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Lv, Maolong; De Schutter, Bart; Yu, Wenwu; Baldi, Simone

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Adaptive Asymptotic Tracking for a Class of Uncertain Switched Positive Compartmental Models with Application to Anesthesia

Maolong Lv, Bart De Schutter, *Fellow, IEEE*, Wenwu Yu, *Senior Member, IEEE*, and Simone Baldi, *Member, IEEE*

***Abstract*—This brief work addresses and solves the adaptive asymptotic tracking for a class of uncertain switched positive linear dynamics (also known in literature as compartmental models) subject to dwell time constraints. Compared to the state of the art, the innovative feature of this method is to attain for the first time asymptotic set-point tracking, while guaranteeing nonnegativity of the systems states. To achieve asymptotic tracking, an interpolated Lyapunov function is adopted, which is non-increasing at the switching instants and decreasing in two consecutive switching instants. Such Lyapunov function results in a novel adaptive law with time-varying adaptive gains, as opposed to state-of-the-art laws with fixed positive adaptive gains. The developed design is applicable to classes of compartmental systems compatible with those proposed in literature: an example involving the infusion of anesthesia is conducted to show that the proposed method can achieve better performance than existing methods.**

***Index Terms*—Switched positive linear dynamics, compartmental systems, dwell-time, adaptive asymptotic control.**

I. Introduction

THE past decade has seen several advances in the field of switched dynamical systems and their applications [1]–[15]. Switched dynamical systems switch among a family of subdynamics using a switching rule, which can be state-driven or time-driven [16], [17]. In some cases, the states of the systems must remain nonnegative, then we have the special class of switched positive systems, which has attracted much attention thanks to its tremendous number of applications, such as biology systems [18], compartmental model [19], traffic flows systems [20], and communication networks [21]. In all these systems, the positivity of system states is always preserved as long as the input and initial states are positive.

Stability and stabilization of switched positive dynamics has been deeply researched. More specifically, a discretized copositive Lyapunov function has been adopted to conduct stability analysis under minimum dwell time constraint in [20]. In [4], a more general class of copositive polynomial Lyapunov function is proposed, making existing classical Lyapunov functions to be special cases of such method. Necessary and sufficient stability criteria have been investigated in [22] under average dwell time switching. More recently, an improved stability condition and corresponding delay control have been proposed in [23], which relaxes the notion of average dwell time. Exponential stability has been studied in [24] for two-dimensional and higher-order switched positive systems based on copositive Lyapunov functions. More studies involving switched positive systems can be found in [25] and in the references therein.

It has to be stressed that most of the aforementioned stability or stabilization approaches assume perfectly known dynamics, i.e. they ignore system parametric uncertainty which is virtually present in all dynamical systems. It is well established that when uncertainty is large adaptive methods are a viable option (in place of fixed-gain robust control [26]). For non-switched positive systems, adaptive control methods have been studied for a special class of dynamics known in literature as compartmental models [19], [27]–[29]. Such systems can be used to represent many medical systems where high uncertainty poses several control challenges. For example, surgery or the infusion of anesthetic drugs are subject to many uncertainties and disturbances coming from the patient. Such dynamics can be often modelled by switching laws, i.e. switching among certain regimes. However, to the best of the authors' knowledge, adaptive switching control for compartmental systems has not been studied. In fact, it is worth remarking that, while some adaptive methods for switched systems have appeared in [30], [31], none of these methods can guarantee the systems states to remain nonnegative. Therefore, such methods could not be applied in all the applications (biology, traffic flows, communication networks, etc) where positivity of system states must be preserved. In other words, to the authors' best knowledge, the development of adaptive methods for uncertain switched positive dynamics or uncertain switched compartmental systems is completely open.

Inspired by the above discussion, this work proposes an adaptive design for a class of switched positive linear dynamics, whose main innovations are:

1) In contrast with non-adaptive methods for switched positive linear systems [4], [20], [22]–[24], a direct adaptive switched method is investigated for the first time, so as to

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M. Lv, and B. De Schutter are with the Delft Center for Systems and Control, Delft University of Technology, Mekelweg 2, Delft 2628 CD, The Netherlands (email: M.Lyu@tudelft.nl and maolonglv@gmail.com; B.DeSchutter@tudelft.nl).

S. Baldi is with the department of Mathematics, Southeast University, Nanjing 210096, China, and also with the Delft Center for Systems and Control, Delft University of Technology, Mekelweg 2, Delft 2628 CD, The Netherlands (email: s.baldi@tudelft.nl).

W. Yu is with the department of Mathematics, Southeast University, Nanjing 210096, China (email: wwyu@seu.edu.cn).

adapt the control gains according to system uncertainty and switching.

2) In contrast with adaptive controllers for switched systems [30], [31], the proposed method can guarantee that the states always remain nonnegative (for nonnegative initial states), which suggests that the strategy developed here find application in all aforementioned fields where switched positive dynamics are crucial.

3) It is well-known in the switched systems literature that asymptotic adaptive state-tracking is challenging [30], and asymptotic adaptive output-tracking is still unsolved [32]. Here, we show that, for the class of switched positive dynamics under consideration, asymptotic output-tracking (set-point tracking) can be solved via an appropriately designed interpolated Lyapunov function. As far as we know, such adaptive output-tracking is established for the first time.

4) State-of-the-art adaptive laws rely on some fixed positive adaptive gains to perform adaptation. In this work we show that the designed Lyapunov function results in a novel adaptive law with time-varying adaptive gains.

The rest of the paper is structured as follows. Section 2 gives the considered system and preliminaries. The switched adaptive control with stability analysis is designed in Section 3. In Section 4, simulation results are given, followed by the conclusions in Section 5.

Notations: We say that a vector $x \in \mathbb{R}^n$ or a matrix $\Lambda \in \mathbb{R}^{n\times m}$ is non-negative or positive if every entry of x or Λ is nonnegative or positive. This will be represented by $x \succeq 0$ and $x \succ 0$ or $\Lambda \succeq 0$ and $\Lambda \succ 0$, respectively. The symbols $\bar{\mathbb{R}}^n_+$ and $\mathbb{R}^n_+$ are used to denote the nonnegative and positive orthants of $\mathbb{R}^n$. A matrix $\Lambda \in \mathbb{R}^{n\times n}$ is said to be a Metzler matrix if its off-diagonal entries are nonnegative. This will be represented by $\Lambda \in \mathbb{M}^n$.

II. Mathematical Preliminaries

Let us consider the continuous-time switched positive system with M subsystems:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t),\ \ \sigma(t) \in \mathcal{M} := \{1, ..., M\} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is a piecewise continuous control input, and the subscript $\sigma(t)$ represents the active subsystem at time t. The switching signal $\sigma(t)$ can be represented in terms of its switching sequences $\mathcal{S} := \big\{(\kappa_0, t_0), (\kappa_1, t_1), \ldots, (\kappa_m, t_m), \ldots \big| \kappa_j \in \mathcal{M}, m \in \mathbb{N}\big\}$ with t_0 indicating the initial time. The notation (κ_m, t_m) indicates that the κ_mth subsystem is activated (i.e. $\sigma(t) = \kappa_m$) for $t \in \big[t_m, t_{m+1}\big)$. As all subsystems in (1) are positive systems, we have $A_{\sigma(t)} \in \mathbb{M}^n$ and $B_{\sigma(t)} \in \mathbb{R}^{n\times m}_+$ for all $\sigma(t) \in \mathcal{M}$. This is clarified by the following lemma.

Lemma 1 [20]: The switched linear dynamics (1) are said to be positive if and only if $A_{\sigma(t)}$ is a Metzler matrix and $B_{\sigma(t)} \succeq 0$, $\forall \sigma(t) \in \mathcal{M}$.

We consider the class considered in [27], [19], [28] and [29], where $B_{\sigma(t)} = \big[B_{\sigma(t)u},\ 0_{(n-m)\times m}\big]^T$ with $B_{\sigma(t)u} = \mathrm{diag}\big\{b_{\sigma(t)1}, ..., b_{\sigma(t)m}\big\}$ and $b_{\sigma(t)i} \in \mathbb{R}_+$ for $i \in \big\{1, ..., m\big\}$. The matrices $A_{\sigma(t)}$ and $B_{\sigma(t)}$ are uncertain in the sense that some of their entires are unknown. Define a desired target state $x_e \triangleq \big[x_d^T,\ x_u^T\big]^T$ where $x_d = \big[x_{d1}, ..., x_{dm}\big]^T$ with x_{di} a desired signal of the ith state $x_i(t)$, and $x_u = \big[x_{u1}, ..., x_{u_{(n-m)}}\big]^T$ can be possibly unknown.

The aim of this work is to design $u(t)$ such that $\lim_{t\to\infty} x_i(t) = x_{di} \succeq 0$ for $i = 1, ..., m \leq n$.

One standard definition and one assumption are recalled.

Definition 1 [30]: The switching signal represented by the switching sequences $\mathcal{S} := \big\{(\kappa_0, t_0), \ldots, (\kappa_m, t_m), \ldots\big\}$ is said to be dwell time admissible if there exists a number $\tau_d > 0$ such that $t_{m+1} - t_m \geq \tau_d$ holds for $\forall m \in \mathbb{N}^+$. The number $\tau_d > 0$ is the *dwell-time* and the set $\mathcal{D}(\tau_d)$ is used to indicate all switching signals satisfying the dwell-time constraint.

Assumption 1 [28]: There exist nonnegative vectors $x_u \in \bar{\mathbb{R}}^{n-m}_+$ and $u_{\sigma(t)e} \in \bar{\mathbb{R}}^m_+$ satisfying $A_{\sigma(t)}x_e + B_{\sigma(t)}u_{\sigma(t)e} = 0$ and there exists a matrix $\Theta_{\sigma(t)g} = \mathrm{diag}\big\{\theta_{\sigma(t)g1}, ..., \theta_{\sigma(t)gm}\big\}$ such that $A_{\sigma(t)s} = A_{\sigma(t)} + B_{\sigma(t)}\widetilde{\Theta}_{\sigma(t)g}$ with $\widetilde{\Theta}_{\sigma(t)g} = \big[\Theta_{\sigma(t)g},\ 0_{m\times(n-m)}\big]$ being asymptotically stable.

Remark 1: It has to be remarked that Assumption 1 has been formulated for non-switched positive systems in [19], [28]. Such an assumption guarantees well-posedness of the tracking problem, thus it will be considered in the switched case as well.

III. Adaptive Control design based on a dwell-time switching method

A novel adaptive controller will now be designed to ensure asymptotic tracking error for the switched positive system (1). The adaptive design is summarized in the following.

Theorem 1: Consider the target state x_d, and the uncertain switched positive linear dynamic (1). Assume there exists a family of diagonal positive-definite matrices $P_{\iota,k} \in \mathbb{R}^{n\times n}$, $\iota \in \mathcal{M}$, $k = 1, ..., \mathcal{K}$, and a sequence $\{\chi_k\}_{k=1}^{\mathcal{K}}$ with $\chi_k > 0$ and $\sum_{k=1}^{\mathcal{K}} \chi_k = \tau_d$ such that the following matrix inequalities are satisfied:

$$P_{\iota,k} \succ 0 \quad (2a)$$

$$\frac{P_{\iota,k+1} - P_{\iota,k}}{\chi_{k+1}} + P_{\iota,k}A_{\iota s} + A_{\iota s}^T P_{\iota,k} \prec 0 \quad (2b)$$

$$\frac{P_{\iota,k+1} - P_{\iota,k}}{\chi_{k+1}} + P_{\iota,k+1}A_{\iota s} + A_{\iota s}^T P_{\iota,k+1} \prec 0 \quad (2c)$$

$$k = 0, ..., \mathcal{K} - 1 \quad (2d)$$

$$P_{\iota,\mathcal{K}}A_{\iota s} + A_{\iota s}^T P_{\iota,\mathcal{K}} \prec 0 \quad (2e)$$

$$P_{\iota,\mathcal{K}} - P_{\varrho,0} \succeq 0 \quad (2f)$$

$$\text{for } \varrho = 0, ..., \iota - 1, \iota + 1, ..., M \quad (2g)$$

with $\mathcal{K}$ an adjustable integer and $\chi_{k+1} = t_{i,k+1} - t_{i,k}$ after defining the sequence $\{t_{i,0}, ..., t_{i,\mathcal{K}}\}$ with $t_{i,0} = t_i$ and $t_{i,\mathcal{K}} - t_{i,0} = \tau_d$. Furthermore, define the time-varying diagonal matrix $P_\iota(t) = \mathrm{diag}\big\{p_{\sigma(t)1}(t), ..., p_{\sigma(t)m}(t)\big\}$ with $P_\iota(t)$ as

$$P_\iota(t) = \begin{cases} P_{\iota,k} + \dfrac{P_{\iota,k+1} - P_{\iota,k}}{\chi_{k+1}}(t - t_{i,k}), & \text{for } t \in \big[t_{i,k}, t_{i,k+1}\big] \\ P_{\iota,\mathcal{K}}, & \text{for } t \in [t_{i,\mathcal{K}}, t_{i+1}) \end{cases} \quad (3)$$

Then, the adaptive controller

$$u_i(t) = \theta_{\sigma(t)i}(t)\big(x_i(t) - x_{di}\big) + \varphi_{\sigma(t)i}(t),\ \ i = 1, ..., m \quad (4)$$

with parameters adaptation laws

$$\dot{\theta}_{\sigma(t)i}(t) = -\rho_{\sigma(t)i} p_{\sigma(t)i}(t)\big(x_i(t) - x_{di}(t)\big)^2 \tag{5}$$

$$\dot{\varphi}_{\sigma(t)i}(t) = \begin{cases} 0, & \text{if } \varphi_{\sigma(t)i}(t) = 0 \text{ and } x_i(t) \geq x_{di} \\ -\widehat{\rho}_{\sigma(t)i} p_{\sigma(t)i}(t)\big(x_i(t) - x_{di}\big), & \text{otherwise} \end{cases} \tag{6}$$

where $\rho_{\sigma(t)i} > 0$, $\widehat{\rho}_{\sigma(t)i} > 0$, $\theta_{\sigma(t)i}(0) \leq 0$ and $\varphi_{\sigma(t)i}(0) \geq 0$, can guarantee that $\lim_{t\to\infty}(x_i(t) - x_{di}(t)) = 0$ for all $x_0 \in \mathbb{R}^n_+$. Moreover, $x(t) \succeq 0$ for $\forall t \geq 0$.

Proof: As a first step we write the error dynamics, invoking (1) and (4), which yields

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}\Theta_{\sigma(t)}(t)\big[\bar{x}(t) - x_d\big] + B_{\sigma(t)}\varphi_{\sigma(t)}(t) \tag{7}$$

where we have used the compact notation for the control input

$$u(t) = \Theta_{\sigma(t)}(t)\big(\bar{x}(t) - x_d\big) + \varphi_{\sigma(t)}(t) \tag{8}$$

with $\Theta_{\sigma(t)}(t) = \mathrm{diag}\big\{\theta_{\sigma(t)1}(t), ..., \theta_{\sigma(t)m}(t)\big\}$.

Using Assumption 1 gives

$$\begin{aligned}
\dot{e}(t) =& \big(A_{\sigma(t)s} - B_{\sigma(t)}\widetilde{\Theta}_{\sigma(t)g}\big)x(t) + B_{\sigma(t)}\varphi_{\sigma(t)}(t) \\
&+ B_{\sigma(t)}\Theta_{\sigma(t)}(t)\big(\bar{x}(t) - x_d\big) \\
=& A_{\sigma(t)s}\big(x(t) - x_e\big) + B_{\sigma(t)}\widetilde{\Theta}_{\sigma(t)g}\Big[x_e - x(t)\Big] \\
&+ B_{\sigma(t)}\big[\Theta_{\sigma(t)}(t)\big(\bar{x}(t) - x_d\big)\big] \\
&+ B_{\sigma(t)}\big(\varphi_{\sigma(t)}(t) - u_{\sigma(t)e}\big) \\
=& A_{\sigma(t)s}e(t) + B_{\sigma(t)}\big(\varphi_{\sigma(t)}(t) - u_{\sigma(t)e}\big) \\
&+ B_{\sigma(t)}\big(\Theta_{\sigma(t)}(t) - \Theta_{\sigma(t)g}\big)\big(\bar{x}(t) - x_d\big)
\end{aligned} \tag{9}$$

where $e(t) = x(t) - x_e$.

Take the quadratic Lyapunov function:

$$\begin{aligned}
V(t) =& e^T(t)P_{\sigma(t)}(t)e(t) + \sum_{\iota=1}^{\mathcal{M}} \mathrm{tr}\big(\Theta_\iota(t) - \Theta_{\iota g}\big)^T Q_\iota^{-1} \cdot \\
&\big(\Theta_\iota(t) - \Theta_{\iota g}\big) + \sum_{\iota=1}^{\mathcal{M}} \big(\varphi_\iota(t) - u_{\iota e}\big)^T \widehat{Q}_\iota^{-1}\big(\varphi_\iota(t) - u_{\iota e}\big)
\end{aligned} \tag{10}$$

that can be equivalently expressed in the form of

$$\begin{aligned}
V(t) =& \sum_{i=1}^{n} P_{\sigma(t)i}\big(x_i(t) - x_{di}\big)^2 + \sum_{\iota=1}^{\mathcal{M}}\sum_{i=1}^{m} \frac{b_{\iota i}}{\rho_{\iota i}} \cdot \\
&\big(\theta_{\iota i} - \theta_{\iota gi}\big)^2 + \sum_{\iota=1}^{\mathcal{M}}\sum_{i=1}^{m} \frac{b_{\iota i}}{\widehat{\rho}_{\iota i}}\big(\varphi_{\iota i}(t) - u_{\iota ei}\big)^2
\end{aligned} \tag{11}$$

with

$$Q_\iota = \begin{bmatrix} \frac{\rho_{\iota 1}}{b_{\iota 1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\rho_{\iota m}}{b_{\iota m}} \end{bmatrix}, \quad \widehat{Q}_\iota = \begin{bmatrix} \frac{\widehat{\rho}_{\iota 1}}{b_{\iota 1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\widehat{\rho}_{\iota m}}{b_{\iota m}} \end{bmatrix}$$

From (6), (7) and (11), the time derivative of the Lyapunov function follows

$$\begin{aligned}
\dot{V}(t) =& e^T(t)\big(A^T_{\sigma(t)s}P_{\sigma(t)}(t) + P_{\sigma(t)}(t)A_{\sigma(t)s} + \dot{P}_{\sigma(t)}(t)\big)e(t) \\
&+ 2e^T(t)P_{\sigma(t)}(t)B_{\sigma(t)}(\Theta_{\sigma(t)}(t) - \Theta_{\sigma(t)g})(\bar{x}(t) - x_d) \\
&+ 2\sum_{\iota=1}^{\mathcal{M}} \mathrm{tr}(\Theta_\iota(t) - \Theta_{\iota g})^T Q_\iota^{-1}\dot{\Theta}_\iota(t) \\
&+ 2\sum_{\iota=1}^{\mathcal{M}} (\varphi_\iota(t) - u_{\iota e})^T \widehat{Q}_\iota^{-1}\dot{\varphi}_\iota(t) \\
&+ 2e^T(t)P_{\sigma(t)}(t)B_{\sigma(t)}(\varphi_{\sigma(t)}(t) - u_{\sigma(t)e})
\end{aligned} \tag{12}$$

which leads to

$$\begin{aligned}
\dot{V}(t) =& e^T(t)\big(A^T_{\sigma(t)s}P_{\sigma(t)}(t) + P_{\sigma(t)}(t)A_{\sigma(t)s} + \dot{P}_{\sigma(t)}(t)\big)e(t) \\
&+ 2\sum_{i=1}^{m} P_{\sigma(t)i}(t)b_{\sigma(t)i}(\theta_{\sigma(t)i}(t) - \theta_{\sigma(t)gi})(x_i(t) - x_{di})^2 \\
&+ 2\sum_{i=1}^{m} P_{\sigma(t)i}(t)b_{\sigma(t)i}(x_i - x_{di})(\varphi_{\sigma(t)i} - u_{\sigma(t)ei}) \\
&+ 2\sum_{i=1}^{m} \left[\frac{b_{\sigma(t)i}}{\rho_{\sigma(t)i}}(\theta_{\sigma(t)i}(t) - \theta_{\sigma(t)gi})\dot{\theta}_{\sigma(t)i}(t)\right] \\
&+ 2\sum_{i=1}^{m} \left[\frac{b_{\sigma(t)i}}{\widehat{\rho}_{\sigma(t)i}}(\varphi_{\sigma(t)i}(t) - u_{\sigma(t)ei})\dot{\varphi}_{\sigma(t)i}(t)\right]
\end{aligned} \tag{13}$$

where we have used the fact that the adaptive laws for inactive subsystems are inactive (i.e. only $\dot{\theta}_{\sigma(t)i}$ and $\dot{\varphi}_{\sigma(t)i}$ are active).

Substituting (5) and (6) into (13) results in

$$\begin{aligned}
\dot{V}(t) =& e^T(t)\underbrace{\big(A^T_{\sigma(t)s}P_{\sigma(t)}(t) + P_{\sigma(t)}(t)A_{\sigma(t)s} + \dot{P}_{\sigma(t)}(t)\big)}_{\omega_{\sigma(t)}(t)}e(t) + \\
&\underbrace{2\sum_{i=1}^{m} b_{\sigma(t)i}(\varphi_{\sigma(t)i} - u_{\sigma(t)ei})\left[P_{\sigma(t)i}(t)(x_i - x_{di}) + \frac{\dot{\varphi}_{\sigma(t)i}(t)}{\widehat{\rho}_{\sigma(t)i}}\right]}_{\zeta(t)}
\end{aligned} \tag{14}$$

In view of (7), two cases should be considered:

Case 1: If $\varphi_{\sigma(t)i}(t) = 0$ and $x_i \geq x_{di}$, then $\dot{\varphi}_{\sigma(t)i}(t) = 0$. Thus we arrive

$$\zeta(t) = -2\sum_{i=1}^{m} b_{\sigma(t)i}u_{\sigma(t)ei}P_{\sigma(t)i}(t)(x_i - x_{di}) \leq 0. \tag{15}$$

Case 2: If Case 1 is not satisfied, then $\dot{\varphi}_{\sigma(t)i}(t) = -\widehat{\rho}_{\sigma(t)i}P_{\sigma(t)i}(x_i - x_{di})$. Thus one has

$$\zeta(t) = 0. \tag{16}$$

In both cases, (15) and (16) imply that

$$\dot{V}(t) \leq e^T(t)\big(A^T_{\sigma(t)s}P_{\sigma(t)}(t) + P_{\sigma(t)}(t)A_{\sigma(t)s} + \dot{P}_{\sigma(t)}(t)\big)e(t). \tag{17}$$

It has to be noticed that $\omega_{\sigma(t)}(t)$ in (14) is continuous for $t \in \big[t_i, t_{i+1}\big)$ thanks to the continuity of $P_{\sigma(t)}(t)$ for $t \in \big[t_i, t_{i+1}\big)$.

According to the definition of $P_{\sigma(t)}(t)$, there are three situations that should be considered:

Situation 1 (before the dwell time) : Let us first consider $t \in [t_{i,k}, t_{i,k+1})$, $k = 0, ..., \mathcal{K}-1$. Then, we can rewrite $\omega_{\sigma(t)}(t)$ as

$$\begin{aligned}\omega_{\sigma(t)}(t) &= A^T_{\sigma(t)s} P_{\sigma(t)}(t) + P_{\sigma(t)}(t) A_{\sigma(t)s} + \dot{P}_{\sigma(t)}(t) \\ &= \gamma_1 \Big\{ \frac{P_{\sigma(t),k+1} - P_{\sigma(t),k}}{\chi_{k+1}} + P_{\sigma(t),k} A_{\sigma(t)s} \\ &+ A^T_{\sigma(t)s} P_{\sigma(t),k} \Big\} + \gamma_2 \Big\{ \frac{P_{\sigma(t),k+1} - P_{\sigma(t),k}}{\chi_{k+1}} \\ &+ P_{\sigma(t),k+1} A_{\sigma(t)s} + A^T_{\sigma(t)s} P_{\sigma(t),k+1} \Big\}\end{aligned} \tag{18}$$

with $\gamma_1 = 1 - \frac{t - t_{i,k}}{\chi_{k+1}}$ and $\gamma_2 = \frac{t - t_{i,k}}{\chi_{k+1}}$. Recalling (2), we have

$$\omega_{\sigma(t)}(t) \prec 0, \ \text{for } t \in [t_{i,k}, t_{i,k+1}). \tag{19}$$

Situation 2 (after the dwell time) : Let us now consider $t \in [t_{i,\mathcal{K}}, t_{i+1})$ subject to the dwell-time constraint $t_{i+1} - t_i > \tau_d$. One has $P_{\sigma(t)}(t) = P_{\sigma(t),\mathcal{K}}$, which suggests from (2) that

$$\omega_{\sigma(t)}(t) = P_{\sigma(t),\mathcal{K}} A_{\sigma(t)s} + A^T_{\sigma(t)s} P_{\sigma(t),\mathcal{K}} \prec 0 \tag{20}$$

when $t \in [t_{i,\mathcal{K}}, t_{i+1})$. Therefore, from (19) and (20), we obtain $\omega_{\sigma(t)}(t) \prec 0$ when $t \in [t_i, t_{i+1})$, which results in

$$\dot{V}(t) = e^T(t) \omega_{\sigma(t)}(t) e(t) < 0, \ \text{for } t \in [t_i, t_{i+1}) \tag{21}$$

Situation 3 (at the switching instant) : We finally consider the switching instant $t = t_{i+1}$. Thanks to the continuity of $e(t)$, $\Theta_{\sigma(t)}(t)$ and $\varphi_{\sigma(t)}(t)$, one gets

$$\begin{aligned}& V_{\sigma(t_{i+1})}\big(x(t_{i+1}), \Theta_{\sigma(t)}(t_{i+1}), \varphi_{\sigma(t)}(t_{i+1})\big) \\ & - V_{\sigma(t^-_{i+1})}\big(x(t^-_{i+1}), \Theta_{\sigma(t^-_{i+1})}(t^-_{i+1}), \varphi_{\sigma(t^-_{i+1})}(t^-_{i+1})\big) \\ & = e^T(t_{i+1}) \Big[P_{\sigma(t_{i+1})} - P_{\sigma(t^-_{i+1})} \Big] e(t_{i+1}) \\ & = e^T(t_{i+1}) \Big[P_{\sigma(t_{i+1}),0} - P_{\sigma(t^-_{i+1}),\mathcal{K}} \Big] e(t_{i+1})\end{aligned} \tag{22}$$

which means that $V(\cdot)$ is non-increasing at switching instants due to (2f). Strict decrease of the Lyapunov function between two consecutive switching instants in conjunction with non-increase at each switching instant allows us to conclude that the equilibrium is Lyapunov stable (i.e. $e \in \mathcal{L}_\infty$). Additionally, it follows from $\int_0^\infty e^T(t) \omega_\iota(t) e(t) dt < \infty$ and from boundedness of $P_\iota(\cdot)$ and $\omega_\iota(\cdot)$ that $e(\cdot) \in \mathcal{L}_2$. Note that $\dot{e}(t) = \dot{x}(t) \in \mathcal{L}_\infty$, thus we can conclude that $e(t) \to 0$, i.e., $\lim_{t\to\infty} \underbrace{||x(t) - x_e||}_{e(t)} = 0$ as $t \to \infty$ in accordance with Barbalats's lemma.

Finally, we will show that the nonnegativity of states $x(t)$ is guaranteed by (4), (5), and (6). Note that

$$\begin{aligned}\dot{x}(t) =& A_{\sigma(t)} x(t) + B_{\sigma(t)} \Big(\Theta_{\sigma(t)}(t) \big(\bar{x}(t) - x_d\big) + \varphi_{\sigma(t)}(t) \Big) \\ =& \big(A_{\sigma(t)} + B_{\sigma(t)} \widetilde{\Theta}_{\sigma(t)}(t)\big) x(t) + B_{\sigma(t)} \varphi_{\sigma(t)}(t) \\ & - B_{\sigma(t)} \big(\Theta_{\sigma(t)}(t) x_d\big) \\ =& \widetilde{\widetilde{\Xi}}_{\sigma(t)} x(t) + \xi(t) + \phi(t)\end{aligned} \tag{23}$$

where

$$\widetilde{\widetilde{\Xi}}_{\sigma(t)} = \begin{bmatrix} \widetilde{\vartheta}_{11}(t) & \cdots & \widetilde{\vartheta}_{\sigma(t)1m} & \cdots & \widetilde{\vartheta}_{\sigma(t)1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widetilde{\vartheta}_{\sigma(t)m1} & \cdots & \widetilde{\vartheta}_{mm}(t) & \cdots & \widetilde{\vartheta}_{\sigma(t)mn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widetilde{\vartheta}_{\sigma(t)n1} & \cdots & \widetilde{\vartheta}_{\sigma(t)nm} & \cdots & \widetilde{\vartheta}_{\sigma(t)nn} \end{bmatrix}$$

$$\xi(t) = \begin{bmatrix} b_{\sigma(t)1} \theta_{\sigma(t)1} x_{d1} \\ \vdots \\ b_{\sigma(t)m} \theta_{\sigma(t)m} x_{dm} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \phi(t) = \begin{bmatrix} b_{\sigma(t)1} \varphi_{\sigma(t)1} \\ \vdots \\ b_{\sigma(t)m} \varphi_{\sigma(t)m} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

with $\widetilde{\vartheta}_{11}(t) = \widetilde{\vartheta}_{\sigma(t)11} + b_{\sigma(t)1} \theta_{\sigma(t)1}(t)$ and $\widetilde{\vartheta}_{mm}(t) = \widetilde{\vartheta}_{\sigma(t)mm} + b_{\sigma(t)m} \theta_{\sigma(t)m}(t)$. From (6) and (7), one obtains that $\theta_{\sigma(t)i}(t) \leq 0$ and $\varphi_{\sigma(t)i}(t) \geq 0$ for $t \geq 0$, $i = 1, ..., m$. Therefore, it holds that $\xi(t) \succeq 0$ and $\phi(t) \succeq 0$, for $t \geq 0$. Hence, according to the Proposition 7.1 of [29], it follows that $x(t) \succeq 0$, $t \geq 0$ for all $x_0 \in \bar{\mathbb{R}}^n_+$.

The proof of Theorem 1 is complete. ■

Remark 2: Differently from the standard proofs for adaptive control of switched dynamics [30], here we have not only proven stability, but also that the adaptive law will guarantee nonnegativity of the states. This is because the adaptive closed-loop should remain a positive switched system. Note that the methods in [30]–[32] cannot in general guarantee nonnegativity of the system states.

Remark 3: It is worth remarking that the developed adaptive controller (8) does not rely on the knowledge of $\Theta_{\sigma(t)g}$, x_u, and $u_{\sigma(t)e}$. In other words, $\Theta_{\sigma(t)g}$, x_u and $u_{\sigma(t)e}$ are assumed to exist, but their knowledge is not needed to achieve asymptotic set-point tracking.

Remark 4: Because the proposed result relies on the selection of some $A_{\sigma(t)s}$, in (2), some observations apply. If the system is fully actuated (i.e. $B_{\sigma(t)}$ is full row rank, $m = n$), then the selection of $A_{\sigma(t)s}$ can be arbitrary. This is because any uncertainty in $A_{\sigma(t)}$ can be matched by some appropriate constant input. However, as the system becomes less actuated ($m < n$), the uncertainty in $A_{\sigma(t)}$ can be only on the diagonal and on the actuated rows. This implies that the non-actuated rows of $A_{\sigma(t)s}$ (except the diagonal) must have the same entries as $A_{\sigma(t)}$. Therefore, there is a link between the dimension of the input of the system and the number of uncertain entries in $A_{\sigma(t)}$ that one can handle.

Remark 5: Differently from [31], [33] where $P_{\sigma(t)}$ is a constant matrix, the proposed method adopts a time-varying $P_{\sigma(t)}(t)$: because such matrix is also diagonal, this can be seen as having a time-varying adaptive gain in (5)-(6). It has also to be noticed that the interpolation approach implies that the adaptive law is coupled with the switching law via (2).

IV. Illustrative examples

To demonstrate the feasibility of the designed method, a potential clinical application for general anesthesia and a numerical example are given.

A. Numerical Example 1

We take the positive switched linear dynamics in (1) with

$$A_1 = \begin{bmatrix} -1.5 & 0.3 & 0.6 \\ 0.2 & -0.3 & 0.2 \\ 0.5 & 0.25 & -0.6 \end{bmatrix}, \ B_1 = \begin{bmatrix} 9.8 \\ 0 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.9 & 0.5 & 0.8 \\ 0.3 & -0.5 & 0.3 \\ 0.2 & 0.35 & -0.8 \end{bmatrix}, \ B_2 = \begin{bmatrix} 9.8 \\ 0 \\ 0 \end{bmatrix}$$

We choose $\tau_d = 5$ and $\mathcal{K} = 1$. To address the fastest dwell-time signal, we select the switching instants such that $t_{i+1} - t_i = \tau_d \ \forall \ i$. Thus the time-varying matrix $P_\iota(t)$ can be computed by $P_\iota(t) = \big(t - \tau_d \cdot \mathrm{floor}(\frac{t}{\tau_d})\big) \cdot \big(\frac{P_{\iota,1} - P_{\iota,0}}{\tau_d}\big) + P_{\iota,0}$. All initial values are $x(0) = [0;\ 0;\ 0]$, $\theta_1(0) = \theta_2(0) = -2.5$, $\rho_1 = 3.5$, $\rho_2 = 4.5$, $\widehat{\rho}_1 = \widehat{\rho}_2 = 0.25$ and $\varphi_1(0) = \varphi_2(0) = 10$. To make the tracking more challenging, we select different target states for each subsystem, i.e. $x_{d1} = 40$ for subsystem 1 and $x_{d1} = 5$ for subsystem 2. Fig. 1 suggests that good tracking performance is achieved in spite of the systems uncertainty. It has to be noted that the tracking has been defined only for x_1, whereas x_2 and x_3 will evolve according to the system dynamics. Figs. 2 and 3 provide the evolutions of the adaptive gains θ_1, θ_2, φ_1, and φ_2. From the figures it can be noted that θ_1 and ϕ_1 are constant when subsystem 1 is inactive, while θ_2 and ϕ_2 are constant when subsystem 2 is inactive.

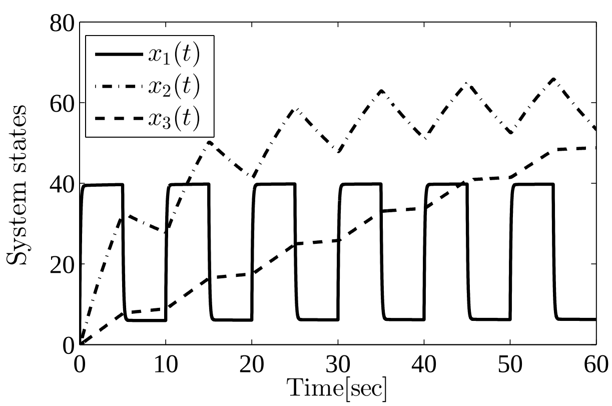

Figure 1: Systems states x_1, x_2 and x_3.

B. Practical Example 2

We consider the hypothetical model for anesthetic propofol in [19]. The overall structure of the model is provided in Fig. 4. Moreover, $\alpha_{ij} \geq 0$, $i \neq j$, $i, j = 1, 2, 3$ are the transfer constant rates for between compartments and $u(t)$ is the infusion rate. Such rates are in practice unknown due to unknown pre-existing diseases or concomitant medication of patients, or other causes. Two subsystems (named A and B) with different constants are considered (summarized in Table I): our goal is to change the propofol concentration in compartment 1 to 4 $\mu g/m\ell$ for subsystem A and 2.5 $\mu g/m\ell$ for subsystem B, respectively. In line with [19], we calculate

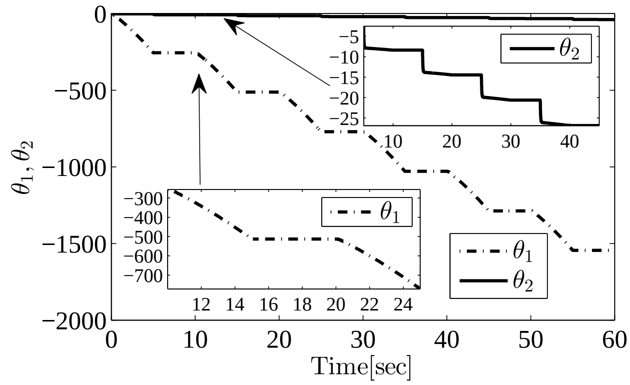

Figure 2: Adaptive gains θ_1 and θ_2.

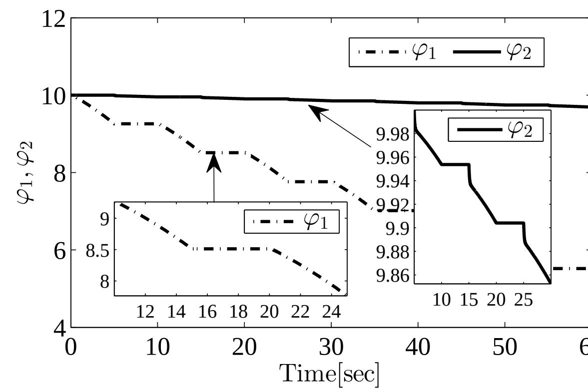

Figure 3: Adaptive gains φ_1 and φ_2.

the propofol concentration of central compartment by $\frac{x_1}{V_1}$ with $V_1 = \big(0.159\ \ell/\mathrm{kg}\big)\big(\mathrm{W\ kg}\big)$ the volume of central compartment and $\mathrm{W} = 70\mathrm{kg}$ the patient mass. Two sets of pharmacokinetic parameters are provided in Table I with dynamics described by (1) with

$$A_\iota = \begin{bmatrix} -(\alpha_{11} + \alpha_{21} + \alpha_{31}) & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & -\alpha_{12} & 0 \\ \alpha_{31} & 0 & \alpha_{13} \end{bmatrix}, B_\iota = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

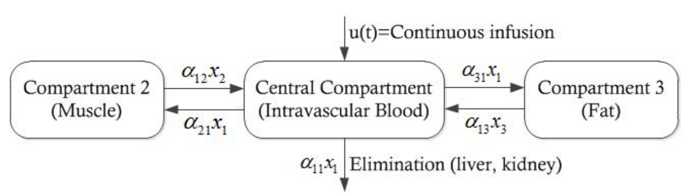

Figure 4: Three-compartment mammillary model

Table I: Pharmacokinetic parameters [19]

Subsystem	α_{11}	α_{21}	α_{12}	α_{31}	α_{13}	Unit
A	0.152	0.207	0.092	0.040	0.0048	min^{-1}
B	0.119	0.114	0.055	0.041	0.0033	min^{-1}

Let the initial values be $x(0) = [0;\ 0;\ 0]$, $\theta_1(0) = \theta_2(0) = 0\ \mathrm{min}^{-1}$, $\rho_1 = \rho_2 = 1000\ \mathrm{g}^{-2}\mathrm{min}^{-2}$, $\widehat{\rho}_1 = \widehat{\rho}_2 = 0.5\ \mathrm{min}^{-2}$

and $\varphi_1(0) = \varphi_2(0) = 0.01$ g/min^{-1}. For comparison purposes, the method in [19] (using a single controller) and the proposed switched strategy are compared for three cases of the dwell time (i.e. Case 1: $\tau_d = 3$ min, $\tau_d = 6$ min, and $\tau_d = 10$ min). For each case we set $t_{i+1} - t_i = \tau_d$ for the switching signal $\sigma(\cdot)$ and $\mathcal{K} = 1$.

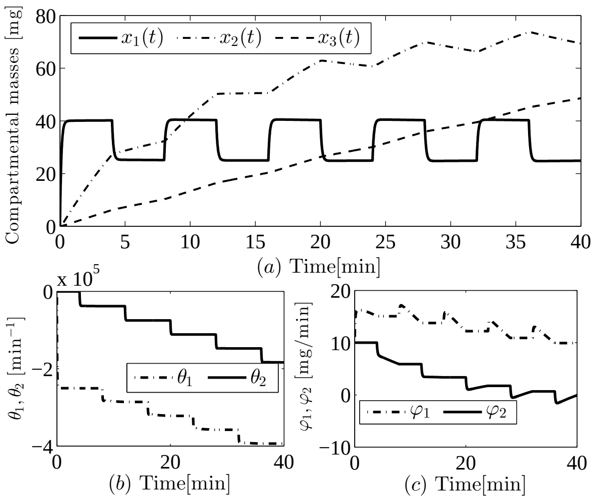

Figure 5: (a): Compartmental masses x_1, x_2 and x_3; (b): Adaptive gains θ_1 and θ_2; (c): Adaptive gains φ_1 and φ_2.

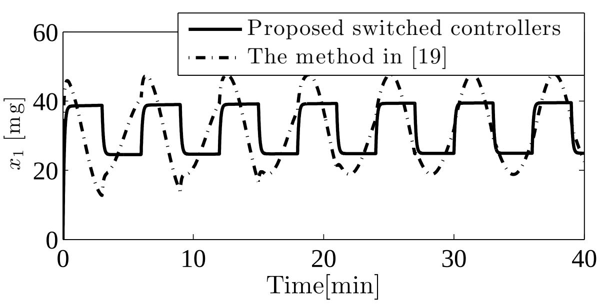

Figure 6: Case 1 with $\tau_d = 3$ min.

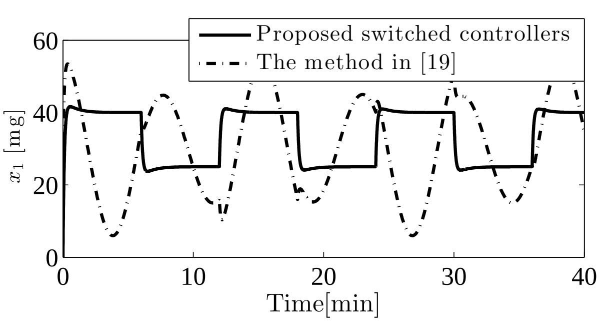

Figure 7: Case 2 with $\tau_d = 6$ min.

Fig. 5 (a) shows the masses of propofol in the three compartments : good anesthetic control performance can be noticed in compartment 1 despite switching. Figs. 5 (b) and (c) depict the curves of the adaptive gains θ_1, θ_2, φ_1, and φ_2, respectively. It can be seen from Figs. 6-8 that the developed switched

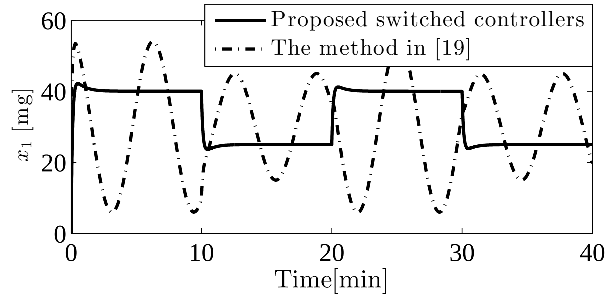

Figure 8: Case 3 with $\tau_d = 10$ min.

controllers work well and possess better control performance than the method with a single controller of [19]. It is clear that the method of [19] can be tuned to have better performance, but, in the presence of switching, its main problem is that the single controller has to continuously adapt to the two different subsystems (instead of switching to different gains), which eventually causes poor tracking.

V. Conclusions

This brief has proposed a direct adaptive control method for a class of time-driven switched positive linear dynamics. The main achievement of the method is to achieve asymptotic set-point tracking, which, to the best of the authors' knowledge, has been proved for the first result for uncertain switched positive linear dynamics subject to dwell time constraints. An application to general anesthesia has been presented to validate the developed strategy. Most notably, the proposed adaptive switched approach works better than adaptive approaches proposed in literature based on a single (non-switched) controller. We believe that the following points are worth investigating in future research: 1) it is still unclear if neural networks [34] or fuzzy logic systems of [35]–[37] can be adopted to handle some continuous uncertainties (e.g. unknown extra disturbances); 2) it is still unclear the proposed method can be adopted in a distributed control setting like in [38], [39] and [40], when the systems have to minimize a consensus error, in place of a tracking error: studying this point would be relevant to address more general systems.

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