Thermal time as a POVM

and a proof of Tomita's theorem

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Abstract

This thesis explores the mathematical foundations of quantum measurements for relativistic particles, focusing on the construction of positive operator-valued measures (POVMs) within the framework of functional analysis. Motivated by recent developments in mathematical physics, particularly the construction of a POVM for a one-dimensional relativistic massless particle, we develop an analogous POVM for a relativistic particle with mass. The mathematical framework is firmly grounded in the Tomita-Takesaki modular theory, and its connection to Connes-Rovelli thermal time is established.

A central component of the thesis is a detailed, self-contained proof of Tomita's theorem, a cornerstone result in the modular theory of von Neumann algebras. This theorem plays a critical role in understanding the structure of operator algebras associated with quantum systems and underpins the rigorous formulation of covariant observables. Furthermore, it is of crucial importance to formulating Rovelli's thermal time hypothesis.

Building upon this theoretical groundwork, we construct a POVM for a relativistic massive particle in one spatial dimension. The resulting POVM satisfies temporal covariance relations and provides a mathematically consistent description of time measurements for massive particles, addressing key challenges in quantum field theory. The thesis concludes by relating the newly found time observable to Connes-Rovelli thermal time.

Contents

Nomenclature

Notation

Notation	Definition
$\mathbb{N}$	Natural numbers (including 0)
$\mathbb{Z}$	Integers
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex numbers
$\mathbf{1}_S$	Indicator function of the set S
$\mathrm{int}(S)$	Interior of the set S
$\mathrm{bd}(S)$	Boundary of the set S
$\overline{S}$, $\overline{A}$	Closure of the set S, closure of the operator A
$D(A)$	Domain of the operator A
$\mathrm{range}(A)$	Range of the operator A

1 Introduction

This master's thesis aims to provide a formulation of the mathematical description of Carlo Rovelli's thermal time, including a detailed proof of Tomita's theorem, the main result of Tomita-Takesaki theory. The mathematical background of this thesis is largely based on Jan van Neerven's book Functional Analysis[14], and the proof of Tomita's theorem is based on a 2024 paper by Jonathan Sorce[21]. The research and writing of this thesis were conducted at the Delft Institute of Applied Mathematics from March 2024 until March 2025.

Chapter 2 gives the mathematical preliminaries needed to understand the rest of this thesis. It includes a short introduction to Lie groups and Lie algebras. The majority of this chapter is devoted to a handful of fundamental proofs in operator algebras.

Chapter 3 gives an outline of the physics framework for which the mathematics of this thesis is relevant. Specifically, it introduces the mathematical formulation of quantum mechanics after recapping classical mechanics, as given by the Dirac-von Neumann axioms. This includes the notions of states, observables, and symmetries.

Chapter 4 provides a rigorous mathematical construction of the framework introduced in Chapter 3. As in the previous chapter, classical mechanics serves as an entryway to the formulation of quantum mechanics. A few theorems essential to quantum mechanics are proved, including the Born rule and Wigner's theorem.

Chapter 5 introduces the idea of thermal time, as formulated by Carlo Rovelli in 1993[19]. It is based on a 2013 paper by Pierre Martinetti[12] and highlights the difference between time in special relativity and in quantum mechanics. It also describes the thermal time hypothesis, an attempt to account for this difference in formulation, put forward by Alaines Connes and Carlo Rovelli in 1994[4].

Chapter 6 broadens the notion of what can be considered an observable in quantum mechanics. While observables are typically described by projection-valued measures, this chapter introduces positive operator-valued measures, a generalization of the former. An example is provided to illustrate the relevance of this concept.

Chapter 7 introduces Tomita-Takesaki theory, a method of constructing modular automorphisms of von Neumann algebras. Tomita-Takesaki theory plays an important role in the thermal time hypothesis. This chapter is largely based on Section 2.5 in Operator Algebras and Quantum Statistical Mechanics I by Ola Bratelli and Derek W. Robinson[18].

Chapter 8 proves Tomita's theorem. This proof is based on a 2024 paper by Jonathan Sorce titled "A short proof of Tomita's theorem."[21] The proof is similar to the one given by Bratelli and Robinson in the case where the modular operator is bounded.

Finally, Chapter 9 gives a proof of a positive operator-valued measure covariant with thermal time. A 2024 paper by Jan van Neerven and Pierre Portal[15] already proves the case for the free relativistic particle with no mass. This thesis generalizes the proof to a time covariant positive operator-valued measure to the free relativistic particle with mass, where it is shown that the defined time can be interpreted as thermal time.

2

Mathematics Preliminaries

2.1. Lie Groups

In the early 1800s, the French mathematician Éraviste Galois determined a necessary and sufficient condition for a polynomial to be solvable by radicals. Essentially, his work classified polynomials in terms of group theory. Later that century, the Norwegian mathematician Sophus Lie made it his life's work to develop a theory of symmetries for differential equations that would accomplish what Galois had done for algebraic equations (polynomials).

Lie's main idea was to construct a theory of **continuous groups**, complementing the theory of discrete groups that developed from Galois theory. Although Lie's hope of unifying the entire field of ordinary differential equations was never fulfilled, the continuous groups he introduced, now known as Lie groups, remain an important area of study in mathematics. Roughly speaking, a Lie group is a group that is also a smooth manifold with group operations that are smooth.

Definition 1 (Lie group)**.** A **Lie group** is a smooth manifold G equipped with a group structure such that the maps $\mu : G \times G \to G, (x, y) \mapsto xy$ and $i : G \to G, x \mapsto x^{-1}$ are smooth.

Some basic examples of Lie groups include $(\mathbb{R}^n, +)$, $(\mathbb{C}^n, +)$, $(\mathbb{R}^\star, \cdot)$, and $(\mathbb{C}^\star, \cdot)$. For all of these, it can easily be seen that they are manifolds and groups. The most general example of a Lie group is the general linear group $GL(n, \mathbb{C})$, which is the set of $n \times n$ matrices with complex coefficients and non-zero determinant, and the group operation is matrix multiplication. This forms a group, as it consists precisely of the invertible matrices.

Definition 2 (One-parameter group)**.** Let G be a Lie group with identity e. A continuous function $\phi : \mathbb{R} \to G,\ t \mapsto \phi_t$ is a **one-parameter group** if:

1. $\phi_0 = e$
2. $\phi_{s+t} = \phi_s \phi_t$ for all $s, t \in \mathbb{R}$

In other words, a one-parameter group is a continuous group homomorphism $\phi : \mathbb{R} \to G$, where $\mathbb{R}$ is viewed as the additive group $(\mathbb{R}, +)$. For example, let $x \in M_n(\mathbb{C})$. Then $\phi : \mathbb{R} \to GL(n, \mathbb{C}),\ t \mapsto \exp(tx)$ is a one-parameter group. Remarkably, it turns out that this is the only one-parameter subgroup of $GL(n, \mathbb{C})$ (Theorem 4.3 in [7].)

Theorem 1. *Let $\phi : \mathbb{R} \to GL(n, \mathbb{C})$ be a one-parameter group. Then there exists a unique $x \in M_n(\mathbb{C})$ such that $\phi_t = \exp(tx)$.*

Proof. Suppose that $\phi_0' = x$. Then

$$\phi_t' = \lim_{h \to 0} \frac{\phi_{t+h} - \phi_t}{h} = \frac{\phi_t \phi_h - \phi_t \phi_0}{h} = \phi_0' \phi_t = x \phi_t$$

So ϕ is a solution to the initial value problem $y'(t) = xy(t), y(0) = I$, the unique solution of which is $\phi_t = \exp(tx)$. $\square$

2.2. Lie Algebras

Interestingly, it turns out that every Lie group gives rise to a particular linear space that encodes many of the properties of the Lie group. These spaces were introduced by Sophus Lie to study the concept of infinitesimal transformations and were originally known as the **algebra of infinitesimal transformations** of a Lie group. After Lie's continuous groups were renamed as Lie groups, these algebras become known as **Lie algebras**.

Consider the Earth, a sphere-like object that can be thought of as a manifold. The surface of the Earth is, of course, curved, but when imposed with a coordinate system (longitude and latitude), a map, which is flat, can be created. A map naturally encodes many (but not all) of the properties of the Earth and is much easier to produce than a globe. Lie wanted something similar for his continuous groups, attempting to encode the group structure in their associated linear spaces. Roughly speaking, a Lie algebra is a linear space together with an operation that measures the failure of commutativity for the associated Lie group.

Definition 3 (Lie algebra)**.** A **Lie algebra** is a vector space L over a field $\mathbb{F}$ with an operation $L \times L \to L, (x, y) \mapsto [x, y]$, called the **Lie bracket**, that satisfies the following conditions: [9]

1. The Lie bracket operation is bilinear
2. $[x, x] = 0$ for all $x \in L$
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

The third condition is known as the **Jacobi identity**, and arises from the associativity of the group operation in groups. Indeed, it should be no surprise that the Lie bracket of a Lie algebra has something to do with the group operation of its associated Lie group. In general, for all Lie algebras of interest in this thesis, the Lie bracket is the commutator bracket $[x, y] = xy - yx$.

As mentioned above, Lie algebras represent infinitesimal transformations of Lie groups. As such, moving from a Lie group to its Lie algebra is in some sense analogous to moving from a differentiable function to its derivative. Consider the function $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$. Its tangent line at any point $a \in \mathbb{R}$ is given by $y = 2ax - a^2$. Although the tangent line is different at each point, it is always a line defined on the real numbers, and so it is isomorphic to the real number line $\mathbb{R}$. Therefore, $\mathbb{R}$ can be thought of as the linearization of the parabola $y = x^2$, or of any other differentiable curve, for that matter.

Lie algebras linearise Lie groups in a similar fashion. Let G be a Lie group. A **differentiable path** through $x \in G$ is a differentiable function $\gamma : (-\varepsilon, \varepsilon) \to G$ for some $\varepsilon > 0$, such that $\gamma(0) = x$. It can be thought of as a differentiable function travelling through G (viewed as a manifold), with x at its origin. We say that two differentiable paths γ_1 and γ_2 are equivalent if $\gamma_1'(0) = \gamma_2'(0)$. That is to say, two such functions are equivalent if their derivative at x is the same.

Definition 4 (Tangent space)**.** Let G be a Lie group. The **tangent space** T_xG to G at x is the set of all equivalence classes of differentiable paths through x.

Physically, the tangent space at $x \in G$ can be viewed as the space of possible velocities of a particle on the manifold moving through x, just like the tangent line to a differentiable function at any point. Since group multiplication is smooth, the tangent plane at any point $x \in G$ is isomorphic to the tangent plane at the identity $e \in G$, and therefore all tangent planes $T_xG, x \in G$ are isomorphic to each other. As such, the Lie algebra associated with a Lie group G is the tangent space at the identity T_eG.

Theorem 2. *Let $G \subset GL(n, \mathbb{C})$ be a Lie group. Then T_eG with the Lie bracket $[x, y] = xy - yx$ is a real Lie algebra.*

Proof. Let $x, y \in T_eG$ with respective differentiable paths γ_x and γ_y, and let $\alpha \in \mathbb{R}$. Then the map $\gamma(t) := \gamma_x(\alpha t)\gamma_y(t)$ is a differentiable path such that $\gamma(0) = e$ and, by the product rule, $\gamma'(0) = \alpha x + y$. Thus, T_eG is a real subspace of $M_n(\mathbb{C})$.
Now, since T_eG is a real subspace of $M_n(\mathbb{C})$, it is closed and so

$$[x, y] = xy - yx = \gamma_x'(0)y\gamma_x(0) + \gamma_x(0)y(\gamma_x^{-1})'(0) = \frac{d}{dt}\gamma_x(t)y\gamma_x^{-1}(t)\Big|_{t=0} = \lim_{h\to 0} \frac{\gamma_x(t)y\gamma_x^{-1} - y}{h} \in T_eG.$$

It is easily checked that $[x, y]$ satisfies the conditions of the Lie bracket. □

2.3. Operator Algebras

The theory of operator algebras originates from the late 1930s when John von Neumann proposed a new framework for studying quantum mechanics using ”rings of operators,” mathematical objects which are now known as von Neumann algebras. Soon after, Soviet-American mathematician Israel Gelfand developed the theory of C*-algebras, which are generalizations of von Neumann algebras. The theory of C*-algebras and von Neumann algebras form a single area that we call **operator algebras**, now considered a branch of functional analysis. In a nutshell, operator algebras concerns the theory of algebras of bounded operators on a Hilbert space. That is to say, it is the study of the mathematical space $\mathcal{B}(\mathcal{H})$ as an algebraic object (or generalizations of this space), where the multiplication operation is given by the composition of operators.

Often, it will be desirable to study only a closed subspace of $\mathcal{B}(\mathcal{H})$, which is an operator algebra in its own right. These subspaces are what we call C*-algebras and von Neumann algebras, but in order to formally introduce them, we first need some concepts from functional analysis.

Recall that a **Banach algebra** is a complete normed algebra and that a $\star$**-algebra** is an algebra A with an **involution** on A, that is, an antilinear map $a \mapsto a^*$ on A, such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$.

Definition 5. A **C*-algebra** A is a Banach $\star$-algebra such that $\|a^*a\| = \|a\|^2$ for all $a \in A$.

It is not clear from the definition that C*-algebras are subspaces of $\mathcal{B}(\mathcal{H})$. In fact, for any C*-algebra A, there could be multiple ways to represent it as bounded operators acting on a Hilbert space, each arising from some linear functional $\tau : A \to \mathbb{C}$. For any such τ, the representation associated with it is known as the **Gelfand-Naimark-Stark representation** (commonly abbreviated to **GNS representation**).

Definition 6 (Positive linear functional)**.** Let A be a C*-algebra. An element $a \in A$ is called **positive** if it is self-adjoint and $\sigma(a) \subset [0, \infty)$. The set of positive elements of A is denoted by A^+.
A linear functional $\tau : A \to \mathbb{C}$ is called **positive** if $\tau(A^+) \subset [0, \infty)$, that is, τ maps positive elements to positive elements.

We will now prove that for a C*-algebra A, every positive linear functional $\tau : A \to \mathbb{C}$ gives rise to a $\star$-homomorphism $\varphi_\tau : A \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}_\tau$.

Theorem 3 (Gelfand-Naimark-Stark representation)**.** *Let A be a C*-algebra A and let $\tau : A \to \mathbb{C}$ be a positive linear functional. Then there exists a Hilbert space $\mathcal{H}_\tau$ and a map $\varphi_\tau : A \to \mathcal{B}(\mathcal{H})$ such that it is a $\star$-homomorphism.*

Proof. Let $N_\tau = \{a \in A \mid \tau(a^*a) = 0\}$, which is a closed left ideal of A, and define $\mathcal{H}_\tau := \overline{A/N}$.
The inner product $A/N_\tau \times A/N_\tau \to \mathbb{C},\ (a + N_\tau, b + N_\tau) \mapsto \tau(b^*a)$ turns $\mathcal{H}_\tau$ into a Hilbert space.
For $a \in A$, define $\varphi(a) : A/N_\tau \to A/N_\tau$ by $\varphi(a)(b + N_\tau) = ab + N_\tau$.
Since $\|\varphi(a)(b + N_\tau)\|^2 = \tau(b^*a^*ab) \le \|a\|^2\tau(b^*b) = \|a\|^2\|b + N_\tau\|^2$, we have $\|\varphi(a)\| \le \|a\|$, and so $\varphi(a)$ is bounded on A/N. The operator $\varphi(a)$ has a unique bounded extension to a bounded operator $\varphi_\tau(a)$ on $\mathcal{H}_\tau$.
Thus, $\varphi_\tau : A \to \mathcal{B}(\mathcal{H}_\tau)$ is a well-defined $\star$-homomorphism. ☐

The GNS representation has an additional property that makes it particularly attractive.

Definition 7 (Cyclic and separating vectors)**.** Let A be a C*-algebra acting on a Hilbert space $\mathcal{H}$ by some representation $\varphi : A \to \mathcal{B}(\mathcal{H})$.

1. A vector $x \in \mathcal{H}$ is called **cyclic** for A if $\overline{Ax} = \mathcal{H}$, where $Ax = \{\varphi(a)x \mid a \in A\}$.
2. A vector $x \in \mathcal{H}$ is called **separating** for A if $\varphi(a)x = 0 \implies \varphi(a) = 0$ for $a \in A$.

One can keep the following intuitive approach in mind. If $x \in \mathcal{H}$ is cyclic for A, then the operator $A \to \mathcal{H},\ a \mapsto \varphi(a)x$ has dense range. If $x \in \mathcal{H}$ is separating for A, then the operator $A \to \mathcal{H},\ a \mapsto \varphi(a)x$ is injective.

It turns out that the GNS representation of a C*-algebra A always admits a cyclic vector for A. Indeed, if A is unital, that is, it has a multiplicative identity $e \in A$, then $e + N_\tau \in A/N_\tau$ is easily seen to be cyclic for $\mathcal{H}_\tau$. It can be shown that every non-unital C*-algebra A admits a unitization $\widetilde{A}$, that is strictly larger than A (Section 2.1 in [13]). It is a unital C*-algebra and can be used to show that every non-unital C*-algebra A admits a cyclic vector as well (page 55 in [18]).

We now turn to von Neumann algebras, which are a special type of C*-algebra. They are defined more directly as closed subspaces of $\mathcal{B}(\mathcal{H})$. Of course, to speak of being "closed" requires that we endow $\mathcal{B}(\mathcal{H})$ with a topology.

Definition 8 (Strong and weak operator topologies)**.** The **strong operator topology** on $\mathcal{B}(\mathcal{H})$ is the coarsest topology on $\mathcal{B}(\mathcal{H})$ with the property that the linear mappings $T \mapsto Tx$ are continuous for all $x \in \mathcal{H}$.
The **weak operator topology** on $\mathcal{B}(\mathcal{H})$ is the coarsest topology on $\mathcal{B}(\mathcal{H})$ with the property that the linear mappings $T \mapsto (Tx|y)$ are continuous for all $x, y \in \mathcal{H}$.

To make these topologies explicit, the strong operator topology is generated by open sets of the form:

$$\{T \in \mathcal{B}(\mathcal{H}) : \|(T - S)x\| < \varepsilon\}$$

and the weak operator topology is generated by open sets of the form:

$$\{T \in \mathcal{B}(\mathcal{H}) : \|((T - S)x|y)\| < \varepsilon\},$$

with $\varepsilon > 0$, $x, y \in \mathcal{H}$ and $S \in \mathcal{B}(\mathcal{H})$.

Recall that a **subalgebra** of $\mathcal{B}(\mathcal{H})$ is a subspace of $\mathcal{B}(\mathcal{H})$ closed under composition. A $\star$**-subalgebra** of $\mathcal{B}(\mathcal{H})$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ closed under taking Hilbert space adjoints. A subalgebra is said to be **unital** if it contains the identity operator. We can now define von Neumann algebras.

Definition 9 (Von Neumann algebra)**.** A **von Neumann algebra** is a unital $\star$-subalgebra of $\mathcal{B}(\mathcal{H})$, that is closed in the strong operator topology.

Equivalently, a von Neumann algebra is a $\star$-subalgebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ with the following properties:

1. $I := \mathrm{id}_{\mathcal{H}} \in \mathcal{M}$
2. if $\{T_i\}_{i\in\mathbb{N}} \subset \mathcal{M}$, $T \in \mathcal{B}(\mathcal{H})$, and $\|(T_i - T)x\| \to 0$ for all $x \in \mathcal{H}$, then $T \in \mathcal{M}$.

A fundamental theorem in the study of operator algebras is that every abelian von Neumann algebra can be represented as essentially bounded functions on some measure space, where the addition and multiplication of these functions is defined pointwise. Formally, if $\mathcal{M}$ is an abelian von Neumann algebra, there is some second countable compact Hausdorff space X and some positive measure $\mu \in M(X)$, so that $\mathcal{M}$ is unitarily equivalent to $L^\infty(X, \mu)$. To prove this, it is useful to first prove another cornerstone theorem in this field, which gives a similar result for C*-algebras.

The corresponding theorem for C*-algebras states that every abelian C*-algebra can be represented as continuous functions on a locally compact Hausdorff space X that vanish at infinity. Due to this analogy, the theory of von Neumann algebras is sometimes called noncommutative measure theory, while the theory of C*-algebras is sometimes called noncommutative topology. This representation of abelian C*-algebras is known as the **Gelfand representation**. Before we prove this theorem, we describe what X and the continuous functions on X look like for this C*-algebra case.

Definition 10 (Character)**.** Let A be an abelian algebra. A **character** on A is a non-zero homomorphism $\tau : A \to \mathbb{C}$. The set of all characters on A, denoted by $\Omega(A)$, is called the **character space** of A.

Of course, in order to speak of the local compactness and Hausdorff property of X, it must be endowed with some topology. Although we do not prove it here, it can be shown that for an abelian C*-algebra A, the character space $\Omega(A)$ is a compact Hausdorff space, with respect to the relative weak*-topology (Theorem 1.3.5 in [13]). Hence, when speaking of the character space $\Omega(A)$ as a topological space, we always mean the set of characters $\Omega(A)$ endowed with the relative **weak*-topology**, that is, the coarsest topology on the dual space A^* (of which $\Omega(A)$ is a subset) such that the linear mappings $\phi \mapsto \phi(a)$ are continuous for all $a \in A$.

The character space of a C*-algebra is, in fact, the domain of the continuous functions used in the Gelfand representation. So, we will see that every $a \in A$ can be realized as some continuous function $\hat{a} : \Omega(A) \to \mathbb{C}$ that vanishes at infinity.

Now, given $a \in A$, we define:

$$\hat{a} : \Omega(A) \to \mathbb{C}, \quad \tau \mapsto \tau(a),$$

which we call the **Gelfand transform** of a. We now prove that the Gelfand transform is a continuous function that vanishes at infinity, formalizing the Gelfand representation.

Theorem 4 (Gelfand representation)**.** *If A is an abelian non-zero C*-algebra, then*

$$\varphi : A \to C_0(\Omega(A)), \quad a \mapsto \hat{a}$$

is an isomorphism of C-algebras.*

Proof. Let $a \in A$. Then $\hat{a}$ is continuous by the definition of the weak*-topology. Also, for every $\varepsilon > 0$, $\{\tau \in \Omega(A) : |\tau(a)| \geq \varepsilon\}$ is weak* closed in the closed unit ball of A^* and weak* compact by the Banach-Alaoglu theorem. Thus, $\hat{a}$ vanishes at infinity, and so $\hat{a} \in C_0(\Omega(A))$, so φ is well defined.
Next, let $\tau \in \Omega(A)$. Since multiplication in $C_0(\Omega(A))$ is defined pointwise and $\varphi(a^*)(\tau) = \tau(a^*) = \bar{\tau(a)} = \varphi(a)^*(\tau)$, it follows that φ is a $\star$-homomorphism.
Now, since $\|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| = r(a^*a) = \|a^*a\| = \|a\|^2$, we know that φ is isometric and therefore injective.
Finally, $\varphi(A)$ is a closed $\star$-subalgebra of $C_0(\Omega(A))$. Since it separates points in $\Omega(A)$ and vanishes nowhere, by Stone-Weierstrass, we have $\varphi(A) = C_0(\Omega(A))$. Hence φ is surjective.
We conclude that φ is a bijective $\star$-homomorphism, and hence, an isomorphism of C*-algebras. □

We are now in a position to prove that every abelian von Neumann algebra can be represented as $L^\infty(X, \mu)$ functions for some second countable compact Hausdorff space X and some positive measure $\mu \in M(X)$. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be an abelian von Neumann algebra. Many aspects of the proof break down when $\mathcal{M} = \{0\}$ or when $\overline{\mathcal{M}\mathcal{H}} \neq \mathcal{H}$. Of course, when $\mathcal{M} = \{0\}$, $0 \in \mathcal{M}$ can be represented as the zero function $X \to \mathbb{C},\ x \mapsto 0$, which is clearly bounded, and so the assertion is trivial. For this reason, from this point onwards, we always assume that $\mathcal{M}$ is a non-zero von Neumann algebra. If $\overline{\mathcal{M}\mathcal{H}} \neq \mathcal{H}$, there is some $y \in \mathcal{H}$ such that $Tx \neq y$ for all $T \in \mathcal{M},\ x \in \mathcal{H}$. In some sense, this means that the Hilbert space is "too big" for the von Neumann algebra, and it truly only acts on some subspace $\mathcal{H}' \subset \mathcal{H}$. If this is the case, simply redefine M to be a subset of $\mathcal{B}(\mathcal{H}')$, and the proof will provide the desired result. So, from this point onwards, we always assume that a von Neumann algebra $\mathcal{M}$ acts nondegenerately on its underlying Hilbert space. In the proof, we will need the Riesz-Markov-Kakutani representation theorem, which we give first.

Theorem 5 (Riesz-Markov-Kakutani representation theorem)**.** *Let Ω be a locally compact Hausdorff space and τ a positive linear functional on $C(\Omega)$. There there exists a unique positive Borel measure μ on Ω such that*

$$\tau(f) = \int_\Omega f \, d\mu, \quad f \in C(\Omega).$$

Proof. See page 40 in [20]. □

Let $\mathcal{M}$ be an abelian von Neumann algebra. At this point, it should be no surprise that the second countable compact Hausdorff space that serves as the domain of the L^∞ functions is the character space $\Omega := \Omega(\mathcal{M})$. The basic idea behind the proof is to first show that $\mathcal{H}$ is isometrically isomorphic to the space of square-integrable functions $L^2(\Omega, \mu)$ for some positive measure $\mu \in M(\Omega)$, after which it can be shown that $\mathcal{M}$ is unitarily equivalent to the von Neumann algebra of multiplication operators M_φ acting on $L^2(\Omega, \mu)$, where $\varphi \in L^\infty(X, \mu)$.

Theorem 6. *Let $\mathcal{M}$ be an abelian von Neumann algebra. Then there exists a second countable Hausdorff space Ω, a positive measure $\mu \in M(\Omega)$, and a unitary $u : \mathcal{H} \to L^2(\Omega, \mu)$, such that $u\mathcal{M}u^*$ is the von Neumann algebra of all multiplcation operators M_φ on $L^2(\Omega, \mu)$, where $\varphi \in L^\infty(\Omega, \mu)$.*

Proof. Let x be a cyclic vector for $\mathcal{M}$, so that $\overline{\mathcal{M}x} = \mathcal{H}$. The closed unit ball of $\mathcal{B}(\mathcal{H})$ is metrisable and separable for the strong topology (Remark 4.4.2 in [13]), so the same is true for the closed unit ball of $\mathcal{M}$. It follows that there is a separable C*-subalgebra A of $\mathcal{M}$ that is strongly dense in $\mathcal{M}$. We may assume $I = \mathrm{id}_{\mathcal{H}} \in A$. Let $\varphi : A \to C(\Omega)$ be the Gelfand representation and note that the compact Hausdorff space Ω is second countable, since $mathcalM$ is countably generated (Remark 4.4.1 in [13]). We define a positive linear functional τ on $C(\Omega)$ by setting $\tau(f) = (\varphi^{-1}(f)(x)|x)$. By the Riesz-Markov-Kakutani theorem, there exists a positive measure $\mu \in M(\Omega)$ such that $\tau(f) = \int f d\mu$ for all $f \in C(\Omega)$. The map

$$\pi : A \to \mathcal{B}(L^2(\Omega, \mu)), \quad v \mapsto M_{\varphi(v)},$$

is an injective $\star$-homomorphism. If $v \in A$, then

$$\int |\varphi(v)|^2 d\mu = \tau(|\varphi(v)|^2) = (\varphi^{-1}\varphi(v^*v)(x)|x) = \|v(x)\|^2.$$

Hence, the map $u : Ax \to C(\Omega)$, $v(x) \mapsto \varphi(v)$ is well-defined and isometric, since $C(\Omega)$ is L^2-dense in $L^2(\Omega, \mu)$, and it is clearly linear. Since $\overline{\mathcal{M}x} = \mathcal{H}$ and A is strongly dense in $\mathcal{M}$, we have $\overline{Ax} = \mathcal{H}$. We may therefore extend $u : Ax \to C(\Omega)$ to a unitary $u : \mathcal{H} \to L^2(\Omega, \mu)$. If $v, w \in A$ then $\pi(v)uw(x) = \varphi(vw) = uvw(x)$. Hence $\pi(v)u = uv$ for all $v \in A$. As a result, uAu^* is the algebra of multiplication operators with continuous symbol. Since A is strongly dense in $\mathcal{M}$, and $C(\Omega)$ is L^2-dense in $L^2(\Omega, \mu)$, $u\mathcal{M}u^*$ is the von Neumann algebra of all multiplication operators $\{M_\varphi \mid \varphi \in L^\infty(\Omega, \mu)\}$ on $L^2(\Omega, \mu)$. $\square$

The above theorem establishes a representation of abelian von Neumann algebras as $L^\infty(X, \mu)$ functions using the unitary equivalence of $\mathcal{M}$ and $\{M_\varphi \mid \varphi \in L^\infty(\Omega, \mu)\}$.

3

Physics Preliminaries

In 1687, Isaac Newton published his book Philosophiæ Naturalis Principia Mathematica, often called the greatest scientific work in history. It established the field of classical mechanics and introduced calculus to mathematically formulate the theory. Classical mechanics is typically concerned with everyday conditions: speeds are much lower than the speed of light, sizes are much larger than that of atoms, and energies are relatively small.

However, it turns out that when we are not confronted with everyday conditions, classical physics fails to accurately explain many physical observations. Indeed, towards the end of the nineteenth century, scientists discovered phenomena on both the macroscopic and microscopic scales that classical physics could not explain. The desire to resolve these issues between observations and theory led to a paradigm shift in the study of physics. The main theories that emerged from this revolution were quantum mechanics and relativity. The study of nature at the atomic and subatomic scales developed into a theory now known as quantum mechanics. Albert Einstein formulated relativity to describe nature at speeds close to the speed of light.

3.1. Classical Mechanics

We start by reviewing some classical physics, specifically mechanics, which is the study of motions of macroscopic bodies under the influence of certain specified forces.

A physical system generally consists of three basic ingredients [5]:

1. **states**, mathematical entities that embody the knowledge of a physical system;
2. **observables**, physical quantities that can be measured;
3. **symmetries**, state transformations that describe the passage of time.

In general, in a classical mechanical system, (pure) states are points in a **state space** formulated by a smooth manifold, usually corresponding to the possible values of certain properties of that system. Examples include $\mathbb{R}^{3n} \times \mathbb{R}^{3n}$, the state space of n free moving point particles, and $\{(x, p) \in \mathbb{R} \times \mathbb{R} : x^2 + p^2 = 1\}$, the state space of the harmonic oscillator. In these examples, the state space corresponds to the possible combinations of position and momentum the particles in these systems could attain. The state of a system gives us complete information about the properties and evolution of that system.

We know that classical observables are single-valued, meaning that they only take on one value at any one time. So, for example, a particle cannot be in two places at once and cannot have two different velocities. Given this observation, it hopefully seems clear that the way to model these physical quantities is with a function. Observables are physical quantities that can be measured, meaning that we can infer their values from the state of the system. Therefore, observables are formulated by **functions on the state space**. Thus, using mathematical notation, given some smooth manifold X, observables are functions of the form $f : X \to \Omega$, for some measurable space Ω. In general, observables are real-valued, which means that they take values in the real number line $\mathbb{R}$. However, there are observables better described by another space, like $\mathbb{C}$.

Since the evolution of a classical mechanical system in time is deterministic, given some state $x \in X$ (where X is the state space), we can determine the state of the system after an amount of time $t \in \mathbb{R}$ has passed. So, for any time $t \in \mathbb{R}$, there is a bijective state transformation $\phi_t : X \to X$, where $\phi_t(x)$ is the state of the system in state x after t time has passed. The passage of time is inherently a passive process, that is, it does not act directly on any physical system. Symmetries are state transformations that describe the passage of time, and so they are functions of the form ϕ_t that preserve the structure of the state space, specifically the smooth property it has. A bijective transformation that preserves the structure between two mathematical objects is known as an isomorphism, and an isomorphism from a mathematical object to itself is known as an automorphism of that object. Therefore, symmetries are formulated by **automorphisms of the state space**.

Summarising, a classical mechanical system is generally described by:

1. **states**, points in state space formulated by a smooth manifold;
2. **observables**, real-valued functions on state space;
3. **symmetries**, automorphisms of the state space.

Let X be the state space of a physical system with its time evolution described by the symmetries $\phi_t,\ t \in \mathbb{R}$. An isomorphism of smooth manifolds is called a diffeomorphism, and hence the group of all diffeomorphisms from X to itself (equivalently, automorphisms of X) is called the **diffeomorphism group** of X and is denoted by $\mathrm{Diff}(X)$. It forms an infinite-dimensional Lie group[1], where the group operation is given by composition. Like $\mathcal{B}(\mathcal{H})$, $\mathrm{Diff}(X)$ has two natural topologies: weak and strong, as defined in Definition 8. We know that physical quantities do not change values suddenly, that is, they vary continuously in time. Therefore, $\phi(x) : \mathbb{R} \to X,\ t \mapsto \phi_t(x)$ should be continuous for every $x \in X$. Thus, by definition, $\phi : \mathbb{R} \to \mathrm{Diff}(X),\ t \mapsto \phi_t$ should be strongly continuous.

Of course, after no time has passed, nothing has happened and so, $\phi_0 = \mathrm{id}_X$. Also, it makes no difference if we determine the state of the system in state x after s amount of time and then t amount of time has passed, or if we simply determine the state of the system in state x after $t + s$ time has passed. That is, the state transformations must satisfy $\phi_{t+s} = \phi_t \phi_s$. According to Definition 2, we conclude that, with respect to the strong topology on $\mathrm{Diff}(X)$, the **time evolution** of a physical system is given by a one-parameter group of symmetries. Usually, for a specific physical system, symmetries take only values in a certain Lie subgroup $G \subset \mathrm{Diff}(X)$. We will see shortly that every possible law of time evolution $\phi : \mathbb{R} \to G$ is uniquely generated by an element of the Lie algebra $T_e G$.

Mathematically, classical mechanics is formulated in terms of ordinary differential equations. These equations can be presented in one of three general formulations: Newtonian, Hamiltonian, or Lagrangian. In this thesis, we use the Hamiltonian formulation, which is most easily connected with quantum mechanics.

3.2. Single Particle

Let us turn to one of the most well-known examples of a classical mechanical system: a single particle in an external field. In Hamiltonian mechanics, the state of a classical particle is generally specified by a point $(x, p) \in \mathcal{P} \doteq \mathbb{R}^3 \times \mathbb{R}^3$, which describes its position and momentum. Thus, $\mathcal{P}$ is the state space of this physical system.

The states of this system completely describe the properties and behaviour of this particle. So, the particle's position and momentum are all the information we need to completely determine its properties (like its kinetic energy) and its evolution through time. The state space embodies the possible states of this physical system.

Next, we know that a classical particle carries with it physical quantities such as position, momentum, and energy. Any such quantity that can be measured is called an observable. For example, since the particle exists in an external field, $F : \mathcal{P} \to \mathbb{R}^d$ is an observable, where $F(x, p)$ is the total force acting on the particle in the state (x, p). We know the internal forces of the field, and so the total force on the particle is something that can be measured. Similarly, if we know the mass of the particle m, the kinetic energy of the particle is easily seen to be an observable of this system as well.

[1] Lie groups are often defined to be finite-dimensional, but there are many groups that resemble Lie groups, except for being infinite-dimensional. Some of the examples that have been studied include the diffeomorphism group of a smooth manifold and infinite-dimensional analogues of general linear groups, unitary groups, etc. In this case, much of the basic theory is similar to that of finite-dimensional Lie groups.

Lastly, as this particle moves in space and interacts with objects, we know that these quantities may change in time. For example, the total force acting on a classical particle due to the passage of time is described by Newton's second law: $m\frac{d^2x}{dt^2} = F(x)$. The way in which the particle (the states) and consequently its properties (the observables) change in time is closely related to the symmetries of the physical system.

For example, suppose that the particle is moving at a constant velocity $v \in \mathbb{R}^3$, such as an electron in absence of potentials. This means that because of the passage of time, the position of the particle changes at a constant rate, and the momentum does not change. Therefore, the symmetries of this system are translations. Clearly, a translation by $a \in \mathbb{R}^3$ acts on $\mathcal{P}$ by $T_a : \mathcal{P} \to \mathcal{P},\ (x, p) \mapsto (x + a, p)$. The set of all translations of X, denoted $\mathbb{T}(X) \subset \mathrm{Diff}(X)$, forms a Lie group. Hence, the time evolution of this particle is given by the one-parameter group of symmetries $T : \mathbb{R} \to \mathbb{T}(\mathcal{P}),\ t \mapsto T_{tv}$. Observe that every velocity $v \in \mathbb{R}^3$ uniquely generates a law of time evolution and corresponds to an element of the Lie algebra $T_0\mathbb{T}(X) \cong \mathbb{R}^3$. Also note that velocity is a **conserved** (constant in time) observable of this system.

As a second example, suppose that the particle is in orbit around the axis $e_3 = (0, 0, 1)$, like an electron orbiting a nucleus. In this case, as time passes, the particle's position and momentum rotate at a constant rate. Therefore, the symmetries of this system are rotations about the origin in $\mathbb{R}^3$, which are given by the Lie group $SO(3) \subset GL(3, \mathbb{R})$. A rotation of $\theta \in \mathbb{R}$ around e_3 acts on $\mathcal{P}$ by $\hat{R}(e_3, \theta) : \mathcal{P} \to \mathcal{P},\ (x, p) \mapsto (R(e_3, \theta)x, R(e_3, \theta)p)$, where $R(e_3, \theta) \in SO(3)$. Suppose that this particle rotates around e_3 with an angular velocity $\omega \in \mathbb{R}$. Then, the time evolution of this particle in orbit is given by the one-parameter group of symmetries $\hat{R} : \mathbb{R} \to SO(3),\ t \mapsto \hat{R}(e_3, t\omega)$. Each angular velocity $\omega \in \mathbb{R}$ uniquely determines an orbit around e_3, which, in turn, generates a law of time evolution. An infinitesimal rotation around e_3 is given by $X_3 := \frac{dR(e_3,\theta)}{dt}\Big|_{\theta=0}$ and so every angular velocity $\omega \in \mathbb{R}$ corresponds to some element ωX_3 of the Lie algebra $T_I SO(3) \subset M_3(\mathbb{R})$. The angular velocity is a conserved observable of this system.

Symmetries of a physical system given rise to a conserved observable is a general principle established by Noether's theorem. It reveals the fundamental relation between the symmetries of a physical system and the conservation laws.[16] It will play an important role in the next section, where we formulate the mathematics of quantum mechanics.

It should be mentioned that this explanation of modelling a classical mechanical system is incomplete and skips a lot of technical details, since this section aims to provide a brief and intuitive understanding of the subject matter. The concepts of states and observables in classical mechanics are made mathematically rigorous in Chapter 4.

3.3. Quantum Mechanics

For quantum mechanics, this model is not sufficient. This is due to the fundamental principle of quantum mechanics: that some attributes of a physical system cannot be specified exactly, but only by a probability density. The theory of quantum mechanics is formulated in a specially developed mathematical formalism. This mathematical formalism uses mainly a part of functional analysis, especially the theory of Hilbert spaces. In this section, we provide a quick introduction to this formulation.

The description of a quantum physical system is often presented in terms of three axioms, corresponding to the quantum mechanical formulation of states, observables, and symmetries, which can be traced back to the Dirac-von Neumann axioms. [5] [10]

Axiom 1. The pure states of the system are described by vectors of norm one in a complex **Hilbert space** $\mathcal{H}$.

We do not distinguish between states that are linearly dependent, that is, two states $\varphi, \psi \in \mathcal{H}$ are considered equivalent if $\alpha\varphi = \psi$ for some $\alpha \in \mathbb{C}$. Thus, it is the equivalence classes of unit vectors which describe the states of the system.

As a motivating example for a single particle, say an electron, the Hilbert space would be $\mathcal{H} = L^2(\mathbb{R}^3)$, with the domain $\mathbb{R}^3$ corresponding to either the particle's position or its momentum. In quantum mechanics, position and momentum are dual to each other (in the sense of Pontryagin duality), and so the state of its position completely determines the state of its momentum and vice versa. For a detailed treatment, see Section 15.5 in [14]. A particle in the state $\psi \in \mathcal{H} = L^2(\mathbb{R}, dx)$, with $\mathbb{R}^3$ corresponding to the position of the particle, is located with a probability distribution $|\psi(x)|^2$, where we refer to $L^2(\mathbb{R}, dx)$ as **position space**. That is, the probability of finding the particle in $B \subset \mathbb{R}^3$ is $\int_B |\psi(x)|^2 dx$. Now we define the **Fourier transform** $\widetilde{\psi} \in L^2(\mathbb{R}, dp)$ as the Fourier-Plancherel transform (on $L^2(\mathbb{R})$) of ψ. Then $|\widetilde{\psi}(p)|^2$ gives the probability distribution for the momentum, where we say that the Fourier transform $\widetilde{\psi}(p)$ is the wave function in **momentum space** $L^2(\mathbb{R}, dp)$. This interpretation of the Fourier transform is not special to this example, but pervades quantum physics. In the Physics literature, this 'duality' between the position and momentum observables is referred to as **complementarity**. Its most famous consequence is the **Heisenberg uncertainty principle**, relating the standard deviation of position σ_x and the standard deviation of momentum σ_p:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2},$$

showing that position and momentum cannot simultaneously be constrained arbitrarily sharply.

Due to this probabilistic nature, it now becomes clear that the state space should consist of functions relating to probability density functions. As a consequence, these spaces will be infinite-dimensional. Also, for two states $\varphi, \psi \in \mathcal{H}$, we would like to be able to express the probability of finding the physical system in state φ, given that it is in state ψ. Therefore, the state space should have some kind of inner product. Lastly, due to a quantum mechanical phenomenon known as **superposition**, we would like (infinite) linear combinations of states, which have norm one, to also be states. Hence, the state space should be complete. In conclusion, the state space of a quantum mechanical system should be a complete infinite-dimensional inner product space, which is exactly what a Hilbert space is.

Axiom 2. Properties of physical measurements of a system correspond to projection operators on $\mathcal{H}$. Physically measurable quantities for a system correspond to **self-adjoint operators** on $\mathcal{H}$.

For example, suppose that we would like to know whether the momentum of the electron is some value in $B \subset \mathbb{R}^3$. The axiom states that there is some projection $P : \mathcal{H} \to \mathcal{H}$ corresponding to this property, so that the probability that the momentum of the electron is some value in B is

$$(\psi|P\psi) = ||P\psi||^2.$$

Of course, we can do this for any (measurable) $B \subset \mathbb{R}^3$, so that $\{E(B)\}$ is a family of projection operators indexed by Borel sets $B \subset \mathbb{R}^3$, and the probability that the momentum of the electron is some value in B is

$$(\psi|E(B)\psi) = ||E(B)\psi||^2.$$

By the spectral theorem, there is a self-adjoint operator $A : \mathcal{H} \to \mathcal{H}$ such that:

$$(\psi|A\psi) = \int \lambda d(\psi|E(\lambda)\psi)$$

where $(\psi, A\psi)$ can be interpreted as the average value of the momentum after repeated measurements. This operator A is the self-adjoint operator associated with momentum, according to the axiom.

Conversely, given some observable, the spectral theorem also states that every self-adjoint operator can be realised as a projection-valued measure as well, where the projections correspond to properties of the observable.

The reason observables are described by operators, as opposed to functions, is due to the intrinsic non-commutative nature of quantum mechanics. At the quantum scale, things are so small, that even measuring a physical quantity impacts the physcial system. Therefore, observables (may) change the state of the system, which is why they are formulated as operators. Self-adjoint operators, then, are a natural choice for observables, as they are, in some sense, the operator-equivalent to real-valued functions, which described observables in classical mechanics. In fact, if $A : \mathcal{H} \to \mathcal{H}$ is self-adjoint, then $(\psi|A\psi)$ is real for every $\psi \in \mathcal{H}$. Since physical quantities are real-valued, $(\psi|A\psi)$ being a real number for every state $\psi \in \mathcal{H}$ is precisely the restriction that we would like for an observable A.

Axiom 3. The time evolution of a system is given by a one-parameter group of **unitary operators** $U(t)$ on $\mathcal{H}$ such that if $\psi \in \mathcal{H}$ is the state of the system at time zero, then $U(t)\psi$ is the state at time t.

Here, we have formulated time evolution so that states evolve in time and operators corresponding to observables are fixed in time. So, the expectation of an observable given by a self-adjoint operator A in a state ψ at time t is $(U(t)\psi, AU(t)\psi)$. This is known as the **Schrödinger picture**. There is also the **Heisenberg picture**, in which the operators evolve in time and the states are fixed.

Although the time evolution in quantum mechanics closely resembles the classical case, it is not clear why symmetries should be unitary operators. The basic idea, similar to the classical case, is that we would like invertible mappings that preserve the general structure of the state space $\mathcal{H}$. To understand why unitary operators fit this description, recall that for two states $\varphi, \psi \in \mathcal{H}$, $|(\varphi, \psi)|^2$ should be interpreted as the probability of finding the physical system in state φ, given that it is in state ψ. So, if the physical system is in state $U(t)\psi$, the probability of finding the system in state $U(t)\varphi$ should also be $|(\varphi, \psi)|^2$. Note that this does not imply that the probability of measuring certain outcomes of measurable quantities cannot change over time. Indeed, for any observable A and any symmetry U it need not be the case that $(\psi|A\psi) = (U\psi|AU\psi)$ for all $\psi \in \mathcal{H}$. It simply means that the passage of time cannot act directly on the state space. Thus, we must have $|(U(t)\varphi, U(t)\psi)|^2 = |(\varphi, \psi)|^2$ for all $t \in \mathbb{R}$. Since we arbitrarily chose $\varphi, \psi \in \mathcal{H}$ and $U(t)$ must be be invertible for all $t \in \mathbb{R}$, $U(t)$ should be a unitary operator for every $t \in \mathbb{R}$. In this sense, unitary operators are to Hilbert spaces what diffeomorphisms are to smooth manifolds, in that they are invertible mappings that preserve the structure of the spaces they operate on.

The group of all unitary operators from a given Hilbert space $\mathcal{H}$ to itself is referred to as the **Hilbert group**, and is denoted by $U(\mathcal{H})$. It forms an infinite-dimensional Lie group, where the group operation is composition, and, as a subset of $\mathcal{B}(\mathcal{H})$, has two natural topologies: weak and strong, as defined in Definition 8. Identical to the classical case, $U(\psi) : \mathbb{R} \to \mathcal{H},\ t \mapsto U(t)\psi$ should be required to be continuous for all $\psi \in \mathcal{H}$, to reflect the continuous nature of physical quantities. Thus, by definition, the time evolution can be realised as some strongly continuous function $U : \mathbb{R} \to U(H),\ t \mapsto U(t)$. Suppose, for a specific physical system, that its symmetries $\{U(t)\}_{t\in\mathbb{R}}$ lie in the Lie subgroup $G \subset U(\mathcal{H})$. Of course, then, according to Definition 2, $U : \mathbb{R} \to G,\ t \mapsto U(t)$ is precisely a one-parameter group of unitary operators. Again, all possible laws of time evolution are uniquely generated by an element of the Lie algebra T_eG.

We continue to build on this notion of **generators** of time evolution. Theorem 1 tells us that one-parameter subgroups of $GL(n, \mathbb{C})$ are uniquely determined by elements $x \in M_n(\mathbb{C})$. Similarly, observe that if $H = H^*$ for some $A : \mathcal{H} \to \mathcal{H}$, then $\exp(-itH)$ is unitary for all $t \in \mathbb{R}$, and so $\phi : t \to U(\mathcal{H}),\ t \mapsto \exp(-itH)$ is a well-defined one-parameter group of unitary operators on $\mathcal{H}$. Conversely, if $U : \mathbb{R} \to U(\mathcal{H}),\ t \mapsto U(t)$ is a one-parameter group of unitary operators on $\mathcal{H}$, it turns out that there is always a self-adjoint operator $H : \mathcal{H} \to \mathcal{H}$ such that $U(t) = \exp(-itH)$ for all $t \in \mathbb{R}$. This is due to Stone's theorem on one-parameter unitary groups. Thus, a law of time evolution in quantum mechanics is uniquely generated by a self-adjoint operator on $\mathcal{H}$.

Whenever a system of physical laws admits a one-parameter group of symmetries, Noether's theorem implies there is a conserved observable, corresponding to the total energy of the system. In the examples in section 3.2, we saw that this conserved observable was a generator of time evolution of the physical system. Similarly, as self-adjoint operators correspond to observables, time evolution is generated by an observable in quantum mechanics as well. This observable is known as the **Hamiltonian** of a quantum mechanical system. To show that the Hamiltonian is conserved, let $U(t) = \exp(-itH)$ be the time evolution for some quantum mechanical system, so that H is the Hamiltonian. Then for any $\psi \in \mathcal{H}$, differentiating both sides yields:

$$\frac{dU(t)\psi}{dt} = -iH\exp(-itH)\psi.$$

After rewriting and rescaling the Hamiltonian by a fundamental constant of nature $\hbar$, we are left with a landmark discovery in the development of quantum mechanics, the **Schrödinger equation**:

$$i\hbar\frac{dU(t)\psi}{dt} = HU(t)\psi.$$

The usual interpretation is that the Hamiltonian represents the total energy of the system, and its conservation represents the law of conservation of energy.

Again, a lot of technical details are skipped in this introduction. The notions of states, observables, and time evolution in quantum mechanics are made mathematically rigorous the next chapter.

4

States and Observables

4.1. States and Observables in Classical Mechanics

As we have seen in Chapter 2, any well-defined physical theory should have some notion of ”states” and ”observables”. In this chapter, we give a rigorous mathematical formulation of these terms, as is done in [14]. Consider any physical system. Intuitively speaking, a state is a mathematical object that represents any given state the physical system might find itself in. An observable, intuitively, is any physical quantity that can be measured by an observer of the physical system.

For example, consider a free-moving massive point particle in space. This physical system can be described by states given by vectors in $\mathbb{R}^3 \times \mathbb{R}^3$. Indeed, such a particle has 3 position coordinates and 3 momentum coordinates completely describing its state. We will later see that the definition of a state is actually much more general. Here, $\mathbb{R}^3 \times \mathbb{R}^3$ is called the **state space** of the physical system.

An observable of this system might be the velocity of the particle. Given any state, we can measure the velocity of the particle by taking its given momentum and dividing coordinate-wise by the mass of the particle. An observable, then, is a function from the state space to some other measurable space. In this example, this measurable space is $\mathbb{R}^3$, as velocities are described by vectors in $\mathbb{R}^3$.

Definition 11 (State)**.** Let $(X, \mathcal{X})$ be a measurable space. A **state** is a probability measure ν on $(X, \mathcal{X})$.

Intuitively, for a measurable set $B \in \mathcal{X}$, one should think of $\nu(B)$ as the probability that the state of the physical system under consideration is described by a point in B. An extreme point of the set of probability measures on $(X, \mathcal{X})$ is called a **pure state**.

Returning to the example of a free-moving particle, suppose that we know with certainty that the particle is at rest and located at the origin. The state of this physical system is given by the probability measure:

$$\nu(B) = \begin{cases} 1 & \text{if } \{0,0,0\} \times \{0,0,0\} \in B \\ 0 & \text{otherwise} \end{cases}$$

Clearly, this is a pure state. Pure states are in one-to-one correspondence with points in the state space. This definition of a state allows for so-called mixed states, where the position or momentum of the particle is not known with certainty. A **mixed state**, then, is a state which is not a pure state.

Definition 12 (Observable)**.** Let $(X, \mathcal{X})$ be a measurable space. An **observable** is a measurable function $f : X \to \Omega$ for some measurable space $(\Omega, \mathcal{F})$. In this case, we call f an Ω-valued observable.

Recall that an observable is any physical quantity that can be measured. Examples may include position, velocity, energy, or angular momentum. Returning to our example, the velocity of the particle is an $\mathbb{R}^3$-valued observable $v : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$v\big((x_1, x_2, x_3) \times (p_1, p_2, p_3)\big) = \frac{1}{m}(p_1, p_2, p_3)$$

An **elementary observable** is a $\{0,1\}$-valued observable. For example, any indicator function $\mathbf{1}_B$ of a measurable set $B \in \mathcal{X}$ is an elementary observable. In fact, these are all the possible elementary observables.

Many popular descriptions of quantum mechanics imply that the concept of mixed states encapsulates the uncertainty inherent to the theory. As an example, let us consider two pure states of a physical system x and y. One could think of these states as two different positions for a free-moving particle in space. Now, let f be an $\mathbb{R}$-valued observable, which takes the value 1 in state x and the value 5 in state y. This observable could represent, say, the particle's speed.

We can now define the mixed state $\frac{1}{3}x + \frac{2}{3}y$, describing an uncertain physical system which has a $\frac{1}{3}$ probability of actually being in state x, and a $\frac{2}{3}$ probability of actually being in state y. Therefore, if we measure f in this mixed state, we have a $\frac{1}{3}$ chance of measuring the value 1, and a $\frac{2}{3}$ chance of measuring the value 5.

Mixed states, however, are not nearly weird enough to capture what is commonly referred to as quantum behaviour. Mixed states are not the same thing as quantum superposition. Mixed states are useful for studying large-scale physical systems made up of an enormous number of particles, where there is no reason to assume that one would have precise information on the state of each individual particle. The most obvious example of this situation is a gas in a box. This idea is often called "classical ignorance."

4.2. States and Observables in Quantum Mechanics

To formulate the ideas of states and observables in quantum mechanics, let us first consider the state space of a quantum mechanical system. Recall that the state space corresponds to all possible pure states. As we have seen in Chapter 2, it turns out to be useful to characterise the state space of a quantum mechanical system by a complex Hilbert space.

A **Hilbert space**, in short, is a Banach space in which the norm comes from an inner product. It allows the methods of linear algebra and calculus to be generalised from finite-dimensional Euclidean vector spaces to spaces that may be infinite-dimensional. One should then think of the pure states as the vectors in a Hilbert space, where we do not distinguish between vectors that differ between a phase. That is, if $\mathcal{H}$ is a Hilbert space and $x, y \in \mathcal{H}$, then x and y correspond to the same pure state if and only if $x = \alpha y$ for some $\alpha \in \mathbb{C}$.

Every Hilbert space in this thesis is assumed to be separable, unless stated otherwise. In mathematics, a topological space is called **separable** if contains a countable, dense set; that is, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence. It can then be proven that a Hilbert is separable if and only if admits a countable orthonormal basis (Theorem 3.22 in [14]). Separability is a mathematically convenient hypothesis, with the physical interpretation that any state is uniquely determined by countably many observations.

Next, recall that in classical mechanics, an elementary observable is of the form $\mathbf{1}_B$ with $B \in \mathcal{X}$. Its range is $\{0,1\}$, unless $B = \varnothing$ or $B = X$, in which case $\mathbf{1}_{\varnothing} = 0$ and $\mathbf{1}_X = 1$. Orthogonal projections in a complex Hilbert space enjoy similar properties. If P is an orthogonal projection in a Hilbert space $\mathcal{H}$, its spectrum is $\sigma(P) = \{0,1\}$ unless $P = 0$ or $P = I$, in which case $\sigma(0) = \{0\}$ and $\sigma(I) = \{1\}$. The basic idea, then, that underlies the mathematical formulation of Quantum Mechanics is to replace elementary observables by orthogonal projections in a complex Hilbert space. The set of all such projections is denoted by $\mathcal{P}(\mathcal{H})$.

We aim to define observables in the quantum setting. Recall that an Ω-valued observable, classically, is a measurable function $f : X \to \Omega$, for some measurable space $(\Omega, \mathcal{F})$. By the definition of measurability, this induces a mapping from $\mathcal{F}$ to $\mathcal{X}$:

$$F \mapsto f^{-1}(F), \quad F \in \mathcal{F}.$$

Identifying sets in $\mathcal{X}$ by their indicator functions and replacing them by orthogonal projections in a complex Hilbert space $\mathcal{H}$, we arrive at our definition of an observable in Quantum Mechanics.

Definition 13 (Observable)**.** Let $\mathcal{H}$ be a Hilbert space. An **observable** is a countably additive mapping $P : \mathcal{F} \to \mathcal{P}(\mathcal{H})$ satisfying $P(\Omega) = I$ for some measurable space $(\Omega, \mathcal{F})$. In this case, we call P an Ω-valued observable.

By countably additive, we mean that $P(F_1 \cup F_2) = P(F_1) + P(F_2)$ for any two disjoint $F_1, F_2 \in \mathcal{F}$. We will see later that this restriction is set in place to preserve the probabilistic nature of states and observables.

Indeed, if ν is a classical state on $(X, \mathcal{X})$ and $f : X \to \Omega$ is an observable for some measurable space $(\Omega, \mathcal{F})$, then for $F \in \mathcal{F}$

$$\nu(f^{-1}(F)) = \nu(\{x \in X : f(x) \in F\})$$

is the probability that measuring f results in a value in F when the physical system is in state ν. Equivalently,

$$F \mapsto \nu(f^{-1}(F)), \quad F \in \mathcal{F}$$

defines a probability measure on $(\Omega, \mathcal{F})$.

We would like to preserve this structure when moving to the quantum setting. We have not yet defined a state in the quantum mechanical setting, so for now, let ϕ denote a mathematical object associated to a state and let $P : \mathcal{F} \to \mathcal{P}(\mathcal{H})$ be an Ω-valued observable. What we would then like to see is that

$$F \mapsto \phi(P(F)), \quad F \in \mathcal{F}$$

defines a probability measure on $(\Omega, \mathcal{F})$. From this observation, one can deduce that states should be something close to the form $\phi : \mathcal{P}(\mathcal{H}) \to [0, 1]$. It turns out the definition can be made more general.

Definition 14. A (normal) **state** is a positive (normal) functional $\phi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ satisfying $\phi(I) = 1$.

Here, $\mathcal{B}(\mathcal{H})$ denotes the space of bounded operators on a Hilbert space. A functional is called **positive** if $\phi(T) \geq 0$ for every positive operator $T \in \mathcal{B}(\mathcal{H})$, and **normal** if

$$\sum_{n\geq 1} \phi(P_n) = \phi(P)$$

whenever $(P_n)_{n\geq 1}$ is a sequence of disjoint orthogonal projections in $\mathcal{H}$ and P is their least upper bound.

We denote the set of all states by $\mathcal{S}(\mathcal{H})$. There are multiple equivalent definitions one could use for states, but this one is the most common. Again, the restrictions of positiveness and normality serve to retain the probabilistic structure of states and observables. This is the mathematical counterpart of the so-called ”Born rule” in quantum mechanics.

Proposition 1 (Born rule). *If $\phi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is a state and $P : \mathcal{F} \to \mathcal{P}(\mathcal{H})$ an Ω-valued observable, the mapping*

$$F \mapsto \phi(P(F)), \quad F \in \mathcal{F}$$

defines a probability measure on $(\Omega, \mathcal{F})$.

Similarly to the definition in classical mechanics, a **pure state** is an extreme point of the convex set of states. Also, like in the classical setting, we can characterise pure states by points in the state space.

Proposition 2. *A state $\phi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is pure if and only if it is a* ***vector state****, that is, there exists a unit vector $h \in \mathcal{H}$ such that*

$$\phi(S) = (Sh|h), \quad S \in \mathcal{B}(\mathcal{H})$$

This unit vector is unique up to a scalar of modulus one.

Note that a real-valued observable P is nothing more than a projection-valued measure on $\mathbb{R}$, and hence, by the spectral theorem, we can associate a unique self-adjoint operator A with P. The spectral theorem also asserts the converse, that every self-adjoint operator arises from a projection-valued measure on $\mathbb{R}$. This leads us to the conclusion that real-valued observables are in one-to-one correspondence with self-adjoint operators on a Hilbert space.

With this in mind, let P be a real-valued observable and A be its associated self-adjoint operator. Then, as a projection-valued measure, P is supported on $\sigma(A)$, and therefore, P can be thought of as a $\sigma(A)$-valued observable. The physical interpretation here is that with probability 1, a measurement of P produces a value belonging to $\sigma(A)$.

In general, most observables will be real-valued, and so, in many cases, all observables of a physical system can be identified by self-adjoint operators. These observables generate an operator algebra, either a C*-aglebra or a von Neumann algebra. This algebra is referred to as the **algebra of observables** and this concept will play an important role in the rest of this thesis. We explain this concept in more detail in section 5.3.

4.3. Symmetries

Now that we have a rigorous formulation of states and observables, we would like to describe how a physical system evolves in time, that is, we would like to formulate a law of time evolution. As seen in Chapter 2, time evolution in classical mechanics is described by a one-parameter group of diffeomorphisms of the state space, while in quantum mechanics it is described by a one-parameter group of unitary operators acting on the state space. However, these conclusions were drawn on the assumption that states are simply elements of the state space. Now we have defined states as probability measures in classical mechanics and functionals on $\mathcal{B}(\mathcal{H})$ in quantum mechanics, we also have a different law of time evolution. Whatever the mathematical object most preferable to describe time evolution turns out to be, the broader concept of a transformation that does not change the outcome of possible experiments formulated by an automorphism of the state space, like the passage of time, is called a **symmetry**.

Returning briefly to the classical setting, let $(X, \mathcal{X})$ be a measurable space. Recall that the idea of a law of time evolution encompassed invertible mappings on the state space $\phi_t : X \to X$ that preserved its structure. Now, given a state v on $(X, \mathcal{X})$, a symmetry must have the property that probabilities are left invariant under the passage of time. That is to say, if there is a probability $p \in [0, 1]$ that the state is in $B \in \mathcal{X}$ at time $t = 0$, there must be a probability p that the state is in $\phi_1(B)$ at time $t = 1$. This is more or less by the definition ϕ_t. It describes exactly how the states evolve in time. With this in mind, we come to the following definition of symmetries in the classical setting.

Definition 15. A **symmetry** of the measure space $(\Omega, \mathcal{F}, \mu)$ is a measurable bijective mapping $g : \Omega \to \Omega$ with measurable inverse that leaves μ invariant, that is,

$$(g(\mu))(F) \coloneqq \mu(g^{-1}(F)) = \mu(F), \quad F \in \mathcal{F}.$$

Now, coming back to the quantum setting, we recall that, in the Schrödinger picture, the idea of time evolution encompassed operators which preserve transition probabilities between pure states. This probability was expressed using the inner product of the Hilbert space. However, given two states $\phi, \psi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$, we have not yet given a way to express their transition probability, that is, the probability of finding the system in state ϕ, given that it is in state ψ. To circumvent this issue, we provide an equivalent way of defining the states of a physical system.

Definition 16. Let $\mathcal{H}$ be a separable Hilbert space and let E be an orthonormal basis of $\mathcal{H}$. The **trace** of an operator $T \in \mathcal{B}(\mathcal{H})$ is

$$\text{Tr}(T) = \sum_{x \in E} (Tx|x)$$

If $\text{Tr}(|T|) < \infty$, we call T **trace class**.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is **positive** if it is self-adjoint and $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$. We denote by $\mathcal{S}(\mathcal{H})$ the set of all positive trace class operators with unit trace on $\mathcal{H}$. We now show that this set is in one-to-one correspondence with the set of all states.

Theorem 7. *The following sets are in one-to-one correspondence:*

1. *positive trace class operators T on H satisfying $Tr(T) = 1$*
2. *positive normal functionals $\phi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ satisfying $\phi(I) = 1$*

Proof. For a proof, see Theorem 15.7 in [14]. □

In essence, the above theorem tells us that the states of a quantum mechanical system can be represented by $\mathcal{S}(\mathcal{H})$, where the transition probability between $S \in \mathcal{S}(\mathcal{H})$ and $T \in \mathcal{S}(\mathcal{H})$ is given by $\text{Tr}(TS)$ With this in mind, it seems natural to define a symmetry in the quantum setting by a mapping $\mathcal{U} : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ that satisfies $\text{Tr}(\mathcal{U}T_1\mathcal{U}T_2) = \text{Tr}(T_1T_2)$. Indeed, these are mappings sending states to states which preserve transition probabilities between pure states. Remarkably, it turns out that for every such mapping, there is an operator $U \in \mathcal{B}(\mathcal{H})$, which is either unitary or antiunitary, which is an antilinear operator $T : \mathcal{H} \to \mathcal{H}$ such that $TT^* = T^*T = I$. This is due to a celebrated theorem by Eugene Wigner, which is considered a cornerstone of the mathematical formulation of quantum mechanics.

Theorem 8 (Wigner)**.** *If $\mathcal{U} : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ is a bijection with the property that*

$$Tr(\mathcal{U}(t)T_1\mathcal{U}(t)T_2) = Tr(T_1T_2), \quad T_1, T_2 \in \mathcal{S}(\mathcal{H}),$$

there exists a mapping $U : \mathcal{H} \to \mathcal{H}$ that is unitary or antiunitary such that

$$\mathcal{U}T = UTU^*, \quad T \in \mathcal{S}(\mathcal{H}).$$

This mapping is unique up to a complex scalar of modulus one.

Essentially, Wigner's theorem tells us that symmetries $\mathcal{U}$ of quantum mechanical systems are given by operators U acting on the underlying Hilbert space $\mathcal{H}$ that are either unitary or antiunitary. Of course, we are mainly interested in one-parameter groups of symmetries indexed by time. Suppose $\{\mathcal{U}(t)\}_{t\in\mathbb{R}}$ is such a group, and let $\{U(t)\}_{t\in\mathbb{R}}$ be the associated group of operators. It need not necessarily be the case that all $U(t)$ are unitary operators. Some may be, and others may be antiunitary. However, a theorem of Bargmann implies the existence of a function $d : \mathbb{R} \to \mathbb{T}$, such that the operators $V(t) := d(t)^{-1}U(t)$ are all unitary and satisfy:

$$\mathcal{U}(t)T = V(t)^*TV(t), \quad V(t)V(s) = V(t+s), \quad V(0) = 1.$$

The map $V : \mathbb{R} \to \text{Hilb}(H),\ t \mapsto V(t)$ can be shown to be strongly continuous and, therefore, $\{\mathcal{U}(t)\}_{t\in\mathbb{R}}$ can be realized as a one-parameter group of unitary operators $\{V(t)\}_{t\in\mathbb{R}} \subset \mathcal{B}(\mathcal{H})$. Thus, our original conception of time evolution, as provided in Chapter 2, is sufficient in this framework as well, and so, by Stone's theorem, $\{V(t)\}_{t\in\mathbb{R}}$ is generated by a self-adjoint operator, which we call the **Hamiltonian** associated with the family $\{\mathcal{U}(t)\}_{t\in\mathbb{R}}$. This motivates the following definitions.

Definition 17 (Symmetry, of a Hilbert space)**.** A symmetry of $\mathcal{H}$ is a unitary operator on $\mathcal{H}$.

Definition 18 (Conservation and covariance)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let U be a symmetry of the Hilbert space $\mathcal{H}$.

1. An observable $P : \mathcal{F} \to \mathcal{P}(\mathcal{H})$ is said to be **conserved** under U if $UP_FU^* = P_F$ for all $F \in \mathcal{F}$.
2. An observable is said to be **covariant** under the pair (g, U), where g is a symmetry of $(\Omega, \mathcal{F}, \mu)$, if $UP_FU^* = P_{g(F)}$ for all $F \in \mathcal{F}$, that is, if the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "F\mapsto g(F)"] \arrow[d, "P"] & \mathcal{F} \arrow[d, "P"] \\
\mathcal{P}(\mathcal{H}) \arrow[r, "P_F\mapsto UP_FU^*"'] & \mathcal{P}(\mathcal{F})
\end{tikzcd}$$

The Hamiltonian is a conserved observable corresponding to the total energy of a physical system.

In conclusion, a quantum mechanical system is mathematically formulated by:

1. a state space given by a separable complex Hilbert space $\mathcal{H}$ and **states**, positive normal functionals $\phi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ satisfying $\phi(I) = 1$;
2. an algebra of **observables**, countably additive mappings $P : \mathcal{F} \to \mathcal{P}(\mathcal{H})$ satisfying $P(\Omega) = I$ for some measurable space $(\Omega, \mathcal{F})$;
3. a one-parameter group of **symmetries**, unitary operators on the state space $\mathcal{H}$, generated by an observable conserved under these symmetries.

5

Thermal time

One of the most important concepts in all of physical theory is the concept of time. Time, by the dictionary definition, is the continued sequence of existence and events that occurs in an apparently irreversible succession from the past, through the present, and into the future. It is one of the seven fundamental physical quantities in the International System of Units (SI). Until Einstein's reinterpretation of the physical concepts associated with time and space in 1907, time was considered to be "the same" everywhere in the universe, with all observers measuring the same time interval for any event. Classical mechanics is based on this Newtonian idea of time, where it is treated as a classical background parameter, external to the system itself.

Einstein, in his special theory of relativity, postulated the constancy and finiteness of the speed of light for all observers. The theory of special relativity finds a convenient formulation in Minkowski spacetime, a mathematical structure that combines three dimensions of space with a single dimension of time. In relativity, time is a geometrical flow in this space-time. Recall from chapter 3 that, in quantum mechanics, time is also something that flows. However, unlike in relativity, it is a flow in an abstract mathematical space, preserving some kind of algebraic structure.

The rest of this chapter is largely based in Pierre Martinetti's 2013 paper "Emergence of time in quantum gravity: is time necessarily flowing?" [12]

5.1. The geometric time of relativity

In special relativity, time evolution has a clear geometrical interpretation: the movement of an observer, that is the evolution of its position as time passes, is described by a **worldline**, namely a one-dimensional trajectory in a mathematical space-time of dimension four. Consider a Minkowski space, which is the flat space-time of special relativity in the absence of curvature, as given in figure 5.1. The temporal evolution of a static observer is described by a line parallel to the T axis, and the surfaces of simultaneity are parallel to the X axis. Meanwhile, for an observer whose speed is constant with respect to the static observer, the temporal evolution as well as the surfaces of simultaneity are no longer parallel to any of the axes.

This means in relativity there is no absolute time, that is to say, there is no global object which flows everywhere in the same way. However, although there is no unique absolute time, each observer along his worldline does experience a single time, called his **proper time**. The proper time of a first observer may not be identical to the proper time of a second observer, but one knows how to go from one to the other. As an analogy, there is no absolute time on Earth: noon-bells do not ring simultaneously in Kuala Lumpur or Hong Kong. In figure 5.1, this means that the origin of time on the worldline $X =$ Kuala Lumpur is not on the same surface of simultaneity as the origin of time on the worldline $X =$ Hong Kong. However, there is a universal time, divided into time zones, and each observer knows how to regulate his clock according to it: noon bells should ring when the Sun is at the highest point in the sky of Kuala Lumpur or Hong Kong.

Summarising, time in relativity can be considered as a geometrical flow where the time evolution of an observer is a locally unique one-dimensional trajectory in Minkowski space.

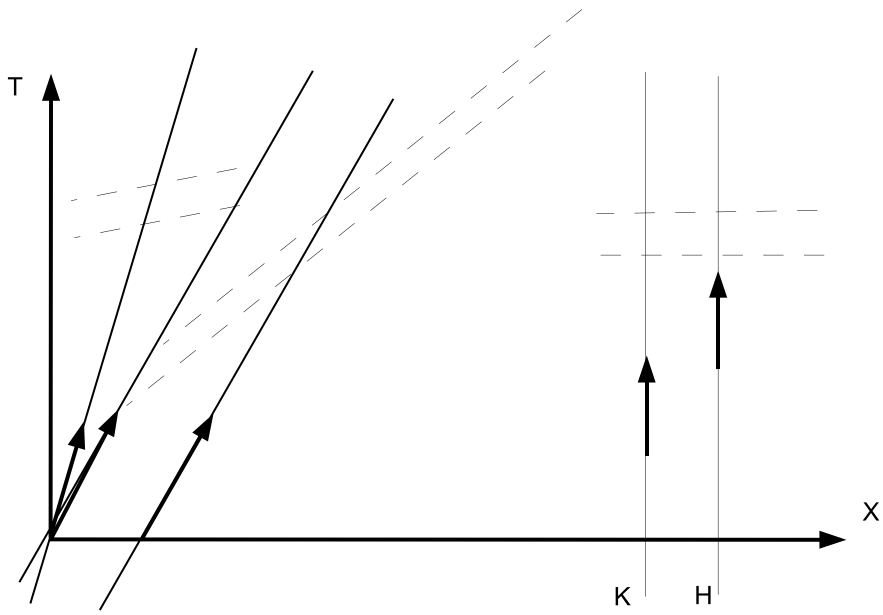

Figure 5.1: The T axis is the worldline of an observer who stays at the same place $X = 0$ at any time. In this space-time, being immobile at a point K or H in space corresponds to worldlines parallel to this T axis. On the left, the worldlines of two observers with the same non-zero constant speed, and a third one with lower speed.[12]

5.2. The algebraic time of quantum mechanics

In quantum mechanics, time is not a geometrical flow. As we have seen in Chapters 3 and 4, time evolution is characterised as a one-parameter group of symmetries. Most of the formulation was done in the Schrödinger picture, where these symmetries were invertible state transformations that preserved the algebraic structure of the state space, and observables were fixed in time. There is also the equivalent Heisenberg picture, in which observables evolve in time and states remain fixed.

In this framework, symmetries are formulated as invertible transformations that preserve the algebraic relations between physical observables. For example, suppose we have a free particle with state space $\mathbb{R}^3 \times \mathbb{R}^3$, corresponding to its position and momentum. Suppose that an observable of this system, say the particle's angular momentum $L \in \mathbb{R}$, is defined as a combination of other observables, say its position $(x_1, x_2, x_3) \in \mathbb{R}^3$ and its momentum $(p_1, p_2, p_3) \in \mathbb{R}^3$, so that

$$L = x_1 p_2 + x_2 p_1.$$

Since this relation is time independent, the symmetries in the Heisenberg picture must preserve the algebraic form of this relation. That is, if α is some symmmetry of this physical system, we must have that

$$\alpha(L) = \alpha(x_1)\alpha(p_2) + \alpha(x_2)\alpha(p_1).$$

Therefore, in this framework, symmetries are **automorphisms of the algebra of observables**.

As mentioned earlier, the Schrödinger and Heisenberg pictures are equivalent, which we show now. Indeed, suppose we have a quantum system with state space $\mathcal{H}$ and algebra of observables $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$. Let $\{U(t) : \mathcal{H} \to \mathcal{H}\}_{t\in\mathbb{R}}$ be its time evolution in the Schrödinger picture and let $\{\alpha_t : \mathcal{M} \to \mathcal{M}\}_{t\in\mathbb{R}}$ be its time evolution in the Heisenberg picture. Then for any state $\psi \in \mathcal{H}$ and any observable $A \in \mathcal{M}$ we have

$$U(t)\alpha_t(A)\psi = AU(t)\psi$$

Indeed, both sides of the equation evolve the system over a time t and measure the observable A after an evolution of the same time t, they just do it in different orders. Hence, the expectation of $A \in \mathcal{M}$ in state $\psi \in \mathcal{H}$ at time $t \in \mathbb{R}$ is

$$\big(\psi|\alpha_t(A)\psi\big) = \big(U(t)\psi|(t)AU(t)\psi\big) = \big(\psi|U^{-1}(t)AU(t)\psi\big)$$

where we have used the fact that $U(t)$ is unitary.

Therefore, every one-parameter group of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ corresponds to a one-parameter group of automorphisms of the algebra of observables $\{\alpha_t\}_{t\in\mathbb{R}}$ and vice versa by the relation

$$\alpha_t(A) = U^{-1}(t)AU(t), \quad A \in \mathcal{M},\ t \in \mathbb{R}. \tag{5.1}$$

Hence, the two pictures are equivalent as they produce the same expectation values.

In any case, time evolution in quantum mechanics is formulated by a one-parameter group of symmetries. It can still be considered a flow, but not a geometrical flow as is the case in relativity. The group property of time evolution in quantum mechanics implies that the evolution is additive. Indeed, if $\{U_t\}_{t\in\mathbb{R}}$ is a one-parameter group of symmetries, then $U(t+t') = U(t)U(t')$, for any $t, t' \in \mathbb{R}$. Since this group is strongly continuous, given any state $\psi \in \mathcal{H}$, we can think of

$$U(s)U(t)\psi, \quad s \in [0, t']$$

as a trajectory from t to t'. In this sense, time is still something that flows, but in an abstract mathematical space, rather than a four-dimensional space-time. In quantum mechanics, time has an algebraic interpretation: the evolution of a system is described by one-parameter group of transformations that preserve the algebraic relations between observables. Furthermore, the time flow is identical for every state, in that it is given by the same one-parameter group. Unlike in relativity, time is universal and absolute; it is the same for all observers.

5.3. The thermal time hypothesis

One of the most important unsolved problems in physics is the formulation of a theory of everything, which would unify the theories of general relativity and quantum mechanics. These are the theories upon which all of modern physics rests. The two theories are considered incompatible in regions of extremely small scale, such as those that exist within a black hole or the moment immediately following the Big Bang. In pursuit of resolving this incompatibility, **quantum gravity** has become an area of active research.

One of the issues quantum gravity addresses is the **problem of time**, referring to the conceptual conflict between general relativity and quantum mechanics in that quantum mechanics regards the flow of time as universal and absolute, whereas general relativity regards the flow of time as malleable and relative. Before examining this conflict, we would like to emphasise that the problem that arises from this conflict is counter-intuitive. Whereas time in quantum mechanics is a unique absolute parameter, the principle of superposition actually offers a multiplicity of time evolutions. Meanwhile, even though there are as many proper times in relativity as there are observers to the system, restricting a physical system to one perspective demands local uniqueness of the time evolution.

Quantum uncertainties appear when one measures an observable: the theory predicts the probability that an observable A lies within some measurable set F (Proposition 1). Before the measurement process, a quantum system is thus described by a state ψ, containing the statistical information of the observables. The state can be thought of as a superposition of all the possible values of A (or any other observable).

In the same way, in quantum gravity, one expects the gravitational field to be described by a superposition of all possible gravitational fields. In general relativity, the gravitational field determines the metric of space-time, and this metric indicates how space can be separated from time. So, even for a single observer, a multiplicity of gravitational fields leads to a multiplicity of time flows. However, we know that in general relativity, time is locally unique. Therefore, a single observer must have a unique time flow.

As an analogy, this would be as if the Sun had no exact mass, but rather, the Solar System is in a superposition of all possible Solar Systems, where each possibility has a Sun with a different mass. A different solar mass produces a different orbital speed of the Earth, and so, there are a multiplicity of clocks (on Earth), with each clock corresponding to a different possible solar mass. However, an observer on Earth with a sundial must have a unique local time.

One potential solution to this problem is known as the **thermal time hypothesis**, put forward by Carlo Rovelli and Alain Connes in 1994[4], and first formulated as a thermodynamical origin of time by Rovelli in 1993[19]. To understand this formulation, we emphasize the difference between a physical system and its states. A famous example of a quantum system is Schrödinger's cat. In Schrödinger's original formulation, a cat, a flask of poison and a radioactive source are placed in a sealed box. If an internal radiation monitor detects radiation, the flask is shattered releasing the poison which kills the cat. The physical system, in this case, is the sealed box containing the cat, and some examples of states include a living cat, a dead cat, or even a superposition of both.

Describing a system by a state is typical of thermodynamics. For example, for a gas in a box, a state is a set of values of the thermodynamical quantities which describe the system, like energy, temperature and pressure. An **equilibrium state** is a state in which the values of the thermodynamical quantities are constant in time. Note that this definition only makes sense with an a priori notion of time. Time defines equilibrium. The thermal time hypothesis reverts this proposition: from the notion of equilibrium, one extracts time. More exactly, starting from a (relativistic and quantum) physical system in a given state, one builds a time flow such that the state one has started with is precisely an equilibrium state. There are two difficulties:

1. We need to characterise those states among all the possible states for which there exists a time flow turning them into an equilibrium state.
2. For a state which can be turned into an equilibrium state, we need to be able to explicitly extract the time flow from the knowledge of this state.

For the first point, a characterisation of the states that could be equilibrium states exists, and is mathematically formulated by Kubo-Martin-Schwinger (KMS) states. They describe the properties of a system in thermal equilibrium, and can be identified as those states satisfying the so-called **KMS conditions**. Consider a timeless quantum mechanical system, that is, a state space $\mathcal{H}$, an algebra of observables $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, and a group of automorphisms $\mathrm{Aut}(\mathcal{M})$, from which we can build a time evolution. The group $\mathrm{Aut}(\mathcal{M})$ in the Heisenberg picture corresponds to the Hilbert group in the Schrödinger picture, in that it represents the group of all possible symmetries of quantum systems. If the KMS conditions are satisfied for a state $\Omega \in \mathcal{H}$, then this state has, with respect to $\mathrm{Aut}(\mathcal{M})$, the same properties as an equilibrium state of a physical system whose quantum time evolution would be given by some one-parameter group $\{\alpha_t\}_{t\in\mathbb{R}} \subset \mathrm{Aut}(\mathcal{M})$.

This brings us to the second point, which mathematics also takes care of. Indeed, given an algebra of observables $\mathcal{M}$ and a state $\Omega \in \mathcal{H}$ (with both requiring some additional technical properties), **Tomita-Takesaki theory** makes it possible to extract a one-parameter group of automorphisms $\{\alpha_t^\Omega\}_{t\in\mathbb{R}} \subset \mathrm{Aut}(\mathcal{M})$, such that Ω is a KMS state with respect to this one-parameter group. Mathematically speaking, in the theory of von Neumann algebras, Tomita-Takesaki theory is a method for constructing modular automorphisms of von Neumann algebras.

Tomita-Takesaki theory lies at the heart of the thermal time hypothesis and was introduced by Minoru Tomita in 1967. Given some von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ and some state $\Omega \in \mathcal{H}$ statsifying certain properties, it proves the existence of a positive operator Δ, which is not necessarily bounded. The main result of the the theory then states that:

$$\Delta^{-it}\mathcal{M}\Delta^{it} = \mathcal{M}, \quad t \in \mathbb{R}.$$

The operators $\{\Delta^{it}\}_{t\in\mathbb{R}}$ are well-defined by the functional calculus and form a one-parameter unitary group, which induces a time flow corresponding to the Schrödinger picture. Alternatively, for every suitable state Ω, there is a one-parameter group of **modular automorphisms** $\{\alpha_t^\Omega\}_{t\in\mathbb{R}}$ defined by $\alpha_t^\Omega(x) = \Delta^{-it}x\Delta^{it}$, which corresponds to the time flow in the Heisenberg picture.

The beauty of this proposition should not be understated. Recall from Chapter 4 that all observables of a physical system can be thought of as self-adjoint operators on the state space. These operators then generate a space called the algebra of observables, which forms a von Neumann algebra. The formal construction of the algebra of observables requires some work, as some observables may be unbounded. However, von Neumann algebras still provide an adequte way of describing quantum systems. In fact, von Neumann algebras were originally developed to study the algebras of observables of quantum systems. As a result, this piece of abstract mathematics, seemingly unrelated to the problem of time in any way, gives us exactly the time flow we are looking for. This time is known as the **thermal time** of the physical system.

It should be stressed that Tomita-Takesaki only produces a ”nice” time flow if we are dealing with a quantum system. Indeed, suppose we are merely in a relativisic setting. Here, there is no superposition of gravitational fields, and hence no multiplicity of time flows. There should already be an apparent time flow, as without the quantum influence, there is no problem of time.

Let us briefly clarify how the thermal time hypothesis proposes to solve the problem of time. In a relativistic quantum system, we are dealing with a superposition of gravitational fields, encoded in some quantum state. If we know the algebra of observables of this system and if this state is potentially an equilibrium state with respect to some conception of time, the thermal time hypothesis associates a unique thermal time to this equilibrium state. The hope, then, is that this thermal time coincides with a proper time in the relativistic formulation of the system.

For the moment, no theory within quantum gravity clearly proposes a definition of the algebra of observables, and so the pertinence of the thermal time hypothesis cannot be tested in this context. However, there are some physical situations where the one-parameter group of modular automorphisms is known, and its interpretation as thermal time makes sense, that is, it corresponds to some relativistic proper time. In any case, the thermal time hypothesis remains an active and promising area of research within the study of quantum gravity.

6

Positive Operator-Valued Measures

As we have seen in the previous chapter, the concept of ”time” at the intersection of quantum mechanics and relativity offers many challenges. There is no quantum theory of time measurement, since relativity is both fundamental to time and difficult to include in quantum mechanics. Although position and momentum are associated with a single particle, time is commonly viewed as a system property. In fact, whereas position, momentum and energy can be given firm mathematical footing in von Neumann’s mathematical foundations of quantum mechanics as observables, this cannot be done for time.

Explicitly, it is a famous observation of Wolfgang Pauli[17] that if H is a semi-bounded Hamiltonian operator, there exists no self-adjoint operator T covariant with the one-parameter unitary group generated by iH. That is, it cannot be true that:

$$e^{itH}Te^{-itH} = T - t, \quad t \in \mathbb{R}.$$

This is a highly undesirable result as it suggests a fundamental difference between position and time as mathematical entities in quantum mechanics, while in relativity, position and time are just different dimensions of spacetime. This notion of realising an ”observable time” remains a challenge for quantum theories. However, some progress has been made using positive operator-valued measures, which are introduced in this chapter.

6.1. Effects

Throughout the rest of this chapter, the Banach space of all bounded measurable functions $f : X \to \mathbb{C}$, endowed with the supremum norm $||f||_\infty = \sup_{x\in X} |f(x)|$, is denoted by $B_b(X)$.

Recall from Chapter 4.2 that the fundamental idea behind the mathematical formulation of quantum mechanics is to replace elementary observables (indicator functions) by orthogonal projections in a Hilbert space. With this in mind, observe the following.

Proposition 3. *Let $(X, \mathcal{X})$ be a measurable space. The closed convex hull $B_b(X)$ of the set of elementary observables $\{\mathbf{1}_B : B \in \mathcal{X}\}$ is*

$$\mathcal{E}(X) := \{f \in B_b(X) : 0 \leq f \leq \mathbf{1} \textit{ pointwise}\}.$$

The extreme points of $\mathcal{E}(X)$ are precisely the elementary observables $\mathbf{1}_B,\ B \in \mathcal{X}$.

Proof. See Proposition 15.19 in [14]. □

In the quantum mechanical counterpart, we naturally characterise the closed convex hull of $\mathcal{P}(\mathcal{H})$ in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. We write $S \leq T$ to express that $T - S$ is a positive operator.

Proposition 4. *The closed convex hull in $\mathcal{B}(\mathcal{H})$ of $\mathcal{P}(\mathcal{H})$ is*

$$\mathcal{E}(\mathcal{H}) := \{E \in \mathcal{B}(\mathcal{H}) : 0 \le E \le I\}.$$

The extreme points of $\mathcal{E}(\mathcal{H})$ are precisely the orthogonal projections.

Proof. See Proposition 15.20 in [14]. □

Elements of this set are known as effects.

Definition 19. An **effect** is an element of the set $\mathcal{E}(\mathcal{H})$.

Effects are self-adjoint, and it is not hard to show that a self-adjoint operator on $\mathcal{H}$ is an effect if and only if its spectrum is contained in the unit interval $[0, 1]$ (Theorem 8.11 in [14]). It can be shown that states are in one-to-one correspondence with affine mappings $\nu : \mathcal{P}(\mathcal{H}) \to [0, 1]$ satisfying $\nu(0) = 0$ and $\nu(I) = 1$ (Theorem 15.7 in [14]).

Similar to how we defined countable additivity for quantum observables in chapter 4.2, a mapping $\nu : \mathcal{E}(\mathcal{H}) \to [0, 1]$ is said to be **finitely additive** if

$$\sum_{n=1}^{N} \nu(E_n) = \nu(E)$$

whenever $E_1, \ldots, E_N, E \in \mathcal{E}(\mathcal{H})$ satisfy $E_1 + \cdots + E_N = E$.

Theorem 9 (Busch)**.** *Every finitely additive mapping $\nu : \mathcal{E}(\mathcal{H}) \to [0, 1]$ satisfying $\nu(I) = 1$ restricts to an affine mapping $\nu : \mathcal{P}(\mathcal{H}) \to [0, 1]$ and hence defines a state.*

Proof. See Theorem 15.22 in [14]. □

6.2. Positive operator-valued measures

The spectral theorem establishes a one-to-one correspondence between self-adjoint operators and projection-valued measures on the real line. A natural generalization of the notion of a projection-valued measure is obtained upon replacing orthogonal projections by effects. The mathematical theory of the resulting positive operator-valued measure had already been developed in the 1940s by Naimark, and its usefulness in quantum mechanics was first advocated by Ludwig[11].

Definition 20 (Positive operator-valued measure)**.** A **positive operator-valued measure** (POVM) on a measurable space $(\Omega, \mathcal{F})$ is a mapping $Q : \mathcal{F} \to \mathcal{E}(\mathcal{H})$ satisfying the following conditions:

1. $Q(\Omega) = I$
2. for all $x \in \mathcal{H}$, the mapping
$$Q_x := F \mapsto \big(Q(F)x|x\big), \quad F \in \mathcal{F}$$
defines a measure on $(\Omega, \mathcal{F})$.

Every projection-valued measure is a POVM. In the converse direction, we have the following simple result.

Proposition 5. *A POVM $Q : \mathcal{F} \to \mathcal{E}(\mathcal{H})$ is a projection-valued measure if and only if $Q(F)Q(F') = Q(F \cap F')$ for all $F, F' \in \mathcal{F}$.*

Proof. See Proposition 15.24 in [14]. □

A POVM which is not a projection-valued measure is called an **unsharp observable**.

Recall the Born rule from Chapter 4.2 (Proposition 1). For every projection-valued measure $P : \mathcal{F} \to \mathcal{P}(\mathcal{H})$, it sets up an affine mapping from $\mathcal{S}(\mathcal{H})$ to the convex set $M_1^+(\Omega)$ of probability measures on $(\Omega, \mathcal{F})$. However, the converse is not necessarily true. That is, if $\Phi : \mathcal{S}(\mathcal{H}) \to M_1^+(\Omega)$ is an affine mapping, it is not necessarily true that there is a projection-valued measure $P : \mathcal{F} \to \mathcal{P}(\mathcal{H})$ such that for every state $\phi \in \mathcal{S}(\mathcal{H})$,

$$F \mapsto \phi(P(F)), \quad F \in \mathcal{F}$$

defines a probability measure on $(\Omega, \mathcal{F})$. This is where POVMs come in.

Theorem 10 (POVMs as unsharp observables)**.** *Let $(\Omega, \mathcal{F})$ be a measurable space. If $\Phi : \mathcal{S}(\mathcal{H}) \to M_1^+(\Omega)$ is an affine mapping, then there exists a unique POVM $Q : \mathcal{F} \to \mathcal{E}(\mathcal{H})$ such that for all $T \in \mathcal{S}(\mathcal{H})$ we have*

$$\big(\Phi(T)\big)(F) = Tr\big(Q(F)T\big), \quad F \in \mathcal{F}.$$

Proof. See Theorem 15.25 in [14]. □

POVMs are the most general kind of measurement in quantum mechanics, and it is for this reason that they are sometimes considered as the ultimate observables.

The assumption that Φ should be affine in the above theorem is reasonable by the following argument. Suppose that we have two quantum mechanical systems represented by the states $T_1, T_2 \in \mathcal{S}(\mathcal{H})$. We use a classical coin to decide which state is going to be observed: if, with probability p, 'heads' comes up, we observe the system corresponding to T_1; otherwise, we observe the system corresponding to T_2. This experiment can be described as observing the state corresponding to the convex combination $pT_1 + (1-p)T_2$. If Φ is the observable to be measured, we expect the probability distribution of the outcomes $\Phi\big(pT_1 + (1-p)T_2\big)$ to be given by $p\Phi(T_1) + (1-p)\Phi(T_2)$.

6.3. Naimark's theorem

In rough analogy, a POVM is to a projection-valued measure what a mixed state is to a pure state. Mixed states are needed to specify the state of a subsystem of a larger system. For example, when considering a gas in a box, the subsystem under consideration, giving values for pressure and temperature for each Borel subset in the state space, approximates the actual larger system, which would describe the physical behaviour of each individual particle in the box. Analogously, POVMs are necessary to describe the effect on a subsystem of a projective measurement (coming from a PVM) performed on a larger system. Let us make this mathematically concrete.

If J is an isometry from $\mathcal{H}$ into another Hilbert space $\hat{\mathcal{H}}$ and $\hat{P}$ is an orthogonal projection in $\hat{\mathcal{H}}$, then $J^*\hat{P}J$ is an effect in $\mathcal{H}$: for all $x \in \mathcal{H}$ we have

$$0 \le (\hat{P}Jx|Jx) = \|\hat{P}Jx\|^2 \le \|x\|^2 = (x|x)$$

and therefore $0 \le J^*\hat{P}J \le I$. This gives a method of producing POVMs from projection-valued measures.

Proposition 6 (Compression)**.** *Let J is an isometry from $\mathcal{H}$ into another Hilbert space $\hat{\mathcal{H}}$. If $\hat{P} : \mathcal{F} \to \mathcal{P}(\hat{\mathcal{H}})$ is a projection-valued measure, then $Q := J^*\hat{P}J : \mathcal{F} \to \mathcal{E}(\mathcal{H})$ is a POVM.*

Proof. By what we just observed, Q maps sets $F \in \mathcal{F}$ to elements of $\mathcal{E}(\mathcal{H})$. It is clear that $Q(\Omega) = J^*J = I$. To see that Q is a POVM, it remains to observe that for all $x \in \mathcal{H}$ and $F \in \mathcal{F}$, we have

$$Q_xF = \big(Q(F)x|x\big) = \big(\hat{P}(F)Jx|Jx\big) = \hat{P}(Jx)F,$$

from which it follows that Q_x is a finite measure on $(\Omega, \mathcal{F})$. □

Conversely, every POVM arises in this way. This is known as **Naimark's theorem**.

Theorem 11 (Naimark)**.** *Let $(\Omega, \mathcal{F})$ be a measurable space and let $Q : \mathcal{F} \to \mathcal{E}(\mathcal{H})$ be a POVM. Then there exists a Hilbert space $\hat{\mathcal{H}}$, a projection-valued measure $\hat{P} : \mathcal{F} \to \mathcal{P}(\hat{\mathcal{H}})$ and an isometry $J : \mathcal{H} \to \hat{\mathcal{H}}$ such that*

$$Q(F) = J^*\hat{P}(F)J, \quad F \in \mathcal{F}.$$

Proof. See Theorem 15.29 in [14]. □

6.4. Phase as an unsharp observable

It was quickly realized that, in contrast to projection-valued measures, covariant POVMs could be constructed that are associated with various time measurements, such as arrival time, times of occurence, screen time, and time of flight. In fact, for some of these, rigorous versions of time-energy uncertainty could be proved.[1][2][3]

Let us provide a concrete example of a POVM associated with time. In quantum mechanics, for systems where the number of particles may not be preserved, the **number operator** N is the observable that counts the number of particles. It can be given as a self-adjoint operator acting on the Hardy space $H^2(\mathbb{D})$ on the open unit disk. Here, $H^2(\mathbb{D})$ is the Hilbert space of all holomorphic functions on $\mathbb{D}$ of the form $f(z) = \sum_{n=1}^{\infty} c_n z^n$ with

$$\|f\|^2 \coloneqq \sum_{n=1}^{\infty} |c_n|^2 < \infty.$$

Let $z_n(z) \coloneqq z^n$ and $e_n(\theta) \coloneqq e^{in\theta}$. It can be shown (section 7.3.d in [14]) that the mapping

$$\sum_{n=1}^{\infty} c_n z_n \mapsto \sum_{n=1}^{\infty} c_n e_n$$

sets up an isometry from $H^2(\mathbb{T})$ to $L^2(\mathbb{T})$. In particular, its range is the closed subspace of $L^2(\mathbb{T})$ whose negative Fourier coefficients vanish. Therefore, $H^2(\mathbb{D})$ can be identified as the range of the **Riesz projection**

$$\sum_{n\in\mathbb{Z}} c_n z_n \mapsto \sum_{n=1}^{\infty} c_n z_n$$

on $L^2(\mathbb{T})$.

The number operator N in $H^2(\mathbb{D})$ is the unbounded self-adjoint operator given by

$$N z_n = n z_n, \quad n \in \mathbb{N}.$$

Its domain is

$$D(N) = \left\{ \sum_{n=1}^{\infty} c_n z_n \in H^2(\mathbb{D}) : \sum_{n=1}^{\infty} n^2 |c_n|^2 < \infty \right\}.$$

We can then construct a POVM for **phase** (in the sense of waves), which is covariant with respect to the one-parameter unitary group generated by the number operator, that is, phase can be defined as a $\mathbb{T}$-valued unsharp observable. To show this, let S be the ”left shift” operator on $H^2(\mathbb{D})$:

$$S \sum_{n=1}^{\infty} c_n z_n \coloneqq c_{n+1} z_n.$$

Now identifying $H^2(\mathbb{D})$ as the range of the Riesz projection in $L^2(\mathbb{T})$ as before, we can similary define the ”left shift” operator $\hat{S}$ on $L^2(\mathbb{T})$ by

$$\hat{S} \sum_{n=1}^{\infty} c_n e_n \coloneqq c_{n+1} e_n.$$

It is then a relatively straightforward verification to show that the projection-valued measure $P : \mathfrak{B}(\mathbb{T}) \to \mathcal{P}(L^2(\mathbb{T}))$ associated with $\hat{S}$ is given by

$$P(B)f = \mathbf{1}_B f, \quad B \in \mathfrak{B}(\mathbb{T}),\ f \in L^2(\mathbb{T}). \tag{6.1}$$

where $\mathfrak{B}(\mathbb{T})$ is the set of all Borel subsets of $\mathbb{T}$. The compression of this projection-valued measure to $H^2(\mathbb{D})$ then defines a POVM $\Phi : \mathbb{T} \to \mathcal{E}(H^2(\mathbb{D}))$ which we call the **phase observable**. The following theorem establishes the covariance property this POVM has with respect to the number operator N, in the sense of Definition 18.

Theorem 12 (Covariance of phase)**.** *The phase observable Φ is covariant with respect to the one-parameter unitary group generated by the number operator N. That is, for all Borel subsets $B \subset \mathbb{T}$ we have*

$$e^{-itN}\Phi(B)e^{itN} = \Phi(e^{it}B), \quad t \in \mathbb{R}$$

where $e^{it}B = \{e^{it}z : z \in B\}$ is the rotation of B over t.

Proof. Since the POVM Φ is the compression of the projection-valued measure P given by equation 6.1, for all $m, n \in \mathbb{N}$ we have

$$(\Phi(B)e^{itN}e_n|e_m) = (P(B)Je^{itN}e_n|Je_m) = e^{int}(\mathbf{1}_B e_n|e_m),$$

while at the same time, with $A = \{\theta \in (-\pi, \pi] : e^{i\theta} \in B\}$,

$$\begin{aligned}(e^{itN}\Phi(e^{it}B)e_n|e_m) = (P(e^{it}B)Je_n|Je^{-itN}e_m) = e^{imt}(\mathbf{1}_{e^{it}B}e_n|e_m) = e^{itm}\frac{1}{2\pi}\int_A e^{i(n-m)(\eta+t)}d\eta \\ = e^{int}\frac{1}{2\pi}\int_A e^{i(n-m)\eta}d\eta = e^{int}(\mathbf{1}_B e_n|e_m).\end{aligned}$$

Since the functions $e_n,\ n \in \mathbb{N}$ have dense span in $H^2(\mathbb{D})$, this completes the proof. □

The number operator N can be viewed as the Hamiltonian of the quantum harmonic oscillator (section 15.6.e in [14]). Therefore, we have essentially proven that POVMs admit a description of time in a way that projection-valued measures cannot account for. Hence, when trying to define time as an observable, this suggests we should be looking for covariant POVMs, rather than covariant projection-valued measures. We return to this idea in Chapter 9, after providing a detailed proof of Tomita's theorem.

7

Tomita-Takesaki theory

7.1. Introduction

In the previous chapter, we introduced the motivation for using Tomita-Takesaki theory to address the problem of time. Indeed, when provided with a state space $\mathcal{H}$ and an algebra of observables $\mathcal{M}$ of some quantum system, we would like to construct a one-paramater group of symmetries corresponding to the time flow of this system. If the algebra of observables is a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ and $\Omega \in \mathcal{H}$ is a state satisfying the KMS conditions, Tomita-Takesaki theory is precisely the mathematical tool that produces a one-parameter group of symmetries. Mathematically, Tomita-Takesaki theory constructs modular automorphisms of von Neumann algebras.

Recall from Theorem 6 that if $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is an abelian von Neumann algebra (with certain conditions), then $\mathcal{H}$ is identifiable as $L^2(X, \mu)$ and $\mathcal{M}$ acts as multiplication operators M_φ for some second countable Hausdorff space X, and some positive measure $\mu \in M(X)$, with $\varphi \in L^\infty(X, \mu)$. This has a satisfying physical interpretation. In Section 3.3 we mentioned that we use generally noncommutative spaces for observables in quantum mechanics, since measurements effect physical systems at the quantum scale. If we are an abelian setting, measuring observables has no impact on the physical system, and we should expect this to align with the mathematical formulation of classical mechanics. In classical mechanics, observables were defined as measurable functions on the state space. So, assuming that X is the state space, $L^\infty(X, \mu)$ are precisely the bounded observables.

As mentioned in the previous chapter, Tomita-Takesaki is not helpful in the abelian setting. As there is no quantum influence, there is no problem of time, and so the time-flow should already be apparent.

7.2. σ-Finite von Neumann algebras

Central to Tomita-Takesaki theory is the space of bounded operators that commute with all elements of a von Neumann algebra, and a certain property that specifically von Neumann algebras have relating to these commuting elements.

Definition 21 (Commutant)**.** Let C be a subset of an algebra A. The **commutant** of C, denoted by C', is the set of elements in A that commute with every element in C, that is, $C' = \{y \in A \mid xy = yx \; \forall x \in C\}$.
The **double commutant** of C, denoted C'', is the commutant of C', that is, $C'' = (C')'$.

Interestingly, it turns out that every von Neumann algebra is equal to its double commutant. In fact, this property completely characterises von Neumann algebras; it provides an alternative way to define them.

Theorem 13 (Von Neumann double commutant theorem)**.** *Let $\mathcal{M}$ be a unital $\star$-subalgebra of $\mathcal{B}(\mathcal{H})$. Then the following are equivalent: [13]*

1. *$\mathcal{M}$ is closed in the strong operator topology,*
2. *$\mathcal{M} = \mathcal{M}''$.*

Proof. 1. $\implies$ 2. Suppose $\mathcal{M}$ is strongly closed. If $y \in \mathcal{M}$, then $xy = yx$ for all $x \in \mathcal{M}'$. So certainly $y \in \mathcal{M}''$ and therefore, $\mathcal{M} \subset \mathcal{M}''$.
Now let $y \in \mathcal{M}''$, $\psi \in \mathcal{H}$ and let $K = \overline{\{x\psi \mid x \in \mathcal{M}\}}$. Then K is a closed subspace of $\mathcal{H}$, which is invariant for all $x \in \mathcal{M}$. Let $p \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection onto K. Since $I \in \mathcal{M}$, $p\psi = \psi$. Suppose $x \in \mathcal{M}$ and $\phi \in K$, so that $\phi = \lim_{n\to\infty} x_n\psi$, with $x_n \in \mathcal{M}$ for all $n \geq 1$. Then $x\phi = \lim_{n\to\infty} xx_n\psi$ with $xx_n \in \mathcal{M}$ for all $n \geq 1$, and therefore $x\phi \in K$. So K is invariant under all $x \in \mathcal{M}$. Similarly, if $x \in \mathcal{M}$ and $\phi \in K^\perp$, then $(x\phi|\xi) = (\phi|x^*\xi) = 0$ for all $\xi \in K$, since $x^* \in \mathcal{M}$. So $x^*\xi \in K$ for all $\xi \in K$, and therefore $K^\perp$ is invariant under $\mathcal{M}$. As a result, for all $x \in \mathcal{M}$ and $\phi \in \mathcal{H}$, we have $xp\phi \in K$ and $x(I-p)\phi \in K^\perp$, so that

$$xp\phi = pxp\phi = px(p\phi + (I-p)\phi) = px\phi.$$

Thus, $xp = px$ for all $x \in \mathcal{M}$ and we conclude that $p \in \mathcal{M}'$. Therefore $py = yp$, and so $y\psi \in K$. Since $\mathcal{M}$ is strongly closed, there is a sequence (x_n) in $\mathcal{M}$ such that $y\psi = \lim_{n\to\infty} x_n\psi$. Fix an orthonormal basis (e_n) for $\mathcal{H}$ and set $\mathcal{H}^{(n)} := \text{span}\{e_1, \dots, e_n\}$. Then elements of $\mathcal{B}(\mathcal{H}^{(n)})$ can be identified as $n \times n$ matrices. For each $n \in \mathbb{N}$ define

$$\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}^{(n)}),\ x \mapsto (\delta_{ij}x),$$

where $(\delta_{ij}x)(e_1, \dots, e_n) := (xe_1, \dots, xe_n)$. Then φ is a unital $\star$-homomorphism, and so $\varphi(\mathcal{M})$ is a $\star$-subalgebra of $\mathcal{B}(\mathcal{H}^{(n)})$ containing $\text{id}_{\mathcal{H}^{(n)}}$. If $z \in (\varphi(\mathcal{M}))'$ and $x \in \mathcal{M}$, then $\varphi(x)z = z\varphi(z) \implies xz_{ij} = z_{ij}x$ for all $1 \leq i,j \leq n$. Thus, $z_{ij} \in \mathcal{M}'$ and so $yz_{ij} = z_{ij}y$ for all $1 \leq i,j \leq n$. Therefore, $\varphi(y)z = z\varphi(y)$. Now let $\psi^{(n)} = (\psi_1, \dots, \psi_n) \in \mathcal{H}^{(n)}$ be the projection of ψ onto $\mathcal{H}^{(n)}$. Again, since $\mathcal{M}$ is strongly closed, there is a sequence (x_m) in $\mathcal{M}$ such that $\varphi(y)\psi^{(n)} = \lim_{m\to\infty} \varphi(x_m)\psi^{(n)}$. Thus, $y\psi_j = \lim_{m\to\infty} x_m\psi_j$ for $1 \leq j \leq n$. If $Y \subset \mathcal{B}(\mathcal{H})$ is a strong neighbourhood of y, then $Y - y$ is a strong neighbourhood of $0 \in \mathcal{B}(\mathcal{H})$. Therefore, there exist $\psi_1, \dots, \psi_n \in \mathcal{H}$ and $\varepsilon > 0$ such that

$$Y - y = \{x \in \mathcal{B}(\mathcal{H}) \mid \|x\psi_j\| < \varepsilon\ \forall 1 \leq j \leq n\}.$$

Hence, there is a sequence (x_m) in $\mathcal{M}$ such that

$$y\psi_j = \lim_{m\to\infty} x_m\psi_j\ \forall 1 \leq j \leq n.$$

Consequently, there is some $N \in \mathbb{N}$ so that $x_N \in Y$. Thus, $Y \cap \mathcal{M} \neq \varnothing$ and therefore, we have shown that every strong neighbourhood of y intersects $\mathcal{M}$. We conclude that y is in the strong closure of $\mathcal{M}$, which, by assumption, is $\mathcal{M}$ itself. Hence, $\mathcal{M}'' \subset \mathcal{M}$.
2. $\implies$ 1. Suppose that $\mathcal{M} = \mathcal{M}''$. Suppose that (x_n) is a sequence in $\mathcal{M}''$ which converges strongly to $x \in \mathcal{B}(\mathcal{H})$. Then x_n commutes with all $y \in \mathcal{M}'$ for all $n \in \mathbb{N}$. Therefore, for any $y \in \mathcal{M}'$ and any $\psi \in \mathcal{H}$,

$$xy\psi = \lim_{n\to\infty} x_ny\psi = \lim_{n\to\infty} yx_n\psi = yx\psi,$$

and so $x \in \mathcal{M}''$. We conclude that $\mathcal{M}''$ is strongly closed, and since $\mathcal{M} = \mathcal{M}''$, $\mathcal{M}$ is closed in the strong operator topology. □

In the previous chapter, we mentioned that the states of a quantum system that could possibly be equilibrium states are described by KMS states. We do not delve into the theory of KMS conditions, but it turns out there is a class of states which are certainly KMS states (section 2.1 in [23]). It is this class of states for which Tomita-Takesaki theory creates a time flow.

Definition 22. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. A state $\phi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is called **faithful** on $\mathcal{M}$ if $\phi(x^*x) = 0 \implies x = 0$ for all $x \in \mathcal{M}$, that is, ϕ is injective on the positive operators of $\mathcal{M}$.

In the next chapters, given a faithful state on a von Neummann algebra $\mathcal{M}$, we will show that there is a one-parameter unitary group of automorphisms of $\mathcal{M}$. Of course, there is no guarantee that every von Neumann algebra has a faithful state to begin with. Therefore, Tomita-Takesaki theory must be restricted to a class of von Neumann algebras, namely those which act on a Hilbert space that admits a faithful state, which we introduce now.

Definition 23 (σ-finite von Neumann algebra)**.** A von Neumann algebra $\mathcal{M}$ is called σ-**finite** if all collections of mutually orthogonal projections have at most a countable cardinality.

All von Neumann algebras encountered in quantum field theory are σ-finite (section 2.5.1 in [18]). In particular, recall from Section 4.2 that all Hilbert spaces in this thesis were taken to be separable. It can be shown that all von Neumann algebras acting on separable Hilbert spaces are, in fact, σ-finite (page 85 in [18]). Our goal in the rest of this section is to show that the σ-finite von Neumann algebras admit a faithful state and that any faithful state can be realised as a vector state for some vector $\Omega \in \mathcal{H}$ which is cyclic and separating for $\mathcal{H}$.

Proposition 7. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$.*

1. *If $\mathcal{M}$ is σ-finite, then $\mathcal{H}$ admits a faithful state.*
2. *If $\mathcal{H}$ admits a faithful state on $\mathcal{M}$, then $\mathcal{M}$ is isomorphic with a von Neumann algebra $\pi(M) \subset \mathcal{B}(\mathcal{H}')$ which admits a cyclic and separating vector $\Omega \in \mathcal{H}'$.*

Proof. 1. Let $\{\psi_\alpha\}$ be a maximal family of vectors in $\mathcal{H}$ such that $P(\overline{\mathcal{M}'\psi_\alpha})$ and $P(\overline{\mathcal{M}'\psi_{\alpha'}})$ are orthogonal projections whenever $\alpha \neq \alpha'$, where $\mathcal{M}'\psi_\alpha = \{x'\psi_\alpha : x' \in \mathcal{M}'\}$. Since $P(\overline{\mathcal{M}'\psi_\alpha})$ is an orthogonal projection in $\mathcal{M}$, and $\mathcal{M}$ is σ-finite, $\{\psi_\alpha\}$ is a countable set. However, since $\{\psi_\alpha\}$ is maximal,

$$\sum_\alpha P(\overline{\mathcal{M}'\psi_\alpha}) = \mathrm{id}_{\mathcal{H}}.$$

Thus $\overline{\cup_\alpha \mathcal{M}'\psi_\alpha} = \mathcal{H}$. Take some $x \in \mathcal{M}''$ such that $\cup_\alpha\{x\psi_\alpha\} = \{0\}$. Then, for any $y' \in \mathcal{M}'$ and any α, we have $xy'\psi_\alpha = y'x\psi_\alpha = 0$. Hence $x\mathcal{H} = x\overline{\cup_\alpha \mathcal{M}'\psi_\alpha} = \{0\}$, and so $x = 0$. Since $\{\psi_\alpha\}$ is countable, we can index it instead by the naturals $\mathbb{N}$ and rescale the elements $\psi_n := 2^{-n/2}\frac{\psi_\alpha}{\|\psi_\alpha\|}$ such that $\sum_n \|\psi_n\|^2 = 1$. Define $\phi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ by

$$\phi(x) = \sum_n (\psi_n|x\psi_n)$$

so that ϕ is a state. By von Neumann's double commutant theorem, $\mathcal{M}'' = \mathcal{M}$ and so, if $\phi(x^*x) = 0$ for some $x \in \mathcal{M}$, then $0 = (\psi_n|x^*x\psi_n) = \|x\psi_n\|^2$ for all n, and therefore, $x = 0$. Hence, ϕ is faithful.
2. Let $\phi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a faithful state on $\mathcal{M}$, and let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}')$ be the GNS representation as constructed in Theorem 3, with cyclic vector $\Omega \in \mathcal{H}'$. Since ψ is faithful, π is an isomorphism of von Neumann algebras. If $\pi(x)\Omega = 0$ for some $x \in \mathcal{M}$, then $\psi(x^*x) = \|\pi(x)\Omega\|^2 = 0$, and so $x^*x = 0$, so that $x = 0$. Therefore, Ω is separating for $\mathcal{H}'$. ☐

Crucially, for every separable Hilbert space, which are the only Hilbert spaces under consideration in this thesis, every von Neumann algebra acting on such a Hilbert space can be realized as a von Neumann algebra which has a cyclic and separating vector. It is this vector, which, as a vector state, corresponds to a KMS state, for which Tomita-Takesaki constructs a time flow. Of course, such a vector need not be unique.

7.3. The modular operator

Throughout, let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a cycling and separating vector $\Omega \in \mathcal{H}$. The goal will be to construct an operator Δ such that $\mathbb{R} \to U(\mathcal{H}),\ t \mapsto \Delta^{it}$ is one-parameter unitary group. Furthermore,

$$\Delta^{-it}\mathcal{M}\Delta^{it} = \mathcal{M}, \quad t \in \mathbb{R},$$

which is known as Tomita's theorem. Consequently, by equation 5.1, there is a one-parameter group of von Neumann algebra automorphisms $\alpha_t(x) = \Delta^{-it}x\Delta^{it}$.

In order to introduce this operator, we need a generalisation of the polar decomposition to unbounded operators, factorising them into a compositions of partial isometries and positive self-adjoint operators.

Theorem 14 (Polar decomposition)**.** *Let $\mathcal{H}$ be a Hilbert space. Let A be a closed, densely defined unbounded operator. Then it has a **polar decomposition***

$$A = U|A|,$$

where $|A| = (A^*A)^{1/2}$ is a (possibly unbounded) positive self-adjoint operator with the same domain as A, and U is a partial isometry vanishing on range$(|A|)^\perp$.

Proof. Since A is closed and densely defined, A^*A is self-adjoint with dense domain (see Theorem 10.46 in [14]), which ensures that $|A| = (A^*A)^{1/2}$ is well-defined by the functional calculus.
By Corollary 10.61 in [14], we have $D(|A|) = D(A)$ and hence,

$$U_0 : \text{range}(|A|) \to \mathcal{H}, \ |A|\varphi \mapsto A\varphi$$

is a well-defined isometric linear map. Therefore, it has a unique linear isometric extension to $\overline{\text{range}(|A|)}$. Define $U \in \mathcal{B}(\mathcal{H})$ by

$$U = \begin{cases} U_0 & \text{on } \overline{\text{range}(|A|)} \\ 0 & \text{on } \overline{\text{range}(|A|)}^\perp \end{cases} .$$

Then U is isometric on $\ker U^\perp$, because $\ker U = \overline{\text{range}(|A|)}^\perp$. Thus, U is a partial isometry and $U|A| = A$, as required. □

By von Neumann's double commutant theorem, it is easily seen that the commutant of a von Neumann algebra is a von Neumann algebra itself. Further, the properties of being a cyclic or separating vector for a von Neumann algebra are in some sense dual with regard to its commutant, in the following way.

Proposition 8. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $\Omega \in \mathcal{H}$. Then the following are equivalent:*

1. *Ω is cyclic for $\mathcal{M}$.*
2. *Ω is separating for $\mathcal{M}'$.*

Proof. 1. $\implies$ 2. Suppose Ω is cyclic for $\mathcal{M}$ and suppose for $x' \in \mathcal{M}'$ that $x'\Omega = 0$. Then for any $x \in \mathcal{M}$, we have $x'x\Omega = xx'\Omega = 0$, and so $x'\mathcal{H} = x'\overline{\mathcal{M}\Omega} = \{0\}$. Hence $x' = 0$, and so Ω is separating for $\mathcal{M}'$.
2. $\implies$ 1. Suppose Ω is separating for $\mathcal{M}'$ and let $p' \in \mathcal{B}(\mathcal{H})$ be the projection onto $\overline{\mathcal{M}\Omega}$. Then $p' \in \mathcal{M}'$ and $(\mathrm{id}_\mathcal{H} - p')\Omega = 0$ and so $(\mathrm{id}_\mathcal{H} - p') = 0$. Therefore, $p' = \mathrm{id}_\mathcal{H}$ and we conclude that $\overline{\mathcal{M}\Omega} = \mathcal{H}$. Hence, Ω is cyclic for $\mathcal{M}$. □

Corollary 1. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $\Omega \in \mathcal{H}$ is cyclic and separating for $\mathcal{M}$, then Ω is also cyclic and separating for $\mathcal{M}'$.*

Proof. Suppose $\Omega \in \mathcal{H}$ is cyclic and separating for $\mathcal{M}$. By Proposition 8, Ω is separating for $\mathcal{M}'$. By von Neumann's double commutant theorem, Ω is separating for $\mathcal{M}''$ and so, also by Proposition 8, Ω is cyclic for $\mathcal{M}'$. □

Now we return to defining our desired operator Δ.

Definition 24 (Antilinear operator)**.** An **antilinear operator** T an a Hilbert space $\mathcal{H}$ is an operator satisfying $T(\psi + \varphi) = T\psi + T\varphi$ and $T(c\psi) = \overline{c}T\psi$ for all $\psi, \varphi \in D(T)$ and $c \in \mathbb{C}$.
The **adjoint** of an antilinear operator T is the unique antilinear operator T^* defined by $T^*\varphi = \xi$ whenever there is a $\xi \in \mathcal{H}$ such that $(\psi|\xi) = (T\psi|\varphi)$ for all $\psi \in D(T)$.

First, we define the antilinear operators S_0 with $D(S_0) = \mathcal{M}\Omega$ and F_0 with $D(F_0) = \mathcal{M}'\Omega$ by

$$S_0(x\Omega) = x^*\Omega, \quad x \in \mathcal{M}, \qquad F_0(x'\Omega) = x'^*\Omega, \quad x' \in \mathcal{M}'.$$

Since Ω is separating for $\mathcal{M}$ and for $\mathcal{M}'$, we have that $x\Omega = y\Omega \implies x = y \implies x^*\Omega = y^*\Omega$ for all $x, y \in \mathcal{M}$ and $x'\Omega = y'\Omega \implies x' = y' \implies x'^*\Omega = y'^*\Omega$ for all $x', y' \in \mathcal{M}'$. Thus, S_0 and F_0 are well-defined. Then, for $x \in \mathcal{M}$ and $x' \in \mathcal{M}'$, we have

$$\begin{aligned}(x'\Omega|S_0x\Omega) &= (x'\Omega|x^*\Omega) = (\Omega|x'^*x^*\Omega) = (\Omega|(xx')^*\Omega) \\ &= (\Omega|(x'x)^*\Omega) = (\Omega|x^*x'^*\Omega) = (x\Omega|x'^*\Omega) = (x\Omega|F_0x'\Omega).\end{aligned}$$

In particular, $D(F_0) \subset D(S_0^*)$. Since, F_0 is densely defined, S_0^* is densely defined, and so S_0 is closable.

Definition 25 (Tomita operator)**.** For a von Neumann algebra $\mathcal{M}$ with cyclic and separating vector Ω, the closure of S_0 as constructed above, denoted by S, is called the **Tomita operator**.

Similarly, for $x \in \mathcal{M}$ and $x' \in \mathcal{M}'$,

$$(x\Omega|F_0x'\Omega) = (x\Omega|x'^*\Omega) = (x'\Omega|x^*\Omega) = (x'\Omega|S_0x\Omega).$$

so that $D(S_0) \subset D(F_0^*)$. As S_0 is densely defined, so is F_0^*, and therefore, F_0 is closable. Just like the Tomita operator, let F denote the closure of F_0.

Lemma 1. *Let S_0 and F_0 be defined as above, and let S and F denote their closures respectively. Then*

$$S_0^* = F \qquad \text{and} \qquad F_0^* = S.$$

Proof. For a proof, see Proposition 2.5.9 in [18]. □

Note that when viewing $\mathcal{M}'$ as a von Neumann algebra, F_0 takes the role of S_0, and by von Neumann's double commutant theorem, S_0 takes the role of F_0 as well. Therefore the same holds true for S and F respectively. Hence, the Tomita operator for $\mathcal{M}'$ can be identified as $F = S^*$.

The Tomita operator S is closed, and since S_0 is densely defined, S is also densely defined. By Theorem 14, S has a polar decomposition $S = J\Delta^{1/2}$, where $\Delta = S^*S$ is self-adjoint and positive (Theorem 10.46 in [14]).

Definition 26 (modular operator)**.** For a von Neumann algebra $\mathcal{M}$ with cyclic and separating vector Ω, the positive self-adjoint operator Δ, as constructed above, is called the **modular operator** associated with $\{\mathcal{M}, \Omega\}$.

It is this operator which is used to prove Tomita's theorem. An important property of the modular operator, which is used in the proof, is that it is an invertible operator, in the sense that it is injective, and its inverse is the modular operator associated with the commutant $\{\mathcal{M}', \Omega\}$.

Proposition 9. *For a von Neumann algebra $\mathcal{M}$ with cyclic and separating vector Ω, the modular operator associated with $\{\mathcal{M}, \Omega\}$ has trivial kernel. Furthermore, the inverse Δ^{-1} is the modular operator associated with $\{\mathcal{M}', \Omega\}$.*

Proof. Next, observe that $\Delta = S^*S = FS$. Clearly $S_0(\mathcal{M}\Omega) = \mathcal{M}\Omega = D(S_0)$, since $\mathcal{M}$ is closed under the taking of adjoints, and $S_0(S_0(x\Omega)) = S_0(x^*\Omega) = x^{**}\Omega = x\Omega$ for all $x \in D(S_0)$. Hence, $S_0S_0 = I$ on $D(S_0)$ and $D(S_0^{-1}) = S_0D(S_0) = S_0(\mathcal{M}\Omega) = D(S_0)$, so that $S_0^{-1} = S_0$. Similarly, we have that $F_0^{-1} = F_0$. Then, by closure, $S^{-1} = \overline{S_0^{-1}} = \overline{S_0} = S$ and $F^{-1} = \overline{F_0^{-1}} = \overline{F_0} = F$. Therefore, $\Delta^{-1} = (FS)^{-1} = S^{-1}F^{-1} = SF$. Now, note that F_0 takes the role of S_0 for $\mathcal{M}'$, and by von Neumann's double commutant theorem, S_0 takes the role of F_0. So, similarly, S takes the role of F. Hence, the modular operator associated with $\{\mathcal{M}', \Omega\}$ is given by $SF = \Delta^{-1}$.
Finally, since Δ^{-1} is the modular operator for $\mathcal{M}'$, it is densely defined by Theorem 10.46 in [14]. Therefore, Δ has dense range, so by Proposition 10.24 in [14] and by the fact that Δ is self-adjoint, we conclude that Δ is injective. □

Since the modular operator is self-adjoint, the operator Δ^{it} is well-defined for all $t \in \mathbb{R}$ by the Borel functional calculus. By Stone's theorem on one-parameter unitary groups, $\mathbb{R} \to U(\mathcal{H}),\ t \mapsto \Delta^{it}$ is a one-parameter unitary group. The fundamental theorem of Tomita is that the time flow of any observable $\Delta^{-it}x\Delta^{it}$ remains in $\mathcal{M}$.

Theorem 15 (Tomita's theorem)**.** *Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω and let Δ be the modular operator associated with $\{\mathcal{M}, \Omega\}$. Then for any $x \in \mathcal{M}$ and $t \in \mathbb{R}$,*

$$\Delta^{-it}x\Delta^{it} \in \mathcal{M}.$$

Once we have proven Tomita's theorem, we automatically have

$$\mathcal{M} = \Delta^{-it}(\Delta^{it}\mathcal{M}\Delta^{-it})\Delta^{it} \subset \Delta^{-it}\mathcal{M}\Delta^{it}.$$

This gives the reverse inclusion and allows us to conclude that

$$\mathcal{M} = \Delta^{-it}\mathcal{M}\Delta^{it}.$$

It can be shown that if $\mathcal{M}$ is abelian, we have $\Delta = I = \mathrm{id}_{\mathcal{H}}$ (page 90 in [18]). The resulting unitary operators $\Delta^{it} = I^{it} = I$ are trivial and so there is no physical time flow associated with abelian von Neumann algebras. Therefore, Tomita-Takesaki is useful specifically to quantum mechanical systems.

In the next chapter, we provide a detailed proof of Tomita's theorem.

8

A proof of Tomita's theorem

8.1. Background material

The **Bochner integral** extends the definition of the Lebesgue integral to functions that take values in a Banach space. Given some measure space $(\Omega, \mathcal{F}, \mu)$, we explicitly define the Bochner integral for a class of functions called μ-simple functions, and then extend the definition to limits of sequences of such functions as well.

Definition 27 (Simple functions, strong measurability)**.** Let $(\Omega, \mathcal{F})$ be a measurable space and let X be a Banach space. A function $f : \Omega \to X$ is called **simple** if it is a finite linear combination of the form $\mathbf{1}_F \otimes x : \omega \mapsto f(\omega)x$ with $F \in \mathcal{F}$ and $x \in X$, and **strongly measurable** if it is the pointwise limit of a sequence of simple functions.

Definition 28 (μ-Simple functions)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, X be a Banach space, let $N \in \mathbb{N}$, and let $F_n \in \mathcal{F}$ and $x_n \in X$ for all $1 \leq n \leq N$. A simple function $f : \Omega \to X,\ \omega \mapsto \sum_{n=1}^{N} \mathbf{1}_{F_n}(\omega)x_n$ is called μ-**simple** if $\mu(F_n) < \infty$ for all $1 \leq n \leq N$. For such functions, we define

$$\int_\Omega f \, d\mu \coloneqq \sum_{n=1}^{N} \mu(F_n)x_n.$$

It is easily verified that $\int_\Omega f \, d\mu$ is well-defined in the sense that it does not depend on the representation of f as a linear combination of functions $\mathbf{1}_{F_n} \otimes x_n$ with $\mu(F_n) < \infty$.

Definition 29 (Bochner integrable)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let X be a Banach space. A strongly measurable function $f : \Omega \to X$ is said to be **Bochner integrable** with respect to μ if there is a sequence of μ-simple functions $f_n : \Omega \to X$ such that

$$\lim_{n\to\infty} \int_\Omega ||f - f_n|| \, d\mu = 0.$$

In that case we define the **Bochner integral** of f by

$$\int_\Omega f d\mu \coloneqq \lim_{n\to\infty} \int_\Omega f_n \, d\mu.$$

It is easily verified that $\int_\Omega f \, d\mu$ is well-defined in the sense that it does not depend on the sequence of approximating functions f_n.

It can be shown that the dominated convergence theorem applies to measurable functions with values in a Banach space (Corolloary III.6.16 in [6]).

Theorem 16 (Dominated convergence)**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, let X be a Banach space, and let $f_n : \Omega \to X$ be a sequence of strongly measurable functions that converges in measure to a function $f : \Omega \to X$, that is,*

$$\lim_{n\to\infty} \mu(\{x \in X : ||f(x) - f_n(x)|| \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$*. Moreover, assume that the sequence* f_n *is dominated by some integrable function* $g : \Omega \to \mathbb{R}$ *in the sense that*

$$\mu(\{x \in X : ||f_n(x)|| > g(x)\}) = 0$$

for all $n \in \mathbb{N}$*. Then* f_n, f *are Bochner-integrable and*

$$\lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega \lim_{n\to\infty} f_n \, d\mu = \int_\Omega f \, d\mu.$$

Throughout, let $\mathcal{H}$ be a Hilbert space.

Definition 30 (Projection-valued measure)**.** Let $(\Omega, \mathcal{F})$ be a measurable space. A **projection-valued measure** on $(\Omega, \mathcal{F})$ is a mapping $\mu : \mathcal{F} \to \mathcal{P}(\mathcal{H})$ satisfying the following conditions:

1. $\mu(\Omega) = \mathrm{id}_\mathcal{H}$
2. for all $\psi \in \mathcal{H}$, the mapping
$$F \mapsto (\mu(F)\psi|\psi), \quad F \in \mathcal{F}$$
defines a measure on $(\Omega, \mathcal{F})$.

For $\psi \in \mathcal{H}$, the measure defined by 2. is denoted by μ_ψ. Thus, for all $F \in \mathcal{F}$,

$$(\mu(F)\psi|\psi) = \mu_\psi(F) = \int_\Omega \mathbf{1}_F \, d\mu_\psi.$$

From

$$\mu_\psi(\Omega) = (\mu(\Omega)\psi|\psi) = (\psi|\psi) = \|\psi\|^2$$

we see that μ_ψ is a finite meaure for each $\psi \in \mathcal{H}$.
The μ**-essential range** of a measurable function $f : \Omega \to \mathbb{C}$ is the set $\mathrm{R}_\mu(f)$ of all $z \in \mathbb{C}$ such that $\mu(\{x \in \Omega \mid |f(x) - z| < r\})$ for all $r > 0$.

Let $(\Omega, \mathcal{F})$ be a measurable space. The Banach space of all bounded measurable functions $f : \Omega \to \mathbb{C}$, endowed with the supremum norm $||f||_\infty = \sup_{x\in\Omega} |f(x)|$, is denoted by $B_b(\Omega)$.

Theorem 17 (Bounded functional calculus)**.** *Let* $\mu : \mathcal{F} \to \mathcal{B}(\mathcal{H})$ *be a projection-valued measure. There exists a unique linear mapping* $\Phi : B_b(\Omega) \to \mathcal{B}(\mathcal{H})$ *with the following properties:*

1. *For all* $F \in \mathcal{F}$*, we have* $\Phi(\mathbf{1}_F) = \mu(F)$
2. *For all* $f, g \in B_b(\Omega)$*, we have* $\Phi(fg) = \Phi(f)\Phi(g)$
3. *For all* $f \in B_b(\Omega)$*, we have* $\Phi(\overline{f}) = (\Phi(f))^*$
4. *For all* $f \in B_b(\Omega)$*, we have* $||\Phi(f)|| \leq ||f||_\infty$
5. *For all* $f_n, f \in B_b(\Omega)$*, if* $\sup_{n\geq 1} ||f_n||_\infty < \infty$ *and* $f_n \to f$ *pointwise on* Ω*, then for all* $\psi \in \mathcal{H}$*, we have* $\Phi(f_n)\psi \to \Phi(f)\psi$*.*

Moreover, for all $\psi \in \mathcal{H}$ *and* $f \in B_b(\Omega)$ *we have*

$$\big(\Phi(f)\psi|\psi\big) = \int_\Omega f \, d\mu_\psi$$

and

$$||\Phi(f)\psi||^2 = \int_\Omega |f|^2 \, d\mu_\psi.$$

The operators $\Phi(f)$ *are normal, and if* f *is real-valued (respectively takes values in* $[0, \infty)$*), they are self-adjoint (respectively positive).*

Proof. See Theorem 9.8 in [14]. □

Theorem 18 (Spectral theorem for bounded normal operators)**.** *Let $A \in \mathcal{B}(\mathcal{H})$ be a normal operator. There exists a unique projection-valued measure μ on $\sigma(A)$ such that*

$$(A\psi|\psi) = \int_{\sigma(A)} \lambda \, d\mu_\psi(\lambda), \quad \psi \in \mathcal{H}.$$

Proof. See Theorem 9.14 in [14]. $\square$

For normal operators $A \in \mathcal{B}(\mathcal{H})$ and functions $f \in B_b(\sigma(A))$, the operator $\Phi(f) \in \mathcal{B}(\mathcal{H})$ defined in terms of the projection-valued measure μ of A by the calculus of Theorem 17 will be denoted by $f(A)$:

$$f(A) \coloneqq \Phi(f).$$

Theorem 19 (Measurable functional calculus)**.** *Let $(\Omega, \mathcal{F})$ be a measurable space, let $\mu : \mathcal{F} \to \mathcal{B}(\mathcal{H})$ be a projection-valued measure, and let $f : \Omega \to \mathbb{C}$ be a measurable function. There exists a unique normal operator $\Phi(f)$ on $\mathcal{H}$ satisfying*

$$D\big(\Phi(f)\big) = \left\{\psi \in \mathcal{H} : \int_\Omega |f|^2 \, d\mu_\psi < \infty\right\}, \qquad \big(\Phi(f)\psi|\psi\big) = \int_\Omega f \, d\mu_\psi, \; \psi \in D\big(\Phi(f)\big).$$

For all $\psi \in D\big(\Phi(f)\big)$ we have

$$||\Phi(f)\psi||^2 = \int_\Omega |f|^2 \, d\mu_\psi.$$

Furthermore, if $f_n, f, g : \Omega \to \mathbb{C}$ are measurable functions, then:

1. *$\Phi(f)\Phi(g) \subset \Phi(fg)$ with $D\big(\Phi(f)\Phi(g)\big) \subset D\big(\Phi(fg)\big) \cap D\big(\Phi(g)\big)$,*
2. *$\Phi(f)^* = \Phi(\bar{f})$,*
3. *if $0 \leq |f_n| \leq |f|$, and $\lim_{n\to\infty} f_n = f$ pointwise on Ω, then $D\big(\Phi(f)\big) \subset D\big(\Phi(f_n)\big)$ and*

$$\lim_{n\to\infty} \Phi(f_n)\psi = \Phi(f)\psi, \quad \psi \in D\big(\Phi(f)\big).$$

The operator $\Phi(f)$ is self-adjoint if and only if it is real-valued μ_ψ-almost everywhere for all $\psi \in \mathcal{H}$.

Proof. See Theorem 10.50 in [14]. $\square$

It follows from 1. that

$$\Phi(f)\Phi(g) = \Phi(fg) \iff D\big(\Phi(fg)\big) \subset D\big(\Phi(g)\big). \tag{8.1}$$

Corollary 2. *Under the above assumptions, it follows that*

$$\Phi(f^n) = \big((\Phi(f)\big)^n, \quad n \in \mathbb{N}.$$

Proof. We prove by induction. The result is trivial for $n = 1$.
So suppose that $\Phi(f^k) = \big((\Phi(f)\big)^k$ for some $k \in \mathbb{N}$. If $\psi \in D\big(\Phi(f^{k+1})\big)$, then $\int_\Omega |f|^{2k+2} \, d\mu_\psi < \infty$, and since μ_ψ is a finite measure, we therefore have $\int_\Omega |f|^{2k} \, d\mu_\psi < \infty$, so that $\psi \in \big(D(\Phi(f^k)\big)$. Thus, $D\big(\Phi(f^{k+1})\big) \subset \big(D(\Phi(f^k)\big)$. By equation 8.1, we then have $\Phi(f^{k+1}) = \big(\Phi(f)\big)^{k+1}$, proving the corollary. $\square$

Theorem 20 (Spectral theorem for normal operators)**.** *For every normal operator A, there exists a unique projection-valued measure μ on $\sigma(A)$ such that*

$$(A\psi|\psi) = \int_{\sigma(A)} \lambda \, d\mu_\psi(\lambda), \quad \psi \in D(A).$$

Proof. See Theorem 10.56 in [14]. $\square$

For normal operators A and measurable functions $f : \sigma(A) \to \mathbb{C}$, the operator $\Phi(f)$ defined in terms of the projection-valued measure μ of A by the calculus of Theorem 19 will be denoted by $f(A)$:

$$f(A) := \Phi(f).$$

Theorem 21 (Spectral mapping theorem). *Let A be normal with projection-valued measure μ, and let $f : \sigma(A) \to \mathbb{C}$ be measurable. Then*

$$\sigma\big(f(A)\big) = R_\mu(f) \subset \overline{f\big(\sigma(A)\big)}.$$

If f is continuous, then

$$\sigma\big(f(A)\big) = \overline{f\big(\sigma(A)\big)}.$$

Proof. See Theorem 10.57 in [14]. □

The following lemma tells us how to think of the domain of the Tomita operator S. Recall that the domain of S_0 is expressed in terms of operators in $\mathcal{M}$ acting on Ω. Similarly, the domain of S can be expressed in terms of operators affiliated to $\mathcal{M}$ acting on Ω.

Definition 31 (Affiliated operator). A closed, unbounded operator T is said to be **affiliated** to $\mathcal{M}$ if it commutes with every operator $x' \in \mathcal{M}'$ on all vectors where both Tx' and $x'T$ are defined, that is, $Tx'\psi = x'T\psi$ for all $\psi \in D(Tx') \cap D(x'T)$.

Definition 32 (Closure, closable operator). Given a linear operator A, not necessarily closed, if the closure of its graph in $\mathcal{H} \oplus \mathcal{H}$ happens to be the graph of some operator, that operator is called the **closure** of A, and we say that A is **closable**. Denote the closure of A by $\overline{A}$. It follows that $A = \overline{A}|_{D(A)}$.

Definition 33 (Core). A **core** of a closable operator A is a subset $C \subset D(A)$ such that $\overline{A|_C} = \overline{A}$.

Lemma 2. *Let Ω be a cyclic and separating vector for a von Neumann algebra $\mathcal{M}$, and let S be the Tomita operator. Then $D(S)$ consists of all vectors of the form $T\Omega$, where T is a closed operator affiliated with $\mathcal{M}$, having $\mathcal{M}'\Omega$ as a core, and for which $\Omega \in D(T) \cap D(T^*)$. The operator S acts as*

$$S(T\Omega) = T^*\Omega.$$

Proof. See Lemma 1.10 in [24]. □

The following lemma tells us how to analytically continue a unitary group generated by a positive operator in the complex plane. It will be needed to show that $\Delta^{-it}x\Delta^{it}$ lies in $\mathcal{M}$ for large $t \in \mathbb{R}$.

Lemma 3. *Let P be a positive, invertible, self-adjoint operator on the Hilbert space $\mathcal{H}$. Let $w \in \mathbb{C}$ such that $\Re(w) > 0$ and fix $x \in \mathcal{B}(\mathcal{H})$.*
If the operator $P^{-w}xP^{w}$ is defined and bounded on a core for P^w, then for every $z \in \mathbb{Z}$ in the strip $\{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq \Re(w)\}$, the operator $P^{-z}xP^{z}$ is bounded on its domain, so that it is closable with bounded closure. The $\mathcal{B}(\mathcal{H})$-valued function

$$z \mapsto \overline{P^{-z}xP^{z}}$$

is analytic in the strip with respect to the norm topology, and continuous on the boundaries of the strip with respect to the strong operator topology.

Proof. See Section 9.24 in [27]. □

The following lemma is due to Takesaki, and appears in almost all general proofs of Tomita's theorem. In this proof, we study a class of operators $x \in \mathcal{M}$ for which the modular operator Δ "looks bounded". Concretely, this will mean that the vector $x\Omega$ lies in a spectral subspace of Δ with bounded spectral range. To produce these vectors, we use operators of the form $(\lambda - \Delta)$ to truncate the spectral subspaces of Δ, and this lemma tells us how to think of $(z - \Delta)$, when z comes from the resolvent.

Lemma 4. *Let Ω be a cyclic and separating vector for a von Neumann algebra $\mathcal{M}$ and let Δ be the associated modular operator. Let z be in the resolvent set of Δ, so that $(z - \Delta)$ is invertible as a bounded operator. Fix $x' \in \mathcal{M}'$.*
Then there exists a unique operator $x \in \mathcal{M}$ satisfying

$$x\Omega = (z - \Delta)^{-1} x'\Omega,$$

and it satisfies the bound

$$||x|| \leq \frac{||x'||}{\sqrt{2(|z| - \Re(z))}}.$$

Proof. See Lemma 2.5.12 in [18]. □

The following lemma allows us to express analytic funtions of bounded operators as residue integrals.

Lemma 5. *Let $x \in \mathcal{B}(\mathcal{H})$ be self-adjoint and let f be a function analytic in a neighbourhood of $\sigma(x)$. Then the operator $f(x)$, defined by the Borel functional calculus, can be written in terms of the norm-convergent Bochner integral*

$$f(x) = \frac{1}{2\pi i} \int_\gamma f(z)(z - x)^{-1} \, dz,$$

where γ is any simple, counter-clockwise oriented, closed contour in the domain of f encircling $\sigma(x)$.

Proof. See Sections 2.25 and 2.29 in [27]. □

In complex analysis, the Phragmén-Lindelöf principle is a generalisation of the maximum modulus principle to holomorphic functions on certain unbounded domains. The version of Phragmén-Lindelöf provided below is specific to this particular proof.

Theorem 22 (Phragmén-Lindelöf)**.** *Let $S = \{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq n\}$ be a vertical strip in the complex plane and let $f : S \to \mathcal{B}(\mathcal{H})$ be a function that is holomorphic on the interior of S and strongly continuous on the boundary of S. Suppose that f is bounded on the boundary of the strip, that is, $\sup_{z \in bd(S)} \|f(z)\| < \infty$, and that it grows at most doubly exponentially in the imaginary direction, that is, there exist constants α, β, γ with $\gamma < \pi/n$, such that*

$$\|f(z)\| \leq \alpha e^{\beta \exp(\gamma |\Im(z)|)}, \quad z \in S.$$

Then f is bounded by its bound on the boundary of the strip, that is, $\sup_{z \in bd(S)} \|f(z)\| < \infty$.

In complex analysis, Carlson's theorem roughly states that two different holomorphic functions which do not grow very fast at infinity cannot coincide at the integers.

Theorem 23 (Carlson's theorem)**.** *Let f be a function defined on the closed right half-plane satisfying the following three conditions:*

1. *f is holomorphic on the open right half-plane, strongly continuous on the imaginary axis, and*

$$\|f(z)\| \leq \alpha e^{\beta |\Re(z)|}, \quad \Re(z) > 0$$

for some $\alpha, \beta \in \mathbb{R}$,

2.

$$\|f(iy)\| \leq \alpha e^{\gamma |y|}, \quad y \in \mathbb{R}$$

for some $\gamma < \pi$,

3. *$f(n) = 0$ for all $n \in \mathbb{N}$.*

Then $f = 0$.

8.2. Introduction

This chapter provides a proof of Tomita's theorem based on a 2024 paper by Jonathan Sorce. [21]

Let Θ be the Heaviside theta function, defined by

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases}.$$

The main idea of this proof is to produce operators in $\mathcal{M}$ for which the modular operator "looks bounded" by starting with a vector $x'\Omega$ and acting on it with the operator $\Theta(\lambda - \Delta)$ for some $\lambda > 0$. We will be able to study the vector $\Theta(\lambda - \Delta)x'\Omega$ by approximating the function $\Theta(\lambda - x)$ with a sequence of sigmoid functions,

$$f_k(z) = \frac{1}{1 + e^{k(z-\lambda)}}.$$

Since f_k is analytic in the complex plane, the operator $f_k(\Delta)$ (defined via the functional calculus from the spectral theorem) can be studied using a contour integral of $f_k(z)$ multiplied by the resolvent $(z - \Delta)^{-1}$. After taking the limit $k \to \infty$, for any $n \in \mathbb{N}$, we can introduce bounded operators $x_{\lambda,n} \in \mathcal{M}$ satisfying

$$x_{\lambda,n}\Omega = g_\lambda(\Delta)x'\Omega.$$

A symmetric argument substituting the modular operator with Δ^{-1} and $\mathcal{M}$ with its commutant (as in Proposition 9) shows that when starting with some vector $x\Omega$ there are operators satisfying

$$x'_{\lambda,n}\Omega = \Delta^n\Theta(\Delta - \lambda)x\Omega.$$

After some manipulation, we can then construct operators restricting the modular operator to acting on its spectral subspace corresponding to the range $[\lambda_1, \lambda_2]$ for any $0 < \lambda_1 < \lambda_2$, that is, operators $x_{[\lambda_1,\lambda_2],n}$ satisfying

$$x_{[\lambda_1,\lambda_2],n}\Omega = g_{\lambda_2}(\Delta)\Theta(\Delta - \lambda_1)x\Omega.$$

It is these operators, which we call tidy operators, for which the modular operator "looks bounded". The space of all tidy operators forms a subspace of $\mathcal{M}$, which we denote $\mathcal{M}_{\text{tidy}}$. Then it can be shown that for any tidy operator x, $\Delta^n x\Delta^{-n}$ is well-defined and bounded on $\mathcal{M}_{\text{tidy}}\Omega$, and that for any real number t, $\mathcal{M}_{\text{tidy}}$ is a core for Δ^t. As a result, we show that Tomita's theorem holds for tidy operators.

The final part of the proof then poses that for any operator x, the operators $\Delta^{it}x\Delta^{-it}$ commute with every operator that is tidy for the commutant $\mathcal{M}'$. This space can be shown to be equal to the commutant $\mathcal{M}'$, and so $\Delta^{it}x\Delta^{-it}$ is in the double commutant $\mathcal{M}''$ for all $t \in \mathbb{R}$. Therefore, by von Neumann's double commutant theorem, $\Delta^{it}x\Delta^{-it}$ is in $\mathcal{M}$ for all $t \in \mathbb{R}$, and so Tomita's theorem holds in general.

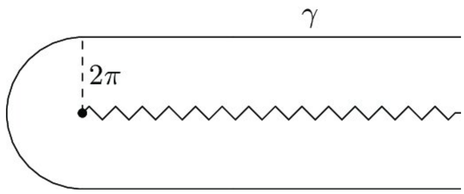

Figure 8.1: A sketch of the contour γ used in Proposition 10 and Theorem 24. The black dot denotes the origin of the complex plane, and the jagged line is the positive real axis.

8.3. Tidy operators

Proposition 10. *Fix $\lambda > 0$, and let $f_k : \mathbb{C} \to \mathbb{C}$ be the sigmoid function*

$$f_k = \frac{1}{1 + e^{k(z-\lambda)}}$$

and let

$$f : [0, \infty) \to \mathbb{C}, \ t \mapsto t^n f_k(t)$$

. Let Δ be an invertible, self-adjoint, positive operator on a Hilbert space $\mathcal{H}$. Let γ be the counter-clockwise contour in the complex plane surrounding the positive real axis, given by combining the half-lines $\{t \pm 2\pi i, \ t \geq 0\}$ with the half-circle of radius $2\pi i$ centered at the origin, as in figure 8.1.
Then for any $n \in \mathbb{N}$, and any $\psi \in \mathcal{H}$ we have

$$f(\Delta)\psi = \frac{1}{2\pi i} \int_\gamma z^n f_k(z)(z - \Delta)^{-1}\psi \, dz,$$

where $f(\Delta)$ is defined by the Borel functional calculus and this integral converges as a Bochner integral.

Proof. Let $m \in \mathbb{N}$. From the spectral theorem for Δ (Theorem 20), there exists a unique projection-valued measure μ on $\sigma(\Delta)$ such that

$$\Delta = \int_{\sigma(\Delta)} t d\mu(t).$$

and let Π_m be the spectral projection $\mu([0, m])$.
By Theorem 10.58 in [14], if $\Pi_m\psi = \psi$, then $\Pi_m\psi = \psi \in D(\Delta)$ and $\Delta\psi = c\psi$, for some $c \in [0, m]$. Otherwise, $\Pi_m\psi = 0$, so $\Pi_m\psi = 0 \in D(\Delta)$ and $\Delta 0 = 0$. Hence, $\Delta_{(m)} := \Delta\Pi_m$ is well-defined and it is precisely the restriction of Δ to the spectral range $[0, m]$. By construction, $\|\Delta_{(m)}\psi\| \leq m\|\psi\|$ for all $\psi \in \mathcal{H}$, and so $\Delta_{(m)}$ is bounded by m.
We know that $\lim_{t\to\infty} t^n f_k(t) = 0$. Hence there is some $c > 0$ so that $|t^n f_k(t)| < 1$ for all $t > c$, and so f is bounded on (c, ∞). Now, since f is continuous on $[0, c]$, we have that f is bounded on $[0, c]$ as well, and so f is bounded. By Corollary 9.18 in [14], since Δ is positive, $\sigma(\Delta) \subset [0, \infty)$, and by construction, $\sigma(\Delta_{(m)}) \subset [0, m]$. Then, by the bounded functional calculus (17), there exist bounded operators $f(\Delta)$ and $f(\Delta_{(m)})$ such that

$$\big(f(\Delta)\psi|\psi\big) = \int_{\sigma(\Delta)} f \, d\mu_\psi, \qquad \big(f(\Delta_{(m)})\psi|\psi\big) = \int_{\sigma(\Delta_{(m)})} f \, d\mu_\psi$$

for all $\psi \in \mathcal{H}$.
Now, consider the vector $\Pi_m\psi$. We have

$$f(\Delta)\Pi_m\psi = f(\Delta_{(m)})\Pi_m\psi$$

by Corollary 10.59 in [14].
Each $\Delta_{(m)}$ is bounded by m by construction, so we may express this equation in terms of a norm-convergent Bochner integral as

$$f(\Delta)\Pi_m\psi = \frac{1}{2\pi i} \int_{\gamma_m + v_m} f(z)(z - \Delta_{(m)})^{-1}\Pi_m\psi \, dz,$$

where v_m is the vertical segment passing through the real axis at $m + 1/2$, oriented in the positive imaginary direction, and with endpoints on the contour γ, and γ_m is the portion of the contour γ lying to the left of this vertical segment.
In fact, since the spectrum of $\Delta_{(m)}$ lies in the range $[0, m]$, we may write this integral for any $m' \geq m$ as

$$f(\Delta)\Pi_m\psi = \frac{1}{2\pi i}\int_{\gamma_{m'}+v_{m'}} f(z)(z-\Delta_{(m)})^{-1}\Pi_m\psi\, dz,$$

where $v_{m'}$ and $\gamma_{m'}$ are defined similarly.
For $z \in \mathbb{C}$ so that $\Re(z) > m$, we have

$$\|(z-\Delta_{(m)})^{-1}\| \leq \sup\left\|\sigma\big((z-\Delta_{(m)})^{-1}\big)\right\| \leq \sup\left\|\big(z-\sigma(\Delta_{(m)})\big)^{-1}\right\| \leq \|(z-m)^{-1}\| \leq \frac{1}{\Re(z)-m}$$

where we used the spectral mapping theorem (Theorem 20) in the second inequality. Also, we have $\|\Pi_m\| \leq 1$. Hence,

$$\left\|\int_{v_{m'}} f(z)(z-\Delta_{(m)})^{-1}\Pi_m\psi\, dz\right\| \leq \int_{v_{m'}} \frac{z^n f_k(z)}{\Re(z)-m}\|\psi\|\, dz \to 0$$

as $m' \to \infty$, since f_k is a sigmoid function. This gives the identity

$$\begin{aligned} f(\Delta)\Pi_m\psi &= \frac{1}{2\pi i}\int_\gamma f(z)(z-\Delta_{(m)})^{-1}\Pi_m\psi\, dz \\ &= \frac{1}{2\pi i}\int_\gamma z^n f_k(z)(z-\Delta)^{-1}\Pi_m\psi\, dz. \end{aligned}$$

So far we have shown that the proposition holds for any vector of the form $\Pi_m\psi$. However, by point 5. in Theorem 17, the sequence Π_m converges strongly to the identity operator, since $\mathbf{1}_{[0,m]}$ converges pointwise to $\mathbf{1}_{[0,\infty)}$. Taking the limit $m \to \infty$ in the above expression gives

$$\Delta^n f_k(\Delta)\psi = \frac{1}{2\pi i}\lim_{m\to\infty}\int_\gamma z^n f_k(z)(z-\Delta)^{-1}\Pi_m\psi\, dz.$$

Since Π_m are projections, $\|\Pi_m\varphi\| \leq \|\varphi\|$. Clearly, $z^n f_k(z)(z-\Delta)^{-1}$ is Bochner-integrable, and so the dominated convergence theorem (Theorem 16) lets us move the limit inside the integral and proves the proposition:

$$\begin{aligned} \Delta^n f_k(\Delta)\psi &= \frac{1}{2\pi i}\int_\gamma \lim_{m\to\infty} z^n f_k(z)(z-\Delta)^{-1}\Pi_m\psi\, dz \\ &= \frac{1}{2\pi i}\int_\gamma z^n f_k(z)(z-\Delta)^{-1}\psi\, dz. \end{aligned}$$

□

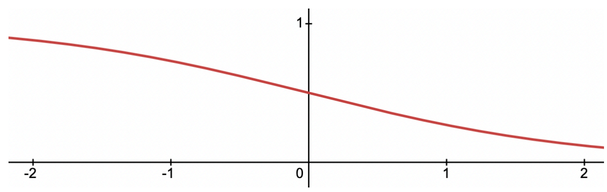

Figure 8.2: A sketch of the sigmoid function used in Proposition 10 and Proposition 11, restricted to the real number line, with $\lambda = 0$ and $k = 1$. The parameters λ and k determine the center and the steepness of the function respectively .

Proposition 11. *Fix $\lambda > 0$, and let $F_k : \mathbb{C} \to \mathbb{C}$ be the sigmoid function*

$$f_k(z) = \frac{1}{1+e^{k(z-\lambda)}},$$

let

$$f_{(k)} : [0, \infty) \to \mathbb{C},\ t \mapsto t^n f_k(t),$$

and let

$$g_\lambda : [0, \infty) \to \mathbb{C}.\ t \mapsto t^n \Theta(\lambda - t).$$

Let Δ be an invertible, self-adjoint, positive operator. Then for any $\psi \in \mathcal{H}$ and any nonnegative integer n, the vector sequence $f_{(k)}(\Delta)\psi$ converges to $g_\lambda(\Delta)\psi$ in the limit $k \to \infty$, where $f_{(k)}(\Delta)$ and $g_\lambda(\Delta)$ are defined by the Borel functional calculus.

Proof. We aim to show the identity

$$\lim_{k\to\infty} \|(f_{(k)}(\Delta) - g_\lambda(\Delta)\psi\| = 0.$$

From the spectral theorem for Δ (Theorem 20), there exists a unique projection-valued measure μ on $\sigma(\Delta)$ such that

$$\Delta = \int_{\sigma(\Delta)} t d\mu(t).$$

In the proof of Proposition 10, we proved that $f_{(k)}$ is bounded. Similarly, since g_λ is continuous on $[0, \lambda]$, it is bounded on $[0, \lambda]$. As $g_\lambda = 0$ on (λ, ∞), it is clear that g_λ is bounded.
Therefore, by the bounded functional calculus (Theorem 17), there are unique bounded operators $f_{(k)}(\Delta), g_\lambda(\Delta)$ such that

$$\big(f_{(k)}(\Delta)\psi|\psi\big) = \int_{\sigma(\Delta)} f_{(k)}\, d\mu_\psi, \qquad \big(g_\lambda(\Delta)\psi|\psi\big) = \int_{\sigma(\Delta)} g_\lambda\, d\mu_\psi.$$

Since the map Φ from the functional calculus is linear, we have that $(f_{(k)} - g_\lambda)(\Delta) = f_{(k)}(\Delta) - g_\lambda(\Delta)$ and we know that $(f_{(k)} - g)(\Delta)$ satisfies

$$\|(f_{(k)} - g)(\Delta)\psi\|^2 = \int_{\sigma(\Delta)} |(f_{(k)} - g)(t)|^2\, d\mu_\psi(t).$$

Substituting, we find

$$\|\big(f_{(k)}(\Delta) - g_\lambda(\Delta)\big)\psi\|^2 = \int_{\sigma(\Delta)} |f_{(k)}(t) - g_\lambda(t)|^2\, d\mu_\psi(t).$$

Since Δ is positive, $\sigma(\Delta) \subset [0, \infty)$, so as the integrand is nonnegative, we have

$$\int_{\sigma(\Delta)} |f_{(k)}(t) - g_\lambda(t)|^2\, d\mu_\psi(t) \le \int_{[0,\infty)} |t^n f_k(t) - t^n\Theta(\lambda - t)|^2\, d\mu_\psi(t).$$

We know that

$$\int_{(\lambda,\infty)} |t^n f_1(t)|\, d\mu_\psi(t) < \infty \qquad \text{and} \qquad |t^n f_k(t) - t^n\Theta(\lambda - t)|^2 \le |t^n f_1(t) - t^n\Theta(\lambda - t)|^2$$

holds for all $k \in \mathbb{N}$. Furthermore, we have the pointwise limit

$$\lim_{k\to\infty} f_k(t) = \Theta(\lambda - t), \quad t \ge 0.$$

Therefore

$$\lim_{k\to\infty} \int_{[0,\infty)} |t^n f_k(t) - t^n\Theta(\lambda - t)|^2\, d\mu_\psi(t) = \int_{[0,\infty)} \lim_{k\to\infty} |t^n f_k(t) - t^n\Theta(\lambda - t)|^2\, d\mu_\psi(t) = 0$$

$\square$

Theorem 24. *Fix $\lambda > 0$. Let Δ be the modular operator associated with a cyclic and separating vector Ω for a von Neumann algebra $\mathcal{M}$. Let $n \in \mathbb{N}$ and fix $x' \in \mathcal{M}'$.*
Let

$$g_\lambda : [0, \infty) \to \mathbb{C}.\ t \mapsto t^n \Theta(\lambda - t).$$

There exists a bounded operator $x_{\lambda,n} \in \mathcal{M}$ satisfying

$$x_{\lambda,n}\Omega = g_\lambda(\Delta)x'\Omega.$$

Proof. From the spectral theorem for Δ (Theorem 20), there exists a unique projection-valued measure μ on $\sigma(\Delta)$.
Fix $n \in \mathbb{N}$ and consider the function

$$f : [0, \infty) \to [0, \infty),\ t \mapsto t^{1/2}.$$

Clearly, $(fg_\lambda)(t) = t^{n+1/2}\Theta(\lambda - t) \leq \lambda^{n+1/2}$ for all $0 \leq t < \lambda$, and so fg_λ is bounded. Therefore, by the bounded functional calculus (Theorem 17), the operator $(fg_\lambda)(\Delta)$ is bounded.
In Proposition 11, we proved that g_λ is bounded. Hence, by the bounded functional calculus, $g_\lambda(\Delta)$ is a bounded operator. Lastly, by the unbounded functional calculus (Theorem 19), there is a normal operator $f(\Delta)$.
Since $g_\lambda(\Delta)$ is bounded, $D\big(g_\lambda(\Delta)\big) = \mathcal{H}$, and so equation 8.1 implies that $f(\Delta)g_\lambda(\Delta) = (fg)(\Delta)$. Therefore, $g_\lambda(\Delta)x'\Omega \in D(f(\Delta)) = D(\Delta^{1/2})$. So by Lemma 2, there exists a closed operator $x_{\lambda,n}$ affiliated to $\mathcal{M}$, with $\mathcal{M}'\Omega$ as a core, satisfying $x_{\lambda,n}\Omega = g_\lambda(\Delta)x'\Omega$. The goal is to show that $x_{\lambda,n}$ is bounded.
Since $\mathcal{M}'\Omega$ is a core for $x_{\lambda,n}$ it suffices to show that $x_{\lambda,n}$ has bounded action on vectors of the form $y'\Omega$ for $y' \in \mathcal{M}'$. Combining Propositions 10 and 11, and once again using f_k to denote the sigmoid function from those propositions, we have

$$\begin{aligned} x_{\lambda,n}y'\Omega &= y'x_{\lambda,n}\Omega \\ &= y'g_\lambda(\Delta)x'\Omega \\ &= y' \lim_{k\to\infty} f_{(k)}(\Delta)x'\Omega \\ &= \frac{1}{2\pi i} y' \lim_{k\to\infty} \int_\gamma z^n f_k(z)(z-\Delta)^{-1}x'\Omega\, dz, \end{aligned} \tag{8.2}$$

where γ is the contour from figure 8.2. By Lemma 4, there exist operators $x_z \in \mathcal{M}$ satisfying

$$(z - \Delta)^{-1}x'\Omega = x_z\Omega \qquad \text{and} \qquad \|x_z\| \leq \frac{\|x'\|}{\sqrt{2(|z| - \Re(z))}}. \tag{8.3}$$

Since y' is a bounded operator, and since the integral in equation 8.2 converges as a Bochner integral, we may move y' through the limit and through the integral symbol to write

$$x_{\lambda,n}y'\Omega = \lim_{k\to\infty} \frac{1}{2\pi i} \int_\gamma z^n f_k(z) x_z y'\Omega\, dz.$$

Taking norms on either side of the equation gives

$$\|x_{\lambda,n}y'\Omega\| \leq \frac{1}{2\pi}\|y'\Omega\| \limsup_{k\to\infty} \int_\gamma |z|^n |f_k(z)| \|x_z\|\, ds,$$

where s is an arclength parameter for the countour γ. Using the bound in (8.3), we may write the inequality

$$\|x_{\lambda,n}y'\Omega\| \leq \frac{\|x'\|}{2\pi}\|y'\Omega\| \limsup_{k\to\infty} \int_\gamma \frac{|z|^n |f_k(z)|}{\sqrt{2(|z| - \Re(z))}}\, ds, \tag{8.4}$$

Our goal is to give an upper bound for the limit in this inequality: $\limsup_{k\to\infty} \int_\gamma \frac{|z|^n|f_k(z)|}{\sqrt{2(|z|-\Re(z))}}$. On either half-line $\{t \pm 2\pi i,\ t \geq 0\}$, we have

$$f_k(t \pm 2\pi i) = \frac{1}{1 + e^{k(t \pm 2\pi i - \lambda)}} = \frac{1}{1 + e^{k(t-\lambda)}},$$

since $e^{2k\pi i} = 1$ for all $k \in \mathbb{N}$. We have the pointwise limit

$$\lim_{k\to\infty} f_k(t) = \Theta(\lambda - t), \quad t \geq 0.$$

Let s be an arclength parameter for the contour γ from figure 8.2. An application of the dominated convergence theorem then gives

$$\begin{aligned}\limsup_{k\to\infty} \int_{\text{half-line}} \frac{|z|^n |f_k(z)|}{\sqrt{2(|z| - \Re(z))}}\, ds &= \int_0^\infty \limsup_{k\to\infty} \frac{(t^2+4\pi^2)^{n/2}\Theta(\lambda - t)}{\sqrt{2((t^2+4\pi^2)^{1/2} - t)}}\, dt \\ &= \int_0^\lambda \frac{(t^2+4\pi^2)^{n/2}}{\sqrt{2((t^2+4\pi^2)^{1/2} - t)}}\, dt.\end{aligned}$$

where, again, the half-line is give by $\{t \pm 2\pi i,\ t \geq 0\}$.
The integrand is monotonically increasing in t, so the integral can be upper bounded by λ times the value of the integrand at $t = \lambda$, giving

$$\limsup_{k\to\infty} \int_{\text{half-line}} \frac{|z|^n |f_k(z)|}{\sqrt{2(|z| - \Re(z))}}\, ds \leq \lambda \frac{(\lambda^2+4\pi^2)^{n/2}}{\sqrt{2((\lambda^2+4\pi^2)^{1/2} - \lambda)}}. \tag{8.5}$$

The other contribution to the contour integral in (8.4) is an integral over a half-circle $\left\{2\pi e^{i\theta},\ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\right\}$ and may be written as

$$\int_{\text{half-circle}} \frac{|z|^n |f_k(z)|}{\sqrt{2(|z| - \Re(z))}}\, ds = \int_{\pi/2}^{3\pi/2} 2\pi \frac{(2\pi)^n |f_k(2\pi e^{i\theta})|}{\sqrt{2(2\pi - 2\pi\cos(\theta))}}\, d\theta = \int_{\pi/2}^{3\pi/2} \frac{(2\pi)^{n+1} |f_k(2\pi e^{i\theta})|}{\sqrt{4\pi(1-\cos(\theta))}}\, d\theta.$$

We have the pointwise limit

$$\lim_{k\to\infty} f_k(2\pi e^{i\theta}) = \lim_{k\to\infty} (1 + e^{k(2\pi\cos(\theta) - \lambda)} e^{ik(2\pi\sin(\theta))})^{-1} = 1,$$

since $2\pi\cos(\theta) - \lambda < 0$ for all $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ and $\lambda > 0$, and another application of the dominated convergence theorem gives

$$\limsup_{k\to\infty} \int_{\text{half-circle}} \frac{|z|^n |f_k(z)|}{\sqrt{2(|z| - \Re(z))}}\, ds = \int_{\pi/2}^{3\pi/2} \frac{(2\pi)^{n+1}}{\sqrt{4\pi(1-\cos(\theta))}}\, d\theta.$$

On the half-circle, $\cos(\theta) \leq 0$, so the denominator is lower-bounded by $\sqrt{4\pi}$, which gives the simple approximation

$$\limsup_{k\to\infty} \int_{\text{half-circle}} \frac{|z|^n |f_k(z)|}{\sqrt{2(|z| - \Re(z))}}\, ds \leq \frac{(2\pi)^{n+1}\pi}{\sqrt{4\pi}}. \tag{8.6}$$

Combining expressions 8.6 and 8.5 with the expression 8.4, we may bound each operator $x_{\lambda,n}$ by

$$\|x_{\lambda,n}\| \leq \frac{\|x'\|}{2\pi}\left(2\lambda \frac{(\lambda^2+4\pi^2)^{n/2}}{\sqrt{2((\lambda^2+4\pi^2)^{1/2} - \lambda)}} + \frac{(2\pi)^{n+1}\pi}{\sqrt{4\pi}}\right).$$

Clearly taking $\alpha_\lambda = \max\{\lambda, \pi\}\|x'\|$ and $\beta_\lambda = \log_\lambda(\lambda + 2\pi) + \log(2\pi)$ satisfies $\|x_{\lambda,n}\| \leq \alpha_\lambda e^{\beta_\lambda n}$. $\square$

Corollary 3. *Fix $\lambda > 0$. Let Δ be the modular operator of a cyclic-separating vector for a von Neumann algebra $\mathcal{M}$ with commutant $\mathcal{M}'$. Let $n \in \mathbb{Z}_{\leq 0}$ and fix $x \in \mathcal{M}$.*
There exists a bounded operator $x'_{\lambda,n} \in \mathcal{M}'$ satisfying

$$x'_{\lambda,n}\Omega = g_{\lambda^{-1}}(\Delta^{-1}) = g_{-\lambda}(-\Delta)x\Omega.$$

Moreover, there exist n-independent constants $\alpha'_\lambda, \beta'_\lambda > 0$ with $\|x'_{\lambda,n}\| \leq \alpha'_\lambda e^{-\beta'_\lambda n}$. In particular, one has the concrete bound

$$\|x'_{\lambda,n}\| \leq \frac{\|x\|}{2\pi}\left(2\lambda^{-1}\frac{(\lambda^{-2}+4\pi^2)^{-n/2}}{\sqrt{2((\lambda^{-2}+4\pi^2)^{1/2}-\lambda^{-1})}} + \frac{(2\pi)^{-n+1}\pi}{\sqrt{4\pi}}\right).$$

Proof. By von Neumann's double commutant theorem (Theorem 16), we know that $\mathcal{M} = \mathcal{M}''$. Recall from Proposition 9 that Δ^{-1} is the modular operator associated with $\{\mathcal{M}', \Omega\}$. Therefore, applying Theorem 24 with the substitutions $\Delta \leftrightarrow \Delta^{-1}$ and $\mathcal{M} \leftrightarrow \mathcal{M}'$ yields an operator $x'_{\lambda,n} \in \mathcal{M}'$ satisfying

$$x'_{\lambda,n}\Omega = g_{\lambda^{-1}}(\Delta^{-1}) = g_{-\lambda}(-\Delta)x\Omega.$$

since $\lambda^{-1} > t^{-1}$ if and only if $-\lambda > -t$.
Moreover, Theorem 24 yields n-indepenedent constants $\alpha'_\lambda, \beta'_\lambda > 0$ satisfying $\|x'_{\lambda,n}\| \leq \alpha'_\lambda e^{-\beta'_\lambda n}$. □

Theorem 25 (Construction of the tidy subspace)**.** *Fix $\lambda_1, \lambda_2 \in \mathbb{R}$ satisfying $0 < \lambda_1 < \lambda_2$ and fix $x \in \mathcal{M}$. Then, for any $n \in \mathbb{Z}$, there exist unique operators*

$$x_{[\lambda_1,\lambda_2],n} \in \mathcal{M}, \quad x'_{[\lambda_1,\lambda_2],n} \in \mathcal{M}'$$

satisfying

$$g_{\lambda_2}(\Delta)\Theta(\Delta-\lambda_1)x\Omega = x_{[\lambda_1,\lambda_2],n}\Omega = x'_{[\lambda_1,\lambda_2],n}\Omega.$$

Furthermore, there exist n-independent constants $\alpha_{[\lambda_1,\lambda_2]}, \beta_{[\lambda_1,\lambda_2]}, \alpha'_{[\lambda_1,\lambda_2]}, \beta'_{[\lambda_1,\lambda_2]} > 0$ so that we have $\|x_{[\lambda_1,\lambda_2],n}\| \leq \alpha_\lambda e^{-\beta_\lambda n}$ and $\|x'_{[\lambda_1,\lambda_2],n}\| \leq \alpha'_\lambda e^{-\beta'_\lambda n}$.

Proof. Let $n \geq 0$. By Corollary 3, there exists an operator $x'_{\lambda_1,0} \in \mathcal{M}'$ such that

$$g_{\lambda_2}(\Delta)\Theta(\Delta-\lambda_1)x\Omega = g_{\lambda_2}(\Delta)x'_{\lambda_1,0}\Omega.$$

Now applying Theorem 24, there exists a bounded operator $x_{[\lambda_1,\lambda_2],n} \in \mathcal{M}$ such that

$$g_{\lambda_2}(\Delta)\Theta(\Delta-\lambda_1)x\Omega = x_{[\lambda_1,\lambda_2],n}\Omega.$$

Now let $n < 0$. First, we introduce a function $h_{\lambda_1,\lambda_2} : [0,\infty) \to \mathbb{C}$ given by $t \mapsto t^n\Theta(\lambda_2 - t)\Theta(t-\lambda_1)$. Clearly, h_{λ_1,λ_2} is continuous on $[\lambda_1, \lambda_2]$, and so it is bounded. Furthermore, we have $g_{\lambda_2}(t)\Theta(t-\lambda_1) = h_{\lambda_1,\lambda_2}(t) = \Theta(\lambda_2 - t)g_{-\lambda}(-t)$ for all $t \in [0,\infty)$. Hence, by point 2. in Theorem 17,

$$g_{\lambda_2}(\Delta)\Theta(\Delta-\lambda_1)x\Omega = h_{\lambda_1,\lambda_2}(\Delta)x\Omega = \Theta(\lambda_2-\Delta)g_{-\lambda_1}(-\Delta)x\Omega.$$

By Corollary 3, there exists an operator $x'_{\lambda_1,n} \in \mathcal{M}'$ so that

$$g_{\lambda_2}(\Delta)\Theta(\Delta-\lambda_1)x\Omega = \Theta(\lambda_2-\Delta)g_{-\lambda_1}(-\Delta)x\Omega = \Theta(\lambda_2-\Delta)x'_{\lambda_1,n}\Omega.$$

Now applying Theorem 24, there exists a bounded operator $x_{[\lambda_1,\lambda_2],n} \in \mathcal{M}$ so that

$$g_{\lambda_2}(\Delta)\Theta(\Delta-\lambda_1)x\Omega = x_{[\lambda_1,\lambda_2],n}\Omega.$$

In both cases, there exist n-independent constants $\alpha_{[\lambda_1,\lambda_2]}, \beta_{[\lambda_1,\lambda_2]} > 0$ so that we have $\|x_{[\lambda_1,\lambda_2],n}\| \leq \alpha_\lambda e^{-\beta_\lambda n}$.
In both cases, since $\Omega \in \mathcal{H}$ is a cyclic and separating vector, there exists an operator $x'_{[\lambda_1,\lambda_2],n} \in \mathcal{M}'$ such that

$$x_{[\lambda_1,\lambda_2],n}\Omega = x'_{[\lambda_1,\lambda_2],n} = \Omega.$$

Furthermore, there exist n-independent constants $\alpha'_{[\lambda_1,\lambda_2]}, \beta'_{[\lambda_1,\lambda_2]} > 0$ such that $\|x'_{[\lambda_1,\lambda_2],n}\| \leq \alpha'_\lambda e^{-\beta'_\lambda n}$, proving the theorem. □

Definition 34 (Tidy operators)**.** The space of operators $x_{[\lambda_1,\lambda_2],0} \in \mathcal{M}$ obtained as in Theorem 25 is called the space of **tidy operators** in $\mathcal{M}$, denoted by $\mathcal{M}_{\text{tidy}}$.
If $x \in \mathcal{M}$ is tidy, we denote by $x' \in \mathcal{M}'$ the operator satisfying

$$x\Omega = x'\Omega$$

and for any $n \in \mathbb{Z}$ we denote by $x_n \in \mathcal{M}$ and $x'_n \in \mathcal{M}'$ the operators satisfying

$$x_n\Omega = \Delta^n x\Omega = \Delta^n x'\Omega = x'_n\Omega.$$

8.4. Tomita's theorem

The following lemma conceptualises the adjoints of tidy operators.

Lemma 6. *Let $x \in \mathcal{M}_{tidy}$ and $n \in \mathbb{Z}$. Then*

$$(x'_{n+1})^*\Omega = (x_n)^*\Omega.$$

Proof. Fix $y \in \mathcal{M}$. Then

$$\begin{aligned}((x'_{n+1})^*\Omega|y\Omega) &= (y^*\Omega|x'_{n+1}\Omega)\\ &= (y^*\Omega|\Delta x_n\Omega)\\ &= (y^*\Omega|S^*Sx_n\Omega)\\ &= (Sx_n\Omega|Sy^*\Omega)\\ &= ((x_n)^*\Omega|y\Omega).\end{aligned}$$

Since $\Omega \in \mathcal{H}$ is cyclic, $\mathcal{M}\Omega$ is dense in $\mathcal{H}$, which proves the lemma. □

Proposition 12. *For any $x \in \mathcal{M}_{tidy}$ and $n \in \mathbb{Z}$, the operator $\Delta^n x\Delta^{-n}$ is defined and bounded on $\mathcal{M}_{tidy}\Omega$, and on that subspace it is equal to x_n.*

Proof. Let $x \in \mathcal{M}_{\text{tidy}}$ and $n \in \mathbb{Z}$, and fix $y \in \mathcal{M}_{\text{tidy}}$. By construction of $\mathcal{M}_{\text{tidy}}$, $y\Omega \in D(\Delta^{-n})$. Furthermore, since

$$x\Delta^{-n}y\Omega = xy_{-n}\Omega,$$

and clearly $xy_{-n} \in \mathcal{M}$, we see that $x\Delta^{-n}y\Omega \in \mathcal{M}\Omega \subset D(S)$. Applying Lemma 6 gives

$$\begin{aligned}Sx\Delta^{-n}y\Omega &= Sxy_{-n}\Omega\\ &= y^*_{-n}x^*\Omega\\ &= y^*_{-n}(x'_1)^*\Omega\\ &= (x'_1)^*y^*_{-n}\Omega\\ &= (x'_1)^*(y'_{-(n-1)})^*\Omega.\end{aligned}$$

So $Sx\Delta^{-n}y\Omega$ is in the domain of the Tomita operator for $\mathcal{M}'$, which is the adjoint of the Tomita operator for $\mathcal{M}$. Thus, $Sx\Delta^{-n}y\Omega \in D(S^*)$, which means that $x\Delta^{-n}y\Omega \in D(S^*S) = D(\Delta)$, since $S^*S = \Delta$. Therefore,

$$\begin{aligned}\Delta x\Delta^{-n}y\Omega &= S^*Sx\Delta^{-n}y\Omega\\ &= S^*(x'_1)^*(y'_{-(n-1)})^*\Omega\\ &= y'_{-(n-1)}x'_1\Omega\\ &= y'_{-(n-1)}x_1\Omega\\ &= x_1y'_{-(n-1)}\Omega\\ &= x_1y_{-(n-1)}\Omega.\end{aligned}$$

Iterating this process n times gives

$$\Delta^n x\Delta^{-n}y\Omega = x_ny\Omega$$

which means that $y\Omega \in D(\Delta^n x\Delta^{-n})$. □

Proposition 13. *For any $t \in \mathbb{R}$, the space $\mathcal{M}_{tidy}\Omega$ is a core for Δ^t.*

Proof. Fix $t \in \mathbb{R}$. To show that $\mathcal{M}_{\text{tidy}}\Omega$ is a core for Δ^t, we must show that $\overline{\{x\Omega \oplus \Delta^t x\Omega \mid x \in \mathcal{M}_{\text{tidy}}\}} = \{\psi \oplus \Delta^t\psi \mid \psi \in D(\Delta^t)\}$. Suppose that $\psi \in D(\Delta^t)$ such that $\psi \oplus \Delta^t\psi$ is orthogonal to all vectors of the form $x\Omega \oplus \Delta^t x\Omega$, where $x \in \mathcal{M}_{\text{tidy}}$. Then, for all $x \in \mathcal{M}_{\text{tidy}}$, we have

$$\begin{aligned} 0 &= (\psi \oplus \Delta^t\psi \mid x\Omega \oplus \Delta^t x\Omega) \\ &= (\psi \mid x\Omega) + (\Delta^t\psi \mid \Delta^t x\Omega) \\ &= (\psi \mid x\Omega) + (\psi \mid \Delta^{2t} x\Omega) \\ &= (\psi \mid (1+\Delta^{2t})x\Omega), \end{aligned}$$

where the third equality is justified by Corollary 2. Our goal is to show that whenever this expression is satisfied, ψ vanishes. Then, $\{x\Omega \oplus \Delta^t x\Omega \mid x \in \mathcal{M}_{\text{tidy}}\}$ must be dense in $\{\psi \oplus \Delta^t\psi \mid \psi \in D(\Delta^t)\}$. It suffices to show that $\overline{(1+\Delta^{2t})\mathcal{M}_{\text{tidy}}\Omega} = \mathcal{H}$.
By construction, each $x \in \mathcal{M}_{\text{tidy}}$ satisfies an equation of the form

$$x\Omega = \Theta(\lambda_2 - \Delta)\Theta(\Delta - \lambda_1)y\Omega \tag{8.7}$$

for some $0 < \lambda_1 < \lambda_2$ and some $y \in \mathcal{M}$.
By the spectral theorem for Δ (Theorem 20), let μ be the spectral measure of Δ on $\sigma(\Delta)$.
Consider the function

$$f : [0,\infty) \to [0,\infty),\ s \mapsto (1+s^{2t})\Theta(\lambda_2 - s)\Theta(s - \lambda_1)$$

Clearly we have $f(s) \leq \max\{1+\lambda_1^{2t}, 1+\lambda_2^{2t}\}$ for all $t \geq 0$, and so f is bounded on $\sigma(\Delta) \subset [0,\infty)$. Therefore, by the bounded functional calculus (Theorem 17), there is a bounded normal operator $\Phi(f) = f(\Delta)$.
We know $[\lambda_1, \lambda_2]$ is a Borel set in $[0,\infty)$ and note that $f(s) \neq 0$ for all $s \in [\lambda_1, \lambda_2]$. So

$$1/f : [0,\infty) \to [0,\infty),\ s \mapsto \begin{cases} 1/f(s) & \text{if } s \in [\lambda_1, \lambda_2] \\ 0 & \text{otherwise} \end{cases}$$

is well-defined and clearly bounded by 1. Hence, by the bounded functional calculus, there is bounded normal operator $\Phi(1/f)$. By point (4.) in Theorem 17, we have

$$\Phi(1/f)\Phi(f) = \Phi(\mathbf{1}_{[\lambda_1,\lambda_2]}) = \mu([\lambda_1, \lambda_2]).$$

which is an orthogonal projection on $\mathcal{H}$. Therefore, its range $\mu([\lambda_1, \lambda_2])\mathcal{H}$ is a subspace of $\mathcal{H}$, and $\mu([\lambda_1, \lambda_2])$ acts as the identity operator on this subspace. Hence, $\Phi(f)$ is invertible on its restriction to $\mu([\lambda_1, \lambda_2])\mathcal{H}$.
Since $\Phi(f)$ is invertible on the spectral subspace of Δ corresponding to the range $[\lambda_1, \lambda_2]$, $\Phi(f)$ has dense range on this subspace. Suppose that $\mathcal{G}$ is dense in $\mathcal{H}$. Then, clearly $\mathcal{G}$ is dense in $\mu([\lambda_1, \lambda_2])\mathcal{H}$ and so $\Phi(f)\mathcal{G}$ is dense in the range of $\Phi(f)$, which as we pointed out before, is dense in the spectral subspace of Δ corresponding to the range $[\lambda_1, \lambda_2]$. Therefore, $\Phi(f)\mathcal{G}$ itself is dense in this subspace.
In particular then, since $\mathcal{M}\Omega$ is dense in $\mathcal{H}$,

$$(1+\Delta^{2t})\Theta(\lambda_2 - \Delta)\Theta(\Delta - \lambda_1)\mathcal{M}\Omega$$

is dense in the spectral subspace of Δ corresponding to the range $[\lambda_1, \lambda_2]$.
Fix $x \in \mathcal{M}$, fix and $0 < \lambda_1 < \lambda_2$. By Theorem 25, there is a tidy operator $x_{[\lambda_1,\lambda_2]} \in \mathcal{M}_{\text{tidy}}$, so that

$$x_{[\lambda_1,\lambda_2]}\Omega = x\Theta(\lambda_2 - \Delta)\Theta(\Delta - \lambda_1).$$

Therefore, $(1+\Delta^{2t})\Theta(\lambda_2 - \Delta)\Theta(\Delta - \lambda_1)\mathcal{M}\Omega$ is a subspace of $(1+\Delta^{2t})\mathcal{M}_{\text{tidy}}\Omega$, hence, $(1+\Delta^{2t})\mathcal{M}_{\text{tidy}}\Omega$ is dense in the spectral subspace of Δ corresponding to the range $[\lambda_1, \lambda_2]$. Note that this applies to any $0 < \lambda_1 < \lambda_2$. So, for all $n \in \mathbb{N}$, $(1+\Delta^{2t})\mathcal{M}_{\text{tidy}}\Omega$ is dense in the spectral subspace of Δ corresponding to the range $[\frac{1}{n}, n]$. As a result, $(1+\Delta^{2t})\mathcal{M}_{\text{tidy}}\Omega$ is dense in the spectral subspace of Δ corresponding to the range $(0,\infty)$. Since Δ is positive, $\sigma(\Delta) \subset [0,\infty)$, and since 0 is not an eigenvalue of Δ, by Theorem 10.58 in [14], $\mu(\{0\}) = 0$. Hence $\mu([0,\infty)) = \mu((0,\infty))$. Therefore, the spectral subspace of Δ corresponding to the range $(0,\infty)$ is $\mu(\sigma(\Delta))\mathcal{H}$. Since $\mu(\sigma(\Delta)) = \mathrm{id}_{\mathcal{H}}$, we have $\mu(\sigma(\Delta))\mathcal{H} = \mathcal{H}$ and we can now conclude that $(1+\Delta^{2t})\mathcal{M}_{\text{tidy}}\Omega$ is dense in $\mathcal{H}$, which gives $\overline{(1+\Delta^{2t})\mathcal{M}_{\text{tidy}}\Omega} = \mathcal{H}$. $\square$

Theorem 26 (Tomita's theorem for tidy operators). *For any $x \in \mathcal{M}_{tidy}$ and $t \in \mathbb{R}$, we have $\Delta^{-it}x\Delta^{it} \in \mathcal{M}$.*

Proof. Let $x \in \mathcal{M}_{\text{tidy}}$ and $t \in \mathbb{R}$. Fix $y' \in \mathcal{M}'$ and $z \in \mathbb{C}$ such that $\Re(z) \geq 0$. Pick $n \in \mathbb{N}$ so that $n > \Re(z)$. By Proposition 12, the operator $\Delta^{-n}x\Delta^{n}$ is defined on the dense subspace $\mathcal{M}_{\text{tidy}}\Omega$ and is equal to x_{-n} on that subspace. By proposition 13, $\mathcal{M}_{\text{tidy}}\Omega$ is a core for Δ^n. Combining these observations with Lemma 3, we have that the operator $\Delta^{-z}x\Delta^{z}$ is bounded on its domain, so that it is closable with bounded closure, and the map

$$F_x : \mathbb{C} \to \mathcal{B}(\mathcal{H}),\ z \mapsto \overline{\Delta^{-z}x\Delta^{z}}$$

is holomorphic on the interior of the strip $\{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq n\}$ and strongly continuous on the strip's boundary. This map is holomorphic on the open right half-plane and strongly continuous on the imaginary axis. Therefore, the function

$$F_{xy'} : \mathbb{C} \to \mathcal{B}(\mathcal{H}),\ z \mapsto \overline{\Delta^{-z}x\Delta^{z}}y' - y'\overline{\Delta^{-z}x\Delta^{z}}$$

is holomorphic on the open right half-plane and strongly continuous on the imaginary axis as well, as the composition of such functions. Furthermore, it is norm-bounded (in the operator norm) by

$$\|F_{xy'}(z)\| \leq 2\left\|\overline{\Delta^{-z}x\Delta^{z}}\right\|\,\|y'\|.$$

Since Δ^{ib} is unitary for any $b \in \mathbb{R}$ can write $\Delta^{z} = \Delta^{\Re(z)}\Delta^{i\Im(z)}$, where $\Delta^{i\Im(z)}$ is unitary. This is justified by equation 8.1, since $\Delta^{i\Im(z)}$ is bounded. Then,

$$\|F_{xy'}(z)\| \leq 2\left\|\overline{\Delta^{-\Re(z)}x\Delta^{\Re(z)}}\right\|\,\|y'\|.$$

For any $n \in \mathbb{N}$ recall that $\overline{\Delta^{-n}x\Delta^{n}} = x_{-n} \in \mathcal{M}$. Therefore, since y' commutes with all of $\mathcal{M}$,

$$F_{xy'}(n) = \overline{\Delta^{-n}x\Delta^{n}}y' - y'\overline{\Delta^{-n}x\Delta^{n}} = x_n y' - y' x_n = 0.$$

For any z in the right half-plane, pick $n \in \mathbb{N}$ so that $\Re(z) \leq n < \Re(z) + 1$.
Since F_z is holomorphic on the interior of the strip $\{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq n\}$ and strongly continuous on its boundary, by the Phragmén-Lindelöf prinicple (Theorem 22), F_z is bounded by the supremum of its norm on the boundary of the strip $\{z \in \mathbb{C} \mid \Re(z) = 0 \vee \Re(z) = n\}$. Then, again using that $\Delta^{z} = \Delta^{\Re(z)}\Delta^{i\Im(z)}$ and $\Delta^{i\Im(z)}$ is unitary for all $z \in \mathbb{C}$,

$$\left\|\overline{\Delta^{-\Re(z)}x\Delta^{\Re(z)}}\right\| = \left\|\overline{\Delta^{-z}x\Delta^{z}}\right\| \leq \sup_{\Re(z)\in\{0,n\}} \left\|\overline{\Delta^{-z}x\Delta^{z}}\right\| = \sup_{\Re(z)\in\{0,n\}} \left\|\overline{\Delta^{-\Re(z)}x\Delta^{\Re(z)}}\right\|$$

Therefore, $\overline{\Delta^{-\Re(z)}x\Delta^{\Re(z)}}$ is upper-bounded by the maximum of $\left\|\overline{\Delta^{0}x\Delta^{0}}\right\| = \left\|x\right\|$ and $\left\|\overline{\Delta^{-n}x\Delta^{n}}\right\| = \left\|x_{-n}\right\|$. From Theorem 25, we know there exist constants $\alpha, \beta > 0$ such that $\|x\| \leq \alpha$ and $\|x_{-n}\| \leq \alpha e^{\beta n}$. Combining, we obtain

$$\|F_{xy'}(z)\| \leq 2\left\|\overline{\Delta^{-\Re(z)}x\Delta^{\Re(z)}}\right\|\,\|y'\| \leq 2\max\{\|x\|, \|x_{-n}\|\}\,\|y'\| \leq 2\alpha e^{\max\{0,\beta n\}}\|y'\| \leq 2\alpha e^{\beta n}\|y'\|.$$

We have that $n < \Re(z) + 1 < |z| + 1$, and so certainly

$$\|F_{xy'}(z)\| \leq 2\alpha e^{\beta n}\|y'\| \leq 2\alpha e^{\beta(|z|+1)}\|y'\| \leq 2e^{\beta}\|y'\|\alpha e^{\beta|z|}.$$

Note that since α and β are n-independent, this holds for all z in the right half-plane. Next, it is immediately clear that for any z on the imaginary axis that $\Re(z) = 0$, and so

$$\|F_{xy'}(z)\| \leq 2\|\overline{\Delta^{-\Re(z)}x\Delta^{\Re(z)}}\|\,\|y'\| \leq 2\|x\|\,\|y'\| \leq 2\alpha\|y'\|.$$

Thus, $F_{xy'}$ is holomorphic in the open right half-plane, strongly continuous on the imaginary axis, bounded by an exponential function in the right half-plane, bounded in the imaginary directions, and it vanishes on the nonnegative integers. By Carlson's theorem (Theorem 23), $F_{xy'}$ is identically zero on the complex plane. Hence, taking $z = it$, we have

$$\overline{\Delta^{-it}x\Delta^{it}}y' - y'\overline{\Delta^{-it}x\Delta^{it}} = x_n y' - y' x_n = F_{xy'}(it) = 0.$$

As a result, $\overline{\Delta^{-it}x\Delta^{it}}y' = y'\overline{\Delta^{-it}x\Delta^{it}}$. Since we fixed $y' \in \mathcal{M}'$ arbitrarily, $\overline{\Delta^{-it}x\Delta^{it}}$ commutes with all $y' \in \mathcal{M}'$, and so certainly $\overline{\Delta^{-it}x\Delta^{it}} \in \mathcal{M}''$. Since Δ^{it} and Δ^{-it} are unitary, we have $\|\Delta^{-it}x\Delta^{it}\| = \|x\|$, and x is bounded. Hence, $\Delta^{-it}x\Delta^{it} = \overline{\Delta^{-it}x\Delta^{it}}$, and by von Neumann's double commutant theorem, $\Delta^{-it}x\Delta^{it} \in \mathcal{M}$, proving the theorem. □

Proposition 14. *We have $\mathcal{M}'_{tidy} = \mathcal{M}'$.*

Proof. Since $\mathcal{M}_{\text{tidy}}$ is a subspace of $\mathcal{M}$, we naturally have $\mathcal{M}_{\text{tidy}} \subset \mathcal{M}$, and so $\mathcal{M}' \subset \mathcal{M}'_{\text{tidy}}$, giving one inclusion. As for the other inclusion, by Proposition 13, $\mathcal{M}_{\text{tidy}}\Omega$ is a core for $\Delta^{1/2}$. Hence, it is also a core for the Tomita operator S, as $D(\Delta^{1/2}) = D(S)$. Consequently, for any $x \in \mathcal{M}$, there exists a sequence of tidy operators $(x_n) \subset \mathcal{M}_{\text{tidy}}$ such that

$$\lim_{n\to\infty} x_n\Omega = x\Omega \qquad \text{and} \qquad \lim_{n\to\infty} Sx_n\Omega = Sx\Omega.$$

Writing out the second equality, we obtain

$$\lim_{n\to\infty} x_n\Omega = x\Omega \qquad \text{and} \qquad \lim_{n\to\infty} x_n^*\Omega = x^*\Omega.$$

Then, for any $y' \in \mathcal{M}'$, we have

$$\lim_{n\to\infty} x_n y'\Omega = \lim_{n\to\infty} y' x_n\Omega = y'x\Omega = xy'\Omega \quad \text{and} \quad \lim_{n\to\infty} x_n^* y'\Omega = \lim_{n\to\infty} y' x_n^*\Omega = y'x^*\Omega = x^*y'\Omega. \tag{8.8}$$

Now suppose that $O \in \mathcal{M}'_{\text{tidy}}$, $x \in \mathcal{M}$, and $(x_n) \subset \mathcal{M}_{\text{tidy}}$ as constructed above. Fix $y', z' \in \mathcal{M}'_{\text{tidy}}$, so that

$$\begin{aligned}
\big((Ox - xO)y'\Omega \mid z'\Omega\big) &= (Oxy'\Omega - xOy'\Omega \mid z'\Omega) \\
&= (xy'\Omega \mid O^*z'\Omega) - (Oy'\Omega \mid x^*z'\Omega) \\
&= \lim_{n\to\infty} \big((x_n y'\Omega \mid O^*z'\Omega) - (Oy'\Omega \mid x_n^* z'\Omega)\big) \\
&= \lim_{n\to\infty} (Ox_n y'\Omega - x_n O y'\Omega \mid z'\Omega) \\
&= \lim_{n\to\infty} \big((Ox_n y' - x_n O y')\Omega \mid z'\Omega\big) \\
&= 0.
\end{aligned}$$

Here the third equality is justified by equation 8.8.
Since $\mathcal{M}'\Omega$ is dense in $\mathcal{H}$, we have that $Ox - xO = 0$. Since $x \in \mathcal{M}$ was chosen arbitrarily, O commutes with all $x \in \mathcal{M}$, that is, $O \in \mathcal{M}'$. Further, as $O \in \mathcal{M}'_{\text{tidy}}$ was also chosen arbitrarily, we have the second desitred inclusion $\mathcal{M}'_{\text{tidy}} \subset \mathcal{M}'$. $\square$

Corollary 4 (Tomita's theorem)**.** *For any $x \in \mathcal{M}$ and $t \in \mathbb{R}$, we have $\Delta^{-it}x\Delta^{it} \in \mathcal{M}$.*

Proof. Let y' be a tidy operator for the von Neumann algebra $\mathcal{M}'$, and fix $\psi, \xi \in \mathcal{H}$. Applying Theorem 26 to tidy operators of $\mathcal{M}'$, we have that $\Delta^{it}y'\Delta^{-it} \in \mathcal{M}'$. Therefore,

$$\begin{aligned}
\big((\Delta^{-it}x\Delta^{it}y' - y'\Delta^{-it}x\Delta^{it})\psi \mid \xi\big) &= (\Delta^{-it}x\Delta^{it}y'\psi - y'\Delta^{-it}x\Delta^{it}\psi \mid \xi) \\
&= (x\Delta^{it}y'\psi \mid \Delta^{it}\xi) - (y'\Delta^{-it}x\Delta^{it}\psi \mid \xi) \\
&= (x\Delta^{it}y'\Delta^{-it}\Delta^{it}\psi \mid \Delta^{it}\xi) - (\Delta^{it}y'\Delta^{-it}x\Delta^{it}\psi \mid \Delta^{it}\xi) \\
&= (x\Delta^{it}y'\Delta^{-it}\Delta^{it}\psi - \Delta^{it}y'\Delta^{-it}x\Delta^{it}\psi \mid \Delta^{it}\xi) \\
&= \big((x\Delta^{it}y'\Delta^{-it}\Delta^{it} - \Delta^{it}y'\Delta^{-it}x)\Delta^{it}\psi \mid \Delta^{it}\xi\big) \\
&= 0.
\end{aligned}$$

Here, the second equality is justified by the fact that Δ^{it} is unitary, and unitary operators leave the inner product invariant.
The third equality is also justified by the fact that Δ^{it} is unitary, and therefore $\Delta^{-it}\Delta^{it} = \mathrm{id}_{\mathcal{H}}$.
As we chose ψ and ξ arbitrarily, we have that $\Delta^{-it}x\Delta^{it}y' - y'\Delta^{-it}x\Delta^{it} = 0$. Hence, as $y' \in \mathcal{M}'$ was also chosen arbitrarily, $\Delta^{-it}x\Delta^{it}$ commutes with every tidy operator $y' \in \mathcal{M}'$. Thus, $\Delta^{-it}x\Delta^{it} \in (\mathcal{M}')'_{\text{tidy}}$, and then by applying Proposition 14 to $\mathcal{M}'$, $\Delta^{-it}x\Delta^{it} \in \mathcal{M}''$. By von Neumann's double commutant theorem, $\Delta^{-it}x\Delta^{it} \in \mathcal{M}$ proving the theorem. $\square$

9

Thermal time as a POVM

At the end of chapter 6, we mentioned that a time observable could be better formulated by a POVM, rather than a projection-valued measure. In 2024, Jan van Neerven and Pierre Portal released a paper ”Thermal Time as an Unsharp Obeservable” proving the existence of POVM covariant with thermal time for the Hamiltonian of a free relativistic particle without mass. In this chapter, we aim to produce a similar result for a free relativistic particle with mass $m > 0$. In order to do so, we first introduce some notions from special relativity.

9.1. Relativity

Since our goal is to solve a problem at the crossroads of quantum mechanics and relativity, we introduce special relativity in a way that is compatible with the mathematical formulation of quantum mechanics, and the following is standard in the quantum field theory literature. The most important consequence of this is that massive particles have energy simply due to their invariant mass, regardless of any kinetic energy they may possess. **Invariant mass** (or rest mass) is a fundamental physical property of matter, independent of velocity. This relationship between mass and the energy is known as **mass-energy equivalence** and is described by Albert Einstein’s famous formula $E = mc^2$.

Suppose we think of one-dimensional space and time as a single entity called 1+1 dimensional **spacetime** and modeled by $\mathbb{R}^2$. Here, 1+1 dimension refers to 1 time dimension and 1 space dimension, where 1+3 dimensions would refer to 1 time dimension and 3 spatial dimensions. A point is labeled $x = (x^0, x^1)$ with $x^0 = ct$ a scaled time with units of distance. The basic postulate of special relativity is that 1+1 dimensional spacetime is to be modeled by $\mathbb{R}^2$, equipped with a so-called **Lorentzian metric**. The main consequence of using this metric, is that it determines an inner product given by

$$v \cdot w = -v^0 w^0 + v^1 w^1$$

for $v = (v^0, v^1), w = (w^0, w^1) \in \mathbb{R}^2$. These notions, and all of the following, naturally extend to the 1+3 dimensional case.

Consider a free relativistic particle with mass $m \geq 0$ and set $\omega(p) := \sqrt{p^2 + m^2}$ for $p \in \mathbb{R}$. Its equations of motion are given by:

$$\frac{dx}{dt} = \frac{cp}{\omega(p)}, \qquad \frac{dp}{dt} = 0.$$

This is a Hamiltonian system with Hamiltonian $c\omega(p)$. It is interpreted as describing a particle of mass m, position x, momentum p, and energy

$$E = c\omega(p) = \sqrt{p^2c^2 + m^2c^4},$$

where the term p^2c^2 is the contribution due to the particle’s kinetic energy, and the term m^2c^4 is the contribution due to the particle’s invariant mass. The value of c, the speed of light, depends on the system of units we are using. By convention, we choose units so that $c = 1$, so that the parameter c disappears from our equations.

Consider the isometries (sometimes called symmetries) of the spacetime $\mathbb{R}^2$. These are maps $y = \kappa(x)$ which preserve the Lorentzian metric, or equivalently, preserve proper time intervals and distances. Thus, they satisfy

$$(\kappa(x_1) - \kappa(x_2)) \cdot (\kappa(x_1) - \kappa(x_2)) = (x_1 - x_2) \cdot (x_1 - x_2)$$

for all $x_1, x_2 \in \mathbb{R}^2$. Translations $y = x + a$ are isometries and linear transformations $y = \Lambda x$, also written $y^k = \Lambda^k{}_\ell x^\ell,\ k, \ell \in \{0, 1\}$, are isometries if $\Lambda x \cdot \Lambda x = x \cdot x$. These (the latter) are called Lorentz transformations. They form a group known as the **Lorentz group**. It turns out these are all the isometries. Thus, a general isometry has the form

$$\{a, \Lambda\}x = \Lambda x + a.$$

The group of all such transformations is called the **Poincaré group**.

Elements of the Lorentz group satisfy $\det \Lambda = \pm 1$ and this condition divides the group into disjoint sets denoted $\mathcal{L}_\pm$. The set $\mathcal{L}_+$ contains the identity and is a subgroup. Furthermore, $\mathcal{L}_+$ is divided into disjoint sets with $\pm\Lambda^0{}_0 > 1$ and denoted $\mathcal{L}_+^\uparrow$ (containing the elements with $\Lambda^0{}_0 > 1$) and $\mathcal{L}_+^\downarrow$ (containing the elements with $-\Lambda^0{}_0 > 1$) respectively. If $-\Lambda^0{}_0 > 1$, then Λ swaps the sign of the time coordinate (the 0th coordinate), so elements of $\mathcal{L}_+^\downarrow$ involve time reversal. The set $\mathcal{L}_+^\uparrow$ contains the identity and is a subgroup known as the **proper Lorentz group**. Correspondingly, there is a **proper Poincaré group** containing all the isometries of spacetime that involve proper Lorentz transformations. Since the proper Lorentz group contains the identity, the proper Poincaré contains all translations. Again, these group properties are easily verified to be retained when extending to 1+3 dimensional spacetime $\mathbb{R}^4$. Mathematically, special relativity is best understood as general relativity with a special metric. For an exhaustive treatment, we refer the reader to Misner *et al.* (1973) [25] and Sachs and Wu (1977) [26].

The Hamiltonian of the free relativistic particle is given by $H(x, p) = \omega(p) = \sqrt{p^2 + m^2}$, where we have used the convention $c = 1$. Recall from Section 3.3 that the Hilbert space of the free particle is $L^2(\mathbb{R})$, where we have a choice of working in position space or momentum space. In this case, we choose for momentum space $L^2(\mathbb{R}, dp)$ with the Hamiltonian $H = \sqrt{p^2 + m^2}$. The two constructions are unitarily equivalent via the Fourier transform.

Further integrating special relativity into the framework of quantum mechanics, we would like to have a **representation** u_0 of the proper Poincaré group on momentum space (or position space) to be able to describe the complete dynamics of the system. The issue that arises is that, while $L^2(\mathbb{R}, dp)$ is invariant under translations, it is not invariant under the proper Lorentz group. That is, there are Lorentz transformations that do not necessarily send elements of $L^2(\mathbb{R}, dp)$ to $L^2(\mathbb{R}, dp)$. To address this, we introduce a new space fundamental to quantum field theory.

Let $V_m^+ = \{p \in \mathbb{R}^2 : p \cdot p = -m^2, p^0 > 0\}$ be the **mass shell** in 1+1 dimensional momentum space $\mathbb{R}^2$, where we write $p = (p^0, p^1)$. The mass shell is invariant under the proper Lorentz group $\mathcal{L}_+^\uparrow$ once we have equipped it with a suitable measure. Then we can define a representation u of the proper Poincaré group $\mathcal{P}_+^\uparrow$ on functions on V_m^+ by

$$(u(a, \Lambda)\psi)(p) = e^{-ip\cdot a}\psi(\Lambda^{-1}p)$$

where $p \cdot a = -p^0 a^0 + p^1 a^1$. Since time translation is included in the proper Poincaré group, the representation contains the time evolution of the system. There is an essentially unique (unique up to a constant multiple) measure on V_m^+ that is invariant under the Lorentz group, which we denote μ_m. Then $u(a, \Lambda)$ is a unitary representation of the proper Poincaré group on $L^2(V_m^+, \mu_m)$, and it turns out to be irreducible.

We can be specific about what this measure looks like. First, note that the map $\phi : \mathbb{R} \to V_m^+$ given by

$$\phi(p) = (\omega(p), p)$$

provides a global set of coordinates for V_m^+. Then, for measurable $B \subset \mathbb{R}$, the measure μ_m is given by:

$$\mu_m(\phi(B)) = \int_B \frac{dp}{2\omega(p)}.$$

Now, $\psi \to \psi \circ \phi$ is a unitary map from $L^2(V_m^+, \mu_m)$ to $L^2(\mathbb{R}, dp/2\omega(p))$ and, consequently, we have a representation of the proper Poincaré group on $L^2(\mathbb{R}, dp/2\omega(p))$ given by

$$(u(a, \Lambda)\psi)(p) = e^{-i\phi(p)\cdot a}\psi(\phi^{-1}\Lambda^{-1}\phi(p)).$$

It is easily checked that the measure $dp/2\omega(p)$ on $\mathbb{R}$ is a Lorentz invariant measure. Furthermore, the space $L^2(\mathbb{R}, dp/2\omega(p))$ is equivalent to momentum space via the unitary operator $V : L^2(\mathbb{R}, dp) \to L^2(\mathbb{R}, dp/2\omega(p))$ defined by

$$(V\psi)(p) = \sqrt{2\omega(p)}\psi(p).$$

If we want a representation of the proper Poincaré group on momentum space, we can simply define

$$u_0(a, \Lambda) := V^{-1}u(a, \Lambda)V.$$

Upon setting $\omega := \sqrt{p^2 + m^2}$ as an energy variable, this relation can be inverted (up to the sign of p) as

$$p = \pm\sqrt{\omega^2 - m^2}.$$

When restricting to positive momenta, we can perform a change of variables to obtain the measure

$$d\mu(\omega) = \frac{d\omega}{2\sqrt{\omega^2 - m^2}}, \quad \omega \geq m.$$

Now, we define the Hilbert space in the energy representation as

$$L^2\big([m, \infty), d\mu(\omega)\big)$$

with the inner product

$$(\psi|\xi) = \int_m^\infty \psi(\omega)\overline{\xi(\omega)}\frac{d\omega}{2\sqrt{\omega^2 - m^2}}$$

which we call **energy space**.

9.2. Weighted Fourier transform

In the massless case [15], a time covariant positive operator-valued measure is constructed in the Hardy space on the upper half-plane $H^2(\mathbb{U})$, where

$$\mathbb{U} := \{(u, v) \in \mathbb{R}^2 : u \in \mathbb{R}, v > 0\}.$$

This space is defined as the Hilbert space of all holomorphic functions $f : \mathbb{U} \to \mathbb{C}$ for which

$$\|f\|_{H^2(\mathbb{U})} := \sup_{v>0} \|f(\cdot + iv)\|_{L^2(\mathbb{R})}$$

is finite. Under convolution with the Poisson kernel for the upper half-plane, this space is isometric to the closed subspace in $L^2(\mathbb{R})$ consisting of all functions whose Fourier-Plancherel transforms vanishes on the negative half-line.

If we substitute $m = 0$ in energy space, we are left with the Hilbert space $L^2\big([0, \infty), d\mu(\omega)\big)$, where the measure reduces to

$$d\mu(\omega) = \frac{d\omega}{2\sqrt{\omega^2 - 0}} = \frac{d\omega}{2\omega}.$$

Since $\omega(p) = \sqrt{p^2 + m^2} = |p|$, energy space for the free massless particle is described by $L^2\big([0, \infty), dp/2\omega(p)\big)$. We know from the previous section that this space is equivalent to $L^2\big([0, \infty), dp\big)$ via the unitary operator V. So, if we view $L^2\big([0, \infty), dp\big)$ as the image of the Fourier transform of some space, then this space is isometric to $H^2(\mathbb{U})$. This is essentially the Paley-Wiener theorem.

Of course, this construction only works because $L^2\big([0, \infty), dp\big)$ is equipped with the Lebesgue measure. Ignoring the use of the map V for now, we will aim to construct a generalization of the Fourier transform for $L^2\big([m, \infty), d\mu(\omega)\big)$ directly. Then, we construct a POVM and then show it is covariant with respect to the time flow of the system, using a change of variables, just as in the massless case [15].

When trying to define the Fourier transform for $L^2\big([m,\infty), d\mu(\omega)\big)$, one might consider the map given by

$$(\mathcal{F}_m\psi)(t) = \int_m^\infty \psi(\omega)e^{-it\omega}\, d\mu(\omega), \qquad \psi \in L^2\big([m,\infty), d\mu(\omega)\big).$$

An issue that arises when trying to define such a map is that it does not necessarily map $L^2\big([m,\infty), d\mu(\omega)\big)$ to $L^2\big([m,\infty), d\mu(\omega)\big)$. That is, given $\psi \in L^2\big([m,\infty), d\mu(\omega)\big)$, it does not need to be the case that $\mathcal{F}_m\psi \in L^2\big([m,\infty), d\mu(\omega)\big)$.

Furthermore, the Plancherel theorem proves that the Fourier transform is a unitary operator on $L^2(\mathbb{R})$. This is fundamental to the mathematical formulation of quantum mechanics, as it establishes the Fourier transform as an automorphism of $L^2(\mathbb{R})$, providing the framework of the duality of position and momentum.

To address these issues, first observe that $C_c^\infty\big((m,\infty)\big)$ is a dense subspace of $L^2\big([m,\infty), d\mu(\omega)\big)$, let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space, the space of smooth functions whose derivatives are rapidly decreasing, and define

$$\rho : [m,\infty) \to (0,\infty),\ \rho(\omega) = \frac{1}{2\sqrt{\omega^2 - m^2}}.$$

We now define the **weighted Fourier transform** $\mathcal{F}_m : C_c^\infty\big((m,\infty)\big) \to \mathcal{S}(\mathbb{R})$ by

$$(\mathcal{F}_m\psi)(t) = \frac{1}{\sqrt{2\pi}} \int_m^\infty e^{-i\omega t}\sqrt{\rho(\omega)}\psi(\omega)\, d\omega, \quad t \in \mathbb{R}.$$

This integral is well-defined for every $t \in \mathbb{R}$, since $\psi \in C_c^\infty\big((m,\infty)\big)$ has compact support in (m,∞).

To establish that $\mathcal{F}_m$ is an isometry, define for each $\psi \in C_c^\infty\big((m,\infty)\big)$ the function

$$\widetilde{\psi}(\omega) = \begin{cases} \sqrt{\rho(\omega)}\psi(\omega), & \omega \geq m \\ 0 & \omega < m. \end{cases}$$

Then it is immediate that

$$\|\widetilde{\psi}\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^\infty |\widetilde{\psi}(\omega)|^2\, d\omega = \int_m^\infty |\psi(\omega)|^2\rho(\omega)\, d\omega = \|\psi\|^2_{L^2([m,\infty),\mu)}.$$

Moreover, note that

$$(\mathcal{F}_m\psi)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega t}\widetilde{\psi}(\omega)\, d\omega$$

so that $\mathcal{F}_m\psi$ coincides with the classical Fourier transform of $\widetilde{\psi}$.

By the classical Plancherel theorem for functions in $L^2(\mathbb{R})$, we know that

$$\|\mathcal{F}_m\psi\|^2_{L^2(\mathbb{R},dt)} = \|\widetilde{\psi}\|^2_{L^2(\mathbb{R},d\omega)} = \|\psi\|^2_{L^2([m,\infty),\mu)}.$$

Since $C_c^\infty\big((m,\infty)\big)$ is a dense subspace of $L^2\big([m,\infty), d\mu(\omega)\big)$, the weighted Fourier transform $\mathcal{F}_m$ extends uniquely by continuity to an isometry

$$\mathcal{F}_m : L^2\big([m,\infty), d\mu(\omega)\big) \to L^2(\mathbb{R}, dt),$$

where we refer to the range $L^2(\mathbb{R}, dt)$ as **time space**.

9.3. Main result

In this section, we prove the existence of a time-covariant POVM for the free relativistic particle with mass $m > 0$. First, we introduce a proposition that enables us to extract bounded operators from a bounded sesquilinar form.

Proposition 15. *Let $\mathfrak{a} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a sesquilinear mapping with the property that there exists a constant $C \geq 0$ such that*

$$|\mathfrak{a}(\psi, \xi)| \leq C\|\psi\|\|\xi\|, \quad \psi, \xi \in \mathcal{H}.$$

Then there exists a unique operator $A \in \mathcal{B}(\mathcal{H})$ such that

$$\mathfrak{a}(\psi, \xi) = (A\psi|\xi), \quad \psi, \xi \in \mathcal{H}.$$

Moreover, $\|A\| \leq C$, where C is the boundedness constant of $\mathfrak{a}$.

Proof. See Proposition 9.15 in [14]. □

Let $\{U(t)\}_{t\in\mathbb{R}}$ be the strongly continuous one-parameter unitary group defined by

$$(U(t)\psi)(\omega) = e^{-it\omega}\psi(\omega).$$

Theorem 27 (Existence and Uniqueness of a Time Covariant POVM for a Massive Particle)**.** *There exists a unique (up to unitary equivalence) positive operator-valued measure (POVM)*

$$E : \mathfrak{B}(\mathbb{R}) \to \mathcal{B}(L^2\big([m, \infty), d\mu(\omega)\big))$$

with the covariance property

$$U(t)E(B)U(t)^{-1} = E(B + t) \;\; \forall t \in \mathbb{R},\, B \in \mathfrak{B}(\mathbb{R}).$$

Proof. Define the weighted Fourier transform as in the previous section

$$\mathcal{F}_m : C_c^\infty\big((m, \infty)\big) \to \mathcal{S}(\mathbb{R}), \quad (\mathcal{F}_m\psi)(t) = \int_m^\infty \sqrt{\rho(\omega)}\, e^{-it\omega}\psi(\omega)\, d\mu(\omega),$$

which can be extended by continuity to a map

$$\mathcal{F}_m : L^2\big([m, \infty), d\mu(\omega)\big) \to L^2(\mathbb{R}, dt).$$

For any Borel set $B \subset \mathbb{R}$ define

$$\mathfrak{a}_B : L^2\big([m, \infty), d\mu(\omega)\big) \times L^2\big([m, \infty), d\mu(\omega)\big) \to \mathbb{C}, \quad (\psi, \xi) \mapsto \int_B (\mathcal{F}_m\psi)(t)\, \overline{(\mathcal{F}_m\xi)(t)}\, dt.$$

It is obvious that this prescription defines a positive sesquilinear form. Furthermore, using the positiveness of $\mathfrak{a}_B$, the Cauchy-Schwarz inequality, and the isometric property of the Fourier transform, it is easily checked that

$$|\mathfrak{a}_B(\psi, \xi)| = \left|\int_B (\mathcal{F}_m\psi)(t)\overline{(\mathcal{F}_m\xi)(t)}\, dt\right| = \big|(\mathbf{1}_B\mathcal{F}_m\psi|\mathcal{F}_m\xi)\big| \leq \|\mathcal{F}_m\psi\|\|\mathcal{F}_m\xi\| = \|\psi\|\|\xi\|.$$

Hence, Proposition 15 implies the existence of a bounded operator $E(B) \geq 0$ such that

$$(E(B)\psi)|\xi) = \int_B (\mathcal{F}_m\psi)(t)\overline{(\mathcal{F}_m\xi)(t)}\, dt.$$

Furthermore,

$$(E(B)\psi|\psi) = (\mathbf{1}_B\mathcal{F}_m\psi|\mathcal{F}_m\psi) = (\mathbf{1}_B\mathcal{F}_m\psi|\mathbf{1}_B\mathcal{F}_m\psi) = \|\mathbf{1}_B\mathcal{F}_m\psi\|^2$$

and

$$0 \leq \|\mathbf{1}_B\mathcal{F}_m\psi\|^2 \leq \|\mathcal{F}_m\psi\|^2 = \|\psi\|^2 = (\psi|\psi)$$

so that

$$0 \leq (E(B)\psi|\psi) \leq (\psi|\psi).$$

Thus $0 \leq E(B) \leq I$.

We claim that $E : \mathfrak{B}(\mathbb{R}) \to \mathcal{B}(L^2\big([m,\infty), d\mu(\omega)\big))$ is a positive operator-valued measure. First, by the isometric property of $\mathcal{F}_m$, we have that

$$(\psi|\xi) = (\mathcal{F}_m\psi|\mathcal{F}_m\xi) = \int_{\mathbb{R}} (\mathcal{F}_m\psi)(t)\overline{(\mathcal{F}_m\xi)(t)}\,dt.$$

So, by the uniqueness property of the operator in Proposition 15, we have that $E(\mathbb{R}) = I$.
Next, countable additivity of E follows from the countable additivity of the Lebesgue integral. Indeed, if $\{B_k\}_{k=1}^{\infty}$ is a countable collection of pairwise disjoint subsets of $\mathbb{R}$, then

$$\left(E\left(\bigcup_{k=1}^{\infty} B_k\right)\psi\middle|\xi\right) = \int_{\bigcup_{k=1}^{\infty} B_k} (\mathcal{F}_m\psi)(t)\overline{(\mathcal{F}_m\xi)(t)}\,dt = \sum_{k=1}^{\infty}\int_{B_k} (\mathcal{F}_m\psi)(t)\overline{(\mathcal{F}_m\xi)(t)}\,dt = \sum_{k=1}^{\infty}(E(B_k)\psi)|\xi),$$

where in the second equality we have used the dominated convergence theorem. Hence, for all $\psi \in L^2\big([m,\infty), d\mu(\omega)\big)$, the mapping

$$B \mapsto (E(B)\psi|\psi), \quad B \in \mathfrak{B}(\mathbb{R})$$

defines a measure.

Let us now check the covariance property. For any $t', t \in \mathbb{R}$ and $\psi \in C_c^{\infty}\big((m,\infty)\big)$ we have

$$(\mathcal{F}_m(U(t')\psi))(t) = \int_m^{\infty} \sqrt{\rho(\omega)}\, e^{-it\omega}\, e^{-it'\omega}\psi(\omega)\,d\mu(\omega)$$

$$= \int_m^{\infty} \sqrt{\rho(\omega)}\, e^{-i(t+t')\omega}\psi(\omega)\,d\mu(\omega) = (\mathcal{F}_m\psi)(t+t').$$

Thus, for any Borel set $B \in \mathfrak{B}(\mathbb{R})$,

$$(E(B)U(t')\psi|U(t')\phi) = \int_B (\mathcal{F}_m\psi)(t+t')\,\overline{(\mathcal{F}_m\phi)(t+t')}\,dt.$$

Perform the change of variable $s = t + t'$ (with $ds = dt$) to obtain

$$(U(t')\psi|E(B)U(t')\phi) = \int_B (\mathcal{F}_m\psi)(t+t')\,\overline{(\mathcal{F}_m\phi)(t+t')}\,dt$$

$$= \int_{B+t'} (\mathcal{F}_m\psi)(s)\,\overline{(\mathcal{F}_m\phi)(s)}\,ds = (\psi|E(B+t')\phi).$$

Since this holds for all $\psi, \phi \in C_c^{\infty}\big((m,\infty)\big)$, and $C_c^{\infty}\big((m,\infty)\big)$ is dense in $L^2\big([m,\infty), d\mu(\omega)\big)$ one concludes by density that

$$U(t')\,E(B)\,U(t')^{-1} = E(B+t'),$$

as required.

Finally, a general result in the theory of covariant observables (see Section 4.8 in [8]) guarantees that any two POVMs satisfying the covariant condition with respect to the unitary group $U(t)$ are unitarily equivalent. Hence, the POVM E constructed above is unique up to unitary equivalence.

$\square$

9.4. Interpretation as thermal time

What remains to be shown is that the physical time evolution generated by the one-parameter unitary group

$$U(t) = e^{itH}, \quad t \in \mathbb{R}$$

coincides with the time flow given by Tomita-Takesaki theory, interpreted as thermal time. Here, the Hamiltonian H is the self-adjoint operator on $L^2\big([m,\infty), d\mu(\omega)\big)$ given by

$$(H\psi)(\omega) = \omega\psi(\omega).$$

This requires suitable choices of von Neumann algebra and cyclic-separating vector to produce the desired one-parameter group of modular automorphisms.

We recall from Chapter 5 that the thermal time hypothesis is two-fold. While it posits the one-parameter group of modular automorphisms arising from Tomita-Takesaki theory as a suitable one-parameter unitary group of time translations, it also identifies which states could conceivably be equilibrium states, the KMS states. The precise definition of KMS states is unimportant for our purposes (for an exhaustive treatment, see [18]). What matters is that this notion extends the notion of a Gibbs state, in that every Gibbs state is KMS; and the notions are equivalent under suitable finiteness conditions. Furthermore, if H is a semibounded Hamiltonian operator on a Hilbert space $\mathcal{H}$ with the property that the bounded operator $e^{-\beta H}$ is of trace class, then the **Gibbs state at inverse temperature** $\beta > 0$ **associated with** H is the state φ on $\mathcal{B}(\mathcal{H})$ given by

$$\varphi(A) = \mathrm{tr}(T_\beta A),$$

where $T_\beta := e^{-\beta H}/\mathrm{tr}(e^{-\beta H})$. It is well-known that Gibbs states are faithful and normal. If this is the case, taking the GNS representation of $\mathcal{B}(\mathcal{H})$ with the faithful state φ results in a von Neumann algebra with a cyclic-separating vector that produces modular automorphisms coinciding with the physical time evolution generated by H.

In the massless case [15], a POVM is constructed in the Hilbert space $H^2(\mathbb{U})$, and the Hamiltonian reduces to $H = \sqrt{p^2 + 0} = |p|$ in momentum space, or equivalently, $H = \sqrt{-\Delta + 0} = |D|$, where $D = \frac{1}{i} d/dx$ with domain

$$D(|D|) := \{f \in H^2(\mathbb{U}) : |D|f \in L^2(\mathbb{R})\}.$$

It is not hard to prove that the spectrum of this operator equals $[0, \infty)$. Therefore, $\mathrm{tr}(e^{-\beta|D|})$ diverges in infinite dimensions, and so the bounded operator $e^{-\beta|D|}$ is not trace class for any $\beta > 0$.

Hence, in the massless case, the Gibbs state approach will not work. Instead of showing that the physical time evolution of the free relativistic massless particle coincides with a thermal time, it is shown in [15] that the defined POVM is covariant with respect to a so-called **modular time**. This notion of time relaxes the definition of thermal time, while retaining the algebraic properties of the modular flow. Crucially, non-commutative spaces $L^2(\mathcal{M}, \tau)$ are introduced, where $\mathcal{M}$ is a von Neumann algebra, and τ is a not necessarily normal trace. These spaces can be thought of as a generalization of the GNS construction.

Let us return to the case of the free relativistic particle with mass $m > 0$. Recall that the Hilbert space is given by

$$L^2\big([m, \infty), d\mu(\omega)\big), \quad \text{with } d\mu(\omega) = \frac{d\omega}{2\sqrt{\omega^2 - m^2}},$$

and the Hamiltonian H is the self-adjoint multiplication operator given by

$$(H\psi)(\omega) = \omega\psi(\omega), \quad \psi \in L^2\big([m, \infty), d\mu(\omega)\big).$$

Let $\beta > 0$ and we will show that the bounded operator $e^{-\beta H}$ is trace class.

Proposition 16. *The bounded operator* $e^{-\beta H}$ *is of trace class and its trace class norm is given by*

$$\|e^{-\beta H}\|_1 = \int_m^\infty e^{-\beta\omega}\, d\mu(\omega)$$

Proof. Writing out the definition of the measure, we have

$$\|e^{-\beta H}\|_1 = \int_m^\infty e^{-\beta\omega}\, d\mu(\omega) = \int_m^\infty e^{-\beta m} \frac{d\omega}{2\sqrt{\omega^2 - m^2}}.$$

Perform the substitution

$$\omega = m \cosh u, \quad u \in [0, \infty).$$

Then

$$d\omega = m \sinh u\, du,$$

and note that

$$\sqrt{\omega^2 - m^2} = \sqrt{m^2 \cosh^2 u - m^2} = m \sinh u.$$

Thus, the measure transforms as follows:

$$\frac{d\omega}{2\sqrt{\omega^2 - m^2}} = \frac{m \sinh u \, du}{2m \sinh u} = \frac{du}{2}.$$

Therefore,

$$\|e^{-\beta H}\|_1 = \int_{\omega=m}^{\infty} e^{-\beta m} \frac{d\omega}{2\sqrt{\omega^2 - m^2}} = \int_{u=0}^{\infty} e^{-\beta m \cosh u} \frac{du}{2} = \frac{1}{2} \int_0^{\infty} e^{-\beta m \cosh u} \, du.$$

A standard result in analysis (Equation 9.6.24 in [22]) is that for any $a > 0$,

$$\int_0^{\infty} e^{a \cosh u} \, du = K_0(a),$$

where K_0 is the modified Bessel function of the second kind. Relevant here is that $K_0(a)$ is finite for all $a > 0$, with $K_0(a)$ approaching $-\ln(a)$ as $a \to 0^+$ and decaying exponentially as $a \to \infty$. Therefore, we conclude that

$$\|e^{-\beta H}\|_1 = \frac{1}{2} K_0(\beta m) < \infty.$$

□

Thus, $e^{-\beta H}$ is indeed trace class. Note that it is crucial to the above proof that $m > 0$. Otherwise, the integrals would diverge logarithmically.

The following closely follows Section IV in [15].

Since $e^{-\beta H}$ is trace class, take the Gibbs state at inverse temperature $\beta > 0$ associated with H

$$\varphi(A) = \text{tr}(T_\beta A), \quad T_\beta = e^{-\beta H}/\text{tr}(e^{-\beta H}).$$

The GNS representation of $\big(\mathcal{B}\big(L^2\big([m,\infty), d\mu(\omega)\big)\big), \varphi\big)$ can be identified as $(\pi, \mathcal{H}_\varphi)$, where $\mathcal{H}_\varphi$ is the space of Hilbert-Schmidt operators on $L^2\big([m,\infty), d\mu(\omega)\big)$ with inner product

$$(A|B) = \text{tr}(B^* A), \quad A, B \in \mathcal{H}_\varphi,$$

and π is given by left multiplication:

$$\pi(A)B = AB, \quad A \in \mathcal{B}\big(L^2\big([m,\infty), d\mu(\omega)\big)\big), B \in \mathcal{H}_\varphi.$$

A canonical choice for the cycling and separating vector is

$$\Omega = T_\beta^{1/2}.$$

By the proposition above, T_β is trace class, and so Ω is Hilbert-Schmidt. It is now immediate that

$$\varphi(A) = (\pi(A)\Omega|\Omega), \quad A \in \mathcal{B}\big(L^2\big([m,\infty), d\mu(\omega)\big)\big).$$

Hence, Ω is cyclic and separating for $\pi\big(\mathcal{B}\big(L^2\big([m,\infty), d\mu(\omega)\big)\big)\big)$.

In what follows, we suppress the notation of π, writing A and B for $\pi(A)$ and $\pi(B)$. On $\mathcal{H}_\varphi$ we consider the unitary operators $U(t)$, $t \in \mathbb{R}$, defined by

$$U(t)S := e^{i\beta t N} S e^{-i\beta t N}, \quad S \in \mathcal{H}_\varphi.$$

The family $(U(t))_{t\in\mathbb{R}}$ is easily seen to be a unitary C_0-group on $\mathcal{H}_\varphi$. By Stone's theorem, the generator of this group is of the from iH with H selfadjoint. By the functional calculus of unbounded selfadjoint operators, the operator $\Delta := e^{-H}$ is injective and selfadjoint (its domain being given by this calculus). By the composition rule of the functional calculus, for $t \in \mathbb{R}$ we obtain

$$\Delta^{-it} = (e^{-H})^{-it} = e^{itH} = U(t),$$

where in the last step we used the fact that the unitary group generated by iH also arises through exponentiation in the functional calculus. In particular, for all $A, B \in \mathcal{B}\big(L^2\big([m,\infty), d\mu(\omega)\big)\big)$,

$$(\Delta^{-it} A\Omega | B\Omega) = \varphi(B^\star e^{i\beta t N} A e^{-i\beta t N}).$$

By spectral theory, for $h \in D(\Delta^{1/2})$ the mapping $z \mapsto \Delta^{-iz} h$ is continuous on the closed strip $\{\Im z \in [0, \frac{1}{2}]\}$ and holomorphic on its interior. This, and standard properties of fractional powers, implies that for all $A, B \in \mathcal{B}\big(L^2\big([m,\infty), d\mu(\omega)\big)\big)$, both sides of the identity

$$(\Delta^{-it} A\Omega | B\Omega) = \varphi(B^\star e^{i\beta t N} A e^{-i\beta t N}) = \mathrm{tr}\,(B^\star e^{i\beta t N} A e^{-\beta(it+1)N})$$

admit a continuous extension to the strip $\{\Im z \in [0, \beta]\}$ which is analytic on the interior, and given by substituting z for t. By the edge of the wedge theorem (Proposition 5.3.6 in [18]) these extensions agree. It follows that for $\Re z \in [0, \frac{1}{2}]$ we have $A\Omega \in D(\Delta^{1/2}) \subseteq D(\Delta^{-iz})$, and taking $z = \frac{1}{2} i$ results in the identity

$$(\Delta^{1/2} A\Omega | B\Omega) = (e^{-\frac{1}{2}\beta H} A\Omega | B\Omega) = \mathrm{tr}\,(B^\star e^{-\frac{1}{2}\beta N} A e^{-\frac{1}{2}\beta N}) = (\Omega A | B\Omega).$$

Since Ω is cyclic for π in $\mathcal{H}_\varphi$, this implies

$$\Delta^{1/2} A\Omega = \Omega A.$$

Now, if J is the antiunitary operator that sends an operator in $\mathcal{H}_\varphi$ to its adjoint, then

$$(J\Delta^{1/2})(A\Omega) = J(\Omega A) = A^*\Omega.$$

Since the Tomita operator and the modular operator associated with a pair $\{\mathcal{M}, \Omega\}$ are unique, we conclude that Δ must be the modular operator associated to $\pi\big(\mathcal{B}\big(L^2\big([m,\infty), d\mu(\omega)\big)\big)\big)$ with cyclic-separating vector Ω. Now recall that we had

$$\Delta^{-it} = (e^{-H})^{-it} = e^{itH} = U(t)$$

for all $t \in \mathbb{R}$, and so the physical time evolution generated by H coincides with a thermal time.

10

Conclusion

A fundamental issue within quantum gravity is that of the problem of time. The principle of superposition offers a multiplicity of time evolutions in quantum mechanics, whereas special relativity demands local uniqueness of time evolution. One potential solution, the thermal time hypothesis, postulates that a law of time evolution in quantum gravity can be obtained using Tomita-Takesaki theory.

The main result of Tomita-Takesaki theory is Tomita's theorem. Given a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ and a cyclic and separating vector $\Omega \in \mathcal{H}$, one can construct a positive, self-adjoint operator Δ known as the modular operator associated to $(\mathcal{M}, \Omega)$. Tomita's theorem then states that:

$$M = \Delta^{-it} \mathcal{M} \Delta^{it}, \quad t \in \mathbb{R}.$$

In words, Tomita-Takesaki theory constructs a one-parameter group of automorphisms of von Neumann algebras. This one-parameter group is the thermal time associated to the quantum system described by $(\mathcal{H}, \mathcal{M})$. It can be shown that the thermal time of some physical systems corresponds to a certain relativistic proper time.

Tommita's theorem was originally proved by Minoru Tomita in 1967, but his work was hard to follow and mostly unpublished. Since then, various proofs have been discovered. The most recent proof was given in a 2024 paper by Jonathan Sorce.

Given a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ with a cyclic and separating vector $\Omega \in \mathcal{H}$, the main idea of the proof is to produce operators in $\mathcal{M}$ for which the modular operator associated to $(\mathcal{M}, \Omega)$ "looks bounded." These operators are known as tidy operators, and they form a subspace of $\mathcal{M}$ known as the tidy subspace. Tomita's theorem is shown to hold for the tidy subspace, and by an argument using von Neumann's double commutant theorem, it is shown to hold for the entire von Neumann algebra $\mathcal{M}$.

The proof as originally given in Jonathan Sorce's paper skips many mathematical details. This thesis gives a detailed extension of Sorce's proof, carefully applying the spectral theorem for bounded and unbounded operators in each proposition used in the proof. Furthermore, one small error in Sorce's proof is corrected (this error has been updated in the most recent version of Sorce's paper).

The thermal time arising from Tomita-Takesaki theory can be formulated as a positive operator-valued measure for certain physical systems, as is shown in a 2024 paper by Jan van Neervan and Pierre Portal. One such system is that of the one-dimensional free moving massless particle. A similar result can be obtained for the one-dimensional free moving massive particle, leveraging the Lorentz invariant measure on the mass shell from quantum field theory.

The massless case uses the Fourier transform to switch from a momentum representation to an energy representation of a physical system. This can be replicated for the massive case, but the Fourier transform has to be generalized somewhat. Time covariance of the defined positive operator-valued measure then still holds in the general case using a density argument. Finally, it is shown that this time observable can be interpreted as a thermal time.

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