dilogarithm:

$$e_b(x) = \exp\left(- \int_{\mathbb{R}-i0} \frac{1}{4t} \frac{e^{-2itx}}{\sinh(bt)\sinh(b^{-1}t)} dt \right),$$

with the contour going around the pole at $t = 0$ in the upper-half plane. We can then define $s_b(x) = e^{\frac{i\pi}{2}x^2 + \frac{i\pi}{24}(b^2+b^{-2})} e_b(x)$. The analytic continuation of $s_b(x)$ to all of $\mathbb{C}$ is a meromorphic function with the following properties:

functional equation: $\frac{s_b(x+\frac{i}{2}b^{\pm 1})}{s_b(x-\frac{1}{2}b^{\pm 1})} = 2\cosh(\pi b^{\pm 1} x)$,

reflection property: $s_b(x)s_b(-x) = 1$,

complex conjugation: $\overline{s_b(x)} = s_b(-\overline{x})$,

poles and zeros: $s_b(x)^{\pm 1} = 0 \iff \pm x \in \{i\frac{Q}{2} + nb + mb^{-1} \,|\, n, m \in \mathbb{N}\}$,

residue: $Res_{x=-i\frac{Q}{2}} (s_b(x)) = \frac{i}{2\pi}$,

asymptotic behvaiour: $s_b(x) \sim \begin{cases} e^{-\frac{i\pi}{2}(x^2+\frac{1}{12}(b^2+b^{-2}))} & \text{as } |x| \to \infty,\ |arg(x)| < \frac{\pi}{2} \\ e^{\frac{i\pi}{2}(x^2+\frac{1}{12}(b^2+b^{-2}))} & \text{as } |x| \to \infty,\ |arg(x)| > \frac{\pi}{2}. \end{cases}$

Now, the *double Sine function* can now be defined in two ways,

$$S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)} = s_b(ix - \frac{iQ}{2}).$$

Another useful function, related to the double Sine function, is

$$G_b(x) = e^{\frac{\pi i}{2}x(x-Q)} S_b(x).$$

These two functions have the following properties:

self-duality: $S_b(x) = S_{b^{-1}}(x), \qquad G_b(x) = G_{b^{-1}}(x)$,

functional equation: $S_b(x+b) = 2\sin(\pi b x) S_b(x), \qquad G_b(x+b) = (1 - e^{2\pi i b x}) G_b(x)$,

reflection property: $S_b(x) S_b(Q-x) = 1, \qquad G_b(x) G_b(Q-x) = e^{\pi i (x^2 - xQ)}$,

analyticity: $S_b(x)$ and $G_b(x)$ are meromorphic functions with poles at $ix = -nb - mb^{-1}$ and zeros at $ix = iQ + nb + mb^{-1}$, $n, m \in \mathbb{N}$,

asymptotic behaviour: $S_b(x) \sim \begin{cases} e^{-\frac{\pi i}{2}(x^2 - xQ)} & \text{as } Im(x) \to \infty \\ e^{+\frac{\pi i}{2}(x^2 - xQ)} & \text{as } Im(x) \to -\infty \end{cases} \qquad G_b(x) \sim \begin{cases} 1 & \text{as } Im(x) \to \infty \\ e^{\pi i (x^2 - xQ)} & \text{as } Im(x) \to -\infty. \end{cases}$

We also have the following integral identity for $G_b(x)$, called the *b-beta integral*:

$$B_b(\alpha, \beta) = \frac{1}{i} \int_{-i\infty}^{i\infty} e^{2\pi i \tau \beta} \frac{G_b(\tau + \alpha)}{G_b(\tau + Q)} d\tau = \frac{G_b(\alpha) G_b(\beta)}{G_b(\alpha + \beta)}.$$

We then get the following function, which is also used to define the Clebsch-Gordan coefficients,

$$\Theta_b(y; \alpha) = \frac{G_b(y)}{G_b(y + \alpha)}.$$

Note that the b-beta integral can be seen as a Fourier-transform for $\Theta_b(y; \alpha)$ as,

$$\Theta_b(y; \alpha) = \frac{1}{iG_b(y)} \int_{-i\infty}^{i\infty} e^{2\pi i \alpha \tau} \Theta_b(\tau + y; Q + y) d\tau.$$

The last special function that we use in the definition of the Clebsch-Gordan coefficients is $\Psi_b(\alpha, \beta; \gamma; y)$, which is also given by an integral:

$$\Psi_b(\alpha, \beta; \gamma; y) = \frac{1}{i} \int_{-i\infty}^{i\infty} e^{2\pi i s \beta} \frac{G_b(s + y) G_b(s + \gamma - \beta)}{G_b(s + y + \alpha) G_b(s + Q)} ds.$$

For the Racah-Wigner coefficients, we will see that we can represent the coefficients with a certain integral relation. This integral can be calculated with the use of the *b-hypergeometric function*. This is a function that is defined via the following contour integral, which seems similar to that of $\Psi_b(\alpha, \beta; \gamma; y)$:

$$F_b(\alpha, \beta; \gamma; y) = \frac{1}{i} \frac{S_b(\gamma)}{S_b(\alpha) S_b(\beta)} \int_{-i\infty} i\infty e^{2\pi i s y} \frac{S_b(\alpha + s) S_b(\beta + s)}{S_b(\gamma + s) S_b(Q + s)} ds.$$

The contour is taken on the right of the poles at $is = -\alpha - nb - mb^{-1}$ and $is = -\beta - nb - mb^{-1}$ and on the left of the poles at $is = nb + mb^{-1}$ and $is = Q - \gamma + nb + mb^{-1}$ with $n, m \in \mathbb{N}$. Moreover, this function is a solution to the equation

$$([\delta_x + \alpha][\delta_x + \beta] - e^{-2\pi bx}[\delta_x][\delta_x + \gamma - Q])F_b(\alpha, \beta; \gamma; -ix) = 0, \qquad \delta_x = \frac{1}{2\pi}\frac{d}{dx}.$$

We end this appendix by given analytic properties of the Clebsch-Gordan coefficients and of the kernels $\Phi^{\delta}_{\alpha_\delta}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}$. For the Clebsch-Gordan coefficients we have the following analytical and asymptotic properties, as given in [1]:

i) $\begin{bmatrix}Q-\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{bmatrix}$ decays exponentially as $e^{-2\pi\alpha_i|x_i|}$ if any $|x_i| \to \infty$ for $i \in \{1,2,3\}$,

ii) The Clebsch-Gordan coefficients are meromorphic in x_1, x_2 and x_3 and have poles with respect to x_1 at:

Upper half plane: $x_1 = x_2 - \frac{i}{2}(\alpha_1 + \alpha_2 - 2\alpha_3) + i(\varepsilon + nb + mb^{-1})$,
$x_1 = x_3 - \frac{i}{2}(\alpha_3 + \alpha_1 - Q) + i(\varepsilon + nb + mb^{-1})$
Lower half plane: $x_1 = x_2 - \frac{1}{2}(Q - \alpha_1 - \alpha_2) - i(Q + nb + mb^{-1})$,
$x_1 = x_3 - \frac{i}{2}(2\alpha_2 - \alpha_3 - \alpha_1) - i(Q + nb + mb^{-1})$,

with $n, m \in \mathbb{N}$ and poles with respect to x_2 at:

Upper half plane: $x_2 = x_1 + \frac{i}{2}(Q - \alpha_1 - \alpha_2) + i(Q + nb + mb^{-1})$,
$x_2 = x_3 - \frac{i}{2}(2\alpha_1 - \alpha_3 - \alpha_2) + i(Q + nb + mb^{-1})$
Lower half plane: $x_2 = x_1 - \frac{1}{2}(2\alpha_3 - \alpha_1 - \alpha_2) - i(\varepsilon + nb + mb^{-1})$,
$x_2 = x_3 - \frac{i}{2}(Q - \alpha_3 - \alpha_2) - i(\varepsilon + nb + mb^{-1})$.

For the kernels $\Phi^{\delta}_{\alpha_\delta}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}(x_4, \mathbf{x})$, $\delta \in \{s, t\}$, we have the following analytical and asymptotic properties:

i) $\Phi^{s}_{\alpha_s}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_\varepsilon(x_4, \mathbf{x})$ is meromorphic with respect to

$$x_1 \text{ in } \{x_1 \in \mathbb{C}\,|\, im(x_1) \in (-Q, b)\}, \qquad x_3 \text{ in } \{x_3 \in \mathbb{C}\,|\, im(x_1) \in (-b, Q)\},$$
$$x_2 \text{ in } \{x_2 \in \mathbb{C}\,|\, im(x_1) \in (-b, Q)\}, \qquad x_4 \text{ in } \{x_4 \in \mathbb{C}\,|\, im(x_1) \in (-b, b)\},$$

with poles at

$$x_1 - x_2 + \tfrac{i}{2}(\alpha_2 + \alpha_1 - 2\alpha_s) - 2i\varepsilon = 0,$$
$$x_1 - x_2 + \tfrac{i}{2}(\alpha_2 + \alpha_1 - 2(Q - \alpha_s)) - i\varepsilon = 0, \qquad x_1 - x_4 + \tfrac{i}{2}(\alpha_1 - \alpha_4) - 2i\varepsilon = 0,$$
$$x_1 - x_3 + \tfrac{i}{2}(\alpha_3 + \alpha_1 - 2(Q - \alpha_4)) - 2i\varepsilon = 0, \qquad x_3 - x_4 + \tfrac{i}{2}(\alpha_4 - \alpha_3) + i\varepsilon = 0,$$

and it decays exponentially as $e^{-\pi Q|x_i|}$ if any $|x_i| \to \infty$ for $i \in \{1,2,3,4\}$,

ii) $\Phi^{t}_{\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_\varepsilon(x_4, \mathbf{x})$ is meromorphic with respect to

$$x_1 \text{ in } \{x_1 \in \mathbb{C}\,|\, im(x_1) \in (-Q, b)\}, \qquad x_3 \text{ in } \{x_3 \in \mathbb{C}\,|\, im(x_1) \in (-b, Q)\},$$
$$x_2 \text{ in } \{x_2 \in \mathbb{C}\,|\, im(x_1) \in (-Q, b)\}, \qquad x_4 \text{ in } \{x_4 \in \mathbb{C}\,|\, im(x_1) \in (-b, b)\},$$

with poles at

$$
\begin{aligned}
&x_3 - x_2 i\tfrac{i}{2}(\alpha_3 + \alpha_2 - 2\alpha_t) + 2i\varepsilon = 0, \\
&x_3 - x_2 - \tfrac{i}{2}(\alpha_3 + \alpha_2 - 2(Q - \alpha_t)) + i\varepsilon = 0, \qquad x_1 - x_4 + \tfrac{i}{2}(\alpha_1 - \alpha_4) - i\varepsilon = 0, \\
&x_1 - x_3 + \tfrac{i}{2}(\alpha_3 + \alpha_1 - 2(Q - \alpha_4)) - 2i\varepsilon = 0, \qquad x_3 - x_4 + \tfrac{i}{2}(\alpha_4 - \alpha_3) + 2i\varepsilon = 0,
\end{aligned}
$$

and it decays exponentially as $e^{-\pi Q|x_i|}$ if any $|x_i| \to \infty$ for $i \in \{1, 2, 3, 4\}$.

Appendix E

This appendix contains the calculations that are needed to simplify the action of the representation $\pi_{\alpha_2,\alpha_1}(C_q - \frac{2}{(q-q^{-1})^2}) = \pi_{21}(C'_q)$ on the Fourier-transformed functions $\mathcal{F}(f(x_2,x_1)$. It starts with the exact defined action and all following segments contain the calculations for each single term. No conclusion is given, as the simplification is exactly as is stated in 8.3.1.

$\pi_{21}(C)f(x_2,x_1)$

$$
\begin{aligned}
&\pi_{21}(C)f(x_2,x_1) = \\
&\left(\pi_2(FE)\otimes \pi_1(K)^2 + \pi_2(FK^{-1})\otimes\pi_1(KE) + \pi_(K^{-1}E)\otimes\pi_1(FK) + \pi_2(K^{-1})^2\otimes\pi_1(FE)\right) f(x_2,x_1) \\
&\qquad - \frac{1}{(q-q^{-1})^2}(q\pi_2(K)^2\otimes\pi_1(K)^2 + q^{-1}\pi_2(K)^{-2}\otimes\pi_1(K)^{-2} + 2)f(x_2,x_1) = \\
&f(x_2+ib,x_1+ib)\left(\frac{e^{\pi ib^2}}{(q-q^{-1})^2} - \frac{q}{(q-q^{-1})^2}\right) + f(x_2-ib,x_1-ib)\left(\frac{e^{-\pi ib^2}}{(q-q^{-1})^2} - \frac{q^{-1}}{(q-q^{-1})^2}\right) \\
&\quad + f(x_2,x_1)\left(\frac{1}{(q-q^{-1})^2}\left[-e^{-\pi ib(2Q-\alpha_2-\alpha_1)}e^{-2\pi bx_2}\otimes e^{2\pi b(x_1+\frac{ib}{2})} - e^{\pi ib(2Q-\alpha_2-\alpha_1)}e^{2\pi b(x_2-\frac{ib}{2})}\otimes e^{-2\pi bx_1}\right]\right. \\
&\qquad \left. - \frac{2}{(q-q^{-1})^2}\right) \\
&\quad + f(x_2,x_1+ib)\left(\frac{1}{(q-q^{-1})^2}\left[-e^{-\pi ib^2}e^{-2\pi ib(\alpha_2-Q)} - e^{\pi ib^2}e^{2\pi ib(\alpha_2-Q)} + e^{\pi ib(\alpha_2-\alpha_1)}e^{-2\pi bx_2}\otimes e^{2\pi b(x_1+\frac{ib}{2})}\right.\right. \\
&\qquad \left.\left. + e^{-\pi ib(\alpha_2-\alpha_1)}e^{2\pi b(x_2-\frac{ib}{2})}\otimes e^{2\pi bx_1}\right]\right) \\
&+ f(x_2-ib,x_1+ib)\left(\frac{1}{(q-q^{-1})^2}\left[e^{-\pi ib^2} - e^{\pi ib(2Q-\alpha_2-\alpha_1)}e^{-2\pi bx_2}\otimes e^{2\pi b(x_1+\frac{ib}{2})}\right.\right. \\
&\qquad \left.\left. - e^{-\pi ib(2Q-\alpha_2-\alpha_1)}e^{2\pi b(x_2-\frac{ib}{2})}\otimes e^{-2\pi bx_1} + e^{\pi ib^2}\right]\right) \\
&\quad + f(x_2-ib,x_1)\left(\frac{1}{(q-q^{-1})^2}\left[-e^{-\pi ib^2}e^{-2\pi ib(\alpha_1-Q)} - e^{\pi ib^2}e^{2\pi ib(\alpha_1-Q)} + e^{-\pi ib(\alpha_2-\alpha_1)}e^{-2\pi bx_2}\otimes e^{2\pi b(x_1+\frac{ib}{2})}\right.\right. \\
&\qquad \left.\left. + e^{\pi ib(\alpha_2-\alpha_1)}e^{2\pi b(x_2-\frac{ib}{2})}\otimes e^{-2\pi bx_1}\right]\right)
\end{aligned}
$$

Immediately note that since $q = e^{\pi ib^2}$, the terms regarding $f(x_2 \pm ib, x_1 \pm ib)$ have no contributions. So we will ignore those two terms from this point on.

$\mathcal{F}(f(x_2 - ib, x_1 + ib))$ term of q-Casimir

$$\mathcal{F}\left(\left(e^{-\pi ib^2} + e^{\pi ib^2} - e^{\pi ib(2Q-\alpha_2-\alpha_1)}e^{-2\pi bx_2} \otimes e^{2\pi b(x_1+\frac{ib}{2})}\right.\right.$$
$$\left.\left. -e^{-\pi ib(2Q-\alpha_2-\alpha_1)}e^{2\pi b(x_2-\frac{ib}{2})} \otimes e^{-2\pi bx_1}\right) f(x_2 - ib, x_1 + ib)\right)$$
$$= \mathcal{F}\left(\left(e^{-\pi ib^2} + e^{\pi ib^2} - e^{\pi ib(2Q-\alpha_2-\alpha_1)}e^{-2\pi b(x_2-x_1-\frac{ib}{2})} \; -e^{-\pi ib(2Q-\alpha_2-\alpha_1)}e^{2\pi b(x_2-\frac{ib}{2}-x_1)}\right) f(x_2 - ib, x_1 + ib)\right)$$
$$= \left(e^{-\pi ib^2} + e^{\pi ib^2}\right) \int_{\mathbb{R}} e^{-\pi i\omega x_+} f\left(\frac{x_+ + x_- - 2ib}{2}, \frac{x_+ - x_- + 2ib}{2}\right) dx_+$$
$$- e^{\pi ib(2Q-\alpha_2-\alpha_1)} \int_{\mathbb{R}} e^{-\pi i\omega x_+} e^{-2\pi b(x_- - \frac{ib}{2})} f\left(\frac{x_+ + x_- - 2ib}{2}, \frac{x_+ - x_- + 2ib}{2}\right) dx_+$$
$$- e^{-\pi ib(2Q-\alpha_2-\alpha_1)} \int_{\mathbb{R}} e^{-\pi i\omega x_+} e^{2\pi b(x_- - \frac{ib}{2})} f\left(\frac{x_+ + x_- - 2ib}{2}, \frac{x_+ - x_- + 2ib}{2}\right) dx_+$$
$$= \left(e^{-\pi ib^2} + e^{\pi ib^2}\right) T_{x_-}^{-2ib}\mathcal{F}(f(x_2, x_1)) - e^{\pi ib(2Q-\alpha_2-\alpha_1)}e^{-2\pi b(x_- - \frac{ib}{2})}T_{x_-}^{-2ib}\mathcal{F}(f(x_2, x_1))$$
$$- e^{-\pi ib(2Q-\alpha_2-\alpha_1)}e^{2\pi b(x_- - \frac{ib}{2})}T_{x_-}^{-2ib}\mathcal{F}(f(x_2, x_1))$$
$$= \left(e^{-\pi ib^2} + e^{\pi ib^2} - e^{\pi ib(2Q-\alpha_2-\alpha_1)}e^{-2\pi bx_-}e^{\pi ib^2} - e^{-\pi ib(2Q-\alpha_2-\alpha_1)}e^{2\pi bx_-}e^{-\pi ib^2}\right) T_{x_-}^{-2ib}\mathcal{F}(f(x_2, x_1))$$
$$= \left(-e^{-\pi ibQ} - e^{\pi ibQ} + e^{\pi ib(3Q-\alpha_2-\alpha_1)}e^{-2\pi bx_-} + e^{-\pi ib(3Q-\alpha_2-\alpha_1)}e^{2\pi bx_-}\right) T_{x_-}^{-2ib}\mathcal{F}(f(x_2, x_1))$$

In the last line, we used that $Q = b + b^{-1}$, so $b \cdot Q = b^2 + 1$ and thus, $e^{\pi ib^2} = e^{\pi i(bQ-1)} = -e^{\pi ibQ}$. Now,

$e^{\pi ib(3Q-\alpha_2-\alpha_1)}e^{-2\pi bx_-} - e^{\pi ibQ} - e^{-\pi ibQ} + e^{-\pi ib(3Q-\alpha_2-\alpha_1)}e^{2\pi bx_-} =$
$e^{\pi ib(3Q-\alpha_2-\alpha_1+2ix)} - e^{\pi ibQ} - e^{-\pi ibQ} + e^{-\pi ib(3Q-\alpha_2-\alpha_1+2ix)} =$
$(e^{\pi ib(2Q-\frac{1}{2}(\alpha_2+\alpha_1)+ix)} - e^{-\pi ib(2Q-\frac{1}{2}(\alpha_2+\alpha_1)+ix)})(e^{\pi ib(Q-\frac{1}{2}(\alpha_2+\alpha_1)+ix)} - e^{-\pi ib(Q-\frac{1}{2}(\alpha_2+\alpha_1)+ix)}) =$
$4[ix - \frac{1}{2}(\alpha_2 - \alpha_1) + Q]_b[ix - \frac{1}{2}(\alpha_2 - \alpha_1) + 2Q]_b = 4[-ix + \frac{1}{2}(\alpha_2 - \alpha_1) - Q]_b[-ix + \frac{1}{2}(\alpha_2 + \alpha_1) + 2Q]_b$

Where we used that $sin(-x) = -sin(x)$.
Thus we see that the terms of the q-Casimir operator resulting in $f(x_2 - ib, x_1 + ib)$ are mapped to the operator $[-ix + \frac{1}{2}(\alpha_2 - \alpha_1) - Q]_b[-ix + \frac{1}{2}(\alpha_2 + \alpha_1) + 2Q]_b T_{x_-}^{-2ib}$ under the Fourier-transform $\mathcal{F}$.

$\mathcal{F}(f(x_2, x_1))$ term

$$\mathcal{F}\left(f(x_2,x_1)\left(\frac{1}{(q-q^{-1})^2}\left[-e^{-\pi ib(2Q-\alpha_2-\alpha_1)}e^{-2\pi bx_2}\otimes e^{2\pi b(x_1+\frac{ib}{2})}-e^{\pi ib(2Q-\alpha_2-\alpha_1)}e^{2\pi b(x_2-\frac{ib}{2})}\otimes e^{-2\pi bx_1}\right]\right.\right.$$
$$\left.\left.+\frac{2}{(q-q^{-1})^2}\right)\right)$$
$$=\mathcal{F}\left(\frac{f(x_2,x_1)}{(q-q^{-1})^2}\left(-e^{-\pi ib(2Q-\alpha_2-\alpha_1)}e^{-2\pi bx_2}\otimes e^{2\pi b(x_1+\frac{ib}{2})}-e^{\pi ib(2Q-\alpha_2-\alpha_1)}e^{2\pi b(x_2-\frac{ib}{2})}\otimes e^{-2\pi bx_1}+2\right)\right)$$

Now,

$$\mathcal{F}\left(f(x_2,x_1)\left(-e^{-\pi ib(2Q-\alpha_2-\alpha_1)}e^{-2\pi bx_2}\otimes e^{2\pi b(x_1+\frac{ib}{2})}-e^{\pi ib(2Q-\alpha_2-\alpha_1)}e^{2\pi b(x_2-\frac{ib}{2})}\otimes e^{-2\pi bx_1}+2\right)\right)$$
$$=\mathcal{F}\left(f(x_2,x_1)\left(-e^{-\pi ib(2Q-\alpha_2-\alpha_1)}e^{-2\pi b(x_2-x_1-\frac{ib}{2})}-e^{\pi ib(2Q-\alpha_2-\alpha_1)}e^{2\pi b(x_1-\frac{ib}{2}-x_1)}+2\right)\right)$$
$$=\mathcal{F}(f(x_2,x_1))\left(-e^{-\pi ib(2Q-\alpha_2-\alpha_1)}e^{-2\pi b(x_--\frac{ib}{2})}-e^{\pi ib(2Q-\alpha_2-\alpha_1)}e^{2\pi b(x_--\frac{ib}{2})}+2\right).$$

This can then be rewritten to:

$$-e^{-\pi ib(2Q-\alpha_2-\alpha_1)}e^{-2\pi b(x_--\frac{ib}{2})}-e^{\pi ib(2Q-\alpha_2-\alpha_1)}e^{2\pi b(x_--\frac{ib}{2})}+2=$$
$$-e^{-\pi ib(2Q-\alpha_2-\alpha_1)}e^{-2\pi bx_-}e^{\pi ib^2}-e^{\pi ib(2Q-\alpha_2-\alpha_1)}e^{2\pi bx_-}e^{-\pi ib^2}+2=$$
$$e^{-\pi ib(2Q-\alpha_2-\alpha_1-Q-2ix_-)}+e^{\pi ib(2Q-\alpha_2-\alpha_1-Q-2ix_-)}=e^{-\pi ib(Q-\alpha_2-\alpha_1-2ix_-)}+e^{\pi ib(Q-\alpha_2-\alpha_1-2ix_-)}+2$$

Which in turn can be rewritten into:

$$[c]_b^2+[-ix-\frac{1}{2}(\alpha_1+\alpha_2-Q)+c]_b[-ix-\frac{1}{2}(\alpha_1+\alpha_2-Q)-c]_b,$$

which follows from

$$(e^c-e^{-c})^2+(e^{a+c}-e^{-(a+c)})(e^{a-c}-e^{-(a-c)})=e^{2a}-e^{-2c}-e^{2c}+e^{-2a}+e^{2c}-1-1+e^{-2c}$$
$$=e^{2a}+e^{-2a}-2,$$

with $2a=\pi ib(-2ix-(\alpha_1-\alpha_2+Q))$.
Note, this does use the fact that $[x]_b=\frac{\sin\pi bx}{\sin\pi b^2}=\frac{e^{\pi ibx}-e^{-\pi ibx}}{q-q^{-1}}$. Now we can also take $c=\alpha_3-\frac{Q}{2}$ to get the desired action that only depends on the variable α_3 of the decomposition.

$\mathcal{F}(f(x_2, x_1 + ib)$ term

$$\mathcal{F}\Big(\Big(-e^{-\pi ib^2}e^{-2\pi ib(\alpha_2-Q)} - e^{\pi ib^2}e^{2\pi ib(\alpha_2-Q)} + e^{\pi ib(\alpha_2-\alpha_1)}e^{-2\pi bx_2} \otimes e^{2\pi b(x_1+\frac{ib}{2})}$$
$$+e^{-\pi ib(\alpha_2-\alpha_1)}e^{2\pi b(x_2-\frac{ib}{2})} \otimes e^{-2\pi bx_1}\Big) f(x_2, x_1+ib)\Big)$$
$$= \Big(-e^{-\pi ib^2}e^{-2\pi ib(\alpha_2-Q)} - e^{\pi ib^2}e^{2\pi ib(\alpha_2-Q)}\Big)\int_{\mathbb{R}} e^{-\pi i\omega x_+} f\left(\frac{x_+ + x_-}{2}, \frac{x_+ - x_- + 2ib}{2}\right) dx_+$$
$$+ e^{\pi ib(\alpha_2-\alpha_1)} \int_{\mathbb{R}} e^{-\pi i\omega x_+} e^{-2\pi b(x_- - \frac{ib}{2})} f\left(\frac{x_+ + x_-}{2}, \frac{x_+ - x_- + 2ib}{2}\right) dx_+$$
$$+ e^{-\pi ib(\alpha_2-\alpha_1)} \int_{\mathbb{R}} e^{-\pi i\omega x_+} e^{2\pi b(x_- - \frac{ib}{2})} f\left(\frac{x_+ + x_-}{2}, \frac{x_+ - x_- + 2ib}{2}\right) dx_+$$
$$= \Big(-e^{-\pi ib^2}e^{-2\pi ib(\alpha_2-Q)} - e^{\pi ib^2}e^{2\pi ib(\alpha_2-Q)}\Big)\int_{\mathbb{R}} e^{-\pi i\omega (x_+-ib)} f\left(\frac{x_+ - ib + x_-}{2}, \frac{x_+ - x_- + ib}{2}\right) dx_+$$
$$+ e^{\pi ib(\alpha_2-\alpha_1)} \int_{\mathbb{R}} e^{-\pi i\omega (x_+-ib)} e^{-2\pi b(x_- - \frac{ib}{2})} f\left(\frac{x_+ - ib + x_-}{2}, \frac{x_+ - x_- + ib}{2}\right) dx_+$$
$$+ e^{-\pi ib(\alpha_2-\alpha_1)} \int_{\mathbb{R}} e^{-\pi i\omega (x_+-ib)} e^{2\pi b(x_- - \frac{ib}{2})} f\left(\frac{x_+ - ib + x_-}{2}, \frac{x_+ - x_- + ib}{2}\right) dx_+$$
$$= \Big(-e^{-\pi ib^2}e^{-2\pi ib(\alpha_2-Q)} - e^{\pi ib^2}e^{2\pi ib(\alpha_2-Q)} + e^{\pi ib(\alpha_2-\alpha_1)}e^{-2\pi b(x_- - \frac{ib}{2})}$$
$$+e^{-\pi ib(\alpha_2-\alpha_1)}e^{2\pi b(x_- - \frac{ib}{2})}\Big) e^{\pi ib(i\omega)} T^{-ib}_{x_-}\mathcal{F}(f(x_2, x_1))$$

In step two we changed the variable $x_+ \to x_+ - ib$.
The terms in front of $T^{-ib}_{x_-}$ can then be rewritten to:

$$\Big(e^{-\pi ib^2}e^{-2\pi ib(\alpha_2-Q)} - e^{\pi ib^2}e^{2\pi ib(\alpha_2-Q)} + e^{\pi ib(\alpha_2-\alpha_1)}e^{-2\pi b(x_- - \frac{ib}{2})} + e^{-\pi ib(\alpha_2-\alpha_1)}e^{2\pi b(x_- - \frac{ib}{2})}\Big)e^{\pi ib(i\omega)} =$$
$$\Big(e^{-\pi ibQ}e^{-2\pi ib(\alpha_2-Q)} + e^{\pi ibQ}e^{2\pi ib(\alpha_2-Q)} - e^{\pi ib(\alpha_2-\alpha_1)}e^{-2\pi bx_-}e^{\pi ibQ} - e^{-\pi ib(\alpha_2-\alpha_1)}e^{2\pi bx_-}e^{-\pi ibQ}\Big)e^{\pi ib(i\omega)} =$$
$$e^{-\pi ib(2\alpha_2-Q-i\omega)} + e^{\pi ib(2\alpha_2-Q+i\omega)} - e^{\pi ib(\alpha_2-\alpha_1+2ix_-+Q+i\omega)} - e^{-\pi ib(\alpha_2-\alpha_1+2ix_-+Q-i\omega)}$$

$\mathcal{F}(f(x_2 - ib, x_1))$ term

$$
\begin{aligned}
&\mathcal{F}\Big(\Big(-e^{-\pi ib^2}e^{-2\pi ib(\alpha_1-Q)} - e^{\pi ib^2}e^{2\pi ib(\alpha_1-Q)} + e^{-\pi ib(\alpha_2-\alpha_1)}e^{-2\pi b(x_2-x_1-\frac{ib}{2})} \\
&\quad + e^{\pi ib(\alpha_2-\alpha_1)}e^{2\pi b(x_2-\frac{ib}{2}-x_1)}\Big) f(x_2-ib,x_1)\Big) \\
&= \Big(-e^{-\pi ib^2}e^{-2\pi ib(\alpha_1-Q)} - e^{\pi ib^2}e^{2\pi ib(\alpha_1-Q)}\Big)\int_{\mathbb{R}} e^{-\pi i\omega x_+} f\left(\frac{x_++x_--2ib}{2},\frac{x_+-x_-}{2}\right)dx_+ \\
&\quad + e^{-\pi ib(\alpha_2-\alpha_1)}e^{-2\pi b(x_--\frac{ib}{2})}\int_{\mathbb{R}} e^{-\pi i\omega x_+} f\left(\frac{x_++x_--2ib}{2},\frac{x_+-x_-}{2}\right)dx_+ \\
&\quad + e^{\pi ib(\alpha_2-\alpha_1)}e^{2\pi b(x_--\frac{ib}{2})}\int_{\mathbb{R}} e^{-\pi i\omega x_+} f\left(\frac{x_++x_--2ib}{2},\frac{x_+-x_-}{2}\right)dx_+ \\
&= \Big(-e^{-\pi ib^2}e^{-2\pi ib(\alpha_1-Q)} - e^{\pi ib^2}e^{2\pi ib(\alpha_1-Q)}\Big)\int_{\mathbb{R}} e^{-\pi i\omega(x_++ib)} f\left(\frac{x_++x_--ib}{2},\frac{x_+-x_-+ib}{2}\right)dx_+ \\
&\quad + e^{-\pi ib(\alpha_2-\alpha_1)}e^{-2\pi b(x_--\frac{ib}{2})}\int_{\mathbb{R}} e^{-\pi i\omega(x_++ib)} f\left(\frac{x_++x_--ib}{2},\frac{x_+-x_--ib}{2}\right)dx_+ \\
&\quad + e^{\pi ib(\alpha_2-\alpha_1)}e^{2\pi b(x_--\frac{ib}{2})}\int_{\mathbb{R}} e^{-\pi i\omega(x_++ib)} f\left(\frac{x_++x_--ib}{2},\frac{x_+-x_--ib}{2}\right)dx_+ \\
&= \Big(-e^{-\pi ib^2}e^{-2\pi ib(\alpha_1-Q)} - e^{\pi ib^2}e^{2\pi ib(\alpha_1-Q)} + e^{-\pi ib(\alpha_2-\alpha_1)}e^{-2\pi b(x_--\frac{ib}{2})} \\
&\quad + e^{\pi ib(\alpha_2-\alpha_1)}e^{2\pi b(x_--\frac{ib}{2})}\Big) e^{-\pi ib(i\omega)}T_{x_-}^{-ib}\mathcal{F}(f(x_2,x_1))
\end{aligned}
$$

We again used a translation of x_+ in the second step, but this time $x_+ \to x_+ + ib$. Then, the terms before $T_{x_-}^{-ib}$ can be rewritten to:

$$
\begin{aligned}
&\Big(-e^{-\pi ib^2}e^{-2\pi ib(\alpha_1-Q)} - e^{\pi ib^2}e^{2\pi ib(\alpha_1-Q)} + e^{-\pi ib(\alpha_2-\alpha_1)}e^{-2\pi b(x_--\frac{ib}{2})} + e^{\pi ib(\alpha_2-\alpha_1)}e^{2\pi b(x_--\frac{ib}{2})}\Big)e^{-\pi ib(i\omega)} = \\
&\Big(e^{-\pi ibQ}e^{-2\pi ib(\alpha_1-Q)} + e^{\pi ibQ}e^{2\pi ib(\alpha_1-Q)} - e^{-\pi ib(\alpha_2-\alpha_1)}e^{-2\pi bx_-}e^{\pi ibQ} - e^{\pi ib(\alpha_2-\alpha_1)}e^{2\pi bx_-}e^{-\pi ibQ}\Big)e^{-\pi ib(i\omega)} = \\
&e^{-\pi ib(2\alpha_1-Q+i\omega)} + e^{\pi ib(2\alpha_1-Q-i\omega)} - e^{-\pi ib(-\alpha_1+\alpha_2-2ix_--Q+i\omega)} - e^{\pi ib(-\alpha_1+\alpha_2-2ix_--Q-i\omega)}
\end{aligned}
$$

We can combine these terms together with the terms obtained from the $f(x_2, x_1 + ib)$ part of the Fourier transform, as both get send to a multiple of the operator $T_{x_-}^{-ib}$. Recall, those terms were:

$$
\begin{aligned}
&e^{-\pi ib(2\alpha_2-Q-i\omega)} + e^{\pi ib(2\alpha_2-Q+i\omega)} - e^{\pi ib(\alpha_2-\alpha_1+2ix_-+Q+i\omega)} - e^{-\pi ib(\alpha_2-\alpha_1+2ix_-+Q-i\omega)} = \\
&e^{-\pi ib(2\alpha_2-Q-i\omega)} + e^{\pi ib(2\alpha_2-Q+i\omega)} - e^{-\pi ib(-\alpha_2+\alpha_1-2ix_--Q-i\omega)} - e^{\pi ib(-\alpha_2+\alpha_1-2ix_--Q+i\omega)}
\end{aligned}
$$

And note that

$$
\begin{aligned}
&- e^{\pi ib(-\alpha_2+\alpha_1-2ix_--Q+i\omega)} - e^{\pi ib(-\alpha_1+\alpha_2-2ix_--Q-i\omega)} + e^{\pi ib(2\alpha_1-Q-i\omega)} + e^{\pi ib(2\alpha_2-Q+i\omega)} \\
&+ e^{-\pi ib(2\alpha_2-Q-i\omega)} + e^{-\pi ib(2\alpha_1-Q+i\omega)} - e^{-\pi ib(-\alpha_1+\alpha_2-2ix_--Q+i\omega)} - e^{-\pi ib(-\alpha_2+\alpha_1-2ix_--Q-i\omega)} = \\
&-[-ix - \frac{1}{2}(\alpha_1+\alpha_2-Q) + (\alpha_3 - \frac{Q}{2})]_b\left(e^{\pi ib(-ix-\frac{1}{2}(\alpha_1+\alpha_2))}\{\alpha_1-\alpha_2+i\omega\}_b - e^{-\pi ib(-ix-\frac{1}{2}(\alpha_1+\alpha_2))}\{\alpha_1-\alpha_2-i\omega\}_b\right)
\end{aligned}
$$

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