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# Elastodynamic Full Wavefield Modelling with Legendre Polynomials

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## Summary

Full Wavefield Migration (FWMig) is an inversion-based seismic imaging modality that incorporates multiple reflections via one-way wave propagation. The flexible Full Wavefield Modelling (FWMod) engine that undergirds FWMig can be extended to address both compressional and converted waves. To take care of the angle-dependent nature of reflection and transmission coefficients, a vast number of unknown subsurface parameters has to be estimated in the FWMig process, especially when elastodynamic wave propagation is considered. This can easily result in a significant null space, potentially hampering the underlying inversion procedure. To restrain the number of unknown parameters, we propose an efficient new parameterization for FWMod by expanding reflection and transmission coefficients in Legendre polynomials, providing us with an orthonormal basis that is expected to benefit FWMig. With the aid of a numerical experiment in a two-dimensional layered elastic medium, we show that a relatively small number of only three or four Legendre polynomials per coefficient per gridpoint is sufficient to model pre-critical seismic data. We prospect that our methodology can be extended to include (spatially-varying) reflector dips, so that it might eventually be used for FWMig in laterally-varying two- and three-dimensional elastic media.

# Elastodynamic Full Wavefield Modelling with Legendre Polynomials

## Introduction

In Full Wavefield Modelling (FWMod), seismic wavefields are generated by recursively applying depth extrapolation and reflection / transmission operators (Berkhout, 2014). These operators that are typically expressed as convolutional matrices in the (horizontal) space-frequency domain, describe the angle-dependent reflection / transmission process that takes place at each depth level in the subsurface. With the help of the slant-stack (or Radon) transform (Stoffa et al., 2006), these matrices can be related to the angle-dependent reflection / transmission coefficients at individual subsurface gridpoints (Davydenko & Verschuur, 2017). In Full Wavefield Migration (FWMig), the FWMod framework is used to invert for the reflection and transmission coefficients by least-squares minimization. As these coefficients are functions of the incident angle, this approach might easily lead to an unacceptably high number of unknown parameters per gridpoint, especially when elastodynamic wave propagation is considered. To mitigate this problem, it has been proposed to relate the reflection and transmission coefficients to physical rock properties with the aid of Shuey's approximation (Shuey, 1985), leading to a manageable problem with three unknown subsurface parameters per gridpoint (Hoogerbrugge & Verschuur, 2021). Since Shuey's approximation is known to be inaccurate at high incident angles and for high-contrast interfaces (which are especially relevant for imaging with multiple reflections), we propose a more flexible mathematical alternative by expanding the reflection and transmission coefficients in Legendre polynomials, providing us with an orthonormal basis that may benefit FWMig.

## Theory

After each extrapolation step in the FWMod framework (Berkhout, 2014), the down- and upgoing wavefields at each depth level are convolved with reflection and transmission operators. In case of elastodynamic wave propagation in a two-dimensional medium, these operators are the sixteen entries of the following scattering matrix $\widehat{\boldsymbol{S}}$, expressed here as a function of depth $z$ and angular frequency $\omega$:

$$\widehat{\boldsymbol{S}}(z,\omega)=\begin{pmatrix} \boldsymbol{I}+\delta\widehat{\boldsymbol{T}}^{\downarrow}_{PP}(z,\omega) & \delta\widehat{\boldsymbol{T}}^{\downarrow}_{PS}(z,\omega) & \widehat{\boldsymbol{R}}^{\cap}_{SS}(z,\omega) & \widehat{\boldsymbol{R}}^{\cap}_{PS}(z,\omega) \\ \delta\widehat{\boldsymbol{T}}^{\downarrow}_{SP}(z,\omega) & \boldsymbol{I}+\delta\widehat{\boldsymbol{T}}^{\downarrow}_{SS}(z,\omega) & \widehat{\boldsymbol{R}}^{\cap}_{SP}(z,\omega) & \widehat{\boldsymbol{R}}^{\cap}_{SS}(z,\omega) \\ \widehat{\boldsymbol{R}}^{\cup}_{PP}(z,\omega) & \widehat{\boldsymbol{R}}^{\cup}_{PS}(z,\omega) & \boldsymbol{I}+\delta\widehat{\boldsymbol{T}}^{\uparrow}_{PP}(z,\omega) & \delta\widehat{\boldsymbol{T}}^{\uparrow}_{PS}(z,\omega) \\ \widehat{\boldsymbol{R}}^{\cup}_{SP}(z,\omega) & \widehat{\boldsymbol{R}}^{\cup}_{SS}(z,\omega) & \delta\widehat{\boldsymbol{T}}^{\uparrow}_{SP}(z,\omega) & \boldsymbol{I}+\delta\widehat{\boldsymbol{T}}^{\uparrow}_{SS}(z,\omega) \end{pmatrix}. \tag{1}$$

Each of the sixteen entries $\delta\widehat{\boldsymbol{T}}^{\downarrow}_{PP}$, $\delta\widehat{\boldsymbol{T}}^{\downarrow}_{SP}$, $\widehat{\boldsymbol{R}}^{\cup}_{PP}$,… is a $N_x\times N_x$ matrix, where $N_x$ is the number of (horizontal) gridpoints at each depth level. These matrices can be related to the transmission and reflection coefficients $\delta\tilde{T}^{\downarrow}_{PP}(z,x,p)$, $\delta\tilde{T}^{\downarrow}_{SP}(z,x,p)$, $\tilde{R}^{\cup}_{PP}(z,x,p)$, ... at the individual horizontal locations $x$ (at depth $z$) and rayparameters $p$ by incorporating an inverse slant-stack transform (Stoffa et al., 2006). More specifically, we may write

$$\begin{aligned} \left(\delta\widehat{\boldsymbol{T}}^{\downarrow}_{PP}\right)_{mn}(z,\omega) &= \omega\int_{-p_{max}-p_{taper}}^{p_{max}+p_{taper}} F(p)\,\delta\tilde{T}^{\downarrow}_{PP}(z,x_m,p)\,e^{i\omega p(x_m-x_n)}\,dp, \\ \left(\delta\widehat{\boldsymbol{T}}^{\downarrow}_{SP}\right)_{mn}(z,\omega) &= \omega\int_{-p_{max}-p_{taper}}^{p_{max}+p_{taper}} F(p)\,\delta\tilde{T}^{\downarrow}_{SP}(z,x_m,p)\,e^{i\omega p(x_m-x_n)}\,dp, \\ \left(\widehat{\boldsymbol{R}}^{\cup}_{PP}\right)_{mn}(z,\omega) &= \omega\int_{-p_{max}-p_{taper}}^{p_{max}+p_{taper}} F(p)\,\tilde{R}^{\cup}_{PP}(z,x_m,p)\,e^{i\omega p(x_m-x_n)}\,dp,\ \ldots, \end{aligned} \tag{2}$$

where $m$ and $n$ refer to matrix row and column indices, respectively. Here, $p_{max}$ is the maximum rayparameter that we consider and $F(p)$ is Tukey (tapered cosine) window to taper the intervals $\pm[p_{max}, p_{max}+p_{taper}]$. We exclude post-critical wave phenomena by choosing $p_{max}+p_{taper}\leq 1/c_{max}$ (where $c_{max}$ is the maximum propagation velocity). To restrain the number of unknowns, we propose to approximate the transmission and reflection coefficients with normalized Legendre polynomials $\bar{P}_k\left(\frac{p}{p_{max}}\right)$. Since all non-converted coefficients are even functions of $p$, we approximate them by a weighted sum of the first $K$ even Legendre polynomials, i.e.

$$\delta\tilde{T}^{\downarrow}_{PP}(z,x,p) = \sum_{k=1}^{K} C^{\downarrow}_{PP,k}(z,x)\,\bar{P}_{2k-2}\left(\frac{p}{p_{max}}\right),$$
$$\tilde{R}^{\cup}_{PP}(z,x,p) = \sum_{k=1}^{K} C^{\cup}_{PP,k}(z,x)\,\bar{P}_{2k-2}\left(\frac{p}{p_{max}}\right), \ldots, \tag{3}$$

where $\{C^{\downarrow}_{PP,1}, C^{\cup}_{PP,1}, \ldots, C^{\cap}_{SS,K}, C^{\uparrow}_{SS,K}\}$ denotes a set of $8 \cdot K$ even Legendre coefficients. Similarly, since all converted coefficients are odd functions of $p$, we approximate them by a weighted sum of the first $K$ odd Legendre polynomials, i.e.

$$\delta\tilde{T}^{\downarrow}_{SP}(z,x,p) = \sum_{k=1}^{K} C^{\downarrow}_{SP,k}(z,x)\,\bar{P}_{2k-1}\left(\frac{p}{p_{max}}\right),$$
$$\tilde{R}^{\cup}_{SP}(z,x,p) = \sum_{k=1}^{K} C^{\cup}_{SP,k}(z,x)\,\bar{P}_{2k-1}\left(\frac{p}{p_{max}}\right), \ldots., \tag{4}$$

where $\{C^{\downarrow}_{SP,1}, C^{\cup}_{SP,1}, \ldots C^{\cap}_{PS,K}, C^{\uparrow}_{PS,K}\}$ denotes a set of $8 \cdot K$ odd Legendre coefficients. Next, we can construct the relevant entries of the scattering matrix $\hat{\boldsymbol{S}}(z,\omega)$ by inverse slant-stack transformation of the system of equations (3)-(4) (Stoffa et al., 2006). The result of this operation can be written as

$$\delta\hat{\boldsymbol{T}}^{\downarrow}_{PP}(z,\omega) = \sum_{k=1}^{K} \mathrm{diag}\left(C^{\downarrow}_{PP,k}(z,x_1), C^{\downarrow}_{PP,k}(z,x_2), \ldots, C^{\downarrow}_{PP,k}(z,x_{N_x})\right)\hat{\boldsymbol{B}}_{2k-2}(\omega),$$
$$\delta\hat{\boldsymbol{T}}^{\downarrow}_{SP}(z,\omega) = \sum_{k=1}^{K} \mathrm{diag}\left(C^{\downarrow}_{SP,k}(z,x_1), C^{\downarrow}_{SP,k}(z,x_2), \ldots, C^{\downarrow}_{SP,k}(z,x_{N_x})\right)\hat{\boldsymbol{B}}_{2k-1}(\omega),$$
$$\hat{\boldsymbol{R}}^{\cup}_{PP}(z,\omega) = \sum_{k=1}^{K} \mathrm{diag}\left(C^{\cup}_{PP,k}(z,x_1), C^{\cup}_{PP,k}(z,x_2), \ldots, C^{\cup}_{PP,k}(z,x_{N_x})\right)\hat{\boldsymbol{B}}_{2k-2}(\omega),$$
$$\hat{\boldsymbol{R}}^{\cup}_{SP}(z,\omega) = \sum_{k=1}^{K} \mathrm{diag}\left(C^{\cup}_{SP,k}(z,x_1), C^{\cup}_{SP,k}(z,x_2), \ldots, C^{\cup}_{SP,k}(z,x_{N_x})\right)\hat{\boldsymbol{B}}_{2k-1}(\omega), \ldots \tag{5}$$

In this formulation, matrices $\hat{\boldsymbol{B}}_k(\omega)$ are constructed by applying the inverse slant-stack transform to the individual Legendre polynomials, yielding akin to equation (2)

$$\left(\hat{\boldsymbol{B}}_k\right)_{mn}(\omega) = \omega \int_{-p_{max}-p_{taper}}^{p_{max}+p_{taper}} F(p)\,\bar{P}_k\left(\frac{p}{p_{max}}\right) e^{i\omega p(x_m - x_n)}\,dp. \tag{6}$$

The system of equations (5) can be used to construct reflection and transmission operators directly from the Legendre coefficients. These operators may then be used to realize elastodynamic FWMod.

## Numerical example

In Figure 1, we show a two-dimensional elastic medium. At all horizontal interfaces between spatial gridpoints, we compute the analytic reflection and transmission coefficients as a function of rayparameter $p$ (Wapenaar & Berkhout, 1989) on the interval $[-p_{max}, p_{max}]$, where we have chosen $p_{max} = 0.28\,\mathrm{s/km}$. As an example, some of these curves are shown in Figure 2 (solid black lines). We find a set of $16 \cdot K$ Legendre coefficients $\{C^{\downarrow}_{PP,1}, C^{\downarrow}_{SP,1}, C^{\cup}_{PP,1}, \ldots, C^{\cap}_{SS,K}, C^{\uparrow}_{PS,K}, C^{\uparrow}_{SS,K}\}$ by least-squares fitting the system of equations (3)-(4). As an example, we show some of the fitted curves in Figure 2 for $K = 1$ (dotted blue lines) and $K = 3$ (dashed red lines). The retrieved set of Legendre coefficients may then serve as model parameters for FWMod. To test this idea, we deploy 401 P-wave sources and 401 collocated P- and S-wave receivers with spacing $dx = 8\mathrm{m}$ at the acquisition surface $z = 0$. As a source signal, we choose a Ricker wavelet with a peak frequency of $f_{peak} = 40\mathrm{Hz}$. For our simulation, we use $N_t = 256$ time samples with $dt = 4\mathrm{ms}$. We employ the theory that was described in the previous section to construct the reflection and transmission operators at each spatial gridpoint, where we choose $p_{taper} = 0.02\,\mathrm{s/km}$. Then, we generate pre-critical PP- and SP-data (i.e. the S-wave data from a P-wave source) by elastodynamic FWMod (Berkhout, 2014). We compare our results with reference responses that were obtained by Kennett modelling (Kennett & Kerry, 1979), after filtering the latter with $F(p)$ for a fair comparison. This comparison is shown in Figures 3 and 4, for the cases that $K = 1$ and $K = 3$ polynomials are used per component per gridpoint, respectively. When we compare the difference plots in Figures 4(c) and (f) with those in Figures 3(c) and (f), it is clear that the FWMod results with $K = 3$ are superior to their counterparts with $K = 1$.

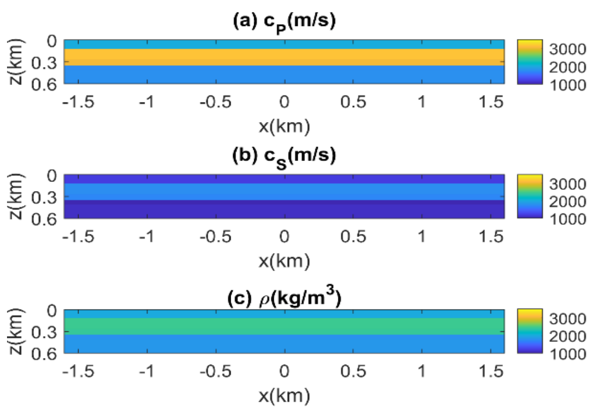


**Figure 1** (a) P-wave velocity, (b) S-wave velocity and (c) density model. All models have $N_z = 121$ and $N_x = 401$ samples with $dz = 5m$ and $dx = 8m$.

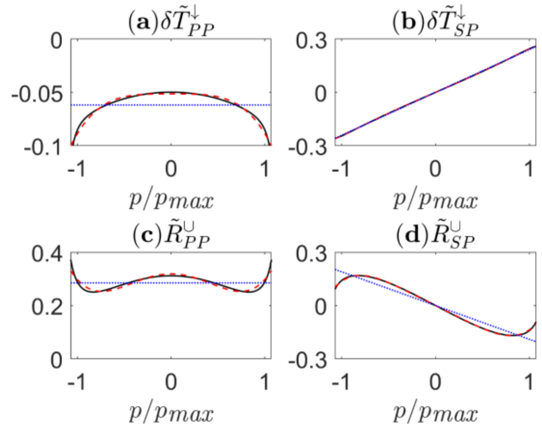


**Figure 2** Transmission and reflection coefficients (a) $\delta\tilde{T}^{\downarrow}_{PP}$, (b) $\delta\tilde{T}^{\downarrow}_{SP}$, (c) $\tilde{R}^{\cup}_{PP}$ and (d) $\tilde{R}^{\cup}_{SP}$ (black lines) just below $(z, x) =$ (0.12km, 0km) (coinciding with the first reflector). The dotted blue and dashed red curves are obtained by least-squares fitting (the system of) equations (3)-(4) with $K = 1$ and $K = 3$, respectively.

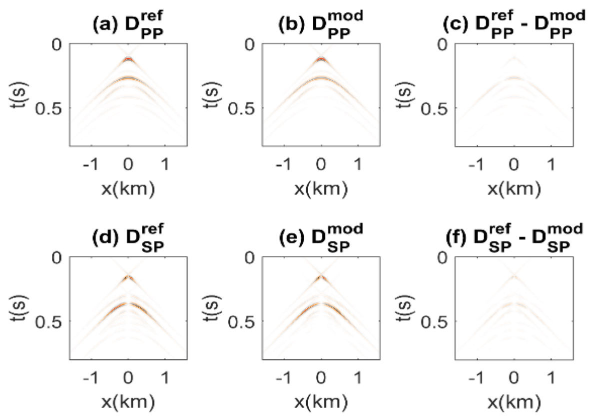


**Figure 3** (a) Reference PP data, (b) FWMod PP data and (c) their difference; (d) reference SP data, (e) FWMod SP data and (f) their difference. In this simulation, we have set $K = 1$.

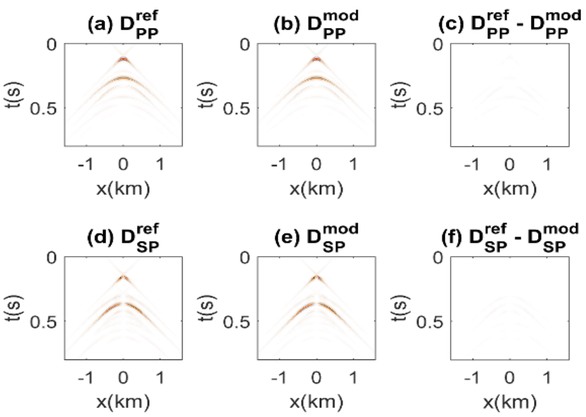


**Figure 4** Same as Figure 3, after setting $K = 3$.

In Figure 5, we show the relative modelling error $\mathcal{E} = \left\|D^{mod} - D^{ref}\right\|^2 / \left\|D^{ref}\right\|^2$ (where $\|\ldots\|$ denotes the Euclidean norm) as a function of $K$. It is clear from these results that a single polynomial per component per gridpoint (as is commonly applied in scalar FWMig) is relatively inaccurate, which is in line with equivalent observations in acoustic media as reported by Davydenko & Verschuur (2019).

## Discussion

The optimal number $K$ of polynomials per component per gridpoint remains a topic for discussion. Raising $K$ increases the computational burden and could easily enlarge the null space of the FWMod operator, which may hinder FWMig. Based on Figures 2(a) and (c), the fitted curves of $\delta\tilde{T}^{\downarrow}_{PP}(p)$ and $\tilde{R}^{\cup}_{PP}(p)$ with $K = 3$ seem suboptimal for the first reflector (while the fitted curves of $\delta\tilde{T}^{\downarrow}_{SP}(p)$ and $\tilde{R}^{\cup}_{SP}(p)$ in Figures 2(b) and (d) are close to perfect in this case). Judging from Figure 5, adding one more Legendre polynomial (i.e. $K = 4$) delivers only a marginal improvement to the PP data, which may or may not be worth the additional computational burden and potential increase of the null space. This observation might be different at higher frequencies and also depends on the medium parameters.

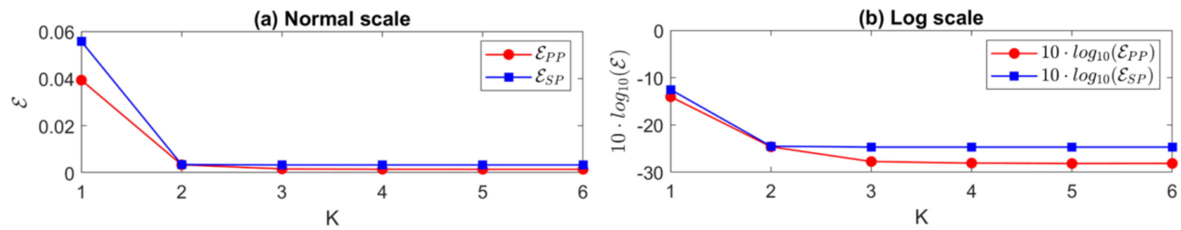


**Figure 5** Relative error $\mathcal{E}$ of modelled PP- and SP-data on a (a) normal and (b) logarithmic scale for varying numbers $K$ of polynomials per component per gridpoint.

## Future prospects

In the future, we aim to recover the Legendre coefficients $\{C^{\downarrow}_{PP,1}, C^{\downarrow}_{SP,1}, C^{\cup}_{PP,1}, \dots, C^{\cap}_{SS,K}, C^{\uparrow}_{PS,K}, C^{\uparrow}_{SS,K}\}$ by inverting the FWMod operator, thus laying the foundation for elastodynamic FWMig. Moreover, we want to extend our model by including (spatially-varying) reflector dips, eventually allowing for applications in arbitrary elastic media.

## Conclusions

We have presented a framework for elastodynamic FWMod by expanding reflection and transmission coefficients in Legendre polynomials. We have used this framework to generate pre-critical seismic reflection data in an elastic medium and compared our results with those of standard Kennett modelling. Choosing an optimal number $K$ of polynomials per component per gridpoint seems to be a trade-off between accuracy and efficiency. We emphasize that our framework is principally designed for inversion, i.e. FWMig. To constrain the null space of the associated inverse problem, we opt for $K$ to be relatively small. For our specific numerical study, $K = 3$ or $K = 4$ seems to be a good choice.

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