Delft University of Technology
Department of Aerospace Engineering

Report LR-379

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SUMMARY

This report deals with a theory of filament wound pressure vessels in which a composite is considered as a material consisting of macroscopically homogeneous and anisotropic layers. The surfaces of the pressure vessels can be chosen in such a way that the requirement that all filaments carry the same load is fulfilled. The shape of the surface depends on the elastic properties of the composite and on one independent geometric parameter. Relations for some characteristic properties of pressure vessels are expressed in this parameter. It is shown that the netting theory in which the matrix is considered as a non load-carrying binder can be simply derived from the continuum theory.

CONTENTS

NOTATION

A	cross section of a (cured) bundle of filaments
C	integration constant
C_1, C_2, C_3	material constants
C_{ij}	elements of the stiffness matrix of the laminate in the principal directions
C_{ijC}	elements of the stiffness matrix of the laminate of the cylindrical part in the principal directions
C^h_{ij}	elements of the stiffness matrix of the hoop layers of the cylindrical part in the principal directions
E(1), E(2)	elliptic integrals of first and second kind
E_α, E_β	Young's moduli of the lamina in the principal directions
F	resultant force in a bundle at strain $\bar{\varepsilon}_\beta$
$G_{\alpha\beta}$	shear modulus of the lamina in the principal direction
l	length of the cylindrical part of a pressure vessel
n	number of bundles of filaments in a cross section perpendicular to the rotation axis
n_h	number of bundles in hoop direction per unit length
n_u	number of bundles, crossing the equator of an isotensoid per unit length
p	internal pressure
q	geometric parameter Y_u^2/Y_i^2
R_1, R_2	meridional and circumferential radius of the curvature
t	local laminate thickness of the isotensoid
t_c	wall thickness of the cylindrical part
t_h	thickness of the hoop layer
t_u	laminate thickness at the equator of an isotensoid
V	volume of the optimum part of the pressure vessel
W	weight of the optimum part of the pressure vessel
x, y	coordinate axis, the x-axis is the rotation axis
y_i, y_u	minimum and maximum radius of the optimum part of a pressure vessel
y_o	radius of the polar opening
X, Y	dimensionless coordinates x/y, and y/y_o
Y = 1	dimensionless radius of the polar opening
Y_b	dimensionless radial distance of the point of inflection of an isotensoid and dimensionless radius of a cylinder with one winding angle
α, β	lamina principal directions, α is the filament direction
α	winding angle (angle between filaments and meridional line)
α_b	winding angle at the point of inflection
α_{cyl}	winding angle of a cylinder with one winding angle
α_u	winding angle at the equator of an isotensoid
γ	specific weight
ε	uniform biaxial strain in the isotensoid pressure vessel or end-closure
$\varepsilon_\ell, \varepsilon_t$	strains of the cylindrical part in longitudinal and circumferential direction
$\bar{\varepsilon}_\beta$	strain at the weeping pressure, limited by the ultimate strain perpendicular to the filaments
$\varepsilon_\xi \ \varepsilon_\eta$	strains in the isotensoid pressure vessel or end-closures in ε- η-directions
Θ_1	amplitude in elliptic integrals and angle in the transformation of y
$\mu_{\alpha\beta}, \mu_{\beta\alpha}$	Poisson's ratios of the lamina
ξ, η	principal directions of the laminate of the isotensoid pressure vessel or end-closure
$\sigma_\xi \ \sigma_\eta$	normal stresses in the isotensoid pressure vessel or end-closure in ε- η-directions
σ_w	stress in the bundles of filaments at the weeping pressure
$\sigma_{\alpha_{ult}}$	strength of a bundle of filaments

1. INTRODUCTION

According to the continuum theory a composite is a homogeneous material with anisotropic elastic properties. The matrix enables the filaments to carry loads and in general only a small part of the loads is carried by the matrix itself. An optimum design condition is that all filaments must be loaded to identical stress levels. This means equal strain in all filaments and consequently uniform biaxial strain in the entire structure. For a pressure vessel with fibers along geodesic lines the shape can be evaluated with this condition and the requirement of equilibrium of a surface element under internal pressure loading.

The netting theory simplifies a composite material to a system of filaments. Strength and stiffness of the matrix are neglected and the filaments are considered as the only load carrying elements. Additionally it is assumed that the filaments have no bending stiffness. According to this theory the only load carrying elements in a filament wound pressure vessel are continuous or closed loop filaments lying along geodesic lines. The tension load will obviously be constant over the total length of these filaments if the vessel is loaded by internal pressure only. Under these assumptions the shape of the vessel can be evaluated from the requirement of equilibrium of a part of a filament.

If a load causes equal strains in all directions first damage of a composite usually will occur in the matrix or in the bond between matrix and fibers, disintegrating the material before the fibers reach their ultimate tensile stress. Hence for a filament wound pressure vessel designed according to the continuum theory an internal pressure can be predicted at which leaking or weeping starts provided the stress or strain level is known at which first cracking of the matrix occurs. If after first damage the composite proceeds to carry loads using the non destructed fibers as a netting the vessel does not have the required geometric properties according to the netting theory. As a result the final failure load of this vessel is unpredictable. On the other hand the netting theory cannot predict the weeping pressure. It is therefore obvious that for the design of vessels with a well defined weeping pressure the continuum theory must be used whereas the netting theory can be used when only the bursting pressure is of interest. It is noted that in both theories the shape of the vessel as well as other geometric parameters follow from the calculations; they are not the starting point for the calculations!

After discussion of some general features of filament wound pressure vessels this report presents a number of formulae which are of engineering interest. They are important for a proper choice of the composite and for the calculation of the number of filaments (bundles, yarns, rovings) required for a given internal pressure and vessel volume. Discontinuity problem arising at the section between a possible cylindrical part and an isotensoid end-closure are discussed briefly. Most numerical results are presented graphically.

The expressions discussed in the current work result from Reference [1] where a general theory on filament wound pressure vessels is treated in detail. A resume of that theory is given in the appendix to make this report selfcontained.

2. THE PRESSURE VESSEL WITH ROTATIONAL SYMMETRY

It is assumed that during the winding process the filaments have no bending stiffness and that there is no friction between the filaments and the winding or mould surface. On a surface of revolution the pre- stressed filaments will then follow a geodesic line given by

$$y \sin \alpha = \text{constant} \tag{2.1}$$

where α is the angle between a filament and a meridional line in a point of the surface and y is the radial distance to the axis of rotation. At the polar opening of a pressure vessel the angle α has a value $\pi/2$, hence for a radius y_o of that opening relation (2.1) becomes

$$\sin \alpha = y_o/y \tag{2.2}$$

Although a filament wound pressure vessel generally is a thin-walled structure for which the membrane theory is applicable the wall consists of several layers. Since for a given radius y_o the winding angle α depends on the radial distance y the angle α will change over the thickness of the wall. The assumption will be made that this change is negligible so all layers are wound alternately under $+ \alpha$ and $- \alpha$ at a certain point of the meridional line.

A further assumption is that the wall consists of an even number of layers of equal thickness and equal elastic properties. The laminate therefore is anti-symmetric with respect to its middle plane which implies coupling between inplane and out of plane forces and displacements. For reasons of symmetry however an element of the vessel (see Figure 1) is unable to warp under the given load and the constitutive equations for such an element simplify to

$$\begin{aligned} \sigma_\xi &= pR_2/2t = C_{11}\, \varepsilon_\xi + C_{12}\, \varepsilon_\eta \\ \sigma_\eta &= pR_2(2 - R_2/R_1)/2t = C_{12}\, \varepsilon_\xi + C_{22}\, \varepsilon_\eta \end{aligned} \tag{2.3}$$

where the thickness t is dependent on the radius y. The coefficients C_{ij} are stiffness elements in ξ- η-directions and R_1 and R_2 are the meridional and circumferential radius of the curvature respectively

$$R_1 = - \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} \Big/ \frac{d^2y}{dx^2}$$

$$R_2 = y \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{1/2} .$$

As already discussed in the introduction an optimum design condition is that all filaments are loaded to identical stress levels which requires uniform strain $\varepsilon_\xi = \varepsilon_\eta = \varepsilon$ in all directions. Substituting ε in equations (2.3) gives

$$pR_2/2t = \varepsilon\left(C_{11} + C_{12}\right)$$
$$pR_2\left(2-R_2/R_1\right)/2t = \varepsilon\left(C_{12} + C_{22}\right) \tag{2.4}$$

Hence

$$\left(C_{22} - 2C_{11} - C_{12}\right)\Big/\left(C_{11} + C_{12}\right) = y\left(\frac{d^2y}{dx^2}\right)\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \tag{2.5}$$

This equation shows that the shape of an optimum pressure vessel depends on the elastic constants C_{ij}. The shape satisfying condition (2.5) is called an isotensoid.

In the appendix it is shown how (2.5) can be transformed into

$$\frac{dX}{dY} = -Y^3\Big/\left\{C\left(Y^2 - C_1\right)^{C_2} - Y^6\right\}^{1/2} \tag{2.6}$$

where C is an integration constant, C_1 and C_2 are material parameters and X and Y are dimensionless coordinates

$$C_1 = \left(1 - \mu_{\beta\alpha}/\mu_{\alpha\beta}\right)\Big/\left(1 + \mu_{\beta\alpha}\right)$$
$$C_2 = 2 - C_1 = \left(1 + 2\mu_{\beta\alpha} + \mu_{\beta\alpha}\Big/\mu_{\alpha\beta}\right)\Big/\left(1 + \mu_{\beta\alpha}\right) \tag{2.7}$$

$$X = x/y_o, \quad Y = y/y_o \tag{2.8}$$

$\mu_{\beta\alpha}$ and $\mu_{\alpha\beta}$ in (2.7) are Poisson's ratios of the constituent layers related to the α- β-axes parallel and perpendicular to the filaments respectively. They can be measured from a unidirectional specimen.

Equation (2.6) can only be solved by numerical methods. C_1 and C_2 represent the influence of the material properties on the shape of the pressure vessel. The constant C will be shown to contain a geometric parameter as well as C_1 and C_2.

3. THE GEOMETRIC PARAMTER q

The general shape of a pressure vessel with equally stressed filaments is shown in Figure 2. In this figure three points should be noted:

- Radius y_u (or Y_u), the maximum value of y
- Radius y_i (or Y_i), the minimum radius of the optimum part of the vessel
- Radius y_o (or $Y = 1$), the radius of the polar opening where $\alpha = \pi/2$

It is important to note that dependent on the constants C, C_1 and C_2 the radius Y_i may be larger or smaller than Y = 1. In the first case the usually small area between Y = 1 and Y_i does not satisfy the optimum requirement and extra reinforcement is needed there. The second case implies that optimum pressure vessels are feasible continuously wound between the polar openings without any additional reinforcement.

With the geometric parameter

$$q = Y_u^{\,2} \Big/ Y_i^{\,2} \tag{3.1}$$

it can be derived from equation (2.6)

$$\begin{aligned} Y_i^{\,2} &= C_1 \left(q^{3/C_2 - 2} - 1\right) \Big/ \left(q^{3/C_2 - 2} - q\right) \\ Y_u^{\,2} &= q\, C_1 \left(q^{3/C_2 - 2} - 1\right) \Big/ \left(q^{3/C_2 - 2} - q\right) \end{aligned} \tag{3.2}$$

$$C = C_1^{\,C_1 + 1} \left(q^{3/C_2 - 2} - 1\right)^3 \Big/ \left\{ \left(q - 1\right)^{C_2} \left(q^{3/C_2 - 2} - q\right)^{C_1 + 1} \right\} \tag{3.3}$$

Apparently the only geometric parameter of an optimum pressure vessel is the ratio of maximum radius to minimum radius of the optimum region.

In Figure 3 isotensoids are presented for a number of q-values corresponding to various maximum radii. The material properties, representative for carbon high modulus fibers and E-glass fibers, both impregnated with an epoxy resin, are given in Table 1. These materials are chosen because they show the differences in shape according to the continuum theory very well. Other composites may result in isotensoids lying between those of Figure 3. In Figure 4 isotensoids are presented normalized to the maximum radius Y_u. Using the same material data the winding angle

$$\alpha_u = \arcsin\,(1/Y_u)$$

at the equator was computed as a function of q, see Figure 5.

In the appendix approximate expressions describing the isotensoid are derived from (2.6). They are expressed in q as well:

$$y = y_o \left\{ \left(q \cos^2\Theta_1 + \sin^2\Theta_1\right) C_1 \left(q^{3/C_2 - 2} - 1\right) \Big/ \left(q^{3/C_2 - 2} - q\right) \right\}^{1/2} \tag{3.4}$$

$$x = y_o \left\{E(2) - \left(q + 1\right) E(1) \Big/ \left(2q + 1\right)\right\} \left\{C_1 \left(q^{3/C_2} - 1\right) \left(2q + 1\right) \Big/ \left(q^{3/C_2} - q\right)\right\}^{1/2} \tag{3.5}$$

where $E(1) = \int_o^{\Theta 1} \left\{1 - \left(q - 1\right) \sin^2\Theta \Big/ \left(2q + 1\right)\right\}^{-1/2} d\Theta$

and $E(2) = \int_o^{\Theta 1} \left\{1 - \left(q - 1\right) \sin^2\Theta \Big/ \left(2q + 1\right)\right\}^{1/2} d\Theta$

are elliptical integrals of the first and second kind respectively. In Reference [4] these integrals are tabulated for different values of the factor $(q - 1)/(2q + 1)$. They facilitate the computation of the contour considerably.

In (3.4) and (3.5) the coordinates x and y are dependent on the material properties by the same factor

$$C_1 \left(q^{3/C_2} - 1\right) \Big/ \left(q^{3/C_2} - q\right).$$

This implies that for a fixed value of q x increases or decreases proportionally with y for different material coefficients C_1 and C_2 resulting in simular contours for different materials. If additionally identical outer radii y_u are chosen the only different geometric quantities are the radius y_o of the polar opening and the winding angle α in corresponding points of the surfaces. Because (3.5) is an approximate formula figures computed with it will differ slightly from the corresponding figures calculated with the exact expression (2.6). For engineering purposes however it is sufficiently accurate which is shown in Figure 3 with the approximated contour for E-glass epoxy and q = 100.

4. THE CYLINDER AS A LIMITING CASE

The meridian described by (2.6) and shown in Figure 2 has a point of inflection where

$$d^2Y / dX^2 = -C \left(Y^2 - C_1\right)^{C_2 - 1} \left\{Y^2\left(C_1 + 1\right) - 3C_1\right\} / Y^7 = 0$$

The radial distance of this point is given by

$$Y_b{}^2 = 3\, C_1 / \left(C_1 + 1\right) \qquad (4.1)$$

and the local winding angle by

$$\sin\ \alpha_b = 1 / Y_b = \left\{ \left(C_1 + 1\right) / 3\, C_1 \right\}^{1/2} \qquad (4.2)$$

Expressions (4.1) and (4.2) are independent of the geometric parameter q which implies that all meridian represented by (2.6) for given material parameters C_1 and C_2 have a point of inflection with equal relative radial distance and equal winding angle. It is obvious that this applies also to a cylinder for which q = 1, resulting in

$$Y_i{}^2 = Y_u{}^2 = Y_b{}^2 = 3\, C_1 / \left(C_1 + 1\right) \qquad (4.3)$$

Formula (4.3) indicates that for the cylinder the correct polar opening required to invert the winding direction has a radius

$$y_{o_{cyl}} = y_{i_{cyl}} / Y_{i_{cyl}} = y_{cyl} \left\{ \left(C_1 + 1\right) / 3\, C_1 \right\}^{1/2} \qquad (4.4)$$

For values of y smaller than y_{cyl} equation (2.6) is not applicable. Since according to (4.4) y_o of a cylinder is always smaller than y_{cyl} a cylinder with end - closures needs additional reinforcement between y_{cyl} and y_o. An infinitely long cylinder without end - closures is a limiting case of the pressure vessels described by (2.6). In Chapter 6 it will be shown how a cylinder with optimum end - closures can be wound by using two different winding angles for the cylindrical part.

The winding angle of an infinitely long cylinder is given by

$$\sin\ \alpha_{cyl} = \left\{\left(C_1 + 1\right) / 3\, C_1\right\}^{1/2} \qquad (4.5)$$

It will be shown later that the value $C_1 = 1$ corresponds to the netting theory with $\alpha_{cyl} = 54.7°$

5. PROPERTIES OF OPTIMUM PRESSURE VESSELS

Formula (2.6) describes the shape of an optimum pressure vessel. Other properties of interest are:

- internal pressure and the required number of filaments
- the volume of the pressure vessel, for instance related to the maximum radius y_u
- the ratio of weight to volume

These relations are derived in the appendix. They will be discussed briefly in this section.

- INTERNAL PRESSURE AND REQUIRED NUMBER OF FILAMENTS

With the additional material constant

$$C_3 = \left(1 - \mu_{\beta\alpha}\ \mu_{\alpha\beta}\right) \Big/ \left(1 + 2\ \mu_{\beta\alpha} + \mu_{\beta\alpha} \Big/ \mu_{\alpha\beta}\right) \tag{5.1}$$

the relation between internal pressure and the required number of filaments can be written as

$$\varepsilon E_\alpha nA \Big/ \pi p y_o^{\ 2} = C_2\ C_3\ Y_u^{\ 3} \left(Y_u^{\ 2} - 1\right)^{1/2} \Big/ \left(Y_u^{\ 2} - C_1\right) \tag{5.2}$$

in which n is the number of bundles of filaments (rovings, yarns, tows) in any cross section of the vessel perpendicular to the rotation axis, all bundles having a cross section A. In (5.2) is $\varepsilon E_\alpha A = F$ the resultant force in longitudinal direction of the bundles of filaments. In order to determine a numerical value of the ultimate pressure p a value of ε is needed, ε being the uniform biaxial strain in the whole vessel. As already discussed ε is generally limited by the ultimate strain $\bar{\varepsilon}_\beta$ transverse to the fibers. Only with a sufficiently ductile matrix material the strength of the fibers can be fully exploited without leakage before the bursting pressure is reached. Both $\bar{\varepsilon}_\beta$ and E_α can be determined with tensile tests on unidirectional specimens.

According to (3.2) the radius Y_u is a function of q. Figure 6 shows expression (5.2) for the E-glass and carbon high modulus fibers.

- THE VOLUME OF THE PRESSURE VESSEL

The expression for the volume of the optimum part of a presssure vessel is

$$V = -\frac{2}{3}\pi\ y_o^{\ 3}\ C\ C_2 \int_{Y_u}^{Y_i} \frac{\left(Y^2 - C_1\right)^{C_2 - 1}\ Y\ d\ Y}{\left\{C\left(Y^2 - C_1\right)^{C_2} - Y^6\right\}^{1/2}} \tag{5.3}$$

The integral in (5.3) must be solved numerically. In the appendix a relation expressed in q is given for the volume of pressure vessels designed according to the netting theory. For engineering purposes this relation has sufficient accuracy for vessels designed according to the continuum theory. The relation is

$$V \Big/ {Y_u}^3 = \frac{2}{3}\pi\left[\left(q^2 + q + 1\right) \Big/ \left\{q^3\left(2q + 1\right)\right\}^{1/2}\right]E \qquad (5.4)$$

where E is the complete elliptical integral of the first kind

$$E = \int_o^{\pi/2} \left\{1 - \left(q - 1\right) \sin^2\Theta \Big/ \left(2\,q + 1\right)\right\}^{-1/2} d\Theta$$

Numerical values of $V \big/ {Y_u}^3$ are presented in Table 2.

- THE RATIO OF WEIGHT TO VOLUME

The expression for the total weight of the optimum part of a pressure vessel contains the same integral as the expression for the volume (5.3).

$$W = -\,\gamma\, C\, C_2\, C_3\, 2\pi {Y_o}^3\, p \Big/ \left(\varepsilon E_\alpha\right) \int_{Y_u}^{Y_i} \frac{\left(Y^2 - C_1\right)^{C_2 - 1}\, Y\, d\, Y}{\left\{C\left(Y^2 - C_1\right)^{C_2} - Y^6\right\}^{1/2}} \qquad (5.5)$$

where γ is the specific weight of the composite. Hence the resulting expression for the ratio of weight to volume is

$$W/V = 3\, p\, C_3\, \gamma \Big/ \varepsilon E_\alpha \qquad (5.6)$$

εE_α is the stress in longitudinal direction of the bundles of filaments. It is obvious that this stress is limited by the ultimate strain $\bar{\varepsilon}_\beta$ as well. $\bar{\varepsilon}_\beta\, E_\alpha$ will be called the stress σ_w in the filaments at the weeping pressure, so one may write

$$(W/V)/p = 3\, C_3\, \gamma/\sigma_w \qquad (5.7)$$

In Table 3 numerical values of this relation are presented for the composite materials listed in Table 1. All these materials will have different $\bar{\varepsilon}_\beta$ -values. For the present calculations however identical allowable strains .5% and 1% have been adopted. Therefore the values of (W/V)/p in Table 3 only have limited significance. Nevertheless they give a good indication for the comparison of the weight to volume ratios.

Relations (5.6) and (5.7) contain no geometric parameter. Apparently the ratio of weight to volume of the optimum part of a pressure vessel with equally stressed fibers only depends on the internal pressure and the properties of the composite and it is independent of the geometry.

For an isotropic material one finds

$$C_3 = (1 - \mu^2)/2(1 + \mu) = (1 - \mu)/2$$

and the uniform biaxial strain is

$$\varepsilon = \sigma(1 - \mu)/E$$

so (5.6) becomes

$$W/V = 3\, p\, \gamma \,/\, 2\, \sigma \tag{5.8}$$

which is the well known formula for an isotropic spherical pressure vessel. Any other shape of isotropic pressure vessels is not optimal and will result in a W/V-expression containing a geometric parameter yielding larger W/V-ratios.

6. THE CYLINDRICAL SECTION WITH TWO WINDING ANGLES

The isotensoids treated in the previous sections can be used as end - closures of cylindrical pressure vessels. The winding angle α_u at the equator of a pressure vessel with $q > 1$ however is smaller than the optimum winding angle of a cylinder with $q = 1$, as shown by Figure 5. If a cylindrical section is wound at y_u the winding angle α_u will remain constant over this section. So, to carry the circumferential load the cylinder has to be reinforced by additional circumferential windings.

For a cylinder with winding angle α_u and additional hoop windings the stress resultants in longitudinal and circumferential directions are

$$\begin{aligned} py_u / 2t_c &= C_{11_c}\, \varepsilon_\ell + C_{12_c}\, \varepsilon_t \\ py_u / t_c &= C_{12_c}\, \varepsilon_\ell + C_{22_c}\, \varepsilon_t \end{aligned} \qquad (6.1)$$

where $t_c = t_u + t_h$

and $$C_{ij_c} = C_{ij}\, t_u / t_c + C_{ij}^{h}\, t_h / t_c \qquad (6.2)$$

t_h is the thickness and C_{ij}^{h} are the stiffness matrix elements of the layer with the hoop windings.

It is obvious that ε_ℓ and ε_t in the cylinder cannot both be identical to the uniform strain ε in the end - closures, since, with the same load in longitudinal direction in the cylinder and in the end - closures at y_u

$$t_u\,(C_{11} + C_{12}) \neq t_c\,(C_{11_c} + C_{12_c})$$

A discontinuity in the strain in hoop direction at the equator of the end-closure would induce bending stresses and must be avoided. Therefore the number of hoop windings has to be evaluated from the requirement that ε_t in the cylindrical part is equal to ε in the end - closures.

From (6.1) it follows

$$\varepsilon_t = \left\{ \left(C_{12_c} - 2\, C_{11_c}\right) \Big/ \left(C_{12_c}^{2} - C_{11_c}\, C_{22_c}\right) \right\} py_u \Big/ 2\, t_c \qquad (6.3)$$

In the appendix it is shown that substitution of this expression in (5.2) yields a quadratic equation for t_h / t_u

$$\left\{\left(t_h/t_u\right)/\left(2\,C_{11}{}^h - C_{12}{}^h\right) + \left(2\,C_{11} - C_{12}\right)\right\}/$$

$$\left\{\left(t_h/t_u\right)^2 \left(C_{11}{}^h\,C_{22}{}^h - C_{12}^{h\,2}\right) + \left(t_h/t_u\right)\left(C_{11}\,C_{22}{}^h + C_{11}{}^h\,C_{22} - \right.\right.$$

$$\left.\left. 2C_{12}\,C_{12}{}^h\right) + \left(C_{11}\,C_{22} - C_{12}{}^2\right)\right\} = C_2\,C_3\,Y_u{}^2 \Big/ \left\{E_\alpha\left(Y_u{}^2 - C_1\right)\right\} \qquad (6.4)$$

The relevant root of this equation, converted into

$$n_h/n_u = t_h/t_u\;\cos\alpha_u$$

is presented in Figure 7 as a function of the q - value of the end - closure. n_h is the number of filaments of the cylindrical section in hoop direction and n_u is the number of filaments crossing the equator of the end - closure at angle α_u, both per unit length. It is noted that in (6.4)

$$C_{11}{}^h = \frac{E_\beta}{1 - \mu_{\alpha\beta}\,\mu_{\beta\alpha}}\;,\; C_{22}{}^h = \frac{E_\alpha}{1 - \mu_{\alpha\beta}\,\mu_{\beta\alpha}}\;\text{ and }\; C_{12}{}^h = \frac{\mu_{\beta\alpha}\,E_\alpha}{1 - \mu_{\alpha\beta}\,\mu_{\beta\alpha}}$$

C_{ij} are the components of the stiffness matrix of the $\pm\,\alpha_u$ layers of the cylinder related to the longitudinal and tangential direction. They can be found by putting $\alpha = \alpha_u$ in formula (A.2).

In the appendix an expression for the wall-thickness t_u of the isotensoids at y_u is derived:

$$t_u = \frac{C_2\,C_3\,Y_u{}^3\,py_o}{2\,\sigma_w\,(y_u{}^2 - 1)}$$

With this relation and a known value of (t_h/t_u) the wall-thickness of the cylindrical part is known as well:

$$t_c = t_u\,\{1 + (t_h/t_u)\}$$

The weight to volume ratio of the cylindrical part with length ℓ then becomes:

$$\left(\frac{W}{V}\right)_{cyl} = \frac{2\pi y_u\,t_c\;\ell\,\gamma}{\pi y_u{}^2\,\ell} = p\,C_2\,C_3\,\frac{\gamma}{\sigma_w}\,\frac{Y_u{}^2}{Y_u{}^2 - C_1}\left\{1 + \left(t_h/t_u\right)\right\} \qquad (6.5)$$

The total weight of the cylindrical vessel with two isotensoid end - closures is

$$W = p\ C_3\ \frac{\gamma}{\sigma_w}\ \left[3V + C_2\ \frac{Y_u^{\ 2}}{Y_u^{\ 2} - C_1}\ \left\{1 + \left(t_h/t_u\right)\right\}\ \pi Y_u^{\ 2}\ \ell\right] \qquad (6.6)$$

7. THE NETTING THEORY AS A SPECIAL CASE

By putting the stiffness transverse to the fibers zero all formula derived with the continuum theory reduce to well-known expressions from the netting analysis, see References [2] and [3].

With $E_\beta = o$ and consequently

$$\mu_{\beta\alpha} = 0 \text{ and } C_1 = C_2 = 1$$

the following expressions are found.

The meridian is described by

$$dY/dX = -\left\{C\left(Y^2 - 1\right) - Y^6\right\}^{1/2} \Big/ Y^3 \tag{7.1}$$

which can be transformed into

$$y^2 = y_u^{\,2} \cos^2\Theta + y_i^{\,2} \sin^2\Theta_1$$

$$x = y_i \left(2q + 1\right)^{1/2} \left\{E(2) - \left(q + 1\right) E(1) \Big/ \left(2q + 1\right)\right\} \tag{7.2}$$

The characteristic radii are given by

$$Y_i^{\,2} = \left(q^2 + q + 1\right) \Big/ \left\{q \left(q + 1\right)\right\}$$

$$Y_u^{\,2} = \left(q^2 + q + 1\right) \Big/ \left(q + 1\right) \tag{7.3}$$

The integration constant C and the expression for the number of bundles of filaments of given strength F are identical

$$C = \left(nF \Big/ \pi\, p\, y_o^{\,2}\right)^2 = \left(q^2 + q + 1\right)^3 \Big/ \left\{q^2 \left(q + 1\right)^2\right\} \tag{7.4}$$

A pressure vessel designed according to the netting analysis has a polar opening radius y_o which is always smaller than y_i. This implies that in the region between y_o and y_i the equally stressed filaments cannot fulfil the condition of equilibrium under internal pressure and an additional reinforcement is needed.

The volume of a netting analysis pressure vessel is

$$V = -\frac{2}{3}\pi y_o^3 C \int_{Y_u}^{Y_i} Y \, d\, Y \Big/ \left\{ C\left(Y^2 - 1\right) - Y^6 \right\}^{1/2} \qquad (7.5)$$

$$\text{or} \quad V/y_u^{\,3} = \frac{2}{3}\pi E \left(q^2 + q + 1\right) \Big/ \left\{ q^{3/2} \left(2q + 1\right)^{1/2} \right\} \qquad (7.6)$$

Formula (7.1) - (7.6) are independent of elastic properties of any kind and only depend on the geometric parameter q. The radius of the point of inflection of a meridian or the radius of a cylinder is

$$Y = (3/2)^{1/2}$$

and the related winding angle is 54.7^o

According to the netting analysis a cylinder with end - closures can be wound without discontinuities in the strains between the cylindrical section and the end - closures. The reason is that with $E_\beta = \mu_{\beta\alpha} = 0$

$$t_u \, (C_{11} + C_{12}) = t_c \, (C_{11_c} + C_{12_c})$$

hence ε_ℓ and ε_t in the cylinder can be identical to ε in the end - closures, all filaments then being loaded to equal stress levels. The relation for the ratio of the number of hoop filaments to the number of filaments at angle α_u reduces to

$$n_h / n_u = t_h / t_u \cos\alpha_u = \left(3 \cos^2\alpha_u - 1\right) \Big/ \cos\alpha_u$$

In all relevant figures the relations according to the netting theory are shown as well.

8. DISCUSSION AND CONSLUSIONS

The expressions describing the geometry of an optimum filament wound pressure vessel, based on the continuum theory, contain one geometric parameter in addition to elastic constants. Hence the shape of such a structure as well as the winding pattern is dependent on the properties of the material being used. The elastic constants do not appear in the netting theory which neglects the presence of a matrix and therefore makes use of an infinitely high ratio E_α/E_β of the impregnated bundles of filaments. Because of the high ratio E_α/E_β of impregnated carbon high modulus fibers the results for these fibers are close to the netting theory results.

It is important to note that two pressure vessels of the same material and same characteristic geometric properties, optimized according to the continuum theory and the netting theory respectively, are different in shape and strength. As already discussed previously the continuum theory can predict the weeping pressure, that is the pressure at which cracking of the matrix and weeping or leaking start. The final failure of a continuum theory pressure vessel cannot be predicted. If the matrix is relatively brittle failure may even occur <u>at</u> the weeping pressure since after failure of the matrix the fibers no longer have the correct winding pattern and cannot maintain equilibrium under internal pressure, provided of course the vessel has a sufficiently ductile inner lining preventing pressure loss.

For a high weeping pressure it is important to use relatively ductile matrix materials. This is shown most clearly in Table 3 where the (W/V)/p values are presented calculated at .5% and 1% allowable strain respectively. Except for the carbon high modulus fibers the ratio values at 1% strain are twice as low as those at .5% strain and obviously allow an increase of the pressure with a factor 2 at the same W/V ratio. The carbon high modulus fibers have an ultimate strain of only .76% and hence with the relatively ductile matrix allowing 1% strain the strength of these fibers is fully exploited. As a result the ultimate pressure according to the continuum theory in this case is a bursting pressure and is even higher than the netting theory ultimate pressure, as can be concluded from Table 3. The reason is that up to the bursting pressure the matrix carries a part of the load and relieves the filaments.

In general however, the ultimate pressure of netting theory pressure vessels is much higher than the weeping pressure of continuum theory pressure vessels at corresponding W/V ratios. The difference can roughly be approximated by the ratio of the ultimate strain of the fibers to the allowable ultimate strain in the matrix. It is remarked that matrix cracking at relatively low pressure levels also accurs in netting theory pressure vessels. These levels however are unpredictable. At pressures exceeding the weeping pressure the filaments will stay in place until their ultimate strength is reached and the vessel completely disintegrates. So for netting theory pressure vessels early matrix cracking is not necessarily catastrophic. Nevertheless ductile matrix materials yielding high weeping pressures are also preferred for netting theory pressure vessels.

The layers composing the wall of the pressure vessel are in a state of unform biaxial strain. It is therefore not entirely correct to use the ultimate uniaxial strain $\bar{\varepsilon}_\beta$ as a failure criterion. For the time being however $\bar{\varepsilon}_\beta$ measured from a simple tensile test seems to be a realistic and reliable criterion for the theory of pressure vessels.

A cylinder with two winding angles can be wound with all filaments loaded to equal stress levels under pressurization. If integrally wound end - closures are used discontinuities in longitudinal and tangential direction will occur between cylindrical section and end - closure. In order to avoid a discontinuity in the strain in hoop direction the number of hoop filaments must be chosen in such a way that the tangential strain in the cylinder is equal to the uniform strain in the end - closure. This results in a strength surplus in the longitudinal direction of the cylinder. Figure 7 shows this surplus for the case q (∞) and $\alpha_u = 0$. In that case the wall of the cylindrical part can be considered as a crossply biaxially loaded with a stress ratio 2. The tangential strain requirement results in n_h/n_u ratios smaller than 2 for the glass and carbon fibers which implies relatively too many fibers in longitudinal direction compared with the results of the classical lamination theory. This theory yields a ratio $n_h/n_u > 2$ for a biaxially loaded crossply where a uniform biaxial strain is required.

CONCLUSIONS:

- Geometry and other properties of optimum filament wound pressure vessels are dependent on the elastic properties of the materials used.
- For a high weeping pressure fibers with a high Young's modulus in combination with a (relatively) ductile matrix must be used. In this respect high modulus carbon fibers are excellent. The pressure vessel must be designed with the continuum theory.
- If early cracking of the matrix is unimportant a high ultimate pressure can be reached with high strength fibers. The (W/V)/p ratio's of the netting theory (see Table 3) show that in this case S-glass fibers are preferable. The pressure vessel must be designed with the netting theory.
- In order to prevent a strain discontinuity in tangential direction between a cylinder and its isotensoid end - closures a little over- strength in longitudinal direction of the cylindrical part must be accepted.

9. REFERENCES

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3. Th. de Jong, Filament Wound Pressure Vessels (In Dutch), Report VTH-166, Technological University Delft.
4. M. Abramowitz, J.A. Stegun Handbook of Mathematical Functions, Dover Publications, Inc. New York.

Table 1:
Mechanical properties used in the calculations (fiber volume content 60%)

	E_α MN/m^2	$\mu_{\alpha\beta}$	$\mu_{\beta\alpha}$
E-glass epoxy	40000	.26	.039
S-glass epoxy	57500	.26	.032
Aramid epoxy	85000	.33	.031
Carbon HT epoxy	146500	.3	.0184
Carbon HM epoxy	210000	.3	.0143

Table 2:
Numerical values of V/y_u^3 for netting theory pressure vessels. y_u is the outer radius, V the volume of the optimum part.

$q = y_u^2/y_i^2$	V/y_u^3
2	3.848
4	3.177
9	2.903
16	2.827
25	2.796
49	2.770
100	2.758
1000	2.747
10000	2.746
∞	2.746

Table 3:
The weight to volume ratio. The weeping stress σ_w i calculated with .5% and 1% uniform biaxial strain in the pressure vessel.

		Continuum Theory				Netting Theory	
		.5%		1%			
	γ $10^4 N/m^3$	σ_w MN/m^2	$(W/V)/p$ m^{-1}	σ_w MN/m^2	$(W/V)/p$ m^{-1}	$\sigma_{\alpha_{ult}}$ MN/m^2	$(W/V)/p$ m^{-1}
E-glass epoxy	2.05	200	$2.4787.10^{-2}$	400	$1.2393.10^{-2}$	2250	$2.7333.10^{-3}$
S-glass epoxy	2.01	287	$1.7552.10^{-2}$	574	$8.7760.10^{-3}$	3000	$2.0100.10^{-3}$
Aramid epoxy	1.35	425	$8.1595.10^{-3}$	850	$4.0798.10^{-3}$	1750	$2.3143.10^{-3}$
Carbon HT epoxy	1.55	732	$5.7528.10^{-3}$	1464	$2.8764.10^{-3}$	2140	$2.1729.10^{-3}$
Carbon HM epoxy	1.62	1050	$4.2821.10^{-3}$	1600*	$2.8101.10^{-3}$	1600	$3.0375.10^{-3}$

*at 1% strain these fibers have already reached their ultimate stress, so $\sigma_w = \sigma_{\alpha_{ult}}$

APPENDIX

The constitutive equations for an element of a pressure vessel under uniform strain ε are (formulae (2.4))

$$pR_2/2t = \varepsilon(C_{11} + C_{12})$$
$$pR_2\ (2 - R_2/R_1)/2t = \varepsilon\ (C_{12} + C_{22}) \qquad (A.1)$$

where R_1 and R_2 are the meridional and the circumferential radius of the curvature respectively. C_{ij} are the components of the stiffness matrix of the laminate related to the ξ- n-axes. Expressed in the stiffness components of the individual layers related to their axes of elastic symmetry they are

$$C_{11} = (E_\alpha \cos^4\alpha + E_\beta \sin^4\alpha + 2\mu_{\beta\alpha} E_\alpha \sin^2\alpha \cos^2\alpha) / (1 - \mu_{\beta\alpha}\mu_{\alpha\beta}) + G_{\alpha\beta} \sin^2 2\alpha$$

$$C_{12} = \left\{(E_\alpha + E_\beta) \sin^2\alpha \cos^2\alpha + \mu_{\beta\alpha} E_\alpha (\sin^4\alpha + \cos^4\alpha)\right\} / (1 - \mu_{\beta\alpha}\mu_{\alpha\beta}) - G_{\alpha\beta} \sin^2 2\alpha$$

$$C_{22} = (E_\alpha \sin^4\alpha + E_\beta \cos^4\alpha + 2\mu_{\beta\alpha} E_\alpha \sin^2\alpha \cos^2\alpha)/(1 - \mu_{\beta\alpha}\mu_{\alpha\beta}) + G_{\alpha\beta} \sin^2 2\alpha \qquad (A.2)$$

E_α, E_β, $\mu_{\alpha\beta}$ and $G_{\alpha\beta}$ are the engineering constants, the α-direction is the direction of the highest Young's modulus. This direction coincides with the direction of the filaments of the individual layer. For filament wound pressure vessels α is also used for the local winding angle.

With (A.2) equations (A.1) can be transformed into

$$\frac{E_\alpha (\sin^2\alpha - 2\cos^2\alpha) + E_\beta (\cos^2\alpha - 2\sin^2\alpha) - \mu_{\beta\alpha} E_\alpha}{E_\alpha \cos^2\alpha + E_\beta \sin^2\alpha + \mu_{\beta\alpha} E_\alpha} = \frac{y \frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2} \qquad (A.3)$$

After introduction of dimensionless parameters

$$X = x/y_o \text{ and } Y = y/y_o$$

equation (2.2) becomes

$$\sin^2\alpha = 1/Y^2$$

and hence

$$\cos^2\alpha = 1 - 1/Y^2$$

Multiplying numerator and denominator by Y^2/E and using Maxwell's law

$$E_\beta / E_\alpha = \mu_{\beta\alpha}/\mu_{\alpha\beta}$$

(A.3) becomes

$$\frac{Y^2 \; (-2 + \mu_{\beta\alpha}/\mu_{\alpha\beta} - \mu_{\beta\alpha}) + 3 \; (1 - \mu_{\beta\alpha}/\mu_{\alpha\beta})}{Y \left\{ Y^2 \; (1 + \mu_{\beta\alpha}) - (1 - \mu_{\beta\alpha}/\mu_{\alpha\beta}) \right\}} = \frac{\dfrac{d^2Y}{dX^2}}{1 + \left(\dfrac{dY}{dX}\right)^2}$$

which after expansion of the lefthand side into partial fractions yields

$$-\frac{3}{Y} + \frac{C_2 \, Y}{Y^2 - C_1} = \frac{\dfrac{d^2Y}{dX^2}}{1 + \left(\dfrac{dY}{dX}\right)^2}$$

Multiplication by $2(dY/dX)$ and integration over X leads to

$$-6 \ln Y + C_2 \ln\left(Y^2 - C_1\right) = \ln\left\{\left(1 + \left(\frac{dY}{dX}\right)^2\right\} + \ln\frac{1}{C}$$

or $$\frac{dY}{dX} = \pm \frac{\left\{C\left(Y^2 - C_1\right)^{C_2} - Y^6\right\}^{1/2}}{Y^3} \qquad \text{(A.4)}$$

The + and - sign indicate that the isotensoid has a plane of axial symmetry. Only the part in the first quadrant of the coordinate system will be considered, hence the + sign will be ignored (see Figure 2).

Since dY/dX is a real quantity, the argument of the square root in (A.4) must be positive or zero. So

$$C\left(Y^2 - C_1\right)^{C_2} - Y^6 \geqslant 0 \qquad (A.5)$$

Figure A.1 showing gaphically (A.5) as a function of Y^2 indicates that only values of Y^2 between Y_i^2 and Y_u^2 satisfy requirement (A.5). It is obvious that complex values of Y^2 satisfying (A.5) are irrelevant.

With $Y^2 = Y_u^2 \cos^2\Theta + Y_i^2 \sin^2\Theta$ (A.6)

(A.5) is approximated with a third degree polynomium in the area where requirement (A.5) is fulfilled.

$$C\left(Y^2 - C_1\right)^{C_2} - Y^6 \approx \left(Y_u^2 - Y^2\right)\left(Y^2 - Y_i^2\right)\left(Y^2 + Y_i^2 + Y_u^2\right)$$

so, with

$$Y\,d\,Y = \left(Y_i^2 - Y_u^2\right) \sin\Theta \cos\Theta \, d\Theta$$

equation (A.4) can be written as

$$X = -\int_{Y_u}^{Y} \frac{Y^2\,Y\,d\,Y}{\left\{\left(Y_u^2 - Y^2\right)\left(Y - Y_i^2\right)\left(Y^2 + Y_i^2 + Y_u^2\right)\right\}^{1/2}}$$

$$= -\int_0^{\Theta_1} \frac{\left(Y_u^2 \cos^2\Theta + Y_i^2 \sin^2\Theta\right)\left(Y_i^2 - Y_i^2\right) \sin\Theta \cos\Theta \, d\Theta}{\left[\left(Y_u^2 - Y_i^2\right)^2 \left\{Y_i^2\left(\sin^2\Theta + 1\right) + Y_u^2\left(\cos^2\Theta + 1\right)\right\} \sin^2\Theta \cos^2\Theta\right]^{1/2}}$$

With $q = Y_u^2/Y_i^2$ this relation can easily be transformed into

$$X = Y_i \int_0^{\Theta_1} \frac{(q \cos^2\Theta + \sin^2\Theta)\, d\Theta}{\left\{\left(\sin^2\Theta + 1\right) + q\left(\cos^2\Theta + 1\right)\right\}^{1/2}}$$

or

$$x = y_i \left(2q + 1\right)^{1/2} \left\{ \int_0^{\Theta_1} \left(1 - \frac{q - 1}{2q + 1} \sin^2\Theta\right)^{1/2} d\Theta \right.$$

$$\left. - \frac{q + 1}{2q + 1} \int_0^{\Theta_1} \left(1 - \frac{q - 1}{2q + 1} \sin^2\Theta\right)^{-1/2} d\Theta \right\} \qquad (A.7)$$

where the integrals are elliptical integrals of the second and first kind respectively.

With transformation (A.6) and approximation (A.7) the shape of an optimum pressure vessel is desdribed by

$$y^2 = y_u^2 \cos^2\Theta_1 + y_i^2 \sin^2\Theta_1 = y_i^2 \left(q \cos^2\Theta_1 + \sin^2\Theta_1\right)$$

$$x = y_i \left(2q + 1\right)^{1/2} \left\{E(2) - \frac{q + 1}{2q + 1} E(1)\right\} \qquad (A.8)$$

Both x and y are linearly dependent on radius Y_i which is the only quantity in (A.8) containing materialparameters C_1 and C_2^1. The approximated shape for glass fiber epoxy and q = 100 is shown in Figure 3.

A relation for the required number of filaments can be derived from the first of equations (2.4) which can be written as

$$\frac{2\varepsilon}{p} = \frac{1}{t\left(C_{11} + C_{12}\right)} y \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{1/2} \qquad (A.9)$$

It is noted that t is the thickness of the laminate in a direction perpendicular to the ξ- n-plane and not in the direction perpendicular to the rotation - axis .t is a function of the radius y.

From (A.2) one deduces

$$C_{11} + C_{12} = \frac{E_\alpha}{1 - \mu_{\beta\alpha}\,\mu_{\alpha\beta}}\,\frac{1}{Y^2}\left(Y^2 - 1 + \mu_{\beta\alpha}\Big/\mu_{\alpha\beta} + \mu_{\beta\alpha}\,Y^2\right)$$

$$= \frac{E_\alpha}{C_2\,C_3}\;\frac{Y^2 - C_1}{Y^2} \qquad \text{(A.10)}$$

Substitution of (A.10) into (A.9) leads with $y = y_o\,Y$ to

$$\frac{2\,\varepsilon\,E_\alpha}{p y_o\,C_2\,C_3} = \frac{Y^3}{t\left(Y^2 - C_1\right)}\left\{1 + \left(\frac{dy}{dX}\right)^2\right\}^{1/2} \qquad \text{(A.11)}$$

For a given q-value relation (A.11) must be satisfied in every point of the surface. Taking $Y = Y_u$ where $t = t_u$ and

$$\left\{1 + \left(\frac{dy}{dX}\right)^2\right\}^{1/2} = \left\{C\,\frac{\left(Y_u^{\,2} - C_1\right)^{C_2}}{Y_u^{\,6}}\right\}^{1/2} = 1$$

one finds

$$\frac{2\,\varepsilon\,E_\alpha}{p y_o\,C_2\,C_3} = \frac{Y_u^{\,3}}{t_u\,(Y_u^{\,2} - C_1)} \qquad \text{(A.12)}$$

The bundles of filaments at Y_u have a width of $b/\cos\alpha_u$ in tangential direction. With a number of n rovings in any cross-section perpendicular to the rotation axis, all rovings having a cross-section A

$$2\pi y_u\,t_u = n\,A/\cos\alpha_u$$

so $t_u = n\,A/2\pi y_o\,Y_u\,\cos\alpha_u = n\,A/2\pi y_o\,(Y_u^{\,2} - 1)^{1/2}$ (A.13)

Substitution of (A.13) into (A.12) results in

$$\varepsilon E_\alpha\,n\,A/\pi p y_o^{\,2} = C_2\,C_3\,Y_u^{\,3}\,(Y_u^{\,2} - 1)^{1/2}\Big/(Y_u^{\,2} - C_1) \qquad \text{(A.14)}$$

With the origin of the coordinate system on the axis of rotation at $Y = Y_u$ the total volume of the optimum part of the pressure vessel is given by

$$V = 2\pi \int_0^{x(y_i)} y^2 \, d\, x = 2\pi y_o^{\,3} \int_{Y_u}^{Y_i} Y^2 \frac{dX}{dY} \, dY$$

or $$V = - 2\pi y_o^{\,3} \int_{Y_u}^{Y_i} \frac{Y^5 \, d\, Y}{\left\{C\left(Y^2 - C_1\right)^{C_2} - Y^6\right\}^{1/2}} \qquad \text{(A.15)}$$

(A.15) can be devided as follows

$$V = \frac{2}{3} \pi \, Y_o^{\,3} \left\{C\left(Y^2 - C_1\right)^{C_2} - Y^6\right\}^{1/2} \Bigg|_{Y_u}^{Y_i}$$

$$- \frac{2}{3} \pi \, Y_o^{\,3} \, C \, C_2 \int_{Y_u}^{Y_i} \frac{\left(Y^2 - C_1\right)^{C_2 - 1} Y \, d\, Y}{\left\{C\left(Y^2 - C_1\right)^{C_2} - Y^6\right\}^{1/2}}$$

where the first term is zero, hence

$$V = - \frac{2}{3} \pi Y_o^{\,3} \, C \, C_2 \int_{Y_u}^{Y_i} \frac{\left(Y^2 - C_1\right)^{C_2 - 1} Y \, d\, Y}{\left\{C\left(Y^2 - C_1\right)^{C_2} - Y^6\right\}^{1/2}} \qquad \text{(A.16)}$$

In the <u>netting theory</u> is $C_1 = C_2 = 1$, hence

$$V = -\frac{2}{3}\pi y_o^{\,3}\, C \int_{Y_u}^{Y_i} \frac{Y\, d\,Y}{\left\{C\left(Y^2 - 1\right) - Y^6\right\}^{1/2}} \qquad \text{A.19)}$$

in which the argument of the square root can be replaced by a third degree expression

$$C\;(Y^2 - 1) - Y^6 = (Y_u^{\,2} - Y^2)\;(Y^2 - Y_i^{\,2})\;(Y^2 + Y_i^{\,2} + Y_u^{\,2})$$

With transformation (A.6) it can be derived from (A.17)

$$V = \frac{2}{3}\pi\; y_u^{\,3}\, \frac{q^2 + q + 1}{q^{3/2}\left(2q + 1\right)^{1/2}} \int_0^{\pi/2} \left(1 - \frac{q - 1}{2q + 1}\;\sin^2\Theta\right)^{-1/2} d\Theta \qquad \text{(A.18)}$$

where $y_u = \dfrac{(q^2 + q + 1)^{1/2}}{(q + 1)^{1/2}}\, y_o$

and the integral is the complete elliptical integral of the first kind.

The weight ΔW of a part of the pressure vessel of width dx and thickness transverse to the rotation axis $t/\cos\phi$ is

$$\Delta W = 2\pi y\;\gamma\;\frac{t}{\cos\phi}\;dx = 2\pi y_o^{\,2}\;\gamma\;\frac{t}{\cos\phi}\;\; Y\,\frac{dX}{dY}\;\;dY \qquad \text{(A.19)}$$

where γ is the specific weight of the composite.

With $\cos\phi = \left\{1 + \left(\dfrac{dY}{dX}\right)^2\right\}^{-1/2}$ equation (A.11) yields

$$t \,/\cos\phi = Y^3 \left\{1 + \left(\frac{dY}{dX}\right)^2\right\} \Big/ \left\{2\varepsilon E_\alpha \left(Y^2 - C_1\right) \Big/ p y_o \; C_2 \; C_3\right\}$$

where $1 + \left(\frac{dY}{dX}\right)^2 = C \, \frac{(Y^2 - C_1)^{C_2}}{Y^6}$

Hence the total weight is found by integration of (A.19)

$$W = - \, \gamma \, C \, C_2 \, C_3 \, 2\pi y_o^{\,3} \, p \Big/ \left(\varepsilon E_\alpha\right) \int_{Y_u}^{Y_i} \frac{\left(Y^2 - C_1\right)^{C_2 - 1} Y \, d \, Y}{\left\{C \left(Y^2 - C_1\right)^{C_2} - Y^6\right\}^{1/2}} \qquad \text{(A.20)}$$

(A.20) contains the same integral as expression (A.16) for the volume.

For the prevention of a discontinuity in tangential strain in the intersection between a cylindrical part and an isotensoid end - closure it is required that

$$\varepsilon_t = \varepsilon$$

where ε_t is the tangential strain in the cylindrical part and ε is the uniform strain in the end - closure.

From (6.1) it follows

$$\varepsilon_t = p y_u \; (C_{12_c} - 2C_{11_c}) \,/\, 2t_c \; (C_{12_c}^{\;2} - C_{11_c} \, C_{22_c})$$

Substituting this expression in (A.12) yields

$$(C_{12_c} - 2C_{11_c}) \,/\, t_c \; (C_{12_c}^{\;2} - C_{11_c} \, C_{22_c}) = C_2 \; C_3 \; Y_u^{\;2} \,/\, E_\alpha \; t_u \; (Y_u^{\;2} - C_1)$$

where $C_{ij_c} = (t_u \,/\, t_c) \; C_{ij} + (t_h \,/\, t_c) \; C_{ij}^{\;h}$

Hence

$$\frac{t_u C_{12} - 2 t_u C_{11} + t_h C_{12}^h - 2 t_h C_{11}^h}{(t_u C_{12} + t_h C_{12}^h)^2 - (t_u C_{11} + t_h C_{11}^h)(t_u C_{22} + t_h C_{22}^h)} =$$

$$C_2 C_3 Y_u^2 / E_\alpha t_u (Y_u^2 - C_1)$$

or

$$\frac{(t_h / t_u)(2 C_{11}^h - C_{12}^h) + (2 C_{11} - C_{12})}{(t_h / t_u)^2 (C_{11}^h C_{22}^h - C_{12}^{h\,2}) + (t_h / t_u)(C_{11} C_{22}^h + C_{11}^h C_{22} - 2C_{12} C_{12}^h) + (C_{11} C_{22} - C_{12}^2)}$$

$$= C_2 C_3 Y_u^2 / E_\alpha (Y_u^2 - C_1) \qquad (A.21)$$

(A.21) is a quadratic equation for (t_h / t_u).

With known (t_h / t_u) the thickness of the wall of the cylinder is

$$t_c = t_h + t_u = t_u \{1 + (t_h / t_u)\}$$

The thickness t_u follows from (A.12). With $\varepsilon E_\alpha = \bar{\varepsilon}_\beta E_\alpha = \sigma_w$ this formula yields

$$t_u = \frac{C_2 C_3 Y_u^3 py_o}{2 \sigma_w (Y_u^2 - C_1)}$$

So the weight of a cylindrical part with length ℓ is

$$W_{cyl} = 2\pi y_u t_c \ell \gamma = \frac{\pi y_u C_2 C_3 Y_u^3 py_o \ell \gamma}{\sigma_w \left(Y_u^2 - C_1\right)} \left\{1 + \left(t_h / t_u\right)\right\}$$

and the weight to volume ratio is

$$\frac{W_{cyl}}{V_{cyl}} = p\, C_2\, C_3\, \frac{\gamma}{\sigma_w}\, \frac{Y_u^{\,2}}{Y_u^{\,2} - C_1} \left\{1 + \left(t_h \middle/ t_u\right)\right\} \tag{A.22}$$

It is noted that $Y_u = y_u/y_o$ is the maximum dimensionless radius of the isotensoid end - closures. It must be calculated with (3.2) and the q-value of the isotensoids.

The total weight of a cylinder with end - closures is

$$W = p\, C_3\, \frac{\gamma}{\sigma_w} \left[3\, V + C_2\, \frac{Y_u^{\,2}}{Y_u^{\,2} - C_1} \left\{1 + \left(t_h \middle/ t_u\right)\right\} \pi Y_u^{\,2}\, \ell\right] \tag{A.23}$$

V is the volume of the isotensoids. It can be calculated (5.3) or approximated with (5.4).

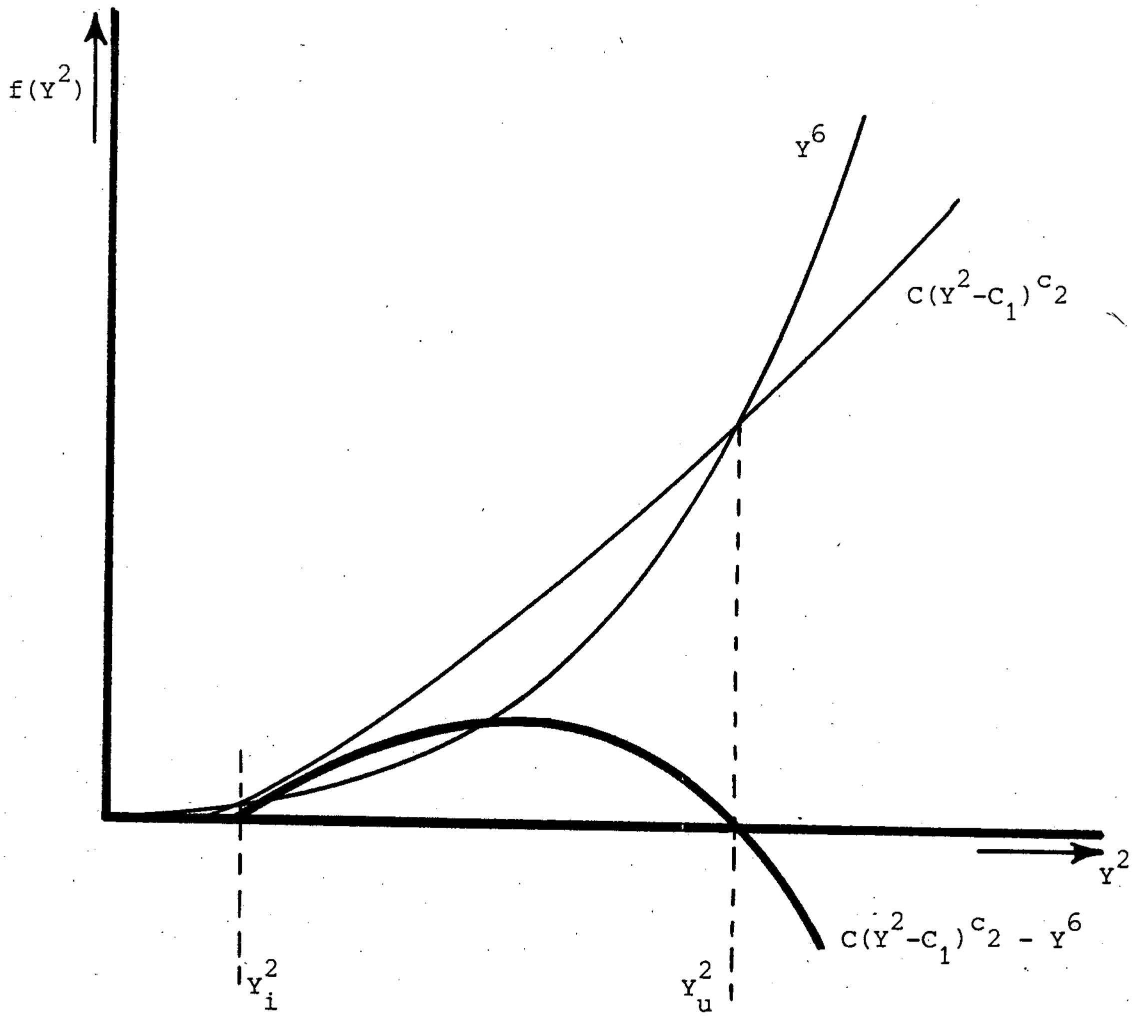

Fig. A.1: the values of Y^2 satisfying condition (A.5)

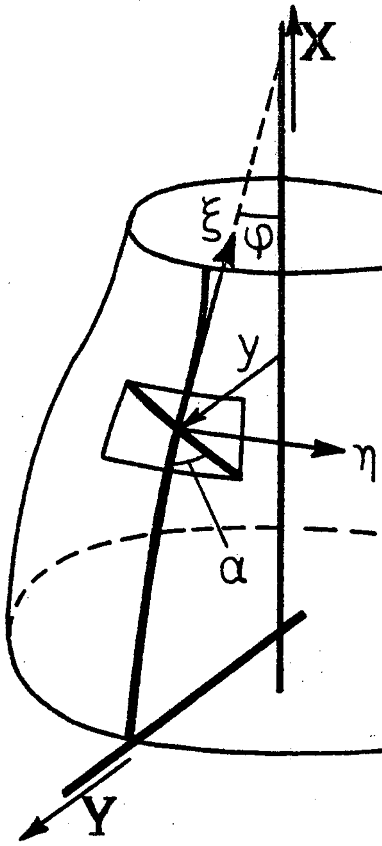

Figure 1. The geometry of a filament wound pressure vessel.

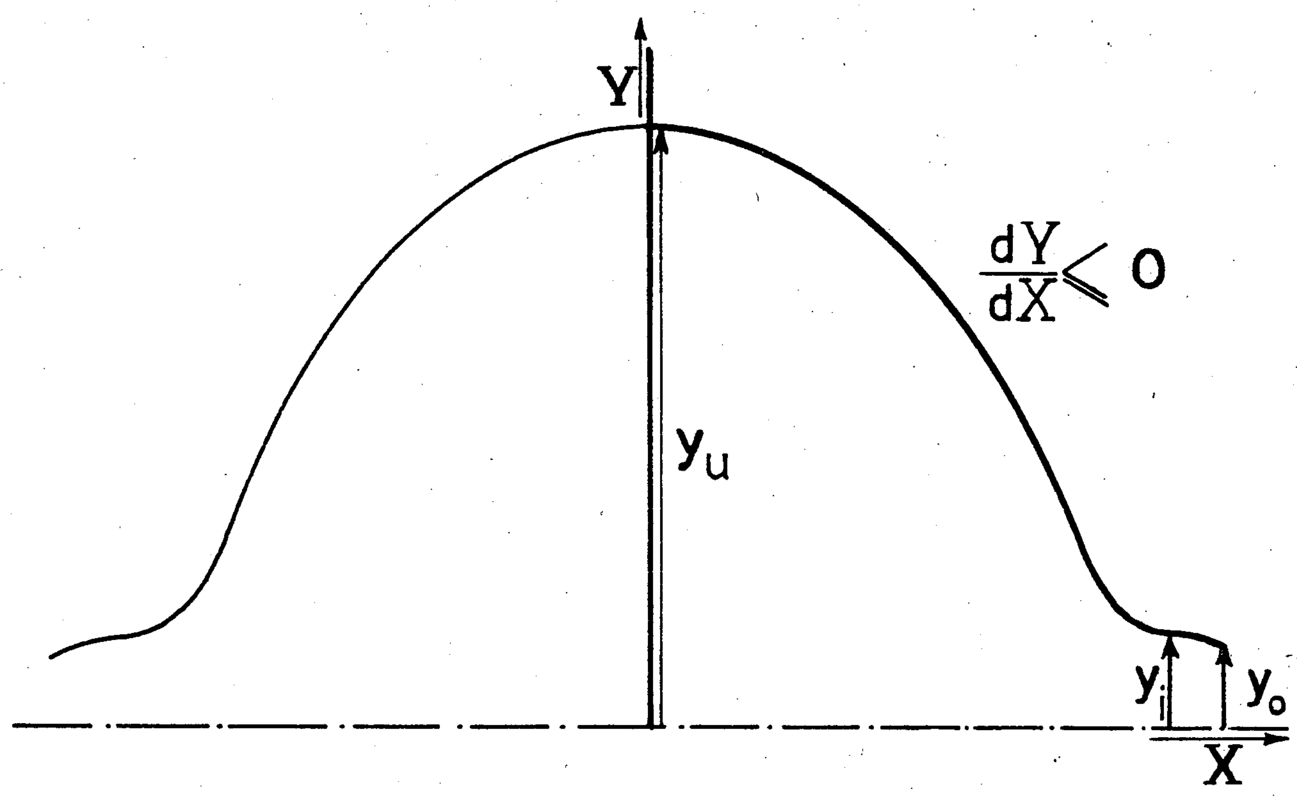

Figure 2. The general shape of an optimum pressure vessel.

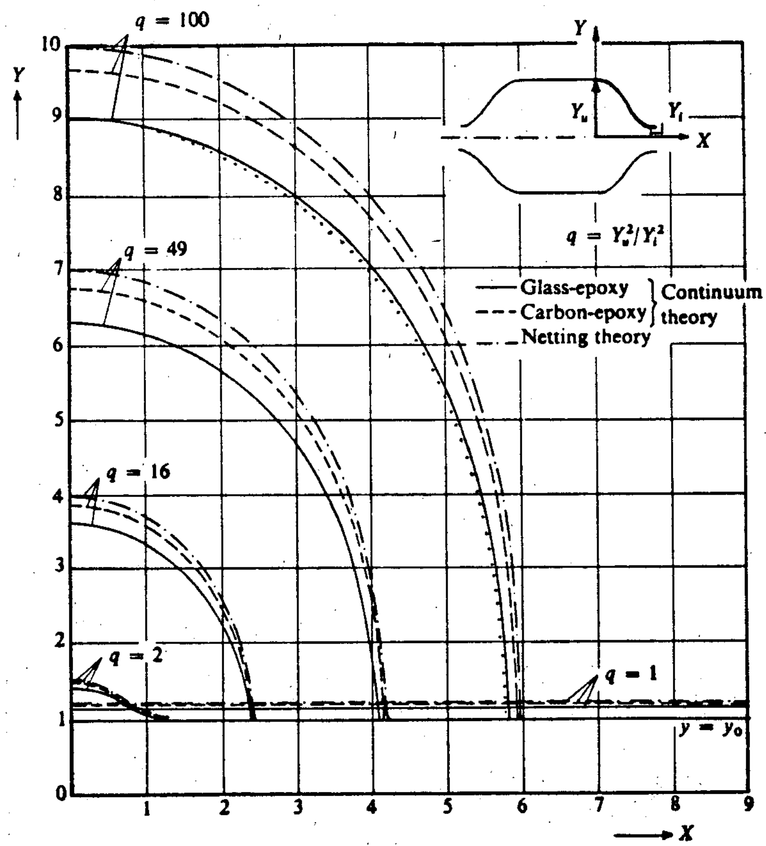

Figure 3. Isotensoids for a number of q-values. The dotted line is the approximation for glass epoxy.

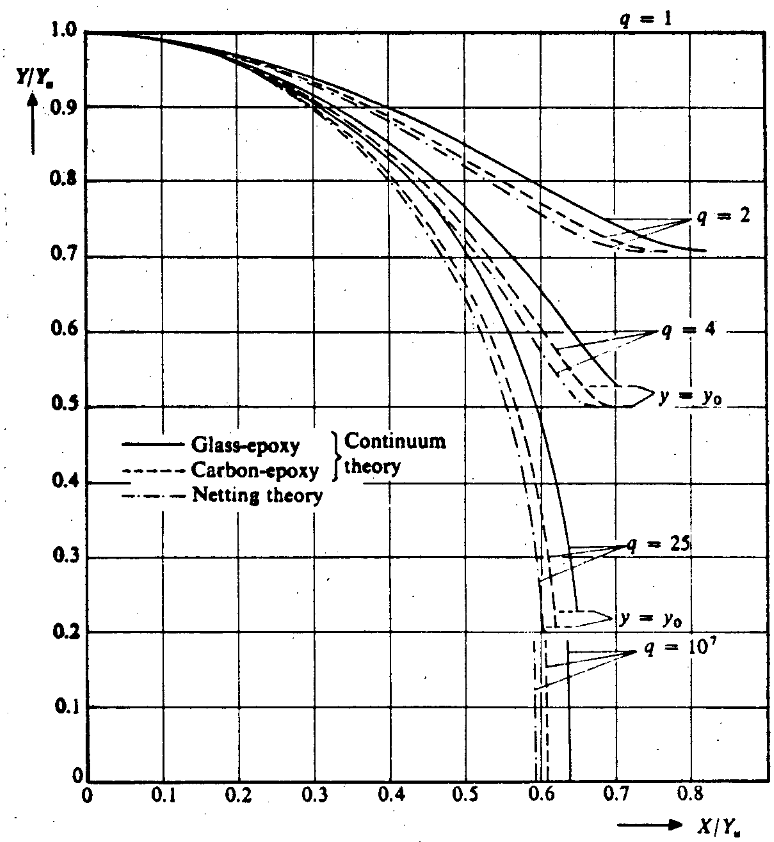

Figure 4. Isotensoids for a number of q-values in coordinates, normalized to the maximum radius Y_u.

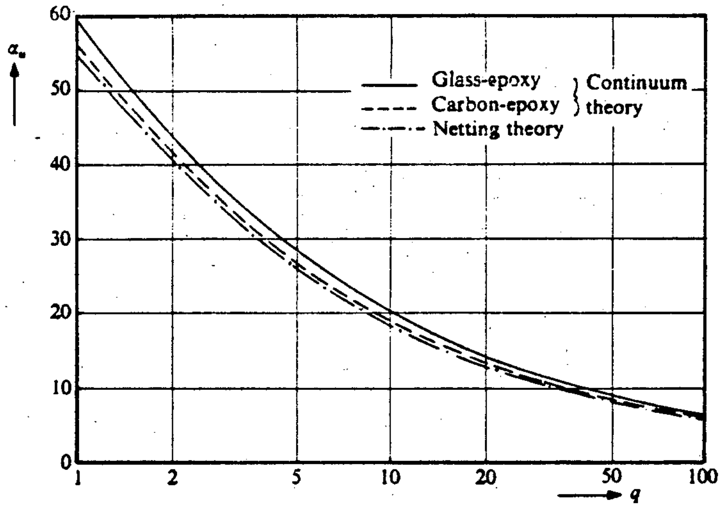

Figure 5. The winding angle α_u at the equator as a function of q.

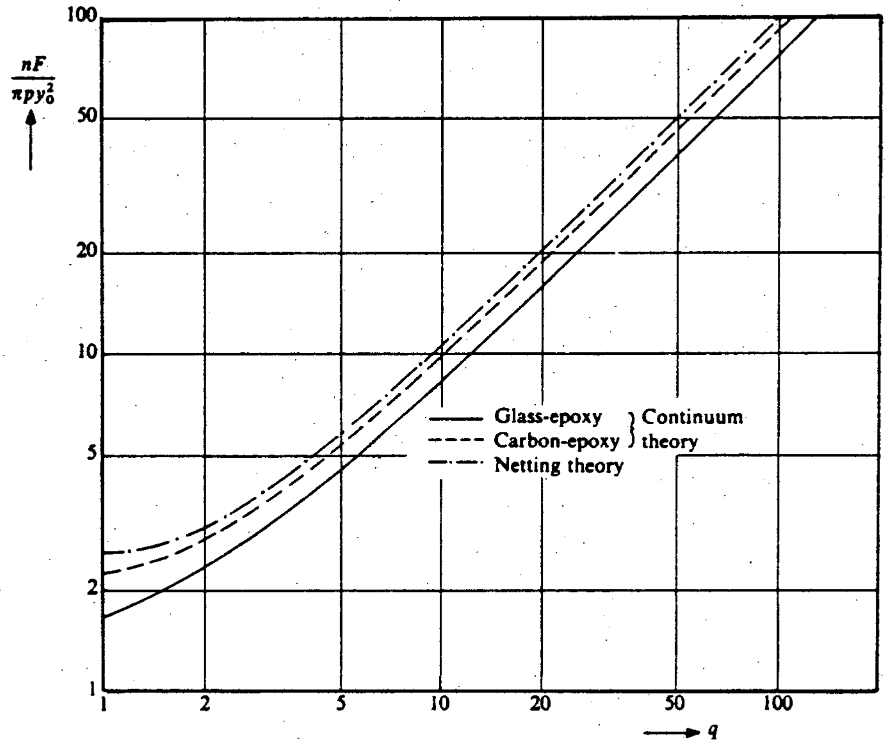

Figure 6. Ratio of the number of bundles of filaments of a given strength to the internal pressure as a function of q.

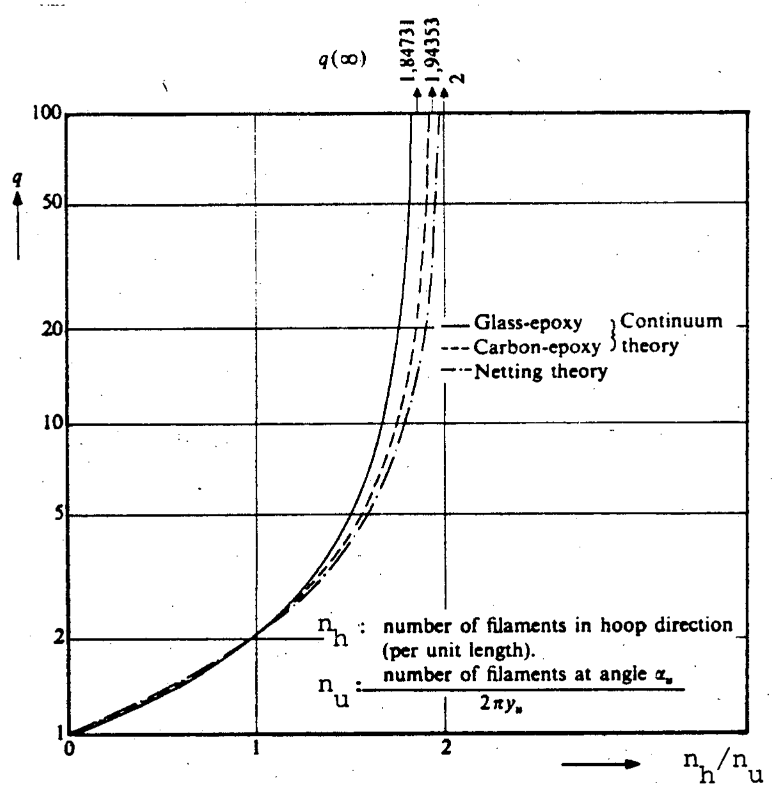

Figure 7. Ratio of the number of hoop filaments to the number of filaments at angle α_u for a cylindrical section as a function of the q-value of the end-closures.