SET FUNCTIONALS

IN

STOCHASTIC GEOMETRY

SET FUNCTIONALS IN STOCHASTIC GEOMETRY

PROEFSCHRIFT

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft
op het gezag van de Rector Magnificus Prof. ir. K. F. Wakker
in het openbaar te verdedigen ten overstaan van een commissie,
door het College van Dekanen aangewezen,
op dinsdag 28 juni te 10.30 uur

door Annoesjka Joberte CABO
doctorandus in de wiskunde

geboren te Kerkrade

Dit proefschrift is goedgekeurd door de promotoren:

Prof. dr. P. Groeneboom

Prof. dr. A. J. Baddeley

Dit onderzoek is uitgevoerd aan de Technische Universiteit Delft en aan het Centrum voor Wiskunde en Informatica te Amsterdam.

Voor Ieke

Ter herinnering aan Anna Scheren–Kockelkoren

VOORWOORD

In de afgelopen vijf jaar hebben velen mij geholpen bij het vervaardigen van dit proefschrift.

Allereerst mijn beide promotoren, Piet Groeneboom en Adrian Baddeley. Van beiden heb ik – zij het op zeer verschillende wijze – veel geleerd. Daarnaast was hun vertrouwen in een goede afloop en hun eindeloos geduld mij tot grote steun.

De medewerkers aan de TU Delft bedank ik voor de aangename tijd die ik aan de TU heb doorgebracht. Mijn collega's op het CWI ben ik zeer dankbaar voor alles wat zij hebben gedaan om er voor te zorgen dat ik daar met zoveel plezier heb gewerkt. Velen hebben mij bovendien geduldig geholpen met mijn problemen met de computer. René wil ik speciaal bedanken voor de leuke tijd die we samen op één kamer hebben doorgebracht en voor de prettige samenwerking.

Verder bedank ik mijn ouders, Carmen en Pépin voor hun voortdurende steun voor en interesse in alles wat ik doe, en Michaëla en Lou Thomas voor de liefdevolle verzorging van Igor.

Aan Ieke tenslotte ben ik zeer veel dank verschuldigd. Zonder zijn steun, zorg, vertrouwen en liefde zou er in de afgelopen jaren heel wat minder mogelijk zijn geweest dan nu tot stand is gebracht.

CONTENTS

INTRODUCTION

This thesis consists of five chapters that are written in such a way that they may be read separately. The main purpose of this introduction is to describe the history of the problems investigated in these chapters and the connection between them. First a brief description of stochastic geometry is given. Then we turn to the subjects treated in this thesis. We survey the results known in the literature and the open questions concerning them. This is done for convex hulls in section 2, for characterisation problems in the third section and for stereological estimation in section 4. Finally the last section summarises the new results obtained in this thesis.

1. Stochastic geometry.

The fundamental idea underlying stochastic geometry is that a probability is related to a geometrical parameter. Its origin is *geometric probability*, the starting point of which is usually considered to be the formulation of a class of 'games' involving geometric objects by Buffon in 1733. One of those was the famous needle problem. The needle problem can be considered as a classical coin-tossing game, where two players throw a coin and guess beforehand whether it will show 'head' or 'tail'. In the needle problem the coin is replaced by a needle of length l say and the guess is about the *position* of the needle on the floor after being thrown. The two players are standing on a floor divided by infinite planks, all of the same width, w say, where w is broader than l. The guess to be made beforehand is whether the needle will cross one of the 'joins' between the planks, or will not cross any of the 'joins'. The probability of the first outcome is $\frac{2}{\pi w}l$. Thus here a probability is linked to a length.

The derivation of the result involves a rather simple integration. This shows that there is a natural probability measure underlying these kind of geometrical games. However, Bertrand's paradox (1888) demonstrated that in more complicated situations it is not at all obvious which measure this is. Bertrand asked for the probability that a chord drawn at random through the unit disk in the plane has length greater than $\sqrt{3}$. In order to perform the computation the chords have to be parametrised. Bertrand showed that this can be done in three different ways. Equipping the parameter spaces with Lebesgue measure – which he considered the only 'natural' measure on the Euclidean space $\mathbb{R}^2$ – this yields three different results – $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ – depending on the parametrisation. To avoid this paradox, one has to consider an invariant measure on the set of all *lines* in the plane. (Here invariance means invariant with respect to Euclidean motions of the plane.) When the chords are viewed as the intercepts of lines with the unit disk, this invariant measure is seen to be the one underlying the distributions of the chords. Invariant measures of this sort and integrals with respect to these measures had already been studied by Crofton, but a thorough investigation originated with the work of Blaschke (1935) and Santaló (see his book (1976); see also Hadwiger (1957)). This new area was called *integral geometry.*

This classical theory of probabilities and integrals was based upon finite numbers of geometrical objects of a fixed form, but at random positions. However, at the beginning of the seventies, it became clear that it was not rich enough to deal with the much more complicated structures encountered in nature. The description of structures as seen e.g. in rocks and the human body or on satelite data needed more complicated models, that would allow not only random positions but also random shapes and random numbers of them. This was the beginning of the field now called *stochastic geometry*, connecting geometry and the theory of stochastic processes, starting with the introduction of *random sets* independently by Kendall (1974) and Matheron (1975). An example of an interesting – and useful – random set is the convex hull of random points in a compact set. A modified version of this set for instance can be used as a 'statistical' estimator for the compact set in which it is contained (Ripley and Rasson (1977)).

The mathematics used to develop stochastic geometry include convex geometry (Schneider (1993)), geometric measure theory (Federer (1969)), the theory of stochastic processes (Daley and Vere-Jones (1988)) and Choquet's theory of capacities (1953).

Spatial statistics and *stereology* are examples of fields in which stochastic geometry and integral geometry provide the theoretical background for the models used. Spatial statistics analyses spatial structures from multidimensional data. Consider for example the configuration of raindrops which have fallen on a roof. A model to describe the drops as well as an estimate for the expected area of the roof covered by them can be given.

In stereology, the estimators are derived from data of a lower dimension than that of the parameter under study. For practical reasons it is not always possible to make observations on the desired object directly. In those cases, it is often possible to take lower dimensional sections of the object and to estimate the desired geometric parameter from the information contained in these sections. The mathematical formulas underlying these results come from integral geometry.

Of course stochastic geometry is a much broader field than described here. For instance I have not mentioned integral geometry on Lie groups, stochastic processes of fibres, stochastic geometry on manifolds, etc. For a more complete account of the subject and its history the reader is referred to the books by Mecke et al. (1990) and Santaló (1976) and to Baddeley (1992). What we have encountered until now however suffices to go on with a summary of this thesis.

Geometric parameters of (random) sets can be considered as functionals of those sets. Several such functionals are investigated in this thesis. This yields probabilistic results, new integral geometric relations, a new stereological estimator for three–dimensional volume and a solution to a characterisation problem. But it also leaves open many other problems...

2. Convex hulls.

First we consider functionals of the convex hull of random points and their asymptotics.

Consider a sample of n independent points drawn uniformly from the interior of a convex compact set C in $\mathbb{R}^d$. Then construct the convex hull C_n of these points. C_n is a random set and we are interested in the behaviour of functionals of C_n as the sample size n tends to infinity. It is clear of course that C_n itself will tend to the containing set C. Also some of the functionals of C_n (like the volume) will tend to the corresponding functionals of C. The question is: what can be said about this convergence?

Since C_n is random, functionals $f : \mathcal{K}(R^d) \to \mathbb{R}$ can be considered as real-valued random variables and so we are interested in asymptotic properties of $f(C_n)$. Examples of f are the number of vertices of C_n, the volume of C_n, the surface area of C_n, etc. Until Groeneboom's pioneering paper (1988), the results about $f(C_n)$ consisted mostly of upper bounds for the expectation $Ef(C_n)$, and some explicit expressions. The results were improved every now and then by applying methods from convex geometry. See the survey paper by Schneider (1988) for an extensive list of references and the beautiful book by Schneider (1993) on convex geometry. See also Brozius (1989). However, no information about higher moments, nor about the limiting distribution was available. With the appearance of the paper by Groeneboom mentioned earlier, it became clear that the use of probability theory and the theory of stochastic processes could solve a geometric problem, rather than the other way around. In this paper uniform samples from the unit square and the unit disk in the plane were studied, and the functional under consideration was the number N_n of vertices of the convex hull. The results contained not only the first two moments of the limiting functionals, but also showed that N_n satisfies a Central Limit Theorem. For the other functionals in the planar case, as in the case of $C_n \subset \mathbb{R}^d$, $d \geq 3$, no such results have been obtained so far.

3. Characterisation problems.

Next let me describe another class of problems concerning functionals of sets in $\mathbb{R}^d$ and their ability to discriminate between sets.

In these problems one studies the *amount* of information contained in functionals of (random) sets. Thus if for two sets these functionals coincide, may we conclude that the sets themselves are equal? Mathematically this is formulated as follows. Let $\phi : \mathcal{A} \to L^1(\mathbb{R}^d)$ be a mapping from a certain class of subsets of $\mathbb{R}^d$ to $L^1(\mathbb{R}^d)$; i.e.

$$\begin{aligned} \phi : \mathcal{A} &\to L^1(\mathbb{R}^d) \\ A &\mapsto \phi_A, \end{aligned}$$

where ϕ_A is a function from $\mathbb{R}^d$ to $\mathbb{R}$, or sometimes from a class of subsets from $\mathbb{R}^d$ to $\mathbb{R}$. When does $\phi_A = \phi_B$ imply $A = B$? Examples of such functions are the distribution function of the Radon transform of a set, the set covariance function, dilation volume and interpoint distance distribution, all to be specified below.

It was Blaschke (1949, p. 51) who asked whether a convex set is uniquely determined by the distribution of its Radon transforms, or chord lengths. To be more specific, denote by μ the invariant measure on the set $\mathcal{L}$ of all lines in the plane. Then the distribution of Radon transforms is defined as

$$\begin{aligned} \phi_A(x) &= \mathbb{P}\{l \in \mathcal{L} :\ \lambda_1(l \cap A) \le x\} \\ &:= \frac{\mu\{l \in \mathcal{L} : l \cap A \neq \emptyset \text{ and } \lambda_1(l \cap A) \le x\}}{\mu\{l \in \mathcal{L} : l \cap A \neq \emptyset\}}, \end{aligned}$$

where λ_d denotes d-dimensional Lebesgue measure, $d \ge 1$. An example by Mallows and Clark (1970, 1971) showed that in general the answer to Blaschke's question was negative, but Waksman (1985) solved the problem in the affirmative for 'generic' polygons satisfying an extra condition. It was not hard to show that this extra condition could be disposed of (see Cabo (1989)). For the case of general convex sets the problem is still open.

The next problem is known as Matheron's conjecture (1986): is a convex set uniquely determined by its *covariance function* (the volume of the intersection of the set with its translate)? This conjecture is closely related to Blaschke's and the situation is similar: Nagel (1993) proved uniqueness (up to translation and reflection) for *all* convex plane polygons, but for other convex sets no solution was known.

Lešanovský and Rataj (1990) solved the *dilation volume* problem. Here ϕ_A is defined on the class of compact subsets of $\mathbb{R}^d$ by

$$\phi_A(K) = \lambda_d(A \oplus K),$$

where $A \oplus K = \{a + k : a \in A,\, k \in K\}$. The authors proved that A is not only determined when one knows $\phi_A(K)$ for all compact sets K, but also that the knowledge of $\phi_A(K)$ for all K containing three points is sufficient to reconstruct A. Finally they provided an example of two nonidentical nonconvex sets that do have the same covariance function. (The example of Mallows and Clark is not a counterexample to Matheron's conjecture.)

In 1989, Pyke asked another closely related question, concerning the determination of a convex set B from its *interpoint distance distribution.* Let X and Y be two independent uniformly distributed points in B. Is B determined (up to translation and rotation) by $\phi_B(\rho) = \mathbb{P}\{\|X - Y\| \leq \rho\}$? It is well known that the covariance function is the density of the vector $X - Y$ (up to a constant). Rost (1989) gave an example of two different nonconvex subsets of the real line for which the distributions of $X - Y$ coincide. Furthermore also the results of Lešanovský and Rataj (1990) can be used to show that for general sets the answer to this problem is negative. However, by the remark made above, Nagel solved Pyke's problem for convex polygons by solving the covariance problem for those sets.

4. Stereological estimation.

Stereology aims at obtaining (quantitative) information about d-dimensional structures, from observations on lower-dimensional sections. For instance, estimating the mean volume of certain cells, while only pictures of plane sections through the cells are available. The first results in this field were obtained for convex particles, which had to be of the same shape or even spherical. Later the methods were extended to nonconvex sets by Miles (1983, 1985) and Jensen and Gundersen (1985). These estimators were based on line sections through the plane section. However it was not clear how to use them in practice, when for instance an image of the data is at hand. Moreover it seems to be more efficient – at least in some situations – to use as much two-dimensional information as possible from the image. Some work in this direction was started by Miles (1979) and Serra (1982).

5. Results.

In this section I describe the main results of this thesis. It will turn out that for each of the problems described above a partial solution is found, but the original questions in the most general case remain open.

In Capter I we show that the martingale approach and the use of the vertex process introduced in Groeneboom (1989) yield asymptotic results for the area and boundary length in the planar case. It is necessary to adapt the definitions of the martingales to the specific functionals. The results have been written down for the case of a uniform sample from the interior of the unit square C. It is shown that the difference between the areas of C

and C_n is asymptotically normal and the limiting expectation and variance are explicitly computed. For the length functional however, the situation is quite different. It is possible to give expressions for the first two moments of the limiting random variable, but no distributional conclusions could be drawn. For the case of a sample from a convex compact set with smooth boundary, we do expect the method to work as well, and to yield Central Limit Theorems for *both* functionals. Recently Hüter has used this technique to derive asymptotic expressions for the number of vertices of the convex hull of random points drawn from a class of spherically symmetric distributions (including the normal distribution) in higher dimensions. The question how to extend this method to other distributions in higher dimensions is still open.

The last four chapters of this thesis resulted from work on characterisation problems. The first (Chapter II) is based on part of my master's thesis and gives a new proof of a well-known integral geometric equality. Let P be a convex plane polygon. The equality relates the integral of a C^1 function of chord lengths over all lines that intersect P, to the sum of two integrals. The first is over all lines intersecting P and integrates the derivative of f and a function of the angles at which the chords cut P. The second is just the sum of the integrals of f over the sides of P. The proof uses Stokes' formula.

In Chapter III we tried to solve Matheron's conjecture following Waksman (1987). The idea was to use the fact that the problem had been solved for convex polygons, together with the fact that every convex set can be approximated by such polygons. For this purpose, a metric seemed to be needed on the space of (convex) sets and the completion of the metric space under this metric had to be investigated. Waksman had introduced such a metric on a class of open subsets of the plane. We modified his construction to obtain a metric on bounded regular closed d-dimensional sets. First a new function is constructed as the 'derivative' of the one-dimensional covariance function of linear transects through a bounded regular closed d-dimensional set. We called it the linear scan transform. The metric is then defined as the L^1 distance between linear scan transforms.

The implications of this paper are twofold. First, the metric is shown to be topologically equivalent to the Hausdorff metric for convex sets. Further it turns out to be an appropriate tool for studying analytic properties of the covariance function. Thus we prove continuity of the covariance function of certain 'regular' sets and find an easy proof for continuity of Lebesgue measure. However this did not solve Matheron's conjecture. For this, inequalities in the reverse direction from those obtained in this paper are needed.

Second, there is an interest in the linear scan transform itself. It turns out that it provides a unification in integral geometry and stereology, clarifying existing relationships between certain quantities, and establishing new ones.

In Chapter IV we discuss the stereological estimation of the volume-weighted mean volume of a population of 'particles' (not-necessarily-convex, compact subsets) in $\mathbb{R}^3$ from plane sections. The standard method is to place test lines in the plane section and measure cubed intercept lengths with the two-dimensional particle profiles. We point out that the integral-geometric identity on which this method is based, can be generalised by replacing linear test lines by r-dimensional plane sections, forming a reproductive family of similar identities. Using this we derive improved unbiased estimators for mean particle volume in a variety of sampling regimes (design-based and model-based) including vertical section designs. We prove that these estimators have smaller variance than the line transect estimators, and indeed are related to them by the Rao-Blackwell process. In the new estimators the cubed intercept length is replaced by a moment of the distance between two points in the section profile; this can be computed in practice as a moment of the set covariance function of the section profile. Finally we present two practical applications. An unexpected result is that the value of the estimator is often close to the area-weighted $\frac{3}{2}$nd moment of the profile areas, which is a lower bound obtained from an isoperimetric-type inequality. We estimate the variance of the technique and the gain in efficiency over line transect techniques; the efficiency improvement appears to be as much as one order of magnitude.

Finally in the last chapter, we actually solve one of the characterisation problems. Together with René Janssen we show that the covariance function does characterise *all* symmetric bounded regular closed sets in $\mathbb{R}^d$, $d \geq 1$. Hence it is seen that convexity nor connectivity play a role, as long as the sets are symmetric (i.e. invariant under reflection with respect to the origin). This was done using Fourier methods, from which an explicit reconstruction procedure immediately follows. We also give a geometric proof of the continuity of the covariance function of a bounded Borel set. Using the continuity of this function and the characterisation result for symmetric sets mentioned above, we solve Pyke's problem for rotation invariant Borel sets in $\mathbb{R}^d$. Finally we introduce a generalisation of the covariance function, dubbed the cross-covariance function of one set with respect to another. This is defined as the volume of the intersection of the first set with a translate of the other one. It is shown that the cross-covariance of a set with respect to its reflection characterises *any* bounded regular closed set. Here again the original question remains open, since for asymmetric convex sets, it is still not known whether the covariance function itself would suffice.

The chapters of this thesis are based on the following papers:

I. A. J. Cabo and P. Groeneboom. *Limit theorems for functionals of convex hulls.* To appear in: Prob. Th. rel. fields.

II. A. J. Cabo (1992). *An elementary proof of the Ambartzumian–Pleijel identity,* Math. Proc. Camb. Phil. Soc. (1992),**112**, pp. 535–538.

III. A. J. Cabo and A. J. Baddeley. *Line transects, covariance functions and set approximation.* To appear in: Adv. Appl. Prob.

IV. A. J. Cabo and A. J. Baddeley. *Estimation of mean particle volume using the set covariance function,* Research Report number 94/14, UWA Department of Mathematics, May 1994.

V. A. J. Cabo and R. H. P. Janssen. *Cross-covariance functions characterise regular compact sets.*

6. Acknowledgements.

A large part of the results presented in this thesis have been obtained in collaboration with others. I want to thank my coauthors for the joy I have taken in doing this work together: A. J. Baddeley, P. Groeneboom, and R. H. P. Janssen. Further it is a pleasure to thank M. Aitkin, K. Dzhaparidze, R. D. Gill, W. Nagel, R. Schneider, E. B. Vedel–Jensen, R. A. Vitale, W. Weil and others for fruitful discussions; G. J. Docter and J. M. Mijnster for generously providing the image data on synaptic boutons (Chapter IV); A. Steenbeek for substantial programming support; Dr. S.P. Justice and the Royal Microscopical Society for permission to use Figure 2 (Chapter IV); and the lecturers of the International Society for Stereology/SCANDEM'92 stereology course for permission to use course materials.

CHAPTER I

LIMIT THEOREMS FOR FUNCTIONALS OF CONVEX HULLS

1. Introduction.

In 1963, Rényi and Sulanke (1963) derived asymptotic expressions for the expected boundary length and the expected area of the convex hull of a uniform sample from the unit square. They stated in their introduction that the computations for more general convex polygons are rather complicated ("ziemlich unübersichtliche Rechnungen"). Moreover they noted that the expected boundary length and the expected area of the convex hull behave surprisingly differently, at least in a first analysis ("Hier ergiebt sich die auf ersten Blick überraschende Tatsache, daß sich Flächeninhalt und Umfang asymptotisch verschieden verhalten").

A paper by Buchta (1984), shows that indeed the computations of the first moment measures become quite complicated for polygons more general than the unit square, and proceeding in this way to the computation of higher moments seems an extraordinarily hard task. We show that the computation of asymptotic expressions for the first moments becomes rather easy, if one looks at the process *locally* instead of *globally*, using an approach which is most conveniently summarized by saying that "everything happens in the corners".

Moreover, we will derive the actual limiting behavior (after rescaling), and show that the area of the region between the convex hull of the sample and the boundary of the convex polygon satisfies a central limit theorem, in contrast to the boundary length of the convex hull. In fact we will show that the dominating asymptotic behavior of the boundary length of the convex hull depends on a number of edges that remains bounded, as the sample size tends to infinity, whereas the dominating asymptotic behavior of the area of the region between the convex hull of the sample and the boundary of the convex polygon will involve a number of edges tending to infinity, as the sample size tends to infinity. In this sense the asymptotic behavior of the boundary length is even *more* local than the behavior of the area.

Another expression of the phenomenon just mentioned, is that there is a natural stationary process, describing the limiting behavior of the area of the region between the convex hull of the sample and the boundary of the convex polygon, whereas no such stationarity holds for the limiting behavior of the boundary length. However, we expect that this striking difference in behavior would disappear when samples from convex figures with a smooth boundary are considered. This would also provide an explanation for the observations in Rényi & Sulanke (1963) on the differences in this respect between samples from the unit square and samples from convex figures with a smooth boundary.

This chapter is structured in the following way. Since we will relate the behavior of the functionals of the finite sample process to the behavior of corresponding functionals of a limiting Poisson point process, we first study the functionals of the limiting process. This is done in section 2. Here we already see the difference in behavior of the boundary length and the area: for the "area functional" of the convex hull of the Poisson point process a central limit theorem is obtained in contrast to the "length functional".

In section 3 we relate the results for the Poisson process to the finite sample behavior. Computations of the relevant second moments are given in the appendix.

Since the computations, involving *second* moments, are considerably more complicated than for first moments, we treat (for reasons of space) the case of uniform samples from convex polygons separately. Moreover, this case has some pecularities which are not present in, say, the case of samples from convex figures with smooth boundaries or samples from absolutely continuous distributions with infinite support, such as a two-dimensional normal distribution. The latter case is studied in Hüter (1992).

2. Functionals of the convex hull of a Poisson point process.

We shall study functionals of the Poisson point process $\mathcal{P}$ on $\mathbb{R}_+^2$, with intensity Lebesgue measure. The functionals will depend on $\mathcal{P}$ via another process, which is defined by Definition 2.2 in Groeneboom (1988). For convenience, this definition is repeated below.

2.1 DEFINITION. For each $a > 0$, $W(a)$ is the point of a realization of the Poisson point process $\mathcal{P}$ on $\mathbb{R}_+^2$, such that all points of the realization lie to the right of the line $x + ay = c$, which passes through $W(a)$. If there are several of these points, we take the point with the smallest y-coordinate.

We first consider the functional corresponding to "area". To this end, we introduce the following process, describing a "growing area", as a function of the parameter a in Definition 2.1.

2.2 DEFINITION. Let $0 < a < b < \infty$. Then $A(a,b)$ is the area of the region bounded on the right and left by vertical lines through the x-coordinates of the points $W(a)$ and $W(b)$, and bounded from below and above by the line $y = 0$ and the (left lower) boundary of the convex hull of $\mathcal{P}$, respectively.

For each $a_0 > 0$, we introduce the increasing filtration $\{\mathcal{F}_{[a_0,a]} : a \geq a_0\}$ of σ-algebras

$$\mathcal{F}_{[a_0,a]} = \sigma\{W(c) : c \in [a_0, a]\}. \tag{2.1}$$

Then the process $\{(W(a), A(a_0,a)) : a \geq a_0\}$ is a Markov process with respect to this filtration.

2.3 THEOREM. *Let C_0 be the set of continuous functions $f : \mathbb{R}_+^3 \to \mathbb{R}$, with compact support contained in $(0,\infty)^2 \times [0,\infty)$, and let, for each $a > 0$, the linear operator $L_a : C_0 \to C_0$ be defined by*

(2.2)
$$[L_a f](x,y,z) = \\ = \int_0^y u\left\{f\left(x+au, y-u, z+\tfrac{1}{2}au^2 + au(y-u)\right) - f(x,y,z)\right\} du,$$

for $(x,y,z) \in (0,\infty)^2 \times [0,\infty)$. Then, for each $f \in C_0$ and each $a_0 > 0$, the process

$$X_f(a) = f\left(W(a), A(a_0,a)\right) - \int_{a_0}^{a} [L_c f]\left(W(c), A(a_0,c)\right) dc,\ a \geq a_0,$$

is a martingale with respect to the filtration $\{\mathcal{F}_{[a_0,a]} : a \geq a_0\}$.

PROOF. We have to show that, for $a > 0$:

$$\lim_{h \downarrow 0} h^{-1} E\left\{ f\left(W(a+h), A(a_0,a+h)\right) - f\left(W(a), A(a_0,a)\right) \mid \left(W(a), A(a_0,a)\right) = (x,y,z)\right\} = \\ = [L_a f](x,y,z).$$

Observe that $A(a, a+h) = 0$ unless the Poisson point process $\mathcal{P}$ has a point in the shaded region $B(h)$ shown in Figure 2.1 below.

Moreover, the probability of getting more than one point in this region is $o(h)$. If $\mathcal{P}$ has a point in $B(h)$, say at $(x,y) + (au, -u) + r(h)$ (where $\|r(h)\| = \mathcal{O}(h)$), then it may be easily verified that $A(a, a+h) = \frac{1}{2}au^2 + au(y-u) + \mathcal{O}(h)$, $h \downarrow 0$. The rest of the proof is the same as that of Theorem 2.1 in Groeneboom (1988). □

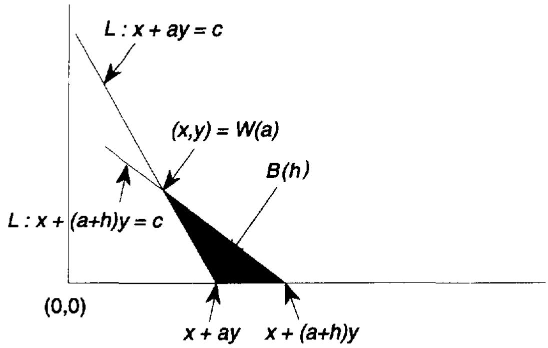


FIGURE 2.1.

It is convenient to write (2.2) in the following form:

$$[L_a f](x,y,z) = \int_{\mathbb{R}^3} \{f(w+z) - f(w)\}\, M(a,w;dz), \tag{2.3}$$

where, for $a > 0$, the jump measure $M(a,w;\cdot)$ is defined by

$$M(a,w;B) = \int_0^y u 1_B\big(au, -u, \tfrac{1}{2}au^2 + au(y-u)\big)\, du, \tag{2.4}$$

where $w = (x,y,z)$ and $B \subset \mathbb{R}^3$ is a Borel set.

We now transform the process

$$\big\{\big(W(a) = (U(a), V(a)), A(a_0,a)\big) : a \geq a_0\big\}$$

into a Markov process with stationary transition probabilities. First, we introduce the process $\{Z(a) : a \in \mathbb{R}\}$, by defining

$$\begin{aligned} X(a) &= \big(U(e^a) + e^a V(e^a)\big) / \exp\left\{\tfrac{1}{2}a\right\}, \\ Y(a) &= e^{\frac{1}{2}a} V(e^a), \\ Z(a) &= \big(X(a), Y(a)\big). \end{aligned}$$

The process $\{Z(a) : a \in \mathbb{R}\}$ has the one-dimensional marginal distributions

$$P\{X(a) \in dx,\, Y(a) \in dy\} = \exp\left\{-\tfrac{1}{2}x^2\right\} dxdy, \tag{2.5}$$

see (2.26) and (2.27) in Groeneboom (1988). Next, defining $\bar{A}(a_0,a)$ by

$$\bar{A}(a_0,a) = A(e^{a_0}, e^a),\ a \geq a_0$$

and using the 1-1 correspondence between $Z(a)$ and $W(e^a)$, we obtain

$$\lim_{h\downarrow 0} h^{-1}E\{\, f\big(Z(a+h),\bar{A}(a_0,a+h)\big) - f(x,y,z) \mid \big(Z(a),\bar{A}(a_0,a)\big) = (x,y,z)\} = \tag{2.6}$$

$$= \int_0^y u\,\{f\big(x,y-u,z+\tfrac{1}{2}u^2+uy\big) - f(x,y,z)\}\,du + \big(y-\tfrac{1}{2}x\big)\frac{\partial}{\partial x}f(x,y,z) + \tfrac{1}{2}y\frac{\partial}{\partial y}f(x,y,z), \qquad a \geq a_0.$$

As a corollary to Theorem 2.3 in Groeneboom (1988), we get the following result.

2.4 THEOREM.
Let for each $a_0 \in \mathbb{R}$ the process $\{\big(X(a),Y(a),\bar{A}(a_0,a)\big) : a \geq a_0\}$ be as defined above. Then, for each $a_0 \in \mathbb{R}$, this process is a Markov process with stationary transition probabilities, and with an infinitesimal generator given by (2.6). Moreover, the process is strongly mixing in the following sense. Defining the σ-algebra

$$\bar{\mathcal{F}}_I = \sigma\{\big(X(c),Y(c)\big) : c \in I\}, \quad \textit{for intervals } I \subset \mathbb{R}$$

we have

$$\big|P(A\cap B) - P(A)P(B)\big| \leq c\cdot e^{-\frac{1}{2}b}, \tag{2.7}$$

if $A \in \bar{\mathcal{F}}_{[a_0,a]}$, $B \in \bar{\mathcal{F}}_{[a+b,\infty)}$, where $c > 0$ is a fixed constant.

PROOF. The statement about the (stationary) Markov structure immediately follows from (2.6), and (2.7) follows from (2.30) in Groeneboom (1988). □

Theorem 2.4 implies that the sequence $X_1, X_2, \ldots$, defined by

$$X_i = \bar{A}(0,i) - \bar{A}(0,i-1),\ i = 1,2,\ldots, \tag{2.8}$$

is a stationary sequence of random variables, satisfying the mixing condition

$$\big|P(A\cap B) - P(A)P(B)\big| \leq c\cdot e^{-\frac{1}{2}n},$$

if $A \in \sigma\{X_1,\ldots,X_k\}$ and $B \in \sigma\{X_{k+m} : m > n\}$, where $k,n \geq 1$ and c is as in (2.7). If we can show that $EX_1^{2+\delta} < \infty$, for some $\delta > 0$, we would get

$$\{\bar{A}(0,n) - nEX_1\}/\sqrt{\mathrm{Var}\big(\bar{A}(0,n)\big)} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1), \tag{2.9}$$

i.e. $\bar{A}(0,n)$ would converge in distribution, after standardization, to a standard normal distribution.

The finiteness of the moments $EX_1^{2+\delta}$ follows from the following lemma, in which we show finiteness of *all* moments of X_1.

2.5 LEMMA. *For each $a \geq 1$ the moment generating function*

$$\lambda \to E\exp\{\lambda A(1,a)\}$$

is finite for λ in a neighborhood of the origin.

PROOF.
Conditionally on $\big(U(1),V(1)\big) = (x,y)$, the area $A(1,a)$ is bounded above by $\frac{1}{2}(a-1)y^2$ (the area of the triangle with vertices (x,y), $(x+y,0)$ and $(x+ay,0)$). In Lemma 2.4 (Groeneboom (1988)), the marginal distribution of the Markov process $\{W(a) : a > 0\}$, was computed. For convenience, the statement of this Lemma is repeated at the beginning of section 4 of the present chapter. By part (i) of this Lemma, we have

$$P\{W(1) \in (dx,dy)\} = \exp\left\{-\frac{(x+y)^2}{2}\right\} dxdy, \quad 0 < x,y < \infty.$$

From this, we obtain

$$E\exp\{\lambda A(1,a)\} \leq \int_{\mathbb{R}_+^2} \exp\left\{-\tfrac{1}{2}(x+y)^2 + \tfrac{1}{2}\lambda(a-1)y^2\right\} dx\, dy, \tag{2.10}$$

and the right side of (2.10) is clearly finite for λ in a neighborhood of zero (depending on a). □

Since $X_1 = A(1,e)$, it follows that the moment generating function of X_1 exists in a neighborhood of the origin. So it is clear that all conditions for a central limit theorem of the form (2.9) are fulfilled, and all that is left to do is to compute first and second moments. We will compute these moments for the process $\{A(a_0,a) : a \geq a_0\}$ in its original parametrization, since the computations are somewhat simpler in appearance, and can be transferred immediately to the process in its stationary form (and hence to properties of the sequence $X_1, X_2, \ldots$).

We have the following result.

2.6 THEOREM. *Let $0 < a < b < \infty$, $\beta = \frac{b}{a}$ and $\alpha = \beta - 1$. Then*

(i) $$EA(a,b) = \tfrac{1}{3}\log\beta,$$

(ii) $$Var\big(A(a,b)\big) = \frac{50}{189}\log\beta - \frac{2}{3\alpha^2} - \frac{4}{3\alpha} + \frac{2}{9}\frac{\beta(4\beta-1)\tan^{-1}\sqrt{\alpha}}{\alpha^{5/2}} + \frac{4}{9}\left\{\tan^{-1}\sqrt{\alpha}\right\}^2 - \frac{68}{135}.$$

The proof of Theorem 2.6 will be given in the appendix. As a corollary we obtain the following central limit theorem.

2.7 Corollary. *For any sequences* (a_n) *and* (b_n) *of positive numbers such that* $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$, *we have*

$$\left\{ A(a_n, b_n) - \tfrac{1}{3} \log \frac{b_n}{a_n} \right\} \Big/ \sqrt{\tfrac{50}{189} \log \frac{b_n}{a_n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1), \quad \text{as } n \to \infty, \tag{2.11}$$

where $\mathcal{N}(0,1)$ *denotes the standard normal distribution.*

The proof follows the same lines as the proof of Corollary 2.3 in Groeneboom (1988).

We briefly comment on the more general situation of a Poisson process $\mathcal{P}_\lambda$ with intensity Lebesgue measure in the region R_λ, defined by

$$R_\lambda = \{(x,y) : y \geq 0, x - \lambda y \geq 0\},\ \lambda \in I\!R.$$

Taking $\lambda = 0$ we get the case we have considered so far. Here we only consider the case $\lambda > 0$, since the case $\lambda < 0$ is quite similar. Let $A'(a,b)$ be the area of the region, bounded on the right and left by lines parallel to the line $x = \lambda y$ through the points $W(a)$ and $W(b)$, and bounded from below and above by the line $y = 0$ and the boundary of the convex hull of $\mathcal{P}_\lambda$, respectively. Here $W(a)$ and $W(b)$ are as defined in Definition 2.1, but now with $a, b \in (-\lambda, \infty)$.

The process $\{(U(a), V(a), A'(a_0, a)) : a \geq -\lambda\}$ can now be transformed into a Markov process with stationary transition probabilities, just as before. This process has the same structure as before, in particular (2.5) and (2.6) are satisfied. It follows that $A'(a,b)$ has the same distribution as as the variable $A(a+\lambda, b+\lambda)$, and we obtain the same central limit theorem as before (with the parameters a and b shifted to $a+\lambda$ and $b+\lambda$).

In section 3 we shall derive from Corollary 2.7 a central limit theorem for the area of the convex hull of a uniform sample from a convex polygon. We now first turn to the other functional: the boundary length. In analogy with the area, we introduce processes, describing the "developing (remaining) boundary length".

2.8 Definition. Denote by $L_{co}(a,b)$ the boundary length of the convex hull of $\mathcal{P}$ between the points $W(a)$ and $W(b)$. Furthermore, we define $L^x(a,b)$, $L^y(a,b)$ and $L(a,b)$ as follows.

(*i*) For $1 \leq a < b < \infty$,

$$L^x(a,b) = L_{co}(a,b) - \{U(b) - U(a)\};$$

(*ii*) For $0 < a < b \leq 1$,

$$L^y(a,b) = L_{co}(a,b) - \{V(a) - V(b)\};$$

(*iii*) For $0 < a \leq 1 \leq b < \infty$,

$$L(a,b) = U(1) + V(1) - L^y(a,1) - L^x(1,b).$$

2.9 Remark. In words: for $1 \leq a < b < \infty$, we define $L^x(a,b)$ as $L_{co}(a,b)$ minus its projection on the x-axis, whereas for $0 < a < b < 1$, we define $L^y(a,b)$ as $L_{co}(a,b)$ minus its projection on the y-axis. In the third case, where we "round the corner", we add to the projections on the x- and y-axis the remaining segments of the boundary of the unit square, meeting at zero, and compare the total length of these segments and the projections with $L_{co}(a,b)$ (note that the total length of projections and segments in bigger than $L_{co}(a,b)$). We will call $L(a,b)$ the *"remaining length"*.

2.10 THEOREM. *Let C_0 be the set of continuous functions $f : \mathbb{R}^3_+ \to \mathbb{R}$, with compact support contained in $(0,\infty)^2 \times [0,\infty)$.*
Let, for each $a \geq 1$, the linear operator $\mathcal{L}_a : C_0 \to C_0$ be defined by

$$[\mathcal{L}_a f](x,y,z) = \\ = \int_0^y u\left\{f\big(x + au, y - u, z + u\sqrt{1+a^2} - au\big) - f(x,y,z)\right\} du,$$

for $(x,y,z) \in (0,\infty)^2 \times [0,\infty)$.

(*i*) *For each $f \in C_0$ and each $a_0 \geq 1$, the process*

$$Y^x_f(a) = f\big(W(a), L^x(a_0,a)\big) - \int_{a_0}^a [\mathcal{L}_c f]\,\big(W(c), L^x(a_0,c)\big)\, dc,\ a \geq a_0$$

is a martingale with respect to the filtration $\left\{\mathcal{F}_{[a_0,a]} : a \geq a_0\right\}$.

(*ii*) *Let, for each $a \geq a_0 \geq 1$, $\tilde{W}$ and $\tilde{L}^y$ be defined by*

$$\tilde{W}(a) = (V(1/a), U(1/a)), \qquad \tilde{L}^y(a_0,a) = L^y(1/a, 1/a_0).$$

Then, for each $f \in C_0$ and each $a_0 \geq 1$, the process

$$Y^y_f(a) = f\big(\tilde{W}(a), \tilde{L}^y(a_0,a)\big) - \int_{a_0}^a [\mathcal{L}_c f]\,\big(\tilde{W}(c), \tilde{L}^y(a_0,c)\big)\, dc,\ a \geq a_0$$

is a martingale with respect to the filtration $\left\{\mathcal{F}_{[1/a,1/a_0]} : a \geq a_0\right\}$.

The proof is completely analogous to the proof of Theorem 2.3 and is therefore omitted.

In this case there is no obvious transformation of the process into a Markov process with *stationary* transition probabilities, as with the number of extreme points or remaining area. Nonetheless Theorem 2.10 is helpful in computing the first and second moment measure of the process.

2.11 THEOREM. *Let, for $a \geq 1$, $L(1/a, a)$ be the remaining boundary length between the slopes $1/a$ and a of the (left-lower) convex hull of the Poisson process $\mathcal{P}$. Then*

(*i*)

$$\begin{aligned} EL(1/a,a) &= EU(1) + EV(1) - EL^x(1,a) - EL^y(1/a,1) \\ &= \sqrt{\frac{\pi}{2}}\left\{1 - \int_1^a \frac{\sqrt{1+s^2}-s}{2s^{3/2}}\,ds\right\}. \end{aligned}$$

(*ii*) *Let the functions χ and ψ be defined in the following way:*

$$\chi(s) := \frac{1}{2(s+1)^2} - \frac{1}{4s(s+1)} + \frac{1}{4s}\frac{\tan^{-1}\sqrt{s}}{\sqrt{s}}, \quad s > 0$$

and

$$\psi(s) := \left(\frac{15}{s^3} + \frac{1}{s^2}\right) - \left(\frac{15}{s^3} + \frac{6}{s^2} - \frac{1}{s}\right)\frac{\tan^{-1}\sqrt{s}}{\sqrt{s}}, \quad s > 0.$$

Then, for $a \geq 1$,

$$\begin{aligned} EL(1/a,a)^2 =& 2 - 4\int_1^a \{\sqrt{1+s^2}-s\}\chi(s-1)\,ds \\ &+ \frac{4}{5}\int_1^a \left\{\frac{\sqrt{1+s^2}-s}{s}\right\}^2 ds \\ &+ \frac{1}{4}\int_1^a \left(\sqrt{1+t^2}-t\right)\int_1^t \frac{\sqrt{1+s^2}-s}{s^3}\,\psi\left(\frac{t}{s}-1\right) ds\,dt \\ &+ \frac{1}{8}\int_1^a\int_1^a \left\{\sqrt{1+s^2}-s\right\}\left\{\sqrt{1+t^2}-t\right\}\psi(st-1)\,ds\,dt. \end{aligned}$$

2.12 Remark. The functions χ and ψ have finite limits, as $s \downarrow 0$. We have:

$$\lim_{s\downarrow 0}\chi(s) = \frac{2}{3}, \quad \text{and} \quad \lim_{s\downarrow 0}\psi(s) = \frac{64}{105}.$$

It is easily seen that $L(1/n, n)$ converges almost surely to a limiting random variable, denoted $L(0,\infty)$. A simple application of the monotone convergence theorem then yields

$$EL(0,\infty) = \lim EL(1/n,n) = \sqrt{\frac{\pi}{2}}\{1 - \int_1^\infty \frac{\sqrt{1+s^2}-s}{2s^{3/2}}\}\,ds \approx 1.06182,$$

and similarly

$$EL(0,\infty)^2 \approx 1.37575.$$

Details are given in Appendix B.

3. Functionals of the convex hull of a uniform sample.

The results of the previous section and the strong approximation result, Lemma 2.2 in Groeneboom (1988), yield limit theorems for the functionals of the finite sample.

First we need to recall the definition of the process, running through the vertices of the left-lower boundary of the convex hull of a uniform sample of size n from the interior of the unit square. (Definition 2.1 in Groeneboom (1988))

3.1 DEFINITION. For each $a > 0$, $W_n(a)$ is the point of the sample, such that all points of the sample lie to the right of the line $x + ay = c$, which passes through $W_n(a)$. If there are several such points, we take the one with the smallest y-coordinate.

The point process $\{\sqrt{n}W_n(a) : a > 0\}$ converges in distribution to the point process $\{W(a) : a > 0\}$ (Groeneboom (1988), Corollary 2.2).

Let A_n denote the difference between the area of the unit square and the area of the convex hull of n uniform points in this square. Analogously, L_n denotes the difference between the boundary length of the unit square and the boundary length of the convex hull of the uniform sample.

To give an idea of the proof of our main theorem, we recall some of the results in Groeneboom (1988), on which the theorem heavily depends.

Consider the region R_n of the unit square, that lies to the left and below the curve $C(\frac{3\log n}{n})$, where $C(\alpha)$ is defined by

$$C(\alpha) = \bigcup_{i=1}^{3} C_i(\alpha),$$

with

$$C_1(\alpha) = \left\{(x,y) : \frac{1}{2} < y \le 1, x = \alpha\right\}$$

$$C_2(\alpha) = \left\{(x,y) : xy = \frac{1}{2}\alpha, \alpha \le x \le \frac{1}{2}\right\}$$

$$C_3(\alpha) = \left\{(x,y) : \frac{1}{2} < x \le 1, y = \alpha\right\}.$$

It was shown in Groeneboom (1988, Corollary 2.1), that the vertices $\{W_n(a) : a > 0\}$ of the convex hull of the uniform sample, belong to the region R_n with a probability tending to 1, as $n \to \infty$.

Now consider a Poisson point process ξ_n on $\mathbb{R}^2_+$ with intensity n times Lebesgue measure and let η_n be the sample point process corresponding to the sample of size n from the unit square. Then, it was also shown in

Groeneboom (1988, Lemma 2.2), that there exists a probability space such that the probability that the realizations from $\eta_n|_{R_n}$ (η_n restricted to the region R_n) and from $\xi_n|_{R_n}$ differ, tends to 0, as $n \to \infty$.

Let us introduce the following notation:
$\phi(\zeta_n)$ is a functional of the convex hull of the process ζ_n, where in our case, ϕ is one of the following

N_n	the number of vertices
A_n	the remaining area
L_n	the remaining length,

where A_n and L_n are defined more precisely at the beginning of this section.

In analogy with the notation $A(a,b)$, we introduce the notation $A_n(a,b)$ to denote the area of the region, bounded on the right and left by vertical lines through the x-coordinates of the points $W_n(a)$ and $W_n(b)$, and bounded from below and above by the line $y = 0$ and the (left lower) boundary of the convex hull of the sample, respectively (see Definition 2.2). Similarly, we will use the notations $L_n(a,b)$, $L_n^x(a,b)$, $L_n^y(a,b)$, which are defined in the same way as in section 2, but with the Poisson process $\mathcal{P}$ replaced by the sample point process. A suffix β_n, as in $\phi_{\beta_n}(\zeta_n)$, is used to express the fact that we consider the functional ϕ only for values of the timeparameter in the interval $[\beta_n, 1/\beta_n]$ and $\beta_n = \frac{\log n}{n}$.

The crucial argument is the following. Summarizing the above mentioned results, we get, loosely speaking,

$$\phi_{\beta_n}(\eta_n) = \phi_{\beta_n}(\eta_n|_{R_n}) = \phi_{\beta_n}(\xi_n|_{R_n}) = \phi_{\beta_n}(\xi_n), \tag{3.1}$$

with the equalities only holding on a set having a probability mass, tending to 1 and for n tending to infinity. Put differently: if we want to study a functional of the convex hull of the sample point process, we might just as well study the same functional of the convex hull of a Poisson point process, as long as we restrict our attention to those lines generating the convex hulls with slopes in the interval $[\beta_n, 1/\beta_n]$ and only for n large enough. (See e.g. Lemma 2.2 and the proof of Corollary 2.4 in Groeneboom (1988), where the approximation argument is given for N_n.)

First, we show that it is sufficient to consider the part $\phi_{\beta_n}$of ϕ. By symmetry, it is enough to prove the following lemma.

3.2 LEMMA.

(1) (*i*) *With* $\beta_n = \frac{\log n}{n}$, *we have*

$$EA_n(0,\beta_n) \sim c_1 \frac{\log\log n}{n}, \quad \textit{as } n \to \infty,$$

for a constant $c_1 > 0$.

(2) *(ii) For each sequence* (β_n), *tending to zero, we have:*

$$EL_n^y(0,\beta_n) \sim c_2\, n^{-1/2}\beta_n^{5/2}, \quad \text{as } n \to \infty,$$

for a constant $c_2 > 0$.

The proof is given in Appendix C.

We get the following result for the remaining area.

3.3 THEOREM. *We have, as* $n \to \infty$,

$$\{A_n - 4b_n\}/2c_n \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

where $b_n = \frac{2}{3}\frac{\log n}{n}$ *and* $c_n = \sqrt{\frac{100}{189}\frac{\log n}{n}}$, *and* $\mathcal{N}(0,1)$ *denotes the standard normal distribution.*

PROOF. From Corollary 2.7 and (3.1), we can deduce that

$$\{A_n(\beta_n, 1/\beta_n) - b_n\}/2c_n \xrightarrow{\mathcal{D}} \mathcal{N}(0,1), \text{as} \quad n \to \infty. \tag{3.2}$$

Furthermore, by Markov's inequality and Lemma 3.2(i),

$$P\left\{\frac{A_n(0,\beta_n)}{c_n} \geq \eta\right\} \leq \frac{1}{\eta}\frac{EA_n(0,\beta_n)}{c_n} \to 0, \quad n \to \infty.$$

Hence

$$\begin{aligned}\frac{A_n - b_n}{c_n} &= \frac{A_n(\beta_n, 1/\beta_n) - b_n}{c_n} + o_P(1)\\ &\xrightarrow{\mathcal{D}} \mathcal{N}(0,1), \quad \text{by (3.2)}.\end{aligned}$$

Of course we may proceed in the same way for the other corners. The only thing left to show is that the random variables $A_n(\beta_n, 1/\beta_n)$, for the different corners, are asymptotically independent. Areas like $A(0,\beta_n)$ give a negligible contribution in view of Lemma 3.2(i). Since we now need to apply exactly the same argument as the one used in the corresponding Corollary 2.4 in Groeneboom (1988), we refer to the last part of the proof of that Corollary. □

It is clear from Theorem 3.3, that A_n is of order $\frac{\log n}{n}$, as $n \to \infty$. The rate for the area A_n is not surprising, since, as was shown in Groeneboom (1988) the expected number EN_n of vertices of the convex hull is of order

$\log n$, for $n \to \infty$. Moreover, by a well-known relation (see e.g. Efron (1965)) between the expected area and the expected number of vertices , we have

$$EA_n = \frac{1}{n} EN_{n-1}.$$

Next, consider a uniform sample from a convex plane polygon with $k(\geq 3)$ vertices. In the same way that led us to Theorem 3.3, we may derive from Corollary 2.7 and the remarks made thereafter, the following more general result.

3.4 THEOREM. *Let $A(C_n)$ denote the area of the convex hull of a uniform sample of size n from the interior of a convex polygon C with $k(\geq 3)$ vertices and area $A(C)$. Then, for the remaining area*

$$A_n := A(C) - A(C_n),$$

we have, as $n \to \infty$,

$$\frac{\left\{A_n - \frac{2}{3} k \frac{\log n}{n}\right\}}{\sqrt{\frac{100}{189} k \frac{\log n}{n}}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ denotes the standard normal distribution.

The behavior of L_n is rather different. Essential in Theorem 3.3 is the asymptotic negligibility (see Feller (1966, p. 177) of the individual components of A_n. However, as is seen from Theorem 2.11, the dominating asymptotic behavior of L_n depends on a bounded number of parts. Another way of expressing this, is saying that the pieces of nA_n constitute a series that diverges as $n \to \infty$, whereas the pieces of $\sqrt{n} L_n$ constitute a series that converges in distribution to a (random) summable infinite series. From Theorem 2.11 and part (*ii*) of Lemma 3.2 we get the following result for the sequence L_n.

3.5 THEOREM. *For the remaining length L_n of the boundary of the convex hull of n uniform points inside the unit square, we have*

$$\sqrt{n} L_n \xrightarrow{\mathcal{D}} L, \text{ as } n \to \infty,$$

where L is a random variable with

$$EL = 2\sqrt{2\pi}\left\{1 - \int_1^\infty \frac{\sqrt{1+s^2} - s}{2s^{3/2}}\, ds\right\} \approx 4.2473.$$

Moreover

$$\operatorname{Var} L = 4\{EL(0,\infty)^2 - (EL(0,\infty)^2\} \approx 0.9932.$$

The analytic expressions for the moments of $L(0,\infty) := \lim_{a\to\infty} L(1/a, a)$ are given in Theorem 2.11.

4. Appendix.

For easy reference, we include the statement of Lemma 2.4 in Groeneboom (1988), which will be needed in the sequel.

4.0 LEMMA. (Lemma 2.4 in Groeneboom (1988)) *Let* $b > a > 0$, *and let* $W(a) = (U(a), V(a))$, $W(b) = (U(b), V(b))$. *Then*

(*i*) $P\{U(a) \in dx,\, V(a) \in dy\} = \exp\left\{-\frac{(x+ay)^2}{2a}\right\} dx\, dy,\ 0 < x, y < \infty.$

(*ii*) $P\{W(b) = W(a) | W(a) = (x, y)\} = \exp\{-\frac{1}{2}(b-a)y^2)\}$, *if* $b \geq a$ *and* $0 < x, y < \infty$.

(*iii*) $P\{U(b) \in dx_2,\, V(b) \in dy_2 | U(a) = x_1,\, V(a) = y_1\}$
$= \exp\{-(x_2 + by_2 - x_1 - ay_1)^2/2(b-a)\} dx_2\, dy_2,$
if $x_2 > x_1 > 0,\ y_1 > y_2 > 0$, *and* $x_1 + ay_1 < x_2 + by_2 < x_1 + by_1$.

4.1. Appendix A. (Proof of Theorem 2.6)

In the proof of Theorem 2.6 we will need to evaluate integrals of the following type

$$I_k(\sigma) := \begin{cases} \int_0^\infty y^k\, e^{-\frac{1}{2}\sigma y^2}\, (\int_y^\infty e^{-\frac{1}{2}t^2} dt)\, dy & , k \text{ even} \\ \int_0^\infty y^k\, e^{-\frac{1}{2}(\sigma+1)y^2} dy & , k \text{ odd}, \end{cases} \tag{4.1}$$

where $\sigma \geq 0$. For these, we have the following lemma.

4.1 LEMMA. *Let the integral* $I_k(\sigma)$ *be defined by (4.1), where* k *is a non-negative integer and* $\sigma \geq 0$. *Then we have*

$$I_0(\sigma) = \begin{cases} \sigma^{-1/2} \tan^{-1} \sqrt{\sigma}, & \sigma > 0, \\ 1, & \sigma = 0, \end{cases}$$

$$I_1(\sigma) = \frac{1}{\sigma+1},$$

and, for $k \geq 1$, *we get the recursive relations*

$$I_{2k}(\sigma) = \frac{2k-1}{\sigma} I_{2k-2}(\sigma) - \frac{1}{\sigma} I_{2k-1}(\sigma),$$

$$I_{2k+1}(\sigma) = \frac{2k}{\sigma+1} I_{2k-1}(\sigma).$$

PROOF. The recursive relations follow from integration by parts. Furthermore, $I_0(\sigma)$ was computed in Groeneboom (1988), (2.37), and $I_1(\sigma)$ is trivial. □

As a consequence of the recursive relations in Lemma 4.1 and Lemma 4.0 we get:

$$\text{(4.2)} \qquad \begin{aligned} EU(1)^2 &= EV(1)^2 = I_2(0) = 2/3, \\ EU(1)^4 &= EV(1)^4 = I_4(0) = 8/5, \\ EU(1)^6 &= EV(1)^6 = I_6(0) = 48/7. \end{aligned}$$

We now proceed to the proof of Theorem 2.6.

ad (i). As shown in the proof of Theorem 2.3, with $f(x,y,z) = z$

$$\text{(4.3)} \qquad \begin{aligned} &\lim_{h\downarrow 0} \frac{1}{h} E\left\{A(a,a+h)|W(a) = (x,y)\right\} \\ &= \int_0^y u(auy - \frac{1}{2}au^2)du \\ &= \frac{5}{24}ay^4. \end{aligned}$$

It follows that, for $a \geq 1$,

$$EA(1,a) = \frac{5}{24}\int_1^a sEV(s)^4 ds = \frac{1}{3}\log a$$

since $EV(s)^4 = s^{-2}EV(1)^4 = \dfrac{8}{5s^2}$, by Lemma 4.0 and (4.2).

ad (ii). We consider the second moment of $A(1,a)$: take $f(x,y,z) = z^2$ in Theorem 2.3. Then

$$\begin{aligned} &\lim_{h\downarrow 0} \frac{1}{h} E\left\{A(1,a+h)^2 - A(1,a)^2|\mathcal{F}_{[1,a]}\right\} \\ &= \lim_{h\downarrow 0} \frac{1}{h} E\left\{A(a,a+h)^2 + 2A(1,a)A(a,a+h)|\mathcal{F}_{[1,a]}\right\} \\ &= \int_0^{V(a)} u\left\{auV(a) - \frac{1}{2}au^2\right\}^2 du + \frac{5}{12}aA(1,a)V(a)^4. \end{aligned}$$

Hence

$$\text{(4.4)} \qquad EA(1,a)^2 = \frac{11}{120}\int_1^a s^2 EV(s)^6 ds + \frac{5}{12}\int_1^a sEA(1,s)V(s)^4 ds.$$

The first integral of (4.4) is easily computed to be $\frac{22}{35}\log a$, using that, by Lemma 4.0 and (4.2),

$$EV(s)^6 = s^{-3}EV(1)^6 = \frac{48}{7}s^{-3}.$$

For the second integral, we use a time reversal argument similar to the one given on page 343 in Groeneboom (1988). It follows that

$$E\left\{A(1,s)|W(s)\right\} = \frac{5}{24}\int_1^s \frac{1}{r^3} E\left\{U(r)^4|W(s)\right\} dr, \tag{4.5}$$

where (4.3) is used.
So we have to compute $EU(r)^4V(s)^4$. We first take $r = 1$. From Lemma 4.0, it follows that

$$\begin{aligned} EU(1)^4V(s)^4 =& \frac{4}{5}EV(1)^4U(1)^4 \exp\left\{-\frac{1}{2}\sigma V(1)^2\right\} \\ &- \frac{4}{5\sigma}EV(1)^2U(1)^4 \exp\left\{-\frac{1}{2}\sigma V(1)^2\right\} \\ &- \frac{8}{5\sigma^2}EU(1)^4\left\{\exp\left\{-\frac{1}{2}\sigma V(1)^2\right\} - 1\right\}, \end{aligned} \tag{4.6}$$

where $\sigma := s - 1$.
Evaluating this term by term using Lemma 4.0(i) leads to expressions containing the integrals $J_k = J_k(y)$, defined by

$$J_k = \int_y^\infty (u-y)^k e^{-\frac{1}{2}u^2}\, du, \tag{4.7}$$

which satisfy

$$\begin{aligned} J_0 &= \int_y^\infty e^{-\frac{1}{2}u^2}\, du \\ J_1 &= e^{-\frac{1}{2}y^2} - y\int_y^\infty e^{-\frac{1}{2}u^2}\, du, \\ J_k &= (k-1)J_{k-2} - yJ_{k-1}, \quad k \geq 2, \end{aligned} \tag{4.8}$$

and

$$I_k(\sigma) = \int_o^\infty y^k e^{\frac{1}{2}\sigma y^2} J_0(y)\, dy.$$

Simple algebra then leads to the following expression for (4.6)

$$\begin{aligned} &\frac{4}{5}\Big\{3I_4 - 5I_5 + 6I_6 - I_7 + I_8 \\ &-\big(3I_2 - 5I_3 + 6I_4 - I_5 + I_6\big)/\sigma \\ &-2\big(3I_0 - 5I_1 + 6I_2 - I_3 + I_4 - EU(1)^4\big)/\sigma^2\Big\}. \end{aligned}$$

Now, using the recursive relations for $I_k = I_k(\sigma)$, we obtain

$$\text{(4.9)}\quad EV(s)^4U(1)^4 = \frac{16}{5}\left(\frac{21}{\sigma^4}+\frac{15}{\sigma^3}\right)\frac{\tan^{-1}\sqrt{\sigma}}{\sqrt{\sigma}} - \frac{16}{5}\left(\frac{21}{\sigma^4}+\frac{8}{\sigma^3}-\frac{4}{5\sigma^2}\right).$$

To simplify the notation slightly, we define the function $\varphi(s) = \varphi(1+\sigma)$ to be equal to the right-hand side of (4.9). Then, by the definition and the stationarity of the process $\{Z(a) : a \in \mathbb{R}\}$, we get

$$EV(s)^4U(r)^4 = \varphi(\frac{s}{r}) \quad , s > r > 0.$$

The second term in (4.4) equals

$$\begin{aligned}&\frac{25}{288}\int_1^a s\,ds\int_1^s \frac{1}{r^3}\varphi(\frac{s}{r})\,dr\\ =&\frac{25}{288}\int_1^a \frac{1}{r}\,dr\int_1^{a/r} u\varphi(u)\,du,\end{aligned}$$

by Fubini's theorem and a change of variables. By straightforward methods or, easier still, by using a computer algebra package, we get the following evaluation of the indefinite integral $\frac{25}{288}\int u\varphi(u)\,du$:

$$\text{(4.10)}\quad \begin{aligned}\frac{25}{288}\int u\varphi(u)\,du =&\frac{2}{9}\log u + \frac{5}{3(u-1)^3} + \frac{31}{9(u-1)^2}\\ &+\frac{16}{9(u-1)} - \frac{25u^2-14u+4}{9(u-1)^{7/2}}\tan^{-1}\sqrt{u-1}.\end{aligned}$$

We note in passing that

$$EU(1)^4V(1)^4 = \lim_{u\to 1}\varphi(u) = \frac{64}{105}.$$

Let $h(u)$ denote the right-hand side of (4.10). Then we get

$$h(1) \stackrel{\text{def}}{=} \lim_{u\to 1} h(u) = 344/945.$$

Next we get

$$\begin{aligned}&\frac{25}{288}\int_1^a r^{-1}\,dr\int_1^{a/r} u\varphi(u)\,du = \int_1^a u^{-1}\{h(u)-h(1)\}\,du\\ &= \frac{1}{9}(\log a)^2 - \frac{344}{945}\log a - \frac{2}{3(a-1)^2} - \frac{4}{3(a-1)}\\ &\qquad + \frac{2}{9}\frac{a(4a-1)\tan^{-1}\sqrt{a-1}}{(a-1)^{5/2}} + \frac{4}{9}\{\tan^{-1}\sqrt{a-1}\}^2 - \frac{68}{135},\end{aligned}$$

and this yields the second term of (4.4), whence the theorem follows.

4.2. Appendix B. (Proof of Theorem 2.11)

To be able to do the computations, we observe that by Definition 2.8

$$L(1/a,a) = U(1) + V(1) - L^y(1/a,1) - L^x(1,a), \quad a \geq 1. \tag{4.11}$$

ad (i). From Theorem 2.10(i) we get for $a \geq 1$

$$\begin{aligned} &\lim_{h\downarrow 0} \frac{1}{h} E\left\{L^x(a,a+h)|W(a) = (x,y)\right\} \\ &= \int_0^y u\left\{u(\sqrt{1+a^2}-a)\right\} du \\ &= \frac{1}{3} y^3 \left(\sqrt{1+a^2}-a\right). \end{aligned} \tag{4.12}$$

Hence, for $a \geq 1$, we have

$$EL^x(1,a) = \frac{1}{3}\int_1^a \left(\sqrt{1+s^2}-s\right) EV(s)^3 ds.$$

By Lemma 4.0(i)

$$EV(s)^3 = \int_0^\infty \int_0^\infty y^3 \exp\{-\frac{(x+sy)^2}{2s}\} dxdy = \frac{3\sqrt{2\pi}}{8s^{3/2}}.$$

This yields

$$EL^x(1,a) = \sqrt{2\pi}\int_1^a \frac{\sqrt{1+s^2}-s}{8s^{3/2}} ds. \tag{4.13}$$

Moreover,

$$EU(1) = \int_0^\infty \int_0^\infty x \exp\left\{-\frac{1}{2}(x+y)^2\right\} dxdy = \frac{1}{4}\sqrt{2\pi}. \tag{4.14}$$

By symmetry, also $EV(1) = \frac{1}{4}\sqrt{2\pi}$. Since $EL^y(\frac{1}{a},1) = EL^x(1,a)$, we obtain from (4.11) to (4.14):

$$EL(\frac{1}{a},a) = \sqrt{\frac{\pi}{2}}\left\{1 - \int_1^a \frac{\sqrt{1+s^2}-s}{2s^{3/2}} ds\right\}.$$

ad (ii).
Using (4.11) and the fact that the processes $\{W(a) = (U(a), V(a)) : a > 1\}$ and $\{(V(\frac{1}{a}), U(\frac{1}{a})) : a > 1\}$ are identically distributed, we obtain for $a \geq 1$:

$$\begin{aligned} EL(\frac{1}{a}, a)^2 =& E\{U(1) + V(1)\}^2 - 4EL^x(1,a)\{U(1)+V(1)\} \\ &+ 2EL^x(1,a)^2 + 2EL^y(\frac{1}{a}, 1)L^x(1,a). \end{aligned} \tag{4.15}$$

First we have

$$E\{U(1)+V(1)\}^2 = \int_0^\infty \int_0^\infty (x+y)^2 e^{-\frac{1}{2}(x+y)^2} dx dy = 2. \tag{4.16}$$

The results of the computations of the other terms in (4.15) will be stated as separate Lemmas.

4.2 LEMMA. *Let the function χ be defined as*

$$\chi(s) := \frac{1}{2(s+1)^2} - \frac{1}{4s(s+1)} + \frac{1}{4s}\frac{\tan^{-1}\sqrt{s}}{\sqrt{s}}, \quad s > 0.$$

Then for $a \geq 1$

$$EL^x(1,a)\{U(1)+V(1)\} = \int_1^a \left\{\sqrt{1+s^2} - s\right\} \chi(s-1)\, ds.$$

PROOF. Observe that

$$E\{L^x(1,a)|W(1)\} = \frac{1}{3}\int_1^a \left(\sqrt{1+s^2} - s\right) E\{V(s)^3|W(1)\}\, ds, \tag{4.17}$$

where we use (4.12). Hence

$$EL^x(1,a)\{U(1)+V(1)\} = \frac{1}{3}\int_1^a \left(\sqrt{1+s^2} - s\right) EV(s)^3\{U(1)+V(1)\}\, ds. \tag{4.18}$$

From Lemma 4.0, we get

$$\begin{aligned} E\left\{V(s)^3|W(1) = (x,y)\right\} =& \frac{3}{4}y^2 e^{-\frac{1}{2}\sigma y^2} \\ &- \frac{3}{4\sigma} y e^{-\frac{1}{2}\sigma y^2} + \frac{3}{4\sigma}\int_0^y e^{-\frac{1}{2}\sigma u^2} du, \end{aligned} \tag{4.19}$$

where $\sigma := s - 1$.

Consequently

$$\begin{aligned} EV(s)^3\{U(1)+V(1)\} =& \frac{3}{4}EV(1)^2\{U(1)+V(1)\}e^{-\frac{1}{2}\sigma V(1)^2} \\ &- \frac{3}{4\sigma}EV(1)\{U(1)+V(1)\}e^{-\frac{1}{2}\sigma V(1)^2} \\ &+ \frac{3}{4\sigma}E\{U(1)+V(1)\}\int_0^{V(1)} e^{-\frac{1}{2}\sigma u^2}du. \end{aligned}$$

Similar computations as in Appendix A yield

$$EV(s)^3\{U(1)+V(1)\} = \frac{1}{2s^2} - \frac{1}{4s(s-1)} + \frac{1}{4(s-1)}\frac{\tan^{-1}\sqrt{s-1}}{\sqrt{s-1}}.$$

Together with (4.18), this yields the Lemma. □

Next, we consider the second moment of $L^x(1,a)$ for $a \geq 1$.

4.3 LEMMA. *Let the function* ψ *be defined as*

$$\psi(s) := \left(\frac{15}{s^3}+\frac{1}{s^2}\right) - \left(\frac{15}{s^3}+\frac{6}{s^2}-\frac{1}{s}\right)\frac{\tan^{-1}\sqrt{s}}{\sqrt{s}}, \quad s>0. \tag{4.20}$$

Then for $a \geq 1$

$$\begin{aligned} EL^x(1,a)^2 =& \frac{2}{5}\int_1^a \left\{\frac{\sqrt{1+s^2}-s}{s}\right\}^2 ds \\ &+ \frac{1}{8}\int_1^a \left(\sqrt{1+s^2}-s\right)\int_1^s \frac{\sqrt{1+r^2}-r}{r^3}\psi\left(\frac{s}{r}-1\right)drds. \end{aligned} \tag{4.21}$$

PROOF.
Taking $f(x,y,z) = z^2$ in Theorem 2.10 and using (4.12) gives

$$\begin{aligned} &\lim_{h\downarrow 0}\frac{1}{h}E\left\{L^x(1,a+h)^2 - L^x(1,a)^2|\mathcal{F}_{[1,a]}\right\} \\ &= \lim_{h\downarrow 0}\frac{1}{h}E\left\{L^x(a,a+h)^2 + 2L^x(1,a)L^x(a,a+h)|\mathcal{F}_{[1,a]}\right\} \\ &= \int_0^{V(a)} u\left\{u(\sqrt{1+a^2}-a)\right\}^2du + \frac{2}{3}(\sqrt{1+a^2}-a)L^x(1,a)V(a)^3. \end{aligned}$$

Hence

$$\begin{aligned} EL^x(1,a)^2 =& \frac{1}{4}\int_1^a (\sqrt{1+s^2}-s)^2 EV(s)^4 ds \\ &+ \frac{2}{3}\int_1^a (\sqrt{1+s^2}-s)EL^x(1,s)V(s)^3 ds. \end{aligned}$$

As noted at the end of the proof of part (i) of Appendix A, $EV(s)^4 = \frac{8}{5s^2}$. This leaves us with

$$EL^x(1,a)^2 = \frac{2}{5}\int_1^a \left\{\frac{\sqrt{1+s^2}-s}{s}\right\}^2 ds + \frac{2}{3}\int_1^a \left\{\sqrt{1+s^2}-s\right\} EL^x(1,s)V(s)^3 ds \tag{4.22}$$

For the second integral we use a time reversal argument as before. It follows that

$$E\left\{L^x(1,s)|W(s)\right\} = \frac{1}{3}\int_1^s \frac{\sqrt{1+r^2}-r}{r^3} E\left\{U(r)^3|W(s)\right\} dr.$$

So we have to compute $EU(r)^3V(s)^3$. First set $r=1$. From (4.19), we get

$$\begin{aligned} EU(1)^3V(s)^3 &= \frac{3}{4}EU(1)^3V(1)^2e^{-\frac{1}{2}\sigma V(1)^2} \\ &- \frac{3}{4\sigma}EU(1)^3V(1)e^{-\frac{1}{2}\sigma V(1)^2} \\ &+ \frac{3}{4\sigma}EU(1)^3\int_0^{V(1)} e^{-\frac{1}{2}\sigma u^2}du. \end{aligned}$$

This is similar to (4.6) and can be evaluated in the same way yielding

$$\begin{aligned} EU(1)^3V(s)^3 &= \frac{9}{16}\left(\frac{15}{(s-1)^3}+\frac{1}{(s-1)^2}\right) \\ &- \frac{9}{16}\left(\frac{15}{(s-1)^3}+\frac{6}{(s-1)^2}-\frac{1}{s-1}\right)\frac{\tan^{-1}\sqrt{s-1}}{\sqrt{s-1}}. \end{aligned} \tag{4.23}$$

With ψ as defined in (4.20) the right-hand side of (4.23) is $\frac{9}{16}\psi(s-1)$. Then it is easily seen that

$$EU(r)^3V(s)^3 = \frac{9}{16}\psi\left(\frac{s}{r}-1\right) \quad , \quad s>r>0. \tag{4.24}$$

Together with (4.22) this proves Lemma 4.3. □

We now turn to the fourth term of (4.15).

4.4 Lemma. *For $a \geq 1$*

$$EL^y(\frac{1}{a},1)L^x(1,a) = \frac{1}{16}\int_1^a\int_1^a \left\{\sqrt{1+s^2}-s\right\}\left\{\sqrt{1+t^2}-t\right\}\psi\left(st-1\right)dsdt, \tag{4.25}$$

where ψ is the function defined in (4.20).

PROOF. First observe that

$$\begin{aligned} EL^y(\frac{1}{a},1)L^x(1,a) &= E\left\{E\{L^y(\frac{1}{a},1)L^x(1,a)|W(1)\}\right\} \\ &= E\left\{E\{L^y(\frac{1}{a},1)|W(1)\}E\{L^x(1,a)|W(1)\}\right\}, \end{aligned}$$

by the conditional independence of $L^y(\frac{1}{a},1)$ and $L^x(1,a)$ given $W(1)$, which follows from Theorem 2.10. Now note

$$E\{L^x(1,a)|W(1)\} = \frac{1}{3}\int_1^a (\sqrt{1+t^2}-t)\,E\{V(t)^3|V(1)\}\,dt$$

and

$$E\{L^y(\frac{1}{a},1)|W(1) = \frac{1}{3}\int_1^a (\sqrt{1+s^2}-s)\,E\{U(s)^3|U(1)\}\,ds.$$

Finally, by integrating with respect to the distribution of the vector $W(1) = (U(1),V(1))$ we get

$$\begin{aligned} &EL^y(\frac{1}{a},1)L^x(1,a) = \\ &\qquad \frac{1}{9}\iint_{1<s,t<a} \left\{\sqrt{1+s^2}-s\right\}\left\{\sqrt{1+t^2}-t\right\} EU(1/s)^3V(t)^3\,dsdt \end{aligned}$$

The result now follows from (4.24). □

The integrals in the preceding lemmas were computed numerically for the case $a=\infty$. For the first and second moment of $L(0,\infty)$ this yields

$$\lim_{a\to\infty} EL(\frac{1}{a},a) = \sqrt{\frac{\pi}{2}}\left\{1-\int_1^\infty \frac{\sqrt{1+s^2}-s}{2s^{3/2}}\,ds\right\} \approx 1.061824,$$

and

$$\lim_{a\to\infty} EL(\frac{1}{a},a)^2 \approx 1.37575.$$

4.3. Appendix C. (Proof of Lemma 3.2)

Let ϕ stand for either A_n or L_n^y. Then

$$E\phi(0,\beta_n) \sim \binom{n}{2} \int_0^{\beta_n} da \iint_{0<x+ay<1} f_a(x,y)(1-A_a(x,y))^{n-1}\,dxdy,$$

where $A_a(x,y)$ denotes the area of the region cut off from the unit square to the left of the line $l_a : x' + ay' = c$ through (x,y), and where $f_a(x,y)$ is defined by

$$f_a(x,y) = \begin{cases} \frac{5}{24} a y^4, & \text{if } \phi(0,\beta_n) = A_n(0,\beta_n), \\ x^3\{\sqrt{1+a^2}-1\}/(3a^3), & \text{if } \phi(0,\beta_n) = L_n^y(0,\beta_n), \end{cases}$$

See (4.5)–(4.7) of Appendix A3 in Groeneboom (1988), where the detailed argument is given for the functional N_n. We now have

$$\begin{aligned} &\int_0^{\beta_n} da \iint_{0<x+ay<1} f_a(x,y)(1-A_a(x,y))^{n-1}\,dxdy \\ (4.26)\qquad &= \int_0^{\beta_n} da \iint_{0<x+ay<a} f_a(x,y)\left\{1-\frac{1}{2}\frac{(x+ay)^2}{a}\right\}^{n-1} dxdy \\ &\quad + \int_0^{\beta_n} da \iint_{a\le x+ay<1} f_a(x,y)\left\{1-x-ay+\frac{1}{2}a\right\}^{n-1} dxdy. \end{aligned}$$

It can be shown that we only need to consider the first integral (see Appendix A3 in Groeneboom (1988)).

(i) Take $\phi = A_n$.

Then the first integral in (4.26) equals

$$\begin{aligned} &\frac{5}{24}\int_0^{\beta_n} a\,da \int_0^a \int_0^{u/a} y^4(1-\frac{u^2}{2a})^{n-1}\,dydu \\ (4.27)\qquad &= \frac{1}{24}\int_0^{\beta_n} a\,da \int_0^a (\frac{u}{a})^5(1-\frac{u^2}{2a})^{n-1}\,du. \end{aligned}$$

To evaluate the inner integral, we set $v = \frac{u^2}{2a}$, yielding

$$\int_0^a (\frac{u}{a})^5(1-\frac{u^2}{2a})^{n-1}\,du = \frac{4}{a^2}\int_0^{\frac{1}{2}a} v^2(1-v)^{n-1}\,dv.$$

Hence (4.27) is

$$\frac{1}{6}\int_0^{\beta_n}\frac{1}{a}\,da\int_0^{\frac{1}{2}a}v^2(1-v)^{n-1}\,dv. \tag{4.28}$$

But

$$\begin{aligned}\int_0^{\frac{1}{2}a}v^2(1-v)^{n-1}\,dv &= -\frac{a^2}{4n}(1-\frac{1}{2}a)^n-\frac{a}{n(n+1)}(1-\frac{1}{2}a)^{n+1}\\ &\quad+\frac{2}{n(n+1)(n+2)}\left\{1-(1-\frac{1}{2}a)^{n+2}\right\}.\end{aligned} \tag{4.29}$$

Hence (4.27) is asymptotically equivalent to

$$\begin{aligned}&-\frac{1}{24}\int_0^{\beta_n}ae^{-\frac{1}{2}na}\,da-\frac{1}{6n(n-1)}\int_0^{\beta_n}e^{-\frac{1}{2}na}\,da\\ &\quad+\frac{1}{3n(n+1)(n+2)}\int_0^{\beta_n}\frac{1}{a}\left\{1-\exp\{-\frac{1}{2}na\}\right\}da\\ &\sim c\frac{\log\log n}{n^3},\ \text{as } n\to\infty.\end{aligned}$$

(See Appendix A3 in Groeneboom (1988).) This yields part (i) of the Lemma, since we now have

$$EA_n(0,\beta_n)\sim c\binom{n}{2}\frac{\log\log n}{n^3}\sim c_1\frac{\log\log n}{n},\ \text{as } n\to\infty.$$

(ii) Take $\phi=L_n^y$.

Then

$$EL_n^y(0,\beta_n)=\binom{n}{2}\int_0^{\beta_n}\frac{\sqrt{1+a^2}-1}{3a^3}\,da\int_{0<x+ay<1}x^3(1-A_a(x,y))^{n-1}\,dxdy.$$

Again, it is enough to consider the following integral:

$$\begin{aligned}&\int_0^{\beta_n}\frac{\sqrt{1+a^2}-1}{3a^3}\,da\int_0^a\int_0^u x^3(1-\frac{u^2}{2a})^{n-1}\,dx\,du\\ &\sim\tfrac{1}{3}n^{-5/2}\int_0^{\beta_n}\frac{\sqrt{1+a^2}-1}{\sqrt{2a}}\,da\int_0^{\frac{1}{2}a}w^{3/2}e^{-nw}\,dw\\ &\sim\frac{\sqrt{2}}{15}\,n^{-5/2}\beta_n^{5/2}\,\Gamma(5/2).\end{aligned}$$

Hence

$$EL_n^y(0,\beta_n)\sim c\cdot n^{-1/2}\beta_n^{5/2},\ \text{where } c=\frac{1}{10}\sqrt{\frac{\pi}{2}}.$$

CHAPTER II

AN ELEMENTARY PROOF OF THE AMBARTZUMIAN–PLEIJEL FORMULA

1. Introduction.

Pleijel (1956) proved an identity relating the area A of a convex plane domain and the length L of its boundary (of class C^1). In particular, it contains the isoperimetric inequality $L^2 - 4\pi A \geq 0$.

Ambartzumian gave two proofs of a generalized version of the Pleijel identity for convex polygons. The first proof (Ambartzumian (1974)) consisted of direct computations. In his book (1982) however, he shows that the identity is an easy consequence of the solution to the Buffon-Sylvester problem.

Pohl (1980) proved an analogous formula for closed convex plane curves with smooth boundary, applying Stokes' theorem to a suitable manifold with boundary.

The aim of this chapter is to show that Stokes' theorem may also be used to prove Ambartzumian's Pleijel-type identity for convex polygons directly. It turns out that the use of differential forms leads to considerable simplifications. The interesting question whether this method may be used to derive a Pleijel-type identity for more general convex domains, remains unanswered.

2. Ambartzumian's Pleijel-type identity for convex polygons.

Throughout the section, let C denote a (bounded) closed convex polygon in the plane. The main idea of the proof is to compute the integral of a differential form over two of the sides of C. Then by a limiting procedure the result follows immediately.

To be able to perform the integration, we have to give an orientation to the sides. Let a and b be two non-intersecting sides of C with distinct endpoints A_1, A_2 and B_1, B_2 respectively. The set $a \times b$ is a two-dimensional submanifold of $\mathbb{R}^4$, which can be parametrized in the following way. Let u

and v be the vectors $A_2 - A_1$ and $B_2 - B_1$ respectively. Then $x \in a$ and $y \in b$ have the representations

$$x = A_1 + \theta_1 \cdot u$$

$$y = B_1 + \theta_2 \cdot v$$

for some numbers $\theta_1, \theta_2 \in [0,1]$.

If dl_1 (dl_2) is the element of length on a (b), directed from A_1 to A_2 (B_1 to B_2), then the 2-form $dl_1 \wedge dl_2$ has the representation

$$dl_1 \wedge dl_2 = |a| \cdot |b| \cdot d\theta_1 \wedge d\theta_2,$$

where $d\theta_1 \wedge d\theta_2$ is the canonical 2-form on $\mathbb{R}^2$ and $|x|$ denotes the length of the side x.

Using this parametrization, we can consider $a \times b$ as an oriented manifold with boundary. Define the mapping ϕ: $[0,1]^2 \to a \times b$ by

$$\phi(\theta_1, \theta_2) = (A_1 + \theta_1 \cdot u \, , \, B_1 + \theta_2 \cdot v).$$

Then we have

$$a \times b = \phi([0,1]^2)$$

and the oriented boundary of $a \times b$ is identified by this mapping with the boundary in $\mathbb{R}^2$ of the unit-square with the usual counter-clockwise orientation. From this identification, it is seen that $a \times \{B_1\}$ and $\{A_2\} \times b$ have the same orientation as a, b respectively, and that $a \times \{B_2\}$ and $\{A_1\} \times b$ have the opposite orientation.

We shall need the following lemma in the proof.

2.1 Lemma. *Let a and b be as described above and let (x, y) be a point on $a \times b$. Let dl_1, dl_2 denote the element of length on a and b respectively and let χ denote the length of the segment joining x and y, that is directed from x to y. Furthermore, let α_1 and α_2 be the angles, lying to the right of χ, formed by χ and the sides a and b respectively.*

Then we have, for fixed y

$$d\alpha_1 = \frac{\sin \alpha_2}{\chi} dl_2 \tag{1}$$

and for fixed x

$$d\alpha_2 = -\frac{\sin \alpha_1}{\chi} dl_1. \tag{2}$$

PROOF. First fix l_1. Let h_x be the length of the perpendicular from x onto b.
Then

$$\frac{h_x}{-l_2} = \tan(\pi - \alpha_2) = -\tan\alpha_2$$

hence

$$\alpha_2 = \arctan\frac{h_x}{l_2}.$$

Consequently

$$\frac{d\alpha_2}{dl_2} = -\frac{h_x}{{l_2}^2 + {h_x}^2} = -\frac{h_x}{\chi^2} = -\frac{\sin\alpha_2}{\chi}.$$

Since clearly $\alpha_1 + \alpha_2 =$ constant, the first assertion follows.

Next, fix l_2 and let h_y be defined similarly to h_x. Then

$$\frac{d\alpha_2}{dl_1} = -\frac{d\alpha_1}{dl_1} = -\frac{d}{dl_1}\arctan(-\frac{h_y}{l_1}) = -\frac{\sin\alpha_1}{\chi}.$$

□

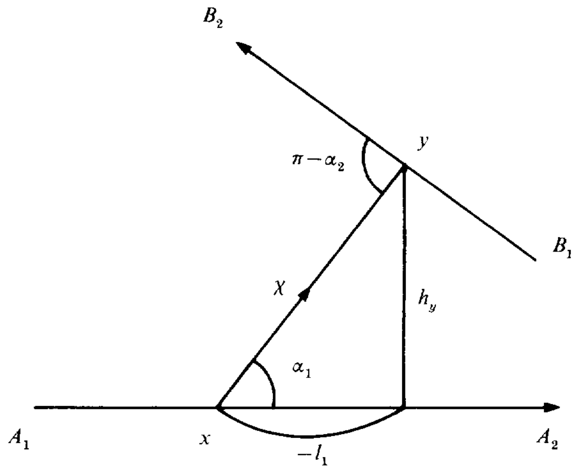


FIGURE 2.1

Observe that if l_2 increases, for l_1 fixed, then the angle α_1 increases. On the other hand, if l_1 increases, for l_2 fixed, then the angle α_2 decreases. As a consequence, we see that the signs of (1) and (2) are correct.

We are now ready to prove the Pleijel-type identity:

2.2 THEOREM. (Ambartzumian-Pleijel) *Let C be a convex polygon with n sides a_i of length $|a_i|$. Suppose that C is oriented as described above. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-function; then*

$$\int_{[C]} f(\chi)\,dg = \int_{[C]} f'(\chi)\chi\cot\alpha_1\cot\alpha_2\,dg + \sum_{i=1}^{n}\int_0^{|a_i|} f(x)\,dx,$$

where dg denotes the element of an invariant measure on the set G of non-oriented lines in the plane and $[C] := \{g \in G : g \cap C \neq \emptyset\}$.

PROOF. First consider two sides a and b with endpoints A_1, A_2 and B_1, B_2 respectively. Suppose that a and b are non-intersecting but not parallel and that they do not share one of their endpoints.

Consider the orientation-preserving differential form

$$dl_1 \wedge dl_2,$$

where dl_1 (dl_2) is the element of length along a (b), as defined above.

Define the 1-form ω on $a \times b$ by

$$\omega(x, y) = \cos\alpha_1 dl_1 + \cos\alpha_2 dl_2$$

then

$$d\omega = -\sin\alpha_1 d\alpha_1 \wedge dl_1 - \sin\alpha_2 d\alpha_2 \wedge dl_2. \tag{3}$$

By Lemma 2.1, (3) may be written as

$$d\omega = -\frac{\sin\alpha_2}{\chi}\sin\alpha_1 dl_2 \wedge dl_1 + \frac{\sin\alpha_1}{\chi}\sin\alpha_2 dl_1 \wedge dl_2$$

whence, by the anti-commutativity of exterior multiplication

$$d\omega = 2\frac{\sin\alpha_1\sin\alpha_2}{\chi} dl_1 \wedge dl_2. \tag{4}$$

Define $\omega_1 := f(\chi)\omega$. Then we may apply Stokes' theorem (see e.g. Guillemin & Pollack (1974)) to the 1-form ω_1 on $a \times b$, since the latter is an oriented 2-manifold with boundary. This yields

$$\begin{aligned} \int_{\partial(a\times b)} \omega_1 &= \int_{a\times b} d\omega_1 \\ &= \int_{a\times b} f'(\chi)\, d\chi \wedge \omega + \int_{a\times b} f(\chi)\, d\omega. \end{aligned} \tag{5}$$

Observe that

$$\frac{-d\chi}{dl_2} = \cos(\pi - \alpha_2) = -\cos\alpha_2$$

hence

$$d\chi = \cos\alpha_2 dl_2.$$

Analogously, we find

$$d\chi = -\cos\alpha_1 dl_1.$$

Consequently

$$\begin{aligned} d\chi \wedge \omega &= d\chi \wedge \cos\alpha_1 dl_1 + d\chi \wedge \cos\alpha_2 dl_2 \\ &= -2\cos\alpha_1 \cos\alpha_2 dl_1 \wedge dl_2. \end{aligned}$$

By (4)

$$\int_{a\times b} f(\chi)\, d\omega = 2\int_{a\times b} \frac{\sin\alpha_1 \sin\alpha_2}{\chi}\, dl_1 \wedge dl_2.$$

Hence (5) may be written as

$$\int_{a\times b} f(\chi)\frac{\sin\alpha_1 \sin\alpha_2}{\chi}\, dl_1 \wedge dl_2 =$$

$$= \int_{a\times b} \cos\alpha_1 \cos\alpha_2 f'(\chi)\, dl_1 \wedge dl_2 + \frac{1}{2}\int_{\partial(a\times b)} f(\chi)\omega. \tag{6}$$

At the beginning of the section, we showed that the boundary $\partial(a\times b)$ of $a \times b$ is

$$\bigcup_{i=1,2} (a \times \{B_i\}) \cup \bigcup_{i=1,2} (\{A_i\} \times b).$$

Consequently

$$\begin{aligned} &\int_{\partial(a\times b)} f(\chi)\omega = \\ &= \int_{\{A_1\}\times b} f(\chi)\omega + \int_{\{A_2\}\times b} f(\chi)\omega + \int_{a\times\{B_1\}} f(\chi)\omega + \int_{a\times\{B_2\}} f(\chi)\omega \\ &= -\int_{\{A_1\}\times b} f(\chi)\cos\alpha_2\, dl_2 + \int_{\{A_2\}\times b} f(\chi)\cos\alpha_2\, dl_2 \\ &\quad + \int_{a\times\{B_1\}} f(\chi)\cos\alpha_1\, dl_1 - \int_{a\times\{B_2\}} f(\chi)\cos\alpha_1\, dl_1, \end{aligned} \tag{7}$$

where one has to take the orientation into consideration.
Equation (7) corresponds with equation (21) in Ambartzumian (1974), in a version for directed lines.

Next, we let the endpoint B_1 of b tend to the endpoint A_2 of a, i.e. the distance between B_1 and A_2 tends to zero. Then in the limit, where $A_2 = B_1$,we get

(8)

$$\begin{aligned} \int_{\partial(a\times b)} f(\chi)\omega &= \int_0^{|a|} f(x)\, dx + \int_0^{|b|} f(x)\, dx \\ &\quad - \int_{a\times\{B_2\}} f(\chi)\cos\alpha_1\, dl_1 - \int_{\{A_1\}\times b} f(\chi)\cos\alpha_2\, dl_2. \end{aligned}$$

Summation of (6) over all sides of C, using (8) as well as Lemma 2.1, completes the proof of the theorem. Observe that indeed terms of the form $\int_0^{|a_i|} f(x)\,dx$ appear twice in the sum. Furthermore, there is a cancellation of terms of the form

$$\int_{\{A_i\}\times a_j} f(\chi)\cos\alpha_2\,dl_2 \quad \text{and} \quad \int_{a_i\times\{A_j\}} f(\chi)\cos\alpha_1\,dl_1$$

as desired. □

CHAPTER III

LINE TRANSECTS, COVARIANCE FUNCTIONS AND A NEW METRIC FOR SETS

Introduction.

This chapter studies the determination of a set $A \subset \mathbb{R}^d$ from information on one-dimensional linear transects $A \cap l$. Three separate issues are discussed:

(a) characterisation: whether a set $A \subset \mathbb{R}^d$ is completely determined by the values of an associated transform (such as the Radon transform or the covariance function);
(b) stereology: whether geometrical parameters of $A \subset \mathbb{R}^d$ can be statistically estimated from randomly-sampled values of the transforms;
(c) approximation: whether good deterministic or stochastic approximation of transforms ensures closeness of the corresponding sets.

Although problems of this kind have received much attention (see the references mentioned below) it is common for the issues (a)–(c) to be considered separately. In this chapter we introduce a construct, the *linear scan transform*, that is serviceable in all three contexts.

Let us briefly sketch the relevant history (see also the introduction of this thesis, §3). The interest in characterisation problems for the set covariance function has recently grown. Nagel (1993) showed that a convex plane polygon is uniquely determined by its set covariance. Lešanovský and Rataj (1990) constructed an example of two distinct *non-convex* sets with the same covariance function. For a more restricted class of 'generic', not necessarily convex polygons, excluding those of the abovementioned counterexample, there is a reconstruction procedure due to Schmitt (1992). In Chapter V of this thesis we show uniqueness of the set covariance function for a certain class of bounded symmetric subsets of $\mathbb{R}^d$.

The related class of problems concerning characterisation of a set from information on linear transects, like the Radon transform, has a much longer history. See e.g. Ambartzumian (1983) and Waksman (1985) and the references therein. For the characterisation of polygons by the distribution of the

Radon transform, see Waksman (1985). The famous example of Mallows and Clark (1970) of two non-congruent convex polygons with the same chord length distribution is not a counterexample for the covariance function (see Nagel (1991)).

In the stereological context (b) above, there are several well known integral geometric identities connecting the chord length distribution of a convex set with its interpoint distance distribution and its set covariance. Moreover an identity of Crofton (1885) concerning the moments of chord lengths was generalised to non-convex sets by Miles (1979) and Jensen and Gundersen (1985) in the construction of an estimator for the total volume of particles. See also Goodey and Weil (1991).

Suppose the intersection $l \cap A$ of a line l with a set $A \subset \mathbb{R}^d$ is a finite union of compact intervals, with ordered endpoints $x_1, x_2, \ldots x_{2n}$. Miles (1983) defined the k-linc as

$$[\sigma(l \cap A)^k] = \sum_{i=1}^{2n-1} \sum_{j=i+1}^{2n} (-1)^{i+j+1}(x_j - x_i)^k, \quad \text{for } k \geq 1.$$

Later Waksman (1987) introduced the glance function of A. In our notation its definition can be written as

$$H_{l \cap A}(t) = \sum_{i=1}^{2n-1} \sum_{j=i+1}^{2n} (-1)^{i+j+1} 1\{x_j - x_i \leq t\}, \quad \text{for } t \geq 0.$$

Waksman used this function to construct a metric on a class of 'regular' subsets of the plane with applications to the approximation problem (c).

The new geometric transform introduced in this chapter is also associated with linear transects of a set $A \subset \mathbb{R}^d$. The transform is a minor modification of Waksman's glance function but turns out to be extremely natural, arising as minus the derivative of the one-dimensional set covariance of a linear transect. It turns out that all the abovementioned integral geometric identities can be rewritten in terms of the linear scan transform.

The chapter can be divided into three parts. The first part (consisting of the first two sections) provides necessary background such as the definition of the covariance function and regular sets in section 1 and the definition of the linear scan transform in section 2. The second part is devoted to stereology and integral geometry, relating the linear scan transform with the k-linc, volume and other quantities. This enables us to directly generalise Crofton's formula to nonconvex sets. Also it is now possible to define stereological estimators in terms of the linear scan transform. This is done in section 3. The third part is concerned with analysis and convex geometry. We pursue the approximation problem in section 4 by constructing a metric η on regular subsets of $\mathbb{R}^d$ defined as the L^1-distance between their linear scan transforms. In the convex case, η is the L^1 distance between Radon transforms. One of the main results of this chapter is that for *convex* sets η is topologically equivalent to the Hausdorff metric.

In the final section we obtain analytic properties of the set covariance function C_A of subsets A of $\mathbb{R}^d$, using the metric η. The most important result is continuity of C_A in A, for A in a class of uniformly bounded regular sets A. We also obtain an elementary proof of Lipschitz continuity of d-dimensional volume of a regular set, due to the definition of η.

1. Preliminaries.

1.1 Notation.

Throughout this chapter λ_d denotes d-dimensional Lebesgue measure defined on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. If it is clear what the dimension is, we write λ without the subscript.

Denote by $\mathcal{L}$ the class of all one-dimensional lines in $\mathbb{R}^d$, and by μ the normalised invariant measure on $\mathcal{L}$ (see Santaló, 1976, pp. 28, 200). For a Borel set $S \in \mathcal{B}(\mathbb{R}^d)$, $[S]$ is the set of lines that intersect S: $[S] = \{l \in \mathcal{L} : l \cap S \neq \emptyset\}$. Furthermore, let $\overline{S}$, S° denote topological closure and interior of S respectively and $B(x,r)$ the open ball with radius r centered at x. Finally, $\mathcal{H}^{d-1}$ denotes $d-1$ dimensional Hausdorff measure ("surface area", or in $\mathbb{R}^2$, "length") and $\kappa_d = \frac{2\pi^{d/2}}{d\Gamma(d/2)}$ is the volume of the unit ball in $\mathbb{R}^d$.

1.2 Covariance functions.

Let $S \in \mathcal{B}(\mathbb{R}^d)$ be a *bounded* Borel set.

1.1 DEFINITION The *covariance function* of S is the function $C_S : \mathbb{R}^d \to \mathbb{R}_+$ defined by

$$C_S(y) := \lambda(S \cap T_y S), \quad y \in \mathbb{R}^d.$$

where we write

$$T_y S = S - y = \{s - y : s \in S\}$$

for the translation of S by a vector $y \in \mathbb{R}^d$.

Observe that the covariance function is measurable as a function of y: the function $g(u,w) = 1_S(u)1_S(u+w)$ is clearly measurable and integrable on $\mathbb{R}^{2d}$ so Fubini's theorem guarantees measurability of $C_S(\cdot)$.

1.2 Examples.

1.2.1 For $l \in \mathcal{L}$, $S \in \mathcal{B}(\mathbb{R}^d)$ as above, view $l \cap S$ as a subset of $\mathbb{R}^1$. Then for $t \geq 0$

$$C_{l\cap S}(t) = \lambda_1\left((l \cap S) \cap T_t(l \cap S)\right).$$

1.2.2 If S is as in the previous example and also *convex*, $C_{l\cap S}(t) = (\lambda_1(l \cap S) - t)_+$.

The following properties are immediate consequences of Definition 1.1:

1.3 LEMMA.
(a) $C_S(0) = \lambda(S)$.
(b) C_S *has compact support:* $C_S(y) = 0$, *for all* $y \in \mathbb{R}^d$, *with* $\|y\| \geq$ *diam*(S).
(c) C_S *is symmetric:* $C_S(y) = C_S(-y)$, *for all* $y \in \mathbb{R}^d$.

Rewriting an old result of Borel (1925) in terms of covariance functions, we get

1.4 LEMMA (BOREL'S OVERLAP FORMULA). *Let* $f : \mathbb{R}_+ \to \mathbb{R}$ *be a measurable function. Denote Euclidean length by* $\|\cdot\|$. *Then*

$$\int_S \int_S f(\|u - v\|)\, du\, dv = \int_{\mathbb{R}^d} f(\|w\|)\, C_S(w)\, dw, \tag{1.1}$$

in the sense that whenever one side of (1.1) exists, so does the other in which case they are equal. In particular

$$\int_{\mathbb{R}^d} C_S(y)\, d\lambda(y) = \lambda(S)^2. \tag{1.2}$$

PROOF. Set $w = u - v$. Then

$$\int_S \int_S f(\|u - v\|)\, du\, dv = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_S(u)\, 1_S(u + w)\, f(\|w\|)\, du\, dw.$$

Now

$$1_S(u + w) = 1_{T_w S}(u)$$

so the inner integral above is

$$\begin{aligned} f(\|w\|) \int_{\mathbb{R}^d} 1_S(u)\, 1_{T_w S}(u)\, du &= f(\|w\|)\, \lambda(S \cap T_w S) \\ &= f(\|w\|)\, C_S(w), \end{aligned}$$

yielding the main result. The second formula is the special case $f \equiv 1$. □

Next we give the relation between the distribution function of the distance between two independent uniformly distributed points in a set and the covariance function of that set. This is a direct consequence of the Borel formula (1.1).

1.5 COROLLARY. *Let X, Y be two independent uniformly distributed points in S. Then*

$$\mathbb{P}\{\|X-Y\| \le \rho\} = \frac{1}{\lambda(S)^2}\int_{B(0,\rho)} C_S(y)\,dy, \quad \textit{for } 0 \le \rho \le \operatorname{diam}(S).$$

The following result is also well-known:

1.6 LEMMA. *For measurable $f : \mathbb{R}_+ \to \mathbb{R}$*

$$(1.3) \quad \int_S \int_S f(\|u-v\|)\,du\,dv = \int_{\mathcal{L}} \int_{l\cap S} \int_{l\cap S} |s-t|^{d-1} f(|s-t|)\,ds\,dt\,d\mu(l),$$

in the same sense as Lemma 1.4.

This is a consequence of the Blaschke-Petkantschin formula (see Santaló (1976), eqn. (4.2) p. 46 and eqn. (12.23) p. 201).

1.3 Regular sets.

A regular closed set is one that satisfies $S = \overline{S^\circ}$. From now on we need to consider a more restricted class $\mathcal{V}$ of regular closed subsets of $\mathbb{R}^d$.

Let $\mathcal{D}$ denote the collection of non-empty, open, convex, relatively compact sets. Then the class $\mathcal{C}$ of closures of sets in $\mathcal{D}$

$$\mathcal{C} = \{\overline{D} : D \in \mathcal{D}\}$$

is the class of convex bodies (i.e. convex, compact with non-empty interior). Let

$$\mathcal{E} = \mathcal{D} \cup \mathcal{C}$$

and let $\mathcal{W}$ be the algebra generated by $\mathcal{E}$, i.e. by finite intersections, unions and differences of subsets of $\mathcal{E}$.

1.7 DEFINITION.

$$\mathcal{V} = \{V \in \mathcal{W} : \overline{V^\circ} = V\}.$$

An element of $\mathcal{V}$ will be called a *regular* set.

Observe that the elements of $\mathcal{V}$ are compact and that $\mathcal{H}^{d-1}(\partial S) < \infty$. For details see Appendix A.

1.8 DEFINITION. For a regular set S and $l \in \mathcal{L}$, write $n(l \cap S)$ for the number of components of $l \cap S$, and $\sigma(l \cap S)$ for the length (1-dimensional Lebesgue measure) of $l \cap S$.

The following lemma is a version of a standard result in integral geometry (e.g. Santaló (1976), pp. 29, 31, 234 and Federer (1969), pp. 173, 258, 294). The proof is given in Appendix A.

1.9 LEMMA. *With respect to the usual σ-algebra on $\mathcal{L}$, we have for $S \in \mathcal{V}$*

(a) *$\sigma(l \cap S)$ is a measurable function of $l \in \mathcal{L}$, and*

$$\int_{\mathcal{L}} \sigma(l \cap S)\, d\mu(l) = d\kappa_d \lambda(S). \tag{1.4}$$

(b) *$n(l \cap S)$ is a measurable function of $l \in \mathcal{L}$ and*

$$\int_{\mathcal{L}} n(l \cap S)\, d\mu(l) = \frac{\kappa_{d-1}}{2} \mathcal{H}^{d-1}(\partial S). \tag{1.5}$$

In particular, for μ-almost all lines l, the transect $l \cap S$ is a finite union of bounded line segments.

2. The linear scan transform.

We now define the linear scan transform of a set S in terms of the one-dimensional covariance function of $l \cap S$.

2.1 DEFINITION. The *linear scan transform* of a regular set $S \in \mathcal{V}$ is defined for given $l \in \mathcal{L}$ as the function $G_{l \cap S}(t)$, $t \geq 0$ such that

$$\int_t^{\operatorname{diam}(S)} G_{l\cap S}(x)\, dx = C_{l \cap S}(t), \quad 0 \leq t \leq \operatorname{diam}(S)$$
$$G_{l \cap S}(t) = 0, \qquad \text{for } t \geq \operatorname{diam}(S).$$

That $G_{l \cap S}$ is well-defined for almost all l will be shown below.

The following properties are immediate.

2.2 LEMMA.

(a) *If $l \cap S = \emptyset$, then $C_{l\cap S} \equiv G_{l \cap S} \equiv 0$.*

(b) *For a compact convex set K the intersection $l \cap K$ is either empty or a compact interval of length $\sigma(l \cap K) \geq 0$ in which case $C_{l \cap K}(t) = (\sigma(l \cap K) - t)_+$ and*

$$G_{l \cap K}(t) = \mathbf{1}\{\sigma(l \cap K) > t\} = \begin{cases} 1 & \text{if } \sigma(l \cap K) > t \\ 0 & \text{otherwise} \end{cases}$$

(c) $C_{l \cap S}(t) = G_{l \cap S}(t) = 0$ *for all* $t \geq \operatorname{diam}(S)$.

According to Lemma 1.9 we may assume $l \cap S$ is a finite union of line segments, and compute $C_{l\cap S}, G_{l \cap S}$.

2.3 LEMMA. *Represent $l \cap S$ isometrically as a subset of $\mathbb{R}$*

$$l \cap S = \bigcup_{i=1}^{n} [x_{2i-1}, x_{2i}] \tag{2.1}$$

where $x_1 < x_2 < \cdots < x_{2n} \in \mathbb{R}$ are the coordinates of the endpoints of the line segments (with respect to an arbitrary origin on l). Then for $t \geq 0$

$$C_{l \cap S}(t) = \sum_{k=1}^{2n} \sum_{i=1}^{2n} (-1)^{i+k+1} (x_k - x_i - t)_+ \tag{2.2}$$

$$G_{l \cap S}(t) = \sum_{k=1}^{2n} \sum_{i=1}^{2n} (-1)^{i+k+1} \mathbf{1}\{x_k - x_i > t\}. \tag{2.3}$$

This representation is independent of the choice of the origin on l and of the choice of the orientation on l.

The fact that it is possible by Lemma 1.9 to represent $l \cap S$ for almost all lines as in (2.1) together with (2.3), shows that the linear scan transform is well-defined almost everywhere.

PROOF. First note that for $J = l \cap S$ as in (2.1) and for any Lebesgue integrable $f : \mathbb{R}_+ \to \mathbb{R}$, trivially

$$\int_J f(t)\,dt = \sum_{k=1}^{2n} (-1)^k F(x_k), \tag{2.4}$$

where F is any primitive of f. Now consider

$$\begin{aligned} C_J(t) = C_J(-t) &= \lambda_1(J \cap (J+t) \\ &= \int_J 1_{J+t}(u)\,du. \end{aligned} \tag{2.5}$$

A primitive of 1_{J+t} is

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} 1_{J+t}(u)\,du \\ &= \int_{-\infty}^{x-t} 1_J(s)\,ds \\ &= \int_J 1_{(-\infty, x-t]}(s)\,ds. \end{aligned} \tag{2.6}$$

But $1_{(-\infty,x-t]}$ has primitive

$$F_1(y) = -\int_y^\infty 1_{(-\infty,x-t]}(s)\,ds = -(x-t-y)_+.$$

So by (2.4)

$$\int_J 1_{(-\infty,x-t]}(s)\,ds = \sum_{i=1}^{2n}(-1)^i F_1(x_i) = \sum_{i=1}^{2n}(-1)^{i+1}(x-t-x_i)_+,$$

whence by (2.6) and (2.4)

$$\begin{aligned} C_J(t) &= \sum_{k=1}^{2n}(-1)^k F(x_k) \\ &= \sum_{k=1}^{2n}\sum_{i=1}^{2n}(-1)^{i+k+1}(x_k-x_i-t)_+. \end{aligned}$$

Independence of the choice of the origin on l follows immediately from these representations since they only depend on the *length* of the intervals in $l\cap S$. □

It is an interesting exercise to check that the alternating sum expression for $C_{l\cap S}(0)$ collapses to $\sigma(l\cap S)$. Setting $t=0$ in (2.2) gives

$$C_{l\cap S}(0) = \sum_i\sum_{k>i}(-1)^{i+k+1}(x_k-x_i);$$

the inner sum collapses to $(x_{i+1}-x_i)$ for left endpoints (i odd) and to 0 for right endpoints (i even) so the total is $\sum(x_{2m}-x_{2m-1}) = \sigma(l\cap S)$.

Observe that for $t\geq 0$, (2.3) can be rewritten as

$$G_{l\cap S}(t) = \sum_{i=1}^{2n-1}\sum_{k=i+1}^{2n}(-1)^{i+k+1}1\{x_k-x_i>t\}.$$

3. Identities concerning the linear scan transform.

3.1 Basic relation.

First we establish a link between the covariance function of a set S in $\mathbb{R}^d$ and the linear scan transform.

Denote by x^k the vector x, viewed as an object in a space of dimension k. If it is clear from the context which dimension we are in, the index is sometimes omitted.

3.1 PROPOSITION. *For a set $S \in \mathcal{V}$*

$$C_S(y) = \int_{l_\omega^\perp} \int_t^\infty G_{l_\omega^x \cap S}(s)\, ds\, dx^{d-1}, \quad \textit{for all } y \in \mathbb{R}^d,$$

where $y = t\omega$, $t = \|y\|$ and $\omega \in S^{d-1}$, l_ω is the line through the origin with direction ω, $l_\omega^\perp$ is the orthogonal complement of l_ω and $l_\omega^x = l_\omega + x^{d-1}$.

PROOF. Let $\omega = y/\|y\| \in S^{d-1}$. By definition

$$C_S(t\omega) = \int_{\mathbb{R}^d} 1_S(z^d) 1_S(z^d + t\omega)\, dz^d. \tag{3.1}$$

Furthermore, z^d and $z^d + t\omega$ lie on a line $l_\omega + x^{d-1}$ with direction ω and their coordinates on this line are z^1 and $(z + t\omega)^1 = z^1 + t$ respectively. Hence the right-hand side of (3.1) is

$$\int_{l_\omega^\perp} \int_{\mathbb{R}} 1_{l_\omega^x \cap S}(z^1) 1_{l_\omega^x \cap S}(z^1 + t)\, dz^1\, dx^{d-1} = \int_{l_\omega^\perp} C_{l_\omega^x \cap S}(t)\, dx^{d-1}$$

This proves the Proposition because by definition

$$C_{l \cap S}(t) = \int_t^\infty G_{l \cap S}(s)\, ds.$$

□

As a corollary to the previous proposition, we can write the expression obtained for the interpoint distance in Corollary 1.5 in terms of G.

3.2 Stereological relations.

In this section we first establish a connection with $n(l \cap S)$ and with the so-called k-linc introduced by Miles (1983).

3.2 DEFINITION.(Miles) Let $l \in \mathcal{L}$ such that $l \cap S^\circ \neq \emptyset$. The *$k$-linc* (for "$k$-th order line section of non-convex domain") of a regular set S in $\mathbb{R}^d$, is

$$[\sigma(l \cap S)^k] = \sum_{i=1}^{2n(l \cap S)-1} \sum_{j=i+1}^{2n(l \cap S)} (-1)^{i+j+1} (x_j - x_i)^k,$$

where $x_1, x_2, \ldots$ are the ordered endpoints of intercepted intervals as before.

By Lemma 1.9, $[\sigma(l \cap S)^k]$ is well-defined for almost all lines. The k-linc is of stereological importance, especially when $d = 2$ and $k = 3$ and also when $d = 3$ and $k = 4$ (see Miles (op.cit.) and Jensen and Gundersen (1985)).

3.3 LEMMA. *Let $l \in \mathcal{L}$. For a regular set S we have for μ-almost all l*

(a) $n(l \cap S) = G_{l \cap S}(0)$.

(b) $[\sigma(l \cap S)^k] = k \int_0^\infty t^{k-1} G_{l \cap S}(t)\, dt$ *for* $k \geq 1$.

(c) $[\sigma(l \cap S)^k] = k(k-1) \int_0^\infty t^{k-2} C_{l \cap S}(t)\, dt$ *for* $k \geq 2$.

PROOF. By Lemma 1.9, $n(l \cap S) < \infty$ for almost all lines. For those lines, the k-linc of S is well-defined and the linear scan transform is defined for almost all lines. Throughout the proof we only consider those lines in $\mathcal{L}$ such that $n(l \cap S) < \infty$ *and* such that $G_{l \cap S}$ is well-defined.

To prove (a), set $t = 0$ in (2.3). This gives

$$G_{l \cap S}(0) = \sum_i \sum_k (-1)^{i+k+1} 1\{x_k > x_i\}$$

and the inner sum collapses to 0 for right endpoints (i even) and 1 for left endpoints (i odd) so that the total is $n(l \cap S)$.

For the second statement observe that by Lemma 2.3

$$G_{l \cap S}(t) = \sum_{i=1}^{2n(l \cap S)-1} \sum_{j=i+1}^{2n(l \cap S)} (-1)^{i+j+1} 1\{x_j - x_i > t\}.$$

Hence the right-hand side of (b) is

$$\begin{aligned} &\int_0^\infty kt^{k-1} \sum_{i=1}^{2n(l \cap S)-1} \sum_{j=i+1}^{2n(l \cap S)} (-1)^{i+j+1} 1\{x_j - x_i > t\}\, dt \\ &= \sum_{i=1}^{2n(l \cap S)-1} \sum_{j=i+1}^{2n(l \cap S)} (-1)^{i+j+1} \int_0^{x_j - x_i} kt^{k-1}\, dt \\ &= \sum_{i=1}^{2n(l \cap S)-1} \sum_{j=i+1}^{2n(l \cap S)} (-1)^{i+j+1} (x_j - x_i)^k \\ &= [\sigma(l \cap S)^k]. \end{aligned}$$

Finally, the third claim follows from the previous one by integration by parts. □

3.4 Examples.

3.4.1. If S is convex, as when $n(l \cap S) = 1$

$$\begin{aligned} [\sigma(l \cap S)^k] &= \int_0^\infty kt^{k-1} 1_{[0, \sigma(l \cap S)]}(t)\, dt \\ &= \int_0^{\sigma(l \cap S)} kt^{k-1}\, dt \\ &= \sigma(l \cap S)^k. \end{aligned}$$

3.4.2. If $k = 1$,

$$[\sigma(l \cap S)^1] = \int_0^\infty G_{l\cap S}(t)\, dt = C_{l\cap S}(0) = \sigma(l \cap S),$$

by part (a) of Lemma 1.3.

3.4.3. From Example 3.4.2 we obtain

$$\sigma(l \cap S) = \int_0^\infty G_{l\cap S}(t)\, dt \tag{3.2}$$

for almost all l. Consequently

$$\int_{\mathcal{L}} \int_0^\infty G_{l\cap S}(t)\, dt d\mu(l) = \int_{\mathcal{L}} \sigma(l \cap S) d\mu(l) = d\kappa_d \lambda(S), \tag{3.3}$$

see Lemma 1.9.

3.5 PROPOSITION. *For $f : I\!R_+ \to I\!R$ integrable on compact sets and $S \in \mathcal{V}$*

$$\begin{aligned}
\int_S \int_S f(\|u - v\|)\, du\, dv &= \int_{\mathcal{L}} \int_{I\!R} |t|^{d-1} f(|t|) C_{l\cap S}(t)\, dt\, d\mu(l) \\
&= 2 \int_{\mathcal{L}} \int_0^\infty t^{d-1} f(t) C_{l\cap S}(t)\, dt\, d\mu(l) \\
&= 2 \int_{\mathcal{L}} \int_0^\infty F_{d-1}(t)\, G_{l\cap S}(t)\, dt\, d\mu(l) \qquad (3.4)
\end{aligned}$$

where $F_{d-1}(t) = \int_0^t s^{d-1} f(s)\, ds$.

PROOF. By Lemma 1.6

$$\int_S \int_S f(\|u - v\|)\, du\, dv = \int_{\mathcal{L}} \int_{l\cap S} \int_{l\cap S} |s - t|^{d-1} f(|s - t|)\, dsdt\, d\mu(l).$$

Now we apply Borel's overlap formula (1.1) to the one-dimensional set $l \cap S$ and the function $|s - t|^{d-1} f(|s - t|)$:

$$\int_{\mathcal{L}} \int_{l\cap S} \int_{l\cap S} |s-t|^{d-1} f(|s-t|)\, dsdt\, d\mu(l) = \int_{\mathcal{L}} \int_{I\!R} |w|^{d-1} f(|w|) C_{l\cap S}(w)\, dw\, d\mu(l).$$

By symmetry of the covariance function the integral over w is

$$2 \int_0^\infty w^{d-1} f(w) C_{l\cap S}(w)\, dw.$$

Let F_{d-1} be as defined in the statement of the Proposition. Integration by parts yields

$$\begin{aligned}\int_0^\infty w^{d-1} f(w) C_{l\cap S}(w)\, dw &= \int_0^\infty C_{l\cap S}(w)\, dF_{d-1}(w) \\ &= \int_0^\infty F_{d-1}(w) G_{l\cap S}(w)\, dw,\end{aligned}$$

since $C_{l\cap S}(w) = 0$ for w large enough and $F_{d-1}(0) = 0$. □

From Proposition 3.5 and part (b) of Lemma 3.3, we obtain a generalisation of Crofton's formula, relating chord length and interpoint distance for non-convex sets cf. Santaló (1975), p. 238, (14.25). (See also Miles (1985).)

3.6 COROLLARY. *For $S \in \mathcal{V}$*

$$(3.5)\quad 2\int_{\mathcal{L}} [\sigma(l\cap S)^k]\, d\mu(l) = k(k-1)\int_S\int_S \|u-v\|^{k-d-1}\, dudv, \quad k \ge d-1.$$

PROOF. Take $f(t) = t^{k-d-1}$ in Proposition 3.5. Then

$$\begin{aligned}\int_S\int_S \|u-v\|^{k-d-1}\, dudv &= 2\int_{\mathcal{L}}\int_0^\infty \left(\int_0^t s^{k-2}\, ds\right) C_{l\cap S}(t)\, dt d\mu(l) \\ &= \frac{2}{k-1}\int_{\mathcal{L}}\int_0^\infty t^{k-1} G_{l\cap S}(t)\, dt d\mu(l).\end{aligned}$$

By Lemma 3.3(b) this is

$$\frac{2}{k(k-1)}\int_{\mathcal{L}} [\sigma(l\cap S)^k]\, d\mu(l)$$

which proves the claim. □

3.7 Remark. In the convex case we recover Crofton's formula, since by Example 3.4.2

$$[\sigma(l\cap S^k)] = \sigma(l\cap S)^k \quad \text{for convex } S.$$

3.8 Remark. Taking $k = d+1$ in Corollary 3.6 or $f \equiv 1$ in Proposition 3.5 we get

$$\begin{aligned}\lambda(S)^2 &= \frac{2}{d(d+1)}\int_{\mathcal{L}} [\sigma(l\cap S)^{d+1}]\, d\mu(l) \\ &= \frac{2}{d}\int_{\mathcal{L}}\int_0^\infty t^d G_{l\cap S}(t)\, dt d\mu(l).\end{aligned}$$

Finally, there is a connection with the chord length distribution

$$F_S(x) = \mathbf{P}\left\{l \in \mathcal{L} : \sigma(l \cap S) \leq x\right\},$$

where $\mathbf{P}$ is a properly normalised probability measure on the set of lines that intersect a *convex* set S.
Recall from Lemma 2.2(b) that $1\{\sigma(l \cap S) > x\} = G_{l \cap S}(x)$. Hence

$$\mathbf{P}\left\{\sigma(l \cap S) \leq x\right\} = 1 - \frac{1}{\mu(S)} \int_{\mathcal{L}} G_{l \cap S}(x)\, d\mu(l).$$

Comparing this with results of Waksman and Pohl, we conclude that $\int_{\mathcal{L}} G_{l \cap S}(x)\, d\mu(l)$ is equivalent to a_S, the so-called *associated* function to S. This function was defined by Pohl (1980) and used by Waksman (1985) to partially solve the problem of characterising convex plane polygons by their chord length distributions. (Extended to a slightly more general class of polygons by Cabo (1989).)

3.3. Glance functions.

Waksman (1987) considered a subclass of *open* subsets of $\mathcal{W}$ with C^2 boundaries made up of finitely many arcs on which the curvature does not change sign. Moreover their diameters are bounded by a fixed constant D. For such a set T, he introduced the *glance function.* In our notation, its definition boils down to

$$H_{l \cap T}(t) := \sum_{i=1}^{2n-1} \sum_{j=i+1}^{2n} (-1)^{i+j+1} 1\{x_j - x_i \leq t\}$$

While the linear scan transform G takes into account only those endpoints which are separated by more than a distance t, the glance function only 'sees' those endpoints that are *not* more than t apart.

Our relations $n(l \cap S) = G_{l \cap S}(0)$ and

$$\sigma(l \cap S) = \int_0^{\operatorname{diam}(S)} G_{l \cap S}(t)\, dt$$

are analogues of Waksman's results $n(l \cap S) = H_{l \cap S}(D)$ and

$$\sigma(l \cap S) = n(l \cap S) \cdot D - \int_0^D H_{l \cap S}(t)\, dt.$$

Since the support of $H_{l \cap S}$ is $[0, \operatorname{diam}(S))$ these are equivalent to

$n(l \cap S) = H_{l\cap S}(\mathrm{diam}(S))$ and

$$\sigma(l \cap S) = n(l \cap S) \cdot \mathrm{diam}(S) - \int_0^{\mathrm{diam}(S)} H_{l\cap S}(t)\, dt.$$

Furthermore, the following general relation trivially holds

$$H_{l\cap S}(t) = n(l \cap S) - G_{l\cap S}(t).$$

Hence $G_{l\cap S}$ and $H_{l\cap S}$ can be considered as dual. The reason we prefer to work with $G_{l\cap S}$ is, among others, that it is possible to express well known stereological entities in terms of the linear scan transform directly, whereas the glance function would always involve some constants depending on l and S.

4. Metrics.

4.1 A new metric for sets.

On the class of regular sets (introduced in § 1) we define a 'stereological' metric, following Waksman. It is defined only in terms of the ***linear*** scan transform G.

4.1 DEFINITION. For $S, T \in \mathcal{V}$, let

$$\begin{aligned} \eta(S,T) &= \int_{\mathcal{L}} \|G_{l\cap S} - G_{l\cap T}\|_1 \, d\mu(l) \\ &= \int_{\mathcal{L}} \int_0^\infty |G_{l\cap S}(t) - G_{l\cap T}(t)| \, dt d\mu(l). \end{aligned} \tag{4.1}$$

4.2 Remark. In fact this definition is a minor modification of the metric defined by Waksman (1987) as the L^1 distance between glance functions. Nevertheless the class of sets on which it was defined was different (see also §3.3).

Measurability and integrability of the linear scan transform are proved in Appendix B; this ensures that η is well-defined.

4.3 PROPOSITION. *η is a metric on $\mathcal{V}$.*

PROOF. Since η is defined in terms of the L^1-distance between linear scan transforms, the only property we have to check is that $\eta(S,T) = 0$

implies $S = T$. Suppose $\eta(S,T) = 0$. Observe that

$$\begin{aligned}\int_{\mathcal{L}} |\sigma(l \cap S) - \sigma(l \cap T)|\, d\mu(l) &= \int_{\mathcal{L}} \Big| \int_0^\infty G_{l\cap S}(t)\, dt - \int_0^\infty G_{l\cap T}(t)\, dt \Big|\, d\mu(l) \\ &\le \int_{\mathcal{L}} \int_0^\infty |G_{l\cap S}(t) - G_{l\cap T}(t)|\, dt\, d\mu(l) \\ &= \eta(S,T) \\ &= 0.\end{aligned}$$

Thus

$$\sigma(l \cap S) = \sigma(l \cap T), \qquad \text{for } \mu\text{-almost lines } l.$$

Consequently the Radon transforms of the indicator functions of S and T are equal for almost all lines. By Helgason (1980), Proposition I.7.5, p. 52 this yields equality of these indicators in L^1. Equivalently

$$\lambda(S \triangle T) = 0. \tag{4.2}$$

Suppose there is an $x \in S \setminus T$. Because T is closed, we can find an open ball $B(x,r)$ around x such that

$$\begin{gathered} B(x,r) \cap T = \emptyset \\ (\text{and } B(x,r) \cap S \neq \emptyset). \end{gathered} \tag{4.3}$$

Recall that by regularity of S, $S = \overline{S^\circ}$. Hence, for all $\epsilon > 0$, we can find $y \in S^\circ$ such that $\|x - y\| < \epsilon$.
Take $\epsilon = \frac{1}{2}r$. Then there is a $\delta < \frac{1}{2}r$ such that

$$B(y,\delta) \subset S.$$

Hence

$$B(y,\delta) \subset B(x,r) \implies B(y,\delta) \cap T = \emptyset, \quad \text{by (4.3)}.$$

However this would imply that $\lambda(S \setminus T) \ge \lambda(B(y,\delta)) > 0$, contradicting (4.2). Thus we conclude $S \setminus T = \emptyset$ and by symmetry also $T \setminus S = \emptyset$, yielding the desired equality of S and T. □

4.4 Examples.

4.4.1. If $K_1, K_2 \in \mathcal{V}$ are both convex

$$\eta(K_1, K_2) = \int_{\mathcal{L}} |\sigma(l \cap K_1) - \sigma(l \cap K_2)|\, d\mu(l),$$

the L^1 distance between their Radon transforms. To see this, observe that for all l

$$G_{l\cap K_1}(t) - G_{l\cap K_2}(t) = 1_{[0,\sigma(l\cap K_1)]}(t) - 1_{[0,\sigma(l\cap K_2)]}(t),$$

by convexity (see Lemma 2.2(b)) so that

$$\begin{aligned}\|G_{l\cap K_1} - G_{l\cap K_2}\|_1 &= \int_0^\infty \left|1_{[0,\sigma(l\cap K_1)]}(t) - 1_{[0,\sigma(l\cap K_2)]}(t)\right| dt \\ &= |\sigma(l\cap K_2) - \sigma(l\cap K_1)|.\end{aligned}$$

4.4.2. If K_1, K_2 are convex and $K_1 \subseteq K_2$ then

$$\eta(K_1, K_2) = d\kappa_d\{\lambda(K_2) - \lambda(K_1)\}.$$

This follows from (1.4) together with the previous example since $\sigma(l\cap K_2) - \sigma(l\cap K_1) \geq 0$.

Apart from this special case, it seems quite hard to give an explicit expression for the metric η. For instance, the other relatively simple case of two non-intersecting convex sets needs results related to the Sylvester problem (see Santaló (1976), p. 63–65). On the other hand, a simple upper bound obtains in the general case.

4.5 LEMMA. *Let $S, T \in \mathcal{V}$. Then*

$$\begin{aligned}\eta(S,T) &\leq \int_{\mathcal{L}} \|G_{l\cap S}\|_1\, d\mu(l) + \int_{\mathcal{L}} \|G_{l\cap T}\|_1\, d\mu(l) \\ &\leq \operatorname{diam}(S)\operatorname{length}(\partial S) + \operatorname{diam}(T)\operatorname{length}(\partial T).\end{aligned} \tag{4.4}$$

PROOF. The first inequality simply follows from the triangle inequality for the L^1 norm.
It is also clear that

$$|G_{l\cap S}(t)| \leq G_{l\cap S}(0) = n(l\cap S) \qquad \text{for all } t. \tag{4.5}$$

Thus we get

$$\begin{aligned}&\int_{\mathcal{L}} \|G_{l\cap S}\|_1\, d\mu(l) + \int_{\mathcal{L}} \|G_{l\cap T}\|_1\, d\mu(l) = \\ &\quad= \int_{\mathcal{L}}\int_0^\infty |G_{l\cap S}(t)|\, dt\, d\mu(l) + \int_{\mathcal{L}}\int_0^\infty |G_{l\cap T}(t)|\, dt\, d\mu(l) \\ &\quad\leq \operatorname{diam}(S)\int_{\mathcal{L}} G_{l\cap S}(0) d\mu(l) + \operatorname{diam}(T)\int_{\mathcal{L}} G_{l\cap T}(0) d\mu(l).\end{aligned}$$

This yields the Lemma by (1.5). (See Santaló (1976) p. 31.) □

4.6 Remark. This bound is sharp in the following sense:
Let $x_n \in \mathbb{R}^d$ be such that $\|x_n\| \to \infty$, $n \to \infty$. Then $\eta(S, T_{x_n}T)$ tends to the first bound in (4.4).

PROOF. Let $R_n := T_{x_n}T$.

$$\begin{aligned}
\eta(S, R_n) &= \int_{\mathcal{L}} \|G_{l\cap S} - G_{l\cap R_n}\|_1 \, d\mu(l) \\
&= \int_{[S]\cap[R_n]} \|G_{l\cap S} - G_{l\cap R_n}\|_1 \, d\mu(l) \\
&\quad + \int_{[S]\setminus[R_n]} \|G_{l\cap S} - G_{l\cap R_n}\|_1 \, d\mu(l) \\
&\quad + \int_{[R_n]\setminus[S]} \|G_{l\cap S} - G_{l\cap R_n}\|_1 \, d\mu(l) \\
&= \int_{[S]\cap[R_n]} \|G_{l\cap S} - G_{l\cap R_n}\|_1 \, d\mu(l) \\
&\quad + \int_{[S]\setminus[R_n]} \|G_{l\cap S}\|_1 \, d\mu(l) \\
&\quad + \int_{[R_n]\setminus[S]} \|G_{l\cap R_n}\|_1 \, d\mu(l).
\end{aligned}$$

As n tends to infinity the first integral tends to zero because $\mu([S]\cap[R_n]) \to 0$ and the linear scan transforms are almost everywhere uniformly bounded by the diameters of S and T. For the second integral, noting that $1_{[S]\setminus[R_n]}$ converges pointwise to $1_{[S]}$ and is dominated by the same function, we get by dominated convergence that

$$\int_{[S]\setminus[R_n]} \|G_{l\cap S}\|_1 \, d\mu(l) \to \int_{[S]} \|G_{l\cap S}\|_1 \, d\mu(l).$$

For the third integral observe that by translation invariance of G

$$\int_{[R_n]\setminus[S]} \|G_{l\cap R_n}\|_1 \, d\mu(l) = \int_{[T]\setminus[T_{-x_n}S]} \|G_{l\cap T}\|_1 \, d\mu(l)$$

Then the same argument as above yields

$$\eta(S, R_n) \to \int_{\mathcal{L}} \|G_{l\cap S}\|_1 \, d\mu(l) + \int_{\mathcal{L}} \|G_{l\cap T}\|_1 \, d\mu(l),$$

as n tends to infinity. □

4.2 Connection with the Hausdorff distance.

It is of interest to compare the metric η introduced in the previous section with the well-known Hausdorff distance between sets.

4.8 DEFINITION. The *Hausdorff distance* between two nonempty sets $A, B \subset \mathbb{R}^d$ is

$$\begin{aligned}\mathcal{H}(A,B) &= \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}\\ &= \inf\{r>0 : A\subset B^r \text{ and } B\subset A^r\},\end{aligned}$$

where $B^r = \{x\in\mathbb{R}^d : \inf_{b\in B}\|x-b\|\le r\}$ denotes the parallel set of B.

The following example shows that the two metrics do not generate the same topology (on $\mathcal{V}$).

4.9 Example. Define

$$X_n = B(0,1)\cup B(x,\frac{1}{n}) \qquad \text{where } n\ge 1 \text{ and } \|x\| = 3.$$

In the Hausdorff metric X_n converges to $B(0,1)\cup\{x\}$. However in η it converges to the unit ball:

$$\begin{aligned}&\eta(X_n, B(0,1)) =\\ &= \int_{\mathcal{L}} |\sigma(l\cap X_n) - \sigma(l\cap B(0,1))|\,d\mu(l)\\ &= \int_{[B(0,1)]\setminus[B(x,\frac{1}{n})]} |\sigma(l\cap X_n) - \sigma(l\cap B(0,1))|\,d\mu(l)\\ &+ \int_{[B(x,\frac{1}{n})]\setminus[B(0,1)]} |\sigma(l\cap X_n) - \sigma(l\cap B(0,1))|\,d\mu(l)\\ &+ \int_{[B(0,1)]\cap[B(x,\frac{1}{n})]} |\sigma(l\cap X_n) - \sigma(l\cap B(0,1))|\,d\mu(l).\end{aligned}$$

The first integral is zero because $\sigma(l\cap X_n)$ equals $\sigma(l\cap B(0,1))$ on the domain of integration. For the second integral we have as n tends to infinity

$$\begin{aligned}\int_{[B(x,\frac{1}{n})]\setminus[B(0,1)]} \sigma(l\cap B(x,\frac{1}{n}))\,d\mu(l) &\le \int_{[B(x,\frac{1}{n})]} \sigma(l\cap B(x,\frac{1}{n}))\,d\mu(l)\\ &= \frac{d\kappa_d^2}{n^d}\to 0.\end{aligned}$$

The third integral tends to zero by observing that the integrand is bounded above by 2 and $\mu([B(0,1)] \cap B(x, \frac{1}{n})) \to \mu([B(0,1)] \cup [\{x\}]) = 0$, for n tending to infinity.

However, for *convex* sets with non-empty interior (i.e. elements of $\mathcal{C}$, see § 1.2) the metrics *are* topologically equivalent. The purpose of the remainder of this section is to prove this.

4.10 THEOREM. *The two metrics η and $\mathcal{H}$ are topologically equivalent on the space of convex bodies.*

Since the proof of Theorem 4.10 is rather lengthy, we treat the statement in two separate parts. The first part is a straightforward application of Steiner's formula.

4.11 PROPOSITION. *Given $S \in \mathcal{C}$ and $\epsilon > 0$ there exists $\delta > 0$ such that for all $T \in \mathcal{C}$*

$$\mathcal{H}(S,T) < \delta \Rightarrow \eta(S,T) < \epsilon.$$

PROOF. Suppose $S, T \in \mathcal{C}$ and $\delta > 0$ satisfy $\mathcal{H}(S,T) < \delta$. Then by definition

$$S \subset T^\delta \quad \text{and } T \subset S^\delta. \tag{4.6}$$

Moreover, the convexity of S and T implies the convexity of their parallel sets S^δ and T^δ. The triangle inequality yields

$$\eta(S,T) \leq \eta(S, S^\delta) + \eta(S^\delta, T).$$

Using (4.6) and Example 4.4.2 gives

$$\eta(S, S^\delta) = d\kappa_d \left(\lambda(S^\delta) - \lambda(S)\right)$$
$$\eta(T, S^\delta) = d\kappa_d \left(\lambda(S^\delta) - \lambda(T)\right).$$

Let $W_r(\cdot)$ for $r = 0, \ldots, d$ denote the Minkowski functionals on $\mathcal{C}$ (see e.g. Santaló (1976), p. 217). In particular $W_0(K) = \lambda(K)$. The Steiner formula (see Santaló (1976), p. 220; Federer (1969), p. 271) states that

$$W_r(K^\delta) = \sum_{s=0}^{d-r} \binom{d-r}{s} W_{r+s}(K)\delta^s$$

Applying the case $r = 0$ to $K = S$ yields

$$\lambda(S^\delta) - \lambda(S) = \sum_{s=1}^{d} \binom{d}{s} W_s(S)\delta^s$$

Applying the same to $K = T$

$$\lambda(T^\delta) = \sum_{s=0}^{d} \binom{d}{s} W_s(T)\delta^s.$$

This yields

$$\begin{aligned}
\eta(S,T) &\le \eta(S,S^\delta) + \eta(S^\delta,T) \\
&= d\kappa_d \left(2(\lambda(S^\delta) - \lambda(S)) + \lambda(S) - \lambda(T)\right) \\
&= d\kappa_d \left(2\sum_{s=1}^{d} \binom{d}{s} W_s(S)\delta^s + \lambda(S) - \lambda(T)\right) \\
&\le d\kappa_d \left(2\sum_{s=1}^{d} \binom{d}{s} W_s(S)\delta^s + \lambda(T^\delta) - \lambda(T)\right) \\
&= d\kappa_d \left(2\sum_{s=1}^{d} \binom{d}{s} (W_s(S) + W_s(T))\delta^s\right)
\end{aligned}$$

Using the Steiner formula for general r, the inclusion $T \subset S^\delta$ and the monotonicity of $W_r(\cdot)$

$$W_r(T) \le W_r(S^\delta) = \sum_{k=0}^{d-r} \binom{d-r}{k} W_{r+k}(S)\delta^k.$$

Collecting together we have

$$\eta(S,T) \le d\kappa_d \left(2\sum_{s=1}^{d} \binom{d}{s} W_s(S)\delta^s + \sum_{s=1}^{d} \binom{d}{s} \sum_{k=0}^{d-s} \binom{d-s}{k} W_k(S)\delta^{s+k}\right)$$

This is a polynomial in δ with zero constant term and finite positive coefficients determined by S. The result follows. $\square$

Next we turn to the second part of the proof of Theorem 4.10.

4.12 PROPOSITION. *Let $\{K_n\}$ be a sequence in $\mathcal{C}$ and let $K \in \mathcal{C}$. If $\eta(K_n, K) \to 0$ then also $\mathcal{H}(K_n, K) \to 0$.*

To prove Proposition 4.12 we need the following result, which shall be proved later.

4.13 PROPOSITION. *Let K, K_n be in $\mathcal{C}$ and suppose $\eta(K_n, K) \to 0$. Then $\bigcup_n K_n$ is bounded.*

PROOF OF PROPOSITION 4.12. We assume that K_n tends to K in η, but *not* in $\mathcal{H}$. Then for all $\epsilon > 0$, there is a subsequence, K_{n_i} say, such that

$$\mathcal{H}(K_{n_i}, K) > \epsilon \quad \text{for all } i. \tag{4.7}$$

Since by assumption $\eta(K_n, K) \to 0$, Proposition 4.13 yields that $\bigcup_i K_{n_i}$ is bounded. However it then follows from Blaschke's selection theorem (see Eggleston (1958)) that there is a sub-subsequence, $K_{n_{i_j}}$ say, that *does* converge in $\mathcal{H}$. Suppose its limit is K^*:

$$\mathcal{H}(K_{n_{i_j}}, K^*) \to 0, \quad j \to \infty.$$

But then

$$\eta(K, K^*) \leq \eta(K, K_{n_{i_j}}) + \eta(K_{n_{i_j}}, K^*) \to 0,$$

by the assumption and Proposition 4.11. This implies $\eta(K, K^*) = 0$. Since η is a metric, $K = K^*$, that is

$$K_{n_{i_j}} \xrightarrow{\mathcal{H}} K.$$

This contradicts (4.7), thus proving that convergence in η implies convergence in $\mathcal{H}$ to the same limit. □

In the sequel we consider the inradius and minimal width of a convex set. Their definitions are given below.

4.14 DEFINITION.(see e.g. Eggleston (1958)). Let $K \in \mathcal{C}$.

(*i*) The *inradius* $r(K)$ of S is the supremum of the radii of all balls contained in K.

(*ii*) Consider all pairs of parallel support hyperplanes at K. The *minimal width* $w(K)$ of K is the minimum of the distances between these planes.

4.15 Remark. For a compact set K, $K^{-\epsilon}$ denotes the set of all points that are centers of balls of radius ϵ contained in K:

$$K^{-\epsilon} = \{y : B(y, \epsilon) \subset K\}.$$

$K^{-\epsilon}$ is called the *erosion* (see Serra (1982), p. 39) of K. From the definition it is clear that $r(K) < \epsilon$ is equivalent to $K^{-\epsilon} = \emptyset$.

We now proceed with the proof of Proposition 4.13. It is divided into three Lemmas.

4.16 LEMMA. *Suppose $\eta(K_n, K) \to 0$ and $\bigcup_{n=1}^{\infty} K_n$ is unbounded. Then $r(K_n) \to 0$ as $n \to \infty$.*

PROOF. Assume without loss of generality that the entire sequence $\{K_n\}$ diverges in the sense that there are points $x_n \in K_n$ such that $0 < \|x_n\| \uparrow \infty$. We prove the result by contradiction. Suppose (again without loss of generality) that $r(K_n) > \epsilon$ for *all* n, where $\epsilon > 0$. Thus by the Remark 4.15 $K_n^{-\epsilon}$ is not empty for *all* n.

We claim that $D(K, K_n^{-\epsilon/2}) \to \infty$, where

$$D(S,T) := \sup_{y \in T} \inf_{x \in S} \|x - y\|.$$

For, either $D(K, K_n^{-\epsilon}) \to \infty$ (which implies the claim) or $D(K, K_n^{-\epsilon}) < M$ for all n (without loss of generality). In the latter case for every n we can find $y_n \in K_n^{-\epsilon}$ such that $d(y_n, K) < M$, so that $B(y_n, \epsilon) \subset K^M$. Put $C_n := \mathrm{co}\,(B(y_n, \epsilon) \cup \{x_n\})$. Then C_n is contained in K_n. Enclosing K in the ball $A_n = B(y_n, \rho)$ where $\rho = \epsilon + 2\mathrm{diam}(K) + M$, we have $K_n \setminus K \supseteq C_n \setminus A_n$. Defining $z_n = (x_n + y_n)/2$ simple trigonometry shows that

$$B_n = B(z_n, \epsilon/2) \subset C_n \subset K_n.$$

Hence $z_n \in K_n^{\epsilon/2}$ and

$$\begin{aligned} d(z_n, K) &\geq \|z_n - y_n\| - d(y_n, K) \\ &\geq \|z_n - y_n\| - M \\ &\to \infty. \end{aligned}$$

Thus $D(K, B_n) \to \infty$, proving the claim.

Consequently we can find balls $B(z_n, \epsilon/2) = B_n \subset K_n$ such that $D(K, B_n) \to \infty$. Now

$$\begin{aligned} \eta(K_n, K) &\geq \int_{[B_n]\setminus[K]} \sigma(l \cap B_n)\, d\mu(l) \\ &= \int_{[B_n]} \sigma(l \cap B_n)\, d\mu(l) - \int_{[B_n]\cap[K]} \sigma(l \cap B_n)\, d\mu(l) \\ &\geq d\kappa_d \lambda(B_n) - \epsilon\, \mu([B_n] \cap [K]), \end{aligned}$$

by (1.4) and the fact that $\sigma(l \cap B_n) \leq \epsilon$. If $B = B(x, \mathrm{diam}(K))$ is the circumsphere of K, then we have

$$\mu([B_n] \cap [B]) = c(\frac{1}{2}\mathrm{diam}(K), \epsilon/2, \|z_n - x\|),$$

where $c(r_1, r_2, s)$ is the measure of all lines intersecting two disjoint balls of radii r_1, r_2 with midpoints separated by a distance s. By standard integral geometric arguments it can be shown that for fixed r_1, r_2, $c(r_1, r_2, s) \to 0$ as $s \to \infty$. Hence $\mu([B_n] \cap [B]) \to 0$, i.e.

$$\lim_{n\to\infty} \eta(K_n, K) > \kappa_d(\epsilon/2)^d > 0.$$

This contradiction proves the Lemma. □

4.17 LEMMA. *For all $\epsilon > 0$ and for every compact convex set K*

$$r(K) < \epsilon \quad \textit{implies } w(K) < \frac{\epsilon}{c_d},$$

where c_d is a constant depending only on the dimension d.

This is a consequence of the following inequality (see Eggleston (1958), p. 112). For a compact convex set K

$$r(K) \geq c_d \cdot w(K), \quad \text{where } c_d = \begin{cases} \frac{1}{2} d^{-\frac{1}{2}}, & d \text{ odd} \\ \frac{(d+2)^{\frac{1}{2}}}{2(d+1)}, & d \text{ even.} \end{cases}$$

4.18 LEMMA. *Let $K \in \mathcal{C}$. Then*

$$\liminf_{L\in\mathcal{C}:w(L)<\delta} \eta(K, L) > 0.$$

PROOF. Fix $0 < \alpha < r(K)$. Observe that for all $0 \leq \alpha < r(K)$, $\lambda(K^{-\alpha}) \neq \emptyset$. Let $0 < \eta_0 < \frac{2}{3}\kappa_d\lambda(K^{-\alpha})$. Then

$$\lambda(K^{-\alpha}) > \frac{\eta_0}{\frac{2}{3}\kappa_d}. \tag{4.8}$$

Now we take $\delta_0 = \delta(\eta_0, K) \leq \min(\alpha, \frac{\frac{2}{3}\kappa_d\lambda(K^{-\alpha})-\eta_0}{2\mu([K])})$ and let L be an element of $\mathcal{C}$ with minimal width smaller than δ_0.

Choose a direction, $\theta^{\perp}_{\min}$ say, normal to two parallel support hyperplanes containing L that are a distance $w(L)$ apart. Write W for the region bounded by the two supporting hyperplanes mentioned above and let $\mathcal{L}_\alpha$ be the set of lines l intersecting $K^{-\alpha}$ whose directions $\theta(l)$ make an angle $\sphericalangle(\theta(l), \theta^{\perp}_{\min})$ with $\theta^{\perp}_{\min}$ that lies in the interval $(-\frac{\pi}{3}, \frac{\pi}{3})$:

$$\mathcal{L}_\alpha := \left\{ l \in [K^{-\alpha}] : \sphericalangle(\theta(l), \theta^{\perp}_{\min}) \in (-\frac{\pi}{3}, \frac{\pi}{3}) \right\}.$$

Let $l \in \mathcal{L}_\alpha$. Then by the assumption on $\sphericalangle(\theta(l), \theta_{\min}^{\perp})$

$$\sigma(l \cap W) = \frac{w(L)}{\cos(\sphericalangle(\theta, \theta_{\min}^{\perp}))} < 2w(L),$$

where we use the fact that $\cos\phi > \frac{1}{2}$ if $\phi \in (-\frac{\pi}{3}, \frac{\pi}{3})$. Thus also

$$\sigma(l \cap L) < 2w(L). \tag{4.9}$$

Furthermore, observe that for lines intersecting $K^{-\alpha}$,

$$\sigma(l \cap K) \geq 2\alpha. \tag{4.10}$$

Hence by (4.9) and (4.10)

$$\sigma(l \cap K) \geq 2\alpha > 2\delta_0 > 2w(L) > \sigma(l \cap L) \quad \text{for } l \in \mathcal{L}_\alpha.$$

Consequently

$$\begin{aligned}\eta(K, L) &\geq \int_{\mathcal{L}_\alpha} |\sigma(l \cap K) - \sigma(l \cap L)| \, d\mu(l) \\ &= \int_{\mathcal{L}_\alpha} \sigma(l \cap K) \, d\mu(l) - \int_{\mathcal{L}_\alpha} \sigma(l \cap L) \, d\mu(l).\end{aligned}$$

By part (a) of Lemma 1.9 and the definition of $\mathcal{L}_\alpha$

$$\int_{\mathcal{L}_\alpha} \sigma(l \cap K) \, d\mu(l) \geq \int_{\mathcal{L}_\alpha} \sigma(l \cap K^{-\alpha}) \, d\mu(l) = \frac{2}{3}\kappa_d \lambda(K^{-\alpha}).$$

For the second integral we have by (4.9)

$$\begin{aligned}\int_{\mathcal{L}_\alpha} \sigma(l \cap L) \, d\mu(l) &< 2w(L) \, \mu(\mathcal{L}_\alpha) \\ &\leq 2w(L) \, \mu([K]).\end{aligned}$$

Summarising

$$\begin{aligned}\eta(K, L) &\geq \frac{2}{3}\kappa_d \lambda(K^{-\alpha}) - 2w(L)\mu([K]) \\ &> \frac{2}{3}\kappa_d \lambda(K^{-\alpha}) - 2\delta_0 \mu([K]) \quad \text{by assumption} \\ &> \eta_0 > 0, \qquad \text{by (4.8)}.\end{aligned}$$

This implies $\inf_{w(L)<\delta} \eta(K, L) > 0$ for $\delta < \delta_0$ arbitrarily small and proves Lemma 4.18. □

To complete the proof of Proposition 4.13, suppose $\eta(K_n, K) \to 0$ and suppose to the contrary that $\cup_{n=1}^{\infty} K_n$ is unbounded. By Lemma 4.16 the inradius of K_n tends to 0. Then by Lemma 4.17, the same is true for the minimal width of K_n. But by Lemma 4.18, $\eta(K_n, K) > 0$. This contradiction implies that $\cup_{n=1}^{\infty} K_n$ is bounded, proving the Proposition.

5. Continuity results.

5.1 Volume.

5.1 LEMMA. *The mapping $S \longmapsto \lambda(S)$ is Lipschitz-continuous with constant 1 on $(\mathcal{V}, \eta)$.*

PROOF. Using Proposition 3.1, expressing the covariance function in terms of G

$$\begin{aligned}|\lambda(S) - \lambda(T)| &= |C_S(0) - C_T(0)| \\ &\leq \int_{\mathcal{L}} \int_0^\infty |G_{l \cap S}(t) - G_{l \cap T}(t)|\, dt\, d\mu(l) \\ &= \eta(S, T).\end{aligned}$$

5.2 The covariance function.

As a corollary to the equivalence of η and $\mathcal{H}$ on the collection of convex bodies we obtain pointwise convergence of an η (or $\mathcal{H}$)-convergent sequence in $\mathcal{C}$.

5.2 COROLLARY. *Let K, $K_n \in \mathcal{C}$ and suppose $\mathcal{H}(K_n, K) \to 0$. Then*

$$C_{K_n}(y) \to C_K(y) \quad \textit{pointwise.}$$

PROOF. Fix $y \in \mathbb{R}^d$ and write $C_K(y) = \int_{\mathbb{R}^d} 1_{K_n}(x)\, 1_{T_y K_n}(x)\, dx$. The result will follow by applying dominated convergence to $1_{K_n}(x)\, 1_{T_y K_n}(x)$. Since $\mathcal{H}(K_n, K) \to 0$

$$K_n \subset K^{\epsilon_n}, \quad \text{where } \epsilon_n \downarrow 0 \text{ as } n \to \infty.$$

But then by convexity, we also have

$$K^{-\epsilon_n} \subset K_n.$$

(See Matheron (1975).) Hence

$$1_{K^{-\epsilon_n}}(x) \leq 1_{K_n}(x) \leq 1_{K^{\epsilon_n}}(x), \quad \forall x.$$

Since the two bounds converge to $1_K(x)$, $\forall x$ as $n \to \infty$, the same is true for $1_{K_n}(x)$. Thus

$$1_{K_n}(x)\, 1_{T_y K_n}(x) \to 1_K(x)\, 1_{T_y K}(x) \text{ pointwise.}$$

Moreover the functions at the left hand-side are bounded above by 1 and have compact support. Hence dominated convergence yields

$$C_K(y) = \int_{\mathbb{R}^d} 1_{K_n}(x)\, 1_{T_y K_n}(x)\, dx \to \int_{\mathbb{R}^d} 1_K(x)\, 1_{T_y K}(x)\, dx = C_K(y).$$

□

The representation of the covariance function in terms of the function G also enables us to prove continuity results of the covariance function with respect to the metric η. As a first step in that direction the following lemma proves useful.

5.3 LEMMA. *For $r > 0$ and $S, T \in \mathcal{V}$*

$$\int_{\|y\|=r} |C_S(y) - C_T(y)|\, d\omega_r \leq \eta(S,T),$$

where ω_r is the spherical measure on a ball with radius r.

PROOF. By Proposition 3.1

$$\begin{aligned} |C_S(y) - C_T(y)| &= |\int_{\mathcal{L}_y} \int_{\|y\|}^{\infty} G_{l\cap S}(t) - G_{l\cap T}(t)\, dt\, d\mu_y(l)| \\ &\leq \int_{\mathcal{L}_y} \int_{\|y\|}^{\infty} |G_{l\cap S}(t) - G_{l\cap T}(t)|\, dt\, d\mu_y(l) \\ &\leq \int_{\mathcal{L}_y} \|G_{l\cap S} - G_{l\cap T}\|_1\, d\mu_y(l). \end{aligned}$$

Integrating this inequality over all directions yields the Lemma. □

We now have enough tools to prove Lipschitz continuity of the covariance function, for sets bounded by a fixed diameter. Denote by $\mathcal{V}(M)$ the subclass of $\mathcal{V}$ consisting of all sets with diameter bounded by M; i.e.

$$\mathcal{V}(M) := \{S \in \mathcal{V} : \operatorname{diam}(S) \leq M\}.$$

5.4 THEOREM. *For all $M > 0$ the mapping $S \longmapsto C_S$ from $(\mathcal{V}(M), \eta)$ into $L^1(\mathbb{R}^d)$ is Lipschitz continuous with constant $\frac{1}{2}M^2$:*

$$\|C_S - C_T\|_1 \leq \frac{1}{2} M^2 \eta(S,T) \qquad \textit{for all } S, T \in \mathcal{V}(M).$$

PROOF. Let $M > 0$ and $S, T \in \mathcal{V}(M)$.
Transforming to polar coordinates we obtain

$$\begin{aligned}
\|C_S - C_T\|_1 &= \int_{\mathbb{R}^d} |C_S(y) - C_T(y)|\, d\lambda(y) \\
&= \int_0^\infty r \int_{\|y\|=r} \big|C_S(y) - C_T(y)\big|\, d\omega_r\, dr \\
&= \int_0^M r \int_{\|y\|=r} |C_S(y) - C_T(y)|\, d\omega_r\, dr \\
&\leq \eta(S,T) \int_0^M r\, dr \qquad \text{by Lemma 5.3} \\
&= \frac{1}{2} M^2\, \eta(S,T).
\end{aligned}$$

This proves the Theorem. □

Lipschitz continuity of the function $C_K : \mathbb{R}^d \to \mathbb{R}_+$ for a fixed convex body has been proved by Matheron (1986).

Recall that the breadth function $b_K : S^{d-1} \to \mathbb{R}$ is defined as the $d-1$-dimensional volume of the projection of K onto a hyperplane orthogonal to the direction u. For $K \in \mathcal{C}$, b_K is a bounded function.

5.5 THEOREM. (Matheron)
The mapping $y \longmapsto C_K(y)$ from $\mathbb{R}^d$ to $\mathbb{R}_+$ is Lipschitz continuous for fixed $K \in \mathcal{C}$:

$$|C_K(y) - C_K(x)| \leq 2\, b\, \|x - y\|,$$

where b is the supremum over S^{d-1} of the breadth function of K.

5.6 COROLLARY. *Let $K_n, K \in \mathcal{C}$ and suppose $\eta(K_n, K) \to 0$. Then*

$$C_{K_n} \to C_K \quad \text{uniformly.}$$

PROOF. This is an application of Dini's Theorem:
By Corollary 5.2

$$C_{K_n}(y) \to C_K(y).$$

As in the proof of Corollary 5.2 there is a sequence $\epsilon_n \downarrow 0$ such that

$$K^{-\epsilon_n} \subset K_n \subset K^{\epsilon_n} \quad \text{by convexity.}$$

This implies

$$C_{K^{-\epsilon_n}} \leq C_{K_n} \leq C_{K^{\epsilon_n}}. \tag{5.1}$$

Observe that the sequences $\{C_{K^{-\epsilon_n}}\}_{n=1}^{\infty}$ and $\{C_{K^{\epsilon_n}}\}_{n=1}^{\infty}$ are both monotone.
Since $\mathcal{H}(K^{\epsilon_n}, K) = \epsilon_n \downarrow 0$, Theorem 4.10 and Corollary 5.2 yield

$$C_{K^{\epsilon_n}}(y) \downarrow C_K(y), \quad \text{for } y \in \mathbb{R}^d.$$

Moreover since $K^{-\epsilon_n} \uparrow K$ then $\mathcal{H}(K^{-\epsilon_n}, \overline{K}) \to 0$ because $\overline{K}$ is compact (see Matheron (1975), Cor. 3, p. 13). This yields $\mathcal{H}(K^{-\epsilon_n}, K) \to 0$ by compactness of $K (\subset \mathbb{R}^d)$. Thus also

$$C_{K^{-\epsilon_n}} \uparrow C_K(y).$$

By Theorem 5.5, all these functions are continuous. Since they all have compact supports, Dini's Theorem asserts that $C_{K^{\epsilon_n}}$ and $C_{K^{-\epsilon_n}}$ converge to C_K *uniformly*. The triangle inequality and other standard arguments together with (5.1) yield the claim. □

Appendix A. (Proof of Lemma 1.9)

To prove Lemma 1.9, recall the following concepts. As before $\mathcal{H}^m$ denotes m-dimensional Hausdorff measure (see Simon (1983), p. 6).

A.1 DEFINITION. Let $E \subset \mathbb{R}^d$. Then E is *$(\mathcal{H}^m, m)$-rectifiable* if $\mathcal{H}^m(E) < \infty$ and there exists a set F containing $\mathcal{H}^m$-almost all of E such that $F = \cup_{n=1}^{\infty} F_n$, where each F_n is the image of a bounded subset of $\mathbb{R}^m$ under a Lipschitz map.
(See Federer (1969) p. 251, 252 or Simon (1983).)

Federer proved that for an $(\mathcal{H}^m, m)$-rectifiable set E

$$\mathcal{H}^m(E) = c(m,d) \int_{\mathcal{L}} \mathcal{H}^{m+1-d}(l \cap E)\, d\mu(l), \tag{A.1}$$

for a certain constant $c(m, d)$.

It is well known that the boundary of a bounded convex set with non-empty interior in $\mathbb{R}^d$ is $(\mathcal{H}^{d-1}, d-1)$-rectifiable. The following result is easy to prove.

A.2 LEMMA. *Every $W \in \mathcal{W}$ (see section 1.2) can be represented (not uniquely) as a disjoint union*

$$W = \bigcup_{j=1}^{m} \left((D_j \cap C_j) \setminus (\cup_{k=1}^{m_j} E_{jk}) \right),$$

where $D_j \in \mathcal{D}$, $C_j \in \mathcal{C}$, $E_{jk} \in \mathcal{E}$.

A.3 COROLLARY. *For the boundary of a set $W \in \mathcal{W}$*

$$\partial W \subset \bigcup_{j=1}^{m} (\partial D_j) \cup (\partial C_j) \cup \bigcup_{k=1}^{m_j} (\partial E_{jk}).$$

Consequently there exist $K_i \in \mathcal{C}$

$$\partial W \subset \bigcup_{i=1}^{n} \partial K_i.$$

In particular, ∂W is $(\mathcal{H}^{d-1}, d-1)$-rectifiable.

PROOF OF LEMMA 1.9.

Let $\mathfrak{A} := \{l \in \mathcal{L} : n(l \cap S) \neq \frac{1}{2}\mathcal{H}^0(\partial S \cap l)\}$. Since $S \in \mathcal{V}$, ∂S is $(\mathcal{H}^{d-1}, d-1)$-rectifiable, hence we can apply (A.1) with $m = d-1$ and $m+1-d = 0$ yielding

$$\int_{\mathcal{L}} \mathcal{H}^0(\partial S \cap l)\, d\mu(l) = c(d-1,d)\, \mathcal{H}^{d-1}(\partial S).$$

So we have to prove that $\mu(\mathfrak{A}) = 0$.

Let $\{K_i\}_i = \{D_j\}_j \cup \{C_j\}_j \cup \{E_{jk}\}_{j,k}$ be the convex sets featuring in the representation of S at Lemma A.2. Observe that

$$2n(l \cap S) \leq \mathcal{H}^0(l \cap S) + \sum_i 1_{\{l \cap K_i \supset \partial K_i\}}$$

since any endpoint of an interval of $l \cap S$ belongs to ∂S.

Fix $0 < \epsilon < \min_i r(K_i)$. Consider an inner approximation S^- to S obtained by replacing each D_j by $D_j^{-\epsilon}$, each C_j by $C_j^{-\epsilon}$, and E_{jk} by E_{jk}^{ϵ} in the representation of S. Write $\{K_i^-\}_i$ for the replacements of the K_i.

Observing that for arbitrary sets $A \subseteq B$

$$1_{[B]} = 1_{[A]} + 1_{[B]\setminus[A]}$$

and applying inclusion-exclusion we obtain

$$n(l \cap S) \geq n(l \cap S^-) - \sum_i 1_{[K_i]\Delta[K_i^-]}(l).$$

Additionally,

$$n(l \cap S^-) \geq \frac{1}{2}\mathcal{H}^0(l \cap S) - \sum_i 1_{[K_i]\Delta[K_i^-]}(l)$$

because intervals of $l \cap S^-$ do not contain points of $l \cap \partial S$, and with each successive pair x_k, x_{k+1} of points of $l \cap \partial S$ we can associate at least one set K_i such that $l \cap K_i \subseteq [x_k, x_{k+1}]$ with the K_i's being disjoint for different k and satisfying either $l \cap [x_k, x_{k+1}] \neq \emptyset$ or $l \cap K_i^- = \emptyset$.

Combining these three inequalities yields

$$\frac{1}{2}\mathcal{H}^0(l \cap S) - 2\sum_i 1_{[K_i]\Delta[K_i^-]}(l) \leq n(l \cap S) \leq \frac{1}{2}\mathcal{H}^0(l \cap S) + \frac{1}{2}\sum_i 1_{\{l \cap K_i \supset \partial K_i\}}.$$

However $\mu([K_i]\Delta[K_i^-]) \downarrow 0$ as $\epsilon \to 0$ and it is well known that $\mu(\{l : l \subset \partial K\}) = 0$ for any $K \in \mathcal{C}$. Hence $n(l \cap S) = \frac{1}{2}\mathcal{H}^0(l \cap S)$ almost everywhere.

Appendix B.

B.1 LEMMA. *The function $G_{l\cap S}$ is (measurable and) integrable simultaneously in l and t, for every $S \in \mathcal{V}$.*

PROOF. We first prove measurability of the transect covariance function. This is done by applying the coarea formula (Federer (1969),3.2.22) to the functions and sets defined below.

First let $\mathbb{V} := \mathbb{R}^d \times S^{d-1}$. Define

$$\begin{aligned} f :& \mathbb{V} \times \mathbb{R} \to \mathbb{R}^d \\ &((x,u),t) \mapsto x + tu, \end{aligned}$$

that is, f maps (x,u,t) onto a point at distance t from x, lying on the line with orientation u through x. Next let

$$\begin{aligned} g :& \mathbb{V} \times \mathbb{R} \to \mathbb{R}^d \times \mathbb{R}^d \\ &((x,u),t) \mapsto (x, f(x,u,t)). \end{aligned}$$

For any measurable set $A \subset \mathbb{R}^d$ with finite Lebesgue measure, the set

$$A^* := g^{-1}(A \times A) = \{(x,u,t) : x \in A,\ f(x,u,t) \in A\}$$

is clearly measurable. Next consider

$$\begin{aligned} h :& \mathbb{V} \to \mathcal{L} \\ &(x,u) \mapsto \{x + au : a \in \mathbb{R}\}. \end{aligned}$$

Identifying $\mathcal{L}$ as usual (Santaló (1976)) with the cylinder $\mathbb{R} \times [0,\pi)$ it is readily seen that h is Lipschitz. Finally we define

$$\begin{aligned} i :& \mathbb{V} \times \mathbb{R} \to \mathcal{L} \times \mathbb{R} \\ &((x,u),t) \mapsto (h(x,u),t). \end{aligned}$$

Then the coarea formula implies that

$$\begin{aligned} s(l,t) :&= \int_{i^{-1}(l,t)} 1_{A^*}((x,u),t)\, d\mathcal{H}^1(x,u) \\ &= 2\, C_{l\cap A}(t) \end{aligned}$$

is measurable in (l,t). Now it is easy to prove Lemma 4.2 for convex bodies. For $K \in \mathcal{C}$

$$G_{l\cap K}(t) = 1\{C_{l\cap K}(t) > 0\}$$

hence G is measurable in (l,t) and integrable since it has compact support. The result for regular sets $A \in \mathcal{V}$ now follows using similar arguments applied to the representation in Lemma A.2.

CHAPTER IV

ESTIMATION OF MEAN PARTICLE VOLUME USING THE SET COVARIANCE FUNCTION

Introduction.

A classical integral geometric result of Crofton and Hadwiger relates the squared volume of a three-dimensional convex body to the fourth moment of its chord length. This has important applications in stereology, to the problem of drawing statistical inferences about a population of three-dimensional objects or 'particles' from information obtained on random two-dimensional plane sections or one-dimensional linear probes of the population. Specifically the result provides an unbiased estimator of the volume-weighted mean volume $\bar{v}$ of the particle population, Haas et al. (1967a,b).

The assumption of convexity is too restrictive for many applications. Miles (1983b, 1985) and Jensen & Gundersen (1983, 1985) have generalised the Crofton-Hadwiger identity to non-convex sets with nonempty interior, yielding estimators of $\bar{v}$ for populations of particles of very general shape. The estimators use information from one-dimensional samples, such as random linear probes and point-sampled intercepts, typically obtained by subsampling a two-dimensional plane section.

A practical drawback of the line intercepts method is the very high sample variability, especially when the particles are elongated rather than approximately spherical. It would be of great interest to develop estimators of $\bar{v}$ which efficiently exploit all the information in a two-dimensional plane section. Miles (1985), p. 123 observed that

> "there is a two-fold source of error — through choice of plane sections, and choice of line sections within plane sections. The latter may be (...) eliminated by integrating (the estimator over all test lines ...), a task theoretically within the capabilities of automatic image analysis."

The aim of the present chapter is to follow this suggestion. We point out that there already exists a generalisation of the abovementioned integralgeometric results to r-dimensional plane sections of an n-dimensional regular compact set $A \subset \mathbb{R}^n$. The k-th moment of distance between two points in A, $J_k^n(A)$ say, satisfies a reproductive or section formula analogous to those holding for the quermass integrals. In case $k = 0$ this yields an identity relating the squared volume of A to the integral of $J_{n-r}^r(A \cap F)$ for all r-dimensional plane sections $A \cap F$ of A. These results are subsumed in earlier work of Miles (1979). Here we also prove counterparts for the case of 'vertical sections'.

Secondly we relate $J_k^n(A)$ to the set covariance function of A. Similar statements were obtained by Borel (1925) and Serra (1982). In applications, the set covariance function can be easily computed for any binary image, so we are able to implement an estimator of $\bar{v}$ based on two-dimensional plane sections of a particle population. We derive explicit estimators for $\bar{v}$ under a variety of sampling regimes, both design-based (area-weighted random sampling and systematic sampling) and model-based (stationary germ-grain models).

Thirdly, echoing the argument of Miles quoted above, we show that these estimators have smaller variance than the estimators based on test lines, by applying the Rao-Blackwell theorem (Baddeley & Cruz-Orive (1993)).

Finally we test the method on some real examples, assess the gain in efficiency with respect to test line methods, and make some proposals for data modelling and variance estimation.

1. Integral geometry of distance moments.

1.1 Definitions.

Fix a compact set $A \subset \mathbb{R}^n$ which is assumed to be *regular closed*, $\overline{A^\circ} = A$, and for brevity call this *regular compact*.

1.1 DEFINITION. For integer $k \geq 1 - n$ define the kth moment of interpoint distance

$$(1.1) \qquad J_k^n(A) = \int_A \int_A ||u - v||^k \, du \, dv$$

where the integrals are with respect to Lebesgue measure λ_n.

In particular

$$J_0^n(A) = \{\lambda_n(A)\}^2 .$$

For the case where A is a convex body, a variety of integralgeometric results is stated in Santaló (1976), pp. 46-49, 237–238. Additionally, Satz I of Carleman (1919) states that among all (not necessarily convex) $A \subset \mathbb{R}^2$ with fixed area $\lambda_2(A)$, the value of $J_k^2(A)$ is maximised when A is a disc.

In one dimension, if A is convex (i.e. a line segment) then trivially for $k \geq 0$

$$J_k^1(A) = \frac{2}{(k+1)(k+2)} \{\lambda_1(A)\}^{k+2} \tag{1.2}$$

while if A is a finite union of line segments

$$A = \bigcup_{i=1}^{N} [x_{2i}, x_{2i+1}]$$

with $x_i < x_{i+1}$ then (cf. Cabo and Baddeley (1993))

$$J_k^1(A) = \frac{2}{(k+1)(k+2)} \sum\sum_{i<j} (-1)^{i+j+1} (x_j - x_i)^{k+2}. \tag{1.3}$$

The following definitions are standard.

1.2 Definition. For $0 < r < n$ let $\mathcal{F}(n,r)$ be the space of r-flats (r-dimensional affine planes) in $\mathbb{R}^n$, and $\mu_{n,r}$ the standard invariant measure on $\mathcal{F}(n,r)$, i.e. the measure invariant under Euclidean motions of $\mathbb{R}^n$ defined in Santaló (1976), (12.18), p. 200. Let $\mathcal{G}(n,r)$ be the space of r-dimensional linear vector subspaces (i.e. through the origin) in $\mathbb{R}^n$, and $\gamma_{n,r}$ the standard rotation-invariant measure on $\mathcal{G}(n,r)$.

The total mass of $\mathcal{G}(n,r)$ is (Santaló (1976), (12.35), p. 203)

$$c_{n,r} = \gamma_{n,r}(\mathcal{G}(n,r)) = \binom{n}{r} \frac{\kappa_n \kappa_{n-1} \cdots \kappa_{n-r+1}}{\kappa_r \kappa_{r-1} \cdots \kappa_1}$$

where κ_n is the volume of the n-dimensional unit ball,

$$\kappa_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}.$$

Useful values are $c_{2,1} = \pi$, $c_{3,2} = c_{3,1} = 2\pi$. For convenience, define $c_{n,0} = 1$ for all n.

1.2 Reproductive formulae.

First we consider line transects. Define for $k \geq 0$

$$I_k^n(A) = \int_{\mathcal{F}(n,1)} J_k^1(\ell \cap A)\, d\mu_{n,1}(\ell). \tag{1.4}$$

where for a given r-flat F, we can define $J_k^r(B)$ for subsets $B \subset F$ naturally by identifying F with $\mathbb{R}^r$ by any isometry. Measurability of the integrand was checked in Cabo and Baddeley (1993) (see Chapter III).

If A is convex then by (1.2), $I_k^n(A)$ is proportional to the $(k+2)$nd moment of the chord length $\lambda_1(\ell \cap A)$ over all lines ℓ intersecting A. This is the much-studied *Sehnenpotenzintegral* (see Blaschke (1949, §§8, 32), Hadwiger (1950), Santaló (1976), pp. 46–49, 237–238).

For general A, the integrand $J_k^1(\ell \cap A)$ can be interpreted using (1.3) as an alternating sum of $(k+2)$nd powers of distances between endpoints of intervals of $\ell \cap A$, whenever this consists of finitely many disjoint intervals (we have shown in (1993) that this holds for a.e. ℓ when A is regular compact).

1.3 PROPOSITION. *For $k \geq 0$*

$$I_k^n(A) = J_{k+1-n}^n(A).$$

In particular $I_{n-1}^n(A) = \{\lambda_n(A)\}^2$.

For convex sets, similar results are stated in Santaló (loc. cit.). The identity $I_{n-1}^n(A) = \{\lambda_n(A)\}^2$ for convex A, i.e.

$$\int_{\mathcal{F}n1} \{\lambda_1(\ell \cap A)\}^{n+1} \, d\mu_{n,1}(\ell) = \frac{n(n+1)}{2} \{\lambda_n(A)\}^2$$

was discovered for $n = 2$ by Crofton (1885), for $n = 3$ by Herglotz (Blaschke (1949), §32, p. 76) and for general n by Hadwiger (1950). The general result for not-necessarily-convex A was derived independently by Miles (1983b) and Jensen & Gundersen (1983,1985). Miles dubbed the alternating sum in (1.3) the $(k+2)$-*linc*.

PROOF. Change the variables of integration in (1.1) by associating with each pair of distinct points $u, v \in \mathbb{R}^n$ the line $\ell = \ell(u,v)$ containing them, and mapping $(u,v) \in \mathbb{R}^n \times \mathbb{R}^n$ to (ℓ, u, v) where $\ell \in \mathcal{F}(n,1)$ and $u, v \in \ell$. By a special case of the Blaschke-Petkantschin formula (Santaló (1976),(12.23), p. 20) the inverse Jacobian of this mapping is $||u-v||^{n-1}$ and we obtain

$$J_k^n(A) = \int_{\mathcal{F}(n,1)} \int_{\ell \cap A} \int_{\ell \cap A} ||u-v||^{k+n-1} \, du \, dv \, d\mu_{n,1}(\ell)$$

i.e. this is $I_{n+k-1}^n(A)$. □

Our key result generalises this to r-dimensional plane sections.

1.4 PROPOSITION. *For compact $A \subset \mathbb{R}^n$ and integers r, k with $1 \leq r < n$, $k \geq n - r$*

$$\int_{\mathcal{F}(n,r)} J_k^r(A \cap F) \, d\mu_{n,r}(F) = c_{n-1,r-1} J_{k+r-n}^n(A). \tag{1.5}$$

In particular

$$\{\lambda_n(A)\}^2 = \frac{1}{c_{n-1,r-1}} \int_{\mathcal{F}(n,r)} J^r_{n-r}(A \cap F)\, d\mu_{n,r}(F). \tag{1.6}$$

These identities are subsumed in a result of Miles ((1979), eq. (16)). A closely related result appears to have been stated in Serra (1982), p. 347.

For the proof we need the following extra definitions.

1.5 Definition. For a given r-flat F and for $1 \le s < r$, write $\mathcal{F}(F,s)$ for the space of s-flats contained in F, and $\mu_{F,s}$ for the standard invariant measure, so that $(\mathcal{F}(F,s), \mu_{F,s})$ is isomorphic to $(\mathcal{F}(r,s), \mu_{r,s})$.

For $r < t < n$ let $\mathcal{F}(n, t[F])$ be the space of all t-flats in $\mathbb{R}^n$ which *contain* a given r-flat F. This is isomorphic to $\mathcal{G}(n-r, t-r)$; let $\gamma_{n,t[F]}$ be the standard rotation-invariant measure on $\mathcal{F}(n, t[F])$ isomorphic to $\gamma_{n-r,t-r}$ (Santaló (1976), p. 202).

Proof of Proposition 1.4. If $r = 1$ then this reduces to Proposition 1.3. For $r > 1$,

$$\begin{aligned}\int_{\mathcal{F}(n,r)} J^r_k(A \cap F)\, d\mu_{n,r}(F) &= \int_{\mathcal{F}(n,r)} I^r_{k+r-1}(A \cap F)\, d\mu_{n,r}(F) \\ &= \int_{\mathcal{F}(n,r)} \int_{\mathcal{F}(F,1)} J^1_{k+r-1}(\ell \cap A)\, d\mu_{F,1}(\ell)\, d\mu_{n,r}(F)\end{aligned}$$

using Proposition 1.3 and (1.4). Change the order of integration using another result of Petkantschin (Santaló (1976), (12.52) p. 207) to obtain

$$\int_{\mathcal{F}(n,1)} \int_{\mathcal{F}(n,r[\ell])} J^1_{k+r-1}(\ell \cap A)\, d\gamma_{n,r[\ell]}(F)\, d\mu_{n,1}(\ell).$$

Now the integrand does not depend on the containing r-flat F, so the last expression becomes

$$c_{n-1,r-1} \int_{\mathcal{F}(n,1)} J^1_{k+r-1}(\ell \cap A)\, d\mu_{n,1}(\ell).$$

The integral is $I^n_{k+r-1}(A)$ by definition. Applying Proposition 1.3 yields (1.5).

1.3 Relation to set covariance function.

1.6 DEFINITION. (Matérn (1960, 1986)) The *set covariance function* of a bounded Borel set $A \subset \mathbb{R}^n$ is

$$C_A(x) = \lambda_n(A \cap T_x A), \quad x \in \mathbb{R}^n$$

where $T_x A = \{a + x : a \in A\}$ is the translation of A by vector x.

Elementary properties are that $C_A(0) = \lambda_n(A)$, and C_A is symmetric, $C_A(-x) = C_A(x)$. See also Serra (1982).

1.7 PROPOSITION. *For compact $A \subset \mathbb{R}^n$ and integer $k \geq 1 - n$*

$$J_k^n(A) = \int_{\mathbb{R}^n} ||x||^k \, C_A(x) \, dx;$$

in particular

$$\int_{\mathbb{R}^n} C_A(x) \, dx = \{\lambda_n(A)\}^2.$$

This is proved by the simple change of variables $(x, y) \mapsto (x, y - x)$ ('Borel's overlap method' (1925)) yielding

$$\begin{aligned}
\int_A \int_A ||x - y||^k \, dx \, dy &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} ||x - y||^k \, 1_A(x) 1_A(y) \, dx \, dy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} ||v||^k \, 1_A(u) 1_A(u + v) \, du \, dv \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} ||v||^k \, 1_A(u) 1_{T_{-v}A}(u) \, du \, dv \\
&= \int_{\mathbb{R}^n} ||v||^k \, \lambda_n(A \cap T_{-v}A) \, dv.
\end{aligned}$$

The case of practical interest is an application of Proposition 1.4 with $n = 3$, $r = 2$, relating $\lambda_3(A)^2$ to the integral of $J_1^2(A \cap F)$ over all section planes F, using Proposition 1.7 to compute $J_1^2(A \cap F)$ in terms of the covariance function of $A \cap F$. We explore this in the sequel.

For an illustration of Proposition 1.7, consider a disc D of radius t. The covariance is

$$C_D(x) = \pi t^2 - 2t^2 \arcsin\left(\frac{||x||}{2t}\right) - ||x|| \sqrt{t^2 - \frac{||x||^2}{4}}, \qquad ||x|| \leq 2t.$$

Then from Proposition 1.7 (or e.g. Santaló (1976), (4.12), p. 48)

$$J_1^2(D) = \frac{2^7}{45} \pi t^5. \tag{1.7}$$

Incidentally it then follows from the result of Carleman (1919, Satz I; see below Definition 1.1) that for arbitrary regular compact sets A in $\mathbb{R}^2$

$$J_1^2(A) \geq \frac{2^7}{45\pi^{3/2}} \lambda_2(A)^{5/2}, \tag{1.8}$$

cf. Santaló (1976), p. 48 where this is stated for convex sets only.

2. Sampling designs and unbiased estimators.

In this section we derive unbiased estimators of particle mean volume in a randomised sampling design (cf. Jensen & Gundersen (1985), Miles (1983)). Consider a fixed, finite collection of disjoint compact regular sets $X_i \subset \mathbb{R}^n$, $i = 1, \ldots, N$ henceforth called 'particles'. Write

$$X = \bigcup_{i=1}^{N} X_i.$$

Define the volume-weighted mean volume of the particle population by

$$\bar{v} = \frac{\sum_{i=1}^{N} \{\lambda_n(X_i)\}^2}{\sum_{i=1}^{N} \lambda_n(X_i)}.$$

This is the weighted mean of the values $\lambda_n(X_i)$ with weights a_i proportional to $\lambda_n(X_i)$. It may also be interpreted as the expected value of the volume of a particle X_I chosen at random with probability proportional to volume, $\mathbb{P}\{I = i\} = \lambda_n(X_i)/\lambda_n(X)$. The latter distribution arises naturally if we generate a random point Z uniformly distributed in the particle phase X, and select X_I to be that (unique) particle which contains Z. See e.g. Davy & Miles (1977), Jensen & Gundersen (1985), Miles (1983).

2.1 Single random section.

2.1 PROPOSITION. *Let F be a random r-flat hitting X with the λ_r-weighted distribution* (cf. Davy and Miles (1977))

$$dP_1(F) = \frac{\lambda_r(F \cap X)}{c_{n,r}\lambda_n(X)}\, d\mu_{n,r}(F). \tag{2.1}$$

Then

$$\widehat{v} = \frac{c_{n,r}}{c_{n-1,r-1}} \frac{\sum_{i=1}^{N} J_{n-r}^{r}(X_i \cap F)}{\sum_{i=1}^{N} \lambda_r(X_i \cap F)} \tag{2.2}$$

is an unbiased estimator of $\bar{v}$.

The case of practical interest is $n = 3$, $r = 2$, for which $c_{n,r}/c_{n-1,r-1} = 2\pi/\pi = 2$.

PROOF.

$$\mathbb{E}\frac{\sum_{i=1}^{N} J^r_{n-r}(X_i \cap F)}{\sum_{i=1}^{N} \lambda_r(X_i \cap F)} = \int_{\mathcal{F}(n,r)} \frac{\sum_{i=1}^{N} J^r_{n-r}(X_i \cap F)}{\sum_{i=1}^{N} \lambda_r(X_i \cap F)}\, dP_1(F)$$
$$= \int_{\mathcal{F}(n,r)} \frac{\sum_{i=1}^{N} J^r_{n-r}(X_i \cap F)}{c_{n,r}\lambda_n(X)}\, d\mu_{n,r}(F)$$
$$= \frac{c_{n-1,r-1}\sum_{i=1}^{N} \{\lambda_n(X_i)\}^2}{c_{n,r}\lambda_n(X)}$$
$$= \frac{c_{n-1,r-1}}{c_{n,r}}\bar{v}$$

where the penultimate line follows from Proposition 1.4, equation (1.6). □

For illustration, consider the case when all particles X_i are spherical. The profiles $X_i \cap F$ are all discs of varying radii t_i (taking $t_i = 0$ if $X_i \cap F = \emptyset$); the estimator is, from (1.7),

$$\widehat{v} = \frac{2^8}{45}\frac{\sum_{i=1}^{N} t_i^5}{\sum_{i=1}^{N} t_i^2}$$

or equivalently

$$\widehat{v} = \frac{2^8}{45\pi^{3/2}}\frac{\sum_{i=1}^{N} a_i^{5/2}}{\sum_{i=1}^{N} a_i}$$

where $a_i = \pi t_i^2$.

For general profile shapes, note that the numerator of (2.2) can be expressed using Proposition 1.7 as follows. Writing $Y_i = X_i \cap F$ and $Y = X \cap F = \bigcup_{i=1}^{N} Y_i$,

$$\sum_{i=1}^{N} J_1^2(Y_i) = \sum_i \int_{\mathbb{R}^2} ||x||\, C_{Y_i}(x)\, dx$$
$$= \int_{\mathbb{R}^2} ||x||\, C_Y^*(x)\, dx$$

where

$$C_Y^*(x) = \sum_{i=1}^{N} C_{Y_i}(x)$$
$$= \lambda_2\{y : y \in Y_i \text{ and } x + y \in Y_i \text{ for some } i\} \tag{2.3}$$

could be called the *within-particle* set covariance.

2.2 Systematic serial sections

In contrast with the previous model, we now consider the intersection of X with a lattice of parallel r-flats. For simplicity we restrict the treatment to planes in $\mathbb{R}^3$, but the general case is analogous. Define the plane with normal vector $u \in S^2$ and displacement $p \in \mathbb{R}$ from the origin by

$$F_{u,p} = \{x \in \mathbb{R}^3 : \langle x, u \rangle = p\} \tag{2.4}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^3$.

Fix $d > 0$ and consider a stack of parallel planes at separation d,

$$F_{[m]} = F_{p+md,u}, \qquad m \in \mathbb{Z}.$$

The section stack is isotropic uniform random (IUR) if p, u are independent and uniformly distributed on S^2_+ and $(-r/2,\ r/2)$ respectively, where $S^2_+ = \{u = (u_1, u_2, u_3) \in \mathbb{R}^3 : ||u|| = 1,\ u_3 \geq 0\}$. Thus

$$dP(p,u) = \frac{1}{2\pi d}\, dp\, du,$$

2.2 PROPOSITION. *If the section stack is IUR then the estimator*

$$\frac{2\sum_{m=-\infty}^{\infty}\sum_{i=1}^{N} J_1^2(X_i \cap F_{[m]})}{\sum_{m=-\infty}^{\infty}\sum_{i=1}^{N} \lambda_2(X_i \cap F_{[m]})}$$

is "ratio-unbiased", i.e.

$$\frac{2\,\mathbb{E}\sum_{m=-\infty}^{\infty}\sum_{i=1}^{N} J_1^2(X_i \cap F_{[m]})}{\mathbb{E}\sum_{m=-\infty}^{\infty}\sum_{i=1}^{N} \lambda_2(X_i \cap F_{[m]})} = \bar{v}.$$

A comparable result appears to be stated in Serra (1982), pp. 237–238.

PROOF.

The numerator is

$$\begin{aligned}\mathbb{E}\sum_m\sum_{i=1}^{N} \lambda_2(X_i \cap F_{[m]}) &= \int_{S^2_+}\int_{-d/2}^{d/2}\sum_m\sum_i \lambda_2(X_i \cap F_{p+md,u})\frac{1}{2\pi d}\,dp\,du \\ &= \frac{1}{2\pi d}\sum_i \int_{S^2_+}\int_{-\infty}^{\infty} \lambda_2(X_i \cap F_{p,u})\,dp\,du \\ &= \frac{1}{d}\sum_i \lambda_3(X_i)\end{aligned}$$

using standard results of integral geometry (Santaló (1976)). Similarly for the denominator, using Proposition 1.4 we get

$$2\,\mathbb{E}\sum_{m=-\infty}^{\infty}\sum_{i=1}^{N} J_1^2(X_i \cap F) = \frac{1}{d}\sum_i \{\lambda_3(X_i)\}^2.$$

Taking the ratio gives the stated result. □

3. Model-based estimation.

In this section we consider a stationary process of compact sets X_i in $\mathbb{R}^3$, sampled by taking the intersection with a *fixed* plane $L \in \mathcal{F}(3,2)$.

We formulate $\{X_i\}$ as a *germ-grain model* following Stoyan et. al (1987). Let $\Phi = \{(\mathbf{x}_i, A_i)\} = \{((x_i, y_i, z_i), A_i)\}_i$ be a first-order stationary and isotropic marked point process in $\mathbb{R}^3$ with marks A_i belonging to the space $\mathcal{K}(\mathbb{R}^3)$ of compact sets in $\mathbb{R}^3$. Define $X_i = T_{\mathbf{x}_i} A_i = T_{(x_i,y_i,z_i)} A_i$. Thus for example if the A_i are spheres of random radius centred at the origin, the X_i are spheres centred at the points of the process $\{\mathbf{x}_i\} = \{(x_i, y_i, z_i)\}$.

Without loss of generality, fix the section plane L to be the (x, y) plane of $\mathbb{R}^3$. We consider the process of nonempty intersections $Y_i = X_i \cap L$. Defining $B_i = A_i \cap T_{(0,0,-z_i)} L$ we have $Y_i = T_{(x_i,y_i,0)} B_i$, so that the Y_i can be represented as the germ-grain model $\Psi = \{((x_i, y_i), B_i)\}$ in $\mathbb{R}^2$ with marks $B_i \in \mathcal{K}(\mathbb{R}^2)$. Ψ is obtained from Φ by restricting Φ to those i satisfying $A_i \cap T_{(0,0,-z_i)} L \neq \emptyset$ and mapping $((x_i, y_i, z_i), A_i) \mapsto ((x_i, y_i), B_i)$. Clearly Ψ is first-order stationary and isotropic in $\mathbb{R}^2$.

Let α be the intensity of (the points of) Φ, and P^0_Φ the Palm distribution of the typical mark A. The volume-weighted mean volume of the X_i is defined in this context as

$$\bar{v} = \frac{\mathbb{E}^0_\Phi \, (\lambda_3(A))^2}{\mathbb{E}^0_\Phi \, \lambda_3(A)}$$

where $\mathbb{E}^0_\Phi$ denotes expectation with respect to P^0_Φ. Clearly $\bar{v}$ may be interpreted either as the ratio of two expectations under the Palm distribution of A or as the expectation of $\lambda_3(A)$ under the corresponding λ_3-weighted distribution.

3.1 PROPOSITION. *Let Φ, Ψ be as above. Then the intensity of Ψ is*

$$\beta = \frac{\alpha}{2\pi} \mathbb{E}^0_\Phi \int_{\mathcal{F}(3,2)} 1\{A \cap F \neq \emptyset\} \, d\mu_{3,2}(F) \tag{3.1}$$

and the Palm distribution P^0_Ψ of the typical section profile B satisfies

$$\mathbb{E}^0_\Psi \, \lambda_2(B) = \frac{\alpha}{2\pi\beta} \mathbb{E}^0_\Phi \, \lambda_3(A) \tag{3.2}$$

$$2\mathbb{E}^0_\Psi \, J_1^2(B) = \frac{\alpha}{2\pi\beta} \mathbb{E}^0_\Phi \, (\lambda_3(A))^2 \tag{3.3}$$

so we have

$$\frac{2\mathbb{E}^0_\Psi \, J_1^2(B)}{\mathbb{E}^0_\Psi \, \lambda_2(B)} = \bar{v}. \tag{3.4}$$

Again the left side of (3.4) may be interpreted as the expectation of $2J_1^2(B)/\lambda_2(B)$ under the λ_2-weighted counterpart of the Palm distribution of B.

PROOF.
α and P_Φ^0 are defined uniquely to satisfy the Campbell-Mecke formula (Stoyan et. al (1987), p. 116)

$$\mathbb{E} \sum_{(\mathbf{x}_i, A_i)\in\Phi} h(\mathbf{x}_i, A_i) = \alpha \int_{\mathbb{R}^3} \mathbb{E}_\Phi^0 h(\mathbf{x}, A)\, d\mathbf{x} \tag{3.5}$$

for any nonnegative measurable function $h : \mathbb{R}^3 \times \mathcal{K}(\mathbb{R}^3) \to [0,\infty)$. This identity is implied by the special case

$$\mathbb{E} \sum_{\substack{(\mathbf{x}_i, A_i)\in\Phi \\ \mathbf{x}_i \in C}} g(A_i) = \alpha\lambda_3(C)\mathbb{E}_\Phi^0\, g(A) \tag{3.6}$$

for all measurable nonnegative $g : \mathcal{K}(\mathbb{R}^3) \to [0,\infty)$, where $C \subset \mathbb{R}^3$ is an arbitrary compact set with nonzero Lebesgue measure. Similarly for β and P_Ψ^0.

Let $g : \mathcal{K}(\mathbb{R}^2) \to [0,\infty)$ be measurable and $\mathbb{P}_\Psi^0$-integrable. Assume $g(\emptyset) = 0$ and that g is invariant under rotations and translations of $\mathbb{R}^2$. For arbitrary compact $C \subset \mathbb{R}^2$ with $\lambda_2(C) > 0$, the analogue of (3.6) for Ψ gives

$$\begin{aligned}\beta\lambda_2(C)\mathbb{E}_\Psi^0\, g(B) &= \mathbb{E} \sum_{\substack{((x_i,y_i),B_i)\in\Psi \\ (x_i,y_i)\in C}} g(B_i) \\ &= \mathbb{E} \sum_{((x_i,y_i,z_i),A_i)\in\Phi} f((x_i,y_i,z_i),A_i)\end{aligned}$$

where

$$\begin{aligned}f(\mathbf{x},A) &= 1\{(x_i,y_i)\in C\}1\{A_i \cap T_{(0,0,-z_i)}L \neq \emptyset\}g(A_i \cap T_{(0,0,-z_i)}L) \\ &= 1\{(x_i,y_i)\in C\}g(A_i \cap T_{(0,0,-z_i)}L)\end{aligned}$$

since $g(\emptyset) = 0$. Then (3.5) yields

$$\begin{aligned}\mathbb{E} \sum_{((x_i,y_i,z_i),A_i)\in\Phi} f((x_i,y_i,z_i),A_i) &= \alpha\mathbb{E}_\Phi^0 \int_{\mathbb{R}^3} f(\mathbf{x},A)\, d\mathbf{x} \\ &= \alpha\lambda_2(C)\mathbb{E}_\Phi^0 \int_{\mathbb{R}} g(A \cap T_{(0,0,-z)}L)\, dz.\end{aligned}$$

Since $\mathbb{P}^0_\Phi$ is isotropic, we may replace A by $r(A)$ in the above expression, where $r \in SO(3)$ is any rotation of $\mathbb{R}^3$. Observe that

$$\begin{aligned} r(A) \cap T_{(0,0,-z)}L &= r(A \cap r^{-1}T_{(0,0,-z)}L) \\ &= r(A \cap F_{\omega,-z}) \end{aligned}$$

where $\omega = r^{-1}(0,0,1)$ and where $F_{\omega,p}$ is defined in (2.4). Since g is r-invariant we get

$$\beta \mathbb{E}^0_\Psi\, g(B) = \alpha \mathbb{E}^0_\Phi \int_{\mathbb{R}} g(A \cap F_{\omega,z})\, dz$$

for any fixed $\omega \in S^2$. Averaging the right side over $\omega \in S^2_+$ and recalling that $d\mu_{3,2}(F_{\omega,p}) \equiv dp d\omega$ we get

$$\beta \mathbb{E}^0_\Psi\, g(B) = \frac{\alpha}{2\pi} \mathbb{E}^0_\Phi \int_{\mathcal{F}(3,2)} g(A \cap F)\, d\mu_{3,2}(F).$$

Taking $g(B) = 1\{B \neq \emptyset\}$ yields the expression (3.1) for β, and taking $g(B) = \lambda_2(B)$, $g(B) = 2J_1^2(B)$ respectively yields (3.2)–(3.3), using (1.6) to evaluate the integral over $\mathcal{F}(3,2)$. □

To estimate $\mathbb{E}^0_\Psi\, J_1^2(B)$, $\mathbb{E}^0_\Psi\, \lambda_2(B)$ from a bounded sample of Ψ one can use standard methods, see Stoyan et al. (1987). However an alternative is suggested by equation (2.3); we have

$$\mathbb{E}^0_\Psi\, J_1^2(B) = \frac{1}{\beta} \int_{\mathbb{R}^2} ||x||\, C^\dagger_Y(x)\, dx$$

where

$$C^\dagger_Y(x) = \mathbb{P}\{0 \in Y_i \text{ and } x \in Y_i \text{ for some } i\}$$

might be called the *within-particle spatial covariance* . Given observations within a fixed compact window $W \subseteq \mathbb{R}^2$, a pointwise unbiased estimator of $C^\dagger_Y(x)$ is

$$\widehat{C^\dagger_Y}(x) = \frac{\sum_i \lambda_2((Y_i \cap W) \cap T_x(Y_i \cap W))}{\lambda_2(Y \cap W)}$$

which is a modification of the standard estimator of the spatial covariance function.

4. Variance comparison and Rao–Blackwell theorem.

In the practical case $\mathbb{R}^3$, Jensen & Gundersen (1983, 1985) proposed an estimator of $\bar{v}$ constructed as follows:

1. Generate a plane F with the area-weighted distribution (2.1) with $r = 2$.
2. Given F, choose one of the profiles $Y_i = X_i \cap F$ with probability proportional to area, say choosing $I \in \{1, \dots, N\}$ with

$$\mathbb{P}\{I = i \mid F\} = \frac{\lambda_2(Y_i)}{\lambda_2(Y)}$$

   where $Y = \bigcup_{i=1}^{N} Y_i$.

3. Given $I = i$, generate a length-weighted random line through Y_i, i.e. a random line in F with distribution

$$d\,P_{F,i}(\ell) = \frac{\lambda_1(\ell \cap Y_i)}{\pi \lambda_2(Y_i)}\, d\mu_{F,1}(\ell).$$

4. Calculate

$$[V(X)]_1 = \frac{2\pi J_2^1(\ell \cap Y_i)}{\lambda_1(\ell \cap Y_i)}.$$

Then $[V(X)]_1$ is an unbiased estimator of $\bar{v}$ Jensen & Gundersen (1983, 1985), Miles (1983b); this is also a consequence of Proposition 1.4 The variance of $[V(X)]_1$ is an important practical consideration; see e.g. Artachó-Pérula & Roldán-Villalobos (1994) for recent practical studies.

Jensen & Gundersen also considered an estimator denoted $[V(X)]_0$ obtained by generating a uniform random point x in X and an isotropic random line through x. They showed ((1985), Proposition 1) that

$$\mathrm{var}[V(X)]_1 \leq \mathrm{var}[V(X)]_0.$$

Here we shall show that, in turn, our estimator $\widehat{v}$ has smaller variance than $[V(X)]_1$ in the present setting.

4.1 PROPOSITION. *When $[V(X)]_1$ is generated according to the construction above and $\widehat{v}$ according to section 2.1*

$$\mathrm{var}\ (\widehat{v}) \leq \mathrm{var}([V(X)]_1).$$

PROOF. Both procedures begin by generating an area-weighted plane F through X. In the Jensen-Gundersen construction above, if we condition on F and on $I = i$ we have

$$\begin{aligned}
\mathbb{E}\left\{[V(X)]_1 \mid F, I = i\right\} &= \int_{\mathcal{F}F1} \frac{2\pi J_2^1(\ell \cap Y_i)}{\lambda_1(\ell \cap Y_i)} \frac{\lambda_1(\ell \cap Y_i)}{\pi \lambda_2(Y_i)} \, d\mu_{F,1}(\ell) \\
&= 2\frac{J_1^2(Y_i)}{\lambda_2(Y_i)}
\end{aligned}$$

using Propositions 1.3 and 1.4. Since the conditional distribution of I given F is area-weighted, we get

$$\begin{aligned}
\mathbb{E}\left\{[V(X)]_1 \mid F\right\} &= \mathbb{E}\left(\mathbb{E}\left\{[V(X)]_1 \mid I, F\right\} \mid F\right) \\
&= \mathbb{E}\left\{ 2\frac{J_1^2(Y_I)}{\lambda_2(Y_I)} \middle| F \right\} \\
&= \frac{2\sum_{i=1}^N J_1^2(Y_i)}{\sum_{i=1}^N \lambda_2(Y_i)} \\
&= \widehat{v}.
\end{aligned}$$

By the stereological Rao-Blackwell theorem (Baddeley & Cruz-Orive (1993)) we obtain the result. □

Comments in Baddeley & Cruz-Orive (1993) suggest that a general variance comparison (holding e.g. also for systematic sampling) may not be available.

5. Vertical section designs.

Here we indicate the existence of counterparts of the integral geometric results of §1 for "vertical sections" (Baddeley et al. (1986)).

Fix arbitrary orthonormal coordinates in $\mathbb{R}^3$, designate the x_3 axis as the "vertical", and identify the "horizontal" (x_1, x_2) plane with $\mathbb{R}^2$. A *vertical plane* is any flat $V \in \mathcal{F}(3,2)$ such that the normal to V is horizontal. Equivalently define $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ to be the standard coordinate projection; then a vertical plane is any $V \in \mathcal{F}(3,2)$ whose horizontal projection is a line, $\pi(V) \in \mathcal{F}(2,1)$. Let $\mathcal{V}(3,2)$ be the class of all vertical planes. Then $L = \pi(V)$, $V = \pi^{-1}(L) = L \times \mathbb{R}$ is a 1-1 correspondence between $\mathcal{F}(2,1)$ and $\mathcal{V}(3,2)$ under which the action of the Euclidean motions in $\mathbb{R}^2$ corresponds to the action of those Euclidean motions in $\mathbb{R}^3$ which preserve the vertical coordinate. Let $\nu_{3,2}$ denote the invariant measure on $\mathcal{V}(3,2)$ obtained by this correspondence.

Our objective is to produce a counterpart to Proposition 1.4 (case $n = 3$, $r = 2$) where $\mathcal{F}(3,2)$ is replaced by $\mathcal{V}(3,2)$.

Consider the integral in (1.4). Note that every line $\ell \in \mathcal{F}(3,1)$ determines a unique vertical plane $V \in \mathcal{V}(3,2)$ such that $\ell \subset V$, by $V = \pi^{-1}(\pi(\ell))$, except in the negligible case where ℓ is parallel to the vertical axis. This leads to the factorisation (Baddeley et al. (1986))

$$d\mu_{3,1}(\ell) = |\sin\,\theta(\ell)|\; d\mu_{V,1}(\ell)\, d\nu_{3,2}(V)$$

where $\theta(\ell)$ is the angle between ℓ and the vertical axis, and as before $\mu_{V,1}$ is the invariant measure on $\mathcal{F}(V,1)$, the space of lines contained in V.

Hence (1.4) becomes

$$\begin{aligned} J^3_{k-2}(A) &= I^3_k(A) \\ &= \int_{\mathcal{F}(3,1)} J^1_k(A\cap\ell)\, d\mu_{3,1}(\ell) \\ &= \int_{\mathcal{V}(3,2)} \int_{\mathcal{F}(V,1)} J^1_k(A\cap\ell)\; |\sin\theta(\ell)|\; d\mu_{V,1}(\ell)\, d\nu_{3,2}(V) \\ &= \int_{\mathcal{V}(3,2)} K^V_k(A\cap V)\, d\nu_{3,2}(V) \qquad (5.1) \end{aligned}$$

where for compact regular $B \subset V$

$$K^V_k(B) = \int_{\mathcal{F}(V,1)} J^1_k(B\cap\ell)\; |\sin\theta(\ell)|\; d\mu_{V,1}(\ell).$$

Next we find an analogue of Proposition 1.7. In the expression above, replace $J^1_k(B\cap\ell)$ by its definition (1.1) and change variables by mapping $\{(u,v,\ell) : u,v \in \ell \in \mathcal{F}(V,1)\}$ to $\{(u,v) : u,v \in V\}$. Observe that

$$\sin\theta(\ell) = \frac{\pi_3(u-v)}{||u-v||}$$

where $\pi_3 : I\!R^3 \to I\!R$ is the projection onto the vertical (x_3) axis. Hence using the Blaschke-Petkantschin formula, cited in the proof of Proposition 1.3

$$K^V_k(B) = \int_B \int_B |\pi_3(u-v)|\, ||u-v||^{k-2}\, du\, dv.$$

By another change of variable $(u,v) \mapsto (v, u-v)$ we obtain

$$K^V_k(B) = \int_{V^*} |\pi_3(x)|\; ||x||^{k-2} C_B(x) dx \qquad (5.2)$$

where $V^* = \{u - v : u,v \in V\}$ is the vector space parallel to V and as usual $C_B(x) = \lambda_2(B \cap T_x B)$ for $x \in V^*$. This is the desired analogue of Proposition 1.7.

In case $k = 2$ we find that

$$\lambda_3(A)^2 = \int_{\mathcal{V}(3,2)} K_2^V(A \cap V)\, d\nu_{3,2}(V) \tag{5.3}$$

where

$$K_2^V(B) = \int_{V^*} |\pi_3(x)|\; C_B(x) dx. \tag{5.4}$$

Estimation formulae can be derived by the same means as in §2–3. For example in the design-based, single section case (§2.1), let V be an area-weighted vertical plane

$$dP(V) = \frac{\lambda_2(X \cap V)}{\pi \lambda_3(X)} d\nu_{3,2}(V);$$

then

$$\tilde{v} = \pi \frac{\sum K_2^V(X_i \cap V)}{\sum \lambda_2(X_i \cap V)}$$

is an unbiased estimator of $\bar{v}$.

6. Applications.

6.1 Remarks on implementation.

A precondition for using the estimators described here is that the section profiles Y_i are fully identifiable, more precisely, that for any given points x, y in the section plane,

(a) it is observable whether x belongs to Y,

(b) it is observable whether x and y belong to the same profile Y_i for some i.

Requirement (b) implies that (disconnected) profiles originating from the same object X_i are identifiable as such. A similar statement was emphasised in Jensen & Gundersen (1985), Miles (1985, Introduction), Serra (1982, p. 257).

In implementations based on image processing technology it would normally be convenient to calculate J_1^2 using (a discrete approximation to) the covariance function C_{Y_i} of each individual profile, or the within-particle covariance function C_Y^* of (2.3). If each profile is available as a binary image, its covariance function can be computed either directly from the definition, or by using the Fast Fourier Transform. If no profiles consist of more than one connected component, C_Y^* can be computed directly from the *connected component transform* of a binary image of Y.

Let s_x, s_y denote the 'real' physical lengths of one pixel step in the x and y directions respectively. To compute J_1^2 for each profile we determine the set covariance of each binary image, expressed as a pixel count multiplied by $s_x s_y$. The integral in Proposition 1.7 is then approximated by $s_x s_y$ times the sum over all pixel displacements (n_x, n_y) of $\sqrt{s_x^2 n_x^2 + s_y^2 n_y^2}$ times the estimated covariance function for this displacement.

6.2 Example 1: Synaptic boutons.

Our first dataset comes from a morphometric study of the effect of neuroleptic treatment on the morphology of encephalinergic synaptic boutons in rat striatum, carried out at the Graduate School of Neuroscience, Free University of Amsterdam, The Netherlands (kindly provided by ir. G.J. Docter and drs. M.J. Mijnster). The data analysed here consist of 33 separate profiles of boutons in ultrathin (60nm) sections, visualised under electron microscopy, from one selected animal. The outlines were manually digitised on an IBAS image analyser to produce 33 separate binary images, each 512×512 pixels resolution. Magnification varied slightly from image to image, and the linear magnifications in the x and y directions were very slightly different. Figure 1 shows a composite of all 33 images: each is roughly 2 microns square in real units (the profile diameters are in the range 0.5 to 1.5 microns).

Taking the model based approach (§3) we treat these images as independent samples B_i from the Palm distribution of the section process Ψ. The standard, approximately unbiased, estimator of $\bar{v}$ is then

$$\widehat{v} = \frac{\sum_{i=1}^m Y_i}{\sum_{i=1}^m X_i} \tag{6.1}$$

with $Y_i = 2J_1^2(B_i)$, $X_i = \lambda_2(B_i)$. Note that from (1.8)

$$Y_i \geq c X_i^{5/2} \tag{6.2}$$

where $c = 2^8/(45\pi^{3/2}) = 1.02165$. The images in Figure 1 have been sorted in ascending order of $Y_i/X_i^{5/2}$ (reading from left to right in rows from top to bottom). The estimate (6.1) is $\widehat{v} = 0.232\,\mu\text{m}^3$. Further analysis is postponed to §7.

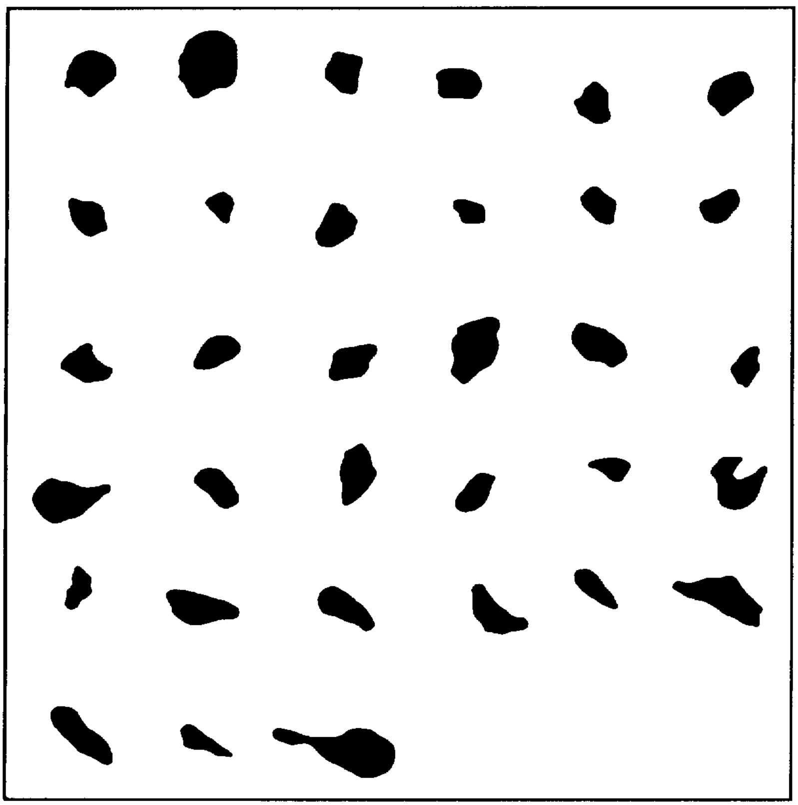


FIGURE 1. Composite of 33 separate binary images of synaptic boutons, as described in the text.

6.3 Example 2: Silicon carbide particles.

The binary image in Figure 2 was obtained from a colour micrograph (magnification ×1570) of a silicon carbide (SiC) composite material published on the front cover of the *Journal of Microscopy*, volume 169 part 2 of 1993 (reproduced by kind permission of Dr. S.P. Justice and the Royal Microscopical Society.) We scanned this 818×736 image directly from the published micrograph. The real dimensions of the image are 131×118 μm so that 1 pixel step corresponds to 0.16 μm.

The total area of profiles was determined by pixel counting the binary image, and J_1^2 using the connected component transform of this image. The estimate (6.1) is $\widehat{v} = 195\,\mu\text{m}^3$.

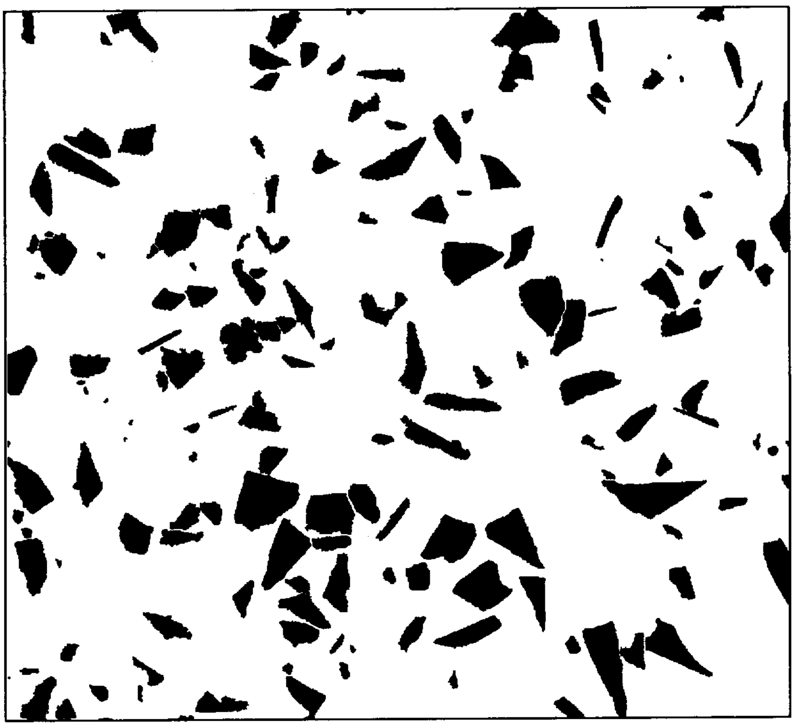


FIGURE 2. Binary image of SiC composite material, plane section. Real dimensions 131 × 118 μm.

7. Estimation of variance.

7.1 Boutons data.

Figure 3 shows the complete set of bivariate data (X_i, Y_i) for the synaptic boutons example, with the dotted line indicating the theoretical lower bound (6.2). It is remarkable that the observed data are very close to this lower bound.

Figure 4 shows the same data after taking logarithms. Apart from linearising the lower bound (6.2) this appears to transform $\log X_i$ to an approximate normal distribution as judged by the histogram and Q-Q plot in Figure 5. Additionally, the 'excess'

$$R_i = \frac{Y_i}{cX_i^{5/2}}$$

seems to be approximately independent of X_i, judging by Figure 6.

Hence we can arguably model

$$Y = cRX^{5/2} \tag{7.1}$$

where X and R are independent, X is lognormally distributed, and R has some unspecified distribution. Then

$$\bar{v} = \frac{\mathbb{E}Y}{\mathbb{E}X} = c\,\mathbb{E}R \cdot \frac{\mathbb{E}\,X^{5/2}}{\mathbb{E}\,X}.$$

Our proposal is to replace (6.1) by

$$\tilde{v} = c\,\bar{r}\widehat{a}$$

where $\bar{r}$ is the sample mean of the values $r_i = y_i/(c\,x_i^{5/2})$ and $\widehat{a}$ is the maximum likelihood estimate of $a = \mathbb{E}[X^{5/2}]/\mathbb{E}[X]$ under the lognormal model. The variance of $\tilde{v}$ is then estimated by combining estimates of the variance of $\bar{r}$ and of $\widehat{a}$ based on the sample variance and on maximum likelihood, respectively. On the present data this procedure yielded $\tilde{v} = 0.221$ with estimated standard deviation 0.037.

For comparison we also applied a bootstrap technique to the statistic (6.1). We generated 10,000 samples of $n = 33$ data points drawn at random with replacement from the data (x_i, y_i), and formed the estimate (6.1) for each sample. The resulting estimates $\widehat{v}$ had mean 0.230 and standard deviation 0.042. The estimates are summarised below.

Method:	$\widehat{v}$	$\widehat{\sigma}$
moment estimator (6.1)	0.232	–
lognormal model	0.221	0.037
bootstrap	0.230	0.042

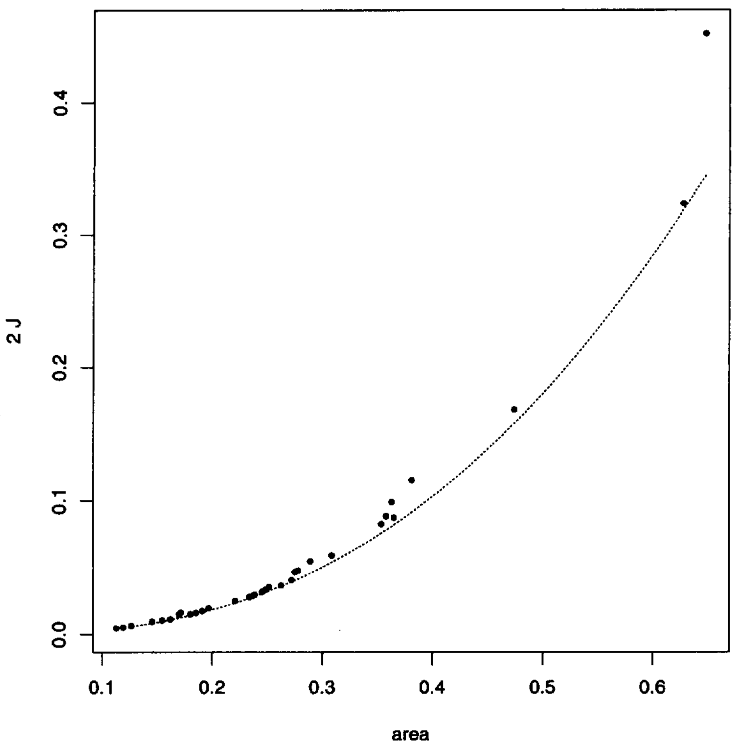


FIGURE 3. Bivariate data (X_i, Y_i) from individual bouton profiles. Dotted line indicates theoretical bound $Y_i \geq cX_i^{5/2}$.

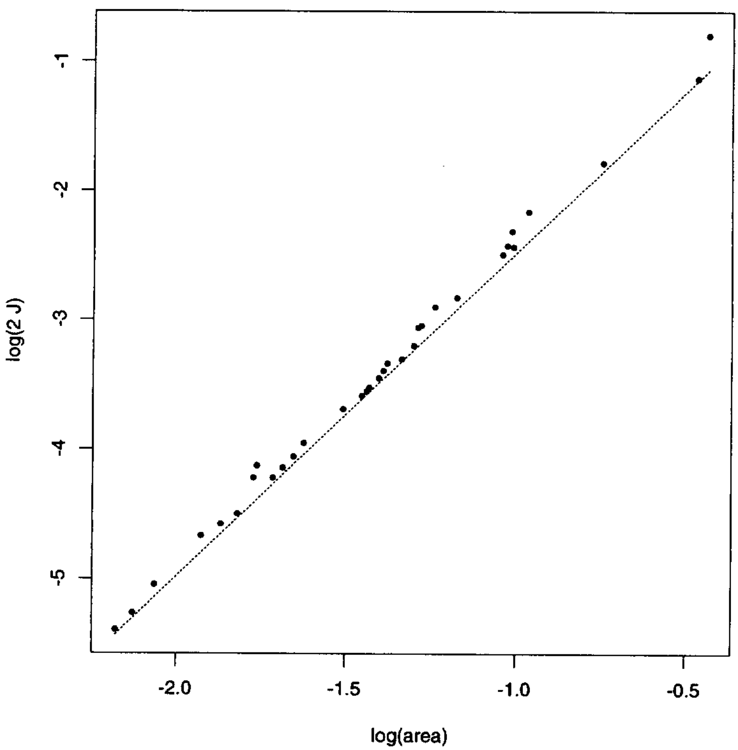
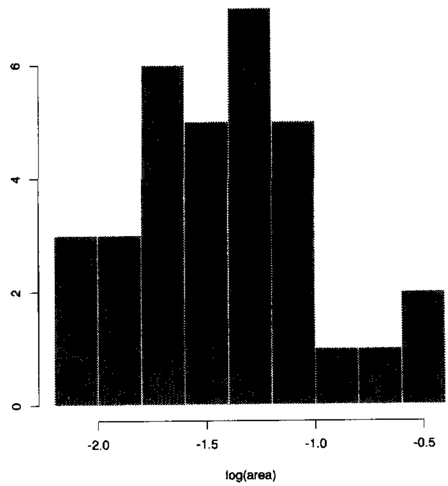


FIGURE 4. Plot of log-transformed data for boutons example.



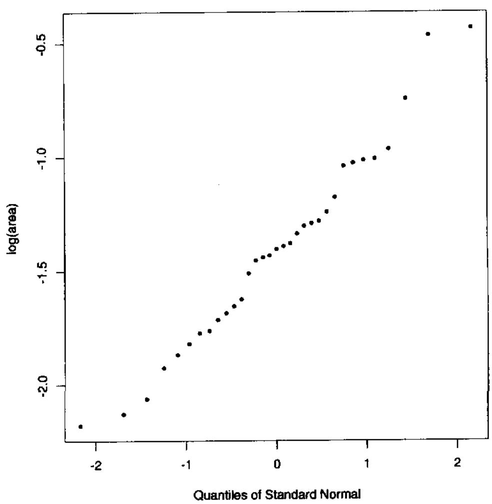


FIGURE 5. Histogram and normal Q-Q plot of $\log(X_i)$ for boutons example

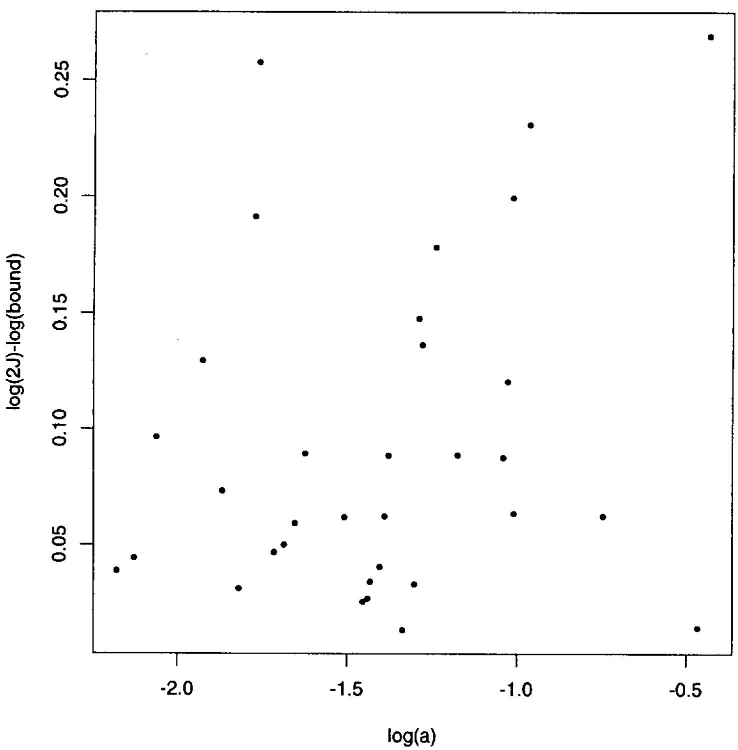


FIGURE 6. Plot of $\log R_i$ against $\log(X_i)$ for boutons example, indicating no strong evidence for dependence of R_i and X_i.

7.2 Silicon carbide data.

Figure 7 shows a logarithmic plot of the data for each particle profile ($n = 170$) in the silicon carbide image of Figure 2. In contrast to the boutons data, there are values which far exceed the lower bound (6.2); the observed r_i ranged from 1.03 to 2.5.

Note that in this case the data points may not be treated as independent if there is spatial dependence between profiles on a section, or if we cannot regard the particle process as ergodic (e.g. if different sections look substantially different). We shall overlook this in the present analysis.

The histogram of $\log(x_i)$ in Figure 8 shows that a lognormal model for X is inappropriate. Furthermore a plot of the r_i analogous to Figure 6 suggests possible dependence between R and X. We therefore abandon the lognormal model.

The bootstrap approach can still be applied. A sample of 10,000 independent resamples with replacement of size $n = 170$ from the data yielded $\widehat{v}$ values (calculated by (6.1)) with mean 193.9 and standard deviation 21.9. Note again that the variance estimate is based on an assumption that the particle profiles are independent, in particular, it ignores between-section variance contributions.

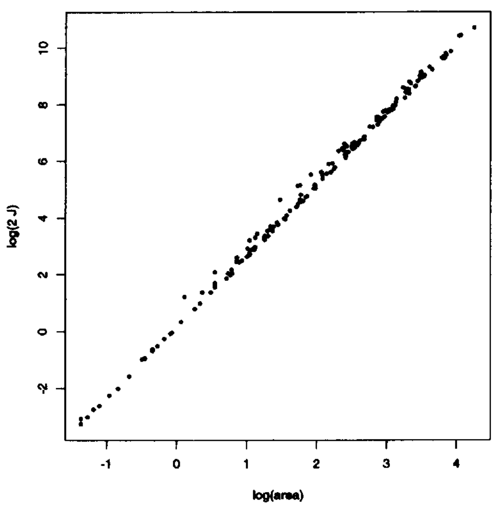


FIGURE 7. Logarithmic plot of bivariate data for silicon carbide example.

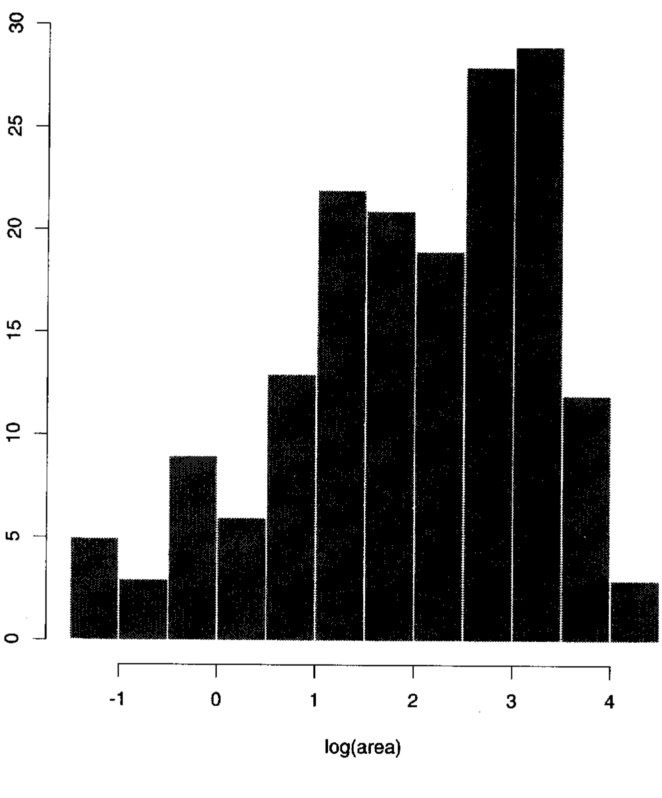


FIGURE 8. Histogram of log areas of silicon carbide particle profiles showing inappropriateness of a lognormal model.

7.3 Comparison with line intercept techniques

In order to evaluate the gain in efficiency compared to the manual point sampled intercept (p.s.i.) technique, we also implemented p.s.i. A laser printer copy of Figure 2 (with further magnification 1.4 relative to the original micrograph) was used. Materials and protocol for p.s.i. were taken from an exercise given at the SCANDEM '92 Stereology Course (Copenhagen, Denmark, 1992) supplied by H.J.G. Gundersen, C.V. Howard, B. Pakkenberg and F.B. Sørensen, and used by kind permission of the authors. A transparency, bearing a grid of equidistant parallel lines and equidistant test points on each line, was superimposed 31 times on the printout. For each test point which hit a SiC profile, the length of the line transect was measured with a logarithmically graduated scale ("ℓ_0^3 ruler"), see Brændgaard & Gundersen (1986).

The $\overline{l_0^3}$-ruler divides lengths into classes $1, \ldots, 15$; the ith class includes lengths in the range a_{i-1} to a_i where $a_0 = 0$ and

$$a_i = L\frac{10^{i/14} - 1}{10^{15/14} - 1} \quad i = 1, \ldots, 15$$

where $L = 35$ mm is the physical length of the graduated scale. The 'average cubed length' in class i is $b_i = (a_{i-1}^3 + a_i^3)/2$.

Average cubed intercept lengths were computed as $\overline{\ell_0^3} = \sum_i n_i b_i$ where n_i was the count of intercept lengths in class i. Then the p.s.i. estimate of $\bar{v}$ was computed from

$$[V(X)]_1 = \frac{\pi}{3} M^{-3} \overline{\ell_0^3} \text{ mm}^3$$

where $M = 1.4 \times 1570 = 2198$ is the real magnification. A histogram of the values obtained is displayed in Figure 9. The sample mean and standard deviation were 146.0 and 84.5 respectively (median 111, quartiles 81, 183).

The efficiency of the new technique relative to the p.s.i. technique is then

$$\frac{84.5^2 + 21.9^2}{21.9^2} = 15.8,$$

a substantial improvement. Again this assumes ergodicity of the particle process, so the above efficiency calculation refers only to variance within a given section.

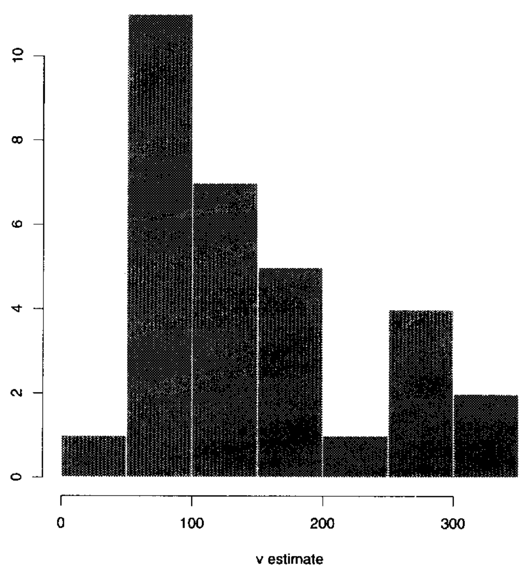


FIGURE 9. Histogram of 31 estimates of $\bar{v}$ (μm^3) for the SiC image obtained by the manual p.s.i. technique

CHAPTER V

CROSS-COVARIANCE FUNCTIONS CHARACTERISE REGULAR COMPACT SETS

Introduction.

The (set) covariance function was introduced by Matérn in 1960 (see Matérn (1985)) and Matheron (1965). It maps a Borel set A and a vector x onto the volume of the intersection of A with the translation of A by x. The covariance function is widely used in spatial statistics to investigate second order properties of the model under study. It can be used for instance to estimate the parameters of a Boolean model (e.g. Hall (1988), chapter 5).

Theoretically however the covariance function seems to have received less attention. Two interesting questions immediately arise:

(1) how much *geometric* information about a set is contained in its covariance function?

(2) what are the *analytic* properties of covariance functions?

Concerning the first problem, Matheron (1986) conjectured that the set covariance function of a *convex* set determines the set uniquely (up to a translation and a reflection). For *planar symmetric convex* sets, he shows the conjecture, using that for *all* sets the support of the covariance function of a set A is equal to the set $A \oplus (-A)$ and for *convex* sets: $A \oplus (-A) = 2A$. The reason Matheron restricted attention to convex sets seemed to be due to the analytic approach he used to describe the boundaries of the sets. Later Lešanovský and Rataj (1990) gave an example of two distinct nonconvex sets with the same covariance functions. For convex polygons in the plane the conjecture has recently been proved by Nagel (1993). (There the covariance function is called the covariogram of a set.) Schmitt (1993) gives a reconstruction procedure that also works for a restricted class of nonconvex polygons. Consequently the role of convexity is not yet completely clarified. In this chapter we show that *any* regular closed *symmetric* Borel subset of $\mathbb{R}^d$, $d \geq 1$, is determined by its covariance function. Here symmetry refers to the fact that $A = -A$. The symmetry forces the statement to be complete in the sense that we obtain the set itself and not a set up to a translation. Observe that no convexity or connectivity assumptions are made. The question whether convexity is enough to ensure uniqueness of a set given its covariance function (up to translation and reflection) remains unanswered.

In the same paper in which Matheron published his conjecture, Lipschitz continuity of the covariance function of a convex set was proved. We give a geometric proof of continuity of the covariance function of a bounded Borel set (which in itself is a direct consequence of a standard result in Fourier theory, see Zaanen (1989)). Continuity of the function considered as a function acting on sets was proved for certain classes of regular closed sets by Cabo and Baddeley (1993).

This chapter is organised in the following way. In section 1 we give the result underlying the uniqueness theorems about the covariance functions. The main idea is to apply Fourier methods to the convolution of two functions to obtain equality almost everywhere of the functions. In section 2 this is applied to the covariance function. Section 3 is devoted to the continuity property. In section 4 we discuss two related problems of Pyke (1989) and Adler and Pyke (1991). In the latter paper, they asked whether a convex set is uniquely determined (up to translation and reflection) by the distribution of the difference vector of two independent uniformly distributed points in its interior. The well known relation between the covariance function and this distribution immediately shows that the problem of characterising a set by this distribution is equivalent to the problem of characterising a set by its covariance function. The results of section 2 and 3 solve this problem for all regular closed symmetric subsets of $\mathbb{R}^d$. Nagel (1993) solves the latter problem for convex polygons, but again for other sets this problem is still open. As a special case of the abovementioned relation, we have a relation between the covariance function and the interpoint distance distribution (i.e. the distribution of the length of the difference vector). This relation is exploited to prove that a rotation invariant, not–necessarily–convex regular closed set is uniquely determined by its interpoint distance distribution. This answers a question of Pyke (1989).

In the last section we introduce a new function, the *cross-covariance* that is shown to characterise *any* regular closed Borel set, thus providing an alternative to the so-called three point covariance (see Nagel (1991,1993)).

1. A uniqueness result for convolutions.

1.1 General finite Borel measures.

Let $\mathcal{M}$ denote the space of all finite Borel measures on $\mathbb{R}$, with non-zero total measure; i.e. $\forall \mu \in \mathcal{M} : 0 < \mu(\mathbb{R}) < \infty$. The *moments* μ_n of $\mu \in \mathcal{M}$

are given by

$$\mu_n = \int_{\mathbb{R}} x^n \, \mu(dx), \quad n = 0, 1, 2, \ldots$$

Let $\mathcal{B}(\mathbb{R})$ denote the Borel σ-algebra on $\mathbb{R}$.

Recall that the convolution $\mu * \nu$ of two measures $\mu, \nu \in \mathcal{M}$ is defined by

$$\mu * \nu(B) = \int_{\mathbb{R}} \mu(B - t)\nu(dt), \quad B \in \mathcal{B}(\mathbb{R})$$

where $B - t = \{y - t : y \in B\}$; see e.g. Dudley (1989).

1.1 THEOREM. *Suppose all moments of μ and ν exist:*

$$\mu_n < \infty \quad \textit{and} \quad \nu_n < \infty \quad \textit{for all } n.$$

Then

$$\mu * \mu = \nu * \nu \tag{1.1}$$

implies

$$\mu_n = \nu_n \textit{ for all } n.$$

PROOF. Let ϕ and ψ be the characteristic functions of μ and ν respectively, that is

$$\phi(t) = \int_{\mathbb{R}} e^{itx} \mu(dx)$$
$$\psi(t) = \int_{\mathbb{R}} e^{itx} \nu(dx), \quad t \in \mathbb{R}.$$

It is well-known that the characteristic function of the convolution of two measures equals the product of their characteristic functions. Consequently the characteristic functions of $\mu * \mu$ and $\nu * \nu$ are given by ϕ^2 and ψ^2 respectively. Thus assumption (1.1) is equivalent to

$$\phi(t)^2 = \psi(t)^2 \quad \text{for all } t.$$

Since all moments exist, the k-th derivative $\phi^{(k)}$ of ϕ exists and is a continuous function that is equal to

$$\phi^{(k)}(t) = i^k \int_{\mathbb{R}} x^k e^{itx} \mu(dx),$$

for all k (Feller (1968), XV.4).

Hence for $t = 0$ we obtain

$$\phi^{(k)}(0) = i^k \mu_k \quad \text{for all } k.$$

Moreover (1.1) yields

$$\frac{\partial^n}{(\partial t)^n}\phi(t)^2 = \frac{\partial^n}{(\partial t)^n}\psi(t)^2 \quad \text{for all } t \text{ and } n, \tag{1.2}$$

and thus in particular this is true for $t = 0$. By Leibniz's rule we have

$$\begin{aligned} \frac{\partial^n}{(\partial t)^n}\phi(t)^2|_{t=0} &= \sum_{k=0}^{n} \binom{n}{k} \phi^{(k)}(0)\phi^{(n-k)}(0) \\ &= i^n \sum_{k=0}^{n} \binom{n}{k} \mu_k \mu_{n-k}, \quad \text{for all } n \end{aligned} \tag{1.3}$$

and a similar expression for ψ. We prove the theorem by induction.
For $n = 0$, (1.2) yields

$$\phi(0)^2 = \psi(0)^2,$$

hence

$$\mu_0 = \phi(0) = \psi(0) = \nu_0,$$

since by definition $\phi(0) = \mu(\mathbb{R}) > 0$ and $\psi(0) = \nu(\mathbb{R}) > 0$. Now suppose

$$\mu_k = \nu_k \quad \text{for all } k < n. \tag{1.4}$$

Then (1.2) and (1.3) yield

$$i^n \sum_{k=0}^{n} \binom{n}{k} \mu_k \mu_{n-k} = i^n \sum_{k=0}^{n} \binom{n}{k} \nu_k \nu_{n-k}$$

which is equivalent to

$$2\mu_n\mu_0 + \sum_{k=1}^{n-1} \binom{n}{k} \mu_k \, \mu_{n-k} = 2\nu_n\nu_0 + \sum_{k=1}^{n-1} \binom{n}{k} \nu_k \, \nu_{n-k}.$$

By (1.4) the sums cancel and by the first induction step $\mu_0 = \nu_0 > 0$, so we can divide both sides by μ_0 to obtain

$$\mu_n = \nu_n.$$

Hence by mathematical induction we have

$$\mu_n = \nu_n \quad \text{for all } n. \qquad \square$$

It is clear that Theorem 1.1 does not depend on the number of dimensions. The analogues for $\mathbb{R}^d$ will be used in the sequel without further comment.

1.2 Remark. Suppose the additional condition is satisfied that the measures are determined by their moments, that is

$$\mu_n = \nu_n \quad \text{for all } n \text{ implies } \mu = \nu. \tag{1.5}$$

A necessary and sufficient condition for this was given by Carleman (see Shohat and Tamarkin (1943)): a finite measure is uniquely determined by its moments *iff*

$$\sum \mu_{2n}^{-\frac{1}{2n}} = \infty. \tag{1.6}$$

(See also Feller (1968), VII.3 for an example of a distribution that is not determined by its moments.) For measures concentrated on a compact set K this condition is trivially satisfied. Indeed we can bound the function x^{2n} on K by its supremum M_0^{2n} say, consequently μ_{2n} is bounded above by $M^{2n} = M_0^{2n}\mu(\mathbb{R})$. Hence

$$\sum \mu_{2n}^{-\frac{1}{2n}} \geq \sum \frac{1}{M_0\mu(\mathbb{R})^{\frac{1}{2n}}} = \frac{1}{M_0}\sum \frac{1}{\mu(\mathbb{R})^{\frac{1}{2n}}}.$$

Since the sequence $\{\mu(\mathbb{R})^{\frac{1}{2n}}\}$ converges to 1 the series $\sum \frac{1}{\mu(\mathbb{R})^{\frac{1}{2n}}}$ diverges.

Theorem 1.1 can be rephrased in several ways. For a probabilistic interpretation suppose X and Y are independent random variables with the same distribution P and let Z and W be independent random variables distributed according to Q. Suppose moreover that P and Q satisfy condition (1.5) of remark 1.2. Then Theorem 1.1 states that if the distribution of $X+Y$ coincides with the distribution of $Z+W$, then $P=Q$.

1.2 Absolutely continuous measures.

If the measure μ has a Radon-Nikodym derivative with respect to Lebesgue measure, $f \in L^1$ say, the characteristic function ϕ is the Fourier transform $\hat{f}$ of f. Moreover the convolution of two such functions is defined as

$$f * g(y) = \int_{\mathbb{R}} f(y-x)g(x)\,dx,$$

and it is well-known that $(f*g)\hat{} = \hat{f}\hat{g}$.

1.3 COROLLARY. *Suppose μ and ν are absolutely continuous with respect to Lebesgue measure λ. Let f and g denote their Radon-Nikodym derivatives and suppose f and g have compact supports. Then $(f * f)\hat{}(t) = (g * g)\hat{}(t)$ for all t implies*

$$f = g \text{ for } \lambda\text{-almost all } x.$$

PROOF. By Theorem 1.1, $(f*f)\hat{}(t) = (g*g)\hat{}(t)$ for all t implies equality of all moments, i.e.

$$\int_{\mathbb{R}} f(x)x^n\,dx = \int_{\mathbb{R}} g(x)x^n\,dx, \text{ for all } n = 0, 1, 2 \ldots.$$

Since f and g have compact supports this immediately implies

$$f = g \text{ for } \lambda\text{-almost all } x.$$

□

1.4 Example. Let A and B be compact Borel sets in $\mathbb{R}$ and let $f = 1_A$, $g = 1_B$. Suppose $1_A * 1_A = 1_B * 1_B$. Then $(1_A * 1_A)\hat{}(t) = (1_B * 1_B)\hat{}(t)$ for all t, thus the corollary yields

$$1_A = 1_B \quad \text{almost everywhere.}$$

In the next section we shall give a geometric interpretation of this example.

In the case of two finite absolutely continuous Borel measures on $[0, \infty)$ the conclusion of Theorem 1.1 can be changed into the direct statement that $\mu * \mu = \nu * \nu$ implies $\mu = \nu$, thus avoiding the somewhat unnatural detour via moments. In fact this is a consequence of the following theorem by Titchmarsh (see Mikusiński (1983), Dieudonné (1960)).

1.5 THEOREM (Titchmarsh). *Let f and g be integrable over $[0, T]$. If the convolution of f and g vanishes almost everywhere in $[0, T]$ then there exist two numbers $T_1 > 0$, $T_2 > 0$, such that $T_1 + T_2 > T$ and such that f vanishes a.e. on $[0, T_1]$ and g vanishes a.e. on $[0, T_2]$*

1.6 THEOREM. *Let μ and ν be two finite Borel measures on $[\alpha, \infty)$ that are absolutely continuous with respect to Lebesgue measure λ on $[\alpha, \infty)$, for some fixed $\alpha \in \mathbb{R}$.*

Then

$$\mu * \mu = \nu * \nu$$

implies

$$\mu = \nu.$$

PROOF. It suffices to give the proof for $\alpha = 0$. Let f and g be the densities of μ and ν respectively. Thus f and g are nonnegative almost everywhere on $[0, \infty)$.
From $\mu * \mu = \nu * \nu$, we immediately obtain

$$f * f = g * g \quad \text{a.e. on } [0, \infty).$$

Equivalently

$$(f - g) * (f + g) = 0 \quad \text{a.e. on } [0, \infty).$$

Since f and g are integrable a.e. on $[0, \infty)$, for all $T > 0$ they are integrable a.e. on the interval $[0, T]$. Then Titchmarsh provides us with $T_1 > 0, T_2 > 0$ such that $T_1 + T_2 > T$ and such that

$$(i) \qquad f - g = 0 \quad \text{a.e. on } [0, T_1]$$

$$(ii) \qquad f + g = 0 \quad \text{a.e. on } [0, T_2].$$

This implies that $f = g$ a.e. on $[0, \max(T_1, T_2)]$. Indeed, if $T_1 = \max(T_1, T_2)$, (i) implies $f = g$ a.e. on $[0, T_1] = [0, \max(T_1, T_2)]$. If $T_2 = \max(T_1, T_2)$, (ii) implies that $f = g = 0$ a.e. on $[0, T_2] = [0, \max(T_1, T_2)]$, because f, $g \geq 0$ a.e. on $[0, \infty)$. Now note that

$$T < T_1 + T_2 = \min(T_1, T_2) + \max(T_1, T_2) < 2 \max(T_1, T_2),$$

thus $f = g$ a.e. on $[0, \frac{1}{2}T]$, for all $T > 0$. Letting $T \to \infty$, this yields $f = g$ a.e. on $[0, \infty)$. □

In Mikusiński (1987) one can find a higher-dimensional analogue of Theorem 1.5 on the convolution ring of continuous functions with supports in a given cone.

2. Covariance functions determine regular compact symmetric sets.

Denote by $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$ the Borel σ-algebra of $\mathbb{R}^d$. A set $A \in \mathcal{B}$ is *regular closed* if it equals the closure of its interior:

$$\overline{A^\circ} = A.$$

If a regular closed set is bounded, we call it *regular compact.*

2.1 DEFINITION. For a bounded Borel set A the *covariance function* of A is defined as

$$C_A(y) = \lambda(A \cap (A - y)), \quad y \in \mathbb{R}^d. \tag{2.1}$$

Here λ is d-dimensional Lebesgue measure. When we wish to stress the dimension, we use a subscript as in λ_d.

The following properties are easy to check:

(1) $C_A(0) = \lambda(A)$;
(2) C_A is symmetric: $C_A(y) = C_A(-y), \quad \forall y \in \mathbb{R}^d$;
(3) C_A has compact support.

An equivalent definition is obtained by writing (2.1) in terms of convolutions.

$$C_A(y) = (1_A * 1_{-A})(y), \tag{2.2}$$

where $-A = \{-x : x \in A\}$.

As a consequence of Corollary 1.3, we immediately obtain the following uniqueness theorem.

2.2 THEOREM. *Let $A, B \in \mathcal{B}(\mathbb{R}^d)$ be regular compact sets such that $A = -A$ and $B = -B$. Then $C_A = C_B$ implies $A = B$.*

Observe that the sets are *completely* determined and not just up to some translation.

PROOF. By (2.2) for all $y \in \mathbb{R}^d$

$$C_A(y) = (1_A * 1_{-A})(y) = (1_A * 1_A)(y) \text{ since } A = -A.$$

Hence the assumption is equivalent to $(1_A * 1_A)(y) = (1_B * 1_B)(y)$ for all $y \in \mathbb{R}^d$. Exactly as in Example 1.4 it follows from Corollary 1.3 that the Fourier transforms of the convolutions are equal and thus $1_A = 1_B$ a.e.. Regularity of the sets now implies $A = B$. (For a proof: see Chapter III, proof of Proposition 4.3.) □

3. Continuity of the covariance function.

In section 4 we need continuity of the covariance function. This is a direct consequence of the continuity of the convolution of of an L^1 function and an essentially bounded function (Zaanen (1989)). Here we give a geometric proof for the special case of the covariance function.

3.1 THEOREM. *Let B be a Borel set with $\lambda_d(B) < \infty$. Then the mapping $x \mapsto C_B(x)$ is continuous on $\mathbb{R}^d$.*

PROOF. From the definition of the covariance function we easily derive

$$|C_B(x) - C_B(y)| \leq 2(C_B(0) - C_B(x-y))$$

(see also Matheron (1986)). Thus it is enough to prove continuity in 0. Let $\epsilon > 0$ be given. By Theorem 11.4 (Billingsley (1979)), there exists a *finite* sequence $\{U_i\}_{i=1}^n$ of hypercubes such that

$$\lambda(B \triangle \cup_i U_i) < \epsilon/2. \tag{3.1}$$

Set $U = \cup_{i=1}^n U_i$. It suffices to prove that

$$\lambda(B) - \lambda(B \cap T_x B) \leq \lambda(U) - \lambda(U \cap T_x U) + \lambda(B \triangle U), \tag{3.2}$$

for all $x \in \mathbb{R}^d$. Indeed, once we have (3.2)

$$\begin{aligned}
\lambda(B) - \lambda(B \cap T_x B) &\leq \lambda(U) - \lambda(U \cap T_x U) + \lambda(B \triangle U) \\
&= \lambda(U \cup T_x U) - \lambda(U) + \lambda(B \triangle U) \\
&\leq \lambda(U^{\|x\|} \setminus U) + \lambda(B \triangle U) \\
&< \lambda(U^{\|x\|} \setminus U) + \epsilon/2, \quad \text{by (3.1)},
\end{aligned}$$

where $U^{\|x\|}$ consists of all points in $\mathbb{R}^d$ at a distance atmost $\|x\|$ from U. Now observe that $\lambda(U^{\|x\|} \setminus U) \leq \sum_i \lambda(U_i^{\|x\|} \setminus U_i)$. Since the U_i are convex, we can use the Steiner formula (see Schneider (1993)) to write each $\lambda(U_i^{\|x\|} \setminus U_i)$ as a polynomial in $\|x\|$ with zero constant term and finite positive coefficients determined by U_i.
Now choose $\delta > 0$, such that $\lambda(U^{\|x\|} \setminus U) < \epsilon/2$ for all $\|x\| < \delta$.
Then for $\|x\| < \delta$

$$C_B(0) - C_B(x) = \lambda(B) - \lambda(B \cap T_x B) < \epsilon.$$

So let us proceed with the proof of (3.2). First observe that

$$B = (B \setminus U) \cup (B \cap U). \tag{3.3}$$

By (3.3)

$$\begin{aligned}
\lambda(B) - \lambda(B \cap T_x B) = \lambda(B \setminus U) &+ \lambda(B \cap U) - \lambda((B \cap U) \cap T_x(B \cap U)) \\
&- \lambda((B \setminus U) \cap T_x(B \setminus U)) \\
&- \lambda((B \setminus U) \cap T_x(B \cap U)) - \lambda((B \cap U) \cap T_x(B \setminus U)
\end{aligned} \tag{3.4}$$

Since the last three terms at the right-hand side of (3.4) are nonnegative, we trivially get

$$\lambda(B) - \lambda(B \cap T_x B) \leq \lambda(B \setminus U) + \lambda(B \cap U) - \lambda((B \cap U) \cap T_x(B \cap U)). \tag{3.5}$$

Now rewrite $\lambda((B\cap U)\cap T_x(B\cap U))$ as $2\lambda(B\cap U)-\lambda((B\cap U)\cup T_x(B\cap U))$, using translation invariance of Lebesgue measure. Then the right-hand side of (3.5) is

$$\lambda(B\setminus U)+\lambda(B\cap U)-2\lambda(B\cap U)+\lambda((B\cap U)\cup T_x(B\cap U)). \tag{3.6}$$

Moreover

$$\begin{aligned}\lambda((B\cap U)\cup T_x(B\cap U)) &\leq \lambda(U\cup T_x(B\cap U))\\ &\leq \lambda(U\cup T_xU),\end{aligned}$$

and

$$\lambda(B\cap U)=\lambda(U)-\lambda(U\setminus B).$$

So (3.6) is not greater than

$$\begin{aligned}\lambda(U\cup T_xU)-\lambda(U)+\lambda(B\setminus U)+\lambda(U\setminus B) &=\\ =\lambda(U)-\lambda(U\cap T_xU)+\lambda(B\Delta U).\end{aligned}$$

□

4. Interpoint distance distributions.

In a sequence of subsequent volumes of the IMS Bulletin (see Pyke (1989)) Pyke asked the following question:

> given two independent points uniformly distributed over the interior of a compact set C, does the distribution of their distance determine C?

Rost (1989) produced an example showing that in general this need not be true and later Lešanovsky and Rataj (1990) published a paper containing among others a result about the structure of these examples. The examples were based on non convex sets and therefore Pyke's question was restricted to *convex* C (see also Adler and Pyke (1991)). However, the following theorem shows that this restriction is not needed as long as symmetric sets are involved. This is a direct consequence of Theorem 2.2 and the continuity of the covariance function.

Recall the relation between interpoint distance distributions and covariance functions derived from Borel's overlap Lemma. (Borel (1925); Sheng (1985). See also Cabo and Baddeley (1993).)

4.1 LEMMA. *For two independent points X, Y uniformly distributed over the interior of a regular compact set $A\in\mathcal{B}(\mathbb{R}^d)$*

$$\mathbb{P}(\|X-Y\|\leq\rho)=\frac{1}{\lambda(A)^2}\int_{B(0,\rho)} C_A(w)\,dw,\quad \rho\geq 0. \tag{4.1}$$

First observe that this implies the equivalence of the problem of characterising a set by the distribution of the *difference vector* of two independent uniformly distributed points X, Y in its interior and the problem of characterising a set by its covariance function. Indeed, from (2.2) it is seen that $\lambda(A)^{-2}\, C_A(\cdot)$ is the density of the distribution of the difference vector, since $X - Y = X + (-Y)$. Nagel's result (1993) thus solves the problem for convex polygons in the plane. Our Theorem 2.2 together with the continuity of the covariance function solves the problem for regular closed symmetric subsets of $\mathbb{R}^d$. In the case of the distribution of the *length* of the difference vector, it is clear that the restriction on the sets to be characterised will have to be more severe, since this distribution throws away the information on the directions. Theorem 2.2 yields the following result.

4.2 PROPOSITION. *Let A and B be regular compact Borel sets that are rotation invariant (and hence centrally symmetric). Let X, Y be independent uniform points in the interior of A and let Z, W be two such points in B. Then*

$$\mathbb{P}(\|X - Y\| \le \rho) = \mathbb{P}(\|Z - W\| \le \rho) \quad \textit{for all } \rho \ge 0 \tag{4.2}$$

implies equality of the sets A and B.

PROOF. Rewrite (4.1) as

$$\mathbb{P}(\|X - Y\| \le \rho) = \frac{1}{\lambda(A)^2} \int_0^\rho \int_{S^{d-1}} C_A(ru)\, du dr. \tag{4.3}$$

Since A is rotation invariant, its covariance function only depends on the distance, hence (4.3) is

$$\frac{d\kappa_d}{\lambda(A)^2} \int_0^\rho C_A(ru_0)\, dr$$

where $d\kappa_d$ is the surface area of the unit sphere S^{d-1} in $\mathbb{R}^d$ and $u_0 \in S^{d-1}$ is arbitrary but fixed. By (4.2) we have

$$\frac{1}{\lambda(A)^2} \int_0^\rho C_A(ru_0)\, dr = \frac{1}{\lambda(B)^2} \int_0^\rho C_B(ru_0)\, dr, \quad \text{for all } \rho.$$

Since by continuity of the covariance function we may take the derivative with respect to ρ on both sides, this is

$$\frac{1}{\lambda(A)^2} C_A(\rho u_0) = \frac{1}{\lambda(B)^2} C_B(\rho u_0) \quad \text{for all } \rho.$$

In particular, if $\rho = 0$ we get $\lambda(A) = \lambda(B)$ thus

$$C_A(\rho u_0) = C_B(\rho u_0) \quad \text{for all } \rho.$$

Since u_0 was arbitrary, Theorem 2.2 yields $A = B$. □

5. Cross-covariance functions determine regular compact sets.

In this section we introduce a generalisation of the covariance function, that is shown to characterise every regular closed subset of $\mathbb{R}^d$.

5.1 DEFINITION. Let $A, B \in \mathcal{B}(\mathbb{R}^d)$ be bounded. The *cross-covariance function of A with respect to B* is defined as

$$C_{A,B}(y) = \lambda(A \cap (B + y)), \quad y \in \mathbb{R}^d. \tag{5.1}$$

The following properties are immediate consequences of the definition:

(1) $C_{A,B}(0) = \lambda(A \cap B)$ $(= 0$ if $A \cap B = \emptyset)$;
(2) $C_{A,B}$ is anti-symmetric in the following sense:

$$C_{A,B}(y) = C_{B,A}(-y), \quad \forall y \in \mathbb{R}^d;$$

(3) $C_{A,B}$ has compact support;
(4) $C_{A,A}(y) = C_A(-y) = C_A(y), \quad y \in \mathbb{R}^d$, in other words Matérn's covariance function can be considered as an 'autocovariance' in this setup.

We call $C_{A,-A}$ *the cross-covariance of A.*

As for covariance functions, an equivalent formulation in terms of convolutions exists:

$$C_{A,B}(y) = (1_A * 1_{-B})(y). \tag{5.2}$$

Using Theorem 1.1, we now immediately derive the uniqueness result.

5.2 THEOREM. *Let $A, B \in \mathcal{B}(\mathbb{R}^d)$ be regular compact sets. Then $C_{A,-A} = C_{B,-B}$ implies $A = B$.*

Again, observe that the sets are *completely* determined and not just up to some translation.

PROOF. Suppose $C_{A,-A} = C_{B,-B}$, or by (5.2)

$$(1_A * 1_A)(y) = (1_B * 1_B)(y), \quad \forall y \in \mathbb{R}^d.$$

Exactly as in the proof of Theorem 2.2 this implies $1_A = 1_B$ a.e., whence $A = B$ by regularity. □

Remark. This result can be extended slightly to the following situation. Suppose we are given the *normed* cross-covariance functions (for example normed by their maximum value); i.e.

$$\alpha^2 C_{A,-A}(x) = \beta^2 C_{B,-B}(x), \quad \forall x \in \mathbb{R}^d, \tag{5.3}$$

where α, $\beta \in \mathbb{R}_+$. Then $A = B$.

Indeed, writing (5.3) in terms of convolutions yields

$$\alpha 1_A * \alpha 1_A(x) = \beta 1_B * \beta 1_B(x), \quad \text{for all } x.$$

Hence from Theorem 1.6 we get

$$\alpha 1_A(x) = \beta 1_B(x) \quad \text{for almost all } x. \tag{5.4}$$

Let $\mathcal{N}$ denote the null set for which (5.4) does not hold. We now first prove that this implies $\alpha = \beta$.

Suppose $x \in A \setminus \mathcal{N}$. Then (5.4) gives $\alpha = \beta 1_B(x)$ thus $x \in B \setminus \mathcal{N}$, implying $\alpha = \beta$. But then $1_A = 1_B$ a.e. which implies $A = B$ by regularity. □

Next we prove continuity of the the cross-covariance function using the result of section 3.

5.3 THEOREM. *Let A and B be Borel sets with finite Lebesgue measure. Then the mapping $x \mapsto C_{A,B}(x)$ is continuous on $\mathbb{R}^d$.*

PROOF. By definition

$$\begin{aligned} \left|C_{A,B}(x) - C_{A,B}(y)\right| &= \left|\int 1_A(t)1_{B+x}(t) - 1_A(t)1_{B+y}(t)\,dt\right| \\ &\leq \int 1_A(t)|1_B(t-x) - 1_B(t-y)|\,dt. \end{aligned}$$

Since obviously $1_A \leq 1$, this is smaller than

$$\int |1_B(t-x) - 1_B(t-y)|\,dt = \int (1_B(t-x) - 1_B(t-y))^2\,dt.$$

By a change of variables and because of the translation invariance of Lebesgue measure this is

$$2(\lambda(B) - C_{B,B}(x-y)) = 2(C_B(0) - C_B(x-y)).$$

Hence the assertion follows from the continuity of the covariance function of B, which was proved in Theorem 3.1. □

Observe that Theorem 5.3 implies *uniform continuity* of the cross-covariance, since $C_{A,B}$ has compact support if A and B are compact.

There does exist another generalisation of the covariance function that also characterises all regular closed sets. It is called the 'three point covariance function' and is defined as follows (see Nagel (1993)):

$$C_A(x,y) = \lambda(A \cap (A-x) \cap (A-y)), \quad x, y \in \mathbb{R}^d.$$

It was proved by Nagel (1991) that it is possible to determine the corresponding regular set from the three point covariance function considered as a function of the two variables x and y *up to translation*. To conclude this paper we would like to remark that these two characterising functions can be considered as special cases of the following unifying entity.

5.5 DEFINITION. For $n = 1, 2 \ldots$ and $A, B \in \mathcal{B}(\mathbb{R}^d)$ bounded let the n-th order *motion covariance function of* A *w.r.t.* B, be defined as

$$\Gamma^{(n)}_{A,B}(\alpha_1, \alpha_2, \ldots \alpha_n; y_1, y_2, \ldots y_n) = \lambda\big(A \cap (\alpha_1 B - y_1) \cap \cdots \cap (\alpha_n B - y_n)\big),$$

where $\alpha_1, \alpha_2, \ldots \alpha_n \in \mathbf{SO}(d)$ denote rotations of $\mathbb{R}^d$ and $y_1, y_2, \ldots y_n \in \mathbb{R}^d$.

Writing the cross-covariance and three point covariance functions in terms of motion covariances, we get

$$C_{A,-A}(x) = \Gamma^{(1)}_{A,A}(\alpha_\pi; -x) = \Gamma^{(2)}_{A,A}(\alpha_\pi, \mathbf{1}; -x, 0);$$
$$C_A(x, y) = \Gamma^{(2)}_{A,A}(\mathbf{1}, \mathbf{1}; x, y),$$

where α_π denotes point reflection w.r.t. the origin and $\mathbf{1}$ is the identity of $\mathbf{SO}(d)$. Hence the results obtained until now may be summarised by saying that for regular closed sets it either suffices to know the motion covariance with two translations, or the one with one 'reflection' and one translation. In these terms Matheron's conjecture is that for convex sets, motion covariance with only one translation – i.e. $\Gamma^{(1)}_{A,A}(\mathbf{1}; x)$ – is sufficient.

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SAMENVATTING

Meetkundige parameters van verzamelingen kunnen worden beschouwd als functionalen van die verzamelingen, dat wil zeggen als functies die aan een verzameling een reëel getal toevoegen. Hierbij kan men bijvoorbeeld denken aan het volume van een verzameling, het aantal hoekpunten van een veelhoek, enzovoort. Als de verzamelingen stochastisch zijn, beschouwt men de functionalen als stochastische variabelen en bekijkt men hun momenten. Bijvoorbeeld: het verwachte aantal eiken dat in één vierkante kilometer bos groeit of de verwachte oppervlakte die bedekt wordt bij de bestraling van een bepaald deel van het menselijk lichaam. In dit proefschrift worden twee soorten functionalen bestudeerd: functionalen van stochastische convexe veelhoeken in het platte vlak en de zogenaamde covariantiefunctie van deterministische verzamelingen in de Euclidische ruimte van dimensie $d \geq 1$. Daarbij worden de volgende problemen behandeld:

(1) Wat zijn de asymptotische eigenschappen van functionalen van stochastische veelhoeken?

(2) Hoeveel informatie over een verzameling A is bevat in zijn covariantiefunctie? (De waarde van de covariantiefunctie van A is het volume van de doorsnede van A met een verschoven versie van A.)

De stochastische veelhoeken in het eerste deel van het proefschrift worden geconstrueerd als convexe omhulsels C_n van een uniforme steekproef ter grootte n uit het eenheidsvierkant (in $\mathbb{R}^2$). Convexe omhulsels zijn onder andere interessant omdat men met behulp van het convexe omhulsel van punten uit een onbekende verzameling die verzameling kan schatten. Resultaten over functionalen van convexe omhulsels in de literatuur bevatten uitdrukkingen voor en afschattingen van hun verwachting. Wij zijn geïnteresseerd in de asymptotische verdeling als de steekproefgrootte naar oneindig gaat.

In Hoofdstuk I van dit proefschrift wordt het asymptotische gedrag van twee functionalen van C_n bestudeerd: de omtrek en de oppervlakte. Met behulp van de kanstheoretische methoden, ontwikkeld in Groeneboom (1988) – een artikel waarin asymptotische normaliteit van het aantal hoekpunten werd bewezen – leiden we expliciete uitdrukkingen af voor de limietstochasten van deze functionalen. Bovendien bewijzen we dat voor de oppervlakte een Centrale Limiet–Stelling geldt. Voor de omtrek is dat niet zo, althans niet als de steekproef uit een veelhoek wordt genomen. Hoe de asymptotische verdeling er in dat geval uitziet weten we niet. Ook voor convexe omhulsels van uniforme steekproeven uit een verzameling in $\mathbb{R}^d$ zijn nog geen resultaten bekend, voor $d \geq 3$. We vermoeden echter dat voor een uniforme steekproef uit een verzamling met gladde rand (als de cirkel in het platte vlak) zowel voor de omtrek als voor de oppervlakte functionaal een Centrale Limiet–Stelling geldt. Op de reden hiervoor wordt in Hoofdstuk I nader ingegaan.

De volgende hoofdstukken van het proefschrift zijn integraalmeetkundig van aard. Dat wil zeggen dat we in eerste instantie deterministische verzamelingen bekijken. (Veel van de behaalde resultaten hebben echter ook een stochastische interpretatie.) In Hoofdstuk II geven we een nieuw bewijs voor een bekende formule voor lijnintegralen van convexe veelhoeken, met behulp van differentiaalmeetkunde.

In de laatste drie hoofdstukken bekijken we de covariantiefunctie van een verzameling. Deze functie wordt gebruikt in de ruimtelijke statistiek om tweede–orde–eigenschappen van statistische modellen te bestuderen. Zo kan de covariantiefunctie van een Boole–model worden gebruikt om de parameters van dat model te schatten. We onderzoeken analytische (Hoofdstuk III), stereologische (Hoofdstuk IV) en eenduidigheidseigenschappen (Hoofdstuk V) van deze functie.

In Hoofdstuk III wordt een nieuwe functie geconstrueerd – de lijnsectie–functionaal (*linear scan transform*) – die beschrijft hoe een lijn een verzameling snijdt. Deze functie is gedefinieerd in termen van de covariantiefunctie van lijnsecties van de verzameling. De lijnsectie–functionaal brengt een aantal bestaande integraalmeetkundige formules met elkaar in verband, en er kunnen nieuwe formules mee worden afgeleid. Met behulp van de lijnsectie–functionaal wordt een nieuw afstandsbegrip op compacte verzamelingen gedefinieerd. Deze metriek levert een goed alternatief voor de Hausdorff–metriek, onder andere doordat de Hausdorff–metriek overgevoelig is voor afstanden tussen punten van de verzameling. We bewijzen dat de twee metrieken op de klasse van convexe lichamen topologisch equivalent zijn.

De nieuwe metriek wordt vervolgens gebruikt om analytische eigenschappen van de covariantiefunctie te bestuderen; zo bewijzen we continuïteit van de covariantiefunctie als functie van de verzameling en als functie van de verschuivingsvector.

De lijnsectie–functionaal (en de covariantiefunctie zelf) hebben als belangrijk kenmerk dat convexiteit van de verzameling voor de berekening niet van belang is. Dit is vooral relevant voor de toepassingen in de stereologie. Hierbij wordt geprobeerd om drie-dimensionale parameters van een object te schatten uit informatie bevat in lager-dimensionale secties van het object. In de praktijk zijn deze objecten (cellen, mineralen en dergelijke) niet noodzakelijk convex. In Hoofdstuk IV worden de resultaten uit het derde hoofdstuk gebruikt om een nieuwe zuivere schatter van het volume van deeltjes te bepalen uit gegevens op vlakke secties van die deeltjes. Hierbij wordt de covariantiefunctie weer gebruikt. We vergelijken deze schatter met een bestaande schatter van Jensen en Gundersen (1985) en concluderen dat de covariantieschatter van het volume een kleinere variantie heeft. Tenslotte is deze schatter eenvoudig te implementeren en dus goed bruikbaar in de praktijk. Dit wordt aan de hand van twee voorbeelden geïllustreerd. Deze voorbeelden komen uit de medische en de bodemkundige wetenschappen.

In het laatste hoofdstuk tenslotte houden we ons bezig met de vraag hoeveel informatie over een verzameling A de covariantiefunctie van A daadwerkelijk bevat. Er is een vermoeden van Matheron (1986) dat zegt dat voor convexe verzamelingen in het platte vlak die informatie volledig is. Voor convexe veelhoeken is dit inderdaad bewezen door Nagel (1993). Dus, als P_1 en P_2 twee zulke veelhoeken zijn en C_{P_1}, C_{P_2} hun covariantiefuncties, dan is $C_{P_1} = C_{P_2}$ dan en slechts dan als $P_1 = P_2$ (op verschuiving en spiegeling na). Het bewijs van deze stelling hangt echter duidelijk af van de dimensie, mede hierdoor is er voor algemene convexe verzamelingen nog niets bekend. Dit geldt zelfs voor convexe verzamelingen met gladde rand in het platte vlak.

In Hoofdstuk V laten wij met behulp van Fourier–methoden zien dat $C_A = C_B$ gelijkheid van A en B impliceert, mits A en B symmetrisch ten opzichte van de oorsprong zijn. De gebruikte methode levert ook meteen een procedure op voor de reconstructie van de verzamelingen. Hierbij zijn de verzamelingen regulier gesloten deelverzamelingen van $\mathbb{R}^d$ (en dus niet noodzakelijk convex of samenhangend). Pyke (1989) vroeg zich af of verzamelingen ook gekarakteriseerd worden door de verdeling van de afstand tussen twee onafhankelijke, uniform gekozen punten in die verzameling. Wegens de bekende relatie tussen die verdeling en de covariantiefunctie, geven we dus ook een antwoord op Pyke's vraag. Het resultaat is dat rotatie-invariante verzamelingen door de afstandsverdeling eenduidig worden vastgelegd. Tenslotte voeren we een uitbreiding van de covariantiefunctie in – de kruiscovariantie – die *alle* regulier gesloten verzamelingen karakteriseert.

CURRICULUM VITAE

geboren:	9 februari 1965 te Kerkrade
VWO:	eindexamen gymnasium β Stedelijk Gymnasium te Haarlem, 1983.
studie:	wiskunde aan de Universiteit van Amsterdam: van september 1983 tot januari 1989.
a.i.o.:	aan de Technische Universiteit Delft: van 1 februari 1989 tot 1 oktober 1990.
o.i.o.:	aan het Centrum voor Wiskunde en Informatica: van 1 oktober 1990 tot 1 juni 1994.