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Max-plus-algebraic hybrid automata

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Dissertation

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by

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To define is to limit.

Oscar Wilde

Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two facilities, which we may call intuition and ingenuity.

Alan Turing

Contents

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This dissertation marks the culmination of over four years of a humbling PhD journey spelunking down the max-plus rabbit hole, and navigating the boundaries of (hybrid) work-life balance. I have been fortunate to be supported by numerous people in various capacities along this long and winding road interspersed with hope and despair.

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Abhimanyu Gupta
Delft, January 2023

Summary

Man-made systems, such as manufacturing and transportation networks, and their interactions with the environment are driven by human-designed operational rules. These rules are most often based on the asynchronous occurrence of discrete events over time, such as the arrival and departure of trains at a station. The modelling, analysis, and control of the system evolution over discrete events result in the discrete-event systems framework. Here, the dynamics are derived from two layers of behaviours: the logical ordering of event occurrences on the one hand, and the timing of events on the other. Automata are untimed finite-state sequential processing machines typically used to study the logical behaviour of the discrete-event system. Here, a state transition diagram encodes the allowed sequences of events, such as the order of successive trains departing a station, resulting in a variable (possibly non-deterministic) schedule of operation. Max-plus algebra, with maximisation and addition as its basic operations, (and associated algebraic structures) conveniently handle the timing aspects of discrete-event systems when the schedule of operation of different tasks, such as the order of trains, is made deterministic.

In this PhD thesis, we develop tools for system-theoretical analysis of discrete-event systems when purely (max-plus) algebraic models, derived from timing constraints among events, are enriched with automata-theoretic conflict resolution schemes to treat variable schedules. We follow the hybrid dynamical systems approach that offers a powerful description of the interplay between the logical and timing aspects of discrete-event systems. On the one hand, the resulting hybrid automata allow a continuous-variable dynamic representation of discrete-event systems analogously to time-driven systems. On the other hand, the framework is convenient when timing constraints are of explicit concern in system dynamics and performance specifications. We address issues related to the stability, reachability, and solvability of discrete-event systems in this PhD thesis.

Firstly, we focus on formalising the discrete-event modelling framework as a novel max- plus-algebraic hybrid automaton analogously to the hybrid automaton framework in conventional algebra. There are mainly two phenomena of concern: synchronisation and choice of event occurrences. We illustrate how the proposed framework offers explicit flexibility in modelling the interplay of synchronisation and choice phenomena among event occurrences. We show that the proposed framework unifies and extends the existing max-plus-algebraic models of discrete-event systems with the variable ordering of events. We derive equivalence relations between the proposed framework and other automata-theoretic models with timing features such as weighted automata.

Stability analysis plays an important role in the operation and control of dynamical systems. There has been considerable research on generalising the notions of stability from linear time-invariant systems to hybrid systems in conventional algebra. The research for the counterpart in max-plus-algebraic systems is still limited. This motivates us to study the stability of discrete-event systems in the second part of the thesis. We present a novel stability analysis framework under the broad setting of max-plus-algebraic hybrid automata. We achieve this by reformulating various notions of stability of discrete-event systems phrased in the classical Lyapunov sense. We then integrate tools from max-plus algebra and Lyapunov theory to demonstrate the decision-making capabilities of the proposed approach.

In the last part of the PhD thesis, we focus on the parametric modelling of constrained discrete-event systems. This allows capturing variations in the timing and ordering of event occurrences within the framework of max-plus-algebraic hybrid automata analogously to the conventional time-driven linear parameter-varying systems. The analysis of the effect of parameter variations on the existence of admissible trajectories is of paramount importance in model-based decision-making for discrete-event systems. Therefore, we focus on validating the coherence of the obtained model in presence of nonlinear implicitness in the system dynamics. In our analysis, we borrow tools from max-plus algebra, monotone functions theory, graph theory, and computational geometry. Finally, we study the application of the proposed approach to an urban railway system.

SAMENVATTING

De interactie tussen door mens gemaakte systemen, zoals productie- en transportnetwerken, en hun omgeving wordt aangestuurd door middel van door mens ontworpen regels. Deze regels zijn veelal gebaseerd op het asynchrone en discrete voorkomen van gebeurtenissen in de tijd, zoals de aankomst en vertrek van treinen op een station. Het modelleren, analyseren en regelen van een system over discrete gebeurtenissen resulteert in het discrete-event-systeemraamwerk. In dit raamwerk wordt de dynamica afgeleid van twee gedragslagen: enerzijds van de volgorde van gebeurtenissen en anderzijds van de timing van gebeurtenissen. Automata zijn niet-getimede sequentiële verwerkingsmachines met eindige toestandvariabelen die doorgaans worden gebruikt om het logische gedrag van systemen met discrete gebeurtenissen te bestuderen. Een toestandsovergangsdiagram codeert de toegestane opeenvolgingen van gebeurtenissen, zoals de volgorde van opeenvolgende treinen die een station verlaten. Dit resulteert in een variabel (mogelijk niet-deterministisch) werkingsschema. Max-plus-algebra, met maximalisatie en optelling als basisoperaties, (en bijbehorende algebraïsche structuren) is een handige methode voor het modelleren van de timingaspecten van discrete-event-systemen wanneer het werkschema van verschillende taken, zoals de volgorde van treinen, deterministisch wordt gemaakt.

In dit doctoraatsproefschrift ontwikkelen we methoden voor systeemtheoretische analyse van systemen met discrete gebeurtenissen wanneer zuiver (max-plus) algebraïsche modellen, afgeleid van timingbeperkingen tussen gebeurtenissen, worden verrijkt met automatisch-theoretische conflictoplossingsschema's om variabele schema's te behandelen. We volgen de hybride dynamische systeembenadering die een krachtige beschrijving biedt van het samenspel tussen de logische en timingaspecten van systemen met discrete gebeurtenissen. Enerzijds maken de resulterende hybride automata een continu variabele dynamische representatie mogelijk van systemen met discrete gebeurtenissen, analoog aan tijdgestuurde systemen. Aan de andere kant is het raamwerk handig wanneer timingsbeperkingen expliciet van belang zijn in de systeemdynamica en prestatiespecificaties. In dit proefschrift behandelen we kwesties die verband houden met de stabiliteit, bereikbaarheid en oplosbaarheid van systemen met discrete gebeurtenissen.

Ten eerste richten we ons op het formaliseren van het modelleringsraamwerk voor discrete gebeurtenissen als een nieuwe max-plus-algebraïsche hybride automaton, analoog aan het hybride automaatraamwerk in conventionele algebra. Hierbij zijn voornamelijk twee fenomenen van belang: synchronisatie en keuze van het voorkomen/ordenen van gebeurtenissen. We illustreren hoe het voorgestelde raamwerk expliciete flexibiliteit biedt bij het modelleren van het samenspel van synchronisatie- en keuzefenomenen tussen gebeurtenissen. We laten zien dat het voorgestelde raamwerk de bestaande max-plus-algebraïsche modellen van systemen met discrete gebeurtenissen verenigt en uitbreidt met de variabele ordening van gebeurtenissen. We leiden equivalentierelaties af tussen het voorgestelde raamwerk en andere automaat-theoretische modellen met timingkenmerken zoals gewogen automata.

Stabiliteitsanalyse speelt een belangrijke rol bij de werking en besturing van dynamische systemen. Veel onderzoek is gedaan naar het generaliseren van stabiliteitsbegrippen van lineaire tijdsinvariante systemen naar hybride systemen in conventionele algebra. Voor hybride system in max-plus-algebra is dit onderzoek nog beperkt. Dit is de motivatie voor het bestuderen van de stabiliteit van systemen met discrete gebeurtenissen in het tweede deel van dit proefschrift. We presenteren een nieuw raamwerk voor stabiliteitsanalyse onder de brede setting van max-plus-algebraïsche hybride automata. We bereiken dit door verschillende begrippen van stabiliteit van systemen met discrete gebeurtenissen te herformuleren in de klassieke Lyapunov-betekenis. Vervolgens integreren we methoden van max-plus algebra en de Lyapunov-theorie om de besluitvormingsmogelijkheden van de voorgestelde benadering te demonstreren.

In het laatste deel van het proefschrift richten we ons op het parametrische modelleren van discrete-event systemen onderhevig aan randvoorwaarden. Dit maakt het mogelijk variaties in de timing en volgorde van gebeurtenissen binnen het kader van max-plus-algebraïsche hybride automata vast te leggen, analoog aan de conventionele tijdgestuurde lineaire parametervariërende systemen. De analyse van het effect van parametervariaties op het bestaan van toelaatbare trajecten is van allergrootste belang bij modelgebaseerde besluitvorming voor systemen met discrete gebeurtenissen. Om deze reden richten we ons op het valideren van de coherentie van het verkregen model onder niet-lineaire impliciteit in de systeemdynamiek. In onze analyse gebruiken we methoden van max-plus algebra, theorie van monotone functies, grafentheorie en computationele meetkunde. Ten slotte bestuderen we de toepassing van de voorgestelde methode op een stedelijk spoorwegsysteem.

1

INTRODUCTION

In this chapter, we first give a brief overview of the discrete-event systems framework in Section 1.1. Then we highlight the role that the max-plus algebra plays in the modelling and analysis of discrete-event systems in Section 1.2. We point out the underlying assumptions and ensuing limitations of the existing tools arising from the max-plus algebra for the analysis of discrete-event systems. This, among other things, sheds light on the steps that need to be taken to better understand the behaviour of more complex discrete-event systems. The dissertation builds upon the hybrid systems framework of [216] for studying system-theoretic properties of discrete-event systems in the max-plus algebra. To this end, we formulate the objectives of this thesis exploiting the hybrid systems perspective to modelling and analysis of discrete-event systems in the max-plus algebra in Section 1.3. We also present the major contributions of this thesis in 1.3. In conclusion, we present an overview of the organisation of the subsequent chapters of the dissertation in Section 1.4.

1.1. DISCRETE-EVENT SYSTEMS

From a classical system-theoretic point of view, any dynamical system can be understood as a collection of its input-output (behaviour) trajectories defined along the time axis [221]. We are often interested in defining a model that specifies the input-output behaviour of the system by adjoining the concept of state to the system. Here, the state is an auxiliary variable that captures the internal configuration of the system by bifurcating the past and future of the input-output behaviour. Thus, the state formalises the concept of memory of the system [221]. The state space is then the set of all possible values of the state variable. The model obtained in this way allows for predicting the future behaviour of the system, as an aid in quantitative analysis, and for an adequate design of control methods to verify or optimise the behaviour of the system such that certain specifications are met [54, 135, 221]. The distinction is then made based on the flow of time (continuous vs discrete) and on the definition of the state space (continuous vs discrete) [42]. We speak of 'hybrid systems' when both continuous and discrete elements are present in the behaviour of the system. A hybrid system description is most relevant in cyber-physical systems [40, 156, 208] where natural phenomena interact with computer-based logic, for instance, via control. Herein, a part of the dynamics are represented using a set of differential or difference equations (progressing in time) as operation modes. Another part of the dynamics are represented by a discrete mechanism that switches the mode of operation in response to (internal or external) events.

Concept of event. In this dissertation, we are interested in system dynamics that are event-driven. An event-driven system, in contrast to a time-driven system, evolves with the occurrences of events causing instantaneous transitions in the state of the system. In general, the events occur asynchronously with the tick of a clock. The formalism of logical (or untimed) discrete-event systems is characterised by an event-driven dynamics and is usually defined on a discrete (potentially infinite) set of states [54]. The formalism is rather abstract as the states can be symbolic rather than numerical. Thus, such systems *cannot* generally be defined using models based on differential/difference equations. The discrete-event system modelling formalism finds applications in the study of man-made systems such as transportation networks, flexible manufacturing systems, telecommunication networks, and so on [17]. These man-made systems are characterised by the completion of tasks/activities (such as the assembly of a product) by co-operation between a finite number of entities (jobs or intermediate manufactured parts) to utilise a finite number of resources (processors or machines). The starting and finishing of different entities at different resources, for instance, constitute the set of events. Time intervals between such events are not necessarily identical. The controller design problem for (untimed) discrete-event systems then consists of determining the optimal routing and/or scheduling of activities of entities on resources.

Concept of time. So far we have discussed the logical behaviour of an (untimed) discrete-event system observed as a string of events over a discrete set of states. The formalism of timed discrete-event system explicitly incorporates the timing (or temporal) features on the occurrence of events. This is achieved by adding one or more clocks to the logical discrete-event system resulting in a timed discrete-event system. In this thesis, we assume that the event occurrences are instantaneous. Therefore, the timing information appears only as durations between the occurrence of events. This timing information is either deterministic, stochastic, or provided as non-deterministic intervals. Also, the clock variables can either take values in a continuous space or they can be synchronised and measured with a single global digital clock. The durations or incurred delays between event occurrences, for instance, are associated to the processing times of entities on different resources. Initially, the concept of time was introduced in the modelling of discrete-event systems for the purpose of performance evaluation (in terms of throughput and makespan) of the system [27, 190, 226]. In this dissertation, we are interested in situations where timing constraints are of explicit and primary importance in the modelling, performance specification, and control of discrete-event systems. For instance, sometimes the aim of the controller design problem is to determine when certain activities (events) should occur in time such that the completion of a task follows a due-date reference.

Models. There are many well-established modelling and analysis techniques for timed discrete-event systems in the literature [24, 26, 39, 54, 190]. The choice of modelling class typically trades off the modelling power against the decision power. In particular, the larger the class of systems that can be modelled under a framework, the less it is amenable to efficient analysis using analytical and mathematical tools. Moreover, timing adds another dimension to discrete-event systems modelling. In particular, if the clock variables are accounted for in the state of the system then it can lead to an infinite state space.

Timed Petri nets obtained by associating clock structures to (untimed) Petri nets form one of the largest and most powerful classes of timed discrete-event systems [37, 184, 190]. These models are typically defined on a hybrid state space and are therefore infinite state. The analysis and synthesis issues are typically studied for the state transition structure of the system utilising graph-theoretic tools. However, even for small systems the number of possible state transitions over a future event sequence can explode combinatorially. Complete analytic solutions are unavailable for a general class of timed Petri nets. Therefore, several problems (such as deadlock analysis that operates on the reachability graph) are rendered undecidable, in that there do not exist generally applicable algorithms to evaluate certain properties in finite time using a computer. In some cases, it is possible to obtain finite-state representations of timed discrete-event systems defined on an infinite state space. The resulting model classes include timed automata [12, 13], timed transition systems [25, 39], and state class graphs [25]. Such models lend themselves to language-theoretic analysis along the lines of Ramadge and Wonham [191, 192, 202].

In this dissertation, we are interested in modelling classes that arise from a continuous-valued event-driven dynamical system representation of timed discrete-event systems. Herein, the concept of time forms the basic aspect for deriving the model of the system. Thereby, the state-space consists of variables that can take values in a continuous space and the dynamics can be expressed algebraically as difference equations. On the one hand, there are special subclasses of timed Petri nets that have a continuous-valued recursive dynamical representation. This includes, but is not limited to, the class of timed-event graphs [17, §2.5] and timed Petri nets with fixed priority rules [16, 60]. On the other hand, there are subclasses of timed discrete-event systems whose dynamics can be represented algebraically using difference equations, that cannot be easily expressed using timed Petri nets [17, §9.6.1], [203, §7]. The obvious disadvantage is that the class of timed discrete-event systems that can be represented algebraically as difference equations is rather restricted. However, the algebraic description allows deriving efficient specialised analysis tools [17, 61].

For a more complete overview of the applications of continuous-valued representations of timed discrete-event systems in control and analysis, the interested reviewer is referred to [17, 59, 65, 85, 137, 138, 217] for manufacturing and queueing networks and [38, 119, 134, 217] for railway networks.

Synchronisation and concurrency. The common feature of timed discrete-event systems of interest, whose dynamics can be expressed as difference equations, is that their dynamics are governed by synchronisation and certain forms of concurrency. Synchronisation, a nonlinear and non-smooth phenomenon, refers to the requirement that several resources be available at the same time to process an entity/job. The first kind of concurrency occurs when an entity initiates multiple activities at different resources at the same time. Another form of concurrency appears when an entity has a choice of resources to visit at a certain time. Choice in the use of common resources requires making decisions to resolve conflict. For instance, a choice can be modelled as a competition among successor resources. Timed-event graphs form a subclass of timed Petri nets where all possible choices have been resolved beforehand. Timed-event graphs have a continuous state space and their dynamics can be expressed explicitly using difference equations in the temporal (clock) variables. In particular, the clock variable updates are governed by synchronisation effects and delays. Therefore, timed-event graphs can model synchronisation but not choice phenomena [17].

Caveat. As stated earlier, conventional hybrid systems arising from cyber-physical systems are characterised by an interaction of time-driven and event-driven dynamics. We note that discrete-event systems can also be obtained from hybrid (cyber-physical) systems by abstracting away information in both time and space. For instance, a high-level abstraction can be obtained by a discrete partition of the state space and attributing the events (such as passage of time) to transitions between the regions of the partition [14, 207]. The issues arising in analysis and control of such systems, mainly due to sampling of continuous time and state space, are conceptually different and out-of-scope of this thesis.

Finally, we note that all of the work recalled from the literature and presented in this dissertation focuses on timed discrete-event systems. For brevity of expression, we often leave out the term ‘timed’ when we speak of discrete-event systems.

1.2. ROLE OF THE MAX-PLUS ALGEBRA

The application of the max-plus algebra to control and performance evaluation of discrete-event systems has been a popular field of research for several decades [17, 61, 138]. The max-plus techniques also find applications to problems outside the field of discrete-event systems [88, 96].

The dynamics of a timed-event graph corresponds directly to a max-plus linear system [17, §2.5]. As mentioned earlier, the dynamics of a timed-event graph is governed by synchronisation. Therein, the starting time of an activity can be expressed as the *maximum* of a set of the earliest times when all resources are available to an entity. These availability times can, in turn, be expressed as the starting time of a preceding activity *plus* the processing time at the corresponding resource. Therefore, the description of timed discrete-event systems under synchronisation but not choice results in state-space models composed of linear expressions in the max-plus algebra [17].

In what follows, we discuss the tools arising from the max-plus algebra and the role they play in performance analysis and control of timed discrete-event systems governed by synchronisation but no choice.

Max-plus algebraic systems theory. There is a considerable amount of literature on system-theoretic linear max-algebraic notions and max-plus algebra tools in parallel with conventional linear algebra [17, 48, 61, 66, 69, 84]. In particular, the max-plus Perron-Frobenius theory allows for analytical study of the asymptotic behaviour of max-plus linear maps [17], [61, §3], and hence discrete-event systems modelled by timed-event graphs. The max-plus Perron-Frobenius theory has since been extended to non-linear expressions in the max-plus algebra (such as the dynamics described by min-max-plus linear functions) [108]. This aforementioned extension is based on the realisation that we only need the underlying assumptions of monotonicity and additive homogeneity on dynamics [89, 109]. The tools for performance evaluation (related to throughput analysis) have also been extended to the dynamics described by a set of max-plus linear maps along with a scheduling mechanism [36, 85, 216]. The extensions of the max-plus linear systems theory for analysis of a more general class of timed Petri nets are rather limited [16, 60, 94].

Graph theory. It is noteworthy that graph theory plays a key role in the control and analysis tools associated with max-plus linear systems theory. Firstly, several results from max-plus Perron-Frobenius theory rely on a graph-theoretical interpretation for efficient evaluation of spectral properties and transient behaviour of timed-event graphs [70, 119, 171]. Secondly, structural analysis tools for validating liveness, deadlock, and boundedness of the associated timed-event graph exploit tools from graph theory [63]. Lastly, graph theory allows a distributed approach to control and scheduling of discrete-event systems [133, 217]. The tools from graph theory associated with the max-plus algebra have also been extended to study the dynamic behaviour of other classes of Petri nets, such as P-time event graphs [205, 225]

Geometric systems theory. The control objective or the desired behaviour of a timed-discrete event system is often specified as a set of acceptable schedules of operations such as a set of sequences of event occurrence times [58]. Firstly, it is necessary to assess whether the desired behaviour can be attained from a given set of initial conditions via forward reachability analysis. Secondly, backward reachability analysis can be used to find the largest subset of initial conditions such that the desired specification is satisfied for all possible trajectories of the system. Max-plus geometry plays an important role in studying several reachability problems for timed-event graphs [61, §4], [90]. This involves the study of images and kernels of max-plus matrices as geometric objects, and operations on them [91]. This also forms the basis of the max-plus geometric control theory analogous to the classical geometric control theory of [223]. The theory of max-plus geometric objects is more involved than its conventional algebra counterpart. Nevertheless, several important results towards the study of controllability [187], observability [99], invariance-based control and estimation [80, 128, 162], and disturbance rejection [114, 148, 199] problems can be found in the literature. The thrust of the research efforts in this direction is towards providing elegant solutions to the algorithmic issues encountered in max-plus control and estimation problems at the cost of restrictive assumptions and hypotheses. The geometric max-plus theory employs tools from residuation theory, which provides analytic solutions to solve equations involving order-preserving (monotone) maps between partially ordered sets [61]. In particular, a solution to max-plus linear equations, if it exists, can be obtained using multiplication in the dual min-plus algebra. Residuation theory then finds applications in resource optimisation and stabilisability of timed-event graphs [61, §3].

Optimisation. The decision problems arising from algebraic and geometric systems-theoretic treatment of max-plus linear systems often reduce to finding the solution (spaces) to a system of equations and inequalities involving only maximisation, minimisation, and addition operations [48, 69, 128]. In the most general case, this can be reduced to solving mixed-integer linear programs [71]. However, developing tools for efficiently finding and analysing solutions to special cases of max-min-plus systems of equations and inequalities is an active area of research (see [48, 98] and references therein). In particular, solutions to (parametric) one-sided and (parametric) two-sided max-plus equations can be finitely generated as max-plus polyhedra, defined analogously to convex polyhedra, [7, 93, 97]. Most importantly, max-plus polyhedra are capable of encoding disjunctive (either-or) constraints among variables [9]. For example, the constraint $z = \max(x, y)$ encodes the linear constraints $x - z \leq 0$ and $y - z \leq 0$, along with the disjunctive information $z = x \vee z = y$. This also leads to closed-form expressions for control and estimation problems arising in timed discrete-event systems when the desired specification is formulated as a max-plus polyhedron [103, 104]. Finally, we note that the notion of (tropical) linear program and its parametric version have been proposed in the literature analogously to the conventional (parametric) linear program [93, 140, 141, 224]. The authors of [93, 140, 141, 224] have also proposed (pseudo) polynomial-time algorithms to solve optimisation problems involving maximisation of max-plus linear costs under max-plus linear constraints.

1.3. OBJECTIVE OF THE RESEARCH

Linear models (even in conventional algebra) form the simplest abstraction of dynamical systems. The classical systems theory has its roots in the analysis of linear systems in conventional algebra [135]. As the summary in the preceding section shows, the focus of much of the existing research towards applications in discrete-event systems has been on developing a mathematical framework for a coherent systems theory for max-plus linear systems that are analogous to the classical systems theory. We intend to develop a systems theory for classes of discrete-event systems obtained as extensions of max-plus linear systems by allowing the choice phenomenon and parametric uncertainties.

The modelling power of a continuous-variable representation of a discrete-event system (governed only by synchronisation but no choice) as a max-plus linear system is restrictive [17]. There has been much work towards extending the class of max-plus linear systems by allowing certain forms of concurrency [85, 214]. The models proposed in [85, 214], in particular, allow modelling a larger class of discrete-event systems by including a discrete-valued state to model the choice phenomenon. There have also been extensions towards modelling more involved constraints between the timing of occurrences of events, such as competition [203] and implicit dependence [127]. The authors of [127] propose an implicit system of difference equations in the max-plus algebra for modelling, control and, analysis of P-time event graphs.

Hybrid systems framework. The (common) objective of the different contributions of this thesis is to extend max-algebraic system-theoretic tools for analysis and performance evaluation to switching and uncertain systems in max-plus algebra. To this end, we demonstrate the relation between hybrid behaviours observed in discrete-event systems governed by both synchronisation and choice phenomena and cyber-physical systems. In particular, we rely heavily on the analysis framework for discrete hybrid systems of [209] to develop tools for performance evaluation (in terms of throughput and makespan) and trajectory computation of discrete-event systems in the max-plus algebra.

A hybrid-systems approach to control and analysis serves the following objectives. Firstly, the multi-modelling capability of the hybrid systems framework of [157] allows exploiting and translating the existing analysis and control methods to hybrid models of discrete-event systems in max-plus algebra. Secondly, the framework of [157] incorporates tools for addressing problems of stability, safety analysis, control, and state estimation for conventional time-driven switching systems [157, 158, 160, 172, 194]. We would like to develop a system-theoretic framework for max-plus discrete-event systems analogous to the hybrid systems framework of [157]. In particular, a hybrid-systems approach to problem solving exploits the modular structure of highly complex systems. This is done by modelling complex systems as a collection of simpler dynamical systems. It is then important to address the difficulties arising from the observed hybrid phenomena due to interactions of the constituent dynamical systems.

1.4. ORGANISATION OF THE DISSERTATION

We present an overview of the dissertation along with key contributions. The structure of this thesis and the connections between the different chapters can be found in Fig. 1.1. The contents of the dissertation are as self-contained as possible. Briefly, Chapter 2 presents the mathematical preliminaries for the rest of the thesis. Chapters 3-5 contain the core contributions of this dissertation. Each core chapter also briefly recalls the existing related work in the respective direction along with a detailed statement of contributions. We present the concluding remarks and suggestions for future work in Chapter 6.

Chapter 2: Mathematical background. We begin this thesis with a concise mathematical overview of the max-plus algebra and of the max-plus systems theory. In doing so, we also present the analogy between the max-plus formalism and the conventional algebra. We present tools from max-plus Perron-Frobenius theory and max-plus convex geometry before presenting the max-plus systems theory for analysing dynamical behaviour. We also highlight the various assumptions underlying the development of the existing analytical tools. Then we recall the stability theory for conventional time-driven systems. In particular, we recall the role Lyapunov functions play in the analysis of asymptotic stability, ultimate boundedness, and the invariance for systems defined on finite-dimensional normed spaces. We then focus our attention on discrete-time

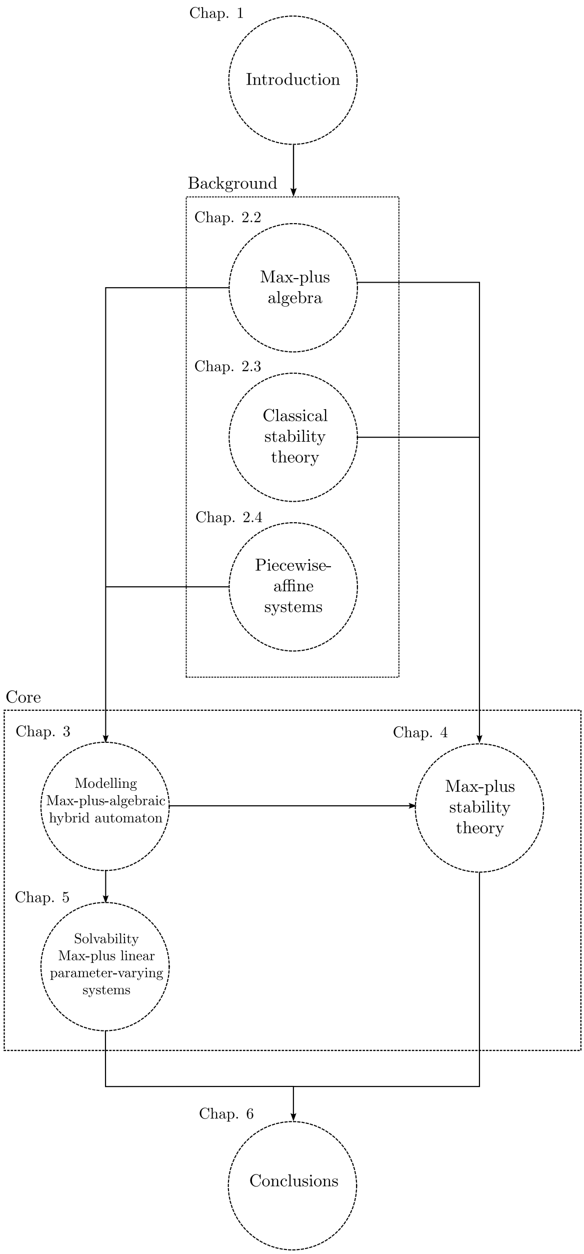


Figure 1.1: Overview of the dissertation.

hybrid systems defined by piecewise-affine systems. The notions and tools discussed for piecewise-affine systems form the fundamental basis for the development of several analysis tools proposed in this thesis. This is in accordance with the relation of the max-plus linear description with the piecewise-affine description of a dynamical system [5, 118, 216]. Finally, we discuss the tools of computational geometry that are necessary for reachability analysis and invariance set computations for discrete-time piecewise-affine systems.

Chapter 3: Modelling and equivalences. The first core chapter of the thesis presents the theoretical foundations for modelling discrete-event systems. We recall the classes of switching max-plus linear systems and max-plus automata from the literature. We identify the discrete phenomena arising in continuous-valued descriptions of discrete-event systems. This, in particular, allows us to form a modelling hierarchy for hybrid system descriptions of discrete-event systems. We introduce the novel class of max-plus-algebraic hybrid automata as a unified modelling language for systems arising from the extensions of max-plus linear systems. We formally present the equivalence relationships of the proposed max-plus-algebraic hybrid automata with the existing classes of switching max-plus linear systems and max-plus automata. In particular, the obtained results allow the comparison of timed and logical behaviours of switching max-plus linear systems and max-plus automata.

Chapter 4: Stability of max-plus-algebraic hybrid automata. In this chapter, we treat the problem of stability of the continuous-valued portion of a max-plus-algebraic hybrid automaton. We restrict ourselves to autonomous (internal) notions of stability for discrete-event systems. The main contribution of this chapter entails the extension of tools from Lyapunov stability, in conventional normed spaces, to systems defined on a special semi-normed space. Conventional Lyapunov theory studies the stability of dynamical systems using certain well-defined functions such as quadratic and piecewise-affine functions [31]. We present theorems and analysis tools for asymptotic stability, ultimate boundedness, and LaSalle-like invariance principle for discrete-event systems modelled by max-plus-algebraic hybrid automata. In this regard, we propose certain well-behaved classes of max-plus Lyapunov functions derived from monotone and additively homogeneous functions. Finally, we present an algorithmic perspective on the stability analysis of switching max-plus linear systems described by a set of max-plus matrices.

Chapter 5: Max-plus linear parameter-varying systems. In this chapter, we deal with the issues, related to the existence and uniqueness of trajectories, arising in implicit parametric descriptions of discrete-event systems in max-plus algebra. We introduce a general class of max-plus linear parameter-varying systems analogously to the linear parameter-varying system descriptions in conventional time-driven systems theory. The class of max-min-plus-scaling systems forms a nonlinear extension of the dynamical system descriptions in the max-plus algebra [75]. We formally establish the equivalence relationship between the proposed max-plus linear parameter-varying systems and the class of max-min-plus-scaling systems and hence continuous piecewise-affine systems.

On the one hand, a (quasi-)linear description in max-plus algebra allows us to provide necessary and sufficient conditions for the existence and uniqueness of trajectories. On the other hand, the proposed equivalence relationship allows us to exploit tools from piecewise-affine analysis to compute the invariant regions of the state space where a (unique) state solution always exists to the max-plus linear parameter-varying system at each event step. Finally, we present an intuitive case study involving a unidirectional urban railway system. This provides an application of the presented sufficient conditions to ascertain the existence and uniqueness of solutions.

Chapter 6: Conclusions & future work. Some concluding remarks and suggestions for future work can be found in the final chapter.

2

Mathematical background

In this chapter, we introduce the notations and preliminaries pertaining to the max-plus algebra, conventional stability theory, and piecewise-affine analysis tools relevant to the rest of the dissertation.

2.1. Introduction

This chapter is organised as follows. We begin with some preliminary mathematical notations common to all sections of this chapter. Section 2.2.1 recalls the basic notion of max-plus algebra (and associated idempotent semirings), and the constituent operations and conventions. Section 2.2.2 discusses max-plus spectral theory and the associated graph-theoretical tools. Section 2.2.3 summarises the notions of distances and geometrical objects defined max-plus algebra analogously to the conventional algebra. Section 2.2.4 lays out the systems theory for the qualitative characterisation of the dynamical behaviour of additively homogeneous monotone functions using max-plus-algebraic tools. Section 2.2.4 also details procedures for finding solutions of certain max-plus equations. Section 2.3 mainly recalls conventional Lyapunov stability theory and its extensions for dynamics defined on normed spaces. Several max-plus-algebraic objects can be studied using polyhedral methods. Therefore, Section 2.4 presents the necessary definitions pertaining to piecewise-affine systems from the literature. In particular, Section 2.4 recalls an algorithm for the computation of invariants for a piecewise-affine dynamics from the literature. The chapter ends with concluding remarks in Section 2.5.

2.1.1. Basic mathematical notations

The sets of all non-negative integers and positive integers are denoted as $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \mathbb{N} \setminus \{0\}$, respectively. The set of all non-negative integers up to n is denoted as $[n] = \{l \in \mathbb{N}_0 \mid l \le n\}$. The set of positive integers up to n is denoted as $\underline{n} = [n] \setminus \{0\}$. The set of all non-negative real numbers is denoted as $\mathbb{R}_+ = \{l \in \mathbb{R} \mid l \ge 0\}$. The vectors of length $n \in \mathbb{N}$ consisting of only zeros and ones are denoted as $\mathbf{0}_n$ and $\mathbf{1}_n$, respectively. The set of unit vectors $\{e_{\{1\}}, e_{\{2\}}, \ldots, e_{\{n\}}\}$ denotes the standard basis of $\mathbb{R}^n$.

We speak of Euclidean topology as the topology induced on the Euclidean space $\mathbb{R}^n$ by the Euclidean metric, $d(x,y) = \sqrt{\sum_{i\in\underline{n}}\left(x_i^2 - y_i^2\right)}$ for $x, y \in \mathbb{R}^n$.

Let P be a finite set. Then $|P|$, 2^P, and P^* denote the cardinality, power set (set of all subsets), and set of non-empty finite sequences of elements from P, respectively. A non-empty finite set of symbols is referred to as an alphabet.

2.2. MAX-PLUS ALGEBRA

This section presents some important notions in max-plus algebra and monotone function theory. For a comprehensive overview of max-plus algebra and its applications, an interested reader is referred to [17, 66].

2.2.1. NOTATIONS AND TERMINOLOGY

The section is based entirely on [17, 75, 115, 119, 178, 180].

Definition 2.2.1 (Semigroup)**.** Let $\mathscr{R}$ be a set and $\diamond$ a binary operation defined on the elements of $\mathscr{R}$. Then $(\mathscr{R},\diamond)$ is a semigroup if $\diamond$ is associative, i.e. $a\diamond(b\diamond c) = (a\diamond b)\diamond c$ for all $a, b, c \in \mathscr{R}$. □

An element $\mathscr{I}_\mathrm{R} \in \mathscr{R}$ is an *identity* if for all $a \in R$, $\mathscr{I}_\mathrm{R}\diamond a = a\diamond\mathscr{I}_\mathrm{R} = a$. An element $0_\mathrm{R} \in R$ is *zero* if for all $a \in \mathscr{R}$, $0_\mathrm{R}\diamond a = a\diamond 0_\mathrm{R} = 0_\mathrm{R}$.

We sometimes refer to $\mathscr{R}$ as a semigroup instead of $(\mathscr{R},\diamond)$ if the binary operation is clear from the context. The semigroup $(\mathscr{R},\diamond)$ is called *commutative* if $a\diamond b = b\diamond a$ for all $a, b \in \mathscr{R}$.

We speak of the conventional algebra as the field of real numbers: set $\mathbb{R}$ equipped with addition $(+)$ and multiplication[1] $(\times)$. In the following, we present the concept of max-plus algebra as an abstract generalisation of the conventional algebraic structure.

Definition 2.2.2 (Max-plus algebra)**.** The max-plus algebra is defined as $\mathbb{R}_{\max} = (\mathbb{R}_\varepsilon, \oplus, \otimes)$, and consists of the set $\mathbb{R}_\varepsilon = \mathbb{R}\cup\{-\infty\}$ endowed with the max-plus addition $(\oplus)$ and the multiplication $(\otimes)$ operations:

$$a \oplus b = \max(a, b)$$
$$a \otimes b = a + b.$$

The zero element is denoted as $\varepsilon = -\infty$ and the unit element as $\mathbb{1} = 0$. The max-plus algebra is a *tropical semiring* and satisfies the following axioms:

- The set $(\mathbb{R}_\varepsilon, \oplus)$ forms a commutative semigroup with ε as identity;
- The set $(\mathbb{R}_\varepsilon, \otimes)$ forms a semigroup with $\mathbb{1}$ as identity;
- Max-plus multiplication distributes over max-plus addition, i.e. $a\otimes(b\oplus c) = (a\otimes b)\oplus(a\otimes c)$ for all $a, b, c \in \mathbb{R}_\varepsilon$;
- The element ε is absorbing for $\otimes$. □

[1]We sometimes denote the conventional multiplication as $\cdot$.

Additionally, max-plus addition is idempotent, i.e. it lacks an additive inverse: $a \oplus b = \varepsilon \Rightarrow a = b = \varepsilon$.

There are several max-plus analogues to classical algebraic definitions. For instance, the set of n-dimensional vectors and $m \times n$ matrices can be defined as $\mathbb{R}_\varepsilon^n$ and $\mathbb{R}_\varepsilon^{m\times n}$, respectively. The partial order $\leq$ is defined such that for vectors $x, y \in \mathbb{R}_\varepsilon^n$, $x \leq y \Leftrightarrow x \oplus y = y \Leftrightarrow x_i \leq y_i$, $\forall i \in \underline{n}$.

A max-plus zero matrix is denoted as $\mathcal{E}_{n\times n}$. A max-plus identity matrix $\mathcal{I}_n^\otimes$ is defined as: $[\mathcal{I}_n^\otimes]_{ii} = 0$ for all $i \in \underline{n}$ and $[\mathcal{I}_n^\otimes]_{ij} = \varepsilon$ for all $i, j \in \underline{n}$ with $i \neq j$. A max-plus permutation matrix is obtained by permuting the rows and columns of a max-plus identity matrix.

The max-plus powers of a matrix are defined recursively as $A^{\otimes^{k+1}} = A^{\otimes^k} \otimes A$ for $k \in \mathbb{N}$ with $A^{\otimes^0} = \mathcal{I}_n^\otimes$. For scalars $\gamma, c \in \mathbb{R}$, we have $\gamma^{\otimes^c} = c \cdot \gamma$.

The (i, j)-th element of a matrix $A \in \mathbb{R}_\varepsilon^{m\times n}$ is denoted as $[A]_{ij} = a_{ij}$ and the i-th element of a vector $x \in \mathbb{R}_\varepsilon^n$ is denoted as x_i. The vector and matrix operations can also be defined analogously to the conventional algebra. Let $A, B \in \mathbb{R}_\varepsilon^{m\times n}$, $C \in \mathbb{R}_\varepsilon^{n\times p}$ be matrices in the max-plus algebra; then

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} = \max\left(a_{ij}, b_{ij}\right)$$
$$[A \otimes C]_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} = \max_k\left(a_{ik} + c_{kj}\right)$$

The matrix A normalised by a scalar $\mu \in \mathbb{R}$ is denoted as $[A_\mu]_{ij} = [A]_{ij} - \mu$.

Definition 2.2.3 (Regular (row-finite) matrix)**.** A matrix $A \in \mathbb{R}_\varepsilon^{m\times n}$ is called *regular* if it has one finite element (different from ε) in each row. □

We denote the set of square regular max-plus matrices of dimension $n \in \mathbb{N}$ as $\mathcal{M}^{(n\times n)}(\mathbb{R}_\varepsilon)$.

Definition 2.2.4 (Max-plus matrix semigroup, [107])**.** A set of regular square matrices in the max-plus algebra of dimension n, denoted as $\mathcal{A} \subseteq \mathcal{M}^{n\times n}(\mathbb{R}_\varepsilon)$, forms a max-plus multiplicative semigroup $(\mathcal{A}, \otimes)$:

$$\Psi(\mathcal{A}) := \left\{A^{(i_1)} \otimes \cdots \otimes A^{(i_k)} \mid A^{(i_j)} \in \mathcal{A},\ j \in \underline{k},\ k \in \mathbb{N}\right\} \qquad \square$$

The max-plus convex hull of $m \in \mathbb{N}$ matrices in $\mathcal{A}$ is defined as

$$\mathrm{conv}_\otimes(\mathcal{A}) = \left\{\bigoplus_{j=1}^{m} \alpha_j \otimes A^{(j)} \,\middle|\, j \in \underline{m},\ A^{(j)} \in \mathcal{A},\ \alpha_j \in \mathbb{R}_\varepsilon,\ \bigoplus_{j=1}^{m} \alpha_j = \mathbb{1}\right\}. \tag{2.1}$$

The identities based on Kronecker products and vectorisation can also be extended to max-plus algebra.

Definition 2.2.5 (Max-plus Kronecker product, cf. [116])**.** Let $A \in \mathbb{R}_\varepsilon^{m\times n}$ and $B \in \mathbb{R}_\varepsilon^{r\times s}$. The max-plus Kronecker product of A and B is defined as

$$A \boxtimes B = \begin{pmatrix} b_{11} \otimes A & b_{12} \otimes A & \cdots & b_{1s} \otimes A \\ b_{21} \otimes A & b_{22} \otimes A & \cdots & b_{2s} \otimes A \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} \otimes A & b_{r2} \otimes A & \cdots & b_{rs} \otimes A \end{pmatrix} \in \mathbb{R}^{mr\times rs}. \tag{2.2}$$

□

The following max-plus Kronecker product identities follow analogously from conventional algebra[2] [115]:

$$(A \boxtimes B) \otimes (C \boxtimes D) = (A \otimes C) \boxtimes (B \otimes D), \tag{2.3}$$

$$(A \boxtimes B)^\top = A^\top \boxtimes B^\top, \tag{2.4}$$

$$\mathcal{I}_m^\otimes \boxtimes \mathcal{I}_n^\otimes = \mathcal{I}_n^\otimes \boxtimes \mathcal{I}_m^\otimes = \mathcal{I}_{m\cdot n}^\otimes. \tag{2.5}$$

The (column-wise) vectorisation of a matrix $[B]_{ij} = b_{ij}$,

$$\mathrm{vec}(B) = \begin{pmatrix} b_{11} & b_{21} & \cdots & b_{12} & b_{21} & \cdots \end{pmatrix}^\top, \tag{2.6}$$

is compatible with the max-plus Kronecker product. The following identities can also be derived analogously for matrices A, B, and C of dimensions $k \times l$, $l \times m$, and $m \times n$ in $\mathbb{R}_\varepsilon$, respectively, and $\tilde{c} = \mathrm{vec}(C)$:

$$\mathrm{vec}(A \otimes B \otimes C) = (\mathcal{I}_n^\otimes \boxtimes A \otimes B) \otimes \mathrm{vec}(C), \tag{2.7}$$

$$\begin{aligned} \mathrm{vec}(A \otimes B) &= (\mathcal{I}_m^\otimes \boxtimes A) \otimes \mathrm{vec}(B) \\ &= (B^\top \boxtimes \mathcal{I}_k^\otimes) \otimes \mathrm{vec}(A), \end{aligned} \tag{2.8}$$

$$C = \mathrm{vec}^{-1}(\tilde{c}) = \left(\mathrm{vec}(\mathcal{I}_n^\otimes)^\top \boxtimes \mathcal{I}_m^\otimes\right) \otimes \left(\mathcal{I}_n^\otimes \boxtimes \tilde{c}\right). \tag{2.9}$$

As mentioned earlier, the max-plus algebra is a special case of a semiring. In the following, we recall other (tropical) semirings important for the scope of this dissertation.

Definition 2.2.6 (Min-plus algebra)**.** The *min-plus algebra*, $\mathbb{R}_{\min} = (\mathbb{R}_\top, \oplus', \otimes')$, defined as a dual of the max-plus algebra acting on the set $\mathbb{R}_\top = \mathbb{R} \cup \{+\infty\}$, is also a tropical semiring. The zero element is $\top = +\infty$ and the unit element is $\mathbb{1}$. The vector and matrix operations are defined analogously as in the max-plus algebra. □

The dual min-plus conjugate of a max-plus vector $x \in \mathbb{R}_\varepsilon^n$ is denoted as $x^- = (-x)^\top \in \mathbb{R}_\top^n$. The relation between max-plus and min-plus algebra is then apparent: $a \oplus b = (a^- \oplus' b^-)^-$ for $a, b \in \overline{\mathbb{R}}_\varepsilon^n = \mathbb{R}_\varepsilon^n \cup \{+\infty\}$.

We then speak of $(\overline{\mathbb{R}}_\varepsilon, \oplus, \otimes)$ as the[3] *completed max-plus algebra*. The set of all vectors in $\overline{\mathbb{R}}_\varepsilon^n$ with at least one finite entry is denoted as $\overline{\mathbb{R}}_\varepsilon^n \setminus \{\varepsilon, \top\}^n$.

Remark. We attribute the terminology *max-plus-algebraic* to the set of elements $\overline{\mathbb{R}}_\varepsilon$. The term max-plus-algebraic indicates that the operations and elements of max-plus algebra take precedence over min-plus and conventional operations and elements. In doing so, we adopt the following conventions:

- $\varepsilon + \top = \top + \varepsilon = \varepsilon$,
- $a \otimes \top = \top \otimes a = \top$ if $a \in \mathbb{R}_\top$,

[2] Assuming that the products $A \otimes C$ and $B \otimes D$ are compatible.

[3] A semiring is said to be complete if it is complete as an ordered set, i.e. any arbitrary subset attains a least upper bound, and the product distributes over infinite sums.

- $\varepsilon^{\otimes^a} = \top$ if $a < 0$, $\varepsilon^{\otimes^a} = \varepsilon$ if $a > 0$, $\top^{\otimes^a} = \top$ if $a > 0$, and $\top^{\otimes^a} = \varepsilon$ if $a < 0$,
- $\varepsilon^{\otimes^0} = \top^{\otimes^0} = 0$.

Definition 2.2.7 (Max-plus Boolean algebra)**.** The max-plus Boolean algebra defined as

$$\mathbb{B}_{\max} = (\mathbb{B}_\varepsilon, \oplus, \otimes), \quad \mathbb{B}_\varepsilon = \{\varepsilon, \mathbb{1}\}.$$

It is isomorphic to the Boolean algebra $\mathbb{B} = (\{\text{false}, \text{true}\}, \text{or}, \text{and})$. □

2.2.2. MAX-PLUS SPECTRAL THEORY

The max-plus spectral theory bears remarkable analogy with the Perron-Frobenius theory for non-negative matrices. The interested reader is referred to [89] for an exhaustive description. Here, we recall certain concepts from the literature pertaining to the dissertation. We begin with graph theory and then describe its connection with max-plus eigenvalue problem.

Definition 2.2.8 (Precedence graph of a max-plus matrix)**.** The directed graph of a matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$, denoted $\mathcal{G}(A) = (N(A), E(A))$, is a directed graph with labelled edges on a vertex set $N(A) = \underline{n}$ and an edge set $E(A) \subseteq n \times n$. Here, an edge $(i, j) \in E(A)$ whenever $[A]_{ji}$ is finite.

A path from a node i_1 to i_k, $k \in \mathbb{N}$, on the graph $\mathcal{G}(A)$ is a directed chain of edges $\omega = ((i_1, i_2), \ldots, (i_{k-1}, i_k))$, such that $(i_l, i_{l+1}) \in E(A)$ for all $l \in \{1, \ldots, k-1\}$. A path is a circuit if $i_k = i_1$. The circuit is elementary if all nodes $i_1, \ldots, i_{k-1}$ are distinct. □

The weight of a path $\omega = ((i_1, i_2), \ldots, (i_{k-1}, i_k))$ is the max-plus product of the edge labels on the path:

$$|\omega|_\text{w} = \sum_{l=1}^{k-1} [A]_{i_{l+1} i_l}.$$

The length of the path, the number of edges in the path, is denoted as $|\omega|_\text{l} = k - 1$. The average weight of the path ω is the ratio $|\omega|_\text{w}/|\omega|_\text{l}$. If ω is a circuit, then we speak of *circuit mean* for the average weight of the circuit. The maximal circuit mean is then the maximal average weight over all the circuits of the graph $\mathcal{G}(A)$. By convention, the weight of an empty path is defined as max-plus unity, $|\omega|_\text{w} = 0$ if $|\omega|_\text{l} = 0$. Vertices $i, j \in N(A)$ are said to *communicate* with each other if either $i = j$ or there exists a path from i to j and a path from j to i.

A circuit σ of $\mathcal{G}(A)$ is said to be *critical* if its circuit mean is maximal. We define the set of all nodes on the critical circuit(s) as the *critical nodes*, $N_\text{c}(A)$. Similarly, the set of all labelled edges belonging to critical circuit(s) are denoted as $E_\text{c}(A)$. The resulting critical (sub-)graph of $\mathcal{G}(A)$ is denoted as $\mathcal{G}_\text{c}(A) = (N_\text{c}(A), E_\text{c}(A))$. If $i, j \in N_\text{c}(A)$ belong to the same critical cycle then they are equivalent, denoted as $i \sim j$.

Max-plus algebraic operations on matrices have corresponding graph-theoretic interpretations. For instance, each (j, i)-th element of the s-th power of a square matrix, $[A^{\otimes^s}]_{ji}$, corresponds to the maximum weight over all paths from i to j of length $s \in \mathbb{N}$. Similarly, the element $[C]_{ij}$ of $C = A \oplus B$ corresponds to the maximum of the weights from i to j over the corresponding precedence graphs of the max-plus summands.

The following concept is important to draw an analogy between Perron-Frobenius theory of max-plus and non-negative matrices.

Definition 2.2.9 (Irreducible matrices)**.** A matrix $A \in \mathbb{R}_{\varepsilon}^{n\times n}$ is said to be *irreducible* if $\mathcal{G}(A)$ is strongly connected i.e., for each $i, j \in N(A)$, there is a path that starts in i and ends in j. Algebraically, the matrix A is irreducible if

$$[\Gamma]_{ij} = [A \oplus A^{\otimes^2} \oplus \cdots \oplus A^{\otimes^{n-1}}]_{ij} \neq \varepsilon, \quad \forall i, j \in \underline{n},\ i \neq j. \qquad \square$$

Note that the element $[\Gamma]_{ij}$ is finite only if there exists at least a path from j to i of length up to $n-1$. As the shortest path between any two connected vertices are of length $n-1$ or smaller, the paths of length n or greater need not be considered in the preceding definition.

If a matrix A is not irreducible, it is possible to partition the set of nodes $N(A)$ to form subgraphs $\mathcal{G}_t = (N_t(A), E_t(A))$, $t \in \underline{r}$ with $r \in \underline{n}$, of $\mathcal{G}(A)$. Here, $E_t(A)$ is defined as the subset of edges that both begin and end in vertices contained in $N_t(A)$. Moreover, no two vertices contained in disjoint partitions communicate with each other. If $N_t(A)$ is not empty then $\mathcal{G}_t(A)$ is said to be a strongly connected subgraph of $\mathcal{G}(A)$.

Definition 2.2.10 (Frobenius normal form, [119])**.** Let $A \in \mathbb{R}_{\varepsilon}^{n\times n}$ be a reducible matrix. Then it can be transformed into the Frobenius normal form by a suitable max-plus permutation matrix:

$$P \otimes A \otimes P^{\otimes^{-1}} = \tilde{A} = \begin{pmatrix} A_{11} & \varepsilon & \dots & \varepsilon \\ A_{21} & A_{22} & \dots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \dots & A_{rr} \end{pmatrix} \tag{2.10}$$

where $A_{11}, \dots, A_{rr}$ are irreducible submatrices of $\tilde{A}$, and pertain to the strongly connected subgraphs of $\mathcal{G}(A)$.

A reducible matrix A can be brought to a *lower-triangular form* if any strongly connected components in $\mathcal{G}(A)$ is a self-loop. In particular, the diagonal matrix blocks $A_{11}, \dots, A_{rr}$ in the Frobenius normal form $\tilde{A}$ are of unit dimension. $\square$

The partition of the subset of vertices (classes) $N(A)$, corresponding to the Frobenius normal form, is denoted as $N_1, \dots, N_r$. An arc from a vertex in N_i to a vertex in N_j, denoted $N_i \to N_j$, exists only if $i \leq j$. The classes of A with no incoming arc are called the *initial classes* and those with no outgoing arcs are called the *final classes*.

The max-plus eigenvalue problem is defined analogously to the conventional algebra albeit using max-plus operations.

Definition 2.2.11 (Max-plus eigenspace, [66])**.** The max-plus eigenspace of a matrix $A \in \mathbb{R}_{\varepsilon}^{n\times n}$ corresponding to a max-plus eigenvalue $\lambda \in \mathbb{R}$ is defined as :

$$\operatorname{eig}(A, \lambda) = \{z \in \mathbb{R}_{\varepsilon}^{n} \setminus \{\varepsilon\}^n \mid A \otimes z = \lambda \otimes z\}. \tag{2.11}$$

Here, λ is called the max-plus eigenvalue and $z \in \operatorname{eig}(A, \lambda)$ is called the corresponding max-plus eigenvector. The set of max-plus eigenvalues λ of the matrix A such that $\operatorname{eig}(A, \lambda) \neq \emptyset$ is denoted as $\Lambda(A)$. $\square$

We note here that the max-plus eigenspace $\operatorname{eig}(A, \lambda(A))$ contains a finite max-plus eigenvector only if the matrix A is regular (row-finite).

The largest max-plus eigenvalue, denoted $\overline{\lambda}(A)$, of a matrix $A \in \mathbb{R}_{\varepsilon}^{n\times n}$ can be interpreted as the maximal circuit mean of the graph $\mathscr{G}(A)$. Let Σ be the set of all elementary cycles of $\mathscr{G}(A)$. Then, we have

$$\overline{\lambda}(A) = \max_{\omega\in\Sigma} \frac{|\omega|_{\mathrm{w}}}{|\omega|_1}.$$

The max-plus spectral characteristics can be computed exactly and efficiently. We refer to [119] for a complete overview. The max-plus eigenvalue problem can also be solved as a linear program:

$$\overline{\lambda}(A) = \min_{x,\lambda} \left\{\lambda \middle| [A]_{ij} + x_j - x_i \leq \lambda,\ \forall\, i,j \in \underline{n} \text{ s.t. } [A]_{ij} \text{ is finite}\right\}. \tag{2.12}$$

Theorem 2.2.1 (Max-plus spectral theorem, [17])**.** *An irreducible matrix $A \in \mathbb{R}_{\varepsilon}^{n\times n}$ attains a unique max-plus eigenvalue. That is, the set $\Lambda(A)$ is a singleton. Moreover, the set* $\mathrm{eig}(A, \lambda(A))$ *then contains only vectors with only finite elements.* ■

In general, the max-plus eigenspace of an irreducible matrix contains several max-plus eigenvectors that are not proportional in the max-plus sense. Note also that some reducible matrices also attain unique max-plus eigenvalues corresponding to finite max-plus eigenvectors.

Lemma 2.2.1 (Finite eigenvectors, [66])**.** *Let $A \in \mathbb{R}_{\varepsilon}^{n\times n}$ be a regular matrix. Then A admits finite max-plus eigenvectors, corresponding to $\overline{\lambda}(A)$, if and only if for every $i \in N(A)$ there is a $j \in N_{\mathrm{c}}(A)$ such that $(j,i) \in E(A)$.* ■

In general, a reducible matrix $A \in \mathbb{R}_{\varepsilon}^{n\times n}$ has several (at most n) distinct max-plus eigenvalues. The set of max-plus eigenvalues can be found using the Frobenius normal form (3.12) as detailed in [119, §6].

The following result presents the notion of periodicity of max-plus linear maps.

Lemma 2.2.2 (Matrix periodicity, [66])**.** *Let $A \in \mathbb{R}_{\varepsilon}^{n\times n}$ be irreducible. Then there exist scalars $k_0 \in \mathbb{N}_0$ and $c \in \mathbb{N}$ such that for all $k \geq k_0$, we have*

$$A^{\otimes^{k+c}} = \lambda^{\otimes^{c}} \otimes A^{\otimes^{k}},$$

where $\lambda \in \mathbb{R}$ is the (unique) max-plus eigenvalue and the smallest non-zero $c \in \mathbb{N}$ satisfying the preceding equation is the cyclicity of the matrix A. □

The notions of irreducibility and max-plus eigenvalues can also be extended to matrix semigroups in the max-plus algebra (see Definition 2.2.4).

Definition 2.2.12 (Irreducible semigroups, [107])**.** Let $\mathscr{A} \subseteq \mathscr{M}_{n\times n}(\mathbb{R}_{\varepsilon})$ be a set of regular matrices. The semigroup $(\mathscr{A}, \otimes)$ is said to be irreducible if there exists an irreducible matrix in the max-plus convex hull of the matrices in $\Psi(\mathscr{A})$. □

In particular, for an irreducible matrix semigroup there exist irreducible matrices $S \in \Psi(\mathscr{A})$ [107]. Equivalently, we have

$$\exists\, c \in \mathbb{N}, \quad S = A^{(i_c)} \otimes A^{(i_{c-1})} \otimes \cdots \otimes A^{(i_1)},\ A^{(i_j)} \in \mathscr{A},\ j \in \underline{c}, \tag{2.13}$$

such that S is irreducible.

Lemma 2.2.3 (Irreducible matrix semigroup, [107])**.** *Let $\mathscr{A} \subseteq \mathscr{M}_{n\times n}(\mathbb{R}_\varepsilon)$ be a closed and bounded*[4] *set of regular matrices. The matrix semigroup $(\mathscr{A}, \otimes)$ is irreducible if and only if there exists an irreducible matrix in $\mathrm{conv}_\otimes(\mathscr{A})$. Equivalently, the semigroup $(\mathscr{A}, \otimes)$ is irreducible if the matrix*

$$\mathscr{S}_A = \bigoplus_{A\in\mathscr{A}} A \tag{2.14}$$

is irreducible. ■

In view of the preceding result, the irreducibility of a matrix semigroup in max-plus algebra can then be evaluated efficiently.

The notion of joint-spectral radius extends the notion of eigenvalues from a matrix to a semigroup of matrices [33].

Definition 2.2.13 (Max-plus spectral radii, [33])**.** Let $\mathscr{A} \subseteq \mathscr{M}_{n\times n}(\mathbb{R}_\varepsilon)$ be a finite set of $L \in \mathbb{N}$ regular matrices. The max-plus joint spectral radius $\rho_{\max}$ and max-plus lower spectral radius $\rho_{\min}$ are defined as

$$\rho_{\max}(\mathscr{A}) = \lim_{k\to\infty} \max_{i_1,i_2,\ldots,i_k\in\underline{L}} \left\| A_{i_1} \otimes A_{i_2} \otimes \cdots \otimes A_{i_k} \right\|^{\otimes 1/k},$$

$$\rho_{\min}(\mathscr{A}) = \lim_{k\to\infty} \min_{i_1,i_2,\ldots,i_k\in\underline{L}} \left\| A_{i_1} \otimes A_{i_2} \otimes \cdots \otimes A_{i_k} \right\|^{\otimes 1/k},$$

where $\|A\| = \max_{i,j\in\underline{n}} [A]_{ij}$ for $A \in \mathbb{R}_\varepsilon^{n\times n}$. □

Note that the max-plus joint spectral radius of a finite set of regular max-plus matrices is readily calculated as stated in the following result.

Lemma 2.2.4 (Max-plus joint spectral radius, [85])**.** *Let $\mathscr{A} \subseteq \mathscr{M}_{n\times n}(\mathbb{R}_\varepsilon)$ be a finite set of regular matrices. Let the maximum of the matrix set $\mathscr{A}$ be defined as in (2.14). Then $\rho_{\max}(\mathscr{A}) = \overline{\lambda}(\mathscr{S}_A)$.* ■

The approximation of lower spectral radius of a given matrix group in max-plus algebra is known to be computationally difficult [33, 85]. Therefore, a complete spectral characterisation of a max-plus matrix semigroup is usually unavailable.

Now we recall the notion related to asymptotes of max-plus matrix powers. The result is particularly important for solving implicit equations in max-plus algebra. The notion of a Kleene star of a matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ is analogous to the matrix inverse $(\mathscr{I} - A)^{-1}$ in conventional algebra.

Lemma 2.2.5 (Kleene plus and Kleene star, [17])**.** *Let $A \in \mathbb{R}_\varepsilon^{n\times n}$. Then the series*

$$A^+ = A \oplus A^{\otimes 2} \oplus \cdots, \quad A^\star = \mathscr{I}_n^\otimes \oplus A \oplus A^{\otimes 2} \oplus \cdots \tag{2.15}$$

converges if and only if $\overline{\lambda}(A) \leq 0$, i.e. the associated directed graph $\mathscr{G}(A)$ has only non-positive circuit weights.

If the limits A^+ and $A^\star$ exist, then we have

$$A^+ = \bigoplus_{k=1}^{n} A^{\otimes k}, \quad A^\star = \bigoplus_{k=0}^{n-1} A^{\otimes k}.$$ ■

[4]Here, the compactness of a set of matrices in max-plus algebra is understood under the topology induced by the metric $d(x, y) = \left|e^x - e^y\right|$, with $x, y \in \mathbb{R}_\varepsilon^n$, where the exponential is understood to be element-wise.

The matrices A^+ and $K = A^\star$, if they exist, are known as the *Kleene plus* and *Kleene star* matrices, respectively. These matrices can be found efficiently using the Floyd-Marshall algorithm [48]. The Kleene star matrix K accumulates the paths with the greatest weights, by definition. Moreover, $\overline{\lambda}(K) = 0$. Equivalently, K is a Kleene star matrix if $K^{\otimes^2} = K$ and $[K]_{ii} = 0$ for all $i \in \underline{n}$. Moreover, if a matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ is irreducible and $\overline{\lambda}(A) \leq 0$, then $A^\star$ has all entries finite [17].

As will become clear in the subsequent subsection, Kleene star matrices form a particularly well-behaved class of max-plus matrices. Therefore, it is important to point out here that Kleene star matrices play a vital role in several aspects of the max-plus algebra.

2.2.3. MAX-PLUS CONVEX GEOMETRY

In this section, we recall the max-plus analogues of linear spaces, convex sets, and cones along with their important geometrical properties. The section is based on [8, 9, 50, 79, 197].

Firstly, we recall the notions of norms and metrics suited to max-plus geometric structures. This will allow us to define notions of distance from and projection onto subsets of $\mathbb{R}_\varepsilon^n$.

Definition 2.2.14 (Supremum norm)**.** Let $x \in \mathbb{R}_\varepsilon^n$. The max-plus algebra is equipped with the conventional ℓ^∞ norm defined as[5]

$$\|x\|_\infty = \max_{i\in\underline{n}} |x_i| = \max(\max_{i\in\underline{n}}(x_i), \max_{j\in\underline{n}}(-x_j)). \tag{2.16}$$

The metric induced by the ℓ^∞ norm is denoted as $d(x,y) = \|x-y\|_\infty$. □

We note that the ℓ^∞ norm on $\mathbb{R}_\varepsilon^n$ can take an infinite value.

The introduction of the next norm requires the notion of the projective space.

Definition 2.2.15 (Max-plus Hilbert's projective space)**.** Let $\sim$ be an equivalence relation on $\mathbb{R}_\varepsilon^n$ such that

$$x \sim y \Leftrightarrow \exists v \in \mathbb{R} \text{ s.t. } x = v + y, \quad \forall x, y \in \mathbb{R}_\varepsilon^n. \tag{2.17}$$

The respective equivalence class denotes the ray $\overline{y} = \{x \in \mathbb{R}_\varepsilon^n \mid x \sim y\}$. The Hilbert projective space $\mathbb{P}\mathbb{R}_\varepsilon^n$ is identified as the quotient space $\mathbb{R}_\varepsilon^n/\sim$. The max-plus Hilbert projective space $\mathbb{P}\mathbb{R}^n$ is similarly defined as the quotient space $\mathbb{R}^n/\sim$. □

While the supremum norm measures the distance from the origin, it is sometimes required to measure distances between rays. The distance between rays is understood as the distance between points in the projective space. In the following, we denote $\overline{\mathbb{1}}_n$ as the equivalence class of the vector $\mathbb{1}_n$.

Definition 2.2.16 (Max-plus Hilbert projective metric, [119])**.** The max-plus Hilbert projective (semi) norm in max-plus algebra is defined as

$$\begin{aligned} \|x\|_{\mathbb{P}} &= \max_{i\in\underline{n}}(x_i) - \min_{j\in\underline{n}}(x_j), \quad x \in \mathbb{R}^n \\ \|A\|_{\mathbb{P}} &= \max\left\{\|[A]_{\cdot i}\|_{\mathbb{P}} \mid i \in \underline{m}\right\}, \quad A \in \mathbb{R}^{n\times m}. \end{aligned} \tag{2.18}$$

[5]Recall that under the adopted convention, $\varepsilon^{\otimes^{-1}} = -\varepsilon = \top$.

The max-plus Hilbert projective (semi-) norm induces the max-plus Hilbert projective (pseudo-) metric as $d_{\mathrm{H}}(x,y) = \|x-y\|_{\mathbb{P}}$ for $x, y \in \mathbb{R}^n$.

The (Hilbert) projective norm satisfies: *i)* Triangle inequality, $\|x+y\|_{\mathbb{P}} \le \|x\|_{\mathbb{P}} + \|y\|_{\mathbb{P}}$, *ii)* Definiteness, $\|x\|_{\mathbb{P}} = 0 \Leftrightarrow x \in \overline{\mathbb{1}}_n$, and *iii)* Absolute homogeneity $\|v \cdot x\|_{\mathbb{P}} = |v| \cdot \|x\|_{\mathbb{P}}$ for $x, y \in \mathbb{R}^{n}$" and $v \in \mathbb{R}$. □

The notion of max-plus Hilbert projective metric can be extended to the space $\mathbb{R}_\varepsilon$ as in [62]. We note that the max-plus Hilbert projective norm and consequently the max-plus Hilbert projective metric fails to satisfy the properties of positive definiteness and indiscernibility, respectively. We have $d_{\mathrm{H}}(x,y) = 0$ if and only if $x \sim y$. Therefore, d_{H} forms a metric on the Hilbert projective space $\mathbb{PR}$ as all points are distinguishable. However, d_{H} does not form a metric on the max-plus Hilbert projective space $\mathbb{PR}_\varepsilon$.

Definition 2.2.17 (Open ball)**.** An open ball of radius $\delta > 0$ centred at $\{\lambda + x\}$, $\lambda \in \mathbb{R}$, with respect to the max-plus Hilbert projective norm is defined as

$$\mathscr{B}_\delta(x) := \left\{ y \in \mathbb{R}^n \middle| \|y - x\|_{\mathbb{P}} < \delta \right\}. \tag{2.19}$$

Additionally, the set $\mathscr{B}_\delta(x)$ for $x \in \overline{\mathbb{1}}_n$ is denoted as $\mathscr{B}_\delta$. If $\mathscr{K} \subseteq \mathbb{R}^n$ is a set, then $\mathscr{B}_\delta(\mathscr{K}) = \bigcup_{x \in \mathscr{K}} \mathscr{B}_\delta(x)$. □

The set $\mathscr{B}_\delta$ is a polytope with $n(n-1)$ facets in the projective space $\mathbb{PR}^n$. To see this, it is convenient to embed the domain $\mathbb{R}^n$ into the affine hyperplane $\mathscr{H} = \{x \in \mathbb{R}^n \mid x_n = 0\}$. Then $\mathbb{PR}^n$ can be identified as a real $n-1$ dimensional space.

It can be noted that the ball $\mathscr{B}_\delta$ is centred around the origin in the max-plus Hilbert projective space $\mathbb{R}^n \cap \mathscr{H}$. The closed unit ball $\mathscr{B}_1$ and its sub-level sets are plotted in Fig. 2.1. Note that in dimension $n = 3$, the representative unit ball and its sub-level sets are hexagons.

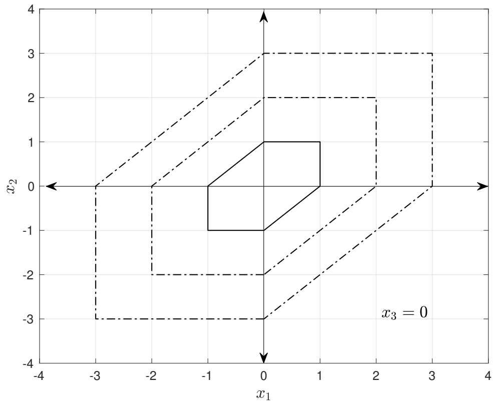


Figure 2.1: Concentric max-plus Hilbert balls $\mathscr{B}_\delta$ of radii $\delta \in \{1,2,3\}$ in $\mathbb{R}^3$ with a representative $x_3 = 0$ in the max-plus Hilbert projective space $\mathbb{R}^2$.

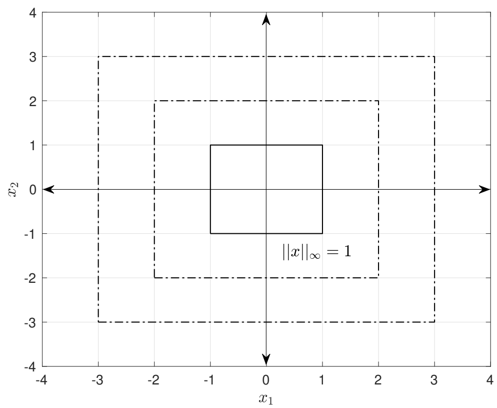


Figure 2.2: Concentric balls in the ℓ^∞ norm of radii $\delta \in \{1,2,3\}$ in $\mathbb{R}^2$.

We deal with finite dimensional spaces in this dissertation. The geometric notions on max-plus (vector) spaces $\mathbb{R}_\varepsilon^n$ can be defined analogously to the Euclidean space.

For the set $\mathcal{V} \subseteq \mathbb{R}^n$, a point $x \in \mathbb{R}^n$ is in its interior, denoted $x \in \text{int}(\mathcal{V})$, if there exists $\delta > 0$ such that $\mathcal{B}_\delta(x) \subseteq \mathcal{V}$. The closure and boundary of the set $\mathcal{V}$ are defined as $\text{cl}(\mathcal{V}) = \{x \in \mathbb{R}^n \mid d_\text{H}(x, \mathcal{V}) = 0\}$, and $\partial\mathcal{V} = \text{cl}(\mathcal{V}) \setminus \text{int}(\mathcal{V})$, respectively.

Definition 2.2.18 (Max-plus cone)**.** A subset $\mathcal{W} \subseteq \mathbb{R}_\varepsilon^n$ is said to be a max-plus cone if it is closed under addition $\oplus$ of its elements and under multiplication $\otimes$ with scalars in $\mathbb{R}_\varepsilon$:

$$\lambda \otimes u \oplus \mu \otimes v \in \mathcal{W} \tag{2.20}$$

for all $u, v \in \mathcal{W}$ and $\lambda, \mu \in \mathbb{R}_\varepsilon$. The subset $\mathcal{W}$ is said to be a convex max-plus cone if (2.20) holds for all $u, v \in \mathcal{W}$ and $\lambda, \mu \in \mathbb{R}_\varepsilon$ such that $\lambda \oplus \mu = \mathbb{1}$. □

A max-plus cone $\mathcal{V}$ is said to be bounded in the max-plus Hilbert projective norm if

$$\exists \delta > 0 \text{ s.t. } \forall x \in \mathcal{V} \Rightarrow \|x\|_\mathbb{P} < \delta \Leftrightarrow \mathcal{V} \subseteq \mathcal{B}_\delta. \tag{2.21}$$

Max-plus cones have several features common with classical convex cones. The notions of max-plus span $\text{span}_\oplus$ and max-plus convex hull $\text{conv}_\oplus$ are defined analogously to the conventional algebra.

Definition 2.2.19 (Finitely generated max-plus cones)**.** A max-plus cone $\mathcal{W} \subseteq \mathbb{R}_\varepsilon^n$ is said to be *finitely generated* if there exists a finite set of vectors $W = \{w_1, w_2, \ldots, w_m\}$ such that

$$\mathcal{W} = \text{span}_\oplus(W) = \left\{ \bigoplus_{i=1}^m \alpha_i \otimes w_i \;\middle|\; \alpha_i \in \mathbb{R}_\varepsilon \right\}, \tag{2.22}$$

or the max-plus combination of the column vectors of W, the generating set of $\mathcal{W}$.

The *max-plus weak dimension* of a finitely generated max-plus cone $\mathcal{W}$ is the cardinality of the minimal generating set of $\mathcal{W}$. □

Here, minimality is understood in the sense of max-plus independence: the set $W = \{w_1, w_2, \dots, w_m\}$ is *dependent* if v can be expressed as a max-plus span of vectors in $W \setminus \{v\}$ for some $v \in W$. Note that a finitely generated max-plus cone of $\mathbb{R}^n_\varepsilon$ is closed [92, Lemma 2.20].

A min-plus cone and a (classical) convex cone can be defined analogously over the min-plus algebra and conventional algebra, respectively.

Definition 2.2.20 (Extremals)**.** Let $\mathscr{W} \subseteq \mathbb{R}^n_\varepsilon$ be a max-plus cone. An element $u \in \mathscr{W}$ is an extremal in $\mathscr{W}$ if

$$u = v \oplus w, \quad v, w \in \mathscr{W} \;\Rightarrow\; u = v \text{ or } u = w.$$

If u is an extremal in $\mathscr{W}$ then $\lambda \otimes u$ is also an extremal in $\mathscr{W}$ for all $\lambda \in \mathbb{R}$. □

We note that there exist several max-plus analogues to results in classical discrete geometry. Here, we only recall the analogue of the Carathéodory's Theorem that is relevant to the contents of this dissertation.

Lemma 2.2.6 (Max-plus Carathéodory's Theorem, [50])**.** *Let $\mathscr{W} \subseteq \mathbb{R}^n_\varepsilon$ be a closed max-plus cone. Then $\mathscr{W}$ is generated by its set of extremals, and any element $w \in \mathscr{W}$ is in the max-plus span of no more than n extremals.*

An important subclass of max-plus cones is obtained when its extremals can be expressed as a Kleene star matrix (see Lemma 2.2.5).

Definition 2.2.21 (Kleene cones)**.** A max-plus cone is said to be a Kleene cone if it is generated as a max-plus column span of a Kleene star matrix. □

We refer the interested reader to [123, 197] for a comprehensive overview of Kleene cones and their relation with convex sets.

Lemma 2.2.7 (Convex max-plus cones, [197])**.** *A max-plus cone generated as a max-plus column span of a Kleene star matrix is also convex in the Euclidean sense.* ■

The preceding result is stronger in the sense that for a max-plus cone that is bounded in the max-plus Hilbert projective norm, min-plus convexity and Euclidean convexity are equivalent [123, Theorem B]. Moreover, a max-plus cone bounded in the max-plus Hilbert projective norm is Euclidean convex if and only if it is generated as the max-plus column span of a Kleene star matrix [123, Corollary C].

It can then be noted that $\mathscr{B}_\delta = \mathrm{span}_\oplus(K^{(\delta)})$ where

$$K^{(\delta)} = \begin{pmatrix} 0 & -\delta & \cdots & -\delta \\ -\delta & 0 & \cdots & -\delta \\ \vdots & \vdots & \ddots & \vdots \\ -\delta & -\delta & \cdots & 0 \end{pmatrix} \tag{2.23}$$

is a Kleene-star matrix with finite columns.

It follows that the sub-level sets of the unit ball $\mathscr{B}_1$ can be obtained by appropriately scaling the off-diagonal elements of the Kleene star matrix, in this case equal to $\lambda_2(K^{(1)})$. This intuition, in principle, can be extended to any max-plus cone generated

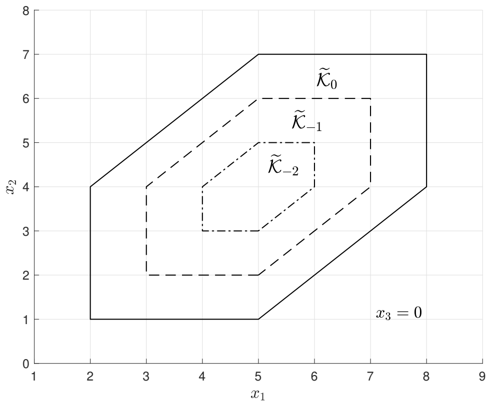


Figure 2.3: The max-plus cone generated by K in (2.24) as $\text{span}_{\oplus}(\widetilde{K}_{\mu})$ are max-plus Hilbert balls for $\mu \in \{0, -1, -2\}$ in $\mathbb{R}^3$ with a representative $x_3 = 0$ in the max-plus Hilbert projective space [196].

by the columns of a Kleene star matrix. The following matrix scaling facilitates a compact representation.

The normalisation $\widetilde{K}_{\mu}$ of a Kleene star matrix[6] $K \in \mathbb{R}_{\varepsilon}^{n\times n}$ by a scalar $\mu \in \mathbb{R}$ is obtained as follows: $[\widetilde{K}_{\mu}]_{ii} = 0$, $[\widetilde{K}_{\mu}]_{ij} = [K]_{ij} - \mu$, for all $i, j \in \underline{n}$. Thus, we subtract μ from the off-diagonal elements.

Then for the max-plus cone generated by a Kleene star matrix $\mathcal{K}$, $\text{span}_{\oplus}(\widetilde{K}_{\mu})$ represents the resulting non-empty sub-level sets for $\mu > -\lambda_2(K)$ [196].

Example 2.2.1 ([196])**.** *Consider a max-plus Hilbert ball $\mathcal{K}$ centred at $x = \begin{pmatrix} 5 & 4 & 0 \end{pmatrix}^{\top}$ with radius $\delta = 3$. Then we have $\mathcal{K} = \text{span}_{\oplus}(K)$ for*

$$K = \begin{pmatrix} 0 & -2 & 2 \\ -4 & 0 & 1 \\ -8 & -7 & 0 \end{pmatrix}. \tag{2.24}$$

The corresponding max-plus Hilbert ball and its sub-level sets are plotted in Fig. 2.3.

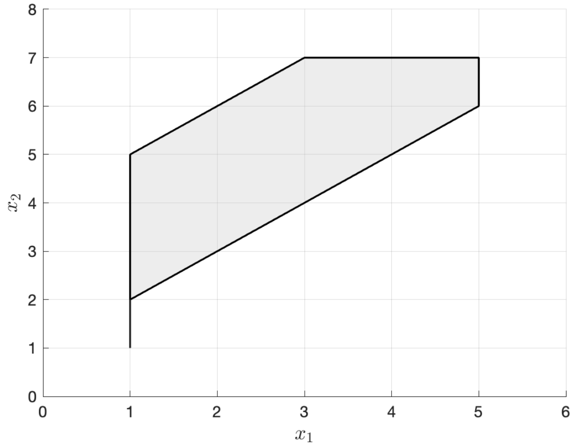
Consider the following matrix and its Kleene star:

$$A = \begin{pmatrix} 0 & -4 & 1 \\ 1 & 0 & 1 \\ -5 & -7 & 0 \end{pmatrix}, \quad A^{\star} = \begin{pmatrix} 0 & -4 & 1 \\ 1 & 0 & 2 \\ -5 & -7 & 0 \end{pmatrix}. \tag{2.25}$$

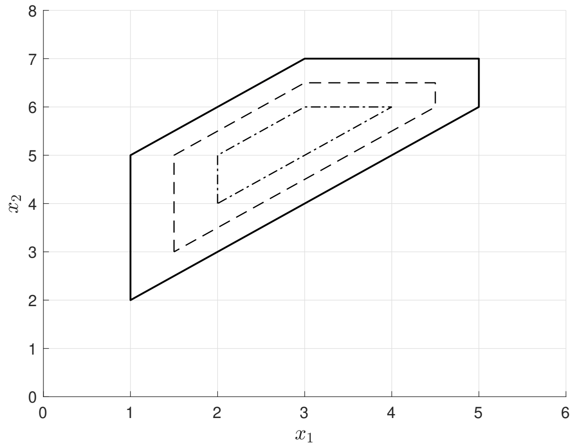
The resulting $\text{span}_{\oplus}(A)$ and $\text{span}_{\oplus}(A^{\star})$ represented in $\mathbb{R}^2$, by projecting out the last coordinate, are plotted[2] in Fig. 2.4. Note that the max-plus cone generated by columns of A is not a convex cone as opposed to that generated by columns of $A^{\star}$.

Finally, it can be observed that all the illustrated max-plus cones can be circumscribed in max-plus Hilbert balls $\mathcal{B}_{\delta}$ for large enough δ's. They are bounded in the max-plus Hilbert projective metric.

[6]Note that for a Kleene star matrix $[K]_{ii} = 0$ for all $i \in \underline{n}$.



(a)



(b)

Figure 2.4: The max-plus cone generated by A and $A^{\star}$ in (2.25) as 2.4a $\text{span}_{\oplus}(A)$ and 2.4b $\text{span}_{\oplus}(\widetilde{A^{\star}\mu})$ for $\mu \in \{0, -0.5, -1\}$ in $\mathbb{R}^3$ with a representative $x_3 = 0$ in the max-plus Hilbert projective space $\mathbb{R}^2$ [196].

Finally, we recall notions of distances and projections under max-plus convex geometry. The distance of $x \in \mathbb{R}^n_\varepsilon$ to a max-plus cone $\mathcal{V} \subseteq \mathbb{R}^n_\varepsilon$ is denoted as:

$$\|x\|_{\mathcal{V},\mathbb{P}} \triangleq d_{\mathrm{H}}(x,\mathcal{V}) = \inf_{v\in\mathcal{V}} d_{\mathrm{H}}(x,v) \tag{2.26}$$

Definition 2.2.22 (Nonlinear projector, [8])**.** Let $\mathcal{V} \subseteq \mathbb{R}^n_\varepsilon$ be a max-plus cone. The nonlinear projection $P_{\mathcal{V}}(x)$ of $x \in \mathbb{R}^n_\varepsilon$ onto $\mathcal{V}$ is defined as:

$$P_{\mathcal{V}}(x) = \max\{v \in \mathcal{V} \mid v \leq x\}. \tag{2.27}$$

The nonlinear projector $P_{\mathcal{V}}(\cdot)$ minimises the max-plus Hilbert projective metric:

$$d_{\mathrm{H}}(x,\mathcal{V}) = d_{\mathrm{H}}(x, P_{\mathcal{V}}(x)). \tag{2.28}$$

The operator $P_{\mathcal{V}}$ is additively homogeneous, monotone, non-decreasing, and continuous. For any $x \in \mathbb{R}^n_\varepsilon$, there exists at least one index $i \in \underline{n}$ such that $[P_{\mathcal{V}}(x)]_i = x_i$. □

For a max-plus cone $\mathcal{V} = \mathrm{span}_\oplus(V)$, where $V \in \mathbb{R}^{n\times m}_\varepsilon$, we have $P_{\mathcal{V}}(x) = V \otimes ((-V)^\top \otimes' x)$ for all $x \in \mathbb{R}^n_\varepsilon$. In the case that $V \in \mathbb{R}^{n\times n}_\varepsilon$ is a Kleene star (see Definition 2.2.5), we have $P_{\mathcal{V}}(x) = (-V)^\top \otimes' x$ for all $x \in \mathbb{R}^n_\varepsilon$. We note that the projection operator $P_{\mathcal{V}}$ is min-plus linear if and only if $\mathcal{V}$ is generated as the max-plus span of a Kleene star matrix [197, Proposition 3.2].

2.2.4. SYSTEMS THEORY

In this subsection, we recall the systems theory arising from the max-plus algebra to study iterates of monotone and additively homogeneous functions. We also recall systems of max-plus linear equations that can be solved efficiently using the tools of the max-plus algebra. [86].

In the first part of this subsection, we are interested in presenting the max-plus algebra tools to analyse the dynamical behaviour of functions as iterates $\{g^k(x) \mid k \in \mathbb{N}\}$ of $g : \mathbb{R}^n \to \mathbb{R}^n$ for arbitrary $x \in \mathbb{R}^n$. One aspect of the study of such dynamics deals with the (qualitative) characterisation of the asymptotic linear growth rate of the function g.

Definition 2.2.23 (Cycle time vector)**.** The cycle time vector for a function $g : \mathbb{R}^n \to \mathbb{R}^n$ is defined as the limit, if it exists,

$$\chi(g,\xi) = \lim_{k\to+\infty} \frac{x(k)}{k}, \tag{2.29}$$

where $x(k) = g(x(k-1))$ and $x(0) = \xi \in \mathbb{R}^n$. □

Another important aspect of the dynamics is concerned with the characterisation of the fixed point set of the function g. The notion of max-plus eigenspace (Definition 2.2.11) can be extended to a general class of functions $g : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\mathrm{eig}(g,\lambda) = \{z \in \mathbb{R}^n \mid g(z) = \lambda \otimes z\}.$$

The corresponding set of max-plus eigenvalues is denoted as $\Lambda(g) = \{\lambda \mid \mathrm{eig}(g,\lambda) \neq \emptyset\}$.

Definition 2.2.24 (Fixed points)**.** The set of fixed points of a function $g : \mathbb{R}^n \to \mathbb{R}^n$ is defined as its max-plus eigenspace:

$$\begin{aligned}\mathrm{Fix}(g) &= \bigcup_{\lambda \in \Lambda(g)} \mathrm{eig}(g, \lambda) \\ &= \left\{x \in \mathbb{R}^n \,\middle|\, g(x) = \lambda \otimes x \text{ for some } \lambda \in \mathbb{R}\right\}.\end{aligned} \tag{2.30}$$

□

Note that when the set of max-plus eigenvalues $\Lambda(g) = \{0\}$, the definition of the fixed-point set boils down to the conventional definition $\mathrm{Fix}(g) = \{x \mid g(x) = x\}$. The modification adopted in Definition 2.2.24 is aligned with the fact that the max-plus eigenvalues represent the asymptotic rate of occurrence of events (or inverse of throughput) in a discrete-event system.

The max-plus spectral theory plays a crucial role in the study of dynamics (behaviour under iteration) of max-plus linear functions. It is important to note that the max-plus spectral theory extends to a larger class of functions than max-plus linear ones [109]. Therefore, we present it here for a general class of tropical functions.

Definition 2.2.25 (Tropical functions)**.** A function $g : \mathbb{R}^n_\varepsilon \to \mathbb{R}^n_\varepsilon$ is said to be *additively homogeneous* if $g(\mu + x) = \mu + g(x)$, for all $\mu \in \mathbb{R}$. The function g is *monotone* if for all $x, y \in \mathbb{R}^n_\varepsilon$, $x \leq y$ implies $g(x) \leq g(y)$. The class of additively homogeneous and monotone functions is called tropical functions. □

Max-plus linear maps $g : x \mapsto A \otimes x$, that can be represented by a regular (row-finite) matrix $A \in \mathbb{R}^{n\times n}_\varepsilon$, fall under the category of tropical functions.

Definition 2.2.26 (Max-min-plus expression, [108])**.** A max-min-plus expression f of variables $x_1, \ldots, x_n$ is defined by the recursive grammar

$$f := x_i | f_k \oplus f_l | f_k \oplus' f_l | f_k + \alpha,\ \alpha \in \mathbb{R},\ i \in \underline{n}, \tag{2.31}$$

where f_k and f_l are again max-min-plus expressions. The symbol | stands for “or". □

Lemma 2.2.8 (Max-min-plus conjunctive form, [108])**.** *A max-min-plus expression f can be placed in the max-min-plus conjunctive form:*

$$\begin{aligned} &f = f_1 \oplus' f_2 \oplus' \cdots \oplus' f_m, \\ &i \neq j \Rightarrow f_i \nleq f_j, \end{aligned} \tag{2.32}$$

where $f_j = (a_{j1} \otimes x_1) \oplus (a_{j2} \otimes x_2) \oplus \cdots \oplus (a_{jn} \otimes x_n)$ is said to be a max-plus projection of f with $a_{ji} \in \mathbb{R}_\varepsilon$ for all $i \in \underline{n}$ and $j \in \underline{m}$. The max-min-plus conjunctive form (2.32) is unique up to reordering of f_j's. ■

It can be noted that max-plus and min-plus expressions are special cases of max-min-plus expressions. We denote the general class of functions obtained from the max-min-plus grammar as *max-min-plus linear functions*. More importantly, any function built on the max-min-plus grammar is also a tropical function [108, 109].

Definition 2.2.27 (Lipschitz continuity)**.** A function $g : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous (with respect to d_H) if there exists a constant $L \geq 0$ such that for all $x_1, x_2 \in \mathbb{R}^n/\sim$ we have

$$d_\mathrm{H}(g(x_1), g(x_2)) \leq L \cdot d_\mathrm{H}(x_1, x_2). \tag{2.33}$$

A function g is said to be contractive in the max-plus Hilbert projective norm if $L < 1$ and non-expansive if $L = 1$. □

Lemma 2.2.9 (Monotonicity and non-expansiveness, [109]). *An additively homogeneous monotone function is non-expansive in the max-plus Hilbert projective metric and the ℓ^∞ metric. Moreover, an additively homogeneous non-expansive function is monotone.* ■

The non-expansiveness property plays a significant role in the characterisation of the steady-state characteristics (asymptotic linear growth rate and existence of fixed points) of a function g.

Lemma 2.2.10 (Uniqueness of cycle time vector, [109]). *Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be an additively homogeneous and non-expansive function. If the cycle time vector $\chi(g,\xi)$ (as the limit defined in* (2.29)*) exists for a particular $\xi \in \mathbb{R}^n$; then the limit exists as the same value for any arbitrarily chosen initial condition $\xi' \in \mathbb{R}^n$:*

$$\chi(g,\xi) \text{ exists} \Rightarrow \chi(g,\xi') \text{ exists and } \chi(g,\xi) = \chi(g,\xi').$$ ■

The preceding result shows that the cycle time vector $\chi(g,\xi) \in \mathbb{R}^n$, if it exists, is a characteristic of the function g and is independent of the choice of the initial condition ξ. Also, all trajectories $\{g^k(x) \mid k \in \mathbb{N}\}$ for arbitrary $x \in \mathbb{R}^n$ are asymptotically equivalent in the supremum metric. Efficient algorithms for computing the cycle time vector of a given max-min-plus linear function can be found in [57, 203].

Lemma 2.2.11 (Existence of fixed point, [89]). *Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be an additively homogeneous monotone function. Then g admits a max-plus eigenvector with eigenvalue $\lambda \in \mathbb{R}$ if and only if there exists an $x \in \mathbb{R}^n$ such that $\left\| g^k(x) - k \cdot \lambda \right\|_\infty$ is bounded as $k \to \infty$.*

Moreover, a max-plus eigenvector $z \in \mathbb{R}^n$ exists, $g(z) = \lambda + z$ for some $\lambda \in \mathbb{R}$, if and only if all trajectories $\{g^k(x) \mid k \in \mathbb{N}\}$ are bounded in the max-plus Hilbert projective norm for arbitrary $x \in \mathbb{R}^n$. ■

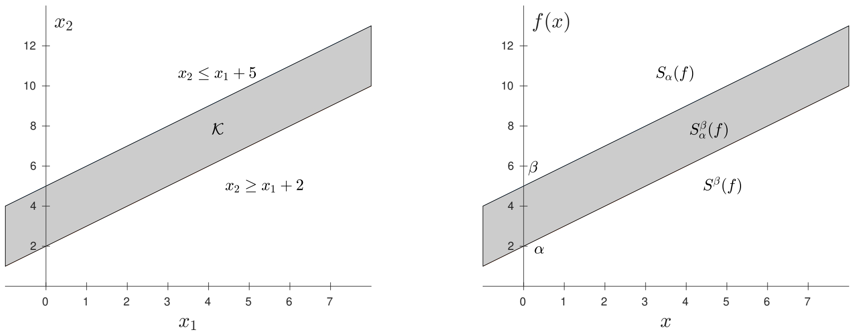
The first implication of the preceding result is that the entries of the cycle time vector of a tropical function g attain the same value, $\chi(g) = [\lambda, \lambda, \ldots, \lambda]^\top$, if and only if g attains a max-plus eigenvector. In particular, this result is stronger than the result presented in Theorem 2.2.1. The graph-theoretic generalisation of the max-plus spectral theorem for max-plus linear maps, in Theorem 2.2.1, to general tropical functions can be found in [89, §3]. The second implication of the preceding result is that if $z \in \operatorname{eig}(g,\lambda)$ then any arbitrary trajectory $\{g^k(x) \mid k \in \mathbb{N}\}$ remains at a bounded distance from $\{z + k \cdot \lambda\}_{k \in \mathbb{N}}$ as k tends to infinity.

Definition 2.2.28 (Max-plus slice spaces). Let $\alpha, \beta \in \mathbb{R}$ with $\beta \geq \alpha$. The max-plus super-eigenspace S^β, max-plus sub-eigenspace S_α, and max-plus slice space S_α^β generated by a function $g : \mathbb{R}^n \to \mathbb{R}^n$ are defined as

$$\begin{aligned} S^\beta(g) &= \left\{ x \in \mathbb{R}^n \mid g(x) \leq \beta + x \right\} \\ S_\alpha(g) &= \left\{ x \in \mathbb{R}^n \mid \alpha + x \leq g(x) \right\} \\ S_\alpha^\beta(g) = S^\beta(g) \cap S_\alpha(g) &= \left\{ x \in \mathbb{R}^n \mid \alpha + x \leq g(x) \leq \beta + x \right\}. \end{aligned} \tag{2.34}$$

For $\alpha \leq \alpha'$ and $\beta' \leq \beta$ with $\alpha' \leq \beta'$, we have $S_{\alpha'}^{\beta'}(g) \subseteq S_\alpha^\beta(g)$. □

Chapter 2



(a) (b)

Figure 2.5: Max-plus cones: (a) Max-plus cone $\mathcal{K} = \{(x_1, x_2)^\top \in \mathbb{R}_\varepsilon^2 \mid x_1 + 2 \leq x_2 \leq x_1 + 5\}$, and (b) Max-plus eigenspace $S_\alpha^\beta(g) = S^\beta(g) \cap S_\alpha(g)$ for $\alpha = 2$ and $\beta = 5$.

We note that the max-plus eigenspaces, as introduced in the preceding definition, are invariant with respect to the iterates of the function g if the function g is tropical. Finally, we recall the following important result that can be derived from the boundedness of the max-plus slice spaces of a tropical function.

Theorem 2.2.2 (Finite fixed-points [86])**.** *Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be an additively homogeneous monotone function. Let there exists a non-empty max-plus slice space $S_\beta^\alpha(g)$, for some $\beta \geq \alpha$, $\alpha, \beta \in \mathbb{R}$. If $S_\beta^\alpha(g)$ is bounded in the max-plus Hilbert projective norm, then g admits a finite max-plus eigenvector.* ■

We remark here that it is difficult to prove boundedness of a given max-plus slice space for a general tropical function [87].

In the following, we restrict ourselves to max-plus linear maps. We recapitulate certain solution methods for certain max-plus system of equations and closed-form generators of the max-plus slice spaces of max-plus linear maps.

The solution to the equation $f(x) = b$ is not unique if the mapping is not injective. Then, we study the solution as an upper bound of the subset of subsolutions of the equation [17]:

$$f^\sharp(b) = \max_{\{x \mid f(x) \leq b\}} x, \quad f(f^\sharp(b)) \leq b. \tag{2.35}$$

Dually, we can study the solution as a lower bound of the subset of supersolutions:

$$f^\flat(b) = \min_{\{x \mid f(x) \geq b\}} x, \quad f(f^\flat(b)) \geq b. \tag{2.36}$$

Lemma 2.2.12 (One-sided max-plus equation, [48])**.** *Let $A \in \mathbb{R}_\varepsilon^{m \times n}$ and $b \in \mathbb{R}_\varepsilon^m$. The greatest subsolution to the equation $A \otimes x = b$ is given using the left division operator $\circ\backslash$:*

$$\begin{aligned} \hat{x} = A \circ\backslash b &= \max\{x \mid A \otimes x \leq b\} \\ &= (-A)^\top \otimes' b. \end{aligned} \tag{2.37}$$

Then, $A \otimes x = b$ has a solution if and only if $\hat{x}$ is a solution, i.e. $A \otimes (A \circ\backslash b) = b$. ■

The solution methods to two-sided (implicit) system of equations in max-plus algebra play an important role in max-plus systems theory. We begin with the simpler cases.

Lemma 2.2.13 (Max-plus sub-eigenpace, [49]). *Let the sub-eigenspace of a regular matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ with respect to $\beta \in \mathbb{R}$ be defined as*

$$V_*(A,\beta) = S^\beta(A) = \left\{x \in \mathbb{R}^n \mid A \otimes x \leq \beta \otimes x\right\}.$$

Then $V_*(A,\beta) \neq \emptyset$ if and only if $\beta \geq \overline{\lambda}(A)$. We have

$$V_*(A,\beta) = \mathrm{span}_\oplus^+(A_\beta^\star) = \left\{\bigoplus_{i=1}^n \alpha_i \otimes [A_\beta^\star]_{\cdot i} \,\middle|\, \alpha_i \in \mathbb{R}\right\}, \quad \beta \geq \overline{\lambda}(A).$$ ■

The characterisation of max-plus super-eigenspace of a max-plus matrix is a more intricate topic [49, 198]. We recall a simpler result relevant to this dissertation.

Lemma 2.2.14 (Max-plus super-eigenpace, [49]). *Let the max-plus super-eigenspace of a regular matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ with respect to $\beta \in \mathbb{R}$ be defined as*

$$V^*(A,\alpha) = S_\alpha(A) = \left\{x \in \mathbb{R}^n \mid A \otimes x \geq \alpha \otimes x\right\}.$$

Consider the matrix A in the Frobenius normal form (3.12) with the corresponding partition of the vertices of the directed graph $\mathcal{G}(A)$ into $\{N_j\}_{j\in\underline{r}}$. Define

$$\lambda^*(A) = \min_{j\in\underline{r}} \left\{\lambda(A_{jj}) \mid N_j \text{ is an initial class}\right\}$$
$$\underline{\lambda}(A) = \min_{i\in\underline{n}} [A]_{ii}.$$

Then $V^*(A,\alpha) \neq \emptyset$ if and only if $\alpha \leq \lambda^*(A)$. Also, $V^*(A,\alpha) = \mathbb{R}^n$ if $\alpha \leq \underline{\lambda}(A)$. ■

The max-plus eigenspace of a matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$, with $\overline{\lambda}(A) = 0$, is generated by the max-plus span of the columns of A^+ pertaining to the critical vertices $N_c(A)$ in the directed graph $\mathcal{G}(A)$ [66]. Here, we present the characterisation of the finite max-plus eigenvectors of the matrix A.

Lemma 2.2.15 (Max-plus eigenspace, [48]). *Let the finite max-plus eigenspace of a regular matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ with respect to $\lambda = \overline{\lambda}(A)$ be defined as*

$$V(A,\lambda) = S_\lambda^\lambda(A) = \left\{x \in \mathbb{R}^n \mid A \otimes x = \lambda \otimes x\right\}.$$

Then we have

$$V(A,\lambda) \neq \emptyset \Leftrightarrow \bigoplus_{j\in N_c(A)} [A_\lambda^+]_{\cdot j} \in \mathbb{R}^n.$$

Moreover, if $V(A,\lambda) \neq \emptyset$ then

$$V(A,\lambda) = \left\{\bigoplus_{j\in V_c^*(A)} \alpha_j \otimes [A_\lambda^+]_{\cdot j} \,\middle|\, \alpha_j \in \mathbb{R},\ j \in \underline{n}\right\},$$

where $V_c^*(A)$ is the maximal set of non-equivalent critical nodes of $\mathcal{G}(A)$. ■

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We note that, in general, the set $V(A,\lambda)$ is not finitely generated. However, if the matrix A is additionally irreducible then $V(A,\lambda)$ is a max-plus cone finitely generated by the columns of A^+ corresponding to the critical vertices [66].

Finally, we recall some results on more general two-sided max-plus linear systems of equations.

Lemma 2.2.16 (Implicit max-plus linear system, [119]). *Let $A \in \mathbb{R}_\varepsilon^{n\times n}$ and $b \in \mathbb{R}_\varepsilon^n$. Then the smallest solution $\hat{x} \in \mathbb{R}_\varepsilon^n$ to the implicit system of equations $x = A \otimes x \oplus b$ is given as $\hat{x} = A^\star \otimes b$.*

The solution $\hat{x} = A^\star \otimes b$ to the implicit system of equations exists if and only if the circuit weights of the associated directed graph $\mathcal{G}(A)$ are non-positive, $\overline{\lambda}(A) \le 0$. Moreover, the solution is unique if the circuit weights are negative, $\overline{\lambda}(A) < 0$. ■

Note that the existence and uniqueness of the solution to an implicit system of equations is inherently connected to the existence of the Kleene star $A^\star$ matrix (as in Lemma 2.2.5).

Theorem 2.2.3 (Homogeneous two-sided systems, [48]). *Let $A, B \in \mathbb{R}_\varepsilon^{m\times n}$. Then the set*

$$S(A,B) = \left\{x \in \mathbb{R}_\varepsilon^n \;\middle|\; A \otimes x = B \otimes x\right\}$$

is a finitely generated max-plus cone in $\mathbb{R}_\varepsilon^n$. ■

Theorem 2.2.4 (Separated two-sided systems, [48]). *Let $A, B \in \mathbb{R}_\varepsilon^{m\times n}$. Then the set*

$$S(A,B) = \left\{[x^\top,\ y^\top]^\top \in \mathbb{R}_\varepsilon^{2n} \;\middle|\; A \otimes x = B \otimes y\right\}$$

is a finitely generated max-plus cone in $\mathbb{R}_\varepsilon^{2n}$. ■

Note that the homogeneous and separated two-sided system of max-plus equations can be transformed into one another [48, §7.4]. An iterative method to find a finite solution to a separated two-sided system of max-plus equations formed by matrices, with at least one finite element in each row and each column, can be found in [48, §7.3].

2.3. CLASSICAL STABILITY THEORY

In this subsection, we recall certain stability properties of conventional nonlinear systems defined on discrete-time and continuous space. We note that, in particular, such systems are assumed to be defined on a normed vector space $(\mathbb{R}, \|\cdot\|)$. In addition to the axioms satisfied by a seminorm on $\mathbb{R}^n$, a norm satisfies positive definiteness: $\|x\| = 0$ if and only if $x = 0$. We first discuss notion of Lyapunov stability (with respect to an equilibrium point) for a general time-varying nonlinear system. Then we discuss the notion of Krasovskii-Lasalle invariance (inside a positively invariant set) for a set-valued dynamical system defined as an autonomous difference inclusion. This section is based entirely on [31, 46, 47, 130, 135].

2.3.1. TIME-VARYING DYNAMICAL SYSTEMS

In this thesis, the dynamical systems of interest are defined analogously to a discrete-time continuous-space dynamical system in conventional algebra. We recall the relevant definitions and properties. Then we discuss notions and relevant theorems on stability of such systems. A (time-varying) discrete-time non-linear dynamics is defined as

$$x(k) = F(k, x(k-1)), \quad k \in \mathbb{N}, \tag{2.38}$$

where $F : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ is an arbitrary time-dependent non-linear function. We assume that $F(k, \cdot)$ is continuous on $\mathbb{R}^n$ for all $k \in \mathbb{N}$. For simplicity, we assume that the solution to the dynamical system exists for all $k \in \mathbb{N}$. We denote this solution starting at $x_0 = x(k_0) \in \mathbb{R}^n$ as $\varphi(k; x_0)$. In addition, we assume that the origin is an equilibrium point (or fixed point): $F(k, 0) = 0$ for all $k \in \mathbb{N}$. The central idea of the classical stability theory is to transfer the stability property of the dynamical system to monotonic decrescent behaviour of certain nice auxiliary functions along the trajectories of the system.

We first provide some definitions useful for further exposition.

Definition 2.3.1 (Positive definiteness)**.** A locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}^n$ is said to be positive definite if $V(x) > 0$ for $x \in \mathbb{R}^n \setminus \{0_n\}$ and $V(0) = 0$. The function is positive semi-definite if $V(x) \geq 0$ for all $x \in \mathbb{R}^n \setminus \{0_n\}$. The function V is negative definite if $-V$ is positive definite. □

Definition 2.3.2 (Positive invariance)**.** A closed set $\mathscr{P} \subseteq \mathbb{R}^n$, containing the origin in its interior, is said to be positively invariant to the dynamics (2.38) if for all $x(k_0) \in \mathscr{P}$, we have $\varphi(k; x_0) \in \mathscr{P}$ for all $k \geq k_0$. □

We say that a set is *compact* if it is both closed and bounded in the normed vector space $(\mathbb{R}, \|\cdot\|)$.

Definition 2.3.3 (Comparison functions, [130])**.** A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathscr{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$.
If $\lim_{s \to +\infty} \alpha(s) = +\infty$, then it is said to be of class $\mathscr{K}_\infty$.
A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathscr{KL}$ if $\beta(\cdot, k)$ is of class $\mathscr{K}$ for each fixed $k \geq 0$ and $\lim_{k \to \infty} \beta(r, k) = 0$ for each $r \geq 0$.
An identity function is denoted by id such that $\mathrm{id}(s) = s$ for all $s \in \mathbb{R}_+$. □

Definition 2.3.4 (Uniform local asymptotic stability, [135])**.** The system (2.38) is said to be uniformly locally asymptotically stable with respect to the origin if there exists a neighbourhood of the origin $\mathscr{P}$,with $0 \in \mathscr{P}$, such that the following conditions hold:

1. *Uniform local stability*: for all $\epsilon > 0$ there exists a scalar $\delta = \delta(\epsilon) > 0$ independent of k_0 such that
$$\|x_0\| \leq \delta \Rightarrow \left\|\varphi(k; x_0)\right\| \leq \epsilon, \quad \forall k \geq k_0.$$

2. *Local uniform attraction*: for each $\eta > 0$, there exists a constant $T = T(\eta) \in \mathbb{N}$ such that
$$\|x_0\| \in \mathscr{P} \Rightarrow \left\|\varphi(k; x_0)\right\| \leq \eta, \quad \forall k \geq k_0 + T(\eta),$$
for every $k_0 \in \mathbb{N}$. □

Definition 2.3.5 (Lyapunov function)**.** Let $\mathscr{P} \subseteq \mathbb{R}^n$ be a neighbourhood of the origin with $0 \in \mathscr{P}$. A continuous function $V : \mathscr{P} \to \mathbb{R}$ is said to be a Lyapunov function for the system (2.38) in $\mathscr{P}$ if the following conditions are satisfied:

1. there exist two functions α_1, α_2 of class $\mathscr{K}$ such that for all $\xi \in \mathscr{P}$, we have

$$\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|);$$

2. there exists a continuous positive definite function α_3 such that

$$\Delta V(x) = V(F(k,x)) - V(x) \leq -\alpha_3(\|x\|),$$

for any $x \in \mathscr{P}$ and for all $k \geq 0$. □

We say that a Lyapunov function V is smooth if it is smooth in x on $\mathscr{P}$.

Theorem 2.3.1 (Asymptotic Lyapunov stability theorem, [135])**.** *Consider the system* (2.38) *and a neighbourhood of the origin* $\mathscr{P} \subseteq \mathbb{R}^n$ *with* $0 \in \mathscr{P}$ *. The origin is uniformly locally asymptotically stable for the system* (2.38) *if the system admits a smooth Lyapunov function* V *in the set* $\mathscr{P}$ *(as in Definition 2.3.5).* ■

It is important to remark here that the equilibrium point (origin) in the preceding definitions and theorem can be replaced by a compact (closed and bounded) positively invariant set $\mathscr{Q}$ [122]. In that case, the distances are measured with respect to the set $\mathscr{P}$ as $\|x\|_{\mathscr{Q}} = \inf_{\xi \in \mathscr{Q}} \|x - \xi\|$.

Definition 2.3.6 (Uniform ultimate boundedness, [31])**.** Let $\mathscr{P} \subseteq \mathbb{R}^n$ be a neighbourhood of the origin with $0 \in \mathscr{P}$. Then the system (2.38) is said to be uniformly ultimately bounded in the set $\mathscr{P}$ if there exists a scalar $c > 0$ and for every scalar $a \in (0, c)$, there is $T = T(a, \mathscr{P}) \in \mathbb{N}$ such that

$$\|x_0\| \leq a \Rightarrow \varphi(k, x_0) \in \mathscr{P}, \quad \forall k \geq k_0 + T(\eta)$$

for all $k_0 \in \mathbb{N}$. □

In what follows, we recall how Lyapunov analysis can be used to study ultimate boundedness of dynamical systems. Briefly, we require that there exist a bounded neighbourhood of the equilibrium (origin) that is positively invariant. Additionally, we require that there exist a continuous positive-definite function that behaves like a Lyapunov function outside the bounded neighbourhood of the equilibrium (origin).

Definition 2.3.7 (Sub-level sets)**.** Given a function $\Psi : \mathbb{R}^n \to \mathbb{R}$ and a scalar $\nu \in \mathbb{R}$, the sub-level set of Ψ is defined as the set

$$\mathscr{N}(\Psi, \nu) = \left\{ x \in \mathbb{R}^n \,\middle|\, \Psi(x) \leq \nu \right\}. \qquad \square$$

Definition 2.3.8 (Lyapunov-like function, [31])**.** Let $\mathscr{P}$ be a neighbourhood of the origin. A continuous positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be a Lyapunov-like function outside $\mathscr{P}$ for the system (2.38) if the following conditions are satisfied:

1. there exists a scalar $\nu > 0$ such that $\mathcal{N}(V,\nu) \subseteq \mathcal{P}$,
2. there exists a function α_3 of class $\mathcal{K}$ such that

$$x \notin \mathcal{N}(V,\nu) \Rightarrow \Delta V(x) = V(F(k,x)) - V(x) \leq -\alpha_3(\|x\|).$$

3. $V(F(k,x)) \leq \nu$ for all $x \in \mathcal{N}(V,\nu)$ and for all $k \in \mathbb{N}$. □

The notion of a Lyapunov-like theorem can now be used to provide stability theorem for ultimate boundedness of (2.38).

Theorem 2.3.2 (Ultimate boundedness stability, [31])**.** *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a bounded neighbourhood of the origin with $0 \in \mathcal{P}$. The system* (2.38) *is uniformly ultimately bounded in the set $\mathcal{P}$ if it admits a Lyapunov-like function outside $\mathcal{P}$ (as in Definition 2.3.8).* ■

To facilitate analysis, a candidate Lyapunov-like function can be generated as the gauge function of a compact convex set containing the origin. This gives rise to set-induced Lyapunov functions.

Definition 2.3.9 (C-set, [31])**.** A compact convex set $\mathcal{P} \subseteq \mathbb{R}^n$ containing the equilibrium (origin) in its interior is called a C-set. □

Given a C-set, we define a class of functions whose sub-level sets are obtained by linearly scaling the C-set.

Definition 2.3.10 (Minkowski gauge function, [31])**.** The Minkowski gauge function induced by a given C-set $\mathcal{P}$ is defined as

$$\Psi_{\mathcal{P}}(x) = \inf\left\{\mu \geq 0 \,\middle|\, x \in \mu\mathcal{P}\right\}.$$

The Minkowski gauge function satisfies the following properties:

- *Positive definiteness*: $\Psi_{\mathcal{P}}(x) \geq 0$, and $\Psi_{\mathcal{P}}(x) = 0 \Leftrightarrow x = 0$,
- *Positive homogeneity*: $\Psi_{\mathcal{P}}(\mu \cdot x) = \mu \cdot x$ for $\mu > 0$ and all $x \in \mathbb{R}^n$,
- *Sub-additivity*: $\Psi_{\mathcal{P}}(x+y) \leq \Psi_{\mathcal{P}}(x) + \Psi_{\mathcal{P}}(y)$ for all $x, y \in \mathbb{R}^n$. □

The Minkowski gauge function of a given C-set is continuous and convex, by definition. Its unit sub-level set results in the C-set. The function is a norm if and only if it is 0-symmetric, $\Psi(x) = \Psi(-x)$.

2.3.2. SET-VALUED DYNAMICAL SYSTEMS

A set-valued dynamics, on $(\mathbb{R}^n, \|\cdot\|)$, is defined as a difference inclusion

$$x(k) \in T(x(k-1)), \quad k \in \mathbb{N}, \tag{2.39}$$

where the (set-valued) map T assigns to each $x \in \mathbb{R}^n$ a (non-empty) set $T(x) \subseteq \mathbb{R}^n$. Note that the system definition allows a non-deterministic evolution of the state. We still speak of a trajectory of the system (2.39) as $\varphi(k; x_0)$ such that $\varphi(0, x_0) = x_0$ and $\varphi(k, x_0) \in T(\phi(k-1, x_0))$ for all $k \in \mathbb{N}$.

A particularly interesting case arises when the set-valued map is generated by a finite number $L \in \mathbb{N}$ of continuous mappings:

$$T(x) = \left\{F(l,x) \,\middle|\, l \in \underline{L}\right\}, \quad x \in \mathbb{R}^n. \tag{2.40}$$

If the map $T = \{F(x)\}$ is a singleton, we retrieve the time-invariant dynamical system.

The notion of continuity extends to such set-valued maps from the continuity of the constituent dynamics $F(l,\cdot)$ for all $l \in \underline{L}$ [46, Lemma 4.2]. A point $x^* \in \mathbb{R}^n$ is said to be am equilibrium point of T if $x^* \in T(x^*)$.

The notion of positive invariance carries over to set-valued dynamics from its time-varying counterpart: A closed set $\mathscr{P} \subseteq \mathbb{R}^n$, containing the origin in its interior, is said to be positively invariant to the dynamics (2.39) if for all $x \in \mathscr{P}$, we have $T(x) \subseteq \mathscr{P}$.

Definition 2.3.11 (Weak positive invariance)**.** A closed set $\mathscr{P} \subseteq \mathbb{R}^n$, containing the origin in its interior, is said to be weakly positive invariant to the dynamics (2.39) if for all $x \in \mathscr{P}$, there exists $y \in \mathscr{P}$ such that $y \in T(x)$. Equivalently, $T(x) \cap \mathscr{P} \neq \emptyset$ for any $x \in \mathscr{P}$. □

Definition 2.3.12 (Attractivty of a set)**.** Consider a set-valued dynamics (2.39) evolving on a positively invariant set $\mathscr{P} \subseteq \mathbb{R}^n$. A trajectory $\varphi : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ is said to *approach* a set $\mathscr{Q} \subseteq \mathscr{P}$ if for every neighbourhood $\mathscr{R}$ of $\mathscr{Q}$, there exists a time $K \in \mathbb{N}$ such that $\phi(k,\cdot) \in \mathscr{R}$ for all $k \geq K$.

The set $\mathscr{Q}$ is said to be *locally attractive* for (2.39) if there exists a neighbourhood $\mathscr{R}$ of $\mathscr{Q}$ such that every evolution with $x(0) \in \mathscr{R}$ approaches the set $\mathscr{Q}$. □

In what follows, we recall the Krasovskii-Lasalle invariance principle for set-valued dynamics (2.39) when the evolution is a priori bounded in a closed positively invariant set. Such a stability notion is interesting because: *i)* it does not require a strict Lyapunov function (as in Definition 2.3.5), *ii)* it allows estimation of attractors of discrete-time systems thus resulting in asymptotic properties of the dynamics.

Theorem 2.3.3 (Krasovskii–LaSalle invariance principle for set-valued maps, [46])**.** *Consider the set-valued dynamics* (2.39) *generated by finitely many continuous mappings* (2.41) *evolving on a compact positively invariant set* $\mathscr{P} \subseteq \mathbb{R}^n$*.*

Assume that there exists a continuous function $U : \mathscr{P} \to \mathbb{R}$ *such that* $U(x') \leq U(x)$ *for all* $x \in \mathscr{P}$ *and* $x' \in T(x)$*. Then there exists a constant* $c \in \mathbb{R}$ *such that each trajectory generated by* (2.39) *starting in* $\mathscr{P}$ *approaches the set* $\mathscr{M} = \mathscr{K} \cap U^{-1}(c)$ *where* $\mathscr{K}$ *is the largest weakly positively invariant set contained in*

$$\left\{x \in \mathscr{P} \,\middle|\, \exists x' \in T(x) \text{ such that } U(x') = U(x)\right\} \qquad \blacksquare$$

It is particularly important to note that the analysis of the time-varying system (2.38) can also be performed on an appropriately defined set-valued dynamical system (2.39). Herein, the evolution of the system (2.38) is over-approximated by considering a larger set of evolutions in (2.39):

$$T_{\mathrm{F}}(x) = \left\{F(k,x) \,\middle|\, k \in \mathbb{N}\right\}. \tag{2.41}$$

Then, any trajectory defined by (2.38), from a given initial time, is contained in the set of trajectories defined by (2.39) and (2.41). We end the subsection with the following lemma.

Lemma 2.3.1 (Over-approximation Lemma, [47])**.** *Consider the time-varying nonlinear dynamics* (2.38) *and its set-valued approximation* (2.39), (2.41). *Let a closed set* $\mathscr{P} \subseteq \mathbb{R}^n$ *be locally attractive with respect to the set-valued dynamics (as in Definition 2.3.12). Then the set* $\mathscr{P}$ *is uniformly locally attractive with respect to time-varying dynamics* (2.38). ■

2.4. PIECEWISE-AFFINE SYSTEMS

This thesis deals with the system-theoretic analysis of discrete-event system descriptions, consisting of both continuous and discrete variables, evolving over a discrete-event counter. In this section, we give an overview of discrete-time hybrid systems that consist of continuous and discrete variables evolving over a discrete-time counter. There is a remarkable analogy between such system descriptions [5, 118] and the continuous-valued description of certain subclasses of discrete-event systems [17, 61]. The methods described in this section form a fundamental basis for several analysis methods derived in this dissertation. In this section, we assume that the dynamics and geometrical objects are defined on a finite-dimensional Euclidean normed space $(\mathbb{R}^n, \|\cdot\|)$.

2.4.1. POLYHEDRA AND FUNCTIONS

We recall the notions of polyhedra and polyhedral functions for completeness. This subsection is based entirely on [28, 73, 75, 105, 211]. The notions for boundedness (or compactness) of sets used in this section are understood to be derived from a finite-dimensional Euclidean normed space. The interior, closure, and boundary of a set Γ are denoted as $\text{int}(\Gamma)$, $\text{cl}(\Gamma)$, and $\partial\Gamma = \text{cl}(\Gamma) \setminus \text{int}(\Gamma)$, respectively.

Definition 2.4.1 (Polyhedra and polytopes)**.** A polyhedron is a closed and convex set generated as an intersection of finitely many half-spaces (Half-space representation):

$$\Omega = \left\{x \in \mathbb{R}^n \mid \alpha_j^\top x \leq \beta_j,\ j \in \underline{r}\right\} = \left\{x \in \mathbb{R}^n \mid Ax \leq b\right\}. \tag{2.42}$$

where $r \in \mathbb{N}$, $\{\alpha_j\}_{j \in \underline{r}}$ is a set of vectors in $\mathbb{R}^n$, and $\{\beta_j\}_{j \in \underline{r}}$ is a set of scalars. Here, $[A]_{j\cdot} = \alpha_j^\top$ and $b_j = \beta_j$ for $j \in \underline{r}$.

A polytope is a bounded polyhedron. □

Briefly, a polyhedral set can also be represented as a union of a finite family of polyhedral sets [105, Proposition 2.3].

A *polyhedral cone* is generated as

$$\Upsilon = \left\{x \in \mathbb{R}^n \mid \alpha_j^\top x \leq 0,\ j \in \underline{r}\right\} = \left\{x \in \mathbb{R}^n \,\middle|\, Wx \leq 0\right\}, \tag{2.43}$$

where $r \in \mathbb{N}$ and $\{\alpha_j\}_{j \in \underline{r}}$ is a set of vectors. Here, $[W]_{j\cdot} = \alpha_j^\top$. A polyhedral cone is said to be a *subspace* if for any $x, y \in \Upsilon$ and $\lambda, \mu \in \mathbb{R}$, we have $\lambda x + \mu y \in \Upsilon$.

A polyhedron can also be represented as a sum of a finitely generated cone and a convex-hull of finitely many points.

Definition 2.4.2 (Minkowski-Weyl representation)**.** A set Ω is polyhedral if and only if there exists a finitely generated cone Υ and a finite set of points $\{v_1, v_2 \ldots, v_m\}$ such that

$$\Omega = \left\{x \in \mathbb{R}^n \,\middle|\, x = \sum_{i=1}^{m} \lambda_i v_i + w,\ \sum_{i=1}^{m} \lambda_i = 1,\ \lambda_i \geq 0,\ i \in \underline{m},\ w \in \Upsilon\right\}. \tag{2.44}$$

The preceding representation is referred to as the *vertex representation*.
A polyhedron is a polytope if $\Upsilon = \emptyset$ in (2.44). □

It is important to note here that several operations on polyhedral sets preserve the polyhedral character.

Lemma 2.4.1 (Operations on polyhedral sets, [28]). *The following operations on polyhedral sets result in a polyhedral set:*

1. *Intersection of polyhedral sets;*
2. *Cartesian product of polyhedral sets;*
3. *Image under a linear transformation;*
4. *Inverse image under a linear transformation.* ■

Definition 2.4.3 (Polyhedral partitioning). A partition of a polyhedron $\Omega \subseteq \mathbb{R}^n$ is a collection of sets $\{\Omega\}_{i\in\underline{m}}$ such that:

1. $\Omega_i \neq \emptyset$ for all $i \in \underline{m}$;
2. $\bigcup_{i\in\underline{m}} \Omega_i = \Omega$;
3. $\mathrm{int}(\Omega_i) \cap \mathrm{int}(\Omega_j) = \emptyset$, for all $i, j \in \underline{m}$ with $i \neq j$.

If $\Omega_i \cap \Omega_j \neq \emptyset$ for $i \neq j$ then the intersection forms an ($n-1$ dimensional) common face of Ω_i and Ω_j.
The partition $\{\Omega\}_{i\in\underline{m}}$ is polyhedral if, in addition, Ω_i are polyhedral for $i \in \underline{m}$. □

Definition 2.4.4 (Projection onto a set).

The (lower) *epigraph* of a function $f : \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^n$ is a subset of $\mathbb{R}^{n+1}$:

$$\mathrm{epi}(f) = \left\{(x, \beta) \mid f(x) \leq \beta,\ x \in \Omega,\ \beta \in \mathbb{R}\right\}. \tag{2.45}$$

The convexity and (semi-) continuity of a function are derived from the convexity and closedness of its epigraph, respectively. Consequently, a function is said to be (lower) polyhedral if its epigraph is a polyhedral subset of $\mathbb{R}^{n+1}$.

Definition 2.4.5 (Piecewise-affine functions). A function $f : \Omega \to \mathbb{R}^n$ is piecewise-affine (over the set $\Omega \subseteq \mathbb{R}^n$) if there exists a partition $\{\Omega_i\}_{i\in\underline{m}}$ of Ω such that

$$f(x) = A_i x + f_i, \quad \forall\, x \in \Omega_i,\ i \in \underline{m}. \tag{2.46}$$

The piecewise-affine function f is polyhedral if $\Omega \subseteq \mathbb{R}^n$ is polyhedral with a polyhedral partition $\{\Omega_i\}_{i\in\underline{m}}$. □

Note that a general piecewise-affine function can be multi-valued due to discontinuity on the boundaries of the partitions. A continuous piecewise-affine function attains several useful equivalent formulations [105, Theorem 3.1].

Lemma 2.4.2 (Convex piecewise-affine functions, [105]). *A (lower) polyhedral function $f : \Omega \to \mathbb{R}$ (defined over a polyhedron $\Omega \subseteq \mathbb{R}^n$) is convex if and only if there exist affine functions $g_i : \Omega \to \mathbb{R}$, $i \in \underline{m}$, such that*

$$f(x) = \max_{i\in\underline{m}} (g_i(x)) = \max_{i\in\underline{m}} (\alpha_i^\top x + \beta_i), \qquad \forall\, x \in \Omega, \tag{2.47}$$

where $\alpha_i \in \mathbb{R}^n$, $\beta_i \in \mathbb{R}$, and $m \in \mathbb{N}$. ■

The preceding result leads to the class of max-plus-scaling functions.

Definition 2.4.6 (Max-plus-scaling functions, [211]). A max-plus-scaling function $f : \Omega \to \mathbb{R}$ (defined over a polyhedron $\Omega \subseteq \mathbb{R}^n$) is a convex polyhedral function such that

$$f(x) = \max_{i\in\underline{m}} (\alpha_i^\top x + \beta_i), \qquad \forall\, x \in \Omega, \tag{2.48}$$

where $\alpha \in \mathbb{R}^n_+$ and $\beta \in \mathbb{R}$ for all $i \in \underline{m}$. □

The collection of max-plus-scaling functions, denoted $\mathscr{S}_{\text{mps}}$, is closed under the operations $(\max, +)$, and multiplication by a non-negative scalar [211, Lemma 1]. Moreover, all functions $f \in \mathscr{S}_{\text{mps}}$ are monotone: $x \geq y \Rightarrow f(x) \geq f(y)$ [211, Lemma 2].

Lemma 2.4.3 (Min-max representation of polyhedral functions, [105]). *A function $f : \Omega \to \mathbb{R}$ (defined over a polyhedron $\Omega \subseteq \mathbb{R}^n$) is (lower) polyhedral if only if there exist affine functions $g_{ij} : \Omega \to \mathbb{R}$, $j \in \underline{m_i}$ and $i \in \underline{k}$, such that*

$$f(x) = \min_{i\in\underline{k}} \left(\max_{j\in\underline{m_i}} (g_{ij}(x)) \right), \qquad \forall\, x \in \Omega.$$ ■

The preceding result leads to the class of min-max-plus-scaling functions that are formed recursively from the max-min-plus-scaling grammar [75, 118].

Definition 2.4.7 (Max-min-plus-scaling expression, [74]). The max-min-plus-scaling expression h of the variables $x_1, \ldots, x_n$ is defined by the recursive grammar

$$h := x_i | \alpha | \max(f_k\, f_l) | \min(f_k, f_l) | f_k + f_l | \beta \cdot f_k, \ \alpha, \beta \in \mathbb{R},\ i \in \underline{n}, \tag{2.49}$$

where f_k and f_l are again max-min-plus-scaling expressions. The symbol | stands for "or". □

Note that if the function f in Lemma 2.4.3 is continuous then it can also be written as (point-wise) maximum of (point-wise) minimum of affine functions [105, Theorem 3.1]. In particular, max-min-plus-scaling functions, formed from a max-min-plus-scaling expression, attain max-min and min-max canonical formulations [75, Theorem 3.1].

Consequently, the functions formed from the max-min-plus-scaling grammar are also continuous piecewise-affine [75, Lemma 2.4].

2.4.2. DYNAMICS

The dynamical systems built from piecewise-affine functions (as in Definition 2.4.5) are called piecewise-affine (PWA) systems. Although the class is rather general, we focus mainly on the case when the partitions of the state-space are defined as convex polyhedra. We say that a dynamical system is autonomous if there are no exogenous inputs to the system. This subsection is based entirely on [73, 75, 118].

Definition 2.4.8 (Piecewise-affine systems, [118])**.** The discrete-time piecewise affine systems form a subclass of affine switched systems when the switching is driven by polyhedral partitioning $\{\Omega_i\}_{i\in \underline{L}}$ of the state-space time at time step $k \in \mathbb{N}$:

$$\begin{aligned} x(k) &= A_i x(k-1) + B_i u(k) + d_i \\ y(k) &= C_i x(k) + e_i \end{aligned} \quad \text{for } \begin{bmatrix} x(k-1) \\ u(k) \end{bmatrix} \in \Omega_i \triangleq \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \middle| H_i x + R_i u \le h_i \right\}. \tag{2.50}$$

where $x(\cdot) \in \mathbb{R}^n$, $u(\cdot) \in \mathbb{R}^{n_\mathrm{u}}$, $y(\cdot) \in \mathbb{R}^{n_\mathrm{y}}$ denote the state, input, output, respectively. □

Piecewise-affine systems are a subclass of hybrid systems and form the 'simplest' extension of linear system and can (arbitrarily) approximate non-linear and non-smooth processes. We say that a given piecewise-affine system is continuous if the constituting piecewise-affine functions are continuous on the polyhedral domain. A piecewise-affine systems is said to be *well-posed* if given $x(k-1)$ and $u(k)$, (2.50) is uniquely solvable for $x(k)$ and $y(k)$.

Definition 2.4.9 (Max-min-plus-scaling systems, [118])**.** A discrete-time constrained max-min-plus-scaling system over a counter $k \in \mathbb{N}$ is defined as

$$\begin{aligned} x(k) &= f_{\mathrm{MMPS}}(x(k-1), u(k)), \\ y(k) &= h_{\mathrm{MMPS}}(x(k), u(k)), \\ g_{\mathrm{MMPS}}(x(k), u(k)) &\le c, \end{aligned} \tag{2.51}$$

where $x(\cdot) \in \mathbb{R}^n$, $u(\cdot) \in \mathbb{R}^{n_\mathrm{u}}$, $y(\cdot) \in \mathbb{R}^{n_\mathrm{y}}$, $c \in \mathbb{R}^{n_\mathrm{c}}$, and where the (vector-valued) functions $f_{\mathrm{MMPS}}, g_{\mathrm{MMPS}}, h_{\mathrm{MMPS}}$ are again max-min-plus-scaling expressions. □

The class of (unconstrained) max-min-plus-scaling systems generalises the framework of max-plus linear systems [74] and max-min-plus linear systems [108]. Finally, the following equivalence relationships can be obtained from literature.

Lemma 2.4.4 (PWA to MMPS [117])**.** *Every well-posed piece-wise affine system can be written as a constrained max-min-plus-scaling system.* ■

Lemma 2.4.5 (Continuous PWA to MMPS[75])**.** *The class of continuous piece-wise affine systems and the class of (unconstrained) max-min-min-plus scaling system coincide.* ■

2.4.3. COMPUTATIONAL GEOMETRY

In this section, we delineate the procedure to compute a positively invariant set for a continuous piecewise-affine system, defined on a convex polyhedral partition. This subsection is based entirely on [30, 132, 188, 189].

Let $\{\Omega_i\}_{i\in\underline{m}}$ be a finite number of convex polyhedra forming a partition of the state-space $\mathbb{X} \subseteq \mathbb{R}^n$. Consider the following autonomous discrete-time piecewise-affine system:

$$x(k) = f(x(k-1)), \quad k \in \mathbb{N}, \tag{2.52}$$

subject to the set of constraints:

$$x(k) \in \mathbb{X} \subseteq \mathbb{R}^n, \quad \forall k \in \mathbb{N}. \tag{2.53}$$

Here, the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be continuous piecewise-affine on the polyhedral partition $\{\Omega_i\}_{i\in\underline{m}}$. The constraint set $\mathbb{X}$ is assumed to be polyhedral. For convenience, we write f_i for the affine restriction of the function f to the polyhedron Ω_i.

We recall again the notion of positive invariance.

Definition 2.4.10 (Positively invariant set)**.** A set $\Psi \subseteq \mathbb{X}$ is said to be positively invariant set for the piecewise-affine dynamics (2.52) subject to the constraints (2.53) if for every $x \in \Psi$, we have $f(x) \in \Psi$. □

In case a given set $\Psi \subseteq \mathbb{X}$ is not positively invariant, we are interested in determining the largest positively invariant set contained in Ψ.

Definition 2.4.11 (Maximal positively invariant set)**.** The set $\mathscr{O}_\infty(\Psi)$ is said to be maximal positively invariant set for the dynamics (2.52) subject to the constraints (2.53) if $\mathscr{O}_\infty(\Psi)$ is positively invariant and it contains all positively invariant sets in Ψ. □

We note that the union of two positively invariant sets is also a positively invariant set. It can then be shown that the maximal robust positively invariant set is unique [30]. Then the preceding definition implies that for any positively invariant set Φ, we have

$$\Phi \subseteq \mathscr{O}_\infty(\Psi) \subseteq \Psi.$$

The computation of the maximal positively invariant set for a given dynamical system involves reachability analysis and set propagation. To this end, we recall the definition of the (backward) reachable set.

Definition 2.4.12 (Pre-image set)**.** Consider the system dynamics (2.52) and a set $\Psi \subseteq \mathbb{R}^n$. Then the non-empty set

$$\mathscr{Q}(\Psi) = \left\{x \in \mathbb{X} \,\middle|\, f(x) \in \Psi\right\} \tag{2.54}$$

consists of all states in $\mathbb{X}$ that evolve into the set Ψ in one step. □

Note that a set Ψ is positively invariant for a given dynamics (2.52) if and only if $\Psi \subseteq \mathscr{Q}(\Psi)$ [132]. This also implies that $\mathscr{Q}(\Psi) \cap \Psi = \Psi$ if Ψ is a positively invariant set.

The following algorithm provides a procedure for computing the maximal positively invariant set for a given discrete-time autonomous dynamics (2.52).

Algorithm 1 Computation of $\mathcal{O}_\infty(\mathbb{X})$, [188].

Input: Dynamics (2.52), Constraint set $\mathbb{X}$
Output: $\mathcal{O}_\infty(\mathbb{X})$

$k \leftarrow -1$
$\Psi^{(0)} \leftarrow \mathbb{X}$
while $\Psi^{(k+1)} \neq \Psi^{(k)}$ **do**
  $k \leftarrow k+1$
  $\Psi^{(k+1)} \leftarrow \mathcal{Q}(\Psi^{(k)}) \cap \Psi^{(k)}$ ▷ (2.54)
end while
$\mathcal{O}_\infty(\mathbb{X}) \leftarrow \Psi^{(k)}$

The preceding algorithm generates a non-increasing sequence of sets $\Psi^{(k+1)} \subseteq \Psi^{(k)}$ and results in the maximal positively invariant set as the fixed point of $\mathcal{Q}(\cdot)$, $\mathcal{O}_\infty(\mathbb{X}) = \cap_{k\in\mathbb{N}} \Psi^{(k)}$ [188, Lemma 2]. Moreover, $\Psi^{(k)} = \emptyset$ for a finite $k \in \mathbb{N}$ implies $\mathcal{O}_\infty(\mathbb{X}) = \emptyset$. In general, there can exist an infinite sequence of sets $\{\Psi^{(k)}\}_{k\in\mathbb{N}}$.

The authors in [188] present sufficient conditions to guarantee the finite termination of Algorithm 1 for a continuous piecewise-affine system. Briefly, Algorithm 1 terminates in finite number of steps if the constraint set $\mathbb{X}$ is compact, $\mathbb{X}$ contains the origin, and the system (2.52) is asymptotically stable with respect to the origin [188, Corollary 1].

Algorithm 1 requires three routines for the computation of the maximal positively invariant set at each step $k \in \mathbb{N}$:

1. Computation of pre-image set $\mathcal{Q}(\Psi^{(k)})$;
2. Computation of intersection $\mathcal{Q}(\Psi^{(k)}) \cap \Psi^{(k)}$;
3. Testing set inclusion $\Psi^{(k)} \subseteq \Psi^{(k+1)}$.

For any given polyhedral sets Ω_1 and Ω_2 (in the half-space representation (2.42)), the set inclusion $\Omega_1 \subseteq \Omega_2$ requires the linear inequalities in Ω_2 to be redundant with respect to the set of inequalities in Ω_1. Efficient algorithms for the computation of the intersection and subset testing of two polyhedra can be found in [132, §3.3].

In the following, we briefly explain the procedure to compute the pre-image set for a continuous piecewise-affine system.

We note that, in case of an autonomous dynamics (2.52), the pre-image set $\mathcal{Q}(\Psi)$ is readily calculated if the set Ψ is a union of finitely many convex polyhedra in the half-space representation [189, Theorem 4].

Theorem 2.4.1 (Pre-image set of a piecewise-affine system, [189])**.** *Consider the continuous piecewise-affine system* (2.52) *defined over a polyhedral partition* $\Omega = \cup_{i\in\underline{m}} \Omega_i$ *of the state-space* $\mathbb{X}$*. Then the pre-image set* $\mathcal{Q}(\Omega)$*, defined in* (5.4)*, is again a union of finitely many convex polyhedra.* ■

The following properties of the pre-image set can then be used to compute the pre-image set.

Lemma 2.4.6 (Properties of $\mathcal{Q}(\Psi)$, [132])**.** *Consider the system* (2.52) *defined over the polyhedral partition* $\Omega = \cup_{j\in\underline{m}} \Omega_j$*.*

For any two sets Ψ_1 and Ψ_2 with $\Psi_1 \subseteq \Psi_2$, we have $\mathcal{Q}(\Psi_1) \subseteq \mathcal{Q}(\Psi_2)$.
If $\Psi = \bigcup_{i \in \underline{r}} \Psi_i$ is given as a union of finitely many polyhedral sets, then we have

$$\mathcal{Q}(\Psi) = \bigcup_{i \in \underline{r}} \mathcal{Q}(\Psi_i).$$

Define the pre-image set for individual affine dynamics over a union of convex polyhedra Ψ as

$$\mathcal{Q}_j(\Psi_i) = \left\{x \in \mathbb{X} \,\middle|\, f(x) \in \Psi_i,\ x \in \Omega_j\right\}, \quad i \in \underline{r},\ j \in \underline{m}.$$

Then we have

$$\mathcal{Q}(\Psi) = \bigcup_{i \in \underline{r}} \bigcup_{j \in \underline{m}} \mathcal{Q}_j(\Psi_i). \qquad \blacksquare$$

Hence, the polyhedral partitioning of $\mathcal{Q}(\Psi)$ can be easily derived. We note here that the union of convex polyhedral sets is not necessarily convex or connected.

Finally, we note that most available software for computational geometry require the sets $\Psi^{(k)}$ (as in Algorithm 1) to be compact [209, 219].

2.5. CONCLUSIONS

In this chapter, we have comprehensively discussed the background knowledge and tools that form the prerequisite for the remainder of the dissertation. This includes a primer on max-plus algebra and the related system-theoretical concepts (in Section 2.2), stability theory for conventional discrete-time systems defined on normed spaces (in Section 2.3), and piecewise-affine dynamics along with geometrical tools for invariant set computations (in Section 2.4).

3

Modelling and equivalences

In this chapter, we introduce the novel framework of max-plus-algebraic hybrid automata as a hybrid modelling language in the max-plus algebra. Then we formulate formal relationships with the existing modelling classes of switching max-plus linear (SMPL) systems and max-plus automata. The contents of this chapter are based entirely on [110].

3.1. Introduction

max-plus-algebraic models are particularly suited for modelling discrete-event systems, with synchronisation but no concurrency or choice, when timing constraints on event occurrences are of explicit concern in system dynamics and performance specifications [17, 54, 138]. The max-plus linear modelling class coincides with that of timed-event graphs. Moreover, the modelling formalism provides a continuous-variable dynamic representation of discrete-event systems analogous to time-driven systems. This similarity has served as the key motivation in the development of max-plus linear systems theory, analogously to classical linear systems theory [17, 61]. The theoretical developments find applications in analysis of production systems, timetabling of transportation networks, queuing systems, and so on [138].

Prerequisites: Max-plus algebra; finite automata; set theory; piecewise-affine functions; max-min-plus-scaling functions. Please refer to Chapter 2.

3.1.1. Related work

We begin with a literature review to highlight limitations and further extensions of the max-plus linear framework.

The major limitation of the max-plus linear modelling framework is rooted in its inability to model *competition* and/or *conflict* among several event occurrences [138]. One way to circumvent the limitation is by the use of dual min-plus operations to model conflict resolution policies explicitly in the algebraic system description [17, 60]. A discrete-event system with conflict can also be modelled using a non-stationary system of max-plus linear equations [3, 121]. Then a routing function is incorporated to resolve the choice phenomenon for evaluating the system performance under deterministic routing policies. Such models are obtained ad-hoc based on applications [3, 121].

Automata-theoretic models for discrete-event systems are particularly suited for modelling conflicts and certain forms of concurrency. To this end, models have been proposed in literature that follow a modular approach by allowing the conflict resolution mechanism be handled by a discrete variable taking values in a finite set (event alphabet) [85, 214]. The resulting hybrid phenomenon due to the interaction of the discrete-valued and continuous-valued dynamics is the focus of this chapter. In this context, there are two layers of behaviour that are studied: logical ordering of the events on the one hand, and the timing of events on the other.

The max-plus automata approach (Section 3.2.4) for modelling the aforementioned hybrid phenomenon forms an extension of finite automata where transitions are given weights in the max-plus algebra [85]. The weight encodes the timing information as the price of making the transition. The output under a given input sequence over an event alphabet is then evaluated, in the max-plus algebra, as the accumulated weight. Such models lend themselves to path-based performance analysis for discrete-event systems [85, 95].

An alternative approach involves the SMPL modelling paradigm (Section 3.2.3). Such models extend the max-plus linear modelling framework by allowing changes in the structure of synchronisation and ordering constraints as the system evolves [2, 214]. This offers a compromise between the powerful description of hybrid systems and the decision-making capabilities in max-plus algebra [74, 214]. Moreover, the SMPL formalism offers the flexibility of explicitly modelling different switching (or, conflict resolution) mechanisms between the operating modes in a single framework (see Section 3.2.3). However, this modelling formalism abstracts the mechanism by which transitions are orchestrated again into discrete events. This, in particular, allows modelling only the aggregated dynamics from one mode to the other. The conservativeness then lies in the difficulty in picking appropriate partitions of the state space for continuous or discrete control using mixed integer programs.

3.1.2. Statement of contribution

We propose a novel *max-plus-algebraic hybrid automata* (Definition 3.3.1) framework to model discrete-event systems analogously to the hybrid automata framework of [158, 160] for conventional time-driven systems. In the proposed framework, the discrete-valued dynamics is represented as a labelled oriented graph and the continuous-valued dynamics is associated to each discrete state. We formally prove that this serves as a unifying framework for studying the aforementioned models and their equivalence relationships in the behavioural framework [124, 218, 222]. We also provide a finite-state discrete abstraction procedure for a subclass of the max-plus-algebraic hybrid automata (Proposition 3.3.1) that preserves the state-transition structure of the underlying discrete-event system.

We show that the modelling framework unifies and extends the switching max-plus linear systems framework (Theorems 3.4.1 and 3.4.2). This also serves as another step towards importing tools for analysis and optimal control from conventional time-driven hybrid systems [209, 216] to discrete-event systems in max-plus algebra. In addition, we show that the framework serves as a bridge between automata-theoretic models in max-plus algebra and switching max-plus linear systems. In doing so, we formalise the relationship between max-plus automata and switching max-plus linear systems in a behavioural sense (Proposition 3.4.3). We also formulate an equivalence relationship between the finite-state discrete abstractions of a max-plus automaton and the proposed max-plus-algebraic hybrid automaton (Theorem 3.4.4).

Due to the affinity of the SMPL and the discrete piecewise-affine modelling framework [118, 216], we generate a modelling hierarchy for SMPL models under different switching mechanisms (see Fig. 3.2) that could be used for further analysis. We then consider the dynamics of a subclass of SMPL systems with max-plus linear mode dynamics where the switching is orchestrated by a partitioning of the state space. Such systems can be rewritten as max-min-plus-scaling (MMPS) systems with both continuous and discrete variables [216, Proposition 4]. This amounts to incorporation of additional (possibly discrete-valued) variables in the system description so that the SMPL system can be written as a constrained system of difference equations in the max-plus algebra. We present an novel procedure to obtain this relationship (Proposition 3.2.1), in that we replace the requirement on the boundedness of the state space (as in [216, Proposition 4]) with a weaker assumption of non-negativity of the state space.

3.1.3. ORGANISATION OF THE CHAPTER

The chapter is organised as follows. Section 3.2 reviews the literature on discrete-event systems in max-plus algebra focusing on SMPL and max-plus automata frameworks. Section 3.3 introduces the unifying modelling framework of max-plus-algebraic hybrid automata and its finite-state discrete abstraction. Section 3.4 establishes the relationships among different modelling classes namely, SMPL, max-plus automata, and the proposed max-plus-algebraic hybrid automata. Section 3.5 illustrates the modelling of a production line in the proposed max-plus-algebraic hybrid automata framework. The chapter ends with concluding remarks in Section 3.6.

3.2. MAX-PLUS-ALGEBRAIC MODELS OF DISCRETE-EVENT SYSTEMS

This section aims at recapitulating models in the max-plus algebra that capture synchronisation as well as certain forms of concurrency in discrete-event systems. For simplification of the exposition and for further systematic comparisons, we also present a common description of the underlying signals in discrete-event systems.

3.2.1. SYNCHRONISATION AND CONCURRENCY

The max-plus-algebraic modelling paradigm characterises the behaviour of a discrete-event system by capturing the sequences of occurrence times of events (or, *temporal evolution*) over a discrete event counter. This, in particular, is useful when the events are ordered by the phenomena of synchronisation (max operation), competition (min operation), and time delay (plus operation) [17]. The phenomenon of concurrency arising due to variable sequencing (and hence variable synchronisation and ordering structure) of events can lead to a semi-cyclic behaviour [217]. Below we discuss two different modelling approaches, namely SMPL systems and max-plus automata, that extend the max-plus linear framework to incorporate such concurrency. Here, the shared characteristic is the introduction of a discrete variable that completely specifies the ordering structure at a given event counter. This interaction of synchronisation and concurrency is, thus, *hybrid* in nature.

3.2.2. SIGNALS IN DISCRETE-EVENT SYSTEMS

We refer to variables with finite or countable valuations as *discrete*, and variables with valuations in $\overline{\mathbb{R}}_\varepsilon$ as *continuous*. An event-driven system with both continuous and discrete variables evolving over a discrete counter k is characterised by the following signals:

- $x(\cdot)$ and $l(\cdot)$: continuous and discrete states, respectively;
- $u(\cdot)$ and $v(\cdot)$: continuous and discrete controlled inputs respectively;
- $y(\cdot)$: continuous output;
- $r(\cdot)$ and $w(\cdot)$: continuous and discrete exogenous inputs respectively. The signal $r(\cdot)$ can represent a reference signal or a max-plus additive/multiplicative uncertainty in the continuous state x. The signal $w(\cdot)$ can represent a scheduling signal or uncertainty in mode switching.

The uncontrolled exogenous inputs, hereafter, are collected into a single signal $\Theta(\cdot)$. This signal is partitioned as $\Theta = [r^\top,\ w^\top]^\top$. Here r denotes the uncertainty in the continuous-state evolution, and w denotes the uncertainty in the discrete-state evolution.

3.2.3. SWITCHING MAX-PLUS LINEAR SYSTEMS

The dynamics of a general discrete-event system model in max-plus algebra in mode $l(k) \in \mathscr{L} \triangleq \underline{n_\mathrm{L}}$ for the continuous state $x(k) \in \overline{\mathbb{R}}_\varepsilon^n$ at event counter $k \in \mathbb{N}$ can be written as follows:

$$\begin{aligned} x(k) &= f(l(k), x(k-1), u(k), r(k)), \\ l(k) &= \phi(l(k-1), x(k-1), u(k), v(k), w(k)), \\ y(k) &= h(l(k), x(k), u(k), r(k)) \end{aligned} \tag{3.1}$$

where the functions $f(\cdot)$ and $h(\cdot)$ represent the evolution of the continuous state and output, respectively, as MMPS functions[1]. The function $\phi(\cdot)$ encodes the switching mechanism.

We refer to an open-loop SMPL system, $\mathscr{S}_\mathrm{O}$, when the functions f and g are max-plus linear in states and inputs for a fixed l and control inputs u and v are absent. On the other hand, we refer to a *controlled* SMPL system, $\mathscr{S}_\mathrm{C}$, when a controller is also part of the system description. The control inputs in (3.1) can then be modelled as outputs of

[1] See section 2.4.1

a control algorithm:

$$\begin{aligned} u(k) &= f_{\mathrm{C}}^{(u)}(z(k), \Theta(k)) \\ v(k) &= f_{\mathrm{C}}^{(v)}(z(k), \Theta(k)). \end{aligned} \tag{3.2}$$

Here, the signal $z(\cdot)$ denotes the performance signal composed of the (past) known values of the continuous and discrete states, and continuous inputs. It is noted here that the functions $f_{\mathrm{C}}^{(\cdot)}$ might not have a closed form. The most popular control algorithms for continuous-valued discrete-event systems in literature are residuation [163] and model predictive control [216].

An important subclass of controlled SMPL systems can be represented using max-min-plus linear functions[2]. This encompasses the class of max-plus linear systems in open-loop and closed-loop with static [214] and certain dynamic feedback controllers (for e.g., via residuation [143]). The max-min-plus linear functions can also be used to model the dynamics of a subclass of timed Petri nets under a first-in first-out policy [180, 203].

A controlled SMPL system can be represented as the connection of an MMPS dynamics f in (3.1) and a controller dynamics f_{C} in (3.2) via a switching mechanism ϕ in (3.1). The controlled SMPL system can then be represented as a modification of discrete hybrid automata proposed in [209] as shown in Fig. 3.1. The major differences between our framework and that of [209] are: *i)* the control algorithm is explicitly included in the model description, and *ii)* the mode selector can also model a discrete dynamic process. The mode dynamics, however, is still piecewise affine due to the equivalence of max-min-plus-scaling and piecewise-affine systems under fairly non-restrictive assumptions on boundedness and well-posedness of the dynamics [118, 216].

In the sequel, we will adopt a more general representation for the transition notation. We denote by (l^+, x^+) the successors of the current global state (l, x). Similarly, we denote by (l^-, x^-) the known state information that could possibly contain some parts of the current global state (l, x) [212].

SWITCHING MECHANISM

The dynamic evolution of the discrete state l can be brought about by either *i)* a discontinuous change in the continuous dynamics $f(\cdot)$ when the states satisfy certain constraints, or *ii)* in a non-autonomous response to an exogenous event occurrence via the signal $w(\cdot)$. We refer to the dynamics as *autonomous* in the absence of exogenous inputs.

We refer to the discrete evolution as *controlled* when the controller (via (3.2)) is incorporated into the system description in (3.1). Now we classify the switching mechanisms due to autonomous/non-autonomous and controlled/uncontrolled behaviour [214, 216]. The notions are then sub-classified in increasing order of complexity, where the first case(s) are special cases of the last one:

1. State-dependent switching: The function ϕ does not depend on exogenous inputs. The switching class can be segregated based on the presence or absence of controllers:

[2] See Section 2.4.1 for a complete definition.

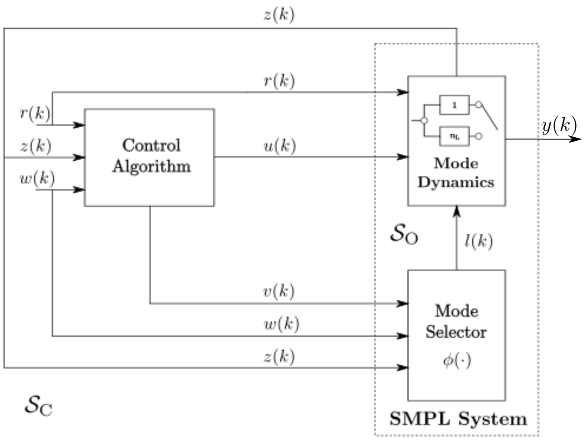

Figure 3.1: The SMPL system in closed loop $\mathscr{S}_{\text{C}}$ with a controller, represented as a subclass of the class of discrete hybrid automata.

1a. Autonomous switching: The controller is either absent or contains only memoryless maps that can be incorporated in the dynamics f:

$$\begin{bmatrix}(l^+)^\top & x^\top\end{bmatrix}^\top = f_\phi(l, x^-). \tag{3.3}$$

1b. Autonomous controlled switching: The control algorithm, in this case, is explicitly part of the system description:

$$\begin{bmatrix}(l^+)^\top & x^\top & (u^+)^\top & (v^+)^\top\end{bmatrix}^\top = f_{\phi,\text{C}}(l, x^-, u, v). \tag{3.4}$$

2. Event-driven switching: The function ϕ depends only on discrete inputs (exogenous or controlled) and discrete states. The class can be subdivided as follows:

2a. Externally driven switching: The switching sequence is completely (arbitrarily) specified by a discrete exogenous input. Therefore, the switching happens uncontrollably in response to exogenous events:

$$l = \phi(w). \tag{3.5}$$

2b. Constrained switching: The switching sequence is driven by a discrete exogenous input with constraints on allowed sequences:

$$l = \phi(l^-, w). \tag{3.6}$$

2c. Constrained controlled switching: A combination of an exogenous discrete input and a discrete control input together describe the switching sequence along with constraints on allowed sequences:

$$l = \phi(l^-, v, w). \tag{3.7}$$

MODELLING HIERARCHY

In this subsection, we classify the discrete-event systems (Eq. (3.1), (3.2)) that arise due to different modelling choices for SMPL systems. The resulting modelling hierarchy consists of classes of models obtained by specifying the switching mechanism (as described in the preceding subsection) in (3.1), (3.2).

The modelling classes are categorised as follows (Fig. 3.2) in the absence of uncertainties:

1. Open-loop: There is no control algorithm. The system has the usual SMPL representation (3.1):
   - (**I**) Non-autonomous switching: The signal $w(\cdot)$ appears as an exogenous input.
   - (**II**) Non-autonomous switching with additive input: The signals $r(\cdot)$ and $w(\cdot)$ appear as exogenous inputs.
2. Closed-loop: There is a control algorithm (3.2) that is considered to be part of the system:
   - (**III**) For some algorithms like output (or state) feedback [177], the closed-loop system can again be formulated as an SMPL system.
   - (**IV**) Under residuation-based control methods, the closed-loop can be explicitly formulated as belonging to a subclass of max-min-plus-scaling systems, i.e. a max-min-plus linear system [143]. The class of max-min-plus-linear systems is known to be a subclass of switching max-plus linear systems [214, Lemma 1].
   - (**V**) For switching based on state-space partitioning, the closed-loop system can be modelled as a max-min-plus-scaling system with both continuous and discrete variables.1

The five broad classes (**I**-**V** in Fig. 3.2) form the basis for analysis (in the subsequent chapter) of max-plus-algebraic hybrid dynamical systems. Each case can be modelled as (3.1) and (3.2) resulting in specific properties of the functions $f(\cdot)$ and $\phi(\cdot)$.

TRANSLATION OF SMPL TO MMPS WITH DISCRETE VARIABLES

In this subsection, we show that an SMPL system with max-plus linear mode dynamics and switching based on state-space partitioning can be rewritten as a max-plus-scaling system with discrete variables. The procedure differs from [216, Proposition 4] as it does not require boundedness of state and input variables. We refer to the resulting modelling class as max-min-plus-scaling system with discrete variables.

Definition 3.2.1. (Autonomous controlled SMPL system [216])**.** An SMPL system with max-plus linear mode dynamics is described for event step $k \in \mathbb{N}$ as

$$\begin{aligned} x(k) &= A^{(l(k))} \otimes x(k-1) \oplus B^{(l(k))} \otimes u(k) \\ y(k) &= C^{(l(k))} \otimes x(k), \end{aligned} \tag{3.8}$$

where $A^{(l)}\mathbb{R}_{\varepsilon}^{n\times n}$, $B^{(l)} \in \mathbb{R}_{\varepsilon}^{n\times 1}$ and $C^{(l)} \in \mathbb{R}_{\varepsilon}^{1\times n}$ are system matrices for mode $l \in \underline{n_{\mathrm{L}}}$.

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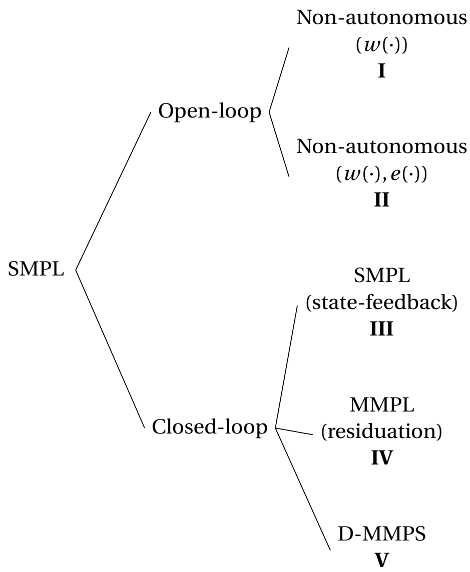

Figure 3.2: Modelling hierarchy for switching max-plus linear systems (left to right). The exogenous signals $w(\cdot)$, and $e(\cdot)$ belong to certain classes $\mathscr{W}$ and $\mathscr{E}$ respectively. MMPL: Max-min-plus linear system [143], and D-MMPS: max-min-plus-scaling system with discrete variables [216].

The switching mechanism of an autonomous controlled SMPL system can be expressed as

$$[l(k) = m] \Leftrightarrow \left[z(k) = \begin{pmatrix} l(k-1) & x^\top(k-1) & u^\top(k) & v^\top(k)\end{pmatrix}^\top \in \Omega_m\right], \tag{3.9}$$

where $v(\cdot)$ is an auxiliary control variable. Here, $\{\Omega_m\}_{m\in\underline{n_\mathrm{L}}}$ is a polyhedral partition of $\Omega \subset \mathbb{R}^{n_\mathrm{z}}$ such that $\bigcup_{m\in\underline{n_\mathrm{L}}} \Omega_m = \Omega$. ♦

We parametrise the elements of the polyhedral partition as

$$\Omega_m = \{z(k) \,|\, R_m z(k) \preceq_m r_m\}, \quad R_m \in \mathbb{R}^{p_m\times n_\mathrm{z}}, \quad r_m \in \mathbb{R}^{p_m} \tag{3.10}$$

where $\preceq_m$ is a vector with entries $\le$ or $<$. Also, p_m denotes the number of supporting hyperplanes of the polyhedron Ω_m.

The following property ensures the existence and uniqueness of the solution to a given autonomous controlled SMPL system.

Definition 3.2.2. (Well-posedness)**.** A autonomous controlled SMPL system as described in Definition 3.2.1 is said to be *well-posed* if the elements of the partition Ω_i and Ω_j are non-overlapping for all $i, j \in \underline{n_\mathrm{L}}$ and $i \neq j$. ♦

A discrete-event system description (4.8) is said to be structurally finite if the states $x(k)$ and output $y(k)$ do not become ε for finite initial states $x(0) \in \mathbb{R}^n$ and finite inputs $u(k) \in \mathbb{R}$. We note that physical systems are typically structurally finite [216].

Lemma 3.2.1. *(Structurally finite SMPL system [216]). An SMPL system representation* (4.8) *is structurally finite if and only if the matrix*

$$H^{(l)} = \begin{pmatrix} A^{(l)} & B^{(l)} & \varepsilon \\ \varepsilon & \varepsilon & C^{(l)} \end{pmatrix} \tag{3.11}$$

contains at least one finite entry in every row for all $l \in \underline{n}_{\mathrm{L}}$*.* ■

An autonomous controlled SMPL system can be rewritten as an equivalent constrained MMPS system by introducing discrete variables [216, Proposition 4]. The derivation involves the "big-M" technique used in [22] to rewrite a piecewise affine model as a mixed-logic dynamical model. This, however, requires the boundedness of the state-space of the underlying system.

For the sake of completeness, we recall the description of a max-min-plus-scaling system.

Definition 3.2.3. (Max-min-plus-scaling expression)**.** A max-min-plus-scaling (MMPS) expression f of the variables $x_1, \dots, x_n$ is defined by the grammar[3]

$$f := x_i | \alpha | f_k \oplus f_l | f_k \oplus' f_l | f_k + f_l | \beta \cdot f_k, \ \alpha, \beta \in \mathbb{R}, \ i \in \underline{n}, \tag{3.12}$$

where f_k and f_l are again MMPS expressions. ♦

Definition 3.2.4. (Constrained MMPS system)**.** A constrained MMPS system is described by a state-space model of the form:

$$x(k) = \mathscr{M}_x(x(k-1), u(k)) \tag{3.13}$$
$$y(k) = \mathscr{M}_y(y(k-1), u(k)) \tag{3.14}$$

where $\mathscr{M}_x$ and $\mathscr{M}_y$ are MMPS expressions.

A constrained MMPS system is defined by (3.13) along with the condition

$$\mathscr{M}_c(x(k-1), u(k)) \preceq c(k), \tag{3.15}$$

where $\mathscr{M}_c$ is again an MMPS expression and $\preceq$ is a vector with entries $\leq$, $=$, or $<$. ♦

We now establish an alternative relationship between an autonomous controlled SMPL system and a constrained MMPS system by rewriting the polyhedral partitions as max-min equations [18]. The procedure does not require the boundedness of the state space. Instead, we make the following assumption.

Assumption 3.2.1. *The finite entries of the system matrices,* $A^{(\cdot)}$, $B^{(\cdot)}$ *and* $C^{(\cdot)}$*, describing the SMPL system* (4.8) *are non-negative. Also the states* $x(\cdot)$ *and input* $u(\cdot)$ *are assumed take non-negative values.* ◊

The preceding assumption is not restrictive in practical situations where the finite elements of the system matrices, and entries of the state and input vectors represent processing/waiting times and event occurrence times, respectively.

[3] The symbol | stands for "or". The definition is recursive.

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Proposition 3.2.1. (SMPL to D-MMPS)**.** *Under Assumption 3.2.1, every well-posed structurally finite autonomous controlled SMPL system (Definition 3.2.1) can be written as a constrained MMPS system with discrete variables.*

Proof. By Assumption 3.2.1, the finite elements of the system matrices are non-negative. Also, the elements of the state $x(\cdot)$ and input $u(\cdot)$ are non-negative.

Consider again the polyhedral partitioning of $\Omega \subset \mathbb{R}^{n_z}$ as in (3.10). These partitions are max-min separable as they are disjoint [18]. An element of the partition Ω_m defines the following max-plus-scaling function

$$\begin{aligned}\varphi_m(z(k)) &= \max_{j\in \underline{p_m}} \left(R_{m,j}z(k) - r_{m,j}\right)\\ &= \bigoplus_{j\in \underline{p_m}} \left(R_{m,j}z(k) - r_{m,j}\right).\end{aligned} \tag{3.16}$$

It can be noted here that $\varphi_m(z(k))$ is a non-positive scalar if $z(k) \in \Omega_m$.

Define a vector with non-negative entries $\tilde{d} \in \mathbb{R}_+^{n_\mathrm{L}}$ such that

$$\tilde{d}(k) = \max(\varphi(k), -\varphi(k)). \tag{3.17}$$

Then given $z(k) \in \mathbb{R}_\varepsilon^{n_z}$, the following statements hold:

- there exists $m \in \underline{n_\mathrm{L}}$ such that $\varphi_m(z(k)) = -\tilde{d}_m(k)$, i.e. $z(k) \in \Omega_m$;
- $\varphi_l(z(k)) = \tilde{d}_l(k)$ for all $l \in \underline{n_\mathrm{L}} \setminus \{m\}$, i.e. $z(k) \notin \Omega_l$ for $l \neq m$.

Define another non-negative scalar using min-plus algebra[4],

$$\bar{x}(k) = \min_{l\in \underline{n_\mathrm{L}}} (\tilde{d}_l(k)) = \tilde{C} \otimes' \tilde{d}(k) \tag{3.18}$$

where $\tilde{C} \in \mathbb{R}_\top^{n_\mathrm{L}}$ is a row vector of zeros (unity element in $\mathbb{R}_\top$). It can be noted that (3.18) is an MMPS expression. Moreover, $\bar{x}(k) \geq 0$. Therefore, for all $l \in \underline{n_\mathrm{L}}$ such that $z(k) \notin \Omega_l$, we have $\bar{x}(k) + \varphi_l(z(k)) > 0$. Then, the following statements hold:

- $\bar{x}(k) \leq \tilde{d}_l(k) = -\varphi_l(z(k))$ for $l = m$;
- $\bar{x}(k) \leq \tilde{d}_l(k) = \varphi_l(z(k))$ for all $l \in \underline{n_\mathrm{L}} \setminus \{m\}$.

The inequality $\bar{x}(k) + \varphi_m(z(k)) \leq 0$ is an MMPS constraint.

Define binary variables $\{\delta_m\}_{m\in\underline{n_\mathrm{L}}} \in \{0, +\infty\}^{n_\mathrm{L}}$ in the min-plus algebra such that

$$[\delta_m(k) = 0] \Leftrightarrow \left[\bar{x}(k) + \varphi_m(z(k)) \leq 0\right] \Leftrightarrow [l(k) = m]. \tag{3.19}$$

Moreover, due to the well-posedness of the SMPL system the condition (3.9) can be rewritten as

$$\bar{x}(k) + \varphi_m(z(k)) \leq \delta_m(k) \tag{3.20}$$

$$\min_{m\in\underline{n_\mathrm{L}}} \delta_m(k) = 0. \tag{3.21}$$

[4] $\mathbb{R}_\top = \mathbb{R} \cup \{+\infty\}$

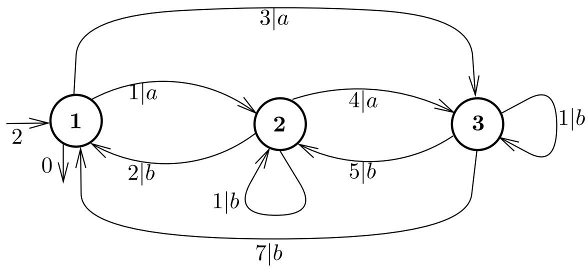

Figure 3.3: A nondeterministic max-plus automaton [85]. An edge label consists of an event label (a letter in Σ) and a weight in the max-plus semiring $\mathbb{R}_{\max}$.

Then the max-plus state space equations in (4.8) can be written as:

$$x(k) = \min_{l(k)\in \underline{n}_{\mathrm{L}}} \left(A^{(l(k))} \otimes x(k-1) \oplus B^{(l(k))} \otimes u(k) \oplus B_2 \otimes \delta_l(k)\right),$$
$$y(k) = \min_{l(k)\in \underline{n}_{\mathrm{L}}} \left(C^{(l(k))} \otimes x(k) \oplus \delta_l(k)\right). \tag{3.22}$$

where $B_2 \in \mathbb{R}_\varepsilon^{n_x\times 1}$ is a vector of zeros. Then, the system (3.22) along with (3.20) and (3.21) is a constrained MMPS system (Definition 3.2.4) representation of the autonomous controlled SMPL system (Definition 3.2.1). This completes the proof. ■

3.2.4. MAX-PLUS AUTOMATA

The max-plus automata are a quantitative extension of finite automata combining the logical aspects from automata/language theory and timing aspects from max-plus linear system [85]. Here, the concurrency is handled at the logical level of the finite automaton. The variable ordering structure in the sequence of events is brought about by the set of accepted input words. The transition labels are augmented with weights in the max-plus semiring. The continuous-variable output dynamics appears as a max-plus accumulation of these weights over the paths accepted by an input word. We now recall the formal definition to elucidate the functioning of a max-plus automaton.

Definition 3.2.5. (Max-plus automata [85]). A max-plus automaton is a weighted finite automaton over the max-plus semiring $\mathbb{R}_{\max}$ and a finite alphabet of inputs Σ represented by the tuple

$$\mathscr{A} = (S, \alpha, \mu, \beta), \tag{3.23}$$

consisting of *i)* S, a finite set states, *ii)* $\alpha : S \to \mathbb{R}_\varepsilon$, the initial weight function for entering a state, *iii)* $\mu : \Sigma \to \mathbb{R}_\varepsilon^{S\times S}$, the transition weight function, and *iv)* $\beta : S \to \mathbb{R}_\varepsilon$, the final weight function for leaving a state. □

A labelled transition between $s, s' \in S$ is denoted as $s \xrightarrow{l|c} s'$ such that $[\mu(l)]_{ss'} = c$ for $l \in \Sigma$. The initial and final transitions are denoted as $\xrightarrow{c_0} s$ and $s' \xrightarrow{c_{\mathrm{f}}}$ such that $\alpha(s) = c_0$ and $\beta(s') = c_{\mathrm{f}}$, respectively. This can be represented by a weighted transition graph (Fig. 3.3).

The discrete (logical) evolution of a max-plus automaton for a given word $\omega_k \in \Sigma^*$ for $k \in \mathbb{N}$ is obtained by concatenating the labelled transitions as an accepting path $\rho_k = (s_0, s_1, \ldots, s_k) \in S^{k+1}$ such that $\alpha(s_0) \neq \varepsilon$, $\beta(s_k) \neq \varepsilon$, and $[\mu(l_i)]_{s_{i-1}s_i} \neq \varepsilon$ for all $i \in \underline{k}$. The language of $\mathscr{A}$ is defined, analogously to that of a finite automaton, as the set of finite words accepted by the max-plus automaton:

$$[\![\mathscr{A}]\!]_{\mathrm{L}} = \{\omega_k \in \Sigma^* \mid \exists \rho_k \in S^* \text{ s.t. } \rho_k \text{ accepts } \omega_k \text{ with } k \in \mathbb{N}\}.$$

The continuous-valued trajectories of a max-plus automaton appear as the maximum accumulated weight over all accepted discrete trajectories. Therefore, it can be expressed completely using max-plus operations on the weights of the transition labels. The output of the max-plus automaton $\mathscr{A}$ for the given word ω_k is obtained over all accepting paths ρ as

$$\begin{aligned} y(\omega_k) := \max_{\rho \in S^{k+1}} \{&\alpha(s_0) + [\mu(l_1)]_{s_0 s_1} + [\mu(l_2)]_{s_1 s_2} + \\ &\cdots + [\mu(l_k)]_{s_{k-1} s_k} + \beta(s_k)\}. \end{aligned} \tag{3.24}$$

Given n states in S, the initial weights $\boldsymbol{\alpha} \in \mathbb{R}_\varepsilon^n$ and final weights $\boldsymbol{\beta} \in \mathbb{R}_\varepsilon^n$ can be identified as vectors and $\boldsymbol{\mu}(l) \in \mathbb{R}_\varepsilon^{n \times n}$ can be identified as a matrix for all $l \in \Sigma$. Then the evolution of the continuous-valued dynamics of the max-plus automaton $\mathscr{A}$ can be represented as [85]:

$$\begin{aligned} x(\omega_k) &= x(\omega_{k-1}) \otimes \boldsymbol{\mu}(l_k), \quad x(\epsilon) = \boldsymbol{\alpha}^\top \\ y(\omega_k) &= \boldsymbol{\alpha}^\top \otimes \boldsymbol{\mu}(l_1) \otimes \boldsymbol{\mu}(l_2) \otimes \cdots \otimes \boldsymbol{\mu}(l_k) \otimes \boldsymbol{\beta} \\ &= x(\omega_k) \otimes \boldsymbol{\beta}. \end{aligned} \tag{3.25}$$

The finite-state discrete abstraction of a max-plus automaton is a finite automaton. It can be obtained by restricting the weights on transitions of $\mathscr{A}$ to the Boolean semiring $\mathbb{B}$ [85]:

$$\mathscr{A}_{\mathrm{T}} = (S, \Sigma, \delta_{\mathscr{A}}, S_0, S_{\mathrm{f}}), \tag{3.26}$$

where the partial transition relation $\delta_{\mathscr{A}} : S \times \Sigma \to 2^S$ is defined such that $s' \in \delta_{\mathscr{A}}(s, l)$ if $[\mu(l)]_{ss'} \neq \varepsilon$. Similarly, we have $s \in S_0$ if $\alpha(s) \neq \varepsilon$ and $s' \in S_{\mathrm{f}}$ if $\beta(s') \neq \varepsilon$. The acceptance condition for a word by the automaton $\mathscr{A}_{\mathrm{T}}$ follows immediately [85]. Moreover, the max-plus automaton and its finite-state discrete abstraction share the same language, i.e. $[\![\mathscr{A}_{\mathrm{T}}]\!]_{\mathrm{L}} = [\![\mathscr{A}]\!]_{\mathrm{L}}$.

Example 3.2.1. *A max-plus automaton (from [85]) with states $S = \{1, 2, 3\}$ over finite alphabet $\Sigma = \{a, b\}$ is depicted in Fig. 3.3. The transition weight functions can be represented as matrices of appropriate dimensions:*

$$\begin{aligned} \boldsymbol{\mu}(a) &= \begin{pmatrix} \varepsilon & 1 & 3 \\ \varepsilon & \varepsilon & 4 \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix}, \quad \boldsymbol{\mu}(b) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ 2 & 1 & \varepsilon \\ 7 & 5 & 1 \end{pmatrix} \\ \boldsymbol{\alpha} &= \begin{pmatrix} 0 & \varepsilon & \varepsilon \end{pmatrix}^\top, \qquad \boldsymbol{\beta} = \begin{pmatrix} 2 & \varepsilon & \varepsilon \end{pmatrix}^\top. \end{aligned} \tag{3.27}$$

The generated language can be obtained from the event labels of the paths originating from the initial state 1 and terminating at the final state 1 in Fig. 3.3. Such words $\omega \in \Sigma^*$ are of the form ab, aab, aabb, and so on.

We can now proceed to the introduction of a unified modelling framework represented by a max-plus-algebraic hybrid dynamical system.

3.3. UNIFIED MODELLING FRAMEWORK

We propose a novel modelling framework of *max-plus-algebraic hybrid automata* for discrete-event systems as hybrid dynamical systems in the max-plus algebra (3.1). The modelling language allows composition with controllers/supervisors and abstraction to refine design problems for individual components. We also propose a finite-state discrete abstraction of the max-plus-algebraic hybrid automaton that preserves the allowed ordering of events of the discrete-event system.

Later, we show (in Section 3.4) that the proposed max-plus-algebraic hybrid automata framework also serves as a link between SMPL systems and max-plus automata. The proposed modelling framework is more descriptive than SMPL systems and max-plus automata in that it allows to capture the different types of interactions between the continuous and discrete evolutions (as presented in Section 3.2.3). Most importantly, the model retains the structure of switching between dynamical systems.

Chapter 3

3.3.1. MAX-PLUS-ALGEBRAIC HYBRID AUTOMATA

A max-plus-algebraic hybrid automaton is presented as an extension of the open hybrid automata in [158, 159] to incorporate max-plus-algebraic dynamics.

Definition 3.3.1. (Max-plus-algebraic hybrid automaton)**.** A max-plus-algebraic hybrid automaton with both continuous and discrete inputs and distinct operating modes can be represented as a tuple

$$\mathcal{H} = (\mathbb{Q}, \mathbb{X}, \mathbb{U}, \mathbb{V}, \mathbb{Y}, \text{Init}, F, H, \text{Inv}, E, G, R, \Lambda) \tag{3.28}$$

where:

- $\mathbb{Q}$ is a finite set of discrete states (or, modes);
- $\mathbb{X} \subseteq \overline{\mathbb{R}}_\varepsilon^n$ is the set of continuous states;
- $\mathbb{U} \subseteq \overline{\mathbb{R}}_\varepsilon^{n_\mathrm{u}}$ is the set of continuous inputs;
- $\mathbb{V}$ is a finite set of discrete inputs;
- $\mathbb{Y} \subseteq \overline{\mathbb{R}}_\varepsilon^{n_\mathrm{y}}$ is the set of continuous outputs;
- $\text{Init} \subseteq \mathbb{Q} \times \mathbb{X}$ is the set of initial states;
- $F : \mathbb{Q} \times \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{X}$ is the continuous-valued dynamics associated to each mode $q \in \mathbb{Q}$;
- $H : \mathbb{Q} \times \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{Y}$ is the continuous-valued output equation;
- $\text{Inv} : \mathbb{Q} \rightarrow 2^{\mathbb{X} \times \mathbb{U} \times \mathbb{V}}$ assigns to each $q \in \mathbb{Q}$ an invariant domain specifying a set of admissible valuations of the state and input variables.

- $G : E \to 2^{\mathbb{X}\times\mathbb{U}\times\mathbb{V}}$ is the collection of guard sets which assigns to each edge $\eta = (q,q') \in E$ the admissible valuation of the state and input variables when transition from the mode q to q' is possible;
- $R := E \times \mathbb{U} \times \mathbb{V} \to 2^{\mathbb{X}\times\mathbb{X}}$ is the collection of reset maps which assigns to each edge $\eta = (q,q') \in E$, $u \in \mathbb{U}$ and $v \in \mathbb{V}$ a destination map specifying the continuous states before and after a discrete transition;
- $\Lambda : \mathbb{Q} \times \mathbb{X} \to 2^{\mathbb{U}\times\mathbb{V}}$ assigns to each state a set of admissible inputs. □

The hybrid state of the max-plus-algebraic hybrid automaton $\mathcal{H}$ is given as $(q,x) \in \mathbb{Q} \times \mathbb{X}$. The hybrid nature stems from the interaction of the discrete-valued state $q \in \mathbb{Q}$ and the continuous-valued state $x \in \mathbb{X}$. Moreover, the valuations of the continuous variables of $\mathcal{H}$ are defined over the completed max-plus semiring $\overline{\mathbb{R}}_\varepsilon$. Therefore, the proposed max-plus-algebraic hybrid automaton forms a novel extension of the hybrid automata framework in [160].

The hybrid state of the max-plus-algebraic hybrid automaton $\mathcal{H}$ is subject to change starting from $(q_0, x_0) \in \text{Init}$ as concatenations of *i)* discrete transitions in the continuous-valued state, to (q_0, x), according to $x = F(q_0, x_0, \cdot)$, as long as the invariant condition of the mode q_0 is satisfied, i.e. $(x,\cdot,\cdot) \in \text{Inv}(q_0)$, and *ii)* discrete transitions in the mode, (q_0, x) to (q', x'), as allowed by the guard set $(x,\cdot,\cdot) \in G(\eta)$, $\eta = (q_0, q')$, while the continuous-valued state changes according to the reset map $(x, x') \in R(\eta,\cdot,\cdot)$.

The exogenous inputs $u \in \mathbb{U}$ and $v \in \mathbb{V}$ allowed by a given hybrid state (q,x), or $(u,v) \in \Lambda(q,x)$, can affect the system evolution through: *i)* the continuous-valued mode dynamics $x' = F(q,x,u)$ and $y = H(q,x,u)$ when $(x,u,v) \in \text{Inv}(q)$, *ii)* the guard sets $(x',u,v) \in G(\eta)$ allowing discrete mode transitions along $\eta = (q,q') \in E$, *iii)* the mode invariants $(x',u,v) \notin \text{Inv}(q)$ forcing discrete mode transitions, and *iv)* the reset maps $(x', x'') \in R(\eta, u, v)$.

3.3.2. FINITE-STATE DISCRETE ABSTRACTION

We now propose a finite-state discrete abstraction of a max-plus-algebraic hybrid automaton (3.28) by embedding it into a finite automaton. The proposed discrete abstraction of $\mathcal{H}$ is a one-step transition system abstracting away valuations of the continuous variables while preserving the state-transition structure of the underlying discrete-event system.

To this end, we define one-step state transition relations corresponding to the mode dynamics F and H based on the underlying directed graph.

Assumption 3.3.1. *The dynamics $F : (q,x,u) \to F(x,q,u)$ and the output function $H : (q,x,u) \to H(x,q,u)$ in (3.28) are max-min-plus functions of the state $x \in \mathbb{X}$ and the input $u \in \mathbb{U}$ for every mode $q \in \mathbb{Q}$. Also, the reset relation is defined such that for a discrete transition allowed by the guard set (i.e. $(x,u,w) \in G(\eta)$ for $\eta = (q,q')$), we have $x = x'$ if $(x,x') \in R(\eta,u,w)$. We denote such a map as $R(\cdot) := R_{\text{id}}(\cdot)$.* ♢

The subclass of max-plus-algebraic hybrid automata modelled using max-min-plus functions is large enough to characterise a broad range of discrete-event systems (see Section 3.2.3). The assumption on the reset relation signifies that the exogenous discrete input via $\mathbb{V}$ does not directly impact the continuous-valued state $x \in \mathbb{X}$.

For convenience, it is also assumed that the functions F and H are in the max-min-plus conjunctive form (2.32). The ambiguity resulting from unspecified ordering of the max-plus projections, in (2.32), is not of consequence to the following analysis. Then, there exist $L, M \in \mathbb{N}$ such that the mode dynamics can be expressed as [108]:

$$\begin{aligned} x^+ = F(q,x,u) &= \min_{l \in \underline{L}} \left(A^{(q,l)} \otimes x \oplus B^{(q,l)} \otimes u \right), \\ y = H(q,x,u) &= \min_{m \in \underline{M}} \left(C^{(q,m)} \otimes x \oplus D^{(q,m)} \otimes u \right). \end{aligned} \tag{3.29}$$

Here, $A^{(q,l)} \in \overline{\mathbb{R}}_{\varepsilon}^{n \times n}$, $B^{(q,l)} \in \overline{\mathbb{R}}_{\varepsilon}^{n \times n_{\mathrm{u}}}$ and $C^{(q,m)} \in \overline{\mathbb{R}}_{\varepsilon}^{n_{\mathrm{y}} \times n}$ for all $q \in \mathbb{Q}$, $l \in \underline{L}$ and $m \in \underline{M}$.

We associate the sets of labels $X_{\mathrm{var}} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$, $U_{\mathrm{var}} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n_{\mathrm{u}}}\}$ and $Y_{\mathrm{var}} = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n_{\mathrm{y}}}\}$ with the continuous-valued state, input and output variables, respectively.

Definition 3.3.2. (One-step state transition graph)**.** Given that Assumption 3.3.1 is satisfied, the one-step state transition graph $\Gamma_F^{(q)} \subseteq (X_{\mathrm{var}} \times X_{\mathrm{var}}) \cup (U_{\mathrm{var}} \times X_{\mathrm{var}})$ of the continuous dynamics $F(q,\cdot,\cdot)$, $q \in \mathbb{Q}$, is defined such that for $(i,j) \in \underline{n}^2$ and $(p,j) \in \underline{n_{\mathrm{u}}} \times \underline{n}$:

$$\begin{aligned} (\mathbf{x}_i, \mathbf{x}_j) \in \Gamma_F^{(q)} &\Leftrightarrow \{\exists l \in \underline{L} \text{ s.t. } [A^{(q,l)}]_{ji} \text{ is finite}\}, \\ (\mathbf{u}_p, \mathbf{x}_j) \in \Gamma_F^{(q)} &\Leftrightarrow \{\exists l \in \underline{L} \text{ s.t. } [B^{(q,l)}]_{jp} \text{ is finite}\}. \end{aligned} \tag{3.30}$$

The one-step state transition graph $\Gamma_H^{(q)} \subseteq (X_{\mathrm{var}} \times Y_{\mathrm{var}}) \cup (U_{\mathrm{var}} \times Y_{\mathrm{var}})$ of $H(q,\cdot,\cdot)$, $q \in \mathbb{Q}$, is defined such that for $(i,j) \in \underline{n} \times \underline{n_{\mathrm{y}}}$ and $(p,j) \in \underline{n_{\mathrm{u}}} \times \underline{n_{\mathrm{y}}}$:

$$\begin{aligned} (\mathbf{x}_i, \mathbf{y}_j) \in \Gamma_H^{(q)} &\Leftrightarrow \{\exists m \in \underline{M} \text{ s.t. } [C^{(q,m)}]_{ji} \text{ is finite}\} \\ (\mathbf{u}_p, \mathbf{y}_j) \in \Gamma_H^{(q)} &\Leftrightarrow \{\exists m \in \underline{M} \text{ s.t. } [D^{(q,m)}]_{jp} \text{ is finite}\}. \end{aligned} \tag{3.31}$$

The transition graph $\Gamma_F^{(\cdot)}$ corresponds to the support of the dynamics F in that the membership of a pair $(\mathbf{x}_i, \mathbf{x}_j)$ in $\Gamma_{\mathrm{F}}^{(\cdot)}$ indicates whether the component F_j is an unbounded function of the coordinate x_i or not. Similarly, the transition graph $\Gamma_H^{(\cdot)}$ corresponds to the support of the output equation H. □

We now propose a finite-state discrete abstraction of the max-plus-algebraic hybrid automaton (3.28). The mode dynamics of the max-plus-algebraic hybrid automaton is abstracted as a one-step transition system. Here, the one-step transition naturally corresponds to the evolution of the discrete-event system in one event step $k \in \mathbb{N}$. Therefore, we denote it with a unique label 1.

Proposition 3.3.1. *(Finite-state discrete abstraction of max-plus-algebraic hybrid automaton)***.** *Consider a max-plus-algebraic hybrid automaton $\mathcal{H}$ (as in (3.28)) under Assumption 3.3.1. We assume that*[5] $\mathbb{X} = \overline{\mathbb{R}}_{\varepsilon}^{n} \setminus \{\varepsilon, \top\}^n$. *Then the max-plus-algebraic hybrid automaton $\mathcal{H}$ generates a finite automaton.*

[5]The set of all vectors in $\overline{\mathbb{R}}_{\varepsilon}^{n}$ with at least one finite entry is denoted as $\overline{\mathbb{R}}_{\varepsilon}^{n} \setminus \{\varepsilon, \top\}^n$.

Chapter 3

Proof. A finite automaton embedding a max-plus-algebraic hybrid automaton can be generated as a one-step transition system:

$$\mathcal{H}_{\mathrm{T}} = (\overline{Q}, \overline{\Sigma}, \delta_{\mathcal{H}}, \overline{Q}_0, \overline{Q}_{\mathrm{f}}), \tag{3.32}$$

that consists of:

- the finite set of states $\overline{Q} = \mathbb{Q} \times (X_{\mathrm{var}} \cup U_{\mathrm{var}})$;
- the input alphabet as a union of mode transition event labels and the one-step transition label denoting state transitions within a mode, $\overline{\Sigma} = \mathbb{V} \cup \{1\}$;
- the set of initial states $\overline{Q}_0$ with $(q, \mathbf{x}_j) \in \overline{Q}_0 \subseteq \mathbb{Q} \times (X_{\mathrm{var}} \cup U_{\mathrm{var}})$ if $(q, x) \in \mathrm{Init}$ and $x_j \neq \varepsilon$, and $(q, \mathbf{u}_p) \in \overline{Q}_0$ if there exists $\mathbf{x}_j \in X_{\mathrm{var}}$ such that $(\mathbf{u}_p, \mathbf{x}_j) \in \Gamma_F^{(q)}$;
- the set of final states $\overline{Q}_{\mathrm{f}}$ with $(q, \mathbf{x}_i) \in \overline{Q}_{\mathrm{f}} \subseteq \mathbb{Q} \times \times (X_{\mathrm{var}} \cup U_{\mathrm{var}})$ if there exists $\mathbf{y}_j \in Y_{\mathrm{var}}$ such that $(\mathbf{x}_i, \mathbf{y}_j) \in \Gamma_H^{(q)}$, and $(q, \mathbf{u}_p) \in \overline{Q}_{\mathrm{f}}$ if there exists $\mathbf{y}_j \in Y_{\mathrm{var}}$ such that $(\mathbf{u}_p, \mathbf{y}_j) \in \Gamma_H^{(q)}$;
- the partial transition function $\delta_{\mathcal{H}} : \overline{Q} \times (\mathbb{V} \cup \{1\}) \to 2^{\overline{Q}}$ is defined as the combination of:
  i) the transition relation corresponding to the one-step evolution inside a mode as $(q, \mathbf{x}_j) \in \delta_{\mathcal{H}}(q, \mathbf{x}_i, 1)$ if $(\mathbf{x}_i, \mathbf{x}_j) \in \Gamma_F^{(q)}$, or $(q, \mathbf{x}_j) \in \delta_{\mathcal{H}}(q, \mathbf{u}_p, 1)$ if $(\mathbf{u}_p, \mathbf{x}_j) \in \Gamma_F^{(q)}$;
  ii) the transition relation corresponding to each edge $\eta = (q, q') \in E$ as $(q', \mathbf{x}_i) \in \delta_{\mathcal{H}}(q, \mathbf{x}_i, w)$ if there exists $w \in \mathbb{V}$. ■

It is noted that the transitions via inputs from $\mathbb{V}$ do not entail transitions in the state $\mathbf{x} \in X_{\mathrm{var}}$. Therefore, the transitions in the mode $q \in \mathbb{Q}$ via $\mathbb{V}$ and one-step state transitions in $\mathbf{x} \in X_{\mathrm{var}}$ are allowed to occur concurrently. Then, the transition $(q', \mathbf{x}_j) \in \delta_{\mathcal{H}}(q, \mathbf{x}_i, w)$ for some $(q, q') \in E$ and $w \in \mathbb{V}$ represents a concatenation of labelled transitions $(q, \mathbf{x}_i) \xrightarrow{1} (q, \mathbf{x}_j)$ and $(q, \mathbf{x}_j) \xrightarrow{w} (q', \mathbf{x}_j)$. A similar statement holds for $(q', \mathbf{x}_j) \in \delta_{\mathcal{H}}(q, \mathbf{u}_p, w)$.

3.4. MODEL RELATIONSHIPS

In this section we formalise the relationships between the classes of SMPL models and max-plus automata described in Section 3.2 and the max-plus-algebraic hybrid automata proposed in Section 3.3.1. To this end, we propose translation procedures among the three modelling classes to further establish partial orders among them.

3.4.1. PRE-ORDER RELATIONSHIPS

We first recall formal notions from literature for comparison of different modelling classes. This subsection is based entirely on [124, 218, 222].

We adopt a behavioural approach towards establishing relationships between different modelling classes, in that the systems are identified as a collection of input-state-output trajectories they allow[6].

[6] The term *recognised* is usually used instead of *allowed* in automata theory [54].

Definition 3.4.1. (Behavioural semantics)**.** The behavioural semantics of a dynamical system is defined as a triple $\Omega = (\mathbb{T}, \mathbb{S}, \mathscr{B})$, where $\mathbb{T}$ is the time axis, $\mathbb{S}$ is the signal space, and $\mathscr{B} \subseteq \mathbb{S}^{\mathbb{T}}$ is the collection of all possible trajectories allowed by the system. The pair $(\mathbb{T}, \mathbb{S})$ is the behavioural *type* of the dynamical system. □

In the context of this chapter, $\mathbb{T} = \mathbb{N}$ represents the event counter axis. The signal space $\mathbb{S}$ is factorised as $\mathbb{S} = \mathbb{D} \times \mathbb{I} \times \mathbb{O}$ into the state space $\mathbb{D}$, input space $\mathbb{I}$, and output space $\mathbb{O}$.

Definition 3.4.2. (Input-output behaviour)**.** Given a behavioural system model $\Omega = (\mathbb{T}, \mathbb{S}, \mathscr{B})$ with $\mathbb{S} = \mathbb{D} \times \mathbb{I} \times \mathbb{O}$ factorised into the state, input and output space, respectively. The input-output behaviour of the system model Ω is the projection of the behaviour $\mathscr{B}$ on the set of input-output signals, $\pi_{\mathrm{IO}}(\mathscr{B}) \subset \mathbb{I}^{\mathbb{T}} \times \mathbb{O}^{\mathbb{T}}$. □

We now proceed to define an input-output behavioural relationship between two dynamical systems.

Definition 3.4.3. (Behavioural equivalence)**.** Consider two dynamical systems $\Omega_i = (\mathbb{T}, \mathbb{D}_i \times \mathbb{I} \times \mathbb{O}, \mathscr{B}_i)$, $i = 1, 2$. The dynamical system Ω_1 is said to be behaviourally included in Ω_2, denoted as $\Omega_1 \preccurlyeq_{\mathrm{B}} \Omega_2$, if $\pi_{\mathrm{IO}}(\mathscr{B}_1) \subseteq \pi_{\mathrm{IO}}(\mathscr{B}_2)$.

The notion of *behavioural equivalence* (denoted as $\Omega_1 \simeq_{\mathrm{B}} \Omega_2$) follows if the said behavioural inclusion is also symmetric. □

The input-output behaviour of a finite automaton can be defined as the collection of all accepted words. In that case, the condition of behavioural equivalence of finite automata implies the equality of their generated languages [124].

We now define pre-order relation that also captures the state transitions structures of two dynamical systems. We first define the concept of a state map.

Definition 3.4.4. (State-map)**.** Given a dynamical system $\Omega = (\mathbb{T}, \mathbb{D} \times \mathbb{I} \times \mathbb{O}, \mathscr{B})$. A *state map* is defined as a map $\varphi : \mathbb{I}^{\mathbb{T}} \times \mathbb{O}^{\mathbb{T}} \times \mathbb{T} \to \mathbb{D}$ such that for every $(x, w, y) \in \mathscr{B}$ and for each $\tau \in \mathbb{T}$ we have $x(\tau) = \varphi(w, y, \tau)$. □

The following notion provides a sufficient condition for demonstrating that an input-output behavioural relationship exists between two dynamical systems.

Definition 3.4.5. (Bisimulation)**.** Consider two dynamical systems $\Omega_i = (\mathbb{T}, \mathbb{D}_i \times \mathbb{I} \times \mathbb{O}, \mathscr{B}_i)$, $i = 1, 2$, and their respective state maps φ_1 and φ_2. A *simulation relation* from Ω_1 to Ω_2, $\Psi : \mathbb{T} \to 2^{\mathbb{D}_1 \times \mathbb{D}_2}$, is defined such that for any $\tau \in \mathbb{T}$ if $(x_1, x_2) \in \Psi(\tau)$ and $(x_1, w_1, y_1) \in \mathscr{B}_1$ where $x_1(\tau) = \varphi_1(w_1, y_1, \tau)$ then there exists $(x_2, w_2, y_2) \in \mathscr{B}_2$ such that $x_2(\tau) = \varphi_2(w_2, y_2, \tau)$, and for all $\tau' \geq \tau$ such that $w_1(\tau') = w_2(\tau')$ we have: *i)* $(\varphi_1(w_1, y_1, \tau'), \varphi_2(w_1, y_2, \tau')) \in \Psi(\tau')$, and *ii)* $y_1(\tau') = y_2(\tau')$.

The dynamical system Ω_1 is said to be *simulated* by Ω_2, $\Omega_1 \preccurlyeq_{\mathrm{S}} \Omega_2$, if a simulation relation exists from Ω_1 to Ω_2.

The notion of *bisimilarity* (denoted as $\Omega_1 \simeq_{\mathrm{S}} \Omega_2$) follows if the said simulation relation is also symmetric. □

Finally, we recall the following result from the literature.

Lemma 3.4.1. *(Simulation to behavioural inclusion [124]). Consider two dynamical systems $\Omega_i = (\mathbb{T}, \mathbb{D}_i \times \mathbb{I} \times \mathbb{O}, \mathscr{B}_i)$, $i = 1, 2$, and their respective state maps φ_1 and φ_2. Then the following implication holds:*

$$\Omega_1 \preceq_{\mathrm{S}} \Omega_2 \Rightarrow \Omega_1 \preceq_{\mathrm{B}} \Omega_2. \qquad \blacksquare$$

We now move on to formalising the relationships between the proposed max-plus-algebraic hybrid automata and the existing frameworks of SMPL systems and max-plus automata.

3.4.2. Equivalent max-plus-algebraic hybrid automata for SMPL systems

In this subsection we show that SMPL systems in open-loop and closed-loop configurations ($\mathscr{S}_{\mathrm{O}}$ and $\mathscr{S}_{\mathrm{C}}$, respectively), are special cases of max-plus-algebraic hybrid automata. To this end, we construct an equivalent restriction of the max-plus-algebraic hybrid automaton. Here, equivalence is expressed in terms of a simulation relation that captures the state transition structure of the SMPL system.

Theorem 3.4.1. *(Max-plus-algebraic hybrid automata simulate open-loop SMPL systems). Given an open-loop SMPL system $\mathscr{S}_{\mathrm{O}}$, there exists a max-plus-algebraic hybrid automaton $\mathscr{H}_{\mathrm{O}}$ that bisimulates it, i.e. $\mathscr{S}_{\mathrm{O}} \simeq_{\mathrm{S}} \mathscr{H}_{\mathrm{O}}$.*

Proof. Consider an open-loop SMPL system $\mathscr{S}_{\mathrm{O}}$ behaviour consisting of states $(l, x) \in \mathbb{D} = \underline{n_{\mathrm{L}}} \times \mathbb{R}_{\varepsilon}^{n}$, inputs $(w, r) \in \mathbb{B}_{\varepsilon}^{n_{\mathrm{L}}} \times \mathbb{R}_{\varepsilon}^{m}$, and output $y \in \mathbb{R}_{\varepsilon}^{d}$ defined on an event counter $k \in \mathbb{N}$. The state maps are defined in (3.1) without the control inputs u and v as $x(k) = f(l(k), x(k-1), r(k))$, $l(k) = \phi(l(k), x(k-1), (w(k), r(k)))$ and $y(k) = h(l(k), x(k), r(k))$. The initial condition is denoted as $x_0 = x(0) \in \mathbb{R}_{\varepsilon}^{n}$.

A max-plus-algebraic hybrid automaton $\mathscr{H}_{\mathrm{O}}$ (as in (3.28)) is constructed with the states $q \in \mathbb{Q} = \underline{n_{\mathrm{L}}}$ and $x_{\mathrm{h}} \in \mathbb{X} = \mathbb{R}_{\varepsilon}^{n}$, the inputs $(w, r) \in \mathbb{I} = \mathbb{V} \times \mathbb{U} = \mathbb{B}_{\varepsilon}^{n_{\mathrm{L}}} \times \mathbb{R}_{\varepsilon}^{m}$, and the output $y_{\mathrm{h}} \in \mathbb{Y} = \mathbb{R}_{\varepsilon}^{d}$. The discrete state characteristics are defined for all $q \in \underline{n_{\mathrm{L}}}$ as: $(q, x_0) \in \mathrm{Init}$, $F(q, \cdot, \cdot) = f(q, \cdot, \cdot)$, $H(q, \cdot, \cdot) = h(q, \cdot, \cdot)$, and $\mathrm{Inv}(q) = \{(x_{\mathrm{h}}, (w, r)) \mid \phi(\cdot, x_{\mathrm{h}}, (w, r)) = q\}$. The edge characteristics are defined for all $(q, q') \in E \subseteq \underline{n_{\mathrm{L}}} \times \underline{n_{\mathrm{L}}}$ as: $G = \{(x_{\mathrm{h}}, (w, r)) \mid \phi(q, x_{\mathrm{h}}, (w, r)) = q'\}$, and $R(\cdot) := R_{\mathrm{id}}(\cdot)$. There are no constraints on the admissible inputs, i.e. $\Lambda(q, x) = 2^{\mathbb{I}}$ for all $(q, x_{\mathrm{h}}) \in \mathbb{X}$.

Note that the two systems share the same state, input and output spaces. An event counter dependent simulation relation can be defined such that for a given $k' \in \mathbb{N}$, if $((l, x), (q, x_{\mathrm{h}})) \in \Psi(k')$ then we have $l(k') = q(k')$ and $x(k') = x_{\mathrm{h}}(k')$. It is now sufficient to show that the two models produce state trajectories, under the same input sequence $(w(k)), r(k))$ for $k \geq k'$, such that $x(k) = x_{\mathrm{h}}(k)$ and $l(k) = q(k)$.

Let $((l, x), (q, x_{\mathrm{h}})) \in \Psi(k')$, $l(k') = l_1$, and $k'' = \inf\{k \in \mathbb{N} \mid \phi(l(k), x(k-1), \cdot) \neq l_1,\ k > k'\}$. We now have that any continuous-valued state trajectory $x(\cdot)$ of the SMPL system inside the mode $l(k) = l_1$, $k \in \{k', k'+1, \ldots, k''-1\}$, also satisfies the invariance condition of the mode $q(k) = l_1$. Then, for the same input sequence we have $x_{\mathrm{h}}(k) = x(k) = f(l_1, \cdot, \cdot)$ as long as $l(k) = q(k) = l_1$. For a mode change $l(k'') = l_2 \neq l_1$ such that $\phi(l_1, x(k''-1), \cdot) = l_2$, the invariance condition of mode $q(k) = l_1$ is also violated in the max-plus-algebraic hybrid automaton resulting in a transition in the state from $(l_1, x_{\mathrm{h}}(k''))$ to $(l_2, x_{\mathrm{h}}(k''))$ with $x(k'') = x_{\mathrm{h}}(k'')$. Thus, $((l, x), (q, x_{\mathrm{h}})) \in \Psi(k)$ for all $k \geq k'$. Moreover, the output function is shared by both the models resulting in $y(k) = y_{\mathrm{h}}(k)$ for all $k \geq k'$.

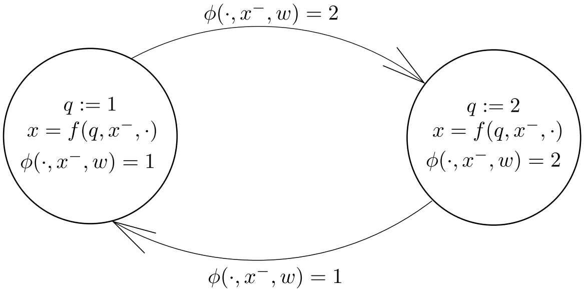

Figure 3.4: A max-plus-algebraic hybrid automaton visualisation of an SMPL system (3.1) with $n_L = 2$ modes. The function $\phi(\cdot)$ encoding the switching mechanism appears in the definition of the mode invariants and as directed edge labels specifying the guard set for mode transition. The reset map is identity.

The simulation relation $\Psi(\cdot)$ is indeed symmetric. Hence, we have $\mathscr{S}_O \simeq_S \mathscr{H}_O$. ■

Theorem 3.4.2. *(Max-plus-algebraic hybrid automata simulate closed-loop SMPL systems)***.** *Given a closed-loop SMPL system $\mathscr{S}_C$, there exists a max-plus-algebraic hybrid automaton $\mathscr{H}_C$ that bisimulates it, i.e. $\mathscr{S}_C \simeq_S \mathscr{H}$.*

Proof. We now consider a closed-loop SMPL system $\mathscr{S}_C$ behaviour consisting of states $(l, z) \in \mathbb{D} = \underline{n_L} \times \mathbb{R}_\varepsilon^{1+n+n_u+n_v}$ where $z(k) = [l(k-1), x^\top(k-1), u^\top(k-1), v^\top(k-1)]^\top$, inputs $(w, r) \in \mathbb{B}_\varepsilon^{n_L} \times \mathbb{R}_\varepsilon^m$, and output $y \in \mathbb{R}_\varepsilon^d$ defined on an event counter $k \in \mathbb{N}$. The state maps are defined as compositions of (3.1) and (3.2) such that $z(k) = f_{\phi,C}(l(k), z(k-1), r(k))$, $l(k) = \phi(l(k), z(k-1), (w(k), r(k)))$ and $y(k) = h_{\phi,C}(l(k), z(k), r(k))$. The initial condition is denoted as $z_0 = z(0) \in \mathbb{R}_\varepsilon^n$.

A max-plus-algebraic hybrid automaton $\mathscr{H}_C$ is constructed with the states $q \in \mathbb{Q} = \underline{n_L}$ and $x_h \in \mathbb{X} = \mathbb{R}_\varepsilon^{1+n+n_u+n_v}$, the inputs $(w, r) \in \mathbb{I} = \mathbb{V} \times \mathbb{U} = \mathbb{B}_\varepsilon^{n_L} \times \mathbb{R}_\varepsilon^m$, and the output $y_h \in \mathbb{Y} = \mathbb{R}_\varepsilon^d$. The discrete state characteristics are defined for all $q \in \underline{n_L}$ as: $(q, z_0) \in \text{Init}$, $F(q, \cdot, \cdot) = f_{\phi,C}(q, \cdot, \cdot)$, $H(q, \cdot, \cdot) = h_{\phi,C}(q, \cdot, \cdot)$, and $\text{Inv}(q) = \{(x_h, (w, r)) \mid \overline{\phi}(\cdot, x_h, (w, r)) = q\}$. The edge characteristics are defined for all $\eta = (q, q') \in E \subseteq \underline{n_L} \times \underline{n_L}$ as: $G(\eta) = \{(x_h, (w, r)) \mid \phi(q, x_h, (w, r)) = q'\}$, and $R(\cdot) := R_{id}(\cdot)$. There are no constraints on the admissible inputs, i.e. $\Lambda(q, x) = 2^{\mathbb{I}}$ for all $(q, x_h) \in \mathbb{X}$.

Then the rest of the proof follows analogously to that of the open-loop case in Theorem 3.4.1. Hence, $\mathscr{S}_C \simeq_S \mathscr{H}_C$. ■

Due to the findings of the preceding theorem, the discrete transition structure of a max-plus-algebraic hybrid automaton can be classified analogously to the switching mechanism of an SMPL system as presented in Section 3.2.3. An open-loop SMPL system with two modes and no continuous-valued inputs is shown in Fig. 3.4.

3.4.3. EQUIVALENT MAX-PLUS-ALGEBRAIC HYBRID AUTOMATA FOR MAX-PLUS AUTOMATA

This section establishes the relationships between max-plus automata and max-plus-algebraic hybrid automata.

We first recall that a max-plus automaton (3.23) provides a finite representation for certain classes of discrete-event systems [85]. A trajectory of a max-plus automaton $\mathscr{A}$

involves transitions among discrete states in S such that a (possibly non-unique) accepting path attains the maximum accumulated weight corresponding to the output (3.24). The auxiliary variable $x(\cdot)$ in (3.25), however, does not constitute the state space. This is in contrast to the SMPL system description (3.1) where the transitions in the hybrid state (l, x) govern the dynamics.

We first treat the problem of generating an equivalent max-plus-algebraic hybrid automaton of a given max-plus automaton behaviourally. We show that a subclass of open-loop SMPL systems (3.1) generates the same input-output behaviour as that of max-plus automata. The required relationship then follows from the notions presented in the preceding section.

Theorem 3.4.3. *(SMPL systems behaviourally include max-plus automata)***.** *Given a max-plus automaton $\mathscr{A}$, there exists an open-loop SMPL system $\mathscr{S}_{\mathrm{OA}}$ that captures its input-output behaviour, i.e. $\mathscr{A} \preccurlyeq_{\mathrm{B}} \mathscr{S}_{\mathrm{OA}}$.*

Proof. We first embed the given max-plus automaton $\mathscr{A} = (S, \Sigma, \alpha, \mu, \beta)$ into a behavioural model consisting of states $s \in \mathbb{D}_1 = S = \{s_1, s_2, \ldots, s_n\}$, inputs $\omega \in \Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, and output $y_a \in \mathbb{R}_\varepsilon$. The input-output behaviour, $\pi_{\mathrm{IO}}(\mathscr{B}_{\mathrm{A}})$, then consists of the language of the max-plus automaton, $[\![\mathscr{A}]\!]_{\mathrm{L}} \subseteq \Sigma^*$, and the output[7], $y_{\mathrm{a}}(\omega_k) = \boldsymbol{\alpha}^\top \otimes \boldsymbol{\mu}(\omega_k) \otimes \boldsymbol{\beta} \in \mathbb{R}$ for $\omega_k = \gamma_1 \gamma_2 \cdots \gamma_k \in [\![\mathscr{A}]\!]_{\mathrm{L}}$, as in (3.25).

We recall that the language of the max-plus automaton is a map $[\![\mathscr{A}]\!]_{\mathrm{L}} : \mathbb{N} \to \Sigma^*$ such that the sequence $\omega_k = \gamma_1 \gamma_2 \cdots \gamma_k \in \Sigma^*$ can be represented as a signal $w(j) = \gamma_j$ for all $j \in \underline{k}$. The output sequence description can be similarly extended and defined along the event counter $k \in \mathbb{N}$.

Consider an open-loop SMPL system $\mathscr{S}_{\mathrm{OA}}$ (as in (3.1)) with the states $(l, x) \in \mathbb{D}_2 = \underline{m} \times \mathbb{R}_\varepsilon^n$, input $w \in \Sigma$, and output $y \in \mathbb{R}_\varepsilon$ defined on an event counter $k \in \mathbb{N}$. The state maps (as in (3.1)) are defined as $x(k) = A^{(l(k))} \otimes x(k-1)$, $l(k) = \phi(\cdot, x(k-1), w(k))$ and output as $y(k) = C \otimes x(k)$ where $C \in \mathbb{R}_\varepsilon^{1\times n}$, $x(0) \in \mathbb{R}_\varepsilon^n$, $A^{(l)} \in \mathbb{R}_\varepsilon^{n\times n}$ for all $l \in \underline{m}$, and

$$\phi(\cdot, x, w) = \left\{ l \in \underline{m} \mid A^{(l)} \otimes x \neq \mathscr{E}_{n\times 1}, w = \sigma_l \right\}. \tag{3.33}$$

For a given initial condition $x(0) \in \mathbb{R}_\varepsilon^n$, the input-output behaviour of the model $\pi_{\mathrm{IO}}(\mathscr{S}_{\mathrm{OA}})$ consists of input sequences $\{w(k)\}_{k\in\mathbb{N}}$ such that $\phi(\cdot, \cdot, w(k)) \neq \emptyset$ and the corresponding output sequences $\{y(k)\}_{k\in\mathbb{N}}$.

It remains to show that for particular valuations of the matrices A and C, the max-plus automaton $\mathscr{A}$ and SMPL system $\mathscr{S}_{\mathrm{OA}}$ generate the same input-output behaviour.

Consider the specifications: *i)* $A^{(l)} = \mu^\top(\sigma_l)$ for all $l \in \underline{m}$, *ii)* $[C]_i = \beta(s_i)$, and *iii)* $x_i(0) = \alpha(s_i)$ for $i \in \underline{n}$. Then using (3.25), given a word $\omega_k = \gamma_1 \gamma_2 \cdots \gamma_k \in [\![\mathscr{A}]\!]_{\mathrm{L}}$ such that $w(j) = \gamma_j$, $j \in \underline{k}$, we have $y_{\mathrm{a}}(\omega_k) = y(k)$ for all $k \in \mathbb{N}$.

Let $x_{\mathrm{a}}(\cdot) \in \mathbb{R}_\varepsilon^{1\times n}$ denote the auxiliary continuous variable satisfying (3.25). Then $x_{\mathrm{a}}(\omega_j) = x_{\mathrm{a}}(\omega_{j-1}) \otimes \boldsymbol{\mu}(\gamma_j) \neq \mathscr{E}_{n\times 1}$ for all $j \in \underline{k}$ when $\omega_k \in [\![\mathscr{A}]\!]_{\mathrm{L}}$. We have $x_{\mathrm{a}}^\top(\omega_j) = x(j) = A^{(l)} \otimes x(j-1) \neq \mathscr{E}_{n\times 1}$. Hence, $l \in \phi(\cdot, x(j-1), w(j))$ in (3.33). Therefore, by induction all finite input sequences ω_k constituting the language of the max-plus automaton also satisfy the condition $\phi(\cdot, \cdot, w(j)) \neq \emptyset$ for $w(j) = \gamma_j$, $j \in \underline{k}$.

For finite input sequences we have $\pi_{\mathrm{IO}}(\mathscr{B}_{\mathrm{A}}) \subseteq \pi_{\mathrm{IO}}(\mathscr{S}_{\mathrm{OA}})$, and hence $\mathscr{A} \preccurlyeq_{\mathrm{B}} \mathscr{S}_{\mathrm{OA}}$. ■

[7] Note that with a slight abuse of notation we use the shorthand $\boldsymbol{\mu}(\omega_k) = \boldsymbol{\mu}(\gamma_1) \otimes \boldsymbol{\mu}(\gamma_2) \otimes \cdots \otimes \boldsymbol{\mu}(\gamma_k)$.

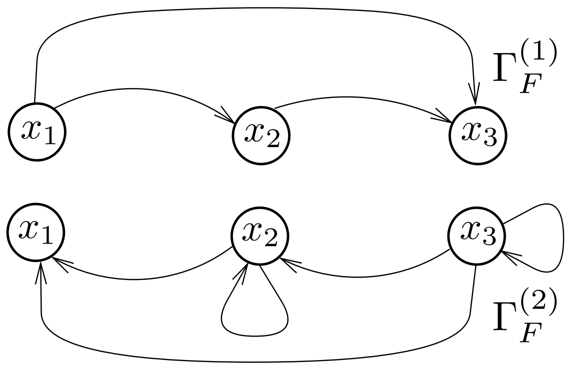

Figure 3.5: The one-step state transition graphs, $\Gamma_F^{(1)}$ and $\Gamma_F^{(2)}$, as defined in Definition 3.3.2, associated to the bimodal open-loop SMPL system of Example 3.4.1.

In the preceding proof, we only considered finite input sequences from the input alphabet Σ. However, the procedure is constructive in that it can be extended to infinite input sequences, by concatenations of finite words from the language $[\![\mathscr{A}]\!]_{\mathrm{L}}$, to establish behavioural equivalence.

The above exposition shows that the subclass of discrete-event systems modelled by SMPL systems is at least as large as the subclass modelled by max-plus automata. The first relation between max-plus automata and max-plus-algebraic hybrid automata then follows from their respective behavioural relations with SMPL systems.

Corollary 3.4.1. *(Max-plus-algebraic hybrid automata behaviourally include max-plus automata)***.** *Given a max-plus automaton $\mathscr{A}$, there exists a max-plus-algebraic hybrid automaton $\mathscr{H}$ (as in (3.28)) that captures its input-output behaviour, i.e. $\mathscr{A} \preccurlyeq_{\mathrm{B}} \mathscr{H}$.*

Proof. The proof follows from Lemma 3.4.1, Theorem 3.4.1, and Theorem 3.4.3. ■

Example 3.4.1. *Consider an open-loop SMPL system (3.1) with three states $n = 3$, two modes $n_{\mathrm{L}} = 2$, discrete input $w \in \Sigma = \{\sigma_1, \sigma_2\}$ with $\sigma_1 = a$ and $\sigma_2 = b$. The mode dynamics are given for $l \in \underline{n_{\mathrm{L}}}$:*

$$\begin{aligned} f(l,x,\cdot) &= \boldsymbol{\mu}^\top(\sigma_l) \otimes x, \quad x(0) = \boldsymbol{\alpha} \\ h(l,x,\cdot) &= \boldsymbol{\beta}^\top \otimes x, \end{aligned} \tag{3.34}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\mu}(\cdot)$ and $\boldsymbol{\beta}$ are given in (3.27). The underlying one-step state-transition graphs for the mode dynamics, $\Gamma_F^{(l)}$ for $l \in \underline{n_{\mathrm{L}}}$, are depicted in Fig. 3.5.

The switching function can be obtained from (3.33) for $m = 2$. Then we have, i) $\left(\boldsymbol{\alpha}^\top \otimes \boldsymbol{\mu}(\sigma_2)\right)^\top = \mathscr{E}_{3\times 1}$, and ii) $\left(\boldsymbol{\mu}(\sigma_1)\right)^{\otimes^2} \neq \left(\boldsymbol{\mu}(\sigma_1)\right)^{\otimes^3} = \mathscr{E}_{3\times 1}$. This also means that for discrete inputs with $w(1) = \sigma_2$ and/or $w(k) = w(k+1) = w(k+2) = \sigma_1$ for $k \in \mathbb{N}$, we have $\phi(\cdot,\cdot,w) = \emptyset$.

It can now be observed that the described SMPL system is behaviourally equivalent to the max-plus automaton in Example 3.2.1 following the arguments in Proposition 3.4.3. The max-plus-algebraic hybrid automaton bisimilar to the provided SMPL system is depicted in Fig. 3.4.

So far we have established that SMPL systems and, by corollary, max-plus-algebraic hybrid automata can encode the input-output characteristics of max-plus automata. We now show that the behaviourally equivalent max-plus-algebraic hybrid automaton also inherits the state transition (logical) structure of the max-plus automaton. To this end, we consider the finite-state discrete abstractions of the two systems (as in (3.31) and (3.26) respectively) that naturally embed their state transition structure. Then, we establish a relationship between a max-plus-algebraic hybrid automaton and max-plus automaton.

Theorem 3.4.4. *Given a max-plus automaton $\mathscr{A}$ with its finite-state discrete abstraction denoted as $\mathscr{A}_{\mathrm{T}}$ (as in (3.26)), there exists a max-plus-algebraic hybrid automaton $\mathscr{H}$ with a finite-state discrete abstraction $\mathscr{H}_{\mathrm{OAT}}$ (as in Definition 3.3.1) such that $\mathscr{H}_{\mathrm{OAT}}$ simulates $\mathscr{A}_{\mathrm{T}}$, i.e. $\mathscr{A}_{\mathrm{T}} \preccurlyeq_{\mathrm{S}} \mathscr{H}_{\mathrm{OAT}}$.*

Proof. Consider a max-plus automaton $\mathscr{A} = (S, \Sigma, \alpha, \mu, \beta)$ (as in (3.23)) with state $s \in \mathbb{D}_1 = S = \{s_1, s_2, \ldots, s_n\}$, input $\omega \in \Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, and output $y_a \in \mathbb{R}_\varepsilon$. We recall that the finite-state discrete abstraction of the max-plus automaton is a tuple $\mathscr{A}_{\mathrm{T}} = (S, \Sigma, \delta_{\mathscr{A}}, S_0, S_{\mathrm{f}})$ with *i)* a partial transition function $\delta_{\mathscr{A}} : S \times \Sigma \to 2^S$ such that $s' \in \delta_{\mathscr{A}}(s, \sigma)$ if $[\mu(\sigma)]_{ss'} \neq \varepsilon$, *ii)* a set of initial states S_0 such that $s \in S_0$ if $\alpha(s) \neq \varepsilon$, and *iii)* a set of final states S_{f} such that $s' \in S_{\mathrm{f}}$ if $\beta(s') \neq \varepsilon$. Moreover, $[\![\mathscr{A}_{\mathrm{T}}]\!]_{\mathrm{L}} = [\![\mathscr{A}]\!]_{\mathrm{L}}$.

We now consider the SMPL system $\mathscr{S}_{\mathrm{OA}}$ that behaviourally includes the max-plus automaton $\mathscr{A}$ as proposed in Theorem 3.4.3. The max-plus-algebraic hybrid automaton $\mathscr{H}_{\mathrm{OA}}$ such that $\mathscr{S}_{\mathrm{OA}} \simeq_{\mathrm{S}} \mathscr{H}_{\mathrm{OA}}$ can be derived using the procedure described in Theorem 3.4.1. Then $\mathscr{H}_{\mathrm{OA}}$ consists of *i)* states $(q, x) \in \mathbb{Q} \times \mathbb{X} = \underline{m} \times \mathbb{R}^n_\varepsilon$, continuous input $\mathbb{U} = \emptyset$, discrete input $w \in \Sigma$, and $(q, x(0)) \in \mathrm{Init}$ for all $q \in \mathbb{Q}$, *ii)* discrete state characteristics for $x \in \mathbb{R}^n_\varepsilon$ and for all $q \in \mathbb{Q}$ as: $F(q, x, \cdot) = A^{(q)} \otimes x$, $H(q, x, \cdot) = C \otimes x$, and $\mathrm{Inv}(q) = \{(x, w) \mid \phi(\cdot, x, w) \neq \emptyset\}$ (as in (3.33)). The edge characteristics are defined for all $(q, q') \in E \subseteq \underline{n_{\mathrm{L}}} \times \underline{n_{\mathrm{L}}}$ as: $G = \{(x, w) \mid \phi(q, x, w) = q'\}$, and $R(\cdot) := R_{\mathrm{id}}(\cdot)$. There are no constraints on the admissible inputs, i.e. $\Lambda(q, x) = 2^{\mathbb{I}}$ for all $(q, x) \in \mathbb{X}$.

Now we derive the finite-state discrete abstraction of the max-plus-algebraic hybrid automaton $\mathscr{H}_{\mathrm{OA}}$ following the procedure described in Section 3.3.2. Recall that the state variables are defined as $X_{\mathrm{var}} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$. The transition graphs (Γ^q_{F} and Γ^q_{H}) for the continuous-variable one-step dynamics (as in Definition 3.3.2) reduce to: for all $(i, j) \in \underline{n}^2$ and $q \in \mathbb{Q}$, we have

$$\begin{aligned}(\mathbf{x}_i, \mathbf{x}_j) \in \Gamma^{(q)}_{\mathrm{F}} &\Leftrightarrow [A^{(q)}]_{ji} \neq \varepsilon,\\ (\mathbf{x}_j, \mathbf{x}_j) \in \Gamma^{(q)}_{\mathrm{H}} &\Leftrightarrow [C]_j \neq \varepsilon.\end{aligned} \tag{3.35}$$

The finite-state discrete abstraction of the max-plus-algebraic hybrid automaton can then be formulated as:

$$\mathscr{H}_{\mathrm{OAT}} = (\overline{Q}, \Sigma, \delta_{\mathscr{H}}, \overline{Q}_0, \overline{Q}_{\mathrm{f}}), \tag{3.36}$$

where $\overline{Q} = \mathbb{Q} \times X_{\mathrm{var}}$; $(q, \mathbf{x}_i) \in \overline{Q}_0$ if $x_i(0) \neq \varepsilon$ and $(q, \mathbf{x}_j) \in \overline{Q}_{\mathrm{f}}$ if $[C]_j \neq \varepsilon$ for all $q \in Q$; the partial transition function $\delta_{\mathscr{H}} : \overline{Q} \times \Sigma \to 2^{\overline{Q}}$ is defined such that for $\eta = (q, q') \in E$ and $\sigma \in \Sigma$, we have that $(q', \mathbf{x}_j) \in \delta_{\mathscr{H}}((q, \mathbf{x}_i), \sigma)$ if $[A^{(q')}]_{ji} \neq \varepsilon$.

It remains to show that there exists a simulation relation from $\mathscr{A}_{\mathrm{T}}$ to $\mathscr{H}_{\mathrm{OAT}}$ that satisfies the properties stated in Definition 3.4.5. The two systems share the same input

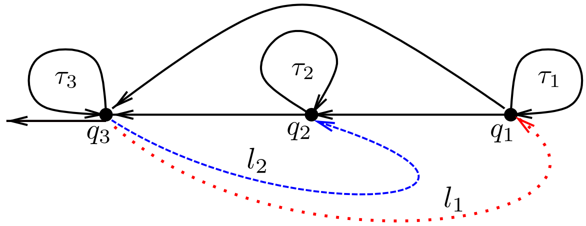

Figure 3.6: A pictorial representation of a production line, adapted from [17, §9.6.1], [203, §7.2]. The nodes q_1-q_3 denote machines and are associated with durations τ_1-τ_3 representing processing/recycling times. The two modes of operation can be distinguished by differently coloured arcs: *i)* Mode l_1 as red dotted line ($\cdots$), and *ii)* Mode l_2 with blue dashed line (- - -).

alphabet Σ. Moreover, $|\Sigma| = |Q|$ and $|S| = |X_{\text{var}}|$. Furthermore, $A^{(l)} = \mu^\top(\sigma_l)$ for $l \in \underline{m}$, and $[C]_j = \beta(s_j)$ and $x_j(0) = \alpha(s_j)$ for $j \in \underline{n}$ (as specified in Theorem 3.4.3).

Recall that words on the input alphabet, $\omega_k = \gamma_1\gamma_2\cdots\gamma_k \in \Sigma^*$, can be identified as a map $\omega : \mathbb{N} \to \Sigma$. Here, $\mathbb{N}$ represents the event counter axis. Also, the partial transition functions, $\delta_{\mathscr{A}}$ and $\delta_{\mathscr{H}}$, can be perceived as state maps (as in Definition 3.4.4).

The simulation relation is defined as a map $\Psi : \mathbb{N} \to S \times \overline{Q}$ that satisfies the following properties for all $k \in \mathbb{N}$: *i)* for every $(s_i, (q, \mathbf{x}_j)) \in \Psi(k)$ we have $i = j$, *ii)* for every $\sigma_l \in \Sigma$ and $(s_i, (q, \mathbf{x}_i)) \in \Psi(k)$, we have that for every state $s_j \in \{s_t \in \delta_{\mathscr{A}}(s_i, \sigma_l) \mid [\mu(\sigma_l)]_{s_i s_t} \neq \varepsilon\}$, there exists $(q', \mathbf{x}_j) \in \{(q', x_t) \in \delta_{\mathscr{H}}((q, \mathbf{x}_i), \sigma_l) \mid [A^{(l)}]_{ti} \neq \varepsilon\}$ such that $(s_j, (q', \mathbf{x}_j)) \in \Psi(k)$, and *iii)* for every $s \in S_0$ and $(q, \mathbf{x}) \in \overline{Q}_0$, we have $(s, (q, \mathbf{x})) \in \Psi(0)$. Note that the provided simulation relation is symmetric.

Therefore, for a given word $\omega_k \in \Sigma^*$ there are equivalent trajectories allowed by $\mathscr{A}_{\text{T}}$ and $\mathscr{H}_{\text{OAT}}$. Finally, for every state $s \in \{s_j \in S_{\text{f}} \mid \beta(s_j) \neq \varepsilon\}$ there exists $(q, \mathbf{x}) \in \{(q, \mathbf{x}_j) \in \overline{Q}_{\text{f}} \mid [C]_j \neq \varepsilon\}$ such that $(s, (q, \mathbf{x})) \in \Psi(k)$, $k \in \mathbb{N}$. Therefore, the final states for the acceptance of the word $\omega_k \in \Sigma^*$ are equivalent in the two models.

Hence, we have $\mathscr{A}_{\text{T}} \simeq_{\text{S}} \mathscr{H}_{\text{OAT}}$. ■

For a max-plus-algebraic hybrid automaton (3.28) with max-plus linear mode dynamics, the finite-state discrete abstraction in (3.32) captures exactly the language of the underlying discrete-event system. The results of the preceding theorem also imply, using Lemma 3.4.1, that the two finite-state discrete abstractions $\mathscr{A}_{\text{T}}$ and $\mathscr{H}_{\text{OAT}}$ and generate the same language, $[\![\mathscr{A}_{\text{T}}]\!]_{\text{L}} = [\![\mathscr{H}_{\text{OAT}}]\!]_{\text{L}}$.

3.5. ILLUSTRATION

In this subsection, we consider the modelling of a production line, as depicted in Fig. 4.3, in the max-plus-algebraic hybrid automata framework.

The network consists of nodes q_1, q_2, and q_3 where activities are performed with processing times $\tau_1, \tau_2, \tau_3 \in \mathbb{N}$, respectively. The buffers between each pair of nodes have zero holding times and are all assumed to have a single product initially. The buffer before q_3 can store at most two incoming products. The other buffers are constrained to hold at most one product at a time. The node q_1 transfers product simultaneously to the buffers before q_2 and q_3. The earliest product[8] arriving at q_3 is processed first.

The product exits node q_3 and then a new cycle is started. This is modelled as a feedback-loop from node q_3 to node q_1. In addition, we introduce a second mode of operation where the product from node q_3 is routed to node q_2 for reprocessing. This is distinguished by differently coloured arcs in Fig 4.3.

The state $x_i(k) \in \overline{\mathbb{R}}_\varepsilon$, for $i \in \{1,2,3\}$ and $k \in \mathbb{N}$, denotes the time when node q_i finishes an activity for the k-th time. The convention is $x_i(k) = +\infty$ if no activity is performed at q_i for the k-th time. It is assumed that all buffers contain a product initially. The dynamics of the production line can be expressed algebraically (as in (3.1)) as follows for mode $\ell(\cdot) = l_1$:

$$\begin{aligned}
x_1(k+1) &= \max(x_1(k)+\tau_1, x_2(k), x_3(k)+\tau_3)\\
x_2(k+1) &= \max(x_1(k)+\tau_1, x_2(k)+\tau_2)\\
x_3(k+1) &= \max(x_1(k)+\tau_1, x_2(k)+\tau_2, x_3(k)+2\tau_3,\\
&\qquad \min(x_1(k)+\tau_1+\tau_3, x_2(k)+\tau_2+\tau_3)).
\end{aligned} \tag{3.37}$$

For the system dynamics in mode $\ell(\cdot) = l_2$, we have:

$$\begin{aligned}
x_1(k+1) &= \max(x_1(k)+\tau_1, x_2(k))\\
x_2(k+1) &= \max(x_1(k)+\tau_1, x_2(k)+\tau_2, x_3(k)+\tau_3),
\end{aligned} \tag{3.38}$$

and the evolution of x_3 follows the same equation as of mode l_1. The initial state and output matrices ($y = C \otimes x$) are chosen as follows:

$$x(0) = \begin{pmatrix} 0 & 0 & \varepsilon \end{pmatrix}^\top, \quad C = \begin{pmatrix} \varepsilon & \varepsilon & 0 \end{pmatrix}. \tag{3.39}$$

The dynamics can be represented in the min-max-plus conjunctive normal form (3.29), for $L = 2$ and $M = 1$, by replacing the expression of $x_3(\cdot)$ in (3.37) with

$$\begin{aligned}
x_3(k+1) = \min\{&\max(x_1(k)+\tau_1+\tau_3, x_2(k)+\tau_2,\\
&\qquad x_3(k)+2\tau_3),\\
&\max(x_1(k)+\tau_1, x_2(k)+\tau_2+\tau_3,\\
&\qquad x_3(k)+2\tau_3)\}.
\end{aligned} \tag{3.40}$$

There are no continuous-valued inputs to the system. The discrete input $w(\cdot) \in \mathbb{V} \triangleq \{l_1, l_2\}$ determines the mode as follows (see (3.1)):

$$\phi(\cdot, x, w) = \left\{ i \in \{1,2\} \,\middle|\, \min_{j \in \underline{L}} A^{(i,j)} \otimes x \in \overline{\mathbb{R}}_\varepsilon^n \setminus \{\varepsilon, \top\}^n, w = l_i \right\}. \tag{3.41}$$

The discrete-event system of the production network under consideration can therefore be expressed as a max-plus-algebraic hybrid automaton as depicted in Fig. 3.4 with continuous-valued dynamics of the form (3.29).

As the system dynamics (3.37)-(3.38) satisfy Assumption 3.3.1, a finite-state discrete abstraction of the max-plus-algebraic hybrid automaton can be obtained using Proposition 3.3.1. The necessity of the restriction of the state space $\mathbb{X}$ is reflected in the definition

[8] The conflict at the buffer before q_3 is resolved here using the so-called first-in first-out policy.

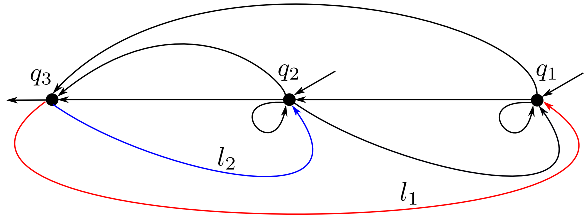

Figure 3.7: The one-step state transition graphs associated to the production network in Fig. 4.3. The finite automaton can be obtained by duplication of the nodes $q_1 - q_3$ for the two modes l_1 and l_2. The black arcs are common to both the modes. The blue arc (—) belongs to mode l_2 and the red arcs (—) belong to mode l_1. The input and output arrows symbolise the initial and final states of the finite automaton.

of the switching function $\phi(\cdot)$ in (3.41). The resulting one-step state transition graphs of the two modes are depicted in Fig. 3.7. Moreover, the reset relation does not entail transitions in continuous-valued state. Then the language of the max-plus-algebraic hybrid automaton model of the production network is contained in the language of the obtained finite automaton. This completes the illustration.

3.6. CONCLUSIONS

In this chapter, we have proposed a unifying max-plus-algebraic hybrid automata framework for discrete-event systems in max-plus algebra. In this context, we have identified the hybrid phenomena due to the interaction of continuous-valued max-plus dynamics and discrete-valued switching dynamics in switching max-plus linear and max-plus automata models. We have formally established the relationship between these two models and their relationships with the proposed max-plus-algebraic hybrid automata framework utilising the notions of behavioural equivalence and bisimilarity. This is achieved in a behavioural framework where the models are seen as a collection of input-state-output trajectories. As a max-plus-algebraic hybrid automaton and a max-plus automaton are defined on different state space, we have also studied their relationship by embedding them into their respective finite-state discrete abstractions. The subclass of max-plus-algebraic hybrid automata that can be simulated by max-plus automata remains to be investigated.

4

Stability of max-plus-algebraic hybrid automata

In this chapter, we present a framework to study stability problems for discrete-event systems modelled as switching max-plus linear (SMPL) systems. Only autonomous notions of stability pertaining to the continuous-valued portion of the dynamics are treated. The presented theory forms the basis for studying non-autonomous stability notions and stability of SMPL systems under constrained switching.

4.1. Introduction

Stability analyses play an important role in the operation and control of dynamical systems. The most important breakthroughs in such analyses rely on finding verifiable conditions to establish boundedness around certain equilibria without the need of computing solutions explicitly [155].

This chapter focuses on the autonomous stability notions of SMPL systems. In particular, we develop novel mathematical principles and tools to ensure the boundedness of the continuous-valued part of SMPL systems. We achieve this analogously to "classical" Lyapunov stability analysis approaches while utilising tools from max-plus algebra and mixed-integer programming. To this end, we adopt the conventional stability analysis roadmap for studying switching systems, i.e. we assume that either the switching sequence is arbitrary [150].

4.1.1. Related work

The stability analysis of a dynamical system is inherently connected to the existence and investigation of its set of equilibria. The study of equilibria (stationary regimes) of a discrete-event system in the max-plus algebra is based on the properties of its underlying timed-event graph [17, 63, 89]. The set of equilibria of interest are linked by the max-plus eigenvalue problem for the max-plus linear system. This existence problem is treated qualitatively i.e., the system dynamics is studied as the event counter approaches infinity and is associated to the unique asymptotic growth rate of the states. The aforementioned analysis tools can be extended to max-min-plus linear systems, if the underlying event graph is fixed [87, 109, 203], owing to the the additive homogeneity and monotonicity of the dynamics. For a suitable comparison with conventional time-driven systems, the interested reader can refer to [15]. The authors of [15] study the stability of an additively homogeneous monotone system. The boundedness of all system trajectories is associated to the qualitative existence and uniqueness of a global equilibrium for a suitable projection of the system dynamics.

The eigenvalue-based analysis of stability in conventional time-driven systems is based on the variational principle, i.e. characterise the most critical switching sequence and prove that it is stabilising [166]. This approach is not suitable for stability analysis of SMPL systems partly due to the high complexity of calculating the spectral characteristics [33]. Moreover, it serves only one aspect of stability analysis concerning performance evaluation in terms of throughput of the system [85].

The existence of stationary regimes of a max-plus matrix semigroup[1] (set of matrices corresponding to the system) can be studied under restrictive assumptions on the structure of the underlying directed graphs of the constituent matrices [17, 119], and/or the contractiveness of the system dynamics [129, 164, 170]: *i)* the matrix semigroup has a fixed ε-structure[2] but its generators form a stationary and ergodic sequence on some probability space [17, 119], *ii)* the matrix semigroup consists of independent and identically distributed matrices that has a unique irreducible element that occurs with a positive probability [164], *iii)* existence of at least one idempotent[3] element in the semigroup that occurs with positive probability or the constituent matrices commute pairwise [129, 170]. The class of SMPL systems is, however, bigger than the ones described in aforementioned references [17, 119, 129, 164, 170]. A generic semigroup of matrices in the max-plus algebra can have multiple stationary regimes that correspond to different asymptotic growth rates of the states, which in turn depend on the variable that orchestrates switching [85]. Therefore, these sufficient conditions for boundedness are restrictive for the purpose of closed-loop analysis and practical applications.

Another stability analysis approach involves proving asymptotic stability in the large [182], i.e. proving that closed-loop system stability is achieved in a finite number of steps for all possible initial states and event trajectories [216, Theorem 1]. Such an ad hoc analysis is performed a posteriori and has limited application in performance evaluation for a general class of timed discrete-event systems. [112, 216].

4.1.2. STATEMENT OF CONTRIBUTION

In what follows, the novel contributions of this chapter are stated.

The first set of contributions involve systematically unifying the notions of internal

[1] A semigroup consists of a set together with an associative binary operation without requiring the existence of an identity element or inverses

[2] The ε-structure in the max-plus description pertains to the structure of the underlying timed-event graph.

[3] Here, an idempotent element refers to a max-plus matrix with a (max-plus) rank 1 such that it maps the entire state space to a unique element in the max-plus Hilbert projective space.

stability from the Lyapunov framework with that of discrete-event systems in the max-plus algebra:

(A) *Stability notions*: We first categorise the notions based on desired types of the boundedness of the states, i.e. pertaining either to the same event counter (max-plus bounded-buffer stability) or consecutive ones (max-plus Lipschitz stability). These notions are further classified based on ultimate boundedness (Definitions 4.2.3 (1) and 4.2.4 (1)) and local asymptotic stability (Definitions 4.2.3 (2) and 4.2.4 (2)). The max-plus bounded-buffer stability is expressed using the max-plus Hilbert projective norm (see (2.32)). The max-plus Lipschitz stability is studied as the growth rate of the state trajectory.

The next set of contributions are bundled under the title *stability analysis*. We provide necessary and sufficient conditions on the proposed notions of stability under arbitrary switching (in Section 4.3):

(B) *Ultimate boundedness*: Inspired by polyhedral Lyapunov functions, we propose max-plus gauge functions (Definition 4.3.4) for analysing ultimate boundedness under max-plus bounded-buffer stability (see Theorem 4.3.1). We also show that max-plus bounded-buffer stability implies max-plus Lipschitz stability for the case of ultimate boundedness (see Proposition 4.3.1).

(C) *Asymptotic stability*: We provide necessary and sufficient conditions for asymptotic max-plus bounded-buffer stability (see Theorem 4.3.3). To this end, we introduce a novel class of max-plus Lyapunov functions (Definition 4.3.6). We show that the dynamics can be decomposed injectively using additively homogeneous, monotone functions (see Lemma 4.3.4). Then asymptotic max-plus Lipschitz stability becomes equivalent to ultimate boundedness under max-plus bounded-buffer stability along with the existence of an asymptotic growth rate (see Theorem 4.3.2).

(D) *Convergence properties*: We develop suitable max-plus versions of the Krasovskii-LaSalle invariance principle to study the convergence properties of SMPL systems (see Theorem 4.3.4 and 4.3.5). To this end, we introduce ω-limit sets in the max-plus Hilbert projective space (Definition 4.3.7) and a weaker version of max-plus Lyapunov functions (Definition 4.3.9). This also allows us to extend the results of [111, Proposition 4.1] anto study attractivity of max-plus eigenspaces under SMPL system dynamics (see Theorem 4.3.6).

Finally, we apply the proposed stability theory to open-loop SMPL systems with linear modes:

(E) *Boundedness of max-plus matrix semigroups*[4]: We consider the application of theoretical results via various numerical examples. In doing so, we provide a constructive necessary and sufficient condition for a given closed max-plus subspace to be positively invariant for the system dynamics (see Theorem 4.4.1). The condition can be checked efficiently using existing algorithms.

[4] A semigroup consists of a set together with an associative binary operation without requiring the existence of an identity element or inverses.

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(F) *Algorithmic perspective*: We provide a mixed-integer linear programming formulation for construction of a positively invariant set common to a given max-plus matrix semigroup (see (5.27a)). We also formulate a condition for existence of a common positively invariant set as a non-homogeneous system of two-sided max-plus linear equations (see Theorem 4.4.2). Finally, we propose a mixed-integer linear programming formulation to determine the region of attraction for asymptotic stability using a max-plus Lyapunov function (see Section 4.4.5).

4.1.3. ORGANISATION OF THE CHAPTER

The chapter is organised as follows. Section 4.2 recalls the modelling framework and introduces the problem statement. Section 4.2 also presents the associated notions of stability along with positive invariance. The theorems and tools for stability analysis are then introduced under arbitrary switching in Section 4.3. Section 4.4 presents the algorithmic aspects for the study of properties of positively invariant sets in the presented stability framework. The technical proofs of the results of the chapter are collected in Section 4.5. The chapter ends with concluding remarks in Section 4.6.

4.2. STABILITY CONCEPTS

This section first presents the model of discrete-event systems in the max-plus-algebraic hybrid automata framework. We formulate the stability analysis problem for discrete-event systems in max-plus algebra. Then we recapitulate the autonomous notions of stability for discrete-event systems. Finally, we formulate these notions in a max-plus-algebraic framework. The importance of distinctions among the different notions is established. We also briefly recall the notion of invariant sets for such systems from [111]. These notions form the basis for further stability analysis.

4.2.1. MODELLING ASSUMPTIONS

Max-plus-algebraic hybrid automata [110] represent a powerful modelling framework for discrete-event systems analogously to the conventional hybrid automata [160]. In this chapter, we study the stability of the continuous-valued portion of the system dynamics of a max-plus-algebraic hybrid system.

To every state (event), say i, a dater function is associated, $x_i(k)$, and it represents the time of the k-th occurrence of the state event i, relative to some arbitrarily chosen origin of time. The dynamics for the continuous state $x(k) \in \mathbb{R}_\varepsilon^n$ at event step $k \in \mathbb{N}$ is written as a prototypical switching system in the multi-modal form:

$$x(k) = f(l, x(k-1)), \quad l \in \underline{n_{\mathrm{L}}}. \tag{4.1}$$

A (possibly infinite) switching sequence $\sigma_k = (l_k)_{k\in\mathbb{N}}$ along with an initial state $x(0)$ and (4.1), completely describes the trajectory of the discrete-event system. We sometimes denote this trajectory as $x_{\boldsymbol{\sigma}}(k)$ to emphasise the dependence on the switching sequence σ_k.

A discrete-event system description (4.1) is said to be structurally finite if the function f maps $\mathbb{R}^n$ to $\mathbb{R}^n$ for each mode $l \in \underline{n_{\mathrm{L}}}$ such that the states do not become ε for finite initial states. The following assumptions on the function $f : \mathbb{N} \times \mathbb{R}^n_\varepsilon \to \mathbb{R}^n_\varepsilon$ hold for the rest of the chapter.

Assumption 4.2.1. *The discrete-event system described by* (4.1) *is structurally finite.* ◊

Assumption 4.2.2. *The function $f(l,x)$ is continuous and additively homogeneous in the state $x \in \mathbb{R}^n$ for every $l \in \underline{n_L}$.* ◊

The latter assumption ensures that any two solutions ($x_2 = \lambda + x_1$, $\lambda \in \mathbb{R}$) that are equivalent in the sense of having identical dynamics ($f(l,x_2) = \lambda + f(l,x_1)$ for a given $l \in \underline{n_L}$) are also indistinguishable when measured in the max-plus Hilbert projective norm (2.32). If the elements of the state x are interpreted as the times of occurrences of events then additive homogeneity can be understood as time-invariance of the system dynamics. If the function $f(l,x)$ is max-plus linear in $x \in \mathbb{R}^n_\varepsilon$ for every $l \in \underline{n_L}$, we refer to the restricted class of open-loop SMPL systems.

4.2.2. PROBLEM STATEMENT

The *buffer level* in discrete-event systems described in max-plus algebra is defined as the time delay between the occurrences of different events in either the same event cycle (k) or the consecutive ones. The notion of stability is associated with the boundedness of these buffer levels [181]. Asymptotic stability then implies that the buffer levels, at an average, take constant values. This is only possible, non-trivially, when the asymptotic growth rates of all the states become equal to each other.

The study of (stable) asymptotic behaviour of trajectories of a max-plus linear system ($n_L = 1$) is intimately connected to the max-plus eigenvalue problem (2.11). The (unique) growth rate is obtained using the max-plus eigenvalue and represents the inverse of throughput of the system. The set of stationary regimes for buffer levels in the same event cycle can be obtained from the associated (finite) max-plus eigenvectors. The extension of this approach to study stationary behaviour of open-loop SMPL systems (or inhomogeneous products of max-plus matrices) is known to be difficult [33, 85]. We therefore follow the framework for studying stability problems in conventional switching systems [150].

In this chapter, we consider the following problem:

- *Problem A.* Formulate conditions for stability, in some suitable sense, of system (4.1) for arbitrary switching sequences.

To this end, we first present what we mean by "stability in some suitable sense". Then we present constructive methods to analyse and compare the various stability concepts.

4.2.3. INVARIANT SETS

Almost all notions of stability addressed in this chapter concern with qualitative characterisation of invariant sets[5]. In the following subsection, we first recall the notions of invariant sets of a dynamical system. Then we subsequently present various notions of stability.

[5]The notions of invariance and attractivity are taken as it is from conventional systems theory as presented in Section 2.3.

Without a loss of generality, we assume that the invariant sets considered henceforth are also invariant with respect to max-plus translation by the vector $\mathbb{1}_n$:

$$x \in \mathcal{K} \Leftrightarrow x + \mu \otimes \mathbb{1}_n \in \mathcal{K}, \quad \forall \mu \in \mathbb{R}. \tag{4.2}$$

The preceding property is not restrictive as the dynamics $f(\cdot, x)$ is additively homogeneous in x under Assumption 4.2.2.

Consider again the discrete-event system dynamics (4.1). We introduce some concepts of invariant sets $\mathcal{K} \subseteq \mathbb{R}^n$ and their attractivity to characterise the behaviour of the system trajectories.

Definition 4.2.1 (Positive invariance, [47])**.** A set $\mathcal{K} \subseteq \mathbb{R}^n$ is positively invariant for the system dynamics (4.1) if $x \in \mathcal{K}$ implies $f(l, x) \in \mathcal{K}$ for all $l \in \underline{n_{\mathrm{L}}}$. If the sets $\mathcal{K}_1$ and $\mathcal{K}_2$ are positively invariant then so are $\mathcal{K}_1 \cap \mathcal{K}_2$ and $\mathcal{K}_1 \cup \mathcal{K}_2$. □

Note that for the general case of the discrete-event system (4.1), we have $\mathcal{K} = \mathbb{R}^n$ under Assumption 4.2.1. For other cases, a positively invariant set is used to characterise attractors (stationary regimes) amounting to certain desirable behaviours of a discrete-event system. The most important case of an invariant set is an equilibrium, $\mathcal{K} = \overline{x}_{\mathrm{e}} \subset \mathbb{R}^n/\sim$: there exist $x_{\mathrm{e}}, x'_{\mathrm{e}} \in \overline{x}_{\mathrm{e}}$ such that $f(l, x_{\mathrm{e}}) = x'_{\mathrm{e}}$ for all $l \in \underline{n_{\mathrm{L}}}$.

Finally, we recall the notion of attractivity of a set with respect to the system dynamics.

Definition 4.2.2 (Attractivity, [47])**.** A non-empty closed set $\mathcal{K} \subseteq \mathbb{R}^n$ is attractive for the state trajectories $\{x(k)\}_{k \in \mathbb{N}}$ of (4.1) if there exists an open neighbourhood $\mathcal{U} \supset \mathcal{K}$ such that for all $x(0) \in \mathcal{U}$ and for all neighbourhoods $\mathcal{V} \supset \mathcal{K}$ there exists a $k_0 \in \mathbb{N}$ such that $x(k) \in \mathcal{V}$ for all $k \geq k_0$. □

4.2.4. AUTONOMOUS NOTIONS OF STABILITY

In this subsection, we recall the following internal notions of stability[6] for a general discrete-event system formulated in max-plus algebra [111, 164].

We recall that $\mathcal{B}_\tau(x)$ denotes an open ball of radius $\tau > 0$ with respect to the max-plus Hilbert projective norm centred at the ray $\overline{x} = \{z \in \mathbb{R}^n \mid \exists \mu \in \mathbb{R},\ x = \mu + z\}$. The set $\mathcal{B}_\tau$ denotes $\mathcal{B}_\tau(\mathbb{1}_n)$.

The first set of notions deals with stability associated to boundedness of buffer levels in the same event cycle.

Definition 4.2.3. (Max-plus bounded-buffer stability)**.** A discrete-event system (4.1) is said to be

1. uniformly max-plus bounded-buffer stable if there exists a constant $\delta > 0$ and for every $\mu > 0$ there exists a constant $T = T(\mu, \delta) > 0$ such that if $x(0) \in \mathcal{B}_\mu$, we have $x(k) \in \mathcal{B}_\delta$ for all $k \geq T(\mu, \delta)$;

2. uniformly locally asymptotically max-plus bounded-buffer stable with respect to a closed set $\mathcal{K} \subseteq \mathcal{B}_\tau$, for some $\tau > 0$, if *i)* for every $\delta > 0$, there is $\mu = \mu(\delta) > 0$ such that if

[6] Please refer to Section 2.3 for stability theory for conventional time-driven systems defined on a suitable normed vector space.

$x(0) \in \mathscr{B}_{\mu}(\mathscr{K})$, we have $x(k) \in \mathscr{B}_{\delta}(\mathscr{K})$ for all $k \geq 0$, and *ii)* there exists a constant $\mu > 0$ and for every $\eta > 0$, there exists a scalar $T = T(\eta) > 0$ such that if $x(0) \in \mathscr{B}_{\mu}(\mathscr{K})$, we have $x(k) \in \mathscr{B}_{\eta}(\mathscr{K})$ for all $k \geq T(\eta)$. □

The notion of max-plus bounded-buffer stability is analogous to ultimate boundedness in conventional time-driven systems (see [135, Definition 4.6]). The asymptotic counterpart, on the other hand, is understood in the sense of Lyapunov (see [122]). Loosely speaking, it is required that a trajectory starting close to a closed set $\mathscr{K}$ should remain close to the set $\mathscr{K}$. Note that when $\mathscr{K} = \{\overline{x}_{\mathrm{e}}\}$ with $\overline{x}_{\mathrm{e}} \in \mathbb{R}^n/\sim$, the asymptotic notion of max-plus bounded-buffer stability requires the existence of a common equilibrium point of the system dynamics. This results in asymptotic trajectories of the form $x(k) = \mu(k) \otimes x(k-1)$ with a (possibly) event-varying growth rate $\mu(k) \in \mathbb{R}$ for a sufficiently large $k \in \mathbb{N}$.

For a max-plus linear system, the upper bound[7] on $\|x(k)\|_{\mathbb{P}}$ can be obtained by analysing the sub-eigencone[8] of the system matrix A [17, Theorem 3.104]. The sub-eigencone is non-empty if there exist finite eigenvectors of the system matrix [66, Lemma 23-2].

The second notion deals with the boundedness of buffer levels associated to time delays in consecutive event cycles of a discrete-event system (See Fig. 4.1b).

Definition 4.2.4. (Max-plus Lipschitz stability)**.** A discrete-event system (4.1) is said to be

1. uniformly max-plus Lipschitz stable if for every $\mu > 0$ there exists a scalar $\delta > 0$ and $T(\mu,\delta) \in \mathbb{N}$ such that if $x(0) \in \mathscr{B}_{\mu}$, we have $x(k) \in \mathscr{B}_{\delta}(x(k-1))$ for all $k \geq T(\mu,\delta)$;

2. uniformly locally asymptotically max-plus Lipschitz stable with a basin of attraction $\mathscr{B}_{\eta}$, for some $\eta > 0$, if the system is stable (as in 1) and there exist scalars $\rho \in \mathbb{R}$ and $T = T(\eta) > 0$ such that if $x(0) \in \mathscr{B}_{\eta}$, $\left\|x(k) - \rho \cdot k\right\|_{\infty}$ is bounded for all $k \geq T(\eta)$. □

The asymptotic notion of max-plus Lipschitz stability suggests that the average growth rate of the state trajectories, of the discrete-event system, become constant. Note that the definition requires that this asymptotic growth rate $\rho \in \mathbb{R}$ exist but no requirement is placed on uniqueness over all system trajectories. As we will prove later (in Section 4.3.1), asymptotic max-plus Lipschitz stability requires that the system be max-plus bounded-buffer stable. The converse is not necessarily true. An interesting special case is when there exists a period $c \in \mathbb{N}$ such that the trajectories are of the form $x(k) = \rho^{\otimes^c} \otimes x(k-c)$ for sufficiently large $k \in \mathbb{N}$.

A different way to study stability and attraction, that does not require a complete knowledge of the attractor, is to compare (adjacent) trajectories with respect to a certain metric. These type of notions are classified under orbital stability [135, Definition 8.2].

Definition 4.2.5. (Max-plus incremental stability)**.** A discrete-event system (4.1) evolving on a positively invariant set $\mathscr{K} \subseteq \mathbb{R}^n$ for a given switching sequence $\sigma_k = \{l(k)\}_{k \in \mathbb{N}}$ is said to be

[7] Please refer to Section 2.2.3 for the definition of various norms and metrics used in this dissertation.

[8] The sub-eigencone of a matrix A is defined as $\underline{\mathrm{eig}}(A, \lambda(A)) = \{z \in \mathbb{R}^n \mid A \otimes z \leq \lambda(A) \otimes z\}$.

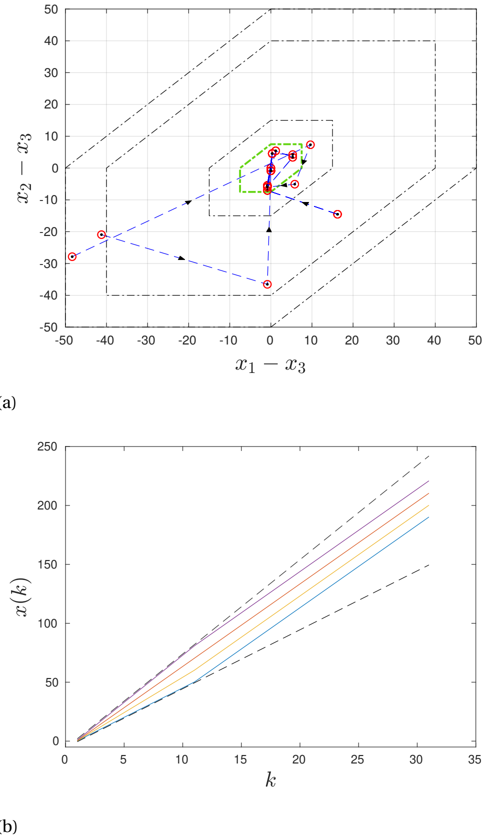

Figure 4.1: Notions of stability: (a) asymptotic max-plus bounded-buffer stability in a set $\mathscr{K}$ (green dashed line - -) and (b) max-plus Lipschitz stability. The asymptotic stability for the first case is understood in the Lyapunov sense. In the latter case, the dashed (- -) lines represent the global bounds on the growth rates trajectories. Then asymptotic max-plus Lipschitz stability can be inferred from the eventual trajectories $x(k)$.

1. uniformly max-plus incrementally stable in $\mathscr{K} \subseteq \mathbb{R}^n$ if for every $\delta > 0$, there is a $\mu = \mu(\delta) > 0$ such that for any $x^{(1)}_{\boldsymbol{\sigma}}(0), x^{(2)}_{\boldsymbol{\sigma}}(0) \in \mathscr{K}$ if $x^{(2)}_{\boldsymbol{\sigma}}(0) \in \mathscr{B}_\mu(x^{(1)}_{\boldsymbol{\sigma}}(0))$, we have $x^{(2)}_{\boldsymbol{\sigma}}(k) \in \mathscr{B}_\delta(x^{(1)}_{\boldsymbol{\sigma}}(k))$ for all $k \geq 0$;

2. uniformly asymptotically max-plus incrementally stable in $\mathscr{K} \subseteq \mathbb{R}^n$ if the system is stable (as in 1), and for each $\eta > 0$, there exists a scalar $T = T(\eta) > 0$ such that for any $x^{(1)}_{\boldsymbol{\sigma}}(0), x^{(2)}_{\boldsymbol{\sigma}}(0) \in \mathscr{K}$, we have $x^{(2)}_{\boldsymbol{\sigma}}(k) \in \mathscr{B}_\eta(x^{(1)}_{\boldsymbol{\sigma}}(k))$ for all $k \geq T(\eta)$. □

The preceding notion of max-plus incremental stability implies that any two trajectories, corresponding to the same switching sequence $\sigma_k = (l(k))_{k\in\mathbb{N}}$, starting close to each other, remain close to each other in the max-plus Hilbert projective metric. The asymptotic notion then implies that all trajectories 'forget' their initial conditions and converge to each other again in the max-plus Hilbert projective metric. Note that the trajectories of a max-plus incrementally stable system can be unbounded in the max-plus Hilbert projective norm.

We note that for a max-plus linear system (or, max-min-plus linear system), monotonicity coupled with additive homogeneity ensures non-expansiveness of the dynamics [109, Proposition 1.1]. This is equivalent to uniform max-plus incremental stability. The non-expansiveness property implies that with no explicit variation of the dynamics over events, all trajectories are asymptotically equivalent to each other [109, §4]. In a more general setting, when the system dynamics (4.1) is varying over the events, the result fails to hold. Furthermore, the limiting behaviour does not necessarily consist of equilibrium points but can rather have a more general event-varying nature.

Definition 4.2.6. (Max-plus convergent dynamics)**.** A discrete-event system (4.1) is said to be uniformly max-plus convergent in a positively invariant set $\mathscr{K} \subseteq \mathbb{R}^n$ for a given switching sequence $\sigma_k = \{l(k)\}_{k\in\mathbb{N}}$ if the following conditions are satisfied:

1. there exists a unique solution $\tilde{x}_{\boldsymbol{\sigma}}(k)$ of system (4.1) defined in $\mathscr{K}$ and bounded in the max-plus Hilbert projective norm for all $k \in \mathbb{N}$;
2. the system is uniformly asymptotically max-plus bounded-buffer stable with respect to the solution $\tilde{x}_{\boldsymbol{\sigma}}(k)$: *i)* for every $\delta > 0$, there is a $\mu = \mu(\delta) > 0$ such that if $x(0) \in \mathscr{B}_{\mu}(\tilde{x}_{\boldsymbol{\sigma}}(0))$, we have $x(k) \in \mathscr{B}_{\delta}(\tilde{x}_{\boldsymbol{\sigma}}(k))$ for all $k \geq 0$, and *ii)* there exists a constant $\mu > 0$ and for each $\eta > 0$, there exists a scalar $T = T(\eta) > 0$ such that if $x(0) \in \mathscr{B}_{\mu}(\tilde{x}_{\boldsymbol{\sigma}}(0))$, we have $x(k) \in \mathscr{B}_{\eta}(\tilde{x}_{\boldsymbol{\sigma}}(k))$ for all $k \geq T(\eta)$. □

The notion of max-plus convergent dynamics requires the existence of a unique and asymptotically max-plus bounded-buffer stable reference solution, possibly dependent on the the switching sequence σ_k. This solution is often referred to as a steady-state solution. Every other solution, in the vicinity, then converges asymptotically to the steady-state solution. This is not necessarily true for max-plus incremental stability where all solutions can be unbounded in the max-plus Hilbert projective norm. Moreover, the convergence of each solution to the steady-state solution does not necessarily imply a uniform convergence of two solutions to each other. Therefore, the two notions of max-plus incremental stability and max-plus convergent dynamics are distinct. A similar relationship between incremental stability and convergent dynamics is very well-known in the literature for conventional time-driven systems [210].

If the matrix describing a max-plus linear system is primitive[9], the steady-state solution is unique in the max-plus Hilbert projective norm [119, §4.3]. Moreover, the convergence to the solution is obtained in finite-step for any finite initial condition.

4.2.5. CONVERGENCE

The notion of max-plus bounded-buffer stability requires finite-step (exponential) convergence to the desired set. This, in particular, is manifested as a turnpike phenomenon: *i)* finite transients to periodic behaviour in max-plus linear systems [59], and *ii)* convergence to a set of stationary regimes for products of max-plus matrices under certain restrictive assumptions [170, 201]. Algorithms for computation of bounds on the

[9] A max-plus matrix $A \in \mathbb{R}_{\varepsilon}^{n\times n}$ is said to be primitive if there exists an integer $N \in \mathbb{N}$ such that for $t \geq N$, $[A^{\otimes^t}]_{ij}$ is finite for all $i, j \in \underline{n}$.

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transients of max-plus linear systems (leading to a periodic behaviour) can be found in [1, 179].

The exponential convergence property is desirable for the purpose of control and stabilisation. However, the a priori computation of bounds on the rate of convergence is not always possible and involves detailed graph-theoretical analysis [131, 175]. Secondly, convergence in the max-plus Lipschitz stability case is studied in an asymptotic sense. This is because the growth rate (inverse of throughput) is an asymptotic property of trajectories of an additively homogeneous and monotone dynamical system [109, §4]. Moreover, it is sometimes possible to achieve a smaller growth rate by (possibly infinitely) increasing the period $c \in \mathbb{N}$ [95, §V.A].

4.3. STABILITY ANALYSIS: PROBLEM A

In this section we derive useful general criteria for stability analysis of max-plus-algebraic hybrid automata. The exposition is largely based on Lyapunov theory and its extensions for conventional time-driven systems [135]. We differentiate between the stability of the discrete-event system (4.1) and that of its individual subsystems corresponding to the different modes of operation. In this section, the notions of stability are studied under arbitrary switching. It is then necessary that all subsystems are stable under the respective notion.

4.3.1. ULTIMATE BOUNDEDNESS

We study the max-plus bounded-buffer stability of discrete-event systems (4.1) (as in Definitions 4.2.3 and 4.2.4) as attractivity and positive invariance of certain sets bounded in the max-plus Hilbert projective norm (2.21). This is carried out analogously to the treatment of uniformly ultimate boundedness of conventional time-driven systems[10] under uncertainty and parameter variations [29, 82, 151]. The notion of stability and the ensuing analysis is therefore robust to unknown and possibly event-varying perturbations.

We begin by presenting a Lyapunov function definition outside an open ball $\mathscr{B}_\delta$ to analyse max-plus bounded-buffer stability (see Definition 1).

Definition 4.3.1. (Sub-level sets)**.** The sub-level sets generated by a continuous function $\Psi : \mathbb{R}^n \to \mathbb{R}$ for $\delta \geq 0$. are denoted as

$$\mathscr{N}(\Psi, \delta) = \{x \in \mathbb{R}^n \mid \Psi(x) \leq \delta\}, \qquad \square$$

Definition 4.3.2. (Lyapunov function outside a set)**.** A positive definite continuous function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function outside $\mathscr{B}_\delta$, $\delta > 0$, for the system (4.1) if

1. there exists $v > 0$ such that $\mathscr{N}(\Psi, v) \subseteq \mathscr{B}_\delta$ and for all $x \notin \mathscr{N}(\Psi, v)$ we have

$$\Delta\Psi(x) = \Psi(f(l,x)) - \Psi(x) \leq -\alpha(\|x\|_{\mathbb{P}} + \delta') \tag{4.3}$$

for some $\delta' \geq 0$, a function α of class $\mathscr{K}$ and for all $l \in \underline{n_{\mathrm{L}}}$.

[10] Please refer to Section 2.3.1 for a brief description of stability analysis tools for conventional time-driven systems.

2. The set $\mathcal{N}(\Psi, \nu)$ is positively invariant, so for all $x \in \mathcal{N}(\Psi, \nu)$ and for all $l \in \underline{n_L}$, we have

$$\Psi(f(l, x)) \le \nu \qquad \square$$

A non-zero value of the scalar δ' in the preceding definition of a Lyapunov function indicates that the ray $\overline{\mathbb{1}}_n$ might not be contained in the set $\mathcal{N}(\Psi, \nu)$. The following result is immediate.

Theorem 4.3.1. (Ultimate buffer boundedness). *A discrete-event system* (4.1) *is uniformly max-plus bounded buffer stable if it admits a Lyapunov function outside* $\mathcal{B}_\delta$*, for a finite* $\delta > 0$*, as in Definition 4.3.2.*

Proof. The proof follows from [31, Theorem 2.50]. ■

The converse of the preceding theorem can be formulated by requiring, for instance, asymptotic convergence to the set $\mathcal{B}_\delta$ and recognising that the max-plus Hilbert projective norm is a set-induced Lyapunov function, $\mathcal{N}(\|\cdot\|_{\mathbb{P}}, \delta) = \mathcal{B}_\delta$ (see [101, Theorem 6.5]). Mathematically, there exist a function β of class $\mathcal{KL}$ and a scalar $\delta > 0$ such that for max-plus bounded-buffer stability under uniform asymptotic convergence, we have

$$\|x(k)\|_{\mathbb{P}} \le \max(\beta(\|x(0)\|_{\mathbb{P}}, k), \delta). \tag{4.4}$$

Now we propose a general candidate Lyapunov function for the discrete-event system (4.1) as a *max-plus gauge function*: *i)* it evaluates the distance of the trajectory from a given closed set $\mathcal{K} \subseteq \mathbb{R}^n$ bounded in the max-plus Hilbert projective norm, and *ii)* its max-plus unit ball (or, 0-level set) represents the set $\mathcal{K}$.

To this end, we present the following definitions analogously to the concept of C-set in set-invariance theory [31]. Again, such a set enforces max-plus bounded buffer stability (as in Definition 4.2.3).

Definition 4.3.3. (Max-plus C-set). A max-plus C-set is defined as a subset $\mathcal{K} \subseteq \mathbb{R}^n$ such that it is a finitely generated max-plus cone[11] and convex cone, and bounded in the max-plus Hilbert projective norm. □

Note that a finitely generated max-plus cone is a convex cone if and only if it is also a min-plus convex cone [123, Theorem B]. Therefore, a max-plus C-set can be obtained by taking a min-plus convex closure of a given finitely generated max-plus cone.

We also note that a max-plus C-set does not necessarily contain the ray $\overline{\mathbb{1}}_n$ in its interior. We now show that a max-plus C-set is generated as max-plus column span[11] of a Kleene star matrix[12].

Lemma 4.3.1. (Kleene star generator). *Given a max-plus C-set* $\mathcal{K} \subseteq \mathbb{R}^n$*, there exists a unique irreducible Kleene star matrix* $K \in \mathbb{R}_\varepsilon^{n\times n}$ *such that* $\mathcal{K} = \mathrm{span}_\oplus(K) = \mathrm{eig}(K, 0)$.

Proof. See Section 4.5. ■

The preceding result also implies that the max-plus weak dimension (see Definition 2.2.19) of a max-plus C-set is at most n. We now define a function associated to a given

[11] Please refer to Section 2.2.3 for an exposition on max-plus convex geometry.
[12] Please refer to Lemma 2.2.5 and the discussion thereafter.

max-plus C-set whose sublevel sets are achieved by scaling the max-plus eigenspace of the generating Kleene star matrix. To this end, we present a normalisation of a Kleene star matrix [196, §4].

A Kleene star matrix[13] $K \in \mathbb{R}_\varepsilon^{n\times n}$ normalised by a scalar $\mu \in \mathbb{R}$ is denoted as $\widetilde{K}_\mu$ and obtained as follows: $[\widetilde{K}_\mu]_{ij} = [K]_{ij} - \mu$ for all $i, j \in \underline{n}$ with $i \neq j$, and $[\widetilde{K}_\mu]_{ii} = 0$. Note that $\widetilde{K}_\mu$ is again a Kleene star matrix.

Definition 4.3.4. (Max-plus gauge function)**.** Given a max-plus C-set $\mathcal{K} \subseteq \mathbb{R}^n = \text{span}_\oplus(K)$, its max-plus gauge function is defined as[14]

$$\Psi_{\mathcal{K}}(x) = \min_{\mu\geq 0}\{\mu \in \mathbb{R} \mid x \in \text{eig}(\widetilde{K}_\mu, 0)\}, \quad \forall x \in \mathbb{R}^n. \tag{4.5}$$

Here, the minimum is attained since the max-plus eigenspace of a Kleene star matrix is finitely generated. □

The following properties of a max-plus gauge function can be verified.

Definition 4.3.5. (Max-plus gauge function properties)**.** Given a max-plus C-set $\mathcal{K} \subseteq \mathbb{R}^n$, the associated max-plus gauge function $\Psi_{\mathcal{K}} : \mathbb{R}^n \to \mathbb{R}$ satisfies the following properties:

a) Max-plus sub-linearity: $\Psi_{\mathcal{K}}(x \oplus y) \leq \Psi_{\mathcal{K}}(x) \oplus \psi_(y)$, for all $x, y \in \mathbb{R}^n$;

b) Scale freeness: $\Psi_{\mathcal{K}}(\mu \otimes x) = \Psi_{\mathcal{K}}(x)$, for all $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$;

c) Positive definiteness: $\Psi_{\mathcal{K}}(x) \geq 0$, $\Psi_{\mathcal{K}}(x) = 0 \Leftrightarrow x \in \mathcal{K}$;

d) Continuity. □

The max-plus convexity of the sublevel-set of a max-plus gauge function, $\mathcal{N}(\Psi_{\mathcal{K}}, \delta)$ for a given $\delta > 0$, follows from [66, Theorem 18-9]. As the max-plus C-set $\mathcal{K}$ is closed, the function $\Psi_{\mathcal{K}}(\cdot)$ is max-plus convex.

Lemma 4.3.2. (Closed-form expression of a max-plus gauge function)**.** *A max-plus gauge function* (4.5)*, for a given max-plus C-set $\mathcal{K} \subseteq \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, can be expressed as*[15]

$$\Psi_{\mathcal{K}}(x) = x^* \otimes K \otimes x, \tag{4.6}$$

where $K \in \mathbb{R}_\varepsilon^{n\times n}$ is the generating Kleene star matrix. The max-plus gauge function provides the minimum max-plus Hilbert projective distance between x and $\mathcal{K}$.

Proof. See Section 4.5. ■

The smallest max-plus Hilbert ball $\mathcal{B}_\delta$ enclosing the set $\text{span}_\oplus(K)$, for a given Kleene star matrix $K \in \mathbb{R}_\varepsilon^{n\times n}$, can be obtained using existing algorithms [68]. This provides a global ultimate upper bound on the buffer-level of the discrete-event system via Theorem 4.3.1.

[13] Note that for a Kleene star matrix $[K]_{ii} = 0$ for all $i \in \underline{n}$.

[14] In conventional algebra, a gauge function or Minkowski function for a convex and compact set $\mathcal{K} \subseteq \mathbb{R}^n$ is defined as $\Psi_{\mathcal{K}}(x) = \inf\{\mu \geq 0 \mid x \in \mu\mathcal{K}\}$.

[15] Recall that $x^* = (-x)^\top$ for $x \in \mathbb{R}^n$.

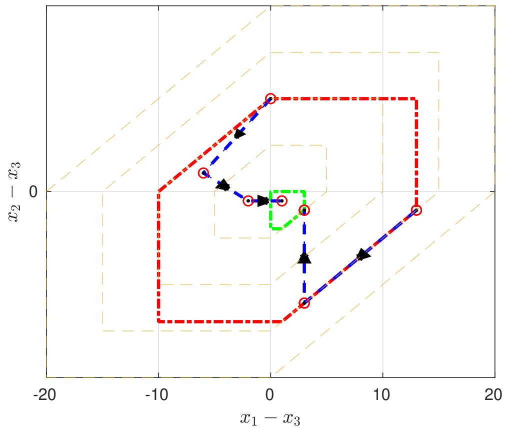

Figure 4.2: Max-plus C-set $\mathscr{K}$ as defined in (4.8) (in green dash-dotted line $-\cdot-$) and its level-set $\widetilde{\mathscr{K}}_\mu$, for $\mu = 10$, (in red dash-dotted line $-\cdot-$) projected on the hyperplane $\{x \in \mathbb{R}^3 \mid x_3 = 0\}$. Sample trajectories (in blue dashed line $--$) of the SMPL system defined by matrices in Example 4.3.1 are also shown. The level-sets of (symmetric) max-plus Hilbert balls centred at $\mathbb{1}_n$ (in mustard $--$) represent practical bounds on the buffer-levels.

Example 4.3.1. *Consider a bimodal open-loop switching max-plus linear system defined by the following matrices:*

$$A^{(1)} = \begin{pmatrix} 4 & 2 & 4 \\ \varepsilon & 3 & 2 \\ 1 & 4 & 1 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 5 & \varepsilon & 5 \\ 4 & 3 & 2 \\ 5 & 1 & 5 \end{pmatrix}. \tag{4.7}$$

We consider a positively invariant max-plus C-set $\mathscr{K} = \mathrm{span}_\oplus(K)$, *where*

$$K = \begin{pmatrix} 0 & 0 & 0 \\ -5 & 0 & -4 \\ -3 & 0 & 0 \end{pmatrix} \tag{4.8}$$

is computed as in Appendix 4.4.4. Consider now the associated max-plus set-induced Lyapunov function $V(x) = x^* \otimes K \otimes x$. *The analysis on attractivity (as in Appendix 4.4.5) of the max-plus C-set shows that max-plus set-induced Lyapunov function, is non-decreasing for all* $x \in \mathbb{R}^n_\varepsilon \setminus \mathscr{K}$ *(check for instance* $x = (13\ -2\ \ 0)^\intercal$*). However, the function* $V(\cdot)$ *is strictly decreasing in two steps. The latter analysis is performed on* $\mathscr{A} = \{A^{(1)}, A^{(2)}\}$, $\mathscr{A}^{\otimes^2} = \{A^{(1)\otimes^2}, A^{(2)\otimes^2}, A^{(1)} \otimes A^{(2)}, A^{(2)} \otimes A^{(1)}\}$.

The max-plus C-set and sample trajectories are plotted in Figure 4.2.

Lastly, the smallest (symmetric) max-plus Hilbert ball centred at $\mathbb{1}_n$ *containing* $\mathscr{K}$ *is of radius* $\delta = 5$. *This gives the ultimate bound on the buffer-levels of the discrete-event system defined by the open-loop switching max-plus linear system defined by* $x(k) = A^{(l(k))} \otimes x(k-1)$ *where* $l(k) \in \{1,2\}$ *for* $k \in \mathbb{N}$. *In particular, for every* $\mu > 0$, *there exists a* $T = T(\mu, 5)$ *such that* $\|x(k)\|_\mathbb{P} \le 5$ *for* $k \ge T$ *if* $\|x(0)\|_\mathbb{P} \le \mu$.

Chapter 4

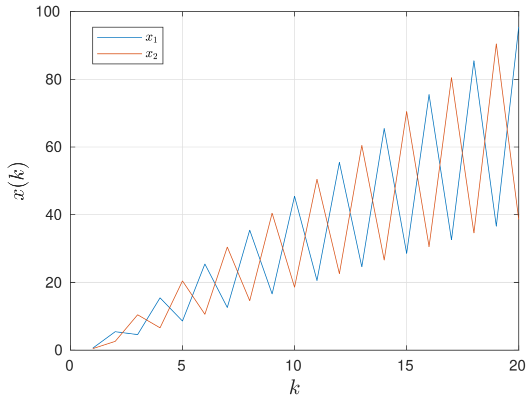

Figure 4.3: Trajectory of the open-loop switching max-plus linear system defined in Example 4.3.2 under periodic switching $\sigma = 1, 2, 1, 2, 1, \ldots$. Note that $\|x(k)\|_{\mathbb{P}}$ and $\max_i (x_i(k) - x_i(k-1))$ diverge along the event step $k \in \mathbb{N}$.

Remark. A semigroup of commuting matrices[16] $\mathscr{A}$ in max-plus algebra leaving a closed subspace $\mathscr{K} \subseteq \mathbb{R}^n$ invariant share a common max-plus eigenvector [200, Theorem 2.1].

The rest of the subsection is devoted to the analysis of max-plus Lipschitz stability of discrete-event systems as boundedness of the growth rate of trajectories.

Proposition 4.3.1. (Uniform Lipschitz stability ultimate bounds)**.** *A uniformly max-plus bounded buffer stable discrete-event system* (4.1) *is also uniformly max-plus Lipschitz stable.*

Proof. See Section 4.5. ■

The converse of the preceding proposition does not however hold: A uniformly max-plus Lipschitz stable discrete-event system is not necessarily uniformly max-plus bounded-buffer stable. We also note that uniform max-plus Lipschitz stability of a discrete-event system can be lost even if the subsystems are max-plus bounded-buffer stable.

Example 4.3.2. *Consider a bimodal open-loop switching max-plus linear system composed of the following matrices:*

$$A^{(1)} = \begin{pmatrix} \varepsilon & 2 \\ 5 & 0 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 0 & 5 \\ 2 & \varepsilon \end{pmatrix}. \tag{4.9}$$

The subsystems are uniformly max-plus bounded-buffer stable. The max-plus eigenspaces are spanned by the vectors $(0\ 1.5)^\top$ *and* $(1.5\ 0)^\top$*, respectively. However, a periodic switching between the two subsystems leads to instability with respect to bounded-buffers and growth rates. A sample trajectory is portrayed in Fig.* 4.3.

[16]A matrix semigroup $\mathscr{A}$ is commutative if $A \otimes B = B \otimes A$ for all $A, B \in \mathscr{A}$.

Hence in the subsequent sections, we often assume that the system is uniformly max-plus Lipschitz stable. Note that this can be achieved on a restricted domain $\mathscr{D} \subseteq \mathbb{R}^n$ by specifying bounds $\alpha, \beta \in \mathbb{R}$, with $\beta \geq \alpha$, on the one-step growth rate of the system:

$$\mathscr{D}_\alpha^\beta = \{x \in \mathbb{R}^n \mid x \in \bigcap_{l \in \underline{n_\mathrm{L}}} S_\alpha^\beta(f(l,\cdot))\}, \tag{4.10}$$

where the slice spaces $S_\alpha^\beta(f(l,\cdot))$ are defined in (2.34).

We now present a qualitative characterisation of max-plus bounded-buffer stability that allows a change of metrics[17] and further analysis of asymptotic max-plus Lipschitz stability of a discrete-event system.

Lemma 4.3.3. (Change of norms). *A discrete-event system (4.1) is max-plus bounded-buffer stable if and only if there exists a function $\gamma : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$ such that the quantity $\left\| x(k) - \gamma(k, x(0)) \otimes \mathbb{1}_n \right\|_\infty$ is bounded as $k \to \infty$.*

Proof. See Section 4.5. ∎

We remark that if the dynamics in (4.1) is such that $n_\mathrm{L} = 1$ and the function f is monotone and additively homogeneous in the state x then the existence of a function $\gamma(k,\cdot) = \lambda(f) \cdot k$ (in Lemma 4.3.3) is necessary and sufficient for the existence of a fixed point (finite eigenvector) of f corresponding to an eigenvalue[18] $\lambda(f)$, or $\mathrm{eig}(f, \lambda(f)) \neq \emptyset$ [109, Lemma 4.2]. In particular, if $\lambda(f)$ exists, it is unique and represents the asymptotic growth rate of any trajectory of the system.

We now recall the following result from literature that allows analysis of system trajectories for a (given) constant asymptotic growth rate.

Lemma 4.3.4. (Projected dynamics [169]). *For any monotone and additively homogeneous function $\Phi : \mathbb{R}^n \to \mathbb{R}$, the function $\Gamma : x \mapsto (\Phi(x), \overline{x})$ is an injective Lipschitz continuous function from $\mathbb{R}^n$ to $\mathbb{R} \times \mathbb{PR}^n$ with a Lipschitz continuous inverse.* ∎

The preceding result implies that the limiting (asymptotic) behaviour of the state trajectory $(x(k))_{k\in\mathbb{N}}$ of a given discrete-event system (4.1) can be studied as a combination of limiting behaviours of the sequences $\{\Phi(x(k))\}_{k\in\mathbb{N}}$ and $\{\overline{x}(k)\}_{k\in\mathbb{N}}$. For example, $\Phi(x) = x_i$, $\Phi(x) = \max_i x_i$, and $\Phi(x) = \min_i x_i$, are interesting monotone and additively homogeneous functions.

Theorem 4.3.2. (Asymptotic Lipschitz stability). *Consider again the discrete-event system in (4.1). Let $\rho \in \mathbb{R}$ be given and let the normalised state be denoted as $x_\rho(k) = x(k) - (\rho \cdot k) \otimes \mathbb{1}_n$. The following statements are equivalent:*

1. *The system is uniformly max-plus bounded-buffer stable.*

   In addition, given any monotone and additively homogeneous function $\Phi : \mathbb{R}^n \to \mathbb{R}$, for every $\mu > 0$ there exists a scalar $T = T(\mu) \in \mathbb{N}$ such that if $x(0) \in \mathscr{B}_\mu$, then $\left|\Phi(x_\rho(k)) - \Phi(x_\rho(k-1))\right|$ is uniformly bounded for all $k \geq T$.

[17] For $x \in \mathbb{R}^n$, if $\alpha \otimes \mathbb{1} \leq x \leq \beta \otimes \mathbb{1}$, for some $\alpha, \beta \in \mathbb{R}$, then $\|x\|_\mathbb{P} \leq \beta - \alpha$ and $\|x\|_\infty \leq \max(|\alpha|, |\beta|)$. Here, the norm equalities are met if $x_i = \alpha$ and $x_j = \beta$ for some $i, j \in \underline{n}$.

[18] Please refer to Section 2.2.4 for an exposition on the max-plus eigenvalue problem and its relation to the fixed-point theory of additively homogeneous and monotone functions.

2. *The system is uniformly asymptotically max-plus Lipschitz stable.*

Proof. The result is immediate from Lemma 4.3.3 and 4.3.4. ■

The scalar additively homogeneous and monotone function Φ in the preceding analysis can be interpreted as an output map of the system. The preceding theorem then suggests that the asymptotic max-plus Lipschitz stability can be equivalently analysed as the boundedness of the output of a (suitably normalised) max-plus bounded-buffer stable discrete-event system.

4.3.2. LYAPUNOV STABILITY

In this section we propose Lyapunov-like theorems to study the asymptotic stability properties of the discrete-event system in (4.1). We follow the approach of [67, 165] to study the asymptotic notions of stability of switching systems under arbitrary switching, in that the switching sequence is not known a priori. The authors of [67, 165] exploit the connection between conventional time-driven nonlinear switched systems and nonlinear systems with disturbances to provide necessary and sufficient conditions for uniform asymptotic stability in the sense of Lyapunov.

We recall the following notions of uniform asymptotic stability with respect to a closed (but not necessarily compact) set from the literature [122, 152], albeit expressed in the max-plus Hilbert projective metric. Note that the following stability notions do not directly ensure stability in the sense of Definitions 4.2.3 and 4.2.4.

Lemma 4.3.5. (Asymptotic bounded-buffer stability, cf. [122])**.** *Consider a discrete-event system* (4.1) *and a closed positively invariant set $\mathscr{K} \subseteq \mathbb{R}^n$. The system is uniformly asymptotically max-plus bounded with respect to $\mathscr{K}$ if there exists a function β of class $\mathscr{K}\mathscr{L}$ and a scalar $\mu > 0$ such that*[19]

$$\|x(k)\|_{\mathscr{K},\mathbb{P}} \leq \beta\left(\|x(k_0)\|_{\mathscr{K},\mathbb{P}}, k\right) \tag{4.11}$$

whenever $\|x(k_0)\|_{\mathscr{K},\mathbb{P}} \leq \mu$ and $k \geq k_0$.

Similarly, the system is uniformly exponentially max-plus bounded with respect to $\mathscr{K}$ if the estimate in (4.11) *can be expressed as*

$$\|x(k)\|_{\mathscr{K},\mathbb{P}} \leq \kappa \cdot \|x(k_0)\|_{\mathscr{K},\mathbb{P}} \cdot \sigma^{-(k-k_0)} \tag{4.12}$$

for some scalars $\kappa > 0$, $\sigma > 1$ whenever $\|x(k_0)\|_{\mathscr{K},\mathbb{P}} \leq \mu$ and $k \geq k_0$. ■

The maximal open neighbourhood $\{x \in \mathbb{R}^n \mid \|x\|_{\mathscr{K},\mathbb{P}} < \mu\}$ allowed in the preceding definition is called the *region of attraction* of $\mathscr{K}$. The Lyapunov characterisation of the preceding acttractiveness properties are very well known in literature for conventioal time-driven systems defined on a suitable normed vector space [122, 152]. We extend the results from literature [122] to the discrete-event system (4.1).

Definition 4.3.6. (Max-plus Lyapunov function)**.** Let $\mathscr{D} \subseteq \mathbb{R}^n$ be an open domain. Let $\mathscr{K} \subset \mathscr{D}$ be a closed set. A continuous function $V : \mathscr{D} \to \mathbb{R}_+$ is said to be a common max-plus Lyapunov function with respect to $\mathscr{K}$ defined on the domain $\mathscr{D}$ for the system dynamics (4.1) if the following conditions hold:

[19] Note that $\|z\|_{\mathscr{K},\mathbb{P}}$ denotes the distance of z to $\mathscr{K}$ in the max-plus Hilbert projective metric (3.27).

1. the function V is scale free, so $V(\mu \otimes x) = V(x)$ for all $x \in \mathscr{D}$ and $\mu \in \mathbb{R}$;
2. there exist two functions[20] α_1, α_2 of class $\mathscr{K}_\infty$ such that
$$\alpha_1(\|x\|_{\mathcal{K},\mathbb{P}}) \leq V(x) \leq \alpha_2(\|x\|_{\mathcal{K},\mathbb{P}}) \tag{4.13}$$
for all $x \in \mathscr{D}$;
3. there exist a continuous, positive definite function α_3 such that
$$V(f(l,x)) - V(x) \leq -\alpha_3(\|x\|_{\mathcal{K},\mathbb{P}}) \tag{4.14}$$
for all $x \in \mathscr{D}$ and for all $l \in \underline{n_{\mathrm{L}}}$. □

Theorem 4.3.3. (Uniform asymptotic max-plus bounded-buffer stability)**.** *Consider a uniformly max-plus Lipschitz stable discrete-event system* (4.1) *and a closed max-plus cone $\mathcal{K} \subset \mathbb{R}^n$ bounded in the max-plus Hilbert projective norm.*

Then the discrete-event system is uniformly locally asymptotically max-plus bounded-buffer stable with respect to the closed set $\mathcal{K}$ if it admits a common max-plus Lyapunov function with respect to $\mathcal{K}$ on a domain $\mathscr{D} \subseteq \mathbb{R}^n$ containing $\mathcal{K}$.

Proof. The proof follows from [122, Theorem 1] by noting that $\mathcal{K}$ is a closed set. ■

Under uniform Lipschitz continuity and uniform max-plus bounded-buffer stability of the constituent subsystems, the conditions of the preceding theorem are necessary as well if the common max-plus Lyapunov function is required to be smooth (once differentiable). This can be proved analogously to [122, Theorem 1].

The following example shows that a max-plus set-induced Lyapunov function can be reused as a max-plus Lyapunov function.

Example 4.3.3. *Consider a bimodal open-loop switching max-plus linear system defined by the following matrices:*
$$A^{(1)} = \begin{pmatrix} 4 & \varepsilon \\ 1 & 1 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 3 & 3 \\ \varepsilon & 6 \end{pmatrix}. \tag{4.15}$$
The max-plus eigenvalues are obtained as $\lambda_1 = 4$ and $\lambda_2 = 6$ for the two subsystems, respectively. Consider a max-plus C-set generated as $\mathcal{K} = \mathrm{span}_\oplus(K)$, where
$$K = \left(A^{(1)}_{\lambda_1} \oplus A^{(2)}_{\lambda_2}\right)^\star = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}. \tag{4.16}$$
The set $\mathcal{K}$ is positively invariant to the system dynamics. We consider the max-plus Lyapunov function[21] $V(x) = x^* \otimes K \otimes x$. The attractivity properties of the set $\mathcal{K}$ can be noted from Fig. 4.4. The plot is obtained using the method explained in Section 4.4.5 for the matrix semigroups $\mathscr{A} = \{A^{(1)}, A^{(2)}\}$, $\mathscr{A}^{\otimes^2} = \{A^{(1)^{\otimes^2}}, A^{(1)} \otimes A^{(2)}, A^{(2)} \otimes A^{(1)}, A^{(2)^{\otimes^2}}\}$, and $\mathscr{A}^{\otimes^3} = \{A^{(1)^{\otimes^3}}, A^{(1)} \otimes A^{(1)} \otimes A^{(2)}, A^{(1)} \otimes A^{(2)} \otimes A^{(1)}, \dots, A^{(2)^{\otimes^3}}\}$. Most importantly, the set $\mathcal{K}$ is uniformly and globally attractive for the trajectories of the open-loop switching max-plus linear system under arbitrary switching sequences.

We can also derive that $\|x(k)\|_{\mathcal{K},\mathbb{P}} \leq 0.45 \cdot \|x(0)\|_{\mathcal{K},\mathbb{P}} \cdot 1.6^{-k}$. This results in a uniform exponential max-plus bounded-buffer stability with respect to $\mathcal{K}$.

[20]For a function α of class $\mathscr{K}$, if $\lim_{s\to+\infty} \alpha(s) = +\infty$, then it is said to be of class $\mathscr{K}_\infty$.
[21]Recall that $x^* = (-x)^\top$ for $x \in \mathbb{R}^n$.

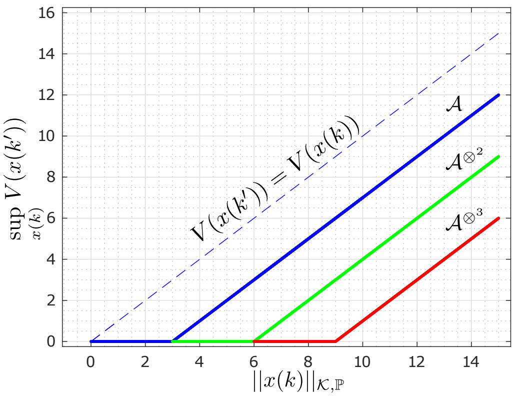

Figure 4.4: The worst-case convergence of trajectories of the open-loop switching max-plus linear system, in Example 4.3.3, to the max-plus C-set $\mathcal{K} = \text{span}_{\oplus} K$. The distance to the set is measured using the max-plus Lyapunov function $V(x(k)) = x^*(k) \otimes K \otimes x(k)$. The change in Lyapunov function, $V(x(k)) - V(x(k'))$, is minimised (as in Section 4.4.5) for $k' = k+1$, $k' = k+2$, and $k' = k+3$. This corresponds to the Lyapunov function decay for matrix semigroups $\mathscr{A}$, $\mathscr{A}^{\otimes^2}$, and $\mathscr{A}^{\otimes^3}$, respectively.

4.3.3. LASALLE-LIKE RELAXATIONS

In this subsection, we consider the non-autonomous discrete-event system dynamics (4.1) evolving over a positively invariant set. The aim is to extend the stability arguments from the classical Krasovskii-LaSalle invariance principle[22] [47, §1.3], [168] to max-plus algebra. This generalisation of the Lyapunov theory from the preceding subsection allows the relaxation of strict decay requirement on Lyapunov functions by exploiting invariance of limit sets.

We first present a definition of an ω-limit set suitable for analysis of discrete-event systems (4.1). These sets play a pivotal role in the characterisation and computation of attractive sets of iterated function systems $\mathscr{G} = \{g_l\}_{l \in \underline{n_\text{L}}}$ consisting of a (possibly infinite) compact set of generating maps $g_l : \mathbb{R}^n \to \mathbb{R}^n$ [20, 41], and multi-valued discrete dynamical systems [47, §1.3]. This involves formulating conditions for existence and uniqueness of attractive sets in general non-contractive [21], contractive [41, 195], or additively homogeneous contractive mappings [35], among others [147].

Definition 4.3.7. (Projective limit set)**.** The ω-limit set $\Omega(\gamma) \subset \mathbb{R}^n/\sim$ of a sequence $\gamma = \{\gamma(k)\}_{k \in \mathbb{N}_0}$ is the set of rays $\overline{y} \subset \mathbb{R}^n/\sim$ for which there exists a subsequence $\{\gamma(k_m)\}_{m \in \mathbb{N}}$ of γ such that the $\lim_{m \to +\infty} d_\text{H}(\gamma(k_m), \overline{y}) = 0$. □

It is noted that the determination of state trajectories, and hence the ω-limit sets, of general non-autonomous dynamical systems requires an additional specification of exogenous input sequence. The important consequence of this non-uniqueness is that the ω-limit set is not invariant. Therefore, we resort to the following notion of weak invariance.

[22] Please refer to Section 2.3.2 for an exposition on the Krasovskii-LaSalle invariance principle for conventional time-driven systems defined on a suitable normed vector space.

Definition 4.3.8. (Weak positive invariance)**.** A set $\mathcal{K} \subseteq \mathbb{R}^n$ is weakly positively invariant for the system (4.1) if for every $x \in \mathcal{K}$ there exists $l \in \underline{n_L}$ such that $f(l,x) \in \mathcal{K}$. □

The subsequent analysis for switching systems (4.1) can also be applied to an over-approximation of general event-varying system dynamics $f : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ as an iterated function system or a multi-valued discrete dynamical system [47, Lemma 1.21]. We begin with the definition of a weak max-plus Lyapunov function in a set.

Definition 4.3.9. (Weak max-plus Lyapunov function)**.** Let $\mathcal{K} \subseteq \mathbb{R}^n$ be any set. A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a weak Lyapunov function in $\mathcal{K}$ for the system dynamics (4.1) if the following conditions hold:

1. $V(\cdot)$ is scale free, so $V(\mu \otimes x) = V(x)$ for all $\mu \in \mathbb{R}$;
2. $V(\cdot)$ is continuous on $\mathcal{K}$;
3. $V(f(l,x)) - V(x) \leq 0$ for any $x \in \mathcal{K}$ and all $l \in \underline{n_L}$.

The function V is positive definite with respect to a set $\mathcal{K}_c \subseteq \mathbb{R}^n$ if

1. $V(x) = 0$ for all $x \in \mathcal{K}_c$;
2. there exists an $\eta > 0$ such that $V(x) > 0$ wherever $x \in \mathcal{B}_\eta(\mathcal{K}_c)$ and $x \notin \mathcal{K}_c$. □

We now present a max-plus version of the Krasovskii-LaSalle invariance principle for discrete-event systems (4.1) under Assumptions 4.2.1 and 4.2.2. Our aim is to characterise the attractive sets of uniformly max-plus bounded buffer stable systems using certain weak max-plus Lyapunov functions.

Theorem 4.3.4. (Max-plus Krasovskii-LaSalle invariance principle I)**.** *Consider a discrete-event system* (4.1) *evolving on a positively invariant set* $\mathcal{K} \subseteq \mathbb{R}^n$. *Suppose that the system dynamics admits a weak max-plus Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}$ *in* $\mathcal{K}$ *(see Definition* 4.3.9*).*

Assume the discrete-event system is uniformly max-plus bounded-buffer stable in $\mathcal{B}_\delta$ *for some* $\delta > 0$. *Then there exists* $c \in \mathbb{R}$ *such that the state trajectories approach a set of the form* $V^{-1}(c) \cap \mathcal{M}$ *where* $\mathcal{M}$ *is the largest weakly positively invariant set contained in* $\{x \in \mathrm{cl}(\mathcal{K}) \mid \exists l \in \underline{n_L} \text{ s.t. } V(f(l,x)) = V(x)\}$.

Proof. See Section 4.5. ■

The preceding result is general in the sense that it lets go of the assumption on non-expansiveness of the constituting maps in (4.1). The assumption on uniform max-plus bounded-buffer stability of the system can be relaxed if the sub-level sets of the weak Lyapunov function are bounded in the max-plus Hilbert projective norm.

Theorem 4.3.5. (Max-plus Krasovskii-LaSalle invariance principle II)**.** *Consider a discrete-event system* (4.1). *Suppose that there exists a continuous function* $V : \mathbb{R}^n \to \mathbb{R}$ *such that*

1. *there exists a scalar* $s \in \mathbb{R}$ *such that* $\mathcal{N}(V,s) = \{x \in \mathbb{R}^n \mid V(x) < s\}$ *is bounded in the max-plus Hilbert projective norm;*

2. *V is a weak Lyapunov function in $\mathcal{N}(V,s)$ (see Definition 4.3.9).*

Then there exists a scalar $c \in \mathbb{R}$ such that the state trajectories approach a set of the form $V^{-1}(c) \cap \mathcal{M}$ where $\mathcal{M}$ is the largest weakly positively invariant set contained in $\{x \in \mathrm{cl}(\mathcal{N}(V,s)) \mid \exists l \in \underline{n_{\mathrm{L}}} \text{ s.t. } V(f(l,x)) = V(x)\}$.

Proof. The proof follows closely the proof of Theorem 4.3.4. A sub-level set of the weak max-plus Lyapunov function provides a positively invariant set that is also bounded in the max-plus Hilbert projective norm. ■

Example 4.3.4. *Consider a bimodal open-loop switching max-plus linear system defined by the following matrices*

$$A^{(1)} = \begin{pmatrix} -2 & 1 & \varepsilon \\ -1 & -1 & -2 \\ -1 & \varepsilon & -2 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 0 & -1 & -1 \\ \varepsilon & 0 & -4 \\ -3 & \varepsilon & 0 \end{pmatrix}. \tag{4.17}$$

We consider a positively invariant max-plus C-set $\mathcal{K} = \mathrm{span}_{\oplus}(K)$, where

$$K = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ -1 & 0 & 0 \end{pmatrix}. \tag{4.18}$$

The unique generator of the max-plus cone $\mathcal{K}$ is a common max-plus eigenvector of the system matrices. We define a weak max-plus Lyapunov function $V(x) = x^* \otimes K \otimes x = u \otimes x - v \,\phi\, x$, where $v = [K]_{\cdot 1}$ and $u = [K]_{1\cdot}$. The sub-level sets of the function $V(\cdot)$ are bounded in the max-plus Hilbert projective norm.

It can be deduced (using the analysis presented in Section 4.4.5) that the Lyapunov function $V(\cdot)$ is non-decreasing along the trajectories of the system.

We now note that $u \otimes A^{(l)} = u$ for $l \in \underline{n_{\mathrm{L}}} \triangleq \{1,2\}$. In accordance with Theorem 4.3.5, we deduce that the trajectories such that $\|x(0)\|_{\mathcal{K},\mathbb{P}} \leq c$, $c \in \mathbb{R}$, approach a set of the form $V^{(-1)}(c) \cap \mathcal{M}$ where $\mathcal{M}$ is the largest weakly positively invariant set contained in

$$\mathcal{C} = \{x \in \mathcal{K} \mid \exists l \in \underline{n_{\mathrm{L}}} \text{ s.t. } v \,\phi\, (A^{(l)} \otimes x) = v \,\phi\, x\}. \tag{4.19}$$

The set $\mathcal{C}$ is generated as solutions to mixed max-min equations. The computation of the solution sets can be reformulated as an extended linear complementarity problem [71, §4.5].

Remark. Under max-plus geometry, the 'bi-vector' $(x,y) \in \mathbb{R}^n_\varepsilon \times \mathbb{R}^n_\varepsilon$ is max-plus orthogonal to $z \in \mathbb{R}^n_\varepsilon$ if $z \,\phi\, x = z \,\phi\, y$ [8, §4]. The set $\mathcal{C}$, in the preceding example, consists of all state vectors $x \in \mathbb{R}^n$ that evolve (max-plus) orthogonally to the common max-plus eigenvector $v \in \mathbb{R}^n$, under the application of either of the modes. This notion can be extended to cases where the dimension of the attractor (max-plus weak dimension[23] of matrix K) is larger than one.

[23] See Definition 2.2.19.

4.3.4. ATTRACTIVITY OF MAX-PLUS EIGENSPACES

In this subsection, we aim to characterise the attractive sets of a discrete-event system (4.1) using the dynamical properties of certain monotone and additively homogeneous functions. This serves as an extension of the work [111] on stability of switching max-plus linear systems.

In the context of discrete-event systems, the max-plus spectral theory of additively homogeneous and monotone functions yields quantitative measures of performance [87, 89, 109]. An additively homogeneous discrete-event system dynamics enjoys a translation property: the occurrence times of all events (continuous state) can be changed by the same amount without affecting the dynamical properties of the system. The monotonicity property, on the other hand, implies that a delay in the occurrence times of some events cannot accelerate the occurrence of any other event. The latter property, however, does not necessarily hold for switching max-plus linear systems [111].

The long-term behaviour of additively homogeneous and monotone functions under iteration is related to desirable periodic trajectories (max-plus eigenvector) and optimal operating conditions (max-plus eigenvalue) of discrete-event systems.

The existence of finite max-plus eigenvector(s) of an additively homogeneous and monotone function $g : \mathbb{R}^n \to \mathbb{R}^n$ is equivalent to the boundedness of its trajectories, denoted $\{g^k(x)\}_{k\in\mathbb{N}}$, in the max-plus Hilbert projective norm for all $x \in \mathbb{R}^n$ [89, Theorem 9]. The function g admits a finite max-plus eigenvector if there exists a non-empty subset $\mathcal{V} \subset \mathbb{R}^n$ that is *i)* invariant under iteration of function g, i.e. $g(\mathcal{V}) \subseteq \mathcal{V}$, and *ii)* bounded in the max-plus Hilbert projective norm [87, Corollary 4.1]. We first recall the definition of a max-plus (sub)eigenspace of a function g that is always invariant under iteration of the function g [89]. For given scalars $\alpha, \beta \in \mathbb{R}$ with $\beta \geq \alpha$, the slice space of a function g is defined as:

$$S_\alpha^\beta(g) = \left\{x \in \mathbb{R}^n \mid \alpha + x \leq g(x) \leq \beta + x\right\}. \tag{4.20}$$

Note that a given slice space $S_\alpha^\beta(g)$ is non-empty for a large enough β and a small enough α with $\beta \geq \alpha$.

In [111] we assume that such a function g exists with a slice space $S_\alpha^\beta(g)$ bounded in the max-plus Hilbert projective norm for some $\alpha, \beta \in \mathbb{R}$. Then the positively invariance of the slice space $S_\alpha^\beta(g)$ for the system dynamics (4.1) implies that any trajectory starting in $S_\alpha^\beta(g)$ remains bounded in the max-plus Hilbert projective norm and satisfies the max-plus Lipschitz stability property [111, Proposition 4.1]. Now we consider the attractivity properties of these slice spaces with respect to the system dynamics (4.1).

Theorem 4.3.6. (Attractivity of max-plus eigenspaces)**.** *Consider a uniformly max-plus bounded buffer stable discrete-event system* (4.1) *evolving over a positively invariant set* $\mathcal{K} \subseteq \mathbb{R}^n$.

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ *be an additively homogeneous and monotone function. Assume that the scale-free function* V *defined by* $V(x) = \|g(x) - x\|_{\mathbb{P}}$ *is a weak max-plus Lyapunov function in* $\mathcal{K}$.

Then there exists a slice space $S_\alpha^\beta(g) \neq \emptyset$*, for some* $\alpha, \beta \in \mathbb{R}$*, such that the state trajectories approach the largest weakly positively invariant set contained in the slice space.*

Proof. See Section 4.5. ■

Chapter 4

The choice of an additively homogeneous and monotone function g in the preceding theorem gives information on the smallest achievable set $\mathscr{M}$ in Theorem 4.3.4. It also facilitates the determination of the largest region of attraction, $S_{\alpha}^{\beta}(g)$, such that for all $x(0) \in S_{\alpha}^{\beta}(g)$ it holds that $\lim_{k\to\infty} x(k) \in V^{-1}(c) \cap \mathscr{M}$.

In accordance with Theorem 4.3.5, the assumption on uniformly max-plus bounded-buffer stability can be relaxed if there exists a slice space $S_{\alpha}^{\beta}(g)$, for some $\alpha, \beta \in \mathbb{R}$, that is *a priori* bounded in the max-plus Hilbert projective norm. It is, in general, difficult to prove boundedness of a given slice space $S_{\alpha}^{\beta}(g)$ [89]. However, there exists a subclass of additively homogeneous and monotone functions for which all slice spaces are bounded in the max-plus Hilbert projective norm [89, §3]. The function V defined by $V(x) = \|g(x) - x\|_{\mathbb{P}}$ is positive definite with respect to the max-plus eigenspace $\mathrm{eig}(g, \lambda(g))$. The boundedness of a non-empty slice space $S_{\alpha}^{\beta}(g)$, for some $\alpha, \beta \in \mathbb{R}$, implies the existence of a finite max-plus eigenvector: $\exists x \in \mathbb{R}^n$ such that $g(x) = \lambda \otimes x$ for some $\lambda \in \mathbb{R}$ [87, Corollary 4.1].

4.4. Algorithmic aspects

We provide some algorithms for stability analysis of an open-loop SMPL system (4.1) defined by a set of matrices $\mathscr{A} = \{A^{(1)}, \dots, A^{(n_{\mathrm{L}})}\}$.

Given a max-plus C-set, we would first like to check if it is a common positively invariant set of the system. Secondly, we propose conditions for the existence of such a max-plus C-set for a given set of matrices $\mathscr{A}$. We then provide an algorithm to construct the largest (bounded) positively invariant C-set common to all subsystems, if it exists. Finally, we provide an optimisation problem to calculate the region of attraction of a given positively invariant set.

4.4.1. Max-plus double description method

Let $\mathscr{K} = \{x \in \mathbb{R}_{\varepsilon}^{n} \mid C \otimes x \le D \otimes x\}$, $C, D \in \mathbb{R}_{\varepsilon}^{m\times n}$, be a given finitely generated max-plus cone expressed as a solution of a system of max-plus inequalities (see Theorem 2.2.3). The explicit set of its generators as columns of a matrix $K \in \mathbb{R}_{\varepsilon}^{n\times q}$, $q \in \mathbb{N}$, can be found using the max-plus double description method [10, §4].

Lemma 4.4.1. (Generator enumeration [10])**.** *Consider a max-plus cone* $\mathscr{G} = \mathrm{span}_{\oplus}(G)$, *where* $G \in \mathbb{R}_{\varepsilon}^{n\times q}$ *with* $q \in \mathbb{N}$, *and a set* $\mathscr{H} = \{x \in \mathbb{R}_{\varepsilon}^{n} \mid c^{\top} \otimes x \le d^{\top} \otimes x\}$, *where* $c, d \in \mathbb{R}_{\varepsilon}^{n}$. *Then the max-plus cone* $\mathscr{G}' = \mathscr{G} \cap \mathscr{H}$ *is generated by the set* $G_0' \cup G_1'$, *where*

$$G_0' = \left\{ [G]_{\cdot i} \;\middle|\; c^{\top} \otimes [G]_{\cdot i} \le d^{\top} \otimes [G]_{\cdot i},\ i \in \underline{q} \right\},$$

and

$$G_1' = \Big\{ (u^{\top} \otimes [G]_{\cdot j}) \otimes [G]_{\cdot i} \oplus (v^{\top} \otimes [G]_{\cdot i}) \otimes [G]_{\cdot j} \;\Big|\; i, j \in \underline{q},\ c^{\top} \otimes [G]_{\cdot i} \le d^{\top} \otimes [G]_{\cdot i},\ \text{and}\ [G]_{\cdot j} > d^{\top} \otimes [G]_{\cdot j} \Big\}$$

■

The elimination algorithm for enumerating the generators of the max-plus cone relies on iteratively applying the preceding result. This is done starting from the max-plus canonical basis[24] of $\mathbb{R}_\varepsilon^n$, $\mathscr{G}^{(0)} = \{e^{(j)} \in \mathbb{R}_\varepsilon^n \mid j \in \underline{n}\}$, and $\mathscr{H}^{(k)} = \{x \in \mathbb{R}_\varepsilon^n \mid [C \otimes x]_k \leq [D \otimes x]_k\}$.

4.4.2. POSITIVE INVARIANCE

We first provide a certificate for validating the positive invariance of a finitely generated max-plus cone with respect to a max-plus linear map.

Lemma 4.4.2 (Positive invariance of max-plus cone [128])**.** *A max-plus cone $\mathscr{K} = \mathrm{span}_\oplus(K)$, $K \in \mathbb{R}_\varepsilon^{n\times m}$, is positively invariant for the system dynamics $x(k) = A \otimes x(k-1)$, $A \in \mathbb{R}_\varepsilon^{n\times n}$, if and only if the following equality holds:*

$$K \otimes (K \,\phi\, (A \otimes K)) = A \otimes K \tag{4.21}$$

■

We now present a constructive theorem for validating the positive invariance of a max-plus C-set.

Theorem 4.4.1 (Max-plus S-Lemma)**.** *Let $\mathscr{A} = \{A^{(1)}, \ldots, A^{(n_\mathrm{L})}\}$, $A^{(l)} \in \mathbb{R}_\varepsilon^{n\times n}$ for $l \in \underline{n_\mathrm{L}}$. Given a max-plus C-set $\mathscr{K} = \mathrm{span}_\oplus(K)$, $K \in \mathbb{R}_\varepsilon^{n\times n}$, then the condition for positively invariance:*

$$x \in \mathscr{K} \Rightarrow A^{(l)} \otimes x \in \mathscr{K}, \ \forall l \in \underline{n_\mathrm{L}} \tag{4.22}$$

holds if and only if

$$K \otimes A^{(l)} \otimes K = A^{(l)} \otimes K, \ \forall l \in \underline{n_\mathrm{L}}. \tag{4.23}$$

Proof See Section 4.5. ■

Remark. For a general case of finitely generated max-plus cones, a certificate for validating an implication between a set of max-plus inequalities can be found in [11, Proposition 13].

Finally we note that for the case of positive invariance with respect to a set of max-plus C-sets (max-plus multi-cones), the preceding theorem can be conveniently extended.

4.4.3. EXISTENCE

We formulate the certificate (4.23), for a max-plus C-set to be positively invariant, as a non-homogeneous system of two-sided max-plus linear equations (see Theorem 2.2.4). This is carried out using vectorisation and max-plus Kronecker product operations[25].

Lemma 4.4.3 (Bilinear vectorisation)**.** *Given matrices $X, T \in \mathbb{R}_\varepsilon^{n\times n}$, we have*

$$\mathrm{vec}(X \otimes T \otimes X) = \left(\mathscr{I}_n^\otimes \boxtimes \mathrm{vec}(T)^\top \boxtimes \mathscr{I}_n^\otimes\right) \otimes (\mathrm{vec}(X) \boxtimes \mathrm{vec}(X)). \tag{4.24}$$

[24]The element $e^{(j)}$ of $\mathbb{R}_\varepsilon^n$ represents a vector of max-plus zeros with the j-th entry equal to zero (max-plus unit element).

[25]Please refer to Section 2.2.1 for identities involving max-plus Kronecker products and vectorisation operations.

Proof. See Section 4.5. ∎

Theorem 4.4.2. (PI reformulation as two-sided max-plus linear system)**.** *Consider a matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ and a max-plus C-set $\mathcal{K} = \mathrm{span}_\oplus(K)$ with $K \in \mathbb{R}^{n\times n}$. The condition for positive invariance of $\mathcal{K}$ with respect to the dynamics $x(k) = A \otimes x(k-1)$, as stated in (4.23), can be expressed as*

$$\begin{pmatrix} M_1 \\ M_3 \end{pmatrix} \otimes z_1 = \begin{pmatrix} M_2 \\ \mathcal{I}^\otimes_{n^2} \end{pmatrix} \otimes z_2, \tag{4.25}$$

where $z_1 = \mathrm{vec}(K) \boxtimes \mathrm{vec}(K) \in \mathbb{R}^{n^4}$, $z_2 = \mathrm{vec}(K) \in \mathbb{R}^{n^2}$, and the remaining matrices are given as

$$M_1 = \mathcal{I}^\otimes_n \boxtimes \mathrm{vec}(A)^\top \boxtimes \mathcal{I}^\otimes_n, \tag{4.26}$$
$$M_2 = \mathcal{I}^\otimes_n \boxtimes A, \tag{4.27}$$
$$M_3 = \mathcal{I}^\otimes_n \boxtimes \mathrm{vec}(\mathcal{I}^\otimes_n)^\top \boxtimes \mathcal{I}^\otimes_n. \tag{4.28}$$

Proof. See Section 4.5. ∎

The results of Theorem 4.4.2 can be extended to a set of matrices $\mathscr{A}$ by augmenting the system (4.25) with matrices M_1 and M_2 defined for each $l \in \underline{n_\mathrm{L}}$. Note that the matrices M_3 and $\mathcal{I}^\otimes_{n^2}$ are regular (each row and column has at least one finite entry). In case the matrices M_1 and M_2 are also regular[26], a finite (non-optimal) solution (z_1, z_2) to (4.25) can be found using existing algorithms (see [48, §7.3]). The algorithm in [48, §7.3] either converges to a solution or provides a certificate that no solution exists.

4.4.4. Construction

We provide a general set-up for construction of a smallest non-empty positively invariant set given an open-loop SMPL system. Here, the radius of the invariant max-plus C-set is measured as the (negative) second largest max-plus eigenvalue of the associated Kleene star matrix [196, Proposition 17].

The optimisation problem can be formulated as follows:

$$\max_{\theta,x} \quad \lambda_2(K) \tag{4.29a}$$
$$\text{subject to} \quad K \otimes A^{(l)} \otimes K \le A^{(l)} \otimes K, \quad \forall l \in \underline{n_\mathrm{L}}, \tag{4.29b}$$
$$K \otimes K \le K, \tag{4.29c}$$
$$[K]_{ij} = -\theta_{ij}, \quad \forall i,j \in \underline{n},\ i \neq j, \tag{4.29d}$$
$$-\theta_{ij} + x_j - x_i \le \lambda_2, \quad \forall i \in \underline{n},\ i \neq j, \tag{4.29e}$$
$$[K]_{ii} = \theta_{ii} = 0, \quad \forall i \in \underline{n}. \tag{4.29f}$$

The constraints in the preceding optimisation problem contain max-plus polynomial expressions of the free variables (θ and x). Therefore, it can be recast as an extended linear complementarity problem and solved as a mixed-integer linear program [71].

[26] If the corresponding rows of both M_1 and M_2 have only ε elements, then they can be removed from the system of equations (4.25) without affecting the solution method.

4.4.5. ATTRACTIVITY

This section provides an algorithm for evaluating the worst case one-step convergence rate of trajectories of a discrete-event system (4.1) to a given positively invariant max-plus C-set.

Let the positively invariant max-plus C-set $\mathcal{K} \subseteq \mathbb{R}^n$ be generated by a Kleene star matrix $K \in \mathbb{R}_\varepsilon^{n\times n}$. The associated max-plus gauge function V can be defined by $V(x) = x^* \otimes K \otimes x$. Then for a given distance from the max-plus C-set $\delta > 0$, the search for the worst-case descent to the max-plus C-set can be formulated as $P(\delta) = \max_{x,\delta'}\{\delta' \mid V(x) \leq \delta,\ V(A^{(l)} \otimes x) \geq \delta',\ \forall l \in \underline{n_\mathrm{L}},\ x \in \mathbb{R}^n\}$. The optimisation problem is given as[27]:

$$P(\delta) \qquad = \max_{\delta',x} \delta' \tag{4.30a}$$

$$\text{subject to} \quad A^{(l)} \otimes x \leq \widetilde{K}_{\delta'} \otimes A^{(l)} \otimes x, \quad \forall l \in \underline{n_\mathrm{L}}, \tag{4.30b}$$

$$\widetilde{K}_\delta \otimes x \leq x. \tag{4.30c}$$

$$\tag{4.30d}$$

The preceding optimisation problem $P(\delta)$ can be recast as a parametric linear complementarity problem [72] and solved efficiently using the multi-parametric toolbox [120]. It is remarkable to note that the preceding optimisation problem for a fixed $\delta \in \mathbb{R}$ is an instance of a tropical linear-fractional programming problem [93]. An efficient algorithm to solve tropical linear-fractional program can be found in [93].

We finally note that the optimal value of $P(\delta)$ can be proved to be monotone non-decreasing on $\delta \geq 0$ using the results presented in [100, Proposition 3.2]. Hence, if the max-plus C-set is uniformly attractive over all possible switching sequences then there exists a function α_1 of class $\mathcal{K}$ such that[28] $\alpha_1(\delta) \leq P(\delta) \leq \mathrm{id}(\delta)$. This implies, in particular, that the set-induced Lyapunov function (see Definition 4.3.2) is non-increasing along the dynamics:

$$V(A^{(l)} \otimes x) - V(x) \leq -\alpha_2(\|x\|_{\mathcal{K},\mathbb{P}}), \quad \forall x \in \mathbb{R}^n,\ \forall l \in \underline{n_\mathrm{L}}, \tag{4.31}$$

where $\alpha_2(\cdot) = \mathrm{id} - \alpha(\cdot)$ is again a function of class $\mathcal{K}$. Therefore, the feasibility of the optimisation problem over $\delta \in [0, \delta_{\max}]$ provides a certificate for attractivity of a given positively invariant max-plus C-set and an estimate of the region of attraction:

$$\mathcal{D} = \mathcal{N}(V, \delta_{\max}) = \{x(0) \in \mathbb{R}^n \mid \lim_{k\to+\infty} x(k) \in \mathcal{K}\}. \tag{4.32}$$

Note that the preceding procedure can be extended to other max-plus Lyapunov functions taking polyhedral forms.

[27] Note that a Kleene star matrix normalised by a scalar $\mu \in \mathbb{R}$ is denoted as $\widetilde{K}_\mu$ and obtained as follows: for all $i, j \in \underline{n}$, $[\widetilde{K}_\mu]_{ij} = [K]_{ij} - \mu$ if $i \neq j$, and $[\widetilde{K}_\mu]_{ii} = 0$.

[28] An identity function is denoted by id such that $\mathrm{id}(s) = s$ for all $s \in \mathbb{R}_+$.

4.5. TECHNICAL PROOFS

In this section, we provide the proofs of various results stated in this chapter.

Proof of Lemma 4.3.1. A finitely generated max-plus cone is a finitely generated convex cone if and only if it is generated as the max-plus column span of a Kleene star matrix [123, Theorem C]. Then, the max-plus C-set $\mathcal{K} = \mathrm{span}_{\oplus}(K)$, for some Kleene star matrix $K \in \mathbb{R}_{\varepsilon}^{n\times n}$, is finitely generated and hence non-empty.

The largest max-plus eigenvalue of a Kleene star matrix is $\overline{\lambda}(K) = 0$. The max-plus column span of a Kleene star matrix is generated as its max-plus eigenspace corresponding to $\overline{\lambda}(K)$ [196, Proposition 7]. The max-plus eigenspace $\mathrm{eig}(K,0)$ of a Kleene star matrix has only vectors with finite entries if and only if it is irreducible [196, Proposition 7]. Therefore, $\mathcal{K}$ is bounded in the max-plus Hilbert projective norm.

For any two Kleene star matrices $K^{(1)}, K^{(2)} \in \mathbb{R}_{\varepsilon}^{n\times n}$ such that $\mathrm{span}_{\oplus}(K^{(1)}) = \mathrm{span}_{\oplus}(K^{(2)})$, we have $K^{(1)} = K^{(2)}$ [196, Proposition 6]. Therefore, the generating Kleene star matrix is unique.

Proof of Lemma 4.3.2. Firstly, the max-plus eigenspace of a Kleene star matrix $K \in \mathbb{R}_{\varepsilon}^{n\times n}$ is given as

$$\mathrm{eig}(K,0) = \{x \in \mathbb{R}_{\varepsilon}^{n} \mid [K]_{ij} + x_j \le x_i,\ i,j \in \underline{n},\ i \ne j\}. \tag{4.33}$$

Here, $x_i + [K]_{ii} \le x_i$ is trivially satisfied as $[K]_{ii} = 0$ for all $i \in \underline{n}$.

Then, given a point $x \in \mathbb{R}^n$, there exists a $\mu \ge 0$ such that $x \in \mathrm{eig}(\widetilde{K}_{\mu}, 0)$:

$$\begin{aligned}
&\max_{i\in\underline{n}}\max_{j\in\underline{n}} ([\widetilde{K}_{\mu}]_{ij} + x_j - x_i) &&\le 0\\
\Leftrightarrow\ &\max_{i\in\underline{n}}\max_{j\ne i} \left([K_{\mu}]_{ij} + x_j - x_i\right) &&\le 0\\
\Leftrightarrow\ &\max_{i\in\underline{n}}\max_{j\ne i} \left([K]_{ij} + x_j - x_i\right) &&\le \mu\\
\Leftrightarrow\ &\quad x^{*} \otimes K \otimes x &&\le \mu.
\end{aligned} \tag{4.34}$$

Here, $[\widetilde{K}_{\mu}]_{ij} = [K_{\mu}]_{ij}$ for $i \ne j$. The max-plus matrix completion in the final step follows by adding the inequality $\mu + x_i - x_i \ge 0$ to each row. Therefore, the expressions (4.5) and (4.6) are equivalent.

Now, we note that the nonlinear projection $P_{\mathcal{K}}(x)$ minimises the distance of a point $x \in \mathbb{R}^n$ to a set $\mathcal{K} \subseteq \mathbb{R}^n$, i.e. $d_{\mathrm{H}}(x,\mathcal{K}) = d_{\mathrm{H}}(x, P_{\mathcal{K}}(x))$, (see Definition 2.2.22).

As the set is generated by a Kleene star matrix, we have $P_{\mathcal{K}}(x) = (-K)^\top \otimes' x$. Therefore,

$$\begin{aligned} d_{\mathrm{H}}(x,\mathcal{K}) &= d_{\mathrm{H}}(x,(-K)^\top \otimes' x) \\ &= \max_{i\in \underline{n}}(x_i - [(-K)^\top \otimes' x]_i) \\ &\quad - \min_{i\in \underline{n}}(x_i - [(-K)^\top \otimes' x]_i) \\ &= \max_{i\in \underline{n}}(x_i - [(-K)^\top \otimes' x]_i) \\ &= \max_{i\in \underline{n}}(x_i + [K^\top \otimes (-x)]_i) \\ &= \max_{i\in \underline{n}}(x_i + \max_{j\in \underline{n}}([K]_{ji} - x_j)) \\ &= \max_{i\in \underline{n}}\max_{j\in \underline{n}}([K]_{ji} + x_i - x_j) \\ &= x^* \otimes K \otimes x. \end{aligned} \tag{4.35}$$

Here, the third equality follows as $P_{\mathcal{K}}(x) \le x$ and the equality holds for at least one index $i \in \underline{n}$. Therefore, $\min(x - P_{\mathcal{K}}(x)) = 0$. This completes the proof.

Proof of Proposition 4.3.1. We first show that the bound on growth rates is intimately connected to the width of a positively invariant set that is bounded in the max-plus Hilbert projective space. Let $\mathcal{B}_\mu$, for some $\mu > 0$, be positively invariant to the system dynamics in (4.1). Consequently, if $x \in \mathcal{B}_\mu$ then $f(l,x) \in \mathcal{B}_\mu$ for all $l \in \underline{n_{\mathrm{L}}}$. From the triangle inequality, we have

$$d_{\mathrm{H}}(f(l,x),x) \le \left\|f(l,x)\right\|_{\mathbb{P}} + \|x\|_{\mathbb{P}} \le 2\cdot\mu. \tag{4.36}$$

Let the system be uniformly max-plus bounded-buffer stable in $\mathcal{B}_\delta$. Then for any trajectory starting with $x(0) \in \mathcal{B}_\mu$, with arbitrary $\mu > 0$, there exist a scalar $T(\mu,\delta) > 0$, such that $x(k) \in \mathcal{B}_\delta$ for all $k \ge T(\mu,\delta)$. Therefore, owing to (4.36) we have $d_{\mathrm{H}}(x(k),x(k-1)) \le 2\cdot\delta$, or $x(k) \in \mathcal{B}_{2\delta}(x(k-1))$ for $k \ge T(\mu,\delta)$. This completes the proof.

Proof of Lemma 4.3.3. ($\Rightarrow$) Take $\gamma(k,x(0)) = \max_{i\in\underline{n}} x_i(k)$. Then we have

$$\begin{aligned} \|x(k) - \max_{i\in \underline{n}} x_i(k) \otimes \mathbb{1}\|_\infty &= |\min_{j\in \underline{n}} x_j(k) - \max_{i\in \underline{n}} x_i(k)| \\ &= \|x(k)\|_{\mathbb{P}}. \end{aligned} \tag{4.37}$$

Therefore, the asymptotic boundedness of the state in the max-plus Hilbert projective norm implies the result.

($\Leftarrow$) We are given a function $\gamma : \mathbb{N}\times\mathbb{R}^n \to \mathbb{R}$ such that $\left\|x(k) - \gamma(k,x(0)) \otimes \mathbb{1}_n\right\|_\infty$ is bounded as $k \to \infty$. Then there exists a finite scalar $M > 0$ such that for $k \to \infty$, we have

$$-M \otimes \mathbb{1}_n \le x(k) - \gamma(k,x(0)) \otimes \mathbb{1}_n \le M \otimes \mathbb{1}_n. \tag{4.38}$$

From the definition of the max-plus Hilbert projective norm, we have $\|x(k)\|_{\mathbb{P}} \le 2M$ as $k \to \infty$. This completes the proof.

Proof of Theorem 4.3.4. Let $\gamma = (x(k))_{k\in\mathbb{N}_0}$ be a state trajectory of the discrete-event system (4.1) with $x(0) \in \mathcal{K}$. Let $\Omega(\gamma) \subset \mathcal{K}/\sim$ denote the ω-limit set of the sequence γ in the projective space. Note that this is possible because the dynamics is additively homogeneous (Assumption 4.2.2). We first prove that the set $\Omega(\gamma)$ is weakly positively invariant for the dynamics (4.1).

As the system is assumed to be max-plus bounded-buffer stable, there exist scalars $\delta, T > 0$ for any given state trajectory γ such that $\|x(k)\|_{\mathbb{P}} < \delta$ for all $k \geq T$. Therefore, $\Omega(\gamma) \subset \mathcal{B}_\delta/\sim$ is bounded in the max-plus Hilbert projective norm. The set is also closed as the system dynamics $f(\cdot, x)$ is assumed to be continuous on $\mathcal{K}$.

For any $\overline{z} \in \Omega(\gamma)$, there exists a subsequence $\{x(k_m)\}_{m\in\mathbb{N}}$ such that $\lim_{m\to+\infty} d_{\mathrm{H}}(x(k_m), \overline{z}') = 0$. Moreover, the subsequence $\{x(k_m+1)\}_{m\in\mathbb{N}}$ has a convergent subsequence by the virtue of boundedness of the ω-limit set in the max-plus Hilbert projective space. Therefore, the limit $\lim_{m\to+\infty} d_{\mathrm{H}}(x(k_m+1), \overline{z}') = 0$ exists and $\overline{z}' \in \Omega(\gamma)$ by definition. As every closed ball in the max-plus Hilbert projective metric is compact in the projective space and the function $f(\cdot, x)$ is continuous in the state x, there exists $l \in \underline{n_{\mathrm{L}}}$ such that $f(l, z) = z'$ for some $z \in \overline{z}$ and $z' \in \overline{z}'$. Therefore, $\Omega(\gamma)$ is weakly positively invariant.

We now prove that the function V along the trajectory γ converges to a level set that coincides with $\Omega(\gamma)$. Consider again that there exist $\delta, T > 0$ such that $x(k) \in \mathcal{B}_\delta$ for $k \geq T$. As the function V is scale free and continuous on $\mathcal{K}$, $V(x(k))$ is lower bounded for all $k \geq T$. Since $V(x(k))$ is also non-increasing along the trajectories γ, there exists $c \in \mathbb{R}$ such that $\lim_{k\to\infty} V(x(k)) = c$. Let $\overline{z} \in \Omega(\gamma)$ such that $\lim_{m\to+\infty} x(k_m) \sim \overline{z}$. Then the limit $\lim_{m\to+\infty} V(x(k_m)) = V(z)$ for all $z \in \overline{z}$ follows from the continuity and scale-free property of the function V. Hence, $V(z) = c$ for all $z \in \overline{z}$. Let the corresponding set be denoted as $V^{-1}(c) = \{z \in \mathbb{R}^n \mid V(z) = c\}$.

Let $\mathcal{K}_{\mathrm{c}} = \mathrm{cl}(\mathcal{K} \cap \mathcal{B}_\delta)$. Noting that the set $\Omega(\gamma) \subset \mathcal{K}_{\mathrm{c}}/\sim$ is weakly positively invariant representing the level set of the function V, we have

$$\Omega(\gamma) \subset \{\overline{z} \in \mathcal{K}_{\mathrm{c}}/\sim \mid \exists l \in \underline{n_{\mathrm{L}}}, \exists z \in \overline{z}, \text{ s.t. } f(l, z) = z' \text{ and } V(z') = V(z)\}. \tag{4.39}$$

Hence, we conclude that there exists a weakly positively invariant set $\mathcal{M}$ contained in $\{x \in \mathcal{B}_\delta \mid \exists l \in \underline{n_{\mathrm{L}}} \text{ s.t. } V(f(l, x)) = V(x)\}$ such that the set $V^{-1}(c) \cap \mathcal{M}$ is attractive in the sense of Definition 4.2.2.

Proof of Theorem 4.3.6. It is sufficient to show that the slice spaces $S_\alpha^\beta(g)$ form sub-level sets of the function V for certain values of $\alpha, \beta \in \mathbb{R}$. The rest of the proof follows from Theorem 4.3.4 as the function g is always continuous.

Let $x \in V^{-1}(c) \neq \emptyset$ for a given $c > 0$. Then $\alpha \leq g(x) - x \leq \beta$ where $\beta - \alpha = c$ and there exist indices $i, j \in \underline{n}$ such that $g_i(x) - x_i = \beta$ and $g_j(x) - x_j = \alpha$. For the given choice of $\alpha, \beta \in \mathbb{R}$, $S_\alpha^\beta(g) \neq \emptyset$ as $V^{-1}(c) \neq \emptyset$. Therefore, the slice spaces $S_\alpha^\beta(g)$ form sub-level sets of the function V for appropriate choices of $\alpha, \beta \in \mathbb{R}$. Moreover, all such slice spaces are positively invariant.

Proof of Theorem 4.4.1. We first recall that the generator of the given max-plus C-set is an irreducible Kleene star matrix (see Lemma 4.3.1). The problem of positive invariance can be restated as the following implication:

(1) $K \otimes x = x \Rightarrow K \otimes A^{(l)} \otimes x = A^{(l)} \otimes x$, for all $x \in \mathbb{R}^n$ and for all $l \in \underline{n_{\mathrm{L}}}$.

We show that statement (1) is equivalent to

(2) $K \otimes A^{(l)} \otimes K \otimes y = A^{(l)} \otimes K \otimes y$, for all $y \in \mathbb{R}^n_\varepsilon$ and for all $l \in \underline{n_{\mathrm{L}}}$.

(2) ⇒ (1) We recall from the literature that for any closed max-plus cone $\mathcal{K} \in \mathbb{R}^n_\varepsilon$, any point in $\mathcal{K}$ is a max-plus combination of at most n generators[29] [50, Proposition 24]. As the max-plus weak dimension[30] of a max-plus C-set is at most n, so for any $z \in \mathcal{K}$, there exists (at least one) $y \in \mathbb{R}^n_\varepsilon$ such that $z = K \otimes y$. Then if (2) holds[31] for all $y \in \mathbb{R}^n_\varepsilon$, then $K \otimes A^{(l)} \otimes z = A^{(l)} \otimes z$, $l \in \underline{n_{\mathrm{L}}}$, holds for all $z \in \mathcal{K}$.

(1) ⇒ (2) An irreducible Kleene star matrix K has only elements with finite entries in its max-plus column span [196, Proposition 7]. Therefore, the matrix K maps the entire space $\mathbb{R}^n_\varepsilon$ into $\mathcal{K}$. It is then sufficient to note that for any $y \in \mathbb{R}^n_\varepsilon \setminus \{\mathcal{E}_n\}$, there exists a $z \in \mathbb{R}^n$ such that $K \otimes y = z = K \otimes z$. Then if the implication (1) holds for all $x \in \mathbb{R}^n$ then (2) holds for all $y \in \mathbb{R}^n_\varepsilon$.

As the equality (2) must hold for every $y \in \mathbb{R}^n_\varepsilon \setminus \{\mathcal{E}_n\}$, then a max-plus C-set $\mathcal{K} = \mathrm{span}_\oplus(K)$ is positively invariant to the system dynamics if and only if (4.23) is satisfied. This completes the proof.

Proof of Lemma 4.4.3. The left-hand side of (4.24) can be rewritten as follows:

$$\begin{aligned}
&\mathrm{vec}(X \otimes T \otimes X) = (\mathcal{I}_n^\otimes \boxtimes X \otimes T)\mathrm{vec}(X) && \text{(using (2.7))}\\
&= \left(\mathcal{I}_n^\otimes \boxtimes \mathrm{vec}^{-1}\left((T^\top \boxtimes \mathcal{I}_n^\otimes) \otimes \mathrm{vec}(X)\right)\right) \otimes \mathrm{vec}(X) && \text{(using (2.8))}\\
&= \Big(\mathcal{I}_n^\otimes \boxtimes \underbrace{(\mathrm{vec}^\top(\mathcal{I}_n^\otimes) \boxtimes \mathcal{I}_n^\otimes) \otimes (\mathcal{I}_n^\otimes \boxtimes (T^\top \boxtimes \mathcal{I}_n^\otimes))}_{M} \otimes (\mathcal{I}_n^\otimes \boxtimes \mathrm{vec}(X))\Big) \otimes \mathrm{vec}(X) && \text{(using (2.9))}\\
&= (\mathcal{I}_n^\otimes \boxtimes M) \otimes (\mathcal{I}_n^\otimes \boxtimes (\mathcal{I}_n^\otimes \boxtimes \mathrm{vec}(X))) \otimes \mathrm{vec}(X) && \text{(using (2.3))}\\
&= (\mathcal{I}_n^\otimes \boxtimes M) \otimes (\mathcal{I}_{n^2}^\otimes \boxtimes \mathrm{vec}(X)) \otimes (\mathrm{vec}(X) \boxtimes \mathbb{1})\\
&= (\mathcal{I}_n^\otimes \boxtimes M) \otimes (\mathcal{I}_{n^2}^\otimes \otimes \mathrm{vec}(X)) \boxtimes \mathrm{vec}(X) && \text{(using (2.3))}\\
&= (\mathcal{I}_n^\otimes \boxtimes M) \otimes (\mathrm{vec}(X) \boxtimes \mathrm{vec}(X)).
\end{aligned}$$

The expression of M can be simplified as:

$$\begin{aligned}
M &= (\mathrm{vec}^\top(\mathcal{I}_n^\otimes) \boxtimes \mathcal{I}_n^\otimes) \otimes (\mathcal{I}_n^\otimes \boxtimes (T^\top \boxtimes \mathcal{I}_n^\otimes))\\
&= (\mathrm{vec}^\top(\mathcal{I}_n^\otimes) \boxtimes \mathcal{I}_n^\otimes) \otimes ((\mathcal{I}_n^\otimes \boxtimes T^\top) \boxtimes \mathcal{I}_n^\otimes)\\
&= \left(\mathrm{vec}^\top(\mathcal{I}_n^\otimes) \otimes (\mathcal{I}_n^\otimes \boxtimes T^\top)\right) \boxtimes (\mathcal{I}_n^\otimes \mathcal{I}_n^\otimes) && \text{(using (2.3))}\\
&= \left((\mathcal{I}_n^\otimes \boxtimes T^\top)^\top \otimes \mathrm{vec}(\mathcal{I}_n^\otimes)\right)^\top \boxtimes \mathcal{I}_n^\otimes\\
&= \left((\mathcal{I}_n^\otimes \boxtimes T) \otimes \mathrm{vec}(\mathcal{I}_n^\otimes)\right)^\top \boxtimes \mathcal{I}_n^\otimes\\
&= \mathrm{vec}^\top(T) \boxtimes \mathcal{I}_n^\otimes. && \text{(using (2.8))}
\end{aligned}$$

This completes the proof.

[29] Max-plus Carathéodory's theorem [91, Proposition 3.3].
[30] See Definition 2.2.19.
[31] Note that the equality (2) is satisfied trivially for $y = \mathcal{E}_n$.

Proof of Theorem 4.4.2. We rewrite left-hand-side of (4.23), for $n_{\mathrm{L}} = 1$, using the result of Lemma 4.4.3. The right-hand-side follows directly from (2.8). This gives the expressions for M_1 and M_2. As we assume that K is a Kleene star matrix, we additionally require that $K \otimes K = K$ (see Lemma 2.2.5 and the discussion thereafter). This is expressed as $M_3 \otimes z_1 = \mathscr{I}_{n^2}^{\otimes} \otimes z_2$ again using Lemma 4.4.3.

4.6. CONCLUSIONS

In this chapter, we have proposed notions of stability and tools for stability analysis for discrete-event systems in max-plus algebra. To this end, we have established a max-plus Lyapunov theoretic framework that can be used to analyse stability properties of switching max-plus linear system operating under arbitrary switching. On one hand, the stability analysis tools let go of the usual assumption of contractiveness of the dynamics in the max-plus Hilbert projective norm. On the other hand, the tools still benefit from and rely on the desirable characteristics of the non-expansive and additively homogeneous dynamics. The presented stability theory studies the positive invariance of certain sets with respect to the discrete-event dynamics. Therefore, we have also presented an algorithmic perspective on construction of such positively invariant sets to further evaluate the stability properties for open-loop switching max-plus linear systems.

5

MAX-PLUS LINEAR PARAMETER-VARYING SYSTEMS

In this chapter, we present a framework for modelling parametric discrete-event systems as linear parameter-varying systems in the max-plus algebra. We present algebraic tools to analyse the consistency of the obtained model, which allows to assess the existence and uniqueness of trajectories of the system. The application of the formalism is motivated using an intuitive case study on an urban railway system.

5.1. INTRODUCTION

Linear parameter-varying systems provide a convenient system-theoretical framework to handle control and analysis problems of conventional time-driven linear systems under explicit parameter dependence [32, 34, 45, 173, 193]. The resulting formalism provides a deeper understanding into the dynamics of the system by preserving certain linearity properties of the dynamics. At the same time, it allows for extension of tools from linear systems theory, for analysis and control, to (nonlinear) parametric systems. We intend to extend the max-plus linear framework to parametric discrete-event systems. Such descriptions of discrete-event systems find applications in production [167, 213], and transportation systems [119].

Parametric modelling of discrete-event systems in max-plus algebra allows capturing variations in timing durations between event occurrences, broadening the class of max-plus linear systems. The obtained model is said to be *consistent* if the existence of a state trajectory that respects all synchronisation and ordering constraints can be guaranteed. We distinguish the concept of *solvability* when the admissible state trajectories are required to be unique for a given realisation of the parameter trajectory. We speak of *invariant solvability* when the system is solvable over an infinite event horizon. The flexibility provided by the introduction of parametric dependence is often offset by the loss of solvability of the obtained model on either the entire state space or for all event steps. This is partly due to inherent implicitness in the state equations of such systems. The problem of ensuring the consistency is fundamental to modelling of parametric discrete-event systems. A complete absence of finite trajectories reveals operational issues in the underlying discrete-event system and also the physical system itself. It is then necessary to study conditions on parameters and past state values that ensure the existence (and possibly uniqueness) of a state solution to the model at a given event step. Thereafter, it is interesting to investigate the set of initial conditions and parameter values such that a given model is solvable on an unbounded event-step domain (or for all event steps).

As an application, we consider the modelling of a unidirectional urban railway system [220] where the states denote the arrival and departure times of trains at the various stations. Such a railway system is usually operated without a timetable. It is assumed that there is no capacity limit for the trains. At any given station, a fraction of the number of passengers on a train disembark and then all passengers present at the platform board the train. Therefore, the dwell time of a train at a given station is conditioned upon the number of passengers present at the platform. This in turn depends on the arrival and departure times of the train at the particular station as well as the ones preceding it. Such dwell times appear as parameters in the state matrices of the max-plus linear model of the dynamical system. Apart from other uncertainties arising due to unwarranted delays, the system evolution can be modelled using a set of implicit max-plus equations in the state variable. The solvability of the obtained model is then the subject of investigation.

5.1.1. RELATED WORK

The analysis and control of uncertain max-plus linear systems has long been an important topic of investigation. The variations in timing durations between event occurrences can be incorporated in different ways in the system dynamics: *i)* polytopic uncertainties in max-plus algebra [106, 167], *ii)* polytopic uncertainties in conventional algebra [211], *iii)* stochastic multiplicative uncertainties [17, 213], or *iv)* non-deterministic holding times at places in the timed-event graph [77, 113, 127, 149, 204].

The analysis of consistency of parametric discrete-event systems using the tools of the max-plus algebra has been a topic of investigation in the literature. The most popular modelling framework, in this respect, is that of P-time event graphs [26]. P-time event graphs form a non-deterministic extension of timed event graphs where the holding times associated to places are given as intervals. Hence, dynamics of the system at each event step $k \in \mathbb{N}$ is captured by a set of inequalities on the state vector $x(k)$ using max-plus and min-plus operations[1] [205, and references therein]:

$$\bigoplus_{\mu=0}^{M} \underline{A_\mu} \otimes x(k-\mu) \leq x(k) \leq \bigoplus_{\mu=0}^{M}{}' \, \overline{A_\mu} \otimes' x(k-\mu). \tag{5.1}$$

Here, the implicitness in the state (in)equalities over the event counter k occurs due to the absence of tokens in certain places in the associated timed-event graph. The consistency problem for the class of P-time event graphs then involves checking the existence of extremal[2] trajectories satisfying the constraints. The author of [76] provides an algo-

[1] Here $(\oplus, \otimes)$ and $(\oplus', \otimes')$ represent max-plus and min-plus operations in $\mathbb{R}_\varepsilon$ and $\mathbb{R}_\top$ respectively (see Section 2.2).

[2] A trajectory is considered extremal if it is either maximal or minimal in the sense of partial order $(\mathbb{R}_+, \leq)$.

rithm for computing extremal trajectories of a P-time event graph over a finite event-step horizon. The algorithmic analysis also leads to an algebraic condition on consistency over an infinite horizon. The existence of 1-periodic[3] state trajectories of a P-time event graph is studied using linear programming methods in [77]. In particular, the linear programming formalism in [77] allows an affine dependence of elements in the matrix $\underline{A}_\mu$ on the state vector $x(\cdot)$.

The authors of [205] provide checkable necessary and sufficient conditions for the existence of (infinite horizon) d-periodic extremal trajectories under the assumption that all the state matrices $\underline{A}_\mu$ and $\overline{A}_\mu$ are periodic (see Lemma 2.2.2). The aforementioned analysis procedures also lead to bounds on achievable throughput of the P-time event graphs operating under a d-periodic regime [78, 205, 225]. Lastly, we note that the implicitness in the (in)equalities describing the system dynamics is circumvented using max-plus and min-plus Kleene star (closure) operations (see Lemma 2.2.5) in the aforementioned articles [76, 205].

We note that our modelling and analysis approach is fundamentally different from the one presented in [76, 77, 205]: *i)* we consider more general (uncertain and state-dependent) parametric variations in the elements of the state matrices, *ii)* we consider a deterministic system and require unique trajectories for any given realisation of the past state variables and exogenous input invariables, *iii)* the invariant solvability property ensures satisfaction of synchronisation and ordering constraints on event occurrences over a possibly infinite horizon without the requirement of periodic behaviour, and *iv)* the model description preserves the incidence structure of the underlying discrete-event system while still exploiting linear programming based methods.

5.1.2. STATEMENT OF CONTRIBUTION

In what follows, the novel contributions of this chapter are stated. The first set of contributions involve modelling of parametric discrete-event systems in the max-plus algebra:

(A) *Taxonomy*: We present an extended description of max-plus linear systems with a linear parameter varying structure analogously to the conventional framework [34, 161], which we designate as MP-LPV systems (in (5.2)). We show (in Section 5.2.1) that the proposed description is general in that it allows modelling various uncertain as well as state-dependent parameter variations in the system, as recalled in the preceding section [17, 77, 106, 113, 127, 167, 205, 211, 213].

(B) *Equivalence relationships*: We show (in Theorem 5.2.1) that under the assumption of piecewise affine dependence on the parameter, the proposed class of MP-LPV systems defined over a discrete event counter is equivalent to that of discrete hybrid systems described by max-min-plus-scaling systems [118]. Under the same assumption, we also present a canonical form of an MP-LPV system (in Lemma 5.2.2) and a procedure to obtain it.

An important aspect to consider while modelling state-dependent parametric variations in the max-plus linear modelling framework is that the inherent implicitness in the state

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[3] A sequence $\{Z(k)\}_{k\in\mathbb{N}}$ is d-periodic if there exist scalars $d \in \mathbb{N}$ and $\rho \in \mathbb{R}$ such that $Z(k+d) = \rho \otimes Z(k)$ for large enough k.

equations cannot always be resolved using the methods of the max-plus algebra. This problem indeed arises due to the lack of an inverse to the maximum and minimum operations [17, 77]. The next set of contributions concern the solvability issues of a given MP-LPV system due to inherent implicitness in the model description. We again assume that the system matrices have a piecewise affine dependence on the parameters.

(C) *Necessity*: We propose necessary (checkable) conditions for the existence of a state solution to the system equations given a realisation of the input signal and past (known) parameter values using tools from max-plus algebra (see Theorem 5.3.1). We achieve this by characterising the set of (admissible) parameter values as a union of polyhedra.

Subsequently, assuming the absence of control inputs, we provide an algorithm to compute the maximal set of admissible parameter values (as a union of polyhedra) such that any trajectory starting in this set remains in the set for all event steps $k \in \mathbb{N}$ (see Section 5.3). The proposed methodology is derived from the tools of computational geometry from piecewise affine analysis, as discussed in Section 2.4.3.

(D) *Practical sufficiency*: We study the effect of coefficients of the piecewise affine dependence on parameters (in the finite matrix entries) on the existence and uniqueness of state trajectories of the MP-LPV system. The proposed conditions, in Theorem 5.3.2, are sufficient for the solvability problem (over a possibly infinite event-step horizon). We provide a physical interpretation of the obtained result in the context of the operation of a unidirectional urban railway system in Section 5.5. The feasible state trajectories can then be evaluated using existing tools from the literature.

5.1.3. Organisation of the chapter

The chapter is organised as follows. Section 5.2 introduces the framework of max-plus linear parameter-varying systems along with its relationship with the max-min-plus-scaling systems, and presents a canonical reformulation suitable for further analyses. Section 5.3 highlights the solvability issues inherent to the MP-LPV modelling class and proposes a modular algebraic approach to ensure existence and uniqueness of the state trajectories. The section provides necessary conditions for solvability of a given MP-LV systems. Section 5.4.1 considers the problem of invariant solvability for MP-LPV system. Section 5.5 presents the case study of an urban railway system to highlight solvability issues. We end the chapter with concluding remarks and future research directions in Section 5.6.

5.2. Parametric discrete-event systems

In this section, we introduce the novel framework of max-plus linear parameter-varying (MP-LPV) systems. We delineate how certain existing modelling classes for parametric and uncertain max-plus linear systems can be reformulated in the proposed MP-LPV systems framework. Consequently, the relationship of the MP-LPV systems with the class of continuous piecewise affine systems under certain assumptions is formalised. Finally, a canonical formulation of the MP-LPV system is presented that aids further investigation into the conditions for the existence and uniqueness of the system trajectory in Section 5.3.

5.2.1. MODELLING AND CLASSIFICATION

We are interested in studying the dynamics of an *implicit* max-plus linear system with (possibly event-varying) parameters or disturbances, contained in $\mathscr{P} \subseteq \mathbb{R}^{n_\mathrm{p}}$, evolving along a discrete-event counter $k \in \mathbb{N}$, $k \geq M$:

$$\begin{aligned} x(k) &= \bigoplus_{\mu=0}^{M} \big(A_\mu(p(k)) \otimes x(k-\mu)\big) \oplus B(p(k)) \otimes u(k), \\ y(k) &= C(p(k)) \otimes x(k), \\ p(k) &\in \mathscr{P}. \end{aligned} \tag{5.2}$$

Here, the order[4] $M \in \mathbb{N}$ is specified along with initial conditions $x(j) \in \mathbb{R}_\varepsilon^n$ for[5] $j \in \underline{M}$. The states $x(k) \in {\mathbb{R}_\varepsilon}^n$, $k \in \mathbb{N}_0$, contain time instants of event occurrences. Note that due to timed-event graph convention [17, §5.4.4], it is assumed that $x(k) = \mathscr{E}_n$ for $k < 0$. The systems matrix functions are defined as: $A_\mu : \mathscr{P} \to \mathbb{R}_\varepsilon^{n\times n}$ for $\mu \in [M]$, $B : \mathscr{P} \to \mathbb{R}_\varepsilon^{n\times n_\mathrm{u}}$ and $C : \mathscr{P} \to \mathbb{R}_\varepsilon^{n_\mathrm{y}\times n}$. This linear parameter-varying description in the max-plus algebra is designated as an MP-LPV system analogously to the conventional LPV framework [173]. The distinction from a max-plus linear system is due to the non-stationary behaviour of the MP-LPV system along the event counter. An event-varying max-plus linear system is distinguished from an MP-LPV system, mainly in analysis and control synthesis, due to the explicit dependence on the counter $k \in \mathbb{N}$ in the former. For instance, a single trajectory of the parameter $p(\cdot)$ in (5.2) would result in an event-varying max-plus linear system. On the contrary, an MP-LPV system framework considers a set of trajectories of the parameter.

The MP-LPV system description (5.2) is general in that it can model different types of multiplicative uncertainties (as introduced in [149, 211, 213, 215]) as well as state-dependent parameter variations. In what follows, we enumerate some important classes of MP-LPV system based on the evolution of the parameter $p(\cdot)$ and the dependence of the state matrices on the parameter.

Max-plus affine MP-LPV systems. The system matrices, $\{A_\mu(\cdot)\}_{\mu\in[M]}$, $B(\cdot)$, and $C(\cdot)$, are considered to be max-plus affine functions of the parameter $p \in \mathscr{P} \subseteq \mathbb{R}^{n_\mathrm{p}}$:

$$\begin{aligned} A_\mu(p(k)) &= A_\mu^{(0)} \oplus \bigoplus_{i=1}^{n_\mathrm{p}} p_i(k) \otimes A_\mu^{(i)}, \quad \mu \in [M], \\ B(p(k)) &= B^{(0)} \oplus \bigoplus_{i=1}^{n_\mathrm{p}} p_i(k) \otimes B^{(i)}, \quad C(p(k)) = C^{(0)} \oplus \bigoplus_{i=1}^{n_\mathrm{p}} p_i(k) \otimes C^{(i)}, \end{aligned} \tag{5.3}$$

with known matrices $A_\mu^{(i)} \in \mathbb{R}_\varepsilon^{n\times n}$ for $\mu \in [M]$, $B^{(i)} \in \mathbb{R}_\varepsilon^{n\times n_\mathrm{u}}$, and $C^{(i)} \in \mathbb{R}_\varepsilon^{n_\mathrm{y}\times n}$ for $i \in [n_\mathrm{p}]$ and p_i is the i-th element of the parameter vector.

[4] The order corresponds to the maximum number of tokens contained initially at a place of the underlying timed-event graph [17].

[5] The set of all non-negative integers up to n is denoted as $[n] = \{l \in \mathbb{N}_0 \mid l \leq n\}$ where $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ and $\mathbb{N} = \mathbb{N}_0 \setminus \{0\}$. The set of positive integers up to n is denoted as $\underline{n}$.

Reference tracking control of parametric max-plus linear systems, with system matrices as in (5.3), when the parameter vector is a (known/measured) exogenous process has been studied in [106, 167].

Polytopic MP-LPV systems. The parameter vector p is assumed to belong to a unit simplex at each event step $k \in \mathbb{N}$:

$$\mathscr{P} := \left\{ p(\cdot) \in \mathbb{R}^{n_\mathrm{p}} \middle| \sum_{i=1}^{n_\mathrm{p}} p_i(\cdot) = 1, \quad p_i(\cdot) \geq 0 \right\}. \tag{5.4}$$

Let each of the matrix functions $A_\mu^{(i)} \in \mathbb{R}_\varepsilon^{n\times n}$ for $\mu \in [M]$, $B^{(i)} \in \mathbb{R}_\varepsilon^{n\times n_\mathrm{u}}$, and $C^{(i)} \in \mathbb{R}_\varepsilon^{n_\mathrm{y}\times n}$ for $i \in \underline{n_\mathrm{p}}$ have fixed finite structures[6].

Then the polytopic description of the system is obtained as [7]:

$$\begin{aligned} A_\mu(p(k)) &= \sum_{i=1}^{n_\mathrm{p}} p_i(k) \cdot A_\mu^{(i)}, \quad \mu \in [M], \\ B(p(k)) &= \sum_{i=1}^{n_\mathrm{p}} p_i(k) \cdot B^{(i)}, \quad C(p(k)) = \sum_{i=1}^{n_\mathrm{p}} p_i(k) \cdot C^{(i)}, \end{aligned} \tag{5.5}$$

where the known matrices $A_\mu^{(i)} \in \mathbb{R}_\varepsilon^{n\times n}$ for $\mu \in [M]$, $B^{(i)} \in \mathbb{R}_\varepsilon^{n\times n_\mathrm{u}}$, and $C^{(i)} \in \mathbb{R}_\varepsilon^{n_\mathrm{y}\times n}$ for $i \in \underline{n_\mathrm{p}}$ share the finite structures of their respective matrix functions. The stabilisability problem of uncertain max-plus linear systems with system matrices as in (5.4), (5.5) has been studied in [215].

Switching max-plus linear systems An MP-LPV system (5.2) subsumes the class of switching max-plus linear systems with max-plus linear modes. The parameter vector p is then assumed to be measurable and at each event step $k \in \mathbb{N}$ belongs to the set

$$\mathscr{P} := \left\{ p(\cdot) \in \{0,1\}^{n_\mathrm{p}} \middle| \sum_{i=1}^{n_\mathrm{p}} p_i(\cdot) = 1 \right\}. \tag{5.6}$$

Then the system matrices of a switching max-plus linear system can be described as (5.5),(5.6), without the requirement of fixed finite structures as in polytopic MP-LPV systems.

Interval MP-LPV system. In some situations, due to the presence of uncertainties, the timing of events is not exactly specified. Instead, the timing of any event is assumed to be non-deterministic, and contained in a (known) bounded interval [149]. This, for instance, is also the case in the modelling of P-Time event graphs [205]. In the same vein, an interval model in the max-plus algebra can be modelled in the MP-LPV framework analogously to the interval models in the conventional LPV framework [56].

[6] The finite structure of a matrix-valued function $A : \mathscr{P} \to \mathbb{R}_\varepsilon^{n\times m}$ is defined as the support of its finite components: $\mathscr{S}_\oplus(A) = \left\{(i,j) \in \underline{n} \times \underline{m} \middle| [A(p)]_{ij} \text{ is finite for all } p \in \mathscr{P}\right\}$.

[7] Recall from Section 2.2 that we follow the convention: $\varepsilon^{\otimes^a} = a \cdot \varepsilon = \varepsilon$ for $a > 0$ and $\varepsilon^{\otimes^0} = 0 \cdot \varepsilon = 0$.

Let the system matrices be defined as

$$\begin{aligned} A_\mu(p(k)) &= A_\mu^{(0)} + \Delta A_\mu(p(k)), \quad \mu \in [M], \\ B(p(k)) &= B^{(0)} + \Delta B(p(k)), \quad C(p(k)) = C^{(0)} + \Delta C(p(k)), \end{aligned} \tag{5.7}$$

where $\{A_\mu^{(0)}\}_{\mu \in [M]}$, $B^{(0)}$, and $C^{(0)}$ are known matrices of appropriate dimensions and fixed finite structures. The matrix functions ΔA_μ, ΔB, and ΔC depend on the (unmeasured) parameter vector $p(\cdot) \in \mathscr{P}$. Then assuming that for all $p(\cdot) \in \mathscr{P}$, we have

$$\begin{aligned} \underline{\Delta A_\mu} \le \Delta A(p(\cdot)) \le \overline{\Delta A_\mu}, \quad \mu \in [M], \\ \underline{\Delta B} \le \Delta B(p(\cdot)) \le \overline{\Delta B}, \quad \underline{\Delta C} \le \Delta C(p(\cdot)) \le \overline{\Delta C}, \end{aligned} \tag{5.8}$$

where $\underline{\Delta M}$ and $\overline{\Delta M}$ are known bounds for $M \in \left\{\{\Delta A_\mu\}_{\mu \in [M]}, \Delta B, \Delta C\right\}$, we obtain an interval model in the max-plus algebra in the MP-LPV framework.

(Quasi) MP-LPV system. The most general MP-LPV system results from abstracting away (max-plus) non-linearity in the system. This results in a max-plus linear, albeit non-stationary dynamical representation. Let the parameter vector be defined as:

$$p(k) = \left[x^\top(k), x^\top(k-1), \ldots, x^\top(k-M), u^\top(k), z^\top(k)\right]^\top \in \mathscr{P}. \tag{5.9}$$

The exogenous input signal $z(\cdot) \in \mathbb{R}^{n_z}$ is assumed to be independent of the state $x(\cdot) \in \mathbb{R}^n$, the control input $u(\cdot) \in \mathbb{R}^{n_u}$, and the output $y(\cdot) \in \mathbb{R}^{n_y}$. Thus, the MP-LPV description (5.2), (5.9) can encode more general constraints on timing of events than a max-plus linear system.

In the context of timed-event graphs, the examples involving state-dependent parametric variations in a max-plus linear system model can be found in [43, 144]. The authors of [43, 144] study the control and performance analysis of timed-event graphs under (known) periodic variations in timing of events along the event counter $k \in \mathbb{N}$.

Chapter 5

5.2.2. MODELLING RELATIONSHIPS

In this subsection, we establish a relationship between the class of implicit MP-LPV systems (5.2), (5.9) and that of max-min-plus-scaling (MMPS) systems (as in Definition 2.4.9). It is noted that MMPS systems are equivalent to continuous piecewise affine systems [75, Proposition 2.5]. We first recall the following result:

Lemma 5.2.1 (Max-min canonical form [75])**.** *An MMPS function $f_{\text{MMPS}} : \mathbb{R}^n \to \mathbb{R}$ can be rewritten into the max-min canonical form:*

$$f_{\text{MMPS}} = \max_{i \in \underline{L}} \min_{j \in \underline{n_i}} \left(\alpha_{(i,j)}^\top x + \beta_{(i,j)}\right), \tag{5.10}$$

for some $L, n_i \in \mathbb{N}$, $\alpha_{(i,j)} \in \mathbb{R}^n$, and $\beta_{(i,j)} \in \mathbb{R}$. ■

The vector-valued MMPS functions satisfy the preceding statement componentwise. We make the following assumption on the MP-LPV system representation (5.2), (5.9).

Assumption 5.2.1. *The finite entries of the system matrices* $\{A_\mu(\cdot)\}_{\mu\in[M]}$, $B(\cdot)$, *and* $C(\cdot)$ *are continuous piecewise affine in the parameter* $p(\cdot)\in\mathscr{P}$*. It is also assumed that the system description in* (5.2) *is structurally finite. Mathematically, the matrix*

$$F(\cdot)=\begin{bmatrix} A_0(\cdot) & A_1(\cdot) & \cdots & A_M(\cdot) & B(\cdot) & \mathscr{E} \\ \mathscr{E} & \mathscr{E} & \cdots & \mathscr{E} & \mathscr{E} & C(\cdot) \end{bmatrix} \tag{5.11}$$

is row finite. The finite structure of the matrix $F(\cdot)$ *is assumed to be independent of the parameter* $p(\cdot)$. ◊

The assumption on affine dependence is not very restrictive as it still allows arbitrarily accurate approximations of non-linear and non-smooth processes. It is noted that a continuous piecewise affine function can be rewritten as an MMPS function [75]. Therefore, functional dependence involving max-plus and min-plus operations (as in [106, 167]) can be conveniently expressed using piecewise affine functions. Secondly, physical systems are typically structurally finite. This ensures that the states do not become ε for finite initial states. This is required, for instance, when the trajectories are non-decreasing as they represent timing of event occurrences.

Theorem 5.2.1 (Equivalence of MMPS and MP-LPV systems)**.** *Under Assumption 5.2.1, the classes of MP-LPV systems* (5.2), (5.9) *and MMPS systems (Definition 2.4.9) coincide.*

Proof. For the first part, it is sufficient to note that the definition of an implicit MP-LPV system (5.2), (5.9) involves only the basic constructors of an MMPS expression (3.12):

- All finite elements of the system matrices in (5.2), such as $[A_\mu(\cdot)]_{ij}=f_{\mu,ij}(\cdot)\neq\varepsilon$, are continuous piecewise affine in the variables $x(\cdot)$, $u(\cdot)$, and $z(\cdot)$. By corollary, they involve only MMPS expressions [75].

- As the definition of the MMPS expressions is recursive, the equations of the form (5.2) again consist of only MMPS expressions.

- The assumption on structural finiteness (Assumption 5.2.1) ensures that state equation in (5.2) maps $\mathbb{R}^{M\cdot n+n_\mathrm{u}}$ into $\mathbb{R}^n$. Similarly, the regularity of the matrix C ensures that the output equation in (5.2) always maps $\mathbb{R}^n$ into $\mathbb{R}^{n_\mathrm{y}}$.

Therefore, the class of MP-LPV systems (5.2) is contained in the class of MMPS systems.

Now we show that the MMPS system (2.51) can be written in the form (5.2). The states $x\in\mathbb{R}^n$ and inputs $u\in\mathbb{R}^{n_\mathrm{u}}$, $z\in\mathbb{R}^{n_\mathrm{z}}$ are first collected as:

$$\begin{aligned} w(k)&=\left[x^\top(k),\ldots,\ x^\top(k-M),\ u^\top(k)\right]^\top\in\mathbb{R}^{n_\mathrm{w}},\\ p(k)&=\left[w^\top(k),\ z^\top(k)\right]^\top\in\mathbb{R}^{n_\mathrm{p}}, \end{aligned} \tag{5.12}$$

where $n_\mathrm{w}=M\cdot n+n_\mathrm{u}$, $n_\mathrm{p}=n_\mathrm{w}+n_\mathrm{z}$. Consider now an MMPS system described as

$$x(k)=f_{\mathrm{MMPS}}(p(k)), \tag{5.13}$$

where the components $f_{\text{MMPS},l}: \mathbb{R}^{n_\text{p}} \to \mathbb{R}$, $l \in \underline{n}$, are MMPS functions. Then from Lemma 5.2.1, for every $l \in \underline{n}$ there exist scalars $N, n_\text{i} \in \mathbb{N}$ such that

$$\begin{aligned} x_l(k) &= \max_{i \in \underline{N}} \min_{j \in \underline{n_i}} \gamma_{l,(i,j)}(p(k)), \\ &= \max_{i \in \underline{N}} g_{l,i}(p(k)) \end{aligned} \tag{5.14}$$

where $\gamma_{l,(i,j)}(\cdot)$ are piecewise affine functions for every $l \in \underline{n}$ and $(i,j) \in \underline{N} \times \underline{n_\text{i}}$. As minimum over piecewise affine functions is again a piecewise affine function [75], the last equality follows by defining piecewise affine functions $g_{l,i}(\cdot) := \min_{j \in \underline{n_\text{i}}} \gamma_{l,(i,j)}(\cdot)$.

It is noted that we can always augment the max expression in (5.14) with “void” terms of the form $\max(s, \varepsilon, \varepsilon, \ldots)$. Therefore, the scalar N can be incremented suitably for the subsequent argument. We now define subsets $N_t \subset \underline{N}$ such that $N_t \neq \emptyset$, $t \in \underline{n_\text{w}}$, of the set $\underline{N}$ such that $N_t \cap N_s = \emptyset$ for $t \neq s$ and $\cup_{t \in \underline{n_\text{w}}} N_t = \underline{N}$. Then from (5.14), we have

$$\begin{aligned} x_l(k) &= \max_{t \in \underline{n_\text{w}}} \left(\max_{i \in N_t} g_{l,i}(p(k)) \right) \\ &= \max_{t \in \underline{n_\text{w}}} \left(\max_{i \in N_t} \left(g_{l,i}(p(k)) - w_t(k) + w_t(k) \right) \right) \\ &= \max_{t \in \underline{n_\text{w}}} \left(f_{lt}(p(k)) + w_t(k) \right). \end{aligned} \tag{5.15}$$

Here, the last equality follows by defining functions $f_{lt}: \mathbb{R}^{n_\text{p}} \to \mathbb{R}_\varepsilon$ for $l \in \underline{n}$ and $t \in \underline{n_\text{w}}$,

$$f_{lt}(p(k)) := \max_{i \in N_t} \left(g_{l,i}(p(k)) - w_t(k) \right). \tag{5.16}$$

Collecting terms in (5.15) and defining a matrix $F(\cdot) \in \mathbb{R}_\varepsilon^{n \times n_\text{w}}$ with elements $[F(\cdot)]_{lt} := f_{lt}(\cdot)$, we have

$$x(k) = F(p(k)) \otimes w(k). \tag{5.17}$$

A suitable partition of the matrix $F(\cdot)$ as

$$F(p(k)) = \begin{bmatrix} A_0(p(k)) & \cdots & A_M(p(k)) & B(p(k)) \end{bmatrix} \tag{5.18}$$

results in an MP-LPV system of the form (5.2). It is noted that the finite elements of the matrix $F(\cdot)$ are continuous piecewise affine in the parameter $p(\cdot)$. By construction, the system (5.17) remains structurally finite even after the introduction of ε elements in the matrix $F(\cdot)$.

A similar argument can be applied to rewrite the output equation in (2.51) as the output equation of (5.2). Therefore, a given MMPS system can be rewritten as an MP-LPV system. ■

It is noted that the MP-LPV model resulting from an MMPS system using the procedure described in the preceding theorem might not be unique. For instance, this loss of uniqueness can arise due to the introduction of ε elements in the expression (5.15). This can lead to different choices for the finite structure[8] of the matrix-valued function $F(\cdot)$ in

[8] See footnote [6].

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(5.17). Therefore, the proof only shows that the system representations can be translated from one to the other but might not result in a model respecting the incidence structure of the underlying discrete-event system. The preceding result allows the tools designed for analysis and control of either class of systems to be applied to the other class of systems.

In the context of uniqueness of the model, we present a canonical form of MP-LPV systems (5.2), (5.9) in the next subsection to simplify the subsequent analysis.

5.2.3. CANONICAL FORM OF MP-LPV SYSTEMS

The Assumption 5.2.1 allows us to rewrite the MP-LPV system (5.2), (5.9) such that it facilitates further analyses. The following canonical form of an MP-LPV system is particularly useful for studying the existence of finite state trajectories.

Lemma 5.2.2 (MP-LPV system canonical formulation)**.** *Consider an implicit MP-LPV system* (5.2)*,* (5.9) *under Assumption 5.2.1. Then the dynamics can be rewritten in the following canonical form*[9] *for* $\mu \in [M]$*:*

$$
\begin{aligned}
x(k) &= \bigoplus_{\mu=0}^{M} \left(A_\mu\left(p^{(\mu)}(k)\right) \otimes x(k-\mu)\right) \oplus B(p^{(1)}(k)) \otimes u(k),\\
y(k) &= C(p(k)) \otimes x(k),\\
p^{(\mu)}(k) &= \left[x^\top(k-\mu), \ldots, x^\top(k-M), u^\top(k), z^\top(k)\right]^\top \in \mathscr{P}^{(\mu)}, \quad \mu \in [M].
\end{aligned}
\tag{5.19}
$$

Here in comparison to (5.2)*, the matrix* $A_\mu(\cdot)$ *is independent of the states* $x(k-m)$ *for* $m < \mu$ *and the matrix* $B(\cdot)$ *is independent of the state* $x(k)$*.*

Proof. Consider the MP-LPV system in (5.2), (5.9). We only show here the transformation of matrices $A_\mu(\cdot)$ for $\mu \in [M]$. The transformation required for the matrix $B(\cdot)$ follows accordingly. Then without a loss of generality, we assume that the inputs $u(\cdot)$ and $z(\cdot)$ are absent from the description as they do not affect the subsequent transformation to the canonical form. For the sake of this proof, we have $n_\mathrm{p} = n(M+1)$ in (5.2).

The property P(M), $M \in \mathbb{N}$, is said to be satisfied if the state dynamics in (5.2) can be rewritten as in (5.19) for orders $\mu \in [M]$. We now prove the result by induction.

It is first noted that the property P(0) holds trivially with $p^{(0)}(k) = p(k)$, $\mathscr{P}^{(0)} = \mathscr{P}$, and $A_0\left(p^{(0)}(k)\right) = A_0(p(k))$.

Assuming that the property P($M-1$) holds for some $M > 1$, we have for indices $i \in \underline{n}$:

$$
x_i(k) = \max_{\mu \in [M-1]} \max_{j \in \underline{n}} \left(f_{ij}^{(\mu)}\left(p^{(\mu)}(k)\right) + x_j(k-\mu)\right), \tag{5.20}
$$

where $f_{ij}^{(\mu)} : \mathscr{P}^{(\mu)} \to \mathbb{R}_\varepsilon$, for all $i, j \in \underline{n}$ and each $\mu \in [M-1]$. Also, $f_{ij}^{(\mu)}(\cdot)$ does not depend on $x(k-m)$ for $m < \mu$.

We now show that given P($M-1$), the property P(M) is satisfied as well. Under Assumption 5.2.1, the finite structures $\mathscr{S}_\oplus(\cdot)$ of the system matrices in (5.2) are fixed. From

[9] With a slight abuse of notation, $A_\mu(p(\cdot)) := A_\mu(p^{(\nu)}(\cdot))$, $\mu, \nu \in [M]$ signifies that the matrix $A_\mu(\cdot)$ is only dependent on the terms in $p^{(\nu)}(\cdot)$.

Lemma 5.2.1, we have for $(i,j) \in \mathscr{S}_{\oplus}(A_\mu(\cdot))$ and orders $\mu \in [M]$:

$$\begin{aligned}
[A_\mu(p(k))]_{ij} = f_{\mu,ij}(p(k)) &= \max_{q \in \underline{Q}} \min_{r \in \underline{n_q}} (\alpha_{(q,r)}^\top p(k) + \beta_{(q,r)}) \\
&= \max(\min(\gamma_{11}(p),\ldots), \min(\gamma_{21}(p),\ldots), \\
&\qquad \ldots, \min(\gamma_{Q1}(p), \gamma_{Q2}(p), \ldots)), \\
&= \max_{q \in \underline{Q}} (g_q(p(k)))
\end{aligned} \tag{5.21}$$

where[10] $\gamma_{qr}(\cdot) := \alpha_{(q,r)}^\top p(\cdot) + \beta_{(q,r)}$, $q \in \underline{Q}$ and $r \in \underline{n_q}$, and $g_q(\cdot) := \min_{r \in \underline{n_q}} (\gamma_{qr}(\cdot))$, $q \in \underline{Q}$. Moreover, $[A_\mu(\cdot)]_{ij} = f_{\mu,ij}(\cdot) \equiv \varepsilon$ for $(i,j) \notin \mathscr{S}_{\oplus}(A_\mu)$. Then from the associativity of the max operation, we have for $i \in \underline{n}$:

$$\begin{aligned}
x_i(k) &= \max_{\mu \in [M]} \max_{j \in \underline{n}} \left(f_{\mu,ij}(p(k)) + x_j(k-\mu) \right) \\
&= \max_{j \in \underline{n}} \Bigg(\max_{\mu \in [M-1]} \left(\bar{f}_{ij}^{(\mu)} \left(p^{(\mu)}(k) \right) + x_j(k-\mu) \right), \\
&\qquad\qquad f_{M,ij}(p(k)) + x_j(k-M) \Bigg).
\end{aligned} \tag{5.22}$$

Here, the last equality follows by invoking P$(M-1)$. Let $\alpha_{ij}(M) = f_{M,ij}(p(k)) + x_j(k-M)$. Now it remains to show that any term dependent on $x(k-m)$, for $m < M$, can be moved from $\alpha_{ij}(M)$ to the rest of the terms in the expression of $x_i(k)$ while preserving the canonical form.

Let $x(k-m)$, for some $m \in [M]$, be the smallest order term present in the expression of $f_{M,ij}(\cdot)$ in (5.21). Then, we have

$$\begin{aligned}
\alpha_{ij}(M) &= \max_{q \in \underline{Q}} \left(g_q \left(p^{(m)}(k) \right) \right) + x_j(k-M) \\
&= \max_{q \in \underline{Q}} \left(g_q \left(p^{(m)}(k) \right) + x_j(k-M) - x_j(k-m) \right) \\
&\qquad\qquad + x_j(k-m), \\
&= \max_{q \in \underline{Q}} \left(\bar{g}_q \left(p^{(m)}(k) \right) \right) + x_j(k-m) \\
&= \bar{f}_{M,ij} \left(p^{(m)}(k) \right) + x_j(k-m).
\end{aligned} \tag{5.23}$$

Let $e_{\{j\}}$ denote the j-th unit vector of $\mathbb{R}^{n(M+1)}$ such that $[e_{\{j\}}]_i = 1$ if $i = j$ and $[e_{\{j\}}]_i = 0$ otherwise. Then the last equality in (5.23) follows by redefining the affine functions[11] in (5.21) for $q \in \underline{Q}$ and $r \in \underline{n_q}$ as

$$\begin{aligned}
\overline{\gamma}_{qr}(p(k)) &= \gamma_{qr}(p(k) + x_j(k-M) - x_j(k-m) \\
&= \left(\alpha_{(q,r)} + e_{\{n \cdot M + j\}} - e_{\{n \cdot m + j\}} \right)^\top p(k) + \beta_{(q,r)},
\end{aligned} \tag{5.24}$$

[10] The indices $i, j \in \underline{n}$ and $\mu \in [M]$ are omitted for the sake of brevity of the expression for the matrix components $[A_\mu(\cdot)]_{ij}$.

[11] Addition '+' distributes over max and min operations in the scalar case.

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and similarly the functions $\bar{g}_q(\cdot)$ in (5.23) for $q \in \underline{Q}$. Substituting the expression for $\alpha_{ij}(M)$ in (5.23) back into (5.22), we have

$$\begin{aligned} x_i(k) = \max_{j\in\underline{n}}\Big(&\max_{\mu\in[m-1]}\Big(\bar{f}_{ij}^{(\mu)}(p^{(\mu)}(k)\Big)+x_j(k-\mu)),\\ &\max\Big(\bar{f}_{ij}^{(m)}(p^{(m)}(k)),\bar{f}_{M,ij}(p^{(m)}(k))\Big)+x_j(k-m),\\ &\max_{\mu\in\{m+1,\dots,M-1\}}\Big(\bar{f}_{ij}^{(\mu)}(p^{(\mu)}(k))+x_j(k-\mu)\Big)\Big)\\ &= \max_{\mu\in[M]}\max_{j\in\underline{n}}\Big(f_{ij}^{(\mu)}(p^{(\mu)}(k))+x_j(k-\mu)\Big). \end{aligned} \tag{5.25}$$

Here, the last equality results by noting that *i)* the maximum of two piecewise affine functions can again be expressed as a piecewise affine function of its variables[12] ([75, 118]), and *ii)* $f_{ij}^{(M)}(\cdot) := \varepsilon$ for all $j \in \underline{n}$ if $m < M$ in (5.23). The new systems matrices $\{A_\mu(\cdot)\}_{\mu\in[M]}$ are updated accordingly as in (5.21).

As the procedure is recursive for all $i \in \underline{n}$, the preceding expression for $x_i(k)$ and hence that of $x(k)$ can be expressed in the canonical form (5.19). As P$(M-1)$ implies P(M), the property $P(M)$ holds for every $M \in \mathbb{N}$ by the principle of induction. The same argument can be extended to rewrite the terms dependent on $x(k)$ in $B(\cdot)\otimes u(k)$ as terms in $A_0(\cdot)\otimes x(k)$ to obtain the final canonical form (5.19). ■

The most important advantage of the MP-LPV canonical form (5.19) is that the implicitness is now concentrated in the $A_0(\cdot)$ matrix. We now look at the analysis to show the existence of state trajectories of the MP-LPV system (5.19).

5.3. SOLVABILITY OF MP-LPV SYSTEM

The dependence of the matrices $\{A_\mu(\cdot)\}_{\mu\in[M]}$ and $B(\cdot)$ on the current state $x(k)$ can result in an implicit system of equations. Unfortunately, such an implicitness cannot always be resolved by the usual Kleene star operation (see Lemma 2.2.5). We would like to investigate the existence and uniqueness of finite trajectories $(x(\cdot), u(\cdot), z(\cdot))$ of the MP-LPV system (5.19). Therefore, we introduce the following definition:

Definition 5.3.1 (Solvability)**.** Consider the MP-LPV system in (5.2). The system is said to be *solvable* at a given event step $k \in \mathbb{N}$, $k > M$, if for every $x(k-j) \in \mathbb{R}^n$ for all $j \in \underline{M}$, $u(k) \in \mathbb{R}^{n_u}$ and $p(k) \in \mathscr{P}$, there exists a unique state $x(k) \in \mathbb{R}^n$ that satisfies (5.2). □

The resulting *solvability problem* then looks at the formulation of *a priori* guarantees to ensure the existence of unique finite state $x(k)$ given the input and parameter values at event step k, and the past state values. The notion of solvability is defined analogously to that of implicit nonlinear discrete-time systems [83], and discrete-time linear systems in the descriptor form [154].

5.3.1. APPROACH

We adopt a modular approach in investigating the solvability problem for MP-LPV systems. The following cases can be identified based on the degree of implicitness present

[12] We note that the argument also holds if $\bar{f}_{ij}^{m}(p^{(m)}(k)) = \varepsilon$ in (5.25) as ε is an identity with respect to the max operation.

in the system model (5.19):

Explicit. For the case $A_0(\cdot) = \mathcal{E}_{n\times n}$, we achieve an explicit MP-LPV system. A cursory inspection of (5.19) reveals that the state $x(k)$ is then only dependent on $p^{(1)}(k)$ and $u(k)$. Therefore, the regularity of system matrices $\{A_\mu\}_{\mu\in\underline{M}}$ (as in Assumption 5.2.1) is sufficient for solvability (as in Definition 5.3.1) of the MP-LPV system.

Single Implicit. For the case $A_0(p(\cdot)) := A_0(p^{(\eta)}(\cdot))$ for $\eta \geq 1$, we achieve a single implicit MP-LPV system. Herein, the implicitness can be resolved using the Kleene star operation resulting in an explicit MP-LPV system. Given that the system matrices are regular (as in Assumption 5.2.1), solvability at each event step $k \in \mathbb{N}$ is ensured if the Kleene star $A_0^\star(p^{(1)})$ exists (see Lemma 2.2.5). The uniqueness is guaranteed if the associated directed graph $\mathcal{G}(A_0(\cdot))$ has negative circuit weights.

Doubly Implicit. In the most general case, the implicitness in the current state also appears due to the parametric dependence of the matrix $A_0(x(k); p^{(1)}(k))$ on $x(k)$. This results in a doubly implicit MP-LPV system. The resulting existence problem for a solution to such a system of equations has not yet been studied in literature. We provide a sufficient condition to ensure solvability under certain practically relevant assumptions
.

In the subsequent subsections, we present algebraic approaches to study the solvability problem for the MP-LPV systems (5.19).

5.3.2. NECESSITY

In this subsection, we restrict our attention to the solvability problem for a single implicit MP-LPV system (5.19) with $A_0(p(\cdot)) := A_0(p^{(\eta)}(\cdot))$ for $\eta \geq 1$. The following result is a direct consequence of Lemma 2.2.16 to the state equation of a single-implicit MP-LPV system.

Lemma 5.3.1 (Solvability of single-implicit MPLPV system)**.** *Consider a single implicit MP-LPV system in* (5.19)*,* $A_0(p(\cdot)) := A_0(p^{(\eta)}(\cdot))$ *for* $\eta \geq 1$*. The state solution to the system* $x(k)$ *at a given event step* $k \in \mathbb{N}$, $k > M$*, exists if and only if* $A_0^\star(p^{(1)}(k))$ *exists. Equivalently, the state solution exists if and only if the largest max-plus eigenvalue of* $A_0(p^{(1)}(k))$ *is nonpositive,* $\overline{\lambda}(A_0(p^{(1)}(k))) \leq 0$. ■

We now show that the solution to the state equation (5.19), for the case of single implicit MP-LPV system, can be computed explicitly if it exists. The following result relies on the fact that $A_0(p(\cdot))$ only depends on past state information and hence can be computed analytically.

Theorem 5.3.1 (Solvability condition)**.** *Consider a single implicit MP-LPV system in* (5.19) *with* $A_0(p(\cdot)) := A_0(p^{(\eta)}(\cdot))$ *for* $\eta \geq 1$*. The set of past state information* $p^{(1)}(k)$ *at any given event step* $k \in \mathbb{N}$, $k \geq M$*, such that a unique state solution* $x(k)$ *to the implicit MP-LPV system exists, is a union of convex polyhedra.*

Proof. Let the finite structure of the matrix $A_0(\cdot)$ be given as:

$$\mathscr{S}_\oplus(A(\cdot)) = \left\{(i,j) \in \underline{n} \times \underline{m} \mid [A(\cdot)]_{ij} \text{ is finite}\right\}. \tag{5.26}$$

Let a denote the vector of finite entries of the matrix $A_0(\cdot)$. The max-plus eigenvalue problem for the matrix $A_0(\cdot)$ is given as a linear program:

$$\min_{\lambda,s} \quad \lambda \tag{5.27a}$$

$$\text{subject to} \quad a_{ij} + s_j - s_i \leq \lambda \quad \forall (i,j) \in \mathscr{S}_\oplus(A(\cdot)). \tag{5.27b}$$

A unique state solution to the single implicit MP-LPV system exists if and only if $\overline{\lambda}(A_0(\cdot)) < 0$ (see Lemma 5.3.1). Then the set of decision variables $s \in \mathbb{R}^n$ and parameter values a such that $\overline{\lambda}(A_0(\cdot)) < 0$, can be expressed as a convex polyhedron for a suitable choice of a matrix S and a slack vector s_ϵ:

$$\Lambda = \left\{[a^\top, s^\top]^\top \,\middle|\, S \cdot [a^\top, s^\top]^\top \leq s_\epsilon\right\}, \tag{5.28}$$

Then the set of all finite entries of matrix $A_0(\cdot)$ such that a unique solution exists to the MP-LPV system can be obtained as a projection of Λ on the a-subspace and expressed in the half-space description:

$$\Lambda_\text{a} = \text{proj}_a(\Lambda) = \{a \mid S_\text{a} \cdot a \leq b_\text{a}\}. \tag{5.29}$$

Under Assumption 5.2.1, used to define the MP-LPV system in (5.19), the finite elements of matrix $A_0(\cdot)$ are continuous piecewise affine functions of the parameter $p(\cdot) \in \mathscr{P}$. Then there exist matrices $G_i \in \mathbb{R}^{n \times n_\text{p}}$, vectors $g_i \in \mathbb{R}^n$, and a partition, the elements of which have non-overlapping interiors, of the state space $\{\Omega_i\}_{i \in \underline{m}}$, $\Omega_i = \{p \mid R_i \cdot p^{(1)}(\cdot) \leq r_i\}$ for $i \in \underline{m}$, such that

$$a(p(\cdot)) = G_i \cdot p^{(1)}(\cdot) + g_i, \quad \text{if } p^{(1)}(\cdot) \in \Omega_i. \tag{5.30}$$

The set $\Lambda_{\text{p},i}$ of vectors $p^{(1)}(\cdot)$ contained in Ω_i such that the corresponding $a(p(\cdot))$ is contained in the subspace Λ_a can be represented as

$$\Lambda_{\text{p},i} = \left\{p^{(1)}(\cdot) \,\middle|\, \begin{bmatrix} S_\text{a} \cdot G_i \\ R_i \end{bmatrix} \cdot p^{(1)}(\cdot) \leq \begin{bmatrix} b_\text{a} - S_\text{a} \cdot g_i \\ r_i \end{bmatrix}\right\}, \quad i \in \underline{m}. \tag{5.31}$$

Note that the intersection of two convex polyhedra is again convex. Then at any given event step $k \in \mathbb{N}$, the set of past states $p^{(1)}(k)$ such that a unique solution $x(k)$ to the MP-LPV system exists is given as:

$$p^{(1)}(k) \in \Lambda_\text{p} \triangleq \bigcup_{i=1}^{m} \Lambda_{\text{p},i}. \tag{5.32}$$

■

The proposed solvability condition in the preceding theorem allows us to readily check for existence of a finite state solution at any given event step.

Corollary 5.3.1. *A single implicit MP-LPV system, with $A_0(p(\cdot)) := A_0(p^{(\eta)}(\cdot))$ for $\eta \geq 1$, is solvable at any given step $k \in \mathbb{N}$, $k > M$, if and only if $p^{(1)}(k) \in \Lambda_\text{p}$ as in (5.32).* ■

We show how the solvability condition in Theorem 5.3.1 can be used to derive conditions for existence of finite trajectories over a possibly infinite-step event horizon in Section 5.4.2.

5.3.3. PRACTICAL SUFFICIENCY

In this subsection, we provide a sufficient condition for the solvability of a doubly-implicit MP-LPV system (5.19) under certain practically relevant assumptions.

We note that at every event step $k \in \mathbb{N}_0$, the state update of a doubly implicit MP-LPV system (5.19) amounts to solving a fixed point problem in the current state $x(k)$ for a (vector-valued) max-min-plus-scaling function. This follows from the equivalence presented in Theorem 5.2.1. We first treat the problem analytically starting with certain assumptions.

Given an index $j \in \underline{n}$, let the state $x(k)$ be partitioned as

$$x(\cdot) = \left[\begin{array}{c} x_1(\cdot) \\ \vdots \\ x_{j-1}(\cdot) \\ \hline x_j(\cdot) \\ \hline x_{j+1}(\cdot) \\ \vdots \\ x_n(\cdot) \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c} x \\ \vdots \\ x \end{array}}\right\} x^{(-j)}(\cdot) \\ \\ \left.\vphantom{\begin{array}{c} x \\ \vdots \\ x \end{array}}\right\} x^{(+j)}(\cdot) \end{array} \tag{5.33}$$

We now make the following assumptions.

Assumption 5.3.1. *The doubly implicit MP-LPV system in* (5.19) *is assumed to satisfy the following assumptions:*

1. *The matrix $A_0(\cdot)$ is reducible and in the lower-triangular form* (3.12)*;*
2. *The finite components of matrix $A_0(\cdot)$ are max-plus-scaling functions (see Definition 2.4.6) of the parameter $p(k)$.*
3. *The finite components of the matrix A_0 are defined such that for $i, j \in \underline{n}$ with $i \le j$, we have $[A_0(\cdot)]_{ij} := [A_0(x^{(-i)}(k), x_i(k), p^{(1)})]_{ij}$.* ◊

The first assumption on the lower-triangular form of the matrix $A_0(\cdot)$ ensures that any circuit present in the underlying directed graph is a self-loop (see Definition 2.2.10). In context of a max-plus linear system (obtained by fixing matrices $\{A_\mu\}_{\mu\in[M]}$, B, and C in (5.2)), the associated timed-event graph is live if the matrix A_0 is strictly lower triangular under a convenient permutation of the coordinates [17, §2.5.3]. The liveness property allows a straight-forward recursive evaluation of the state variables $x_j(k)$ for all $j = 1, \cdots, n$ and $k \in \mathbb{N}$. The first assumption is then less restrictive than usually made in the literature on timed-event graphs. The last assumption ensures that state equations for $x_j(\cdot)$, $j \in \underline{n}$, are causal. Therefore, the first and the last assumptions together allow us to recursively define the evolution equations for the states x_j for $j \in \underline{n}$. These state equations can still be implicit as opposed to that obtained for live timed-event graphs. The practical relevance of these assumptions in the context of the case study of a unidirectional urban railway line will be discussed in Section 5.5.

Under Assumption 5.3.1, the state at event step k can be expressed as

$$\begin{aligned} x_i(k) = g_i(x_i(k)) &= \max_{j\le i} \left[A_0(p^{(0)}(k))\right]_{ij} \otimes x_i(k) \oplus r_i(k) \\ &= \max\left(f^{(i)}_{\mathrm{MPS}\varepsilon}\left(x^{(-i)}(k), x_i(k), p^{(1)}(k)\right), r_i(k)\right), \end{aligned} \tag{5.34}$$

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where the function $f^{(i)}_{\mathrm{MPS}\varepsilon}(\cdot)$ is an extended real-valued max-plus scaling function (see Definition 2.4.6) in $x_i(k)$ for some $N \in \mathbb{N}$:

$$f^{(i)}_{\mathrm{MPS}\varepsilon}(\cdot) = \max_{l \in \underline{N}} \Big(\alpha^{(i)}_l x_i(k) + \beta^{(i)}_l (x^{(-i)}(k), p^{(1)}(k)) \Big) + \delta^{(i)}_0, \tag{5.35}$$

where $\delta^{(i)}_0 = \varepsilon$ if $[A_0(\cdot)]_{ij} = \varepsilon$ for all $j \in \underline{n}$, and $\delta^{(i)}_0 = 0$ otherwise, irrespective of the parameter $p(k)$. The known terms are collected in $\beta^{(i)}_l(\cdot)$ and $r(k) \in \mathbb{R}^n_\varepsilon$ as

$$r(k) = \bigoplus_{\mu=1}^{M} \Big(A_\mu(p^{(\mu)})(k)) \otimes x(k-\mu) \Big) \oplus B(p^{(1)}(k)) \otimes u(k). \tag{5.36}$$

Therefore, the states $x_i(k)$ can be evaluated in increasing order of indices $i \in \underline{n}$ as a univariate fixed-point problem.

We now note that under Assumption 5.3.1 2, the function $f^{(i)}_{\mathrm{MPS}\varepsilon}(\cdot)$ and in turn the expression for $x_i(k)$ in (5.34) are monotone functions (see Definition 2.4.6 and subsequent discussion). This allows us to apply results from fixed-point theory of monotone functions.

Lemma 5.3.2 (Existence of fixed points, [4])**.** *Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be a monotone function. If the set $\{x \in \mathbb{R}^n \mid h(x) \le x\}$ is bounded from below, the function h attains the smallest fixed point that satisfies:*

$$x_* = h(x_*) = \inf \left\{ x \in \mathbb{R}^n \,\middle|\, h(x) \le x \right\}. \qquad \blacksquare$$

The boundedness of the set $\{x \in \mathbb{R}^n \mid h(x) \le x\}$ for a monotone function h is not ensured by the existence of the smallest fixed point [4]. Therefore, the preceding condition is not necessary for the existence of fixed points of a monotone function. We now present the main result of this subsection.

Theorem 5.3.2 (Practical solvability)**.** *For every $i \in \underline{n}$, there exists a unique solution $x_i(k)$ to (5.34) if for every $l \in \underline{N}$, the function $f^{(i)}_{\mathrm{MPS}\varepsilon}(\cdot)$ in (5.35) satisfies $0 < \alpha^{(i)}_l < 1$.*

Proof. We leverage the monotonicity of the functions $f_i(\cdot)$, $i \in \underline{n}$, in (5.34) to show that there exists a unique solution $x(k)$ for a given past state and input data under the proposed condition. Due to the problem setup, it is sufficient to prove the result for an arbitrary index $i \in \underline{n}$.

We first remark that under Assumption 5.2.1 on structural finiteness of the system description in (5.19), $r_i(k)$ is finite if $f^{(i)}_{\mathrm{MPS}\varepsilon}(\cdot) = \varepsilon$, and vice versa. For the former case, we have $x_i(k) = r_i(k)$. Therefore, a unique solution to (5.34) exists trivially. We now treat the problem of existence and uniqueness when $\delta^{(i)}_0 = 0$ in (5.35).

Uniqueness. For the case when $\delta^{(i)}_0 = 0$ in (5.35), we require

$$\begin{aligned} F_i(x_i(k)) &= \max\Big(f^{(i)}_{\mathrm{MPS}\varepsilon}(x_i(k)), r_i(k) \Big) - x_i(k) \\ &= \max_{l \in [N]} \Big((\alpha^{(i)}_l - 1)\, x_i(k) + \beta^{(i)}_l(\cdot) \Big) \\ &= \max_{l \in [N]} \Big(\overline{\alpha}^{(i)}_l\, x_i(k) + \beta^{(i)}_l(\cdot) \Big) \\ &= 0, \end{aligned} \tag{5.37}$$

where $\alpha_0^{(i)} = 0$ and $\beta_0^{(i)}(\cdot) = r_i(k)$. The function $F_i(\cdot)$ is again a continuous piecewise affine function. Under the given condition $\alpha_l^{(i)} < 1$, we have $\overline{\alpha}_l^{(i)} < 0$ for all $l \in [N]$. Hence, $F_i(\cdot)$ is a strictly decreasing continuous function. Therefore, if there exists a zero-crossing then it is unique.

Existence. Under the assumption $\alpha_l^{(i)} > 0$, for all $l \in \underline{N}$, in (5.35), the function $f_{\mathrm{MPS}\varepsilon}^{(i)}$ is a monotone function in $x_i(\cdot)$. As (pointwise) maximisation preserves monotonicity, we have that the function $g_i(\cdot)$ in (5.34) is also monotone in $x_i(\cdot)$.

We show that the set $\Gamma_i = \{t \in \mathbb{R} \mid g_i(t) \le t\}$ is lower bounded for g_i defined in (5.34) when $\alpha_l^{(i)} > 0$ in (5.35). The required result then follows from Lemma 5.3.2.

In the case $r_i(k)$ is finite, then the function $g_i(\cdot)$ is bounded from below, i.e. $g_i(x_i(k)) \ge r_i(k)$ for all $x_i(k) \in \mathbb{R}$. Therefore, a solution exists.

Lastly, we consider the case when $r_i(k) = \varepsilon$ such that $g(\cdot) = f_{\mathrm{MPS}\varepsilon}(\cdot)$. It is then sufficient to show that the set Γ_i is non-empty and bounded from below. Let

$$x_* = \max_{l \in \underline{N}} \frac{\beta_l^{(i)}(\cdot)}{1 - \alpha_l^{(i)}}. \tag{5.38}$$

Then we have $f_{\mathrm{MPS}\varepsilon}^{(i)}(x_*) \le x_*$ for all $\alpha_l^{(i)} \in (0,1)$. Also, x_* is the smallest element of Γ_i. This completes the proof. ■

The condition provided in the preceding theorem is only sufficient for existence and uniqueness. It is indeed possible to obtain a solution to (5.34) for $\alpha_l^{(i)} > 1$ depending on the values of $\beta_l^{(i)}(\cdot)$, $i \in \underline{n}$. This, however, can lead to zero or multiple zero-crossings of the function $F_i(\cdot)$. We find that such cases are not of any physical importance to the case study presented in Section 5.5. The generalisation of the result obtained on practical solvability to other applications is still a topic of investigation.

The solution to the fixed point equation (5.37) can be found using an interval interpolation line-search algorithm, Newton's gradient descent [81], Kleene star (monotone) iteration [64], policy iteration [4], or mixed-integer programming [71].

5.4. INVARIANT SOLVABILITY

In this section, we describe the *invariant solvability problem* that extends the notion of solvability over an infinite event-step horizon starting from admissible initial conditions by restricting parameter values.

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5.4.1. PROBLEM STATEMENT

We first note that a free choice of the initial conditions $\{x(j)\}_{j \in \underline{M}}$ can lead to inconsistencies in the system equations (5.2). We call a given initial condition admissible if the implicit algebraic relation (5.2), relating the states $\{x(k-j)\}_{j \in \underline{M}}$, the control input $u(k)$, and the parameter value $p(k)$, holds for $k = M$.

To simplify the treatment, we only consider the *autonomous* case in the absence of control inputs, i.e. $u(k) = \mathcal{E}_{n_u}$ for all $k \in \mathbb{N}_0$. Note that causal feedback control inputs can still be treated in our framework.

Definition 5.4.1 (Invariant solvability)**.** Consider the MP-LPV system in (5.2) in the absence of exogenous and control inputs. The system is said to be *invariantly solvable* if there exist sets $\mathscr{X} \subseteq \mathbb{R}^n$ and $\mathscr{P} \subseteq \mathbb{R}^{n_p}$ such that given an admissible initial set of states $\{x(j)\}_{j\in\underline{M}}$, there exists a unique state trajectory $x(k) \in \mathscr{X}$ with $p(k) \in \mathscr{P}$ for all $k \in \mathbb{N}$ that satisfies (5.19). □

Note that in the preceding definition, $\mathscr{X}$ is a robust positively invariant set for all parameter values in $\mathscr{P}$. Moreover, the set of admissible initial conditions are also required to satisfy $x(j) \in \mathscr{X}$ for $j \in \underline{M}$. The characterisation of existence of solutions to conventional implicit discrete-time systems using invariant subspaces can be found in [19].

The invariant solvability problem for the case of a (quasi) MP-LPV systems (5.19) boils down to the characterisation of the set $\mathscr{P}$ as it also contains the set of initial conditions $\mathscr{X}_0$. In the following, we treat the autonomous case of the invariant solvability problem, in the absence of exogenous and control inputs.

5.4.2. BACKWARD REACHABILITY

This subsection is an extension of the necessary solvability conditions provided in Section 5.3.1 for an autonomous single-implicit MP-LPV systems, (5.19) with $A_0(p(\cdot)) := A_0(p^{(\eta)}(\cdot))$ for $\eta \geq 1$. We say that the system dynamics is autonomous if it does not depend on the exogenous and control input signals $z(\cdot)$ and $u(\cdot)$ respectively. We treat the problem of invariant solvability for the general case of a single-implicit MP-LPV system with $A_0(p(\cdot)) := A_0(p^1(\cdot))$. The section utilises tools from piecewise affine analysis as detailed in Section 2.4.3. Recall that the maximal positively invariant set for an autonomous dynamics is defined as the largest set containing all positively invariant sets of the dynamics contained in a given set (see Definition 2.4.11).

We intend to compute the maximal positively invariant set contained in Λ_p (derived in Theorem 5.3.1) such that the solution exists for all $k \in \mathbb{N}$. Equivalently, we compute the largest set $\Omega_p \subseteq \Lambda_p$ such that there exists a unique trajectory $p^{(1)}(k) \in \Lambda_p$ for all $k \in \mathbb{N}$, $k > M$, if $p^{(1)}(M) \in \Omega_p$.

Lemma 5.4.1 (Single-implicit to explicit MP-LPV)**.** *Consider an autonomous single-implicit MP-LPV system in* (5.19) *with* $A_0(p(\cdot)) := A_0(p^{(\eta)}(\cdot))$ *for* $\eta = 1$. *Assume a counter* $k \in \mathbb{N}$ *such that* $p^{(1)}(k) \in \Lambda_p$*, as defined in* (5.31) *and* (5.32). *Then the explicit state-solution of the single implicit MP-LPV system for the fixed* $k \in \mathbb{N}$ *is given as*

$$x(k) = A_0^{\star}(p^{(1)}(k)) \otimes \bigoplus_{\mu=1}^{M} \big(A_\mu\big(p^{(\mu)}(k)\big) \otimes x(k-\mu)\big), \tag{5.39}$$

where

$$A_0^{\star}(p^{(1)}(k)) = \bigoplus_{i=0}^{n-1} A_0^{\otimes^i}(p^{(1)}(k)). \tag{5.40}$$

Proof. We first note that if $p^{(1)}(k) \in \Lambda_p$, then the largest max-plus eigenvalue of $A_0(p^{(1)}(k))$ is non-positive and a unique state-solution to the single implicit MP-LPV system exists due to Lemma 5.3.1 and Theorem 5.3.1.

In particular, the matrix $A_0^{\star}(p^{(1)}(\cdot))$ can be readily expressed in a closed form (5.40) using the result presented in Lemma 2.2.5 for an arbitrary $p^{(1)}(\cdot) \in \Lambda_p$.

The preceding observation allows us to employ the result of Lemma 2.2.16 to solve an implicit system of equations in max-plus algebra. The state-solution to the single-implicit MP-LPV system is then given by (5.39). This completes the proof. ■

We note that the formulation (5.40) of the matrix $A_0^\star(p^{(1)}(\cdot))$ is a max-min-plus-scaling expression (Definition 2.4.7) in the parameter vector $p^{(1)}(\cdot)$. Hence, the system dynamics of an autonomous single-implicit MP-LPV system, derived from (5.19), can be expressed as an autonomous MMPS system evolving over a discrete counter $k \in \mathbb{N}$, $k \geq M$:

$$p^{(1)}(k+1) = f(p^{(1)}(k)), \quad p^{(1)}(k) \in \Lambda_p, \tag{5.41}$$

where $f : \mathbb{R}^{n_\mathrm{p}} \to \mathbb{R}^{n_\mathrm{p}}$ is an MMPS function and Λ_p is obtained from (5.32).

We recall here that the class of MMPS systems coincides with continuous piecewise affine systems [75, Proposition 2.5]. Therefore, the results of Section 2.4.3 can be applied to compute the maximal positively invariant set for the dynamics (5.41) contained in Λ_p. Here, we adapt Algorithm 1 to compute the maximal positively invariant for a given single-implicit MP-LPV system.

Algorithm 2 Computation of $\mathcal{O}_\infty(\mathbb{X})$.

Input: Dynamics (5.41), Constraint set $\mathbb{X} = \Lambda_\mathrm{p}$
Output: $\mathcal{O}_\infty(\mathbb{X})$

$k \leftarrow -1$
$\Psi^{(0)} \leftarrow \mathbb{X}$
while $\Psi^{(k+1)} \neq \Psi^{(k)}$ **do**
  $k \leftarrow k+1$
  $\Psi^{(k+1)} \leftarrow \mathcal{Q}(\Psi^{(k)}) \cap \Psi^{(k)}$ ▷ (5.42)
end while
$\mathcal{O}_\infty(\mathbb{X}) \leftarrow \Psi^{(k)}$

As discussed in Section 2.4.3, we are required to compute the pre-image set, i.e. the set of states that map into a given set $\Psi \subseteq \mathbb{R}^n$ under the dynamics:

$$\mathcal{Q}(\Psi) = \left\{ p \in \Lambda_\mathrm{p} \;\middle|\; f(p) \in \Psi \right\}. \tag{5.42}$$

We recall that an MMPS dynamics can be rewritten as a piecewise affine dynamics [75]. Then there exists a polyhedral partition $\{\Upsilon_j\}_{j \in \underline{J}}$ of the state-space Λ_p, such that the MMPS dynamics in (5.41) can be equivalently written as:

$$p^{(1)}(k+1) = T_j \cdot p^{(1)}(k) + t_j, \quad p^{(1)}(k) \in \Upsilon_j \triangleq \left\{ p^{(1)}(\cdot) \;\middle|\; S_j \cdot p^{(1)}(\cdot) \leq s_j \right\}, \quad j \in \underline{J}. \tag{5.43}$$

Lemma 5.4.2 (Pre-image of piecewise affine systems)**.** *Consider the piecewise affine dynamics in* (5.43)*. Let* Ψ *be defined as a union of polyhedra. Then the pre-image set* $\mathcal{Q}(\Psi)$*, as given in* (5.42)*, is again a union of polyhedral sets.*

Proof. We prove the lemma for the case when $\Psi = \bigcup_{i=1}^m \Lambda_{\mathrm{p},i}$, where $\Psi_i = \Lambda_{\mathrm{p},i}$ is defined in (5.31). The proof follows along the lines of [189, Theorem 4].

The required pre-image set (5.42) for (5.43) can be computed as (see Lemma 2.4.6)

$$\mathcal{Q}(\Psi) = \bigcup_{i \in \underline{m}} \bigcup_{j \in \underline{J}} \mathcal{Q}_j(\Psi_i). \tag{5.44}$$

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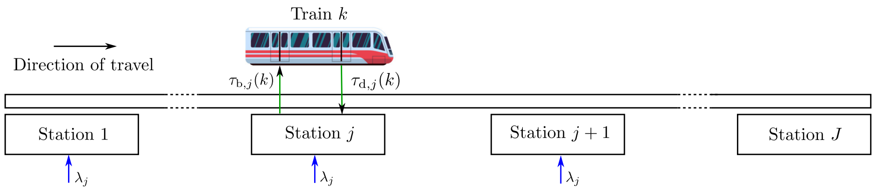


Figure 5.1: Unidirectional urban railway line with J stations and K trains. The boarding and disembarking times for each train $k \in \underline{K}$ at any given station $j \in \underline{J}$ are denoted as $\tau_{\mathrm{b},j(k)}$ and $\tau_{\mathrm{d},j(k)}$, respectively. Passengers arrive at a constant rate λ_j per unit time at all the stations. The passengers alight the train at a constant rate b per unit time.

where

$$\mathscr{Q}_j(\Psi_i) = \{p \in \Lambda_\mathrm{p} \mid p^+ = f(p) \in \Psi_i,\ p \in \Upsilon_j\} \tag{5.45}$$

$$= \{p \mid p^+ = f(p) \in \Psi_i\} \cap \{p \mid p \in \Upsilon_j\} \tag{5.46}$$

$$= \left\{ p \,\middle|\, \begin{bmatrix} S_\mathrm{a} \cdot G_i \cdot T_j \\ R_i \cdot T_j \\ S_j \end{bmatrix} \cdot p \le \begin{bmatrix} b_\mathrm{a} - S_\mathrm{a} \cdot g_i - S_\mathrm{a} \cdot G_i \cdot t_j \\ r_i - R_i \cdot t_j \\ s_j \end{bmatrix} \right\}. \tag{5.47}$$

Finally, since each non-empty set $\mathscr{Q}_j(\Psi_i)$ is a polyhedron, the set $\mathscr{Q}(\Psi)$ is a union of polyhedra. ■

Now we note that Algorithm 2 generates a non-increasing sequence of sets $\Psi^{(k+1)} \subseteq \Psi^{(k)}$. If the sequence of sets $\Psi^{(k)}$ converges, we obtain the maximal positively invariant set $\mathscr{O}_\infty(\Lambda_\mathrm{p})$ for the system dynamics (5.43). Moreover, if $\Psi^{(k)} = \emptyset$ for a finite $k \in \mathbb{N}$ then $\mathscr{O}_\infty(\Lambda_\mathrm{p}) = \emptyset$. In general, there can exist an infinite sequence of sets $\{\Psi^{(k)}\}_{k\in\mathbb{N}}$. Therefore, the algorithm might not converge.

Theorem 5.4.1. *Consider Algorithm 2 such that there exists a finite $j \in \mathbb{N}$ for which $\Psi^{(j+1)} = \Psi^{(j)}$. Then $\Psi^{(j)}$ is a maximal positively invariant set for the dynamics (5.41).*

Proof. The proof is trivial [188, Lemma 2]. ■

The set $\mathscr{O}_\infty(\Lambda_\mathrm{p})$ obtained from finite termination of Algorithm 2, satisfies the condition for invariant solvability for the given autonomous single implicit MP-LPV system.

Finally, we note that the chosen approach to obtain the positively invariant sets can also be extended to non-autonomous single implicit MP-LPV systems (5.43) analogously to the conventional framework presented in [132, 188] for piecewise affine systems.

5.5. CASE STUDY

In this section we first derive the dynamics for a unidirectional urban railway system as an implicit MP-LPV system. Then we study the application of the proposed approach to study the problem of existence and uniqueness of trajectories using the obtained model.

5.5.1. SYSTEM DESCRIPTION

Consider an urban railway line as given in Figure 5.1 with J stations and K trains with unlimited capacities. Each station can only accommodate a single train and the trains are not allowed to overtake along the direction of travel.

We assume there is no timetable and there are no input delays via $u(\cdot)$. Each train $k \in \underline{K}$ arrives empty at station 1 with a minimum headway interval of τ_0. It stops at each station $j \in \underline{J}$ for passengers to first disembark, and then departs when all passengers on the platform have boarded the train. We denote the arrival and departure time of a train k at station j by $a_j(k)$ and $d_j(k)$, respectively. The dwell time at each station is the sum of the time for disembarking and boarding the train. In this model the boarding time is given as $\tau_{\mathrm{b},j}(\cdot) = c_j\,(d_j(k) - d_j(k-1))$, with $c_j = \lambda_j / b$ where λ_j is the number of passengers entering the station per second and b is the number of people that can enter the train per second. This dependence leads to an implicit system of equations.

The number of passengers in a train k arriving at station j is therefore proportional to the differences of the departure times of train k and $k-1$ from preceding stations $i < j$. We also assume that the number of people leaving any train at a particular station j is a fixed fraction $\beta_j \in [0,1]$, $j \in \underline{J}$, of the number of the passengers in the train along with $\beta_0 = 0$ and $\beta_J = 1$.

We now show that the disembarking time $\tau_{\mathrm{d},j}(\cdot)$ is an affine function of the state. Let the number of passengers in a train $k \in \underline{K}$ when leaving station $j \in \underline{J}$ be denoted as $\rho_j(k)$ with $d_j(0) = 0$ for all $j \in \underline{J}$. Recall that λ_j is the number of passengers entering the station per second, b is the number of people that can enter the train per second, and β_j is the fraction of the number of people on a train that disembark at station j. We have

$$
\begin{aligned}
\rho_1(k) &= \lambda_1(d_1(k) - d_1(k-1)) \\
\rho_2(k) &= (1-\beta_2)\lambda_1(d_1(k) - d_1(k-1)) + \lambda_2(d_2(k) - d_2(k-1)) \\
\rho_3(k) &= (1-\beta_3)(1-\beta_2)\lambda_1(d_1(k) - d_1(k-1)) + (1-\beta_3)\lambda_2(d_2(k) - d_2(k-1)) \\
&\quad + \lambda_3(d_3(k) - d_3(k-1)) \\
&\ \ \vdots \\
\rho_j(k) &= \sum_{i=1}^{j-1} \Big(\prod_{m=i+1}^{j} (1-\beta_m)\lambda_i(d_i(k) - d_i(k-1)) \Big) + \lambda_j(d_j(k) - d_j(k-1)).
\end{aligned}
$$

Therefore, the disembarking time, $\tau_{\mathrm{d},j} = \beta_j \rho_{j-1}$, is affine in the departure times $d_i(k)$, $i < j$. A similar result can be obtained for the passenger boarding times $\tau_{\mathrm{b},j}$.

We assume that the trains have unlimited capacity. The minimum dwell time $\tau_{\min}$ at each station, the running times $\tau_{\mathrm{r},j}$ from station $j-1$ to j, and the minimum headway time τ_{h} between successive train are assumed to be fixed. The evolution of the discrete-event system can then be modelled in max-plus algebra as:

$$
\begin{aligned}
a_j(k) &= \max\big(d_{j-1}(k) + \tau_{\mathrm{r},j}, d_j(k-1) + \tau_{\mathrm{h}}\big) \\
d_j(k) &= a_j(k) + \max\big(\tau_{\mathrm{d},j}(d(k), d(k-1)) + \tau_{\mathrm{b},j}(d(k), d(k-1)), \tau_{\min}\big) \\
&= a_j(k) + \gamma_j\,(d(k), d(k-1))\,.
\end{aligned}
\tag{5.48}
$$

for $j = 2, \ldots, J$ along with $a_1(k) = \tau_0 k$ and $d_1(k) = a_1(k)/(1-c_1)$ for $k \in \underline{K}$. Here, the fixed parameter τ_0 represents the constant rate of arrival of trains at the first station. Note that here $\gamma_j\,(d(k), d(k-1))$ is dependent only on $d_i(k)$ for $i < j$.

5.5.2. ANALYSIS

Table 5.1: Urban railway fixed parameters, based on [220].

Property		Value
Minimum dwell time,	$\tau_{\min}$ [s]	30
Minimum headway time,	τ_{h} [s]	90
Passenger boarding rate,	b [passenger/s]	1
Number of trains,	K	4
Number of stations,	J	8

The urban railway system as derived in (5.48) can be expressed as a doubly implicit MP-LPV system (5.19) with $M = 1$:

$$x(k) = A_0(p(k)) \otimes x(k) \oplus A_1(p^{(1)}(k)) \otimes x(k-1). \tag{5.49}$$

The states are defined as $x(k) = \left[a_1(k), d_1(k), a_2(k), d_2(k), \ldots, a_J(k), d_J(k)\right]^\intercal \in \mathbb{R}^{2J}$. The varying parameter is then defined as $p(k) = [d^\intercal(k), d^\intercal(k-1)]^\intercal$ and partitioned such that $p^{(1)}(k) = d^\intercal(k-1)$. The corresponding system state matrices can then be obtained for system description (5.48) as[13]

$$\begin{bmatrix} a_1 \\ d_1 \\ \vdots \\ d_{j-1} \\ a_j \\ d_j \\ \vdots \\ d_{J-1} \\ a_J \\ d_J \end{bmatrix}(k) = \begin{bmatrix} \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ c_1/(1-c_1) & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \cdots & * & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \cdots & \tau_{\mathrm{r,j}} & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \cdots & \varepsilon & \gamma_j(\cdot) & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \cdots & * & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon & \cdots & \tau_{\mathrm{r,J}} & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon & \gamma_J(\cdot) & \varepsilon \end{bmatrix} \otimes \begin{bmatrix} a_1 \\ d_1 \\ \vdots \\ d_{j-1} \\ a_j \\ d_j \\ \vdots \\ d_{J-1} \\ a_J \\ d_J \end{bmatrix}(k)$$

$$\oplus \begin{bmatrix} \tau_0 & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \tau_{\mathrm{h}} & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \tau_{\mathrm{h}} \\ \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \end{bmatrix} \otimes \begin{bmatrix} a_1 \\ d_1 \\ \vdots \\ d_{j-1} \\ a_j \\ d_j \\ \vdots \\ d_{J-1} \\ a_J \\ d_J \end{bmatrix}(k-1).$$

Firstly, we note that the system description is structurally finite and the finite matrix

[13] The $*$ entries in the system matrices denote finite elements different from ε.

Table 5.2: Urban railway sample operating parameters, based on [220].

Station j	Passenger arrival rate, λ_j [passenger/s]	Passenger alighting proportion, β_j	Minimum running time, $\tau_{r,j}$ [s]
1	0.3	0	-
2	0.05	0.05	90
3	0.3	0.3	90
4	0.4	0.38	120
5	0.04	0.4	130
6	0.4	0.32	130
7	0.4	0.5	90
8	0	1	90

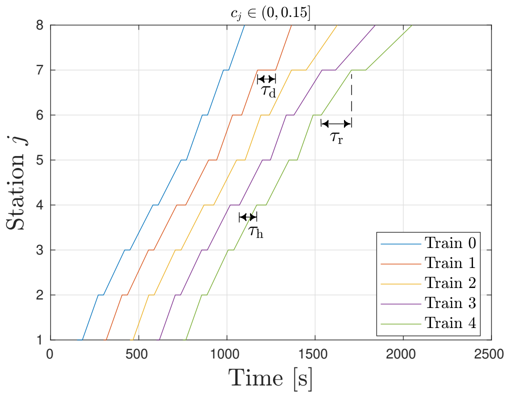
elements are continuous piecewise affine in the chosen parameter. Therefore, Assumption 5.2.1 is satisfied. The system matrix $A_0(\cdot)$ is strictly lower triangular. Moreover the function $\gamma_j(\cdot)$ are only dependent on d_{j-1} for each $j \in \underline{J}$. It can also be noted that the state updates in (5.48) can be expressed as (5.34) because affine functions with non-negative coefficients ($\tau_{d,j}(\cdot)$ and $\tau_{b,j}(\cdot)$) are a subclass of max-plus-scaling functions (2.48). Therefore, the system description satisfies Assumption 5.3.1.

We now consider the solvability of (5.48) as proposed in Section 5.3.3. The state equations (5.48) can be restated as (5.35). We find that the coefficients in (5.35) are given as $\alpha_N^{(j)} = 0$ and $\alpha_N^{(j+J)} = \lambda_j / b$ for $j \in \underline{J}$ with $N = 1$. These values are all smaller than one if the number of people entering the station per second is smaller than the number of people that can enter the train per second for every station. Therefore, the resulting doubly implicit MP-LPV system is solvable, as in Definition 5.3.1, if $c_j = \lambda_j / b < 1$ for all $j \in \underline{J}$. This follows from Theorem 5.3.2.

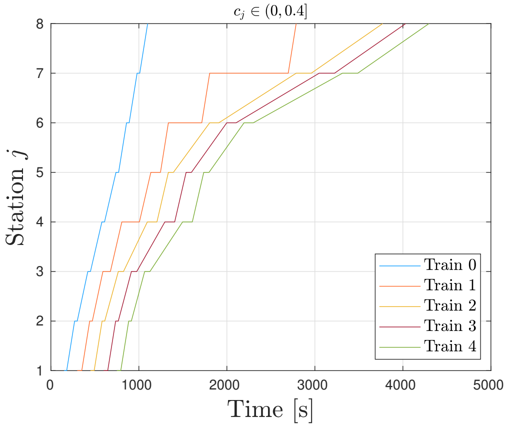
To highlight the numerical effect of variation of the parameter $\{c_j\}_{j\in\underline{J}}$ on the train schedule, we fix the unidirectional urban railway line with eight stations ($J = 8$) and four trains ($K = 4$) in the MP-LPV framework. A set of nominal values of various fixed and operating parameters are taken from [220, §5.6] and recalled in Tables 5.1 and 5.2. The system is initialised with train 0 arriving as $a_1(0) = 0$, $a_j(0) = d_{j-1}(0) + \tau_{r,j}$ for $j = 2, \ldots, J$ and departing as $d_j(0) = a_j(0) + \tau_0$ for $j \in \underline{J}$. Here, the dwell time of train 0 at each station is fixed at $\tau_0 = 150$ s. The trains arrive at the first station at a constant rate of $\tau_0 = 150$ s.

We recall that the function $f(\cdot)$ describing the system dynamics is monotone in the state: $x^{(1)} \geq x^{(2)}$ implies $f(x^{(1)}, \cdot) \geq f(x^{(2)}, \cdot)$. The implicitness in the system of equations (5.48) can therefore be resolved numerically using Kleene iterations: For a given $k \in \underline{K}$, let $x^{(0)}(k) = x^{(0)}(k-1)$. Then compute $x^{(i+1)}(k) = f(x^{(i)}(k), p^{(1)}(k))$ for $i = 0, 1, 2, \ldots$. It is noted that if the iteration terminates, $x^{(m+1)}(k) = x^{(m)}(k)$ for a finite m, the limit $x^{(m)}(k)$ is the least solution to the state equation at the given $k \in \underline{K}$.The procedure is, in general, inefficient [64]. More efficient approaches for fixed point computation of monotone functions can be found in [4, 64, 71, 81].

It can now be observed from Figure 5.2 that an increase in the number of passengers arriving at a given station with respect to the number of passengers that can enter the train in unit time can result in diverging state trajectories. Consequently, the dwell times



(a)



(b)

Figure 5.2: The train schedules for an urban railway line with 4 trains and 8 stations [220, §5.6] obtained by simulating the MP-LPV model (5.48). The effect of increasing the upper limit on the parameter c_j from (a) 0.15 to (b) 0.4 can be observed on the effective dwell times τ_d, and running times τ_r.

of the train at the stations increase from their minimum value (τ_{min}), and the travel times between stations $d_j(k) - d_{j-1}(k)$ increase along with the total travel time $a_J(k) - d_1(k)$. As the parameter c_j approaches 1, the system looses the solvability property.

5.6. CONCLUSIONS

In this chapter, we have presented linear parameter-varying models of discrete event systems in max-plus algebra analogously to the conventional linear parameter-varying systems. We have shown that for the case of continuous piecewise affine dependence of the system matrices on the varying parameter, the class of max-plus linear parameter-varying systems is equivalent to max-min-plus-scaling systems. We have first considered the problem of solvability to ensure existence and uniqueness of the solution to implicit state equation at a given event step. We have provided a necessary condition for the solvability problem for the restricted case of autonomous single-implicit max-plus linear parameter-varying systems. A sufficient condition that ensures the existence and uniqueness of the state trajectories of double-implicit max-plus linear-parameter varying systems has also been proposed. The state trajectories can then be evaluated using tools already studied in the literature. Subsequently, we have studied the invariant solvability problem for autonomous max-plus linear parameter-varying systems that ensure existence and uniqueness of the system trajectory over a possibly infinite event horizon by restricting the state space. We have shown how tools from computational geometry for piecewise affine systems can be used to determine the maximal region of the state space that is invariant to the system dynamics. Finally, we have motivated and illustrated our modelling and solvability approach using an example of a unidirectional urban railway system.

6

Conclusions & Future Work

In this chapter, we summarise the key learning points and major contributions of this thesis. We also provide some interesting directions for future research.

6.1. Overview of conclusions

In the field of discrete-event systems, synchronisation and linearity (in the sense of max-plus algebra) play an important role in deriving tools for performance evaluation and control problems under the max-plus-algebraic system theory. This thesis has focused on developing mathematical theories for the analysis of discrete-event systems in the max-plus algebra when synchronisation can be broken and linearity is lost due to the presence of scheduling and ordering variables. In particular, we have treated problems related to modelling, stability, and reachability of discrete-event systems modelled in the max-plus algebra. Firstly, we pointed out the importance of the assumptions on monotonicity and additive homogeneity of the discrete-event system dynamics in the max-plus linear systems theory. Through various results presented in this thesis as listed below, we have provided insights that help overcome these assumptions to allow extension of the max-plus algebraic tools to a broader class of discrete-event systems. In particular, the class of discrete-event systems studied in this thesis exhibit the phenomenon of choice apart from synchronisation. Another important realisation was that the principles and approach for modelling and analysis of conventional time-driven (cyber-physical) systems need not be relinquished. We, therefore, have relied heavily on classical systems theory in shaping the analysis tools developed in this thesis.

In what follows, we first provide a concise list of main contributions and key learning points of this thesis:

1. We have proposed a novel class of max-plus-algebraic hybrid automata, obtained as a novel reformulation of conventional hybrid automata of [156], extending the class of switching max-plus linear systems of [216]. We have established a formal relationship between a max-plus automaton and a switching max-plus linear system. Moreover, we have found that the class of systems modelled by a max-plus

automaton of [85] is contained in the class of systems modelled by switching max-plus linear systems;

2. We have formulated a max-plus Lyapunov stability theory for discrete-event systems analogous to the Lyapunov stability theory for conventional time-driven systems;

3. We have presented a novel class of max-plus linear parameter-varying systems for modelling parametric discrete-event systems. We also formulated an analysis framework for evaluating existence and uniqueness of the trajectories of a max-plus linear parameter-varying system using tools from the max-plus algebra and piecewise-affine systems.

We now classify and elaborate on the main contributions based on the chapters:

Modelling. In Chapter 3, we have introduced the novel class of max-plus-algebraic hybrid automata as a unifying modelling framework for obtaining hybrid models of discrete-event systems in the max-plus algebra. The proposed modelling framework is a reformulation of the conventional hybrid automata framework of [156] and an extension of the class of switching max-plus linear systems of [216]. The hybrid phenomena due to the interaction of continuous-valued and discrete-valued dynamics have been identified and a modelling hierarchy has been generated. We have formally established equivalence relationships between the proposed max-plus-algebraic hybrid automata and the classes of switching max-plus linear systems and max-plus automata. We have noted the difficulties arising in the direct comparison of switching max-plus linear systems and max-plus automata due to incompatible definitions of state space. Therefore, we have resorted to the behavioural framework where the similarity is studied for the collection of generated input-output trajectories of the systems.

We want to remark that the obtained equivalence relationships allow bridging the knowledge gap between hybrid systems theory and weighted automata theory. This in particular will allow interpreting and solving problems for max-plus automata using tools for max-plus-algebraic hybrid systems developed in Chapters 4 and 5 of this dissertation.

Stability. In Chapter 4, we have developed a framework for studying stability problems for the continuous part of a max-plus-algebraic hybrid system. On the one hand, the stability notions have been carried over from general discrete-event systems described in the max-plus algebra. On the other hand, the stability framework has been developed analogously to that of time-driven switched systems defined on normed vector spaces [150]. The key observation has been that the Hilbert projective (semi-)norm provides a convenient substitute for a vector norm to study the stability notions for discrete-event systems defined in the max-plus algebra. In doing so, we have justified the assumptions on continuity and additive homogeneity of the mode dynamics and have relaxed the assumption on monotonicity. We have provided necessary and sufficient conditions for evaluating stability properties of autonomous discrete-event systems under the assumption of additive homogeneity and continuity of the dynamics.

We have showcased the capability of the proposed stability framework by studying boundedness and convergence of the trajectories of open-loop switching max-plus linear systems. Firstly, the proposed approach allows performance evaluation (as ultimate bounds on the makespan) of discrete-event systems modelled as switching max-plus linear systems. Secondly, we have shown how the associated optimisation problems (such as computation of minimal positively invariant sets and their region of attraction) can be reformulated as mixed-integer linear programs. The key observation was that monotone and additively homogeneous functions (including the ones obtained from Kleene star matrices) can be used to obtain max-plus Lyapunov functions for analysis.

Parametric DES. In Chapter 5 of this dissertation, we have dealt with problems of existence and uniqueness of trajectories of parametric descriptions of discrete-event systems in the max-plus algebra. In doing so, we have introduced the novel class of max-plus linear parameter-varying systems analogously to linear parameter-varying systems in conventional algebra. Under the assumption of piecewise-affine dependence of the finite elements of the system matrices on the varying parameters, we have established an equivalence relationship with the class of max-min-plus-scaling systems.

A max-plus linear description of discrete-event systems is usually implicit in the state of the system. We have pointed out that this implicitness in the state cannot completely be resolved for max-plus linear parameter-varying systems using the tools of max-plus algebra. Therefore, we have provided necessary and sufficient conditions for the existence and uniqueness of trajectories (or solvability) of a max-plus linear parameter-varying system. To ensure this property over the entire event horizon, we have extended the tools from piecewise-affine systems analysis to obtain positively invariant sets where a unique solution to the state equation always exists. The key observation is that the successful treatment of the solvability problem for max-plus linear parameter-varying systems requires a combination of tools from the max-plus algebra and piecewise-affine analysis. Finally, the effectiveness of the proposed theory has been used for assessing solvability property of the model of a uni-directional urban railway system.

6.2. SUGGESTIONS FOR FUTURE WORK

There are still a lot of opportunities for improvement in the proposed system-theoretical analysis framework for discrete-event systems in the max-plus algebra. In what follows, we propose several topics for further investigation and research directions to extend the results of the dissertation:

Modelling and equivalences. A considerable amount of research focuses on establishing equivalence relationships between different modelling classes to study control synthesis and verification problems [207]. The formal analysis approach in hybrid systems concerns checking whether a given hybrid system satisfies certain specifications. The relationship with finite-state systems can then be exploited to study verification and control problems for large infinite-state systems. Such relationships between finite-state and infinite-state systems also provide a framework to trade off tractability (of algorithms) with modelling power, a recurring issue in the hybrid systems literature [14, 207].

Firstly, the reformulation of the switching max-plus linear framework into the max-plus-algebraic hybrid automata framework (Chapter 3) allows borrowing abstraction procedures from conventional hybrid automata [14] to study verification problems for discrete-event systems. Secondly, conditions for finite-state abstraction would help to identify the subclass of max-plus-algebraic hybrid automata that correspond to safe timed Petri nets, readily modelled by max-plus automata [95, 136, 145]. This would, at the same time, allow exploiting tools from the supervisory control theory of Ramadge and Wonham [191, 206] to study reachability and control synthesis problems for switching max-plus linear systems.

Stability and stabilisability. There is still much work to be done before a complete Lyapunov stability framework for timed discrete-event systems comes to fruition. We have developed a max-plus Lyapunov stability framework for the continuous-valued part of max-plus-algebraic hybrid automata under arbitrary switching in Chapter 4.

In many discrete-event applications, the switching sequence is not completely arbitrary but constrained based on the (hybrid) states, and on exogenous and control inputs (see [85, 216]). The uniform stability notions presented in Chapter 4 can be conservative for practical applications with constrained switching sequences. A Lyapunov function or a positively invariant set common to all subsystems might not even exist in the presence of unstable subsystems. In this light, it is important to extend the stability analysis approach of Chapter 4 by possibly employing multiple Lyapunov functions and also to identify interesting classes of switching signals as for conventional time-driven systems [150, Chapter 3]. Logical constraints on the switching sequence appear in several discrete-event system applications [85]. In presence of logical constraints on switching sequences, the admissible switching sequences can be encoded using a finite automaton [185]. Therefore as a subsequent step, the max-plus Lyapunov framework can gain considerably from the stability theory presented in [6, 125, 183, 185]. The authors of [6, 125, 183, 185] provide several constructive theorems and tools to study stability of conventional time-driven switched systems. The extension of max-plus Lyapunov stability theory along the lines of [6, 125, 183, 185] for control and performance evaluation of discrete-event systems is an interesting direction of research.

A consolidation of the max-plus Lyapunov framework with language-theoretic Lyapunov framework presented in [142, 181] to study stability and stabilisability of the discrete-valued part of the dynamics is another interesting direction of research.

Parametric modelling and analysis of discrete-event systems. We showed that the proposed max-plus linear parameter-varying systems are equivalent to max-min-plus-scaling systems, which are in turn equivalent to continuous piecewise-affine systems. Firstly, the description can be extended to include switching behaviour in the system where the structure of the underlying incidence graph is allowed to change over events. This could possibly lead to an equivalence with discontinuous piecewise-affine systems.

Secondly, we have proposed methods to analyse existence and uniqueness of trajectories of max-plus linear parameter-varying systems (Chapter 5). The methods based on piecewise-affine analysis can suffer from exponential computation time. It is then interesting to find conditions for which the presented algorithm either terminates at a solution in finitely many steps or provides an approximate solution subject to error bounds. The analysis procedure can gain immensely from interval analysis [44, 153]. For instance, convex polyhedra can be approximated using union of hyper-rectangles [23] that are amenable to representation by intervals [174] and also difference-bound matrices [53]. The analysis of interval descriptions in the max-plus algebra is an ongoing field of research [44, 113, 153]. In particular, the tools can be employed to obtain (numerically) efficient representations of the pre-image of uncertain max-plus linear maps [52, 53, 176].

The application of the obtained tools to closed-loop feasibility (constraint satisfaction) analysis and robustness analysis of discrete-event systems represented by max-min-plus-scaling systems is an interesting direction of research. The underlying theory can also be directly extended to obtain robust and controlled invariant sets of max-plus linear parameter-varying systems analogously to piecewise-affine systems [132].

Inter-chapter research directions. Finally, we present directions for future work laid out by exploiting relations across the different chapters of this dissertation.

Lyapunov theory plays an important role in designing effective tools for control system design and analysis of conventional time-driven and hybrid systems [51, 55, 126, 146, 186]. The development of the counterpart max-plus Lyapunov theory to study design and analysis problems for discrete-event systems in the max-plus algebra is a long-term goal.

Firstly, Lyapunov theory has been used to obtain finite-state abstraction of switching systems locally (in a given compact set) [102]. This, for instance, can be utilised to obtain stabilising switching laws. Secondly, the results of Chapter 4 can be extended in this direction to study robustness to disturbances, as input-to-state stability and dissipativity, in discrete-event systems. The max-plus Lyapunov stability framework presented in Chapter 4 forms an excellent starting point to address such problems. We have also found that parametric descriptions of discrete-event systems in the max-plus algebra can have states that model quantities (such as number of passengers present at a platform). The interaction of states/parameters representing both timed and untimed quantities has implications for modelling as well as stability analysis of discrete-event systems. An interesting opportunity arises when the discrete-event system can be compartmentalised such that any individual component involves only quantities with the same units. For example, we can compartmentalise the model of a uni-directional railway system to obtain one component consisting of only timing variables and another component modelling the flow of passengers. The stability can then be studied by analysing individual components along with the interactions among them.

P-time event graphs form an extension of timed event graphs where holding times are allowed to take values from an interval in a non-deterministic fashion [205]. We have pointed out (in Chapter 5) how these systems can be modelled in the proposed max-plus linear parameter-varying systems framework. Similarly, interval-weighted automata have been proposed as an extension of max-plus automata for modelling the choice phenomena observed in P-time Petri nets [139]. Therefore, we can aim at finding abstraction procedures that relate the proposed class of max-plus linear parameter-varying systems with the classes of P-time event graphs, P-time Petri nets, and interval-weighted automata. For instance, the tools for performance evaluation of P-time event graphs can be exploited to study performance evaluation of max-plus linear-parameter-varying systems via abstraction.

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