APPLICATIONS OF THE HILBERT PROBLEM TO PROBLEMS OF MATHEMATICAL PHYSICS

PROEFSCHRIFT

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I N T R O D U C T I O N

Problems arising in mathematical physics, can in general be classified from different points of view. From the physical formulation a classification to several branches of physics and mechanics is obvious. From a mathematical point of view, problems, originating in different parts of physics, can sometimes be submitted to a uniform treatment by the same mathematical method. The problems dealt with in this thesis have in common, that they all can be formulated in terms of the Hilbert problem. This problem can be treated by the use of sectionally holomorfic functions (ref. 1), which concept is based on a set of formulae derived by Plemelj. These functions are regular in the whole complex domain with the exception of a discontinuity on a curve. In chapter 1 we give a short survey of the notations and results of this theory.

The chapters 2-5 are concerned with Wiener-Hopf type integral and integro-differential equations. Usually these equations are solved under the condition that there exists some strip of convergence for the Fourier-transformations (ref. 2,3). Using however, the theory of sectionally holomorfic functions it becomes clear that a strip is not essential and we need only to demand convergence on a line. In chapter 3 we shall discuss an application of the theory on a problem of shrink-fit stresses. The stresses are calculated by a method which is equivalent to the procedure for obtaining approximate solutions described in an article by W.T. Koiter (ref. 4). In the next chapter we discuss the homogeneous Wiener-Hopf type integro-differential equation with first order derivatives. We conclude the treatment of Wiener-Hopf type equations by considering an integro-differential equation with a fourth order derivative which occurs under the sign of integration. This equation is a result of considerations about the anomalous skin - effect of electrons in a metal (ref. 5) and is a generalisation of an equation discussed by Reuter and Sondheimer (ref. 6).

In the chapters 6 and 7 we discuss two problems on the motion of water with a free surface. First the two dimensional problem of the reflection and transmission of progressing waves, when a part of the surface of the water is fixed by introducing on the surface a rigid strip of infinite length. This is the so called finite dock problem. A proof of the existence of the solution has been given by H. Rubin (ref. 7), while Mac Camy (ref. 8) discusses the pressure under the dock. Here we have to solve a Hilbert problem for a function which possesses some prescribed discontinuity on the strip. The second problem of this part considers the forces exerted by the water when the strip executes a vertical flexural vibration of high frequency. The results are compared with the forces at a half immersed cylinder which executes a vibration with shear deformation. This has been done in order to obtain a check on the three dimensional correction coefficient for the added mass, used in naval architecture (ref. 9). The question arose whether this correction coefficient, derived from the added mass of a vibrating ellipsoid of revolution, would be accurate enough for ships of shallow draught. We have to consider a singular integral equation which can be reduced, by the theory of sectional holomorfic functions, to an integral equation of the Fredholm type.

Chapter I

THE HILBERT PROBLEM

The formulae of Plemelj, the Hilbert problem and singular integral equations are discussed thoroughly in ref. 1. For direct reference, however, we shall state some results. We shall not enter into details but consider the theory to an extent necessary for understanding the applications.

1.1 THE FORMULAE OF PLEMELJ

Let L be a smooth arc, in the complex plane, defined by

$$x=x(s), \quad y=y(s), \quad s_a \leq s \leq s_b, \qquad (1.1.1)$$

where s is a parameter and x(s) and y(s) have continuous first order derivatives which do not vanish simultaneously. Also we assume L to be simple; this means that never $x(s_1) = x(s_2)$ and

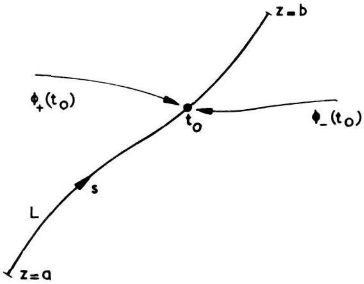

Fig.1.1.1 Arc of discontinuity for a sectionally holomorfic function.

$y(s_1) = y(s_2)$ for values $s_a \leq s_1 \neq s_2 \leq s_b$.

On this arc we consider a function $\varphi=\varphi(t)$ which satisfies the Hölder condition

$$|\varphi(t_2)-\varphi(t_1)| < A|t_2-t_1|^{\mu}, \qquad (1.1.2)$$

where t=t(s) is a point of L which corresponds to the parameter value s and A and μ are positive constants. Then we form the Cauchy integral

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{(t-z)}\,dt. \qquad (1.1.3)$$

From this definition we see that $\Phi(z)$ is an analytic function in the entire complex domain with the exception of the arc L. At L the values of $\Phi(z)$ exhibit a jump by passing from one side of L to the other. Introducing on L a positive direction for increasing values of the parameter s we call the left hand side of L the positive "+" side and the right hand side the negative "-" side. The limiting values of $\Phi(z)$ are denoted respectively by $\Phi_+(t)$ and $\Phi_-(t)$ (fig. 1.1.1). The values of $\Phi(z)$ are continuous up to L, with exception of the ends of L for which $\varphi(t) \neq 0$ and satisfy, as is proved in ref. 1, the following relations of Plemelj

$$\Phi_+(t_o) - \Phi_-(t_o) = \varphi(t_o) \tag{1.1.4}$$

and

$$\Phi_+(t_o) + \Phi_-(t_o) = \frac{1}{\pi i} \oint_L \frac{\varphi(t)}{t-t_o}\, dt. \tag{1.1.5}$$

The integral is to be taken in the sense of Cauchy

$$\oint_L \frac{\varphi(t)}{t-t_o}\, dt = \lim_{\ell \to o} \int_{L-\ell} \frac{\varphi(t)}{t-t_o}\, dt, \tag{1.1.6}$$

where ℓ is a part of L with ends t_1 and t_2 in such a way that $s_1 < s_o < s_2$ and $|t_2-t_o| = |t_o-t_1|$.

Formulae (1.1.4) and (1.1.5) can be verified directly in the case that $\varphi(t)$ represents the values on L of a function $\varphi(z)$ analytic in a neighbourhood of L. In this case we may deform L slightly (fig. 1.1.2) in order to calculate for

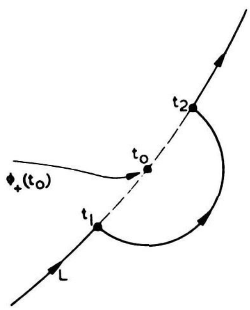

Fig.1.1.2. Deformation of L when $\varphi(z)$ is analytic.

instance the limit value

$$\Phi_+(t_o) = \frac{1}{2\pi i} \oint_L \frac{\varphi(t)}{t-t_o}\, dt + \tfrac{1}{2}\varphi(t_o). \tag{1.1.7}$$

Analogous

$$\Phi_-(t_o) = \frac{1}{2\pi i} \oint_L \frac{\varphi(t)}{t-t_o}\, dt - \tfrac{1}{2}\varphi(t_o). \qquad (1.1.8)$$

Subtracting these formulae yields (1.1.4), adding yields (1.1.5).

Now we come to the definition of a sectionally holomorfic function. A function will be called sectionally holomorfic when it is holomorfic in each finite region which does not contain points of some smooth line L, while it is continuous up to L with possible exception of the ends of L.

It is proved in ref. 1. that $\Phi(z)$ is such a function.

1.2 THE HILBERT PROBLEM FOR AN ARC

The problem is to find a sectionally holomorfic function $\Phi(z)$ which satisfies the relation

$$\Phi_-(t) - G(t)\, \Phi_+(t) = g(t) \qquad (1.2.1)$$

on a smooth arc L, where G(t) and g(t) are given functions, which satisfy the Hölder condition and $G(t) \neq 0$ on L.

First we consider the homogeneous equation

$$\Psi_-(t) = G(t)\, \Psi_+(t). \qquad (1.2.2)$$

We solve this by trying to find a sectionally holomorfic function $\Psi(z)$ without zero's in the whole complex domain. Then $\ln \Psi(z)$ is also sectionally holomorfic and has to satisfy

$$\{\ln \Psi(t)\}_+ - \{\ln \Psi(t)\}_- = -\ln G(t) \qquad (1.2.3)$$

on L. Comparing (1.2.3) and (1.1.4) we find as a solution of (1.2.2)

$$\Psi(z) = \exp - \frac{1}{2\pi i} \int_L \frac{\ln G(t)}{(t-z)}\, dt. \qquad (1.2.4a)$$

However by multiplying this function by an arbitrary rational function $P_1(z)$, which may possess poles only at the ends of L, we do not disturb relation (1.2.2). Hence

$$\Psi(z) = \{ \exp - \frac{1}{2\pi i} \int_L \frac{\ln G(t)}{(t-z)}\, dt \}\, P_1(z) \qquad (1.2.4b)$$

is a more general solution of (1.2.2).

In order to deal with the inhomogeneous equation, we write (1.2.1) in the form

$$\frac{\Phi_-(t)}{\Psi_-(t)} - \frac{\Phi_+(t)}{\Psi_+(t)} = \frac{g(t)}{\Psi_-(t)}\, . \qquad (1.2.5)$$

Hence again by (1.1.4) we obtain

$$\Phi(z) = -\frac{\Psi(z)}{2\pi i}\left\{\int_L \frac{g(t)}{\Psi_-(t)(t-z)}\,dt + P_2(z)\right\}, \qquad (1.2.6)$$

where it is assumed that $\Psi(z)$ is chosen in such a way that the integral converges, $P_2(z)$ is again an arbitrary rational function with the same restriction as $P_1(z)$ in (1.2.4b).

In ref. 1 it is shown that (1.2.6) and (1.2.4b) are the general solutions of the problems (1.2.1) and (1.2.2) when the behaviour at infinity is prescribed to be algebraic.

When the arc L becomes an infinite line, for instance parallel to the real axis, the sectionally holomorfic function $\Psi(z)$ in (1.2.4) is cut into two separate functions $\Psi_+(z)$ and $\Psi_-(z)$ which are analytic in the half planes S_+ and S_-, situated at the + and - side of L. Assuming that the integral in (1.2.4) is convergent for this line we see that the solution of (1.2.2) yields the "factorisation" of a function G(t) defined on L

$$G(t) = \Psi_-(t)/\,\Psi_+(t) \qquad (1.2.7)$$

into two functions $\Psi_+(t)$ and $\Psi_-(t)$ which are boundary values of functions $\Psi_+(z)$ and $\Psi_-(z)$ regular and without zero's in S_+ and S_- respectively and continuous up to L. It is clear that we have to take z in S_+ or S_- when we calculate $\Psi_+(z)$ or $\Psi_-(z)$ with the use of (1.2.4).

The next step done in (1.2.6) is to form, under the assumption of convergence, the integral

$$\Theta_\pm(z) = \frac{1}{2\pi i}\int_L \frac{g(t)}{\Psi_-(t)(t-z)}\,dt, \quad z \text{ in } S_\pm. \qquad (1.2.8)$$

This means for an infinite line L that we "split" the function $g(t)/\Psi_-(t)$ defined on L

$$g(t)/\Psi_-(t) = \Theta_+(t) - \Theta_-(t) \qquad (1.2.9)$$

into two functions $\Theta_+(t)$ and $\Theta_-(t)$ which are boundary values of functions $\Theta_+(z)$ and $\Theta_-(z)$ regular in S_+ and S_- and continuous up to L.

Sometimes it is necessary to multiply G(t) by a simple function in order to produce convergence of the integral in (1.2.4). This will be demonstrated later on (para. 2.3).

The solution of (1.2.1) for an infinite line can now be described very briefly as follows. First, factorize G(t), this yields (1.2.5). Second, split the right hand side of (1.2.5). Third, compare the parts analytic in the same half planes, with the result

$$\Phi_\pm(z) = -\frac{\Psi_\pm(z)}{2\pi i}\left\{\int_L \frac{g(t)}{\Psi_-(t)(t-z)}\,dt + P_2(z)\right\}, \quad z \text{ in } S_\pm. \qquad (1.2.10)$$

1.3 SINGULAR INTEGRAL EQUATIONS

At last we discuss in this chapter the relation between singular integral equations and the Hilbert problem. In chapter 7 we shall have to treat a singular integral equation of the form

$$\oint_{-1}^{+1}\left\{\frac{1}{\pi i(t-t_o)}+k(t-t_o)\right\}\varphi(t)dt = f(t_o), \tag{1.3.1}$$

where $\varphi(t)$ is the unknown function and $k(t)$ and $f(t)$ are given functions, $k(t)$ is bounded at the interval $|t| < 2$. First we consider the "dominant" part of this equation, defined as

$$\frac{1}{\pi i}\oint_{-1}^{+1}\frac{\varphi(t)}{(t-t_o)}dt = f(t_o), \tag{1.3.2}$$

which is closely related to the theory of sectional holomorfic functions. Introducing

$$\Phi(z) = \frac{1}{2\pi i}\int_{-1}^{+1}\frac{\varphi(t)}{(t-z)}dt \tag{1.3.3}$$

we find from (1.1.5) and (1.3.2)

$$\Phi_+(t) + \Phi_-(t) = f(t). \tag{1.3.4}$$

Substitution of $G(t) = -1$ in (1.2.4) shows that we may take as a solution $\Psi(z)$ of the homogeneous part of (1.3.4)

$$\Psi(z) = \frac{1}{(1-z^2)^{\frac{1}{2}}}\ . \tag{1.3.5}$$

The solution of (1.3.4) becomes by using (1.2.6)

$$\Phi(z) = \frac{1}{2\pi i(1-z^2)^{\frac{1}{2}}}\left\{\int_{-1}^{+1}\frac{f(t)(1-t^2)^{\frac{1}{2}}dt}{(t-z)} + P(z)\right\}, \tag{1.3.6}$$

where we have fixed the values of $\Psi(z)$ by taking $\Psi_-(t) = -|(1-t^2)^{\frac{1}{2}}|$, $t \leq 1$. Then by (1.1.4)

$$\varphi(t_o) = \frac{1}{\pi i(1-t_o^2)^{\frac{1}{2}}}\left\{\int_{-1}^{+1}\frac{f(t)(1-t^2)^{\frac{1}{2}}}{(t-t_o)}dt + A\right\}. \tag{1.3.7}$$

Here we have chosen for the arbitrary polynomial $P(z)$ the arbitrary constant

$$A = i\int_{-1}^{+1}\varphi(t)dt, \tag{1.3.8}$$

in order that (1.3.7) satisfies (1.3.2).

Now we return to (1.3.1) which we write in the form

$$\frac{1}{\pi i}\oint_{-1}^{+1}\frac{\varphi(t)}{(t-t_o)}\,dt = f(t_o)-\int_{-1}^{+1}\varphi(t)\,k(t-t_o)dt. \qquad (1.3.9)$$

Then apparently each solution of

$$\varphi(t_o)=\frac{B}{\pi i(1-t_o^2)^{\frac{1}{2}}}+\varphi_d(t_o)+\frac{1}{\pi i(1-t_o^2)^{\frac{1}{2}}}\int_{-1}^{+1}\frac{\int_{-1}^{+1}\varphi(t)k(t-\tau)dt(1-\tau^2)^{\frac{1}{2}}d\tau}{(\tau-t_o)} \qquad (1.3.10)$$

where $\varphi_d(t_o)$ is a solution of the dominant part and B is some arbitrary constant, satisfies also (1.3.9). This can be verified by deviding both sides of (1.3.10) by $(t-t_o)$ and integrating from -1 to $+1$ with respect to t_o. Instead of (1.3.10) we may consider

$$f(t_o)=\frac{B}{\pi i(1-t_o^2)^{\frac{1}{4}}}+\varphi_d(t_o)(1-t_o^2)^{\frac{1}{4}}+\int_{-1}^{+1}f(t)\,K(t,t_o)dt \qquad (1.3.11)$$

where

$$f(t_o) = \varphi(t_o)(1-t_o^2)^{\frac{1}{4}}$$

and

$$K(t,t_o)=\frac{1}{\pi i(1-t_o^2)^{\frac{1}{4}}(1-t^2)^{\frac{1}{4}}}\int_{-1}^{+1}\frac{k(t-\tau)(1-\tau^2)^{\frac{1}{2}}}{(\tau-t_o)}\,d\tau. \qquad (1.3.12)$$

This integral equation is of the second kind and the known function and the kernel are quadratic integrable. Hence the theories on the Neumann expansion of the solution and the replacement of the kernel by approximating kernels of a simpler type can be applied.

Chapter 2.

THE WIENER-HOPF TYPE INTEGRAL EQUATION.

We now shall give a treatment of Wiener-Hopf type integral equations, which resembles the classical procedure. It deviates however at one point since we shall not demand a strip of convergence of the Fourier integrals to be used, but only a line. This is an immediate consequence of the fact that we start from the concept of the sectionally holomorfic function.

2.1 TRANSFORMATION OF THE EQUATIONS

The following three equations are considered, the homogeneous equation of the second kind

$$f(x) - \int_0^\infty k(x-\xi)\, f(\xi)d\xi = 0, \qquad (2.1.1)$$

the inhomogeneous equation of the second kind

$$f(x) - \int_0^\infty k(x-\xi)\, f(\xi)d\xi = h(x), \qquad (2.1.2)$$

and the equation of the first kind

$$\int_0^\infty k(x-\xi)\, f(\xi)d\xi = h(x), \qquad (2.1.3)$$

where $k(x-\xi)$ is the kernel of the equations, $h(x)$ is a function known for $x > 0$ and $f(x)$ is the unknown function. The equations are valid for all values $-\infty < x < +\infty$, hence we shall have to determine in equation (2.1.3) also the values of $h(x)$ for $x < 0$.

We introduce the notation for the Fourier transformation of a function $f(x)$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)e^{i\lambda x}\, dx = F(\lambda) \qquad (2.1.4)$$

and for the one sided transforms

$$\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) e^{i\lambda x} \, dx = F_+(\lambda), \qquad (2.1.5)$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(x) e^{i\lambda x} \, dx = F_-(\lambda). \qquad (2.1.6)$$

In order to apply a Fourier - transformation to (2.1.1), (2.1.2) and (2.1.3) it is sufficient to assume that $e^{-\mu x} k(x)$, $e^{-\mu x} h(x)$ and $e^{-\mu x} f(x)$ belong to L $(-\infty, +\infty)$. On these assumptions the Fourier transform $K(\lambda)$ of $k(x)$ exists for Im $\lambda = \mu$ and the one sided transforms of $k(x)$ and $f(x)$ denoted by $H_+(\lambda)$, $F_+(\lambda)$ and $H_-(\lambda)$, $F_-(\lambda)$ exist for Im $\lambda = \mu$ and are regular in S_+ with Im $\lambda > \mu$ resp. S_- with Im $\lambda < \mu$.

By the convolution theorem equations (2.1.1), (2.1.2) and (2.1.3) are equivalent to the following Hilbert - problems

$$F_-(\lambda)+\{1-\sqrt{2\pi}\, K(\lambda)\}\, F_+(\lambda) = 0, \qquad (2.1.7)$$

$$F_-(\lambda)+\{1-\sqrt{2\pi}\, K(\lambda)\}F_+(\lambda)=H_+(\lambda)+H_-(\lambda), \qquad (2.1.8)$$

$$F_+(\lambda)\, \sqrt{2\pi}\, K(\lambda) - H_-(\lambda) = H_+(\lambda), \qquad (2.1.9)$$

holding on the line L of infinite length with Im $\lambda = \mu$, where $F_-(\lambda)$, $F_+(\lambda)$ and $H_-(\lambda)$ are unknown functions. In (2.1.8) we consider $F_-(\lambda) - H_-(\lambda)$ as one unknown function.

We assume that the known functions $K(\lambda)$ and $H_+(\lambda)$ in (2,1,7), (2,1,8) and (2,1,9) satisfy the Hölder condition (1,1,2) on L, in which case we can apply the theory of chapter 1.

2.2 SOLUTION OF THE HOMOGENEOUS INTEGRAL EQUATION OF THE SECOND KIND

We first consider the Hilbert problem (2.1.7) which corresponds to the homogeneous equation (2.1.1) and assume a strip β of convergence for the integral (2.1.4) for $k(x)$. The case of a line of convergence will be discussed at the end of this paragraph. In β we choose a line L parallel to the real axis in the λ plane, on which no zero of $\{1-\sqrt{2\pi}\, K(\lambda)\}$ lies. The line L with a positive direction (viz. Re $\lambda \to +\infty$) defines the half planes S_+ and S_-.

Of course we have no knowledge a priori whether the Fourier-transformation of (2.1.1) actually holds on L. However, if the transformation holds on some line L in β we may determine this line afterwards and construct the solution to our problem (2.1.1).

In order to be able to apply the theory of para. 1.2. it is necessary that the integral in (1.2.4) converges for the line L of infinite length. This is the case when

$$\lim_{\mathrm{Re}\,\lambda\to\pm\infty} \ln\{1-\sqrt{2\pi}\,K(\lambda)\} = 0. \tag{2.2.1}$$

On first sight this seems to be true because $K(\lambda)$ is a Fourier-transform and hence $K(\lambda)\to 0$ for $\mathrm{Re}\,\lambda \to \pm\infty$. It remains however possible that

$$\lim_{\mathrm{Re}\,\lambda\to+\infty} \ln\{1-\sqrt{2\pi}\,K(\lambda)\} - \lim_{\mathrm{Re}\,\lambda\to-\infty} \ln\{1-\sqrt{2\pi}\,K(\lambda)\} = -2n\pi i \tag{2.2.2}$$

where n is an integer. If we consider instead of (2.1.7) the problem

$$\Psi_-(\lambda) + \{1-\sqrt{2\pi}\,K(\lambda)\}\frac{(\lambda-a)^n}{(\lambda-b)^n}\,\Psi_+(\lambda)=0 \tag{2.2.3}$$

where a is a point in S_+ and b a point in S_-, then (2.2.1) is satisfied. We can write down at once the solution of (2.2.3) with the aid of (1.2.4) and the discussion above (1.2.7)

$$\Psi_\pm(\lambda)=\pm\{\exp - \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\ln[\{1-\sqrt{2\pi}\,K(\eta)\}\frac{(\eta-a)^n}{(\eta-b)^n}]}{(\eta-\lambda)}\,d\eta\}P(\lambda),\ \lambda \text{ in } S_\pm. \tag{2.2.4}$$

Hence also the functions

$$F_+(\lambda)=(\lambda-b)^{-n}\,\Psi_+(\lambda),\quad F_-(\lambda)=(\lambda-a)^{-n}\,\Psi_-(\lambda) \qquad (2.2.5)(2.2.6)$$

are analytic in S_+ and S_- respectively and are solutions of the Hilbert problem (2.1.7)

We wish to obtain functions $F_+(\lambda)$ and $F_-(\lambda)$ which are Fourier-transforms, this restricts $P(\lambda)$, because for $\mathrm{Re}\,\lambda \to \pm\infty$ $F_+(\lambda)$ and $F_-(\lambda)$ must tend to zero.

Now suppose that, for the case of a strip β of convergence of the transform of the kernel $k(x)$, the function $\{1-\sqrt{2\pi}\,K(\lambda)\}$ has zero's λ_1; λ_2;.... in the S_+ part of β, arranged according to increasing imaginary parts. We investigate the functions

$$F_{1+}(\lambda)=\frac{F_+(\lambda)}{(\lambda-\lambda_1)},\qquad F_{1-}(\lambda)=\frac{F_-(\lambda)}{(\lambda-\lambda_1)}. \qquad (2.2.7)(2.2.8)$$

These functions are regular in the regions S_{1+} and S_{1-}, which are situated each on one side of a line L_1, which is parallel to L but lies between the zero's λ_1 and λ_2. That $F_{1+}(\lambda)$ is holomorfic in S_{1+} is obvious and that $F_{1-}(\lambda)$ is holomorfic in S_{1-} follows from equation (2.1.7) because

$F_-(\lambda)$ and $\{1-\sqrt{2\pi}\ K(\lambda)\}$ must have the same zero's in the S_+ part of β. So it is possible, by moving up L parallel to the real axis, to construct new functions $F_{1+}(\lambda)$ and $F_{1-}(\lambda)$...., which are of lower order for $|\lambda| \to \infty$ and which satisfy equation (2.1.7) on new lines L_1, L_2 .... .

In the case that $F_+(\lambda)$ and $F_-(\lambda)$ are not Fourier-transforms, it is possible that F_{1+}, F_{1-} or F_{2+}, F_{2-} .... will be. So the most direct way to obtain all solutions is to take the line L above the zero with the largest imaginary part within the strip of regularity of $K(\lambda)$ and to evaluate $f(x)$ by the inverse transformation. However it may be more convenient to take for L, if possible, the real axis, in view of the evaluation or approximation of the integrals in (2.2.4) and to translate L afterwards into the correct position.

When the transformation of $k(x)$ is only permitted on a line L, while $\{1-\sqrt{2\pi}\ K(\lambda)\}$ has no zero on L we can use the above theory. If in this case $\{1-\sqrt{2\pi}\ K(\lambda)\}$ does possess zero's on L we can also solve the problem, this will be discussed in para. 4.4.

2.3 THE INHOMOGENEOUS EQUATION OF THE SECOND KIND AND THE EQUATION OF THE FIRST KIND

We are now in a position to solve the inhomogeneous equation (2.1.8). It is assumed that the line L, on which we consider the Hilbert problem, is within the strip β and above the zero of $\{1-\sqrt{2\pi}\ K(\lambda)\}$ with the largest imaginary part within β.

First we consider the case that the integer n defined in (2.2.2) is positive or zero. We select a suitable solution of the homogeneous equation (2.1.7) from the set provided by (2.2.5) and (2.2.6) by taking for $P(\lambda)$ some polynomial of degree n without zero's on L. Denoting this solution by $Y_+(\lambda)$ and $Y_-(\lambda)$ we may assume

$$\lim_{\mathrm{Re}\ \lambda\to\pm\infty} Y_\pm(\lambda) = 1. \tag{2.3.1}$$

We can rewrite (2.1.8) in the form

$$\frac{\{F_-(\lambda)-H_-(\lambda)\}}{Y_-(\lambda)} - \frac{F_+(\lambda)}{Y_+(\lambda)} = \frac{H_+(\lambda)}{Y_-(\lambda)}. \tag{2.3.2}$$

This equation has by (1.2.10) the solution

$$F_+(\lambda) = -\frac{Y_+(\lambda)}{2\pi i}\int_L \frac{H_+(\eta)}{Y_-(\eta)(\eta-\lambda)}\,d\eta,\ \lambda \text{ in } S_+, \tag{2.3.3}$$

$$F_-(\lambda)-H_-(\lambda) = -\frac{Y_-(\lambda)}{2\pi i}\int_L \frac{H_+(\eta)}{Y_-(\eta)(\eta-\lambda)}\,d\eta,\ \lambda \text{ in } S_-. \tag{2.3.4}$$

In this case ($n \geqq 0$) the solutions (2.3.3), (2.3.4) can be interpreted as Fourier-transforms and we find the solution of (2.1.2) by the inverse transformation.

For $n < 0$ we cannot find functions $\Psi_+(\lambda)$ and $\Psi_-(\lambda)$ with the property (2.3.1) and in general we cannot interpret in this case (2.3.3) and (2.3.4) as a Fourier transform. Only for special functions $H_+(\eta)$ (2.3.3) and (2.3.4) will tend to zero for Re $\lambda \to \pm\infty$.

We now shall treat equation (2.1.9) which reads

$$F_+(\lambda)\sqrt{2\pi}\,K(\lambda)-H_-(\lambda) = H_+(\lambda). \qquad (2.3.5)$$

This equation differs from (2.1.8) in a rather significant way, viz. the function $\sqrt{2\pi}\,K(\lambda)$ which is here the factor of $F_+(\lambda)$ tends to zero when Re $\lambda \to \pm\infty$. This means that we cannot use (1.2.4) because the integral will not exist for the line L which is infinite. We assume the existence of a function $K^*(\lambda)$ with the properties

$$\lim_{\text{Re}\,\lambda \to \pm\infty} K(\lambda)/K^*(\lambda) = 1, \qquad (2.3.6)$$

and

$$|M(\lambda)-1| = |K(\lambda)/K^*(\lambda)-1| < \varepsilon < 1, \qquad (2.3.7)$$

for λ on L. Further we assume that the function $K^*(\lambda)$ can be "factorized" by inspection in the following way

$$K^*(\lambda) = K^*_+(\lambda)/K^*_-(\lambda), \qquad (2.3.8)$$

where $K^*_+(\lambda)$ and $K^*_-(\lambda)$ are functions which behave algebraicly at $|\lambda| \to \infty$ and are regular and without zero's in S_+ and S_- respectively. Then we consider the equation

$$\sqrt{2\pi}\,X_+(\lambda)\,M(\lambda) - X_-(\lambda) = H_+(\lambda)\,K^*_-(\lambda) \qquad (2.3.9)$$

which is of the type (2.1.8), while the integer n (2.2.2) is zero. Hence the problem can be treated in the indicated way. The solutions of the original equation (2.3.5) then become

$$F_+(\lambda)=X_+(\lambda)/K^*_+(\lambda) \quad , \quad F_-(\lambda)=X_-(\lambda)/K^*_-(\lambda). \qquad (2.3.10)$$

When ε, defined in (2.3.7) can be made sufficiently small we can use this method for obtaining approximate solutions. This will be demonstrated in chapter 3, where we discuss a shrink-fit problem.

2.4 EXAMPLES

First we consider an equation which is solved in ref. 2 by the method of Wiener and Hopf

$$f(x) = m \int_0^\infty e^{-|x-\xi|} f(\xi)d\xi. \tag{2.4.1}$$

We assume $0 < m < \frac{1}{2}$. After transformation we find

$$F_+(\lambda) \{1 - \frac{2m}{(1+\lambda^2)}\} + F_-(\lambda) = 0. \tag{2.4.2}$$

The zero's of $\{1 - \frac{2m}{(1+\lambda^2)}\}$ are $\lambda_1 = -(2m-1)^{\frac{1}{2}}$, $\lambda_2 = +(2m-1)^{\frac{1}{2}}$,

where we assume Im $\lambda_2 > 0$. The strip of convergence for the Fourier-transformation of the kernel is $-1 <$ Im $\lambda < +1$. In this strip we choose the real axis as the line L, on which we have to solve the Hilbert problem (2.4.2). On this line $\{1 - \frac{2m}{1+\lambda^2}\} > 0$ and hence the number n in (2.2.2) is zero. The

integral in (2.2.4) becomes

$$\int_{-\infty}^{+\infty} \frac{\ln\{1 - \frac{2m}{1+\eta^2}\}}{(\eta-\lambda)} d\eta = \ln(\eta-\lambda) \ln\{1 - \frac{2m}{1+\eta^2}\} \Big/_{-\infty}^{+\infty} -$$

$$- \int_{-\infty}^{+\infty} \frac{\ln(\eta-\lambda)}{\{1 - \frac{2m}{1+\eta^2}\}} \frac{4m\,\eta}{(1+\eta^2)^2} d\eta = -4m \int_{-\infty}^{+\infty} \frac{\eta \ln|\eta-\lambda|}{(1+\eta^2)(1+\eta^2-2m)} d\eta.$$

This last integral can be calculated by residues, where we have to take care that our contour does not enclose the branch point $\eta = \lambda$. Substitution of the results in (2.2.4) yields

$$F_+(\lambda) = \frac{(\lambda+i)}{(\lambda-\lambda_1)} \cdot P(\lambda), \quad \lambda \text{ in } S_+, \tag{2.4.3}$$

$$F_-(\lambda) = \frac{-(\lambda-\lambda_2)}{(\lambda-i)} \cdot P(\lambda), \quad \lambda \text{ in } S_-. \tag{2.4.4}$$

These two functions do not tend to zero for Re $\lambda \to \pm\infty$ and hence cannot be interpreted as Fourier-transforms. However, by translating L upwards over a distance between Im λ_2 and 1, and dividing $F_+(\lambda)$ and $F_-(\lambda)$ by $(\lambda-\lambda_2)$ we obtain

$$F_+(\lambda) = \frac{(\lambda+i)\ P(\lambda)}{(\lambda-\lambda_1)(\lambda-\lambda_2)}\ , \quad F_-(\lambda) = -\frac{P(\lambda)}{(\lambda-i)}\ . \tag{2.4.5}$$

It is obvious that $P(\lambda)$ must be a constant. By the inverse transformation we find the solution of (2.4.1).

$$\begin{aligned} f(x) &= C\ \{\cos\ ((2m-1)^{\frac{1}{2}}\ x) + \frac{\sin\ ((2m-1)^{\frac{1}{2}} x)}{(2m-1)^{\frac{1}{2}}}\}\ , \quad x > 0, \\ f(x) &= C\ e^{x}\ , \quad x < 0. \end{aligned} \tag{2.4.6}$$

As a second example we consider the equation

$$f(x) - m\int_0^{\infty} \frac{f(\xi)}{1+(x-\xi)^2}\, d\xi = h(x),\ m < \frac{1}{\pi}\ . \tag{2.4.7}$$

This equation cannot be solved with the method of Wiener and Hopf because the Fourier-transform of the kernel converges only on the real axis.

Fourier-transformation yields the equation

$$F_+(\lambda)\{1-m\pi e^{-|\lambda|}\} + F_-(\lambda) = H_+(\lambda) + H_-(\lambda). \tag{2.4.8}$$

The function $\{1-m\pi e^{-|\lambda|}\}$ is positive on the real axis and hence n (2.2.2) is zero. The solution of the homogeneous part of (2.4.8) is obtained from (2.2.4) and this solution can not be interpreted as a Fourier-transform. However, by taking $P(\lambda) \equiv 1$, we may obtain the solution of the inhomogeneous equation by the procedure outlined in para.2.3. In the present case

$$Y_{\pm}(\lambda) = \exp -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\ln\{1-m\pi e^{-|\eta|}\}}{(\eta-\lambda)}\, d\eta,\ \lambda \text{ in } S_{\pm}. \tag{2.4.9}$$

Then the solutions (2.3.3) and (2.3.4) may be interpreted as a Fourier-transform. The solution of (2.4.7) can now be evaluated by the inverse transformation.

Chapter 3

A SHRINK-FIT PROBLEM FOR A HALF INFINITE RANGE OF CONTACT

We consider an infinite elastic tube which is shrunk onto a semi infinite rigid shaft. In dealing with this problem we aim exclusively at the contact pressure between shaft and tube, for which an integral equation of the Wiener-Hopf type is established. The integral representing the contact pressure is approximated numerically by a method equivalent to the one developped in ref. 4, which rests on approximating the kernel of the governing integral equation.

3.1 FORMULATION OF THE PROBLEM, DETERMINATION OF GREEN'S FUNCTION FOR THE TUBE

Let $(\bar{r},\theta,\bar{x})$ be cylindrical coordinates such that the $\bar{x}$ axis coincides with the axis of the tube (fig.3.1.1). Let $\bar{a}$ and $\bar{b}$ be the inner and outer diameter of the tube, respectively. Assume that the uniform radial shrinkage of the tube is $\bar{\vartheta}$. The stress distribution in the tube for the case to be considered here is then governed by the following boundary conditions:

$$\text{for} \quad \bar{r} = \bar{b}, \quad -\infty < \bar{x} < +\infty, \quad \tau_{rx} = 0, \; \sigma_r = 0, \tag{3.1.1}$$

$$\text{for} \quad \bar{r} = \bar{a}, \quad \bar{x} < 0, \quad \tau_{rx} = 0, \; \sigma_r = 0, \tag{3.1.2}$$

$$\text{for} \quad \bar{r} = \bar{a}, \quad \bar{x} > 0, \quad \tau_{rx} = 0, \; \bar{u} = \bar{\vartheta}, \tag{3.1.3}$$

where $\sigma_r(\bar{r},\bar{x})$ and $\tau_{rx}(\bar{r},\bar{x})$ are the normal and the tangential stresses respectively, and $\bar{u}$ is the radial displacement. We seek $\sigma_r(\bar{a},\bar{x})$ for $0 < \bar{x} < \infty$ appropriate to the stress distribution governed by (3.1.1), (3.1.2) and (3.1.3). To this end we note that this normal stress must satisfy the integral equation

$$\bar{\vartheta} = \int_0^\infty \bar{u}_0(\bar{x}-\xi)\sigma_r(\bar{a},\xi)\,d\xi, \tag{3.1.4}$$

where $\bar{u}_o(\bar{x})$ is a Green function which will be defined presently.

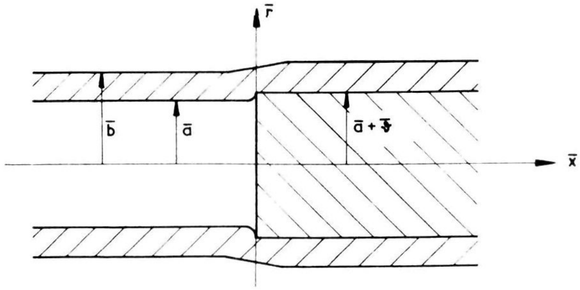

Fig.3.1.1. Infinite elastic tube shrunk onto a semi-infinite rigid shaft.

The function $\bar{u}_o(\bar{x})$ is the radial displacement of the inner wall of the tube corresponding to the following singular loading conditions:

$$\text{for} \quad \bar{r} = \bar{b}, \quad -\infty < \bar{x} < +\infty, \quad \tau_{rx} = 0, \sigma_r = 0, \tag{3.1.5}$$

$$\text{for} \quad \bar{r} = \bar{a}, \quad -\infty < \bar{x} < +\infty, \quad \tau_{rx} = 0, \sigma_r = \delta(x), \tag{3.1.6}$$

$$\text{for} \quad \bar{a} \leq \bar{r} \leq \bar{b}, \qquad \bar{x} \to \pm\infty, \ \tau_{rx}, \sigma_r, \sigma_x, \sigma_\theta \to 0, \tag{3.1.7}$$

in which $\delta(x)$ is the delta function of Dirac. The rotational symmetry of this problem suggests an approach by means of Love's stress function (ref. 10) $\varphi(r,x)$ which satisfies the differential equation

$$\Delta^2\varphi \equiv \left(\frac{\partial^2}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}} + \frac{\partial^2}{\partial\bar{x}^2}\right)^2 \varphi = 0. \tag{3.1.8}$$

The associated stress field is given by

$$\begin{aligned} \sigma_r(\bar{r},\bar{x}) &= \frac{\partial}{\partial\bar{x}}\left[\nu\Delta - \frac{\partial^2}{\partial\bar{r}^2}\right]\varphi(\bar{r},\bar{x}), \\ \sigma_x(\bar{r},\bar{x}) &= \frac{\partial}{\partial\bar{x}}\left[(2-\nu)\Delta - \frac{\partial^2}{\partial\bar{x}^2}\right]\varphi(\bar{r},\bar{x}), \end{aligned} \tag{3.1.9}$$

$$\sigma_\theta(\bar{r},\bar{x}) = \frac{\partial}{\partial \bar{x}} [\nu\Delta - \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}]\, \varphi(\bar{r},\bar{x}),$$
$$\tau_{rx}(\bar{r},\bar{x}) = \frac{\partial}{\partial \bar{r}} [(1-\nu)\Delta - \frac{\partial^2}{\partial \bar{x}^2}]\, \varphi(\bar{r},\bar{x}), \tag{3.1.9}$$

where ν designates Poisson's ratio, which we take equal to 0,25 in numerical calculations.

The corresponding radial and axial displacements appear as

$$\bar{u}(\bar{r},\bar{x}) = -\frac{(1+\nu)}{E}\frac{\partial^2\varphi(\bar{r},\bar{x})}{\partial\bar{r}\partial\bar{x}}, \tag{3.1.10}$$

$$\bar{v}(\bar{r},\bar{x}) = \frac{(1+\nu)}{E}[(1-2\nu)\Delta + \frac{\partial^2}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}]\,\varphi(\bar{r},\bar{x}). \tag{3.1.11}$$

Thus we need to determine a function $\varphi_o(\bar{r},\bar{x})$ which meets (3.1.8) and is such that the stresses (3.1.9) conform to (3.1.5), (3.1.6) and (3.1.7). The desired kernel $\bar{u}_o(\bar{x})$ in (3.1.4) is then obtained from

$$\bar{u}_o(\bar{x}) \equiv \bar{u}(\bar{a},\bar{x}) = -\frac{(1+\nu)}{E}\frac{\partial^2\varphi_o(\bar{r},\bar{x})}{\partial\bar{r}\partial\bar{x}}\bigg|_{\bar{a},\bar{x}}. \tag{3.1.12}$$

We now establish $\varphi_o(\bar{r},\bar{x})$ with the aid of the Fourier - transform. Let $e^{i\lambda\bar{x}}\varphi(\bar{r},\bar{x})$ be absolutely integrable with respect to $\bar{x}$ in the interval $(-\infty, +\infty)$ for $\mu_1 < \operatorname{Im}\lambda < \mu_2$, then the Fourier transform

$$\Phi(\bar{r},\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\lambda\bar{x}}\varphi(\bar{r},x)\,d\bar{x}, \quad \mu_1 < \operatorname{Im}\lambda < \mu_2, \tag{3.1.13}$$

exists and (3.1.8) is carried into

$$(\frac{\partial^2}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}} - \lambda^2)^2\, \Phi(\bar{r},\lambda) = 0. \tag{3.1.14}$$

The general solution of (3.1.14) admits the representation

$$\Phi_o(\bar{r},\lambda) = A_1(\lambda)K_o(\lambda\bar{r}) + A_2(\lambda)\lambda\bar{r}K_1(\lambda\bar{r}) + B_1(\lambda)I_o(\lambda\bar{r}) + B_2(\lambda)\lambda\bar{r}I_1(\lambda\bar{r}), \tag{3.1.15}$$

where $I_o(\lambda\bar{r})$, $I_1(\lambda\bar{r})$ and $K_o(\lambda\bar{r})$, $K_1(\lambda\bar{r})$ are modified Besselfunctions of first and second kind, respectively, Throughout this work we shall use the definitions for these functions given in ref. 11. The arbitrary functions $A_1(\lambda)$, $A_2(\lambda)$, $B_1(\lambda)$ and $B_2(\lambda)$ are to be determined consistent with the transforms of the boundary conditions (3.1.5) and (3.1.6). This process yields four linear equations in the four unknowns A_1, A_2, B_1, B_2. Thus $\Phi_o(\bar{r},\lambda)$ is completely determined.

Upon inversion of $\Phi_o(\bar{r},\lambda)$ and substitution in (3.1.12) we obtain after some computation

$$\bar{u}_o(\bar{x}) = -\frac{(1-\nu^2)}{\pi E}\int_{-\infty}^{+\infty} e^{i\lambda\bar{x}}\,\frac{N(\bar{a},\bar{b},\lambda)}{\lambda D(\bar{a},\bar{b},\lambda)}\,d\lambda, \qquad (3.1.16)$$

with

$$N(\bar{a},\bar{b},\lambda) = \lambda\bar{b}[K_1(\lambda\bar{a})I_o(\lambda\bar{b}) + K_o(\lambda\bar{b})I_1(\lambda\bar{a})]^2 -$$

$$-[\frac{2(1-\nu)}{\lambda\bar{b}} + \lambda\bar{b}][(K_1(\lambda\bar{a})I_1(\lambda\bar{b}) - K_1(\lambda\bar{b})I_1(\lambda\bar{a})]^2 - \frac{1}{\lambda\bar{b}}, \qquad (3.1.17)$$

$$D(\bar{a},\bar{b},\lambda) = \lambda\bar{b}\,[\frac{2(1-\nu)}{\lambda\bar{a}} + \lambda\bar{a}][K_1(\lambda\bar{a})I_o(\lambda\bar{b}) + K_o(\lambda\bar{b})I_1(\lambda\bar{a})]^2 +$$

$$+ \lambda\bar{a}\,[\frac{2(1-\nu)}{\lambda\bar{b}} + \lambda\bar{b}][K_1(\lambda\bar{b})I_o(\lambda\bar{a}) + K_o(\lambda\bar{a})I_1(\lambda\bar{b})]^2 -$$

$$- \lambda^2\,\bar{a}\bar{b}\,[K_o(\lambda\bar{b})I_o(\lambda\bar{a}) - K_o(\lambda\bar{a})I_o(\lambda\bar{b})]^2 -$$

$$-[\frac{2(1-\nu)}{\lambda\bar{a}} + \lambda\bar{a}]\,[\frac{2(1-\nu)}{\lambda\bar{b}} + \lambda\bar{b}]\,.$$

$$.[K_1(\lambda\bar{b})I_1(\lambda\bar{a}) - K_1(\lambda\bar{a})I_1(\lambda\bar{b})]^2 - [\frac{4(1-\nu)}{\lambda^2\,\bar{a}\bar{b}} + \frac{\bar{a}}{\bar{b}} + \frac{\bar{b}}{\bar{a}}]. \qquad (3.1.18)$$

3.2 SOLUTION OF THE INTEGRAL EQUATION FOR THE CONTACT PRESSURE.

At this stage it is convenient to introduce the dimensionless variables

$$x=\bar{x}/\bar{a},\quad \xi=\bar{\xi}/\bar{a},\quad b=\bar{b}/\bar{a},\quad \vartheta=\bar{\vartheta}/\bar{a},\quad p(x) = -\frac{2(1-\nu^2)}{E}\,\sigma_r(\bar{a},\bar{x}). \qquad (3.2.1)$$

Then (3.1.4) can be written

$$\vartheta = \frac{1}{\sqrt{2\pi}} \int_0^\infty k(x-\xi)\, p(\xi)\, d\xi \,, \tag{3.2.2}$$

where

$$k(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\lambda x} \frac{N(1,b,\lambda)}{\lambda D(1,b,\lambda)}\, d\lambda \,. \tag{3.2.3}$$

The integral equation (3.2.2) is of the Wiener-Hopf type. We first investigate whether the Fourier transform $K(\lambda)$ of the kernel $k(x)$ possesses a strip of regularity in the complex λ-plane. From (3.2.3) we obtain

$$K(\lambda) = \frac{1}{\lambda} \frac{N(1,b,\lambda)}{D(1,b,\lambda)} \,. \tag{3.2.4}$$

Using known expansions of the modified Bessel-functions, we find

$$K(0) = \frac{1}{2(1-\nu^2)} \frac{b^2(1+\nu) + (1-\nu)}{(b^2-1)} \,, \tag{3.2.5}$$

$$K'(0) = 0 \tag{3.2.6}$$

and

$$K(\lambda) = \frac{1}{|\lambda|} + O(\lambda^{-2}) \text{ as } \operatorname{Re}\lambda \to \pm\infty \,. \tag{3.2.7}$$

The function $K(\lambda)$ has no poles on the real axis of the λ-plane, because otherwise a cosine loading of the innerwall of the tube would cause infinite displacements. A point of singularity which can be expected, however, in view of the branch points of $K_0(\lambda)$ and $K_1(\lambda)$ is the origin $\lambda = 0$. This, however, is not the case. Consider for instance a typical part of the function $N(1,b,\lambda)$,

$$K_1(\lambda)I_0(\lambda b) + K_0(\lambda b)I_1(\lambda) \,. \tag{3.2.8}$$

Using here a representation of $K_0(\lambda)$ and $K_1(\lambda)$ in the neighbourhood of the origin (ref. 11), we find that the logarithmic singularities of $K_0(\lambda)$ and $K_1(\lambda)$ cancel each other. Hence $K(\lambda)$ possesses a strip of regularity around the real axis. This means that the kernel $k(x)$ decreases exponentially for $x \to \pm\infty$. Because the lefthand side of (3.2.2) is a constant for $x > 0$ we take $\operatorname{Im}\lambda > 0$ and sufficiently small when we apply a Fourier-transformation to this equation. Then we obtain the Hilbert problem

$$P_+(\lambda)\, K(\lambda) - \Theta_-(\lambda) = \frac{i\vartheta}{\sqrt{2\pi}\,\lambda} \tag{3.2.9}$$

valid on a line L just above the real axis in the λ-plane. This equation is of the form (2.3.5) and will be treated by the same procedure. First of all we have to find the function $K^*(\lambda)$. Because $k(x)$ is an even function, it is clear that $K(\lambda)$ is also even. We now consider in correspondence to ref.4

$$K^*(\lambda)=\frac{1}{(\lambda^2+s^2)^{\frac{1}{2}}}\,\frac{T_1(\lambda^2)}{T_2(\lambda^2)}\,, \tag{3.2.10}$$

where s is a real constant and $T_1(\lambda^2)$ and $T_2(\lambda^2)$ are polynomials in λ^2 with the same term of the highest degree.

From this we see that

$$K^*(\lambda)=\frac{1}{|\lambda|}+O(|\lambda|^{-3}) \text{ as } \mathrm{Re}\,\lambda\to\pm\infty, \tag{3.2.11}$$

where $|\mathrm{Im}\,\lambda| < s$. Hence by (3.2.7) the leading term of the asymptotic expansions of $K(\lambda)$ and $K^*(\lambda)$ is the same. Next we choose s and the coefficients of $T_1(\lambda^2)$ and $T_2(\lambda^2)$ so that

$$|M(\lambda)-1| = |K(\lambda)/K^*(\lambda)-1| < \varepsilon < 1, \tag{3.2.12}$$

where ε is some prescribed positive quantity. That this is possible follows from the theorem of Tschebyscheff on the approximation of continuous functions by rational functions (ref. 12, pages 55 and 65). Then we have to solve in accordance to para. 2.3.

$$X_+(\lambda)\,M(\lambda) - X_-(\lambda) = \frac{i\vartheta}{\sqrt{2\pi}\,\lambda}\,K^*_-(\lambda). \tag{3.2.13}$$

The homogeneous part of (3.2.13) yields as a solution which satisfies condition (2.3.1)

$$\Psi_\pm(\lambda)=\exp -\frac{1}{2\pi i}\int_L \frac{\ln M(\eta)}{(\eta-\lambda)}\,d\eta,\ \lambda \text{ in } S_\pm. \tag{3.2.14}$$

Herewith the inhomogeneous equation has the solution

$$X_\pm(\lambda)=\frac{\Psi_\pm(\lambda)}{2\pi i}\int_L \frac{i\vartheta}{\sqrt{2\pi}}\,\frac{K^*_-(\eta)}{\eta\,\Psi_-(\eta)(\eta-\lambda)}\,d\eta,\ \lambda \text{ in } S_\pm. \tag{3.2.15}$$

and the solution of (3.2.9) becomes

$$P_+(\lambda) = X_+(\lambda)/K^*_+(\lambda),\qquad \Theta_-(\lambda) = X_-(\lambda)/K^*_-(\lambda). \tag{3.2.16}$$

The path of integration in (3.2.15) can be closed in the way

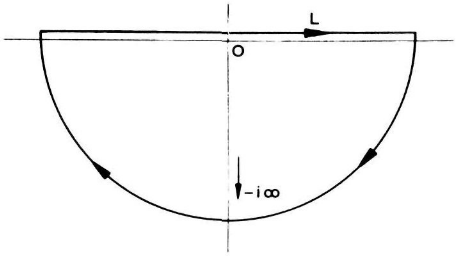

Fig.3.2.1. Contour of integration.

indicated in fig. 3.2.1. Because $K^*_-(\eta)$ and $\Psi_-(\eta)$ are analytic in S_- we find with (3.2.16)

$$P_+(\lambda) = \frac{i\vartheta}{\sqrt{2\pi}} \frac{K^*_-(0)}{\Psi_-(0)} \cdot \frac{\Psi_+(\lambda)}{\lambda\ K^*_+(\lambda)}, \tag{3.2.17}$$

hence

$$p(x) = \frac{i\vartheta}{2\pi} \frac{K^*_-(0)}{\Psi_-(0)} \int_L \frac{e^{-i\lambda x}\ \Psi_+(\lambda)}{\lambda\ K^*_+(\lambda)} d\lambda. \tag{3.2.18}$$

Because of the occurrence of $\Psi_+(\lambda)$ in the integrand, this integral is too involved for numerical calculations. It can however, be expected (ref. 4) that, if ε is sufficiently small, we obtain a good approximation $p^*(x)$ of $p(x)$ when we put $\Psi_+(\lambda)$ and $\Psi_-(\lambda)$ equal to unity for all λ, this yields

$$p^*(x) = \frac{i\vartheta}{2\pi}\ K^*_-(0) \int_L \frac{e^{-i\lambda x}}{\lambda\ K^*_+(\lambda)} d\lambda. \tag{3.2.19}$$

Even when we choose more specifically

$$K^*(0) = K(0) \tag{3.2.20}$$

we can easily show

$$p(x) - p^*(x) \approx O(x^0), \quad x \to 0 \tag{3.2.21}$$

and

$$p(x) - p^*(x) \to 0 \quad , \quad x \to \infty. \tag{3.2.22}$$

Consider

$$p(x)-p^*(x) = \frac{i\vartheta}{2\pi} K^*_-(0) \int_L \frac{e^{-i\lambda x}}{\lambda\ K^*_+(\lambda)} \left\{ \frac{\Psi_+(\lambda)}{\Psi_-(0)} - 1 \right\} d\lambda. \tag{3.2.23}$$

Then it can be shown by estimating (3.2.14) that the function between brackets under the integral sign in (3.2.23) is absolutely integrable. This implies (3.2.21) and (3.2.22).

3.3 NUMERICAL CALCULATION OF THE SHRINK-FIT STRESSES

As has been suggested in para. 3.2. we shall use instead of the exact function $K(\lambda)$ (3.2.4) an approximate function

$$K^*(\lambda) = \frac{1}{(\lambda^2+s^2)^{\frac{1}{2}}} \left\{ \frac{a_0+a_2\lambda^2+a_4\lambda^4+a_6\lambda^6}{b_0+b_2\lambda^2+b_4\lambda^4+b_6\lambda^6} \right\}, \qquad (3.3.1)$$

where the coefficients are given in table I for several values of the dimensionless outer diameter b. The values of s are assumed, while the values of a_n and b_n are computed

TABLE I, coefficients of $K^*(\lambda)$

b	s	a_0	a_2	a_4	a_6	b_0	b_2	b_4	b_6
3	1.25	1.00	-0.725	0.290	0.00553	1.00	-0.756	0.334	0.00553
2	5.00	5.11	2.32	0.332	0	1.00	0.342	0.332	0
1.5	6.00	9.12	0.244	0.0285	0	1.00	0.0178	0.0285	0
1.2	10.0	30.9	0.0963	0.00303	0	1.00	0.00685	0.00303	0

by collocation. The relative deviation of $K^*(\lambda)$ from $K(\lambda)$ is less than 4% for the whole real axis. By determining the roots of the polynomials in (3.3.1) we can easily obtain the factorization (2.3.8) of $K^*(\lambda)$. Using these functions in the integral (3.2.19) we find the values of table II for the approximate shrink-fit stresses.

TABLE II, $p^*(x)/\vartheta$.

b \ x	0.03125	0.0625	0.125	0.1875	0.25	0.375	0.5	1	∞
3		2.83	2.19		1.75		1.42	1.30	1.25
2		2.40	1.80		1.39		1.01	0.921	0.984
1.5		1.77	1.18		0.743		0.538	0.619	0.658
1.2	1.41	0.716	0.242	0.133	0.149	0.253	0.308	0.324	0.324

From (3.2.19) we obtain by (3.2.21) and (3.2.22)

$p(x)$, $p^*(x)$ =

$$= \frac{\vartheta}{\sqrt{x}} \left\{ \frac{2(1-\nu^2)}{\pi} \frac{(b^2-1)}{\{b^2(1+\nu) + (1-\nu)\}} \right\}^{\frac{1}{2}} + \text{const. as } x \to 0, \qquad (3.3.2)$$

and

$$p(x),\ p^*(x) \rightarrow \frac{2\vartheta(1-\nu^2)\ (b^2-1)}{\{b^2(1+\nu)\ +\ (1-\nu)\}} \text{ as } x \rightarrow \infty\ . \qquad (3.3.3)$$

Formula (3.3.3) agrees with the result which we obtain from the elementary theory when an infinite tube is shrunk on to an infinite shaft of uniform diameter and contact occurs for their whole length.

3.4 THE LIMITING CASE OF AN INFINITELY THICK TUBE

We now consider the limit as $b \rightarrow \infty$. We shall again have to determine Green's function $u_o(x)$ of the inner wall. In this case we shall have to take the unknown functions $B_1(\lambda)$ and $B_2(\lambda)$ in (3.1.15) equal to zero in order to avoid a singularity at infinity. This problem was discussed in ref. 13 so that we may write down at once the integral equation with the dimensionless variables (3.2.1)

$$\vartheta = \frac{1}{\sqrt{2\pi}} \int_0^\infty k(x-\xi)\ p(\xi)\ d\xi, \qquad (3.4.1)$$

where

$$k(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-i\lambda x}\ K_1^2(|\lambda|)}{-\lambda^2 K_o^2(|\lambda|)\ + \{\lambda^2+2(1-\nu)\}\ K_1^2(|\lambda|)}\ d\lambda. \qquad (3.4.2)$$

Here a complication arises. The kernel $k(x)$, for finite b, possesses a strip of regularity around the real axis in the λ plane. This is no longer true in the limit as $b \rightarrow \infty$. The function

$$K(\lambda) = \frac{K_1^2(|\lambda|)}{-\lambda^2 K_o^2(|\lambda|)\ +\ \{\lambda^2+2(1-\nu)\}K_1^2(|\lambda|)} \qquad (3.4.3)$$

is not analytic.

From the point of view of mechanics, however, it is not necessary to consider these difficulties in detail. For it is evident that the shrink fit stresses of an infinitely thick tube are the limit of the stresses produced by tubes of increasing thickness. Then, from continuity considerations we can calculate the approximate stresses of the infinitely thick tube by approximating the function $K(\lambda)$ in (3.4.3). Also the asymptotic relations (3.3.2) and (3.3.3) remain valid.

Now the course of the calculation is the same as before; the coefficients of (3.3.1) are

s	a_o	a_2	a_4	a_6	b_o	b_2	b_4	b_6
1.5	1	1.83	0.0853	0	1	2.16	0.0853	0

the error in this case being less than 1.5%. We find for the stresses

x =	0.0625	0.125	0.25	0.5	1	∞
$p^*(x)/\vartheta$ =	3.13	2.44	1.97	1.73	1.58	1.50.

3.5 DISCUSSION OF THE RESULTS

The contact pressures are plotted in fig. 3.5.1, which shows $p^*(x)/\vartheta$ for various thickness ratios b/a. The ratios chosen are $b/a = 1.2$; 1.5; 2; 3 and ∞. The stresses tend to infinity as $x \to 0$, the order of the singularity being independent of the thickness of the tube, as is clear from (3.3.2). The interesting region for the variable x, in which the stresses change rapidly, is approximately $0 \leqq x \leqq 1$. Here we see from fig. 3.5.1. that for $b/a = 2$, 3 and ∞ the shrink-fit pressure is a monotonically decreasing function for increasing values of $x < 1$. However, for $b/a = 1.2$ and 1.5 there exists a minimum which becomes negative for sufficiently thin tubes. Since negative shrink-fit stresses cannot exist, this means that our theory ceases to be applicable. This behaviour is to be expected from the theory of beams on an elastic foundation.

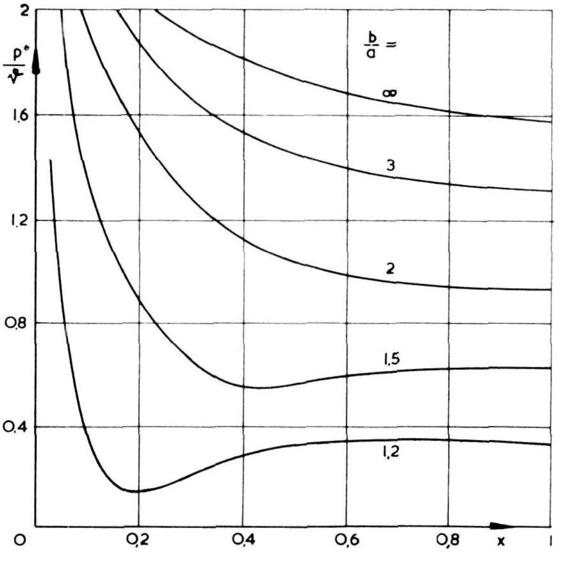

Fig.3.5.1. The contact pressure $p^*(x)/\vartheta$ for various thickness ratios b/a.

Finally we want to make a remark on the accuracy of the numerical results given here. On the basis of observations made in ref. 4 the relative deviation of the approximate values $p^*(x)$ from the exact values $p(x)$ ought to be approximately one half of the corresponding relative deviation of $K^*(\lambda)$ from $K(\lambda)$. For this reason the calculated shrink - fit stress $p^*(x)$ is apt to be accurate within 3%.

Chapter 4

THE HOMOGENEOUS FIRST ORDER INTEGRO-DIFFERENTIAL EQUATION OF THE WIENER-HOPF TYPE.

We shall discuss an integro-differential equation with first order derivatives of the unknown function, which occur both under and outside the integral sign. Equations of this kind sometimes arise in physics (ref. 5). In this chapter the method of solution is exposed for this simple form, it is an extension of methods described in chapter 2. A solution can be obtained even when the kernel $k(x)$ decreases algebraicly for $|x| \to \infty$ while at the same time the solution does not tend to zero for $x \to +\infty$.

The following cases are treated separately:
Case 1. The kernel $k(x)$ decreases exponentially $|k(x)| < e^{-\alpha x}$ with $\alpha > 0$, for $|x| \to \infty$ and the solution has the asymptotical behaviour $|f(x)| = \mathcal{O}(e^{\gamma x})$, with $\gamma < \alpha$ for $x \to +\infty$.
Case 2. The kernel decreases algebraicly $k(x) = \mathcal{O}|x|^{-n}$, $n > 1$ for $|x| \to \infty$.

Whenever we need explicite information about the singularity of the kernel at $x = 0$ we shall assume that $k(x)$ is an even function with respect to x (para. 4.2.).

4.1 THE GENERAL EQUATION, CASE 1

Our integro-differential equation has the following form

$$a_0 f(x)+a_1 f'(x) = \int_0^\infty \{b_0 f(\xi)+b_1 f'(\xi)\} k(x-\xi)d\xi. \qquad (4.1.1)$$

When we have found a function $f(x)$ for $x > 0$ which satisfies this equation we have to solve a simple first order differential equation in order to find $f(x)$ for $x < 0$. From the assumption of the exponential behaviour of $k(x)$ at infinity we find

$$|a_0 f(x)+a_1 f'(x)| < C e^{-\alpha|x|}, \; x \to -\infty. \qquad (4.1.2)$$

Hence if we consider the left hand side of (4.1.1) as one function $h(x)$ we can apply a Fourier-transformation to this equation. We use the notations of para. 2.1. and assume that

λ remains on a line L with Im $\lambda = \mu$, within the strip β, $\gamma < \text{Im}\,\lambda < \alpha$.

Applying a Fourier-transformation to (4.1.1) we obtain

$$H_-(\lambda)+F_+(\lambda)\{a_o-ia_1\lambda - \sqrt{2\pi}\,K(\lambda)(b_o-ib_1\lambda)\}=\frac{f(0)}{\sqrt{2\pi}}\{a_1-b_1\sqrt{2\pi}\,K(\lambda)\} \qquad (4.1.3)$$

where

$$H_-(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\{a_o\,f(x)+a_1\,f'(x)\}e^{i\lambda x}\,dx. \qquad (4.1.4)$$

First we consider the related homogeneous equation

$$Y_-(\lambda)+Y_+(\lambda)\,\{a_o-ia_1\lambda - \sqrt{2\pi}\,K(\lambda)(b_o-ib_1\,\lambda)\} = 0. \qquad (4.1.5)$$

This means that we seek solutions of (4.1.1) with f(0) = 0. We assume further that L has been chosen in such a way that there lies no zero of the factor of $Y_+(\lambda)$ on L.

Analogous to the procedure in para 2.2, we consider the expression

$$\ln G(\lambda)=\ln\left[\frac{i}{a_1}\{a_o-ia_1\lambda-\sqrt{2\pi}\,K(\lambda)(b_o-ib_1\lambda)\}\,\frac{(\lambda-a)^n}{(\lambda-b)^{n+1}}\right], \qquad (4.1.6)$$

where a is in S_+ and b in S_-. The integer n can be determined in such a way that the principal value of ln $G(\lambda)$ for Re$\lambda\to\pm\infty$ on L tends to zero.

We then find for the solution of (4.1.5)

$$Y_+(\lambda) = \frac{i}{a_1}(\lambda-b)^{-n-1}\,\Psi_+(\lambda)\,P(\lambda), \qquad (4.1.7)$$

$$Y_-(\lambda) = (\lambda-a)^{-n}\,\Psi_-(\lambda)\,P(\lambda), \qquad (4.1.8)$$

where

$$\Psi_\pm(\lambda)= \pm\exp - \frac{1}{2\pi i}\int_L \frac{\ln G(\eta)}{(\eta-\lambda)}\,d\eta,\ \lambda \text{ in } S_\pm\,, \qquad (4.1.9)$$

and $P(\lambda)$ is an arbitrary polynomial. Again we consider the zero's, $\lambda = \lambda_m$ (m=1 .... q) of

$$\{a_o-ia_1\lambda - \sqrt{2\pi}\,K(\lambda)(b_o-ib_1\lambda)\} \qquad (4.1.10)$$

within the strip β and above the line L. Then also the functions

$$\underset{q}{Y_+}(\lambda)=\frac{Y_+(\lambda)}{\prod_{m=1}^{q}(\lambda-\lambda_m)}\,,\quad \underset{q}{Y_-}(\lambda)=\frac{Y_-(\lambda)}{\prod_{m=1}^{q}(\lambda-\lambda_m)}\,, \qquad (4.1.11)(4.1.12)$$

are solutions of (4.1.5), but now on a line L_q with $\text{Im}\,\lambda_q < \text{Im}\,\lambda < \beta$. In the case $(n+q) \geqq 1$ we find a solution of (4.1.5) which can be interpreted as a Fourier-transform, while an arbitrary polynomial $P(\lambda)$ of degree $(n+q-1)$ can be taken as a factor.

Now we return to the inhomogeneous equation (4.1.3) and try to obtain a solution. The general solution can then be found by adding solutions of the homogeneous equation. We choose some solution $Y_+(\lambda)$ and $Y_-(\lambda)$, assuming $P(\lambda) \equiv 1$ in (4.1.7) and (4.1.8) in order to avoid zero's of $Y_\pm(\lambda)$ in $S_\pm(\lambda)$, from the set provided by (4.1.11) and (4.1.12) and write instead of (4.1.3)

$$\frac{H_-(\lambda)}{Y_-(\lambda)} - \frac{F_+(\lambda)}{Y_+(\lambda)} = \frac{f(0)}{\sqrt{2\pi}} \; \frac{\{a_1 - b_1\sqrt{2\pi}\,K(\lambda)\}}{Y_-(\lambda)} = \Theta_+(\lambda) - \Theta_-(\lambda). \quad (4.1.13)$$

Here we have introduced the functions $\Theta_+(\lambda)$ and $\Theta_-(\lambda)$ which are discussed in para. 1.2 and which result from splitting the right hand side of (4.1.13) into two parts regular in S_+ and S_- respectively. They are determined within an additive term consisting of an arbitrary polynomial and can easily be calculated without recourse to the general integral representation (1.2.8), in terms of $Y_+(\lambda)$ and $Y_-(\lambda)$. By definition we have

$$Y_-(\lambda) + Y_+(\lambda)\;\{a_0 - ia_1\lambda - \sqrt{2\pi}\,K(\lambda)(b_0 - ib_1\lambda)\} = 0 \qquad (4.1.14)$$

hence

$$\frac{\sqrt{2\pi}\,K(\lambda)}{Y_-(\lambda)} = \frac{1}{(b_0 - ib_1\lambda)}\left\{\frac{1}{Y_+(\lambda)} + \frac{(a_0 - ia_1\lambda)}{Y_-(\lambda)}\right\}. \qquad (4.1.15)$$

Combining (4.1.13) and (4.1.15) and assuming for instance $-\frac{i\,b_0}{b_1}$ in S_+, we find, since $Y_\pm(\lambda)$ have no zero's in $S_\pm$

$$\Theta_-(\lambda) = -\frac{f(0)}{\sqrt{2\pi}(b_0 - ib_1\lambda)}\left\{\frac{(a_1 b_0 - a_0 b_1)}{Y_-(\lambda)} - \frac{b_1}{Y_+(-i\,\frac{b_0}{b_1})}\right\} + Q(\lambda) \qquad (4.1.16)$$

$$\Theta_+(\lambda) = \frac{f(0)}{\sqrt{2\pi}(b_0 - ib_1\lambda)}\left\{\frac{-b_1}{Y_+(\lambda)} + \frac{b_1}{Y_+(-i\,\frac{b_0}{b_1})}\right\} + Q(\lambda), \qquad (4.1.17)$$

where $Q(\lambda)$ is an arbitrary polynomial. The case $-\frac{i\,b_0}{b_1}$ in S_- can be treated entirely analogous. The solution of (4.1.13) and hence of (4.1.3) becomes

$$H_-(\lambda)=-\theta_-(\lambda)Y_-(\lambda)=\frac{f(0)}{\sqrt{2\pi}(b_o-ib_1\lambda)}\{a_1b_o-a_ob_1-\frac{b_1\ Y_-(\lambda)}{Y_+(-i\frac{b_o}{b_1})}\}-Q(\lambda)Y_-(\lambda) \quad (4.1.18)$$

$$F_+(\lambda)=-\theta_+(\lambda)Y_+(\lambda)=-\frac{f(0)}{\sqrt{2\pi}(b_o-ib_1\lambda)}\{-b_1+\frac{b_1\ Y_+(\lambda)}{Y_+(-i\ \frac{b_o}{b_1})}\}-Q(\lambda)Y_+(\lambda). \quad (4.1.19)$$

In this result we may take $Q(\lambda)\equiv 0$ because this term only furnishes a solution of the homogeneous equation. Apparently when $Y_-(\lambda)$ is bounded at infinity we may interprete (4.1.18) and (4.1.19) as Fourier-transforms.

4.2 CASE 1 WITH $a_o \neq 0$, $a_1 = 0$ AND $b_1 \neq 0$

Equation (4.1.3) now reads

$$H_-(\lambda)+F_+(\lambda)\{a_o-\sqrt{2\pi}\ K(\lambda)(b_o-ib_1\lambda)\}=-\frac{f(0)}{\sqrt{2\pi}}\ b_1\ \sqrt{2\pi}\ K(\lambda) \quad (4.2.1)$$

and the homogeneous equation

$$Y_-(\lambda)+Y_+(\lambda)\{a_o-\sqrt{2\pi}\ K(\lambda)(b_o-ib_1\lambda)\}=Y_-(\lambda)+Y_+(\lambda)T(\lambda)=0, \quad (4.2.2)$$

where we have introduced the abbreviation $T(\lambda)$. There is a difference between the equations (4.1.5) and (4.2.2). In (4.1.5) the factor of $F_+(\lambda)$ is of the order λ when $\text{Re}\ \lambda \to \pm\infty$ while in (4.2.2) the asymptotic behaviour of this function, $T(\lambda)$, depends on the behaviour of $K(\lambda)$ for $\text{Re}\ \lambda \to \pm\infty$. We now remind the assumption that in such cases we shall consider only kernels $k(x)$ which are even functions of x, hence also $K(\lambda)$ is even in λ.

The asymptotic behaviour of $K(\lambda)$ for $\text{Re}\ \lambda \to \pm\infty$ depends on the behaviour of $k(x)$ in the neighbourhood of $x = 0$. We assume $k(x)$ to be continuously differentiable for $0 < \varepsilon_1 < x$ for each ε_1. We consider several possibilities for the behaviour of $k(x)$ in the neighbourhood of $x = 0$.

a) Let $k(x)$ be continuously differentiable for $0 \leq x < \varepsilon_2$, $\varepsilon_1 < \varepsilon_2$, then

$$\lim_{\text{Re}\ \lambda\to\pm\infty} K(\lambda) = \mathcal{O}\,|\lambda|^{-m},\ 1 < m. \quad (4.2.3)$$

b) Let $k(x)$ be the sum of a logarithm and a continuously differentiable function for $0 \leq x < \varepsilon_2$, $\varepsilon_1 < \varepsilon_2$ then

$$\lim_{\text{Re } \lambda \to \pm\infty} K(\lambda) \approx A|\lambda|^{-1} . \tag{4.2.4}$$

c) Let $k(x)$ be the sum of $x^{-\delta}$ $(0 < \delta < 1)$ and a continuously differentiable function for $0 \leqq x < \varepsilon_2$, $\varepsilon_1 < \varepsilon_2$ then

$$\lim_{\text{Re } \lambda \to \pm\infty} K(\lambda) \approx B|\lambda|^{-1+\delta} . \tag{4.2.5}$$

The formulae (4.2.3) and (4.2.5) follow directly from ref. 14 while (4.2.4) can be deduced by subtracting from $k(x)$ the function $C\,K_o(x)$ where $K_o(x)$ is a modified Bessel-function and C some suitable constant, then we arrive again at case a.

The difficulties which can occur in the cases a, b and c are concentrated in finding appropriate functions analogous to (4.1.6). We shall now state these functions.

a) In this case the factor $T(\lambda)$ of $Y_+(\lambda)$ in (4.2.2) tends to a_o for Re $\lambda \to \pm\infty$, hence instead of (4.1.6) we can use

$$\ln G(\lambda) = \ln\left\{ \frac{1}{a_o} T(\lambda) \frac{(\lambda-a)^n}{(\lambda-b)^n} \right\}, \tag{4.2.6}$$

b) $T(\lambda)$ tends to $a_o \pm i\sqrt{2\pi}\, A\, b_1$ for Re $\lambda \to \pm\infty$, we take

$$\ln G(\lambda) = \ln\left\{ \frac{T(\lambda)}{a_o + i\sqrt{2\pi}\, Ab_1} \left(\frac{\lambda-a}{\lambda-b}\right)^{n - \frac{1}{2\pi i} \ln\left(\frac{a_o + i\sqrt{2\pi}\, Ab_1}{a_o - i\sqrt{2\pi}\, Ab_1}\right)} \right\}, \tag{4.2.7}$$

c) $T(\lambda)$ tends to $\pm i\sqrt{2\pi}\, b_1\, B|\lambda|^{\delta}$, $0 < \delta < 1$ for Re $\lambda \to \pm\infty$, we take

$$\ln G(\lambda) = \ln\left\{ \frac{T(\lambda)}{i\sqrt{2\pi}\, b_1 B} \frac{(\lambda-a)^{n + \frac{(1-\delta)}{2}}}{(\lambda-b)^{n + \frac{(1+\delta)}{2}}} \right\}. \tag{4.2.8}$$

The values of the multivalued functions which are the factors of $T(\lambda)$ are to be fixed in such a way that the principal value of $\ln G(\lambda)$ for Re $\lambda \to \pm\infty$ on L tends to zero.

The essence of the formulae above is that we have multiplied $T(\lambda)$ by functions which can be factorised by inspection into functions regular and without zero's in S_+ and S_-. We find the following solutions for the homogeneous equation (4.2.2)

a) $$\left.\begin{array}{l} Y_+(\lambda) = \\ Y_-(\lambda) = \end{array}\right. (\lambda - {}^{b}_{a})^{-n}\ \Psi_{\pm}(\lambda)\ P(\lambda) , \tag{4.2.9, 4.2.10}$$

b)

$$Y_+(\lambda) = \qquad\qquad\qquad\qquad\qquad (4.2.11)$$

$$(\lambda - \tfrac{b}{a})^{-n+\frac{1}{2\pi i}\ln\frac{(a_0+i\sqrt{2\pi}\,Ab_1)}{(a_0-i\sqrt{2\pi}\,Ab_1)}}\ \Psi_\pm(\lambda)\ P(\lambda),$$

$$Y_-(\lambda) = \qquad\qquad\qquad\qquad\qquad (4.2.12)$$

c)

$$Y_+(\lambda) = \qquad\qquad\qquad\qquad\qquad (4.2.13)$$

$$(\lambda - \tfrac{b}{a})^{-n-\frac{(1\pm\delta)}{2}}\ \Psi_\pm(\lambda)\ P(\lambda),$$

$$Y_-(\lambda) = \qquad\qquad\qquad\qquad\qquad (4.2.14)$$

where $\Psi_+(\lambda)$ is defined in each case by the integral (4.1.9) with the corresponding $G(\lambda)$.

Following the treatment of the preceeding paragraph below form (4.1.9) we arrive again at formulae analogous to (4.1.18) and (4.1.19).

The case $a_1=0$, $b_1=0$, $a_0 \neq 0$, $b_0 \neq 0$ is the homogeneous Wiener-Hopf equation with exponentially decreasing kernel and is discussed already in chapter 2.

4.3 THE GENERAL EQUATION CASE 2

The treatment given in para. 4.1 is also largely applicable to the case of equations with kernels which are only transformable on a line L. However, there are differences, for in the latter case we cannot translate L and hence (4.1.11) and (4.1.12) cannot be used. Further if there is a zero of (4.1.10) on L this cannot be avoided by translating L slightly.

We now discuss a method to obtain solutions of (4.1.3) when (4.1.10) has zero's on L. To demonstrate the procedure we shall treat the case of one zero $\lambda = \nu$ of the first order. An extension to several zero's and zero's of higher order does not offer principal difficulties.

From (4.1.11) and (4.1.12) we see that a zero λ_m of (4.1.10) introduces a term of the form $c_1 e^{-i\lambda_m x}$ in the solution of the integro-differential equation. In our case with a zero ν on the line of transformation it seems reasonable that the solution of the integral behaves as $c_2 e^{-i\nu x}$ for $x \to +\infty$. Hence we try to find a constant A in order that the function

$$v(x) = f(x) - A\,e^{-i\nu x} \to 0, \quad x \to +\infty. \qquad (4.3.1)$$

The integro-differential equation for $v(x)$ can be derived easily from (4.1.1), we find

$$a_0 v(x) + a_1 v'(x) = \int_0^\infty \{b_0 v(\xi) + b_1 v'(\xi)\}\, k(x-\xi)\,d\xi \; - \qquad (4.3.2)$$

$$- A(a_0 - ia_1\nu)e^{-i\nu x} + A(b_0 - ib_1\nu)\int_0^\infty e^{-i\nu\xi}\, k(x-\xi)\,d\xi\ , \quad x > 0.$$

Under the assumption that ν is a zero from (4.1.10) we see that the last two terms of the right hand side of (4.3.2) cancel each other for $x \to +\infty$. Applying a Fourier-transformation to (4.3.2) we obtain

$$W_-(\lambda)+V_+(\lambda)\ \{a_0-ia_1\lambda - \sqrt{2\pi}\ K(\lambda)(b_0-ib_1\ \lambda)\} = \tag{4.3.3}$$

$$= \frac{v(0_+)}{\sqrt{2\pi}}\ \{a_1-b_1\ \sqrt{2\pi}\ K(\lambda)\}+\frac{Ai}{\sqrt{2\pi}(\lambda-\nu)}\{a_0-ia_1\nu-\sqrt{2\pi}\ K(\lambda)(b_0-ib_1\nu)\},$$

where $v(0_+)$ is the value of $v(x)$ when x tends to zero through positive values and

$$W_-(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-i\lambda x}\{a_0v(x)+a_1v'(x)+A(a_0-ia_1\nu)e^{-i\nu x}\}dx. \tag{4.3.4}$$

The solutions $Y_\pm(\lambda)$ of the homogeneous equation

$$Y_-(\lambda)+Y_+(\lambda)\ \frac{\{a_0-ia_1\lambda-\sqrt{2\pi}\ K(\lambda)(b_0-ib_1\lambda)\}}{(\lambda-\nu)} = 0 \tag{4.3.5}$$

are

$$Y_\pm(\lambda) = (\lambda - \tfrac{b}{a})^{-n}\quad \Psi_\pm(\lambda), \tag{4.3.6}$$

where

$$\Psi_\pm(\lambda)=\pm\exp-\frac{1}{2\pi i}\int_L \frac{\ln i\{\frac{a_0-ia_1\eta-\sqrt{2\pi}\ K(\eta)(b_0-ib_1\eta)}{a_1(\eta-\nu)}\ \frac{(\eta-a)^n}{(\eta-b)^n}\}}{(\eta-\lambda)}d\eta. \tag{4.3.7}$$

Herewith (4.3.3) can be written in the form

$$\frac{W_-(\lambda)}{Y_-(\lambda)}-\frac{a_1(\lambda-\nu)V_+(\lambda)}{i\ Y_+(\lambda)} = \frac{1}{Y_-(\lambda)}\ [\frac{v(0)}{\sqrt{2\pi}}\ \{a_1-b_1\sqrt{2\pi}\ K(\lambda)\}+\frac{Ai}{\sqrt{2\pi}(\lambda-\nu)}\{a_0\cdot$$

$$-ia_1\nu - \sqrt{2\pi}\ K(\lambda)(b_0-ib_1\ \nu\ )\}] = \Theta_+(\lambda) - \Theta_-(\lambda), \tag{4.3.8}$$

where we have introduced $\Theta_+(\lambda)$ and $\Theta_-(\lambda)$ which result from the splitting of the right hand side. Under the assumption that $\rho = i\ b_0/b_1$, for instance, in S_+ we find analogous to the procedure in para. 4.1

$$V_+(\lambda)=\frac{\{v(0)b_1(\lambda-\nu)+Ai(b_0-ib_1\nu)\}\{1-Y_+(\lambda)/Y_-(\rho)\}}{\sqrt{2\pi}\ (\lambda-\nu)(b_0-ib_1\lambda)}+\frac{i\ P(\lambda)Y_+(\lambda)}{a_1\ (\lambda-\nu)}, \tag{4.3.9}$$

$$W_-(\lambda)=\frac{1}{\sqrt{2\pi}}\,(A-v(0))\frac{(a_0b_1-a_1b_0)}{(b_0-ib_1\lambda)}+\frac{Y_-(\lambda)}{\sqrt{2\pi}\,(b_0-ib_1\lambda)Y_+(\rho)}\,\cdot$$

$$\cdot\{v(0)ib_1\;a_1(\lambda-\nu)-A\;a_1(b_0-ib_1\nu)\}+P(\lambda)\;Y_-(\lambda). \qquad (4.3.10)$$

When $n > 0$ we can interprete (4.3.9) and (4.3.10) as **Fourier**-transforms while we can admit an arbitrary polynomial of the degree $(n-1)$. For the case $n = 0$ we can consider these functions as Fourier transforms when

$$P(\lambda) \equiv \frac{v(0)\;a_1}{\sqrt{2\pi}\;Y_+(\rho)}\,. \qquad (4.3.11)$$

From (4.3.9) we observe that we have to choose A in such a way that the singularity $\lambda = \nu$ disappears, this yields

$$A = \frac{-\sqrt{2\pi}\;P(\nu)\;Y_+(\nu)\;Y_-(\rho)a_1}{(Y_-(\rho)-Y_+(\nu))}\,. \qquad (4.3.12)$$

Hence we have determined a solution of (4.3.3) and by the inverse transformation and (4.3.1) also a solution of (4.1.1) with in general $f(0) \neq 0$. An analogous treatment can be given for $\rho = -i\,b_0/b_1$ in S_-.

If we want to solve an equation with several and higher order zero's of (4.1.10) the analysis becomes much more complicated. Assuming zero's $\lambda = \nu_r$ $(r=1 \ldots N)$ of the order p_r, we shall have to consider the integro-differential equation for the function

$$v(x) = f(x) - \sum_{r=1}^{N}\Big(\sum_{m=1}^{p_r} A_{rm}\,x^m\Big)e^{-i\nu_r x}\,. \qquad (4.3.13)$$

By an analogous reasoning as before we can obtain relations for the coefficients A_{rm}.

The discussion of para. 4.2 can also be extended in the way of para. 4.3 to kernels which decrease algebraicly at infinity, this we shall not do here. However, to show the applicability of the ideas of this paragraph we shall discuss an example.

4.4 EXAMPLE

We consider the ordinary Wiener-Hopf integral equation which we have discussed to some extent in para. 2.4

$$f(x) = m\int_0^\infty \frac{f(\xi)}{1+(x-\xi)^2}\,d\xi, \qquad (4.4.1)$$

however, the range of the parameter m is different we take here $m > 1/\pi$. Formal transformation of (4.4.1) yields

$$F_-(\lambda) + F_+(\lambda) \{1 - m\pi e^{-|\lambda|}\} = 0. \qquad (4.4.2)$$

The function $\{1 - m\pi e^{-|\lambda|}\}$ possesses two zero's on the real axis which we call $\lambda = \pm\nu$. The asymptotical behaviour of the solution is then assumed to be

$$f(x) \approx Ae^{-i\nu x} + Be^{i\nu x}, \ \lim x \to +\infty. \qquad (4.4.3)$$

Hence we consider instead of (4.4.1) the equation for the unknown function

$$v(x) = f(x) - (Ae^{-i\nu x} + Be^{i\nu x}), \qquad (4.4.4)$$

which reads

$$v(x) = m\int_0^\infty \frac{v(\xi)d\xi}{1+(x-\xi)^2} - (Ae^{-i\nu x} + Be^{i\nu x}) + m\int_0^\infty \frac{(Ae^{-i\nu\xi} + Be^{i\nu\xi})}{1+(x-\xi)^2} d\xi. \qquad (4.4.5)$$

Transformation of this equation yields

$$W_-(\lambda) + V_+(\lambda)\{1 - m\pi e^{-|\lambda|}\} = -\{\frac{Ai}{\sqrt{2\pi}(\lambda-\nu)} + \frac{Bi}{\sqrt{2\pi}(\lambda+\nu)}\}\{1 - m\pi e^{-|\lambda|}\}, \qquad (4.4.6)$$

where

$$W_-(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{+i\lambda x} \{v(x) + Ae^{-i\nu x} + Be^{i\nu x}\} dx. \qquad (4.4.7)$$

We now write (4.4.6) in the form

$$\frac{W_-(\lambda)}{Y_-(\lambda)} - \frac{V_+(\lambda)}{Y_+(\lambda)} (\lambda^2 - \nu^2) = -\frac{1}{Y_-(\lambda)}\{\frac{Ai}{\sqrt{2\pi}(\lambda-\nu)} + \frac{Bi}{\sqrt{2\pi}(\lambda+\nu)}\}\{1 - m\pi e^{-|\lambda|}\}. \qquad (4.4.8)$$

where $Y_\pm(\lambda)$ are the solutions of

$$Y_-(\lambda) + \{\frac{1 - m\pi e^{-|\lambda|}}{(\lambda^2 - \nu^2)}\} Y_+(\lambda) = 0, \qquad (4.4.9)$$

$$Y_+(\lambda) = (\lambda + i)\, \Psi_+(\lambda), \quad Y_-(\lambda) = (\lambda - i)^{-1}\, \Psi_-(\lambda) \qquad (4.4.10)(4.4.11)$$

and

$$\Psi_{\pm}(\lambda) = \pm \exp - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\ln\{(1-m\pi e^{-|\eta|})\frac{(\eta^2+1)}{(\eta^2-\nu^2)}\}}{(\eta-\lambda)} d\eta,\ \lambda \text{ in } S_{\pm}. \tag{4.4.12}$$

From (4.4.8) we obtain with the use of (4.4.9)

$$\frac{W_-(\lambda)}{Y_-(\lambda)} - \frac{V_+(\lambda)}{Y_+(\lambda)} (\lambda^2-\nu^2) = \frac{i}{\sqrt{2\pi}} \{A(\lambda+\nu) + B(\lambda-\nu)\}. \frac{1}{Y_+(\lambda)}. \tag{4.4.13}$$

Hence

$$V_+(\lambda) = \frac{i}{\sqrt{2\pi}(\lambda^2-\nu^2)} \{A(\lambda+\nu)+B(\lambda-\nu)+P(\lambda)\ Y_+(\lambda)\}, \tag{4.4.14}$$

$$W_-(\lambda) = - \frac{P(\lambda)}{\sqrt{2\pi}} Y_-(\lambda). \tag{4.4.15}$$

In connection with (4.4.11) we see that we have to take $P(\lambda) \equiv C$. Further we have to compensate the singularities $\lambda = \pm \nu$ in (4.4.14). This yields

$$A = - \frac{C}{2\nu} Y_+(\nu), \tag{4.4.16}$$

$$B = \frac{C}{2\nu} Y_+(-\nu). \tag{4.4.17}$$

Hence the solution of (4.4.1) is determined.

We have given two different discussions of equation (4.4.1), depending on the values of m. For $m < 1/\pi$ the equation was discussed in para. 2.4, while here we considered $m > \frac{1}{\pi}$. The case $m = 1/\pi$ cannot be solved directly, however, it is possible to give an asymptotic expansion of A (4.4.16) and B (4.4.17) for small values of ν.

Chapter 5

A WIENER-HOPF TYPE INTEGRO-DIFFERENTIAL EQUATION WITH FOURTH ORDER DERIVATIVES

We shall discuss an integro-differential equation, which is an extension of an equation discussed by Reuter and Sondheimer (ref. 6) which arises in considerations about the anomalous skinn effect (ref. 5). Although the derivatives occurring in this equation are of the second and fourth order we can follow closely the line of thought of para.4.1 and para 4.2. It was asked to expand the quotient of the solution and its first derivative for the value zero of the independent variable in terms of a small parameter β. The case that this parameter is zero is discussed by Reuter and Sondheimer. It turns out that this quotient does not depend analytically on β, there arises a term of the form $\beta \ln \beta$.

5.1 FOURIER TRANSFORMATION OF THE EQUATION

The integral equation has the form

$$f^{(2)}(x)=i\alpha \int_0^\infty \{f(t)-i\beta f^{(4)}(t)\}k(x-t)dt \qquad (5.1.1)$$

where α and β are positive real numbers. By $f^{(2)}(x)$ and $f^{(4)}(x)$ are denoted the second and fourth order derivatives of $f(x)$. The kernel $k(x)$ reads

$$k(x) = \int_1^\infty (\frac{1}{s} - \frac{1}{s^3}) \exp(-c|x|s)ds, \qquad (5.1.2)$$

where c is in general a complex number with a positive real part. We shall determine the physically important quantity

$$f(0)/f^{(1)}(0) , \qquad (5.1.3)$$

and obtain its asymptotical expansion for $\beta \to 0$.

It will be assumed that $f(x)$, $f^{(2)}(x)$ and $f^{(4)}(x)$ are bounded for $x \to +\infty$. From (5.1.1) we deduce that $f^{(2)}(x)$ is of the order e^{cx} when $x \to -\infty$. Then it is allowed to apply a Fourier-transformation to (5.1.1) when λ remains on a line L above and sufficiently close to the real axis in the complex λ plane. We find

$$H_-(\lambda)-F_+(\lambda)\{\lambda^2+i\alpha\sqrt{2\pi}\ K(\lambda)(1-i\beta\ \lambda^4)\} =$$

$$= \frac{1}{\sqrt{2\pi}}\{f^{(1)}(0)-i\lambda f(0)\}+\alpha\beta\ K(\lambda)\{-f^{(3)}(0)+i\lambda f^{(2)}(0)+\lambda^2 f^{(1)}(0)-i\lambda^3 f(0)\}\ , \tag{5.1.4}$$

where

$$H_-(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} f^{(2)}(x)\ e^{i\lambda x}\ dx, \tag{5.1.5}$$

and

$$K(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} k(x)e^{i\lambda x}\ dx=\frac{1}{\sqrt{2\pi}}\{\frac{-2c}{\lambda^2}+i(\frac{1}{\lambda}+\frac{c^2}{\lambda^3})\ \ln(\frac{\lambda+ic}{-\lambda+ic})\}, \tag{5.1.6}$$

the logarithm is defined by

$$\ln\ (\frac{\lambda+ic}{-\lambda+ic}) \to -\pi i,\quad \mathrm{Re}\ \lambda\to+\infty,\quad \lambda \text{ on L}. \tag{5.1.7}$$

First we consider the homogeneous part of (5.1.4)

$$Y_-(\lambda)-Y_+(\lambda)\{\lambda^2+i\alpha\sqrt{2\pi}\ K(\lambda)(1-i\beta\lambda^4)\} = Y_-(\lambda)-Y_+(\lambda)\ T(\lambda)=0, \tag{5.1.8}$$

where, as in para. 4.2, we have introduced the abbreviation $T(\lambda)$. In the following we shall use the notations of that paragraph. The asymptotic behaviour of $K(\lambda)$ is

$$K(\lambda)\approx\sum_{n=1}^{4} a_n\lambda^{-n}=\frac{1}{\sqrt{2\pi}}\{\frac{\pi}{\lambda}-\frac{4c}{\lambda^2}+\frac{\pi c^2}{\lambda^3}-\frac{4c^3}{3\lambda^4}\},\mathrm{Re}\ \lambda\to+\infty,\lambda \text{ on L}, \tag{5.1.9}$$

hence

$$T(\lambda)\approx\ \alpha\beta\pi\lambda^2\,|\lambda|,\quad \mathrm{Re}\ \lambda\to\pm\infty \text{ on L}. \tag{5.1.10}$$

The function $\ln G(\lambda)$ can be taken in the form

$$\ln G(\lambda) = \ln\{\frac{\lambda^2+i\alpha\ \sqrt{2\pi}\ K(\lambda)(1-i\beta\ \lambda^4)}{\alpha\beta\pi(\lambda^2+p^2)\ (\lambda^2+p^2)^{\frac{1}{2}}}\}\ , \tag{5.1.11}$$

where p is an arbitrary real positive number. We find for the solutions of (5.1.8)

$$Y_+(\lambda)=\frac{\Psi_+(\lambda)}{\alpha\beta\pi(\lambda+ip)\sqrt{\lambda+ip}},\quad Y_-(\lambda)=\Psi_-(\lambda)(\lambda-ip)\sqrt{\lambda-ip}, \qquad (5.1.12)$$

where

$$\Psi_\pm(\lambda) = \exp - \frac{1}{2\pi i}\int_L \frac{\ln G(\eta)}{(\eta-\lambda)}\,d\eta. \qquad (5.1.13)$$

We now write (5.1.4)

$$\frac{H_-(\lambda)}{Y_-(\lambda)} - \frac{F_+(\lambda)}{Y_+(\lambda)} = \frac{1}{Y_-(\lambda)}\Big[\frac{1}{\sqrt{2\pi}}\{f^{(1)}(0)-i\lambda f(0)\} + \alpha\beta\,K(\lambda)\{-f^{(3)}(0) +$$

$$+ i\lambda f^{(2)}(0) + \lambda^2 f^{(1)}(0) - i\,\lambda^3 f(0)\}\Big] = \Theta_+(\lambda) - \Theta_-(\lambda). \qquad (5.1.14)$$

The functions $\Theta_+(\lambda)$ and $\Theta_-(\lambda)$ can be calculated exactly in the same way as in para. 4.1 by splitting $K(\lambda)/Y_-(\lambda)$ with the aid of (5.1.8). We find

$$H_-(\lambda)=-\frac{1}{\sqrt{2\pi}\,\beta(\lambda^4-\mu^4)}\{-\lambda^2\beta f^{(3)}(0)+i\lambda^3\beta f^{(2)}(0)+ i f^{(1)}(0)+\lambda f(0)\} +$$

$$+\frac{Y_-(\lambda)}{4\mu\sqrt{2\pi}}\Big\{\frac{-R_1(-\mu)}{Y_-(-\mu)(\lambda+\mu)} + \frac{+i\,R_1(-i\mu)}{Y_-(-i\mu)(\lambda+i\mu)} + \frac{+R_2(\mu)}{Y_+(\mu)\mu^2(\lambda-\mu)} +$$

$$+\frac{i\,R_2(i\mu)}{Y_+(i\mu)\mu^2(\lambda-i\mu)}\Big\}, \qquad (5.1.15)$$

$$F_+(\lambda)=-\frac{1}{\sqrt{2\pi}(\lambda^4-\mu^4)}\{-f^{(3)}(0)+i\lambda f^{(2)}(0)+\lambda^2 f^{(1)}(0)-i\lambda^3 f(0)\} +$$

$$+\frac{Y_+(\lambda)}{4\mu\sqrt{2\pi}}\Big\{\frac{-R_1(-\mu)}{Y_-(-\mu)(\lambda+\mu)}+ \frac{i\,R_1(-i\mu)}{Y_-(-i\mu)(\lambda+i\mu)}+ \frac{+R_2(\mu)}{Y_+(\mu)\mu^2(\lambda-\mu)} +$$

$$+\frac{i\,R_2(i\mu)}{Y_+(i\mu)\mu^2(\lambda-i\mu)}\Big\}, \qquad (5.1.16)$$

where

$$R_1(\lambda)= -\lambda^2\beta f^{(3)}(0)+i\,\lambda^3\beta f^{(2)}(0)+i\,f^{(1)}(0)+\lambda f(0), \qquad (5.1.17)$$

$$R_2(\lambda) = -f^{(3)}(0) + i\lambda f^{(2)}(0) + \lambda^2 f^{(1)}(0) - i\lambda^3 f(0) \qquad (5.1.18)$$

and $\mu = \beta^{-\frac{1}{4}} e^{i\frac{3}{8}\pi}$ is a zero of the function $(1-i\beta\lambda^4)$. The solution, (5.1.15) and (5.1.16) are essentially the same as (4.1.18) and (4.1.19). Because we want to obtain a function $f(x)$ whose fourth order derivative $f^{(4)}(x)$ possesses a one sided Fourier-transform for $0 \leq x \leq \infty$

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty f^{(4)}(x) e^{i\lambda x}\, dx = \frac{1}{\sqrt{2\pi}} \{-f^{(3)}(0) + i\lambda f^{(2)}(0) + \lambda^2 f^{(1)}(0) - i\lambda^3 f(0)\} + \lambda^4 F_+(\lambda) \qquad (5.1.19)$$

it follows that the second term of the right hand side of (5.1.16) must tend to zero more quickly than λ^{-4} for $\mathrm{Re}\,\lambda \to \pm\infty$. In the general form in which it stands it tends to zero as $|\lambda|^{-5/2}$. Hence we shall have to satisfy the following conditions

$$\frac{-R_1(-\mu)}{Y_-(-\mu)} + \frac{i R_1(-i\mu)}{Y_-(-i\mu)} + \frac{R_2(\mu)}{Y_+(\mu)\mu^2} + \frac{i R_2(i\mu)}{Y_+(i\mu)\mu^2} = 0 \qquad (5.1.20)$$

$$\frac{R_1(-\mu)}{Y_-(-\mu)} + \frac{R_1(-i\mu)}{Y_-(-i\mu)} + \frac{R_2(\mu)}{Y_+(\mu)\mu^2} - \frac{R_2(i\mu)}{Y_+(i\mu)\mu^2} = 0. \qquad (5.1.21)$$

When (5.1.20) and (5.1.21) are satisfied, the second term of the right hand side of (5.1.16) tends to zero as $|\lambda|^{-9/2}$ for $\mathrm{Re}\,\lambda \to \pm\infty$. This, however, generates a singularity of the form $x^{-\frac{1}{2}}$ for $f^{(4)}(x)$ as $x \to 0_+$, which cannot be tolerated for physical reasons, hence we have to go one step further and demand

$$\frac{-R_1(-\mu)}{Y_-(-\mu)} - \frac{i R_1(-i\mu)}{Y_-(-i\mu)} + \frac{R_2(\mu)}{Y_+(\mu)\mu^2} - \frac{i R_2(i\mu)}{Y_+(i\mu)\mu^2} = 0. \qquad (5.1.22)$$

These three linear equations, in the four unknowns $f(0)$, $f^{(1)}(0)$, $f^{(2)}(0)$ and $f^{(3)}(0)$, determine the ratio

$$\mu \frac{f(0)}{f^{(1)}(0)} = \frac{\mu^2 \{Y_+(i\mu) - i\, Y_+(\mu)\} + \mu^{-2}\{Y_-(-i\mu) + i\, Y_-(-\mu)\}}{\mu^2 \{Y_+(i\mu) - Y_+(\mu)\} - \mu^{-2}\{Y_-(-i\mu) + Y_-(-\mu)\}}. \qquad (5.1.23)$$

From (5.1.12), (5.1.13) and (5.1.23) we see that the asymptotical expansion for $\beta \to 0$ of $f(0)/f^{(1)}(0)$ can be calculated directly from the asymptotical expansion of

$$E(\lambda)=\int_{-\infty}^{+\infty} \frac{\ln\left\{\dfrac{\eta^2+i\alpha\sqrt{2\pi}\,K(\eta)(1-i\beta\,\eta^4)}{\alpha\beta\pi\,(\eta^2+p^2)\,(\eta^2+p^2)^{\frac{1}{2}}}\right\}}{(\eta-\lambda)}\,d\eta, \qquad (5.1.24)$$

where we take instead of L the real axis in the λ plane as line of integration.

5.2 THE ASYMPTOTICAL EXPANSION OF $E(\lambda)$

Instead of (5.1.24) we can write

$$E(\lambda)=2\lambda\int_{0}^{\infty} \frac{\ln\left\{\dfrac{\eta^2+i\alpha\sqrt{2\pi}\,K(\eta)(1-i\beta\eta^4)}{\pi\,\alpha\,\beta\,\eta^3}\right\}}{(\eta^2-\lambda^2)}\,d\eta \qquad (5.2.1)$$

where we have taken for p the value zero. This limit process must also be made in (5.1.12). We now consider

$$E(ue^{i\varphi})=2e^{i\varphi}u\int_{0}^{\infty}\frac{[-\ln(\alpha\pi u^{-4})-\ln\eta+\ln\{1+i\alpha\sqrt{2\pi}\,K(\eta)\eta^{-2}(1-iu^{-4}\eta^4)\}]}{\eta^2-u^2e^{2i\varphi}}d\eta, \qquad (5.2.2)$$

where $u=\beta^{-\frac{1}{4}}$ and φ assumes the values $\frac{3\pi}{8}$, $\frac{7\pi}{8}$, $\frac{-\pi}{8}$ or $-\frac{5\pi}{8}$ and it is asked to determine $E(u\,e^{i\varphi})$ for large values of u.

We shall treat the three terms of the integrand of (5.2.2) separately. First we consider

$$-2\,e^{i\varphi}u\,\ln(\alpha\pi u^{-4})\int_0^\infty\frac{d\eta}{\eta^2-u^2e^{2i\varphi}}=\mp\,i\pi\ln\alpha\pi u^{-4},\quad ue^{i\varphi}\text{ in } S_{\pm}, \qquad (5.2.3)$$

next

$$-2\,e^{i\varphi}u\int_0^\infty\frac{\ln\eta}{\eta^2-u^2e^{2i\varphi}}\,d\eta=-e^{i\varphi}u\int_{-\infty}^{+\infty}\frac{\ln\eta}{\eta^2-u^2e^{2i\varphi}}\,d\eta+i\pi ue^{i\varphi}\int_{-\infty}^{0}\frac{d\eta}{\eta^2-u^2e^{2i\varphi}}. \qquad (5.2.4)$$

The path of integration in the first integral on the right-hand side of (5.2.4) is the real axis with a small semicircle above the point $\eta = 0$. We find

$$-2\,e^{i\varphi}u\int_0^\infty\frac{\ln\eta\,d\eta}{\eta^2-u^2e^{2i\varphi}}=\pi(\pm\varphi-\tfrac{\pi}{2})\mp i\pi\ln u,\quad ue^{i\varphi}\text{ in } S_{\pm}. \qquad (5.2.5)$$

The last term in the integrand of (5.2.2) causes more trouble. We divide the interval of integration as follows:

$$\int_0^\infty \frac{\ln\{1+i\alpha\sqrt{2\pi}\ K(\eta)\eta^{-2}(1-iu^{-4}\eta^4)\}}{\eta^2-u^2e^{2i\varphi}}\,d\eta =$$

$$= \int_0^{u^s} \ldots\ldots\ d\eta + \int_{u^s}^{u^r} \ldots\ldots\ d\eta + \int_{u^r}^{\infty} \ldots\ldots\ d\eta = I + II + III, \qquad (5.2.6)$$

where $s = \frac{1}{2} + \varepsilon$ and $r = 3 + \delta$, ε and δ being arbitrary but sufficiently small real quantities. We shall treat the integrals I, II and III separately

$$I = \int_0^{u^s} \frac{\ln\{1+i\alpha\sqrt{2\pi}K(\eta)\eta^{-2}(1-iu^{-4}\eta^4)\}}{\eta^2-u^2e^{2i\varphi}}\,d\eta =$$

$$= -u^{-2}e^{-2i\varphi}\int_0^{u^s} [\ln\{1+i\alpha\sqrt{2\pi}K(\eta)\eta^{-2}\} +$$

$$+ \ln\{1+\frac{\alpha\sqrt{2\pi}u^{-4}K(\eta)\eta^2}{1+i\alpha\sqrt{2\pi}K(\eta)\eta^{-2}}\}]\ \sum_{n=0}^{\infty}(\frac{\eta^2}{u^2}\,e^{-2i\varphi})^n\,d\eta. \quad (5.2.7)$$

The expansion of the denominator is possible on account of the special range of integration. The first logarithm of the integrand in (5.2.7) multiplied by the general term of the expansion gives

$$\frac{-e^{-2(n+1)i\varphi}}{u^{2(n+1)}}\int_0^{u^s}\eta^{2n}\ \ln\{1+i\alpha\sqrt{2\pi}K(\eta)\eta^{-2}\}d\eta. \qquad (5.2.8)$$

For a further reduction of this form we need the expansion of the logarithm for $\eta \to +\infty$; using (5.1.9) we obtain

$$\ln\{1+i\alpha\sqrt{2\pi}\ K(\eta)\eta^{-2}\} \sim \sum_{n=3}^{\infty} b_n\,\eta^{-n},\ \eta \text{ real} \qquad (5.2.9)$$

with $b_3 = i\alpha\pi$; $b_4 = -i\alpha 4c$; $b_5 = i\alpha\pi c^2$; ...... Hence we can write instead of (5.2.8)

$$\frac{-e^{-2(n+1)i\varphi}}{u^{2(n+1)}}[\int_0^\infty \eta^{2n}\{\ln(1+i\alpha\sqrt{2\pi}K(\eta)\eta^{-2}) - \sum_{m=3}^{2n} b_m\eta^{-m} - \frac{b_{2n+1}}{\eta^{2n}(\eta+1)}\}d\eta +$$

$$+ \int_0^{u^s} \eta^{2n}\{\sum_{m=3}^{2n} b_m\eta^{-m} + \frac{b_{2n+1}}{\eta^{2n}(\eta+1)}\}\,d\eta -$$

$$- \int_{u^s}^{\infty} \eta^{2n} \{ \ln(1+i\alpha\sqrt{2\pi}\ K(\eta)\eta^{-2}) - \sum_{m=3}^{2n} b_m \eta^{-m} - \frac{b_{2n+1}}{\eta^{2n}(\eta+1)} \} d\eta], \tag{5.2.10}$$

where we have supposed $n \geqq 2$. For $n = 0$ or $n = 1$ we have to change (5.2.10) slightly, because then we need fewer convergence-producing terms or none at all, see (5.2.13) and (5.2.14). The first integral is independent of u. After integration we find for the second integral terms of the form

$$\eta^l \Big|_0^{u^s},\ 1 \leqq l \leqq 2n-2 \text{ and } \ln(\eta+1) \Big|_0^{u^s} \tag{5.2.11}$$

or

$$u^{sl} \text{ and } \ln(u^s+1) = s \ln u - \sum_{m=1}^{\infty} (-1)^m u^{-sm}. \tag{5.2.12}$$

Because the final result must be independent of s, we can omit these terms from the beginning. If now we expand the logarithm in the third integral of (5.2.10) and integrate, we see that only terms of the type of (5.2.12) arise, hence we can also neglect this integral. It will turn out that for our purpose it is sufficient to consider quantities up to the order u^{-6}; then we obtain from (5.2.10) for the values n = 0 1 and 2

$$-e^{-2i\varphi}u^{-2} \int_0^{\infty} \ln \{1+i\alpha\sqrt{2\pi}\ K(\eta)\eta^{-2}\} d\eta = -e^{-2i\varphi}u^{-2}A,\ n=0, \tag{5.2.13}$$

$$-e^{-4i\varphi}u^{-4} \int_0^{\infty} [\eta^2 \ln\{1+i\alpha\sqrt{2\pi}K(\eta)\eta^{-2}\} - \frac{i\alpha\pi}{\eta+1}] d\eta = -e^{-4i\varphi}\ u^{-4}\ B,\ n=1 \tag{5.2.14}$$

$$-e^{-6i\varphi}u^{-6} \int_0^{\infty} [\eta^4 \ln\{1+i\alpha\sqrt{2\pi}K(\eta)\eta^{-2}\} - i\alpha\pi\eta + i\alpha 4c - \frac{i\alpha\pi c^2}{\eta+1}] d\eta = -e^{-6i\varphi}u^{-6}C,\ n=2. \tag{5.2.15}$$

The second logarithm in (5.2.7) gives, with the general term of the expansion of the denominator

$$\frac{-e^{-2(n+1)i\varphi}}{u^{2(n+1)}} \int_0^{u^s} \eta^{2n} \ln\{1 + \frac{\alpha u^{-4}\sqrt{2\pi}K(\eta)\eta^2}{1+i\alpha\sqrt{2\pi}K(\eta)\eta^{-2}}\} d\eta. \tag{5.2.16}$$

For the range $0 \leq \eta \leq u^s$ it is allowed to expand the logarithm, from which results.

$$\frac{e^{-2(n+1)i\varphi}}{u^{2(n+1)}} \int_0^{u^s} \eta^{2n} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} u^{-4m} \left\{ \frac{\alpha\sqrt{2\pi}K(\eta)\eta^2}{1+i\alpha\sqrt{2\pi}K(\eta)\eta^{-2}} \right\}^m d\eta \tag{5.2.17}$$

Now we need the expansion

$$\left\{ \frac{\alpha\sqrt{2\pi}K(\eta)\eta^2}{1+i\alpha\sqrt{2\pi}K(\eta)\eta^{-2}} \right\}^m \sim \sum_{n=-m}^{\infty} c_{m,n} |\eta|^{-n} \quad (\eta \text{ real}), \tag{5.2.18}$$

where

$$c_{1,-1} = \alpha\pi \; ; \; c_{1,0} = -4\alpha c; \; c_{1,1} = \alpha\pi c^2; \ldots\ldots \tag{5.2.19}$$

Using (5.2.18) and (5.2.19), the term with $m = 1$ of (5.2.17) reads

$$\frac{-e^{-2(n+1)i\varphi}}{u^{2(n+3)}} \left[\int_0^{\infty} \eta^{2n} \left\{ \frac{\alpha\sqrt{2\pi}K(\eta)\eta^2}{1+i\alpha\sqrt{2\pi}K(\eta)\eta^{-2}} - \sum_{l=-1}^{2n} c_{1,l}\eta^{-1} - \frac{c_{1,2n+1}}{(\eta+1)\eta^{2n}} \right\} d\eta \; + \right.$$

$$\int_0^{u^s} \eta^{2n} \left\{ \sum_{l=-1}^{2n} c_{1,l}\eta^{-1} + \frac{c_{1,2n+1}}{(\eta+1)\eta^{2n}} \right\} d\eta \; -$$

$$\left. - \int_{u^s}^{\infty} \eta^{2n} \left\{ \frac{\alpha\sqrt{2\pi}K(\eta)\eta^2}{1+i\alpha\sqrt{2\pi}K(\eta)\eta^{-2}} - \sum_{l=-1}^{2n} c_{1,l}\eta^{-1} - \frac{c_{1,2n+1}}{(\eta+1)\eta^{2n}} \right\} d\eta \right]. \tag{5.2.20}$$

These integrals have the same character as those of (5.2.10) hence we can conclude that only the first one contributes to the final result. For the value $n = 0$ we find

$$\frac{-e^{-2i\varphi}}{u^6} \int_0^{\infty} \left\{ \frac{\alpha\sqrt{2\pi}K(\eta)\eta^2}{1+i\alpha\sqrt{2\pi}K(\eta)\eta^{-2}} - \alpha\pi\eta + 4\alpha c - \frac{\alpha\pi c^2}{\eta+1} \right\} d\eta =$$

$$= -e^{-2i\varphi}u^{-6}\, D. \tag{5.2.21}$$

The contributions of (5.2.17) for larger values of n or of m can be neglected.

We will now treat the second range of integration

$$II = \int_{u^s}^{u^r} \frac{\ln\{1+i\alpha\sqrt{2\pi}K(\eta)\eta^{-2}(1-iu^{-4}\eta^4)\}}{\eta^2 - u^2\, e^{2i\varphi}} \, d\eta. \tag{5.2.22}$$

We observe that for the whole interval of integration the argument of the logarithm is in the neighbourhood of 1; this neighbourhood decreases with increasing values of u. Expansion of the logarithm and of $K(\eta)$ and using the identity $i=e^{-4i\varphi}$ results in

$$e^{-2i\varphi}u^{-2}\int_{u^s}^{u^r}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\left(i\alpha\sqrt{2\pi}\sum_{n=1}^{\infty}a_n\eta^{-n-2}\right)^m\left(1+u^{-2}\eta^2e^{-2i\varphi}\right)^m .$$

$$\cdot\left(1-u^{-2}\eta^2e^{-2i\varphi}\right)^{m-1}d\eta . \quad (5.2.23)$$

The integration can be carried out and we obtain terms of the form of (5.2.12). Hence it is allowed to omit (5.2.22).
Finally we investigate the last integral in (5.2.6):

$$III=\int_{u^r}^{\infty}\frac{\ln\{1+i\alpha\sqrt{2\pi}K(\eta)\eta^{-2}(1-iu^{-4}\eta^4)\}}{\eta^2-u^2e^{2i\varphi}}d\eta= \quad (5.2.24)$$

$$=e^{-2i\varphi}u^{-2}\int_{u^r}^{\infty}\Big[\ln(1+u^{-4}\alpha\pi\eta)+$$

$$+\ln\Big\{1+\frac{i\alpha\sqrt{2\pi}\sum_{n=2}^{\infty}a_n\eta^{-n}(\eta^{-2}-iu^{-4}\eta^2)+i\alpha\pi\eta^{-3}}{(1+u^{-4}\alpha\pi\eta)}\Big\}\Big]\sum_{m=1}^{\infty}\frac{u^{2m}e^{2mi\varphi}}{\eta^{2m}}d\eta . \quad (5.2.25)$$

The first logarithm in the integrand of (5.2.25), with the general term of the expansion of the denominator as factor, gives after partial integration

$$\frac{u^{2(m-1)}e^{2(m-1)i\varphi}}{(-2m+1)}\Big\{\eta^{-2m+1}\ln(1+u^{-4}\alpha\pi\eta)\Big|_{u^r}^{\infty}-\int_{u^r}^{\infty}\eta^{-2m+1}\frac{u^{-4}\alpha\pi}{(1+u^{-4}\alpha\pi\eta)}d\eta\Big\} . \quad (5.2.26)$$

If we now expand the first expression between the square brackets in (5.2.26), we see that there is one term for the lower boundary u^r which is independent of r, viz.

$$+\frac{u^{2(m-1)}e^{2(m-1)i\varphi}}{(-2m+1)}\frac{(u^{-4}\alpha\pi)^{2m-1}}{(-2m+1)}=\frac{u^{-6m+2}}{(2m-1)^2}e^{2(m-1)i\varphi}(\alpha\pi)^{2m-1} , \quad (5.2.27)$$

which gives a contribution within the desired order of accuracy for m = 1

$$\alpha\pi u^{-4} . \quad (5.2.28)$$

The integrand of the integral which occurs in (5.2.26) can be split into partial fractions, in general

$$\frac{1}{\eta^{l}(1+u^{-4}\alpha\pi\eta)} = \frac{(-1)^{l}(u^{-4}\alpha\pi)^{l}}{1+u^{-4}\alpha\pi\eta} + \frac{(-1)^{l-1}(u^{-4}\alpha\pi)^{l-1}}{\eta} + \dots \qquad (5.2.29)$$

The first two terms only are of importance, the other terms cannot contribute to the final result (see 5.2.12). Taking $l = (2m-1)$ in (5.2.29) we find for the integral in (5.2.26) including the minus sign:

$$\frac{-u^{2(m-1)}e^{2(m-1)i\varphi}}{(-2m+1)} u^{-4}\alpha\pi(u^{-4}\alpha\pi)^{2m-2} \ln\left(\frac{\eta}{1+u^{-4}\alpha\pi\eta}\right)\Bigg|_{u^{r}}^{\infty}. \qquad (5.2.30)$$

The upper boundary ∞ yields a term which is independent of r ; for $m = 1$ we obtain

$$-u^{-4}\alpha\pi \ln(u^{-4}\alpha\pi); \qquad (5.2.31)$$

larger values of m can be neglected. We now investigate the second part of the integral (5.2.25). Expanding the logarithm and taking the general term of the expansion of the denominator we obtain

$$-e^{2(m-1)i\varphi}u^{2(m-1)} \int_{u^{r}}^{\infty} \eta^{-2m} \Big[\sum_{l=1}^{\infty} \frac{(-1)^{l}}{l} \cdot$$

$$\Big\{ \frac{i\alpha\sqrt{2\pi} \sum_{n=2}^{\infty} \alpha_{n}\eta^{-n}(\eta^{-2}-iu^{-4}\eta^{2})+i\alpha\pi\eta^{-3}}{1+u^{-4}\alpha\pi\eta} \Big\}^{l}\Big]d\eta. \qquad (5.2.32)$$

If we consider the case $m = l = 1$ we see, by using (5.2.29) that the result is of the order u^{-8}, which can be neglected; also larger values of m and l give results of higher order than u^{-6}.

So we find for our expansion of (5.2.2), if we make use of the relation $e^{-4i\varphi} = i$,

$$E(ue^{i\varphi}) \approx (5.2.3)+(5.2.5)+2e^{i\varphi}u\{(5.2.13)+(5.2.14)+(5.2.15) +$$

$$+(5.2.21)+(5.2.28)+(5.2.31)\} =$$

$$=\pi(\pm\varphi-\tfrac{\pi}{2})\mp i\,\pi \ln(\alpha\pi\, u^{-3}-2e^{-i\varphi}u^{-1}A-2u^{-3}e^{i\varphi}.\{iB+\alpha\pi \ln(e^{-1}u^{-4}\alpha\pi)\} +$$

$$-\, 2u^{-5}\, e^{-i\varphi}\{iC + D\}, \quad ue^{i\varphi} \text{ in } S_{\pm}. \qquad (5.2.33)$$

5.3 THE EXPANSION OF $f(0)/f^{(1)}(0)$

The expansion of

$$\Psi_{\pm}(ue^{i\varphi}) = \exp - \frac{1}{2\pi i} E(ue^{i\varphi}) \qquad (5.3.1)$$

can now be written as

$$\Psi_{\pm}(ue^{i\varphi}) \approx (\alpha\pi u^{-3})^{\pm\frac{1}{2}} e^{i(\pm\frac{1}{2}\varphi-\frac{1}{4}\pi)}[1-ie^{-i\varphi}(\pi u^{-1})A+e^{i\varphi}(\pi u^{3})^{-1}(B-i\alpha\pi.$$

$$.\ln(\alpha\pi u^{-4}e^{-1}))+e^{-i\varphi}(\pi u^{5})^{-1}(C-iD)+\tfrac{1}{2}\{-e^{-2i\varphi}(\pi^{2}u^{2})^{-1} A^{2} -$$

$$-2i(\pi^{2}u^{4})^{-1} A(B-i\alpha\pi.\ \ln(\alpha\pi u^{-4}e^{-1}))\}+ \tfrac{1}{6}\{-e^{i\varphi}\pi^{-3}u^{-3} A^{3}-3e^{-i\varphi}\pi^{-3}u^{-5}A$$

$$\cdot(B-i\alpha\pi \ln(\alpha\pi u^{-4}e^{-1}))\} +$$

$$+ \frac{1}{24}\{i\ \pi^{-4}u^{-4} A^{4}\} + \frac{1}{120}\{e^{-i\varphi}\pi^{-5}u^{-5} A^{5}\}],\ ue^{i\varphi} \text{ in } S_{\pm}. \qquad (5.3.2)$$

With (5.3.2) we can calculate the quantities $Y_{\pm}(ue^{i\varphi})$(5.1.12) which must be substituted into (5.1.23). This calculation can be simplified by showing that only the terms of (5.3.2) between square brackets which are independent of φ contribute to the numerator of (5.1.23), while those with a factor $e^{-i\varphi}$ contribute to the denominator. The result is

$$f(0)/f^{(1)}(0)=-\frac{\pi}{A}[1-\beta\{\frac{(D+iC)}{A} + \frac{iA}{2\pi^{2}}(B-i\alpha\pi\ln(\alpha\pi\beta e^{-1}))- \frac{iA^{4}}{30\pi^{4}}\}\ldots]. \qquad (5.3.3)$$

The first term of (5.3.3) agrees with the result found in ref. 6 where the same problem with $\beta = 0$ was solved.

Chapter 6

THE FINITE DOCK

This chapter deals with two dimensional surface waves of a deep sea in the presence of a strip or dock of finite extent. The dock is supposed to be in rest on the level of the undisturbed water surface. At one side at infinity there is a prescribed incoming wave with crests parallel to the edges of the dock. This wave is pa rtly reflected and can partly pass under the dock. We shall calculate the reflection and transmission coefficients.

Following H. Rubin (ref. 7) the problem is transformed by a Babinet principle into a problem for the motion of water between two semi-infinite docks. This can be formulated as a Hilbert problem for a segment of finite length. Its solution yields a Fredholm integral equation of the second kind,which can be solved numerically.

6.1 FORMULATION OF THE PROBLEM

We assume the water to occupy the half space $\bar{y} < 0$. The

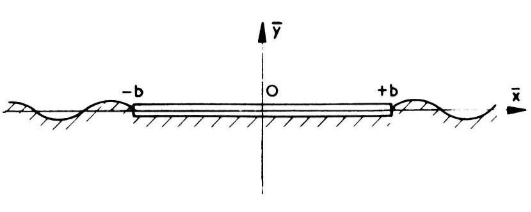

Fig.6.1.1. Situation of the dock on the watersurface.

dock is on the level of the undisturbed water surface (Fig. 6.1.1) and extends from $\bar{x} = -b$ to $\bar{x} = +b$. Because we assume the motion of the water to be non rotational, two dimensional and simply periodic, this motion is governed (ref.15) by a potential $\Theta(\bar{x},\bar{y},t) = \vartheta(\bar{x},\bar{y})e^{i\omega t}$, vanishing at $\bar{y} \to -\infty$, with

$$\left(\frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2}\right) \vartheta(\bar{x},\bar{y}) = 0. \qquad (6.1.1)$$

The velocities in the x and y direction u and v respectively are found by differentiation

$$u = \frac{\partial \Theta}{\partial \bar{x}} \quad , \quad v = \frac{\partial \Theta}{\partial \bar{y}} \; .$$

The boundary conditions for $\bar{y} = 0$ are

$$\frac{\partial \vartheta}{\partial \bar{y}} = 0, \; |\bar{x}| < b; \quad -\omega^2 \vartheta + g \frac{\partial \vartheta}{\partial \bar{y}} = 0, \quad |\bar{x}| > b \, , \qquad (6.1.2)$$

where g is the acceleration of gravity. Introducing the dimensionless quantities

$$x = \bar{x}/b, \quad y = \bar{y}/b \text{ and } \nu = \omega^2 \, b/g \qquad (6.1.3)$$

we find, that the harmonic function $\vartheta(x,y)$ has to satisfy ,for y = 0,the boundary conditions

$$\frac{\partial \vartheta}{\partial y} = 0, \; |x| < 1; \quad \frac{\partial \vartheta}{\partial y} - \nu\vartheta = 0, \quad |x| > 1. \qquad (6.1.4)$$

We consider the pressure function

$$\psi(x,y) = \frac{\partial \vartheta}{\partial y}(x,y) - \nu\vartheta(x,y) \qquad (6.1.5)$$

which has often been used in problems of this type (ref.15). The boundary conditions for this harmonic function for y= 0 are

$$-\nu \frac{\partial \psi}{\partial y} + \frac{\partial^2 \psi}{\partial x^2} = 0, \quad |x| < 1; \quad \psi = 0, \quad |x| > 1. \qquad (6.1.6)$$

By introducing the conjugate function $\varphi(x,y)$ related to $\psi(x,y)$ by

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \quad , \quad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x} \, , \qquad (6.1.7)$$

we replace (6.1.6) after integration in the x direction, for y = 0 by

$$\nu\varphi + \frac{\partial \varphi}{\partial y} = 0, \quad |x| < 1; \quad \frac{\partial \varphi}{\partial y} = 0, \quad |x| > 1. \qquad (6.1.8)$$

where a constant of integration is absorbed in $\varphi(x,y)$.

At last we add two remarks about notations to be used. Firstly the boundary values for y = 0 of any function $\varphi(x,y)$ are denoted by $\varphi(x)$, secondly the notation φ_e or φ_o denote that the function is an even or odd function of x.

6.2 DERIVATION OF THE INTEGRAL EQUATION

We shall now derive the integral equation for the auxiliary function $\varphi(x,y)$. When $\varphi(x,y)$ is harmonic for $y < 0$, then the same is true for $\partial\varphi/\partial y$ and we may assume

$$\frac{\partial\varphi}{\partial y} = \text{Im}\ \Phi(z). \tag{6.2.1}$$

Then $y = 0$ we have, by introducing a positive direction on the segment $|x| < 1$ of the real axis in the direction of increasing values of x,

$$\text{Im}\ \Phi_-(x) = -\nu\varphi(x),\ |x| < 1;\quad \text{Im}\ \Phi_-(x) = 0,\ |x| > 1, \tag{6.2.2}$$

in accordance with the notation used in para. 1.1. Extending the definition of $\Phi(z)$ to va lues of z with $\text{Im}\ z > 0$ by the principle of reflection

$$\Phi(z) = \overline{\Phi(\bar{z})}\ ,\quad \text{Im}\ z > 0, \tag{6.2.3}$$

we obtain a sectionally holomorfic function $\Phi(z)$ for which the part $|x| < 1$ of the real axis is a line of discontinuity. This gives rise to the following Hilbert problem

$$\Phi_+(x)-\Phi_-(x)=-2\ i\ \text{Im}\ \Phi_-(x)=2i\ \nu\varphi(x),\quad y = 0,\ |x| < 1. \tag{6.2.4}$$

The solution follows immediately from (1.1.4) and (1.1.3)

$$\Phi(z) = \frac{\nu}{\pi}\int_{-1}^{+1} \frac{\varphi(\xi)}{(\xi-z)}\,d\xi + \frac{P(z)}{(z-1)^{m_1}(z+1)^{m_2}}\ , \tag{6.2.5}$$

where m_1 and m_2 are arbitrary integers and P(z) is an arbitrary polynomial. The addition of the rational function in (6.2.5) introduces singularities of the function $\varphi(x,y)$ at the points $z = \pm 1$. It turns out that by taking

$$\Phi(z) = \frac{\nu}{\pi}\int_{-1}^{+1} \frac{\varphi(\xi)}{(\xi-z)} - \frac{\alpha}{(z-1)} - \frac{\beta}{(z+1)} + c, \tag{6.2.6}$$

where c is some real constant, the behaviour of $\varphi(x,y)$ is

$$\varphi(x,y)\rightarrow \frac{\alpha}{2}\ \ln\{(x-1)^2+y^2\}\ \text{for}\ x \rightarrow 1,\ y \rightarrow 0 \tag{6.2.7}$$

and

$$\varphi(x,y)\rightarrow \frac{\beta}{2}\ \ln\{(x+1)^2+y^2\}\ \text{for}\ x\rightarrow -1,\ y \rightarrow 0. \tag{6.2.8}$$

From the point of view of hydrodynamics we may tolerate singularities of this strength because then the in- or outflow of mass at these points remains finite.

Now suppose

$$\varphi(x,y) = \mathrm{Im}\, H(z) \tag{6.2.9}$$

then

$$\partial\varphi/\partial y = \mathrm{Im}\, i\, H'(z) = \mathrm{Im}\, \Phi(z) \tag{6.2.10}$$

and by (6.2.6), integrating from an arbitrary point $z_o = x_o + i\, y_o$

$$\varphi(x,y)-\varphi(x_o,y_o)=\mathrm{Im}\ -i\int_{z_o}^{z}\left\{ \frac{\nu}{\pi}\int_{-1}^{+1} \frac{\varphi(\xi)}{(\xi-\zeta)}\, d\xi - \frac{\alpha}{\zeta-1} - \frac{\beta}{\zeta+1}\right\} d\zeta + c(x-x_o). \tag{6.2.11}$$

It will turn out, that the integral in (6.4.4) is only convergent if c = 0. Changing the order of integration in (6.2.11) we find

$$\varphi(x,y)=\frac{\nu}{2\pi}\int_{-1}^{+1} \varphi(\xi)\ln\{(\xi-x)^2+y^2\}d\xi+\frac{\alpha}{2}\ln\{(x-1)^2+y^2\}+\frac{\beta}{2}\ln\{(x+1)^2+y^2\}+\gamma, \tag{6.2.12}$$

where γ is a constant of integration. If y tends to zero for an arbitrary value of x, we obtain the desired integral equation for the function $\varphi(x)$,

$$\varphi(x)= \frac{\nu}{\pi}\int_{-1}^{+1} \varphi(\xi)\ \ln |x-\xi|\, d\xi + \alpha \ln |1-x| + \beta \ln |1+x| + \gamma. \tag{6.2.13}$$

This is an inhomogeneous Fredholm equation of the second kind, with a quadratic integrable kernel. Equation (6.2.13) can also be derived by using Green's function for a bipole.

The homogeneous part of the integral equation (6.2.13) possesses no positive eigenvalues. To prove this x) we show

$$I = \int_{-1}^{+1}\int_{-1}^{+1} \ln |x-\xi|\ \varphi(x)\ \varphi(\xi)dx\ d\xi < 0 \tag{6.2.14}$$

for all functions $\varphi(x)$. We expand ln |x| into a cosine series

$$\ln |x| = \sum_{0}^{\infty} a_n \cos n\frac{\pi}{2}x, \quad |x| \leq 2, \tag{6.2.15}$$

It is easily seen that the coefficients $\alpha_n < 0$. Substitution of (6.2.15) into (6.2.14) yields the required result

$$I= \sum_{0}^{\infty} a_n [\{\int_{-1}^{+1}\varphi(x)\cos nx\ dx\}^2+\{\int_{-1}^{+1} \varphi(x)\sin nx\ dx\}^2] \leq 0. \tag{6.2.16}$$

x) This proof is due to a remark of Prof. A.C. Zaanen.

From this it follows that (6.2.13) has for each value $\nu>0$ a unique solution, which will be calculated numerically later on (para.6.6). Each solution $\varphi(x)$ of (6.2.13) can be split into one odd function $\varphi_o(x)$ and two even function $\varphi_{e,1}(x)$ and $\varphi_{e,2}(x)$,

$$\varphi(x) = \frac{(\alpha-\beta)}{2}\varphi_o(x) + \frac{(\alpha+\beta)}{2}\varphi_{e,1}(x) + \gamma\varphi_{e,2}(x) \qquad (6.2.17)$$

which satisfy respectively

$$\varphi_o(x) = \frac{\nu}{\pi}\int_{-1}^{+1} \ln|x-\xi|\varphi_o(\xi)d\xi + \ln|1-x| - \ln|1+x|, \qquad (6.2.18)$$

$$\varphi_{e,1}(x) = \frac{\nu}{\pi}\int_{-1}^{+1} \ln|x-\xi|\ \varphi_{e,1}(\xi)d\xi + \ln|1-x| + \ln|1+x|, \qquad (6.2.19)$$

$$\varphi_{e,2}(x) = \frac{\nu}{\pi}\int_{-1}^{+1} \ln|x-\xi|\ \varphi_{e,2}(\xi)d\xi + 1. \qquad (6.2.20)$$

By reasoning in the opposite direction, we can show that the solutions of (6.2.18), (6.2.19) and (6.2.20) are the boundary values of potential functions which satisfy (6.1.8).

6.3 DETERMINATION OF $\psi(x)$

We assume that the solutions of (6.2.18), (6.2.19) and (6.2.20) are known for several values of $\nu > 0$. Since our aim is to calculate $\vartheta(x,y)$ we shall have to determine first $\psi(x,y)$, defined in terms of $\varphi(x,y)$ by (6.1.7). The functions $\varphi(x,y)$ and $\psi(x,y)$ are related by

$$\frac{\partial\psi}{\partial s} = -\frac{\partial\varphi}{\partial n} \qquad (6.3.1)$$

where the directions s and n are orthogonal and can be obtained from the x and y directions by a rotation in the x,y plane over the same angle.

For $|x| < 1$ we find for $y = 0$

$$\psi(x) = -\int_0^x \frac{\partial\varphi}{\partial y}(\xi,0)d\xi = \nu\int_0^x \varphi(\xi)d\xi, \qquad (6.3.2)$$

where the lower boundary is chosen arbitrarily equal to zero.

We shall have to be careful when the path of integration approaches the points $x = +1$ or $x = -1$, which we encircle in the way of fig. 6.3.1. On account of the singular behaviour of $\varphi(x,y)$ for these points the small semi-circles with radius r give a contribution for $x = +1$ and $x = -1$ resp.

$$-\int_{\Gamma_+} \frac{\partial\varphi}{\partial n} ds = \int_0^\pi \frac{\alpha}{r} r\ d\tau = \pi\alpha, \quad -\int_{\Gamma_-} \frac{\partial\varphi}{\partial n} ds = -\int_0^\pi \frac{\beta}{r} r\ d\tau = -\pi\beta, \qquad (6.3.3)$$

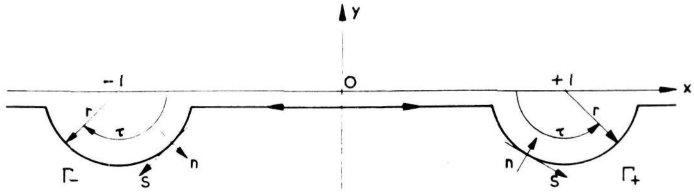

Fig.6.3.1. Path of integration for the determination of $\Psi(x)$.

For $|x| > 1$ $\partial\varphi/\partial y\ (x,0) = 0$ and hence $\psi(x)$ remains constant. The construction of functions $\psi(x)$ satisfying the second equation of (6.1.6) can be done by using (6.2.18), (6.2.19) and (6.2.20).

First we consider the odd solution $\varphi_o(x)$ of (6.2.18). From (6.3.2) it follows that the resulting $\psi(x)$ is an even function denoted by $\bar{\psi}_e(x)$, the general behaviour of which is given in fig. 6.3.2. Hence by subtracting from $\bar{\psi}_e(x)$ an appropriate constant

$$k_e = \nu \int_0^1 \varphi_o(\xi)d\xi + \pi \qquad (6.3.4)$$

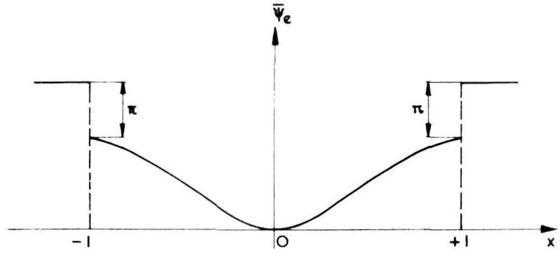

Fig.6.3.2. General behaviour of $\bar{\psi}_e(x)$.

we have found the boundary values of a harmonic function which satisfies the conditions (6.1.6)

$$\psi_e(x) = \bar{\psi}_e(x) - k_e. \qquad (6.3.5)$$

Now we consider the equations (6.2.19) and (6.2.20). From these we obtain two odd functions $\psi_{o,1}(x)$ and $\psi_{o,2}(x)$ the general behaviour of which is drawn in fig. 6.3.3. Because we have in (6.2.20) a continuous inhomogeneous part, it is clear that $\psi_{o,2}(x)$ remains continuous for $x = \pm 1$. Again we have to

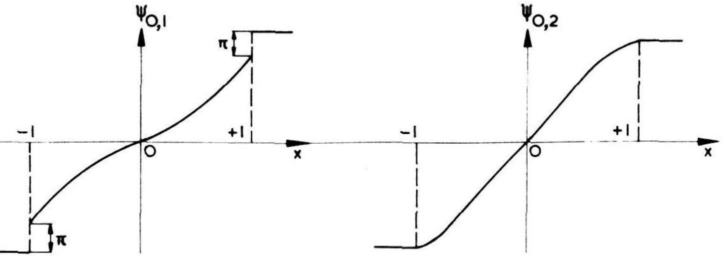

Fig.6.3.3. General behaviour of $\psi_{o,1}(x)$ and $\psi_{o,2}(x)$.

satisfy the second equation of (6.1.6). We combine $\psi_{o,1}(x)$ and $\psi_{o,2}(x)$

$$\psi_o(x) = \psi_{o,1}(x) + k_o\, \psi_{o,2}(x), \qquad (6.3.6)$$

where k_o follows from

$$\nu \int_0^1 \varphi_e(\xi) d\xi = \nu \int_0^1 \{\varphi_{e,1}(\xi) + k_o\, \varphi_{e,2}(\xi)\} d\xi = -\pi. \qquad (6.3.7)$$

In this way for each value of $\nu > 0$ an even and an odd function $\psi_e(x)$ and $\psi_o(x)$ are found, which are boundary values of harmonic functions satisfying (6.1.6).

6.4 CALCULATION OF $\vartheta(x)$

From (6.1.5) we determine the boundary values $\vartheta(x)$ of the desired potential function. This has been done already in ref. 1. The method used here reduces (6.1.5) to an ordinary differential equation in the complex domain.

Because $\vartheta(x,y)$ and $\psi(x,y)$ are harmonic there exist functions $F_1(z)$ and $F_2(z)$ so that

$$\vartheta(x,y) = \text{Re}\, F_1(z), \quad \psi(x,y) = \text{Im}\, F_2(z), \qquad (6.4.1)$$

and from (6.3.1) $F_2(z) = \varphi(x,y) + i\,\psi(x,y)$. The equation (6.1.5) becomes

$$i\, F_1'(z) - \nu F_1(z) = -i\, F_2(z) + ic \qquad (6.4.2)$$

where c is an arbitrary real constant, which can be neglected. The solution of (6.4.2) then becomes

$$F_1(z) = C\, e^{-i\nu z} - e^{-i\nu z}\int_0^z e^{i\nu\zeta}\, F_2(\zeta)\,d\zeta, \qquad (6.4.3)$$

where $C = c_1 + c_2 i$ is a constant of integration. Using (6.4.1) we obtain for $y = 0$

$$\vartheta(x) = c_1 \cos\nu x + c_2 \sin\nu x - \int_0^x \{\varphi(\xi)\cos\nu(x-\xi) + \psi(\xi)\sin\nu(x-\xi)\}\,d\xi. \qquad (6.4.4)$$

The unknown constants c_1 and c_2 are determined by putting

$$\frac{\partial\vartheta}{\partial y} = \frac{\partial^2\vartheta}{\partial x\partial y} = 0 \quad \text{for} \quad x = y = 0. \qquad (6.4.5)$$

This yields

$$c_1 = -\frac{1}{\nu}\,\psi(0,0), \qquad c_2 = 0. \qquad (6.4.6)$$

Equations (6.4.4) and (6.4.6) are equivalent with the result given in ref. 7. We can write (6.4.4) in the form

$$\vartheta(x) = -\frac{\psi(x)}{\nu}, \quad |x| < 1, \qquad (6.4.7)$$

and

$$\vartheta(x) = \frac{\pi}{\nu}\cos\nu(x-1) - \int_1^x \varphi(\xi)\cos\nu(x-\xi)\,d\xi, \quad x > 1. \qquad (6.4.8)$$

In order to investigate the convergence of the integral in (6.4.8) when $x \to \pm\infty$, we observe from (6.2.18)

$$\varphi_o(x) \to \frac{\nu}{\pi}\int_{-1}^{+1} \varphi_o(\xi)\,\{\ln|x| \mp \frac{\xi}{x}\ \ldots\}\,d\xi + $$
$$+ \ln\left|1 - \frac{2}{x}\ \ldots\right| \approx \mathcal{O}(x^{-1}), \quad x \to \pm\infty. \qquad (6.4.9)$$

and from (6.3.7), (6.2.19) and (6.2.20)

$$\varphi_e(x) \to \frac{\nu}{\pi}\int_{-1}^{+1} \varphi_e(\xi)\,\{\ln|x| \mp \frac{\xi}{x}\ \ldots\}\,d\xi + \ln|x^2-1| + k_o, \quad x \to \pm\infty, \qquad (6.4.10)$$

hence we obtain by (6.3.7)

$$\varphi_e(x) \to k_o, \quad x \to \pm\infty. \qquad (6.4.11)$$

Obviously for $x \to \pm\infty$ (6.4.8) represents a harmonic oscillating function, as can be expected from the physical situation.

With the aid of (6.4.9) and (6.4.11) we calculate the asymptotic behaviour of the function $\vartheta(x)$. First we consider the pair of functions $\varphi_o(x)$ and $\psi_e(x)$, these yield by (6.4.4) and (6.4.6) an even function $\vartheta_{e,1}(x)$ with the following asymptotical behaviour

$$\vartheta_{e,1}(x) \to \pm A_1 \sin \nu x + B_1 \cos \nu x \ , \ x \to \pm \infty, \quad (6.4.12)$$

where

$$A_1 = + \frac{\pi}{\nu} \sin \nu - \int_1^\infty \varphi_o(\xi) \sin \nu\xi \, d\xi, \quad (6.4.13)$$

$$B_1 = \frac{\pi}{\nu} \cos \nu - \int_1^\infty \varphi_o(\xi) \cos \nu\xi \, d\xi. \quad (6.4.14)$$

The asymptotical behaviour of the odd function $\vartheta_{o,2}(x)$, related to $\varphi_e(x)$ and $\psi_o(x)$, becomes

$$\vartheta_{o,2}(x) \to + A_2 \sin \nu x \pm B_2 \cos \nu x, \ x \to \pm \infty, \quad (6.4.15)$$

where

$$A_2 = \frac{\pi}{\nu} \sin \nu - k_o \frac{\cos \nu}{\nu} - \int_1^\infty \{\varphi_e(\xi) - k_o\} \sin \nu\xi \, d\xi \quad (6.4.16)$$

$$B_2 = k_o \frac{\sin \nu}{\nu} + \frac{\pi}{\nu} \cos \nu - \int_1^\infty \{\varphi_e(\xi) - k_o\} \cos \nu\xi \, d\xi. \quad (6.4.17)$$

From (6.4.7) and (6.4.8) we see that both $\vartheta_{e,1}(x)$ and $\vartheta_{o,2}(x)$ are bounded for $x \to \pm 1$, because $\varphi_o(x)$ and $\varphi_e(x)$ possess only logarithmic singularities. We can find from $\vartheta_{e,1}(x)$ and $\vartheta_{o,2}(x)$ by derivation with respect to x other solutions $\vartheta_{o,3}(x) = \vartheta'_{e,1}(x)$ and $\vartheta_{e,4}(x) = \vartheta'_{o,2}(x)$, which are odd and even respectively and which possess logarithmic singularities for $x = \pm 1$. Their asymptotic behaviour follows also by derivation with respect to x from (6.4.12) and (6.4.15).

6.5 CONSTRUCTION OF THE SOLUTION FOR PRESCRIBED INCOMING WAVES

We have the general solution

$$\Theta(x,0,t) = \sum_{n=1}^{4} \vartheta_n(x) \{a_n \sin \omega t + b_n \cos \omega t\}, \quad (6.5.1)$$

where we have dropped the subscripts e and o of $\vartheta_{e,1}(x)$....

$\vartheta_{e,4}(x)$ under the sign of summation. This is a superposition of standing waves. We impose the following condition at infinity for $x \to +\infty$

$$\Theta(x,0,t) \to A \cos(\nu x+\omega t)+R_1 \cos(\nu x-\omega t)+R_2 \sin(\nu x-\omega t), \tag{6.5.2}$$

where A is prescribed and determines the incoming wave, while R_1 and R_2 are unknown and determine the reflected wave, for $x \to -\infty$

$$\Theta(x,0,t) \to T_1 \cos(\nu x+\omega t) + T_2 \sin(\nu x+\omega t), \tag{6.5.3}$$

where T_1 and T_2 are the coefficients of the unknown transmitted wave. No incoming wave excists at this side.

The asymptotic behaviour of $\vartheta_{e,1}(x) \ldots \vartheta_{e,4}(x)$ gives the following relations for the coefficients a_n and b_n of(6.5.1)

$$B_1 a_1 + A_1 b_1 + B_2 a_2 + A_2 b_2 + A_1 a_3 - B_1 b_3 + A_2 a_4 - B_2 b_4 = 0 \tag{6.5.4}$$

$$-A_1 a_1 + B_1 b_1 - A_2 a_2 + B_2 b_2 + B_1 a_3 + A_1 b_3 + B_2 a_4 + A_2 b_4 = 2A \tag{6.5.5}$$

$$-B_1 a_1 - A_1 b_1 + B_2 a_2 + A_2 b_2 + A_1 a_3 - B_1 b_3 - A_2 a_4 + B_2 b_4 = 0 \tag{6.5.6}$$

$$-A_1 a_1 + B_1 b_1 + A_2 a_2 - B_2 b_2 - B_1 a_3 - A_1 b_3 + B_2 a_4 + A_2 b_4 = 0 \tag{6.5.7}$$

There are four linear equations for the eight unknowns a_n, b_n $(n=1 \ldots 4)$. For complete determination of the solution additional information is needed.

By a simple experiment it is seen that, when the amplitude of the incoming wave is sufficiently small no breaking occurs at the edges of the dock. This can be formulated mathematically (ref. 15) by the requirement that there are no singularities at the points $(\pm 1,0)$.

This gives:

$$a_3 = b_3 = a_4 = b_4 = 0. \tag{6.5.8}$$

Then the remaining coefficients a_1, b_1, a_2 and b_2 are determined uniquely from the equations (6.5.4)...(6.5.7).

When A increases, breaking is seen to occur only at the leading edge of the dock (+ 1,0). For this it is necessary and sufficient

$$a_3 = a_4 \quad , \quad b_3 = b_4 \tag{6.5.9}$$

because then the singularities of $\vartheta_{o,3}(x)$ and $\vartheta_{e,4}(x)$ cancel each other at the trailing edge. In this case however, further experimental information is needed about the relation between strength and phase of the singularity and the amplitude and phase of the incoming wave.

When the coefficients a_n and b_n are determined, we can calculate the reflected and transmitted waves by comparing (6.5.1) with (6.5.2) and (6.5.3). We find

$$2R_1 = +A_1 a_1 + B_1 b_1 + A_2 a_2 + B_2 b_2 - B_1 a_3 + A_1 b_3 - B_2 a_4 + A_2 b_4, \qquad (6.5.10)$$

$$2R_2 = -B_1 a_1 + A_1 b_1 - B_2 a_2 + A_2 b_2 - A_1 a_3 - B_1 b_3 - A_2 a_4 - B_2 b_4, \qquad (6.5.11)$$

$$2T_1 = +A_1 a_1 + B_1 b_1 - A_2 a_2 - B_2 b_2 + B_1 a_3 - A_1 b_3 - B_2 a_4 + A_2 b_4, \qquad (6.5.12)$$

$$2T_2 = B_1 a_1 - A_1 b_1 - B_2 a_2 + A_2 b_2 - A_1 a_3 - B_1 b_3 + A_2 a_4 + B_2 b_4 . \qquad (6.5.13)$$

The reflection coefficient ρ and the transmission coefficient τ can then be defined by

$$\rho = \frac{(R_1^2 + R_2^2)^{\frac{1}{2}}}{A}, \quad \tau = \frac{(T_1^2 + T_2^2)^{\frac{1}{2}}}{A}. \qquad (6.5.14)$$

6.6 NUMERICAL CALCULATION OF THE REFLECTED AND TRANSMITTED WAVE FOR THE CASE THAT NO BREAKING OCCURS

In the case of no breaking $a_3=b_3=a_4=b_4 = 0$ and we can solve (6.5.4) ... (6.5.7), for a_1, b_1, a_2 and b_2. Substitution of these values in (6.5.10) ... (6.5.13) yields

$$R_1 = -A(A_1^2 A_2^2 - B_1^2 B_2^2) . D^{-1} \qquad (6.6.1)$$

$$R_2 = +A\{A_1 B_1 (A_2^2+B_2^2) + A_2 B_2 (A_1^2+B_1^2)\} . D^{-1} \qquad (6.6.2)$$

$$T_1 = -A (A_1^2 B_2^2 - A_2^2 B_1^2) . D^{-1} \qquad (6.6.3)$$

$$T_2 = -A\{A_1 B_1 (A_2^2+B_2^2) - A_2 B_2 (A_1^2+B_1^2)\} . D^{-1} \qquad (6.6.4)$$

where

$$D = (A_1^2 + B_1^2)(A_2^2 + B_2^2). \qquad (6.6.5)$$

We shall not give the numerical calculation in full detail but discuss only the procedure followed. First of all are solved the integral equations (6.2.18)(6.2.19) and (6.2.20). Because the first two of these equations have solutions with logarithmic singularities we have introduced new continuous unknown functions

$$f_o(x) = \varphi_o(x) - \ln|1-x| + \ln|1+x|, \qquad (6.6.6)$$

$$f_{e,1}(x) = \varphi_{e,1}(x) - \ln|1-x| - \ln|1+x|, \qquad (6.6.7)$$

satisfying integral equations which can easily be determined. Then we have approximated the functions $f_o(x)$, $f_{e,1}(x)$ and $\varphi_{e,2}(x)$ by odd or even polynomials with five unknown coefficients, calculated by collocation.

Next (6.3.7) yields the number k_o. Hence the functions $\varphi_o(x)$ and $\varphi_e(x)$ are known for $|x| < 1$. These functions are now to be calculated for $x > 1$. This can be done respectively with the integral equation (6.2.18) and with

$$\varphi_e(x) = \frac{\nu}{\pi}\int_{-1}^{+1} \ln|x-\xi|\varphi_e(\xi)d\xi + \ln|1-x| + \ln|1+x| + k_o, \qquad (6.6.8)$$

which is a linear combination of (6.2.19) and (6.2.20).

Now we can calculate by (6.4.13), (6.4.14), (6.4.16) and (6.4.17) the quantities A_1, B_1, A_2 and B_2 which determine the asymptotic behaviour of the elementary solutions. Finally (6.6.1) ... (6.6.5) and (6.5.14) yield the reflection and transmission coefficients ρ and τ.

In order to obtain a representation of ρ and τ in the neighbourhood of $\nu = 0$ we have also calculated the asymptotical expansions of these quantities for $\nu \to 0$. This can be done by a straight forward computation of the first terms of the Neumann series expansions of $\varphi_o(x)$ and $\varphi_e(x)$ from the integral equations (6.2.18) and (6.6.8). Using (6.4.13), (6.4.14), (6.4.16) and (6.4.17) we obtain

$$A_1 \approx +2\pi, \quad B_1 \approx \frac{\pi}{\nu}, \quad A_2 \approx +\frac{\pi}{\nu^2}, \quad B_2 \approx \mathcal{O}(\nu^o). \qquad (6.6.9)$$

Herewith we find

$$\rho = 2\nu \quad , \quad \tau \approx 1-2\nu^2. \qquad (6.6.10)$$

The calculated reflection and transmission coefficients are given in the following table, while they are plotted in fig. 6.6.1.

Table III, τ and ρ .

$\nu=$	3π	$1,5\pi$	π	$0,5\pi$	$0,25\pi$
$\tau=$	0,02	0,14	0,22	0,47	0,59
$\rho=$	0,99	0,99	0,98	0,88	0,81

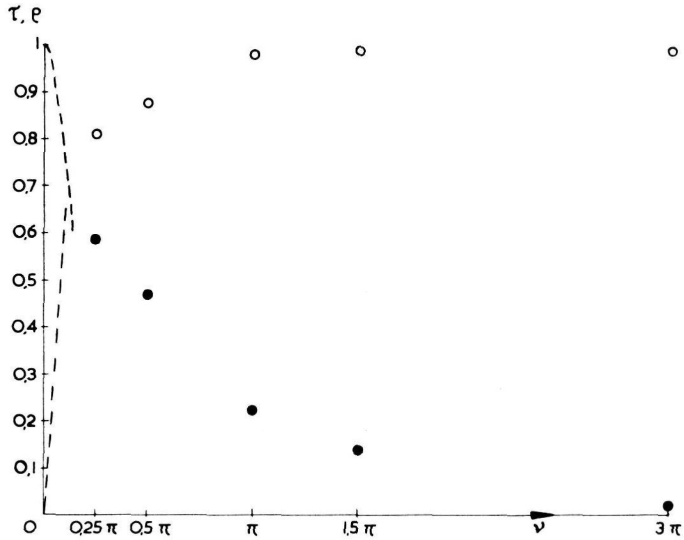

Fig.6.6.1. Calculated values of, the reflection coefficient ρ denoted by O, the transmission coefficient τ, denoted by ● and of the asymptotic expansions- - -.

We can interprete the parameter ν as π. width of dock/ length of incoming wave. This means that when $\nu = \pi$ the length of the incoming wave equals the width of the dock. In this case only 4,8 % of the energy can pass under the dock. We have not drawn an interpolating line through the calculated points because of the unknown extrema which this curve is apt to possess. However, the plotted points give a rough information about the influence of the finite dock on incoming waves.

Chapter 7

ON THE INFLUENCE OF THE CROSS-SECTION FORM OF A SHIP ON THE ADDED MASS FOR HIGHER ORDER VIBRATIONS.*

The main cause of ship vibrations is the engine. A difficulty in the calculation of these vibrations is caused by the motion of the water, generated by the vibrating hull. The influence of this motion is effectively an additional mass. Generally, naval architects assume a two-dimensional water motion. Calculations made with this two-dimensional model are corrected with a coefficient, obtained from the three-dimensional flow around a vibrating ellipsoid of revolution. (ref. 9). However, ship forms occur with shallow draught. The purpose of the following calculation is to investigate a possible difference between the three dimensional correction coefficient of ships with shallow draught and ships with a full cross-section. This is done by comparing the results of the three-dimensional theory for a vibrating flat strip, situated at the surface of the water and a vibrating circular cylinder which is half immersed. Strip and cylinder are both infinite, hence three-dimensional effects caused by the bow and the stern are neglected. It seems that these latter effects are negligible for vibrations with more than about five nodes.

7.1 FORMULATION OF THE PROBLEM FOR THE CASE OF A STRIP.

The water occupies the half space $\bar{y} < 0$. The strip is situated at $\bar{y} = 0$ for $|\bar{x}| < b$ and stretches along the $\bar{z}$ axis.

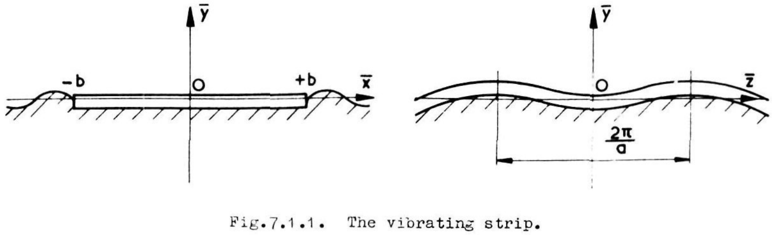

Fig.7.1.1. The vibrating strip.

* This chapter belongs to research carried out for the "Studiecentrum T.N.O. voor Scheepsbouw en Navigatie".

We assume that the motion of the water is non rotational and simple periodic and hence can be described by a potential $\Theta(\bar{x},\bar{y},\bar{z},t) = \bar{\vartheta}(\bar{x},\bar{y},\bar{z})e^{i\omega t}$ with

$$\left(\frac{\partial^2}{\partial\bar{x}^2} + \frac{\partial^2}{\partial\bar{y}^2} + \frac{\partial^2}{\partial\bar{z}^2}\right) \bar{\vartheta}(\bar{x},\bar{y},\bar{z}) = 0, \qquad (7.1.1)$$

and which vanishes for $\bar{y} \to -\infty$. The velocities u,v and w in the $\bar{x}$, $\bar{y}$ and $\bar{z}$ direction are

$$u = \frac{\partial\Theta}{\partial\bar{x}}, \quad v = \frac{\partial\Theta}{\partial\bar{y}}, \quad w = \frac{\partial\Theta}{\partial\bar{z}}. \qquad (7.1.2)$$

On the water surface $\bar{y} = 0$ we prescribe the displacement in the $\bar{y}$ direction $\eta(\bar{x},\bar{z},t)$, for $|\bar{x}| < b$

$$\eta(\bar{x},\bar{z},t) = \varepsilon \cos a\,\bar{z}\, e^{i\omega t}, \quad |\bar{x}| < b, \qquad (7.1.3)$$

and the pressure $P(\bar{x},\bar{z},t)$, for $|\bar{x}| > b$

$$P(\bar{x},\bar{z},t) = -\rho\left(\frac{\partial\Theta}{\partial t} + g\eta\right) = 0, \quad |\bar{x}| > b, \qquad (7.1.4)$$

where ρ is the specific mass of the water and g the acceleration of gravity. Introducing the dimensionless quantities

$x=\bar{x}/b, \quad y=\bar{y}/b, \quad z=\bar{z}/b, \quad \alpha=ab, \quad \nu=\omega^2 b/g,$

$$\vartheta(x,y) \cos a\,\bar{z} = \bar{\vartheta}(\bar{x},\bar{y},\bar{z})/i\omega b\varepsilon, \qquad (7.1.5)$$

$$p(x,y) \cos a\,\bar{z}\, e^{i\omega t} = P(\bar{x},\bar{y},\bar{z},t)/\rho\varepsilon g,$$

we obtain the following boundary value problem

$$\frac{\partial^2\vartheta}{\partial x^2} + \frac{\partial^2\vartheta}{\partial y^2} - \alpha^2\vartheta = 0 \qquad (7.1.6)$$

$$\frac{\partial\vartheta}{\partial y} = 1, \ |x| < 1; \quad \frac{\partial\vartheta}{\partial y} - \nu\vartheta = 0, \ |x| < 1; \quad y = 0,$$

$$\vartheta \to 0, \quad y \to -\infty. \qquad (7.1.7)$$

An estimate of the values of the parameters α and ν which occur in practice, follows from

length of ships	200 - 50 meter	(7.1.8)
width of ships	25 - 10 meter	(7.1.9)
frequency of the vibration	1 - 15 Herz .	(7.1.10)

From 7.1.8 follows the range of the relevant wave-length of the strip, from 200 meter for the vibration with two nodes for the largest length to 25 meter for the vibration with four nodes for the smallest length. Hence

$$\text{wave length strip} \qquad 200 - 25 \text{ meter.} \tag{7.1.11}$$

The dimensionless parameters α^2 and ν then assume the following values

$$\alpha^2 = 0{,}6 - 6, \tag{7.1.12}$$

$$\nu = 100 - 9000. \tag{7.1.13}$$

Since the parameter α, which is a measure for the three dimensional behaviour of the flow, is of the order of magnitude of unity, we can expect some influence of this behaviour at the added mass of the strip. Further, ν being much larger than unity, we replace the second boundary condition in (7.1.7) by the high frequency approximation

$$\vartheta = 0 \,, \quad |x| > 1 \,, \quad y = 0. \tag{7.1.14}$$

7.2 DERIVATION OF THE INTEGRAL EQUATION

The boundary value problem, posed in the preceding paragraph

$$\frac{\partial^2\vartheta}{\partial x^2} + \frac{\partial^2\vartheta}{\partial y^2} - \alpha^2\vartheta = 0 \tag{7.2.1}$$

$$\begin{gathered} \frac{\partial\vartheta}{\partial y} = 1, \quad |x| < 1; \quad \vartheta = 0 \,, \; |x| > 1; \; y = 0, \\ \vartheta \to 0, \quad y \to -\infty, \end{gathered} \tag{7.2.2}$$

is solved by application of a Fourier-transformation to (7.2.1). This gives for the solution of this problem the representation

$$\vartheta(x,y) = \frac{1}{2\pi} \int_{-1}^{+1} \vartheta(\xi)\,d\xi \int_{-\infty}^{+\infty} e^{-i\lambda(x-\xi)}\, e^{(\lambda^2+\alpha^2)^{\frac{1}{2}} y}\, d\lambda, \quad y < 0, \tag{7.2.3}$$

where $\vartheta(x)$ is the unknown limit of $\vartheta(x,y)$ for $y \to 0$ and $|x| < 1$.

Replacement of the first exponential by a cosine, differentation with respect to y and integration with respect to x yields

$$\int_0^x \frac{\partial \vartheta}{\partial y}(\xi,y)d\xi = \frac{1}{\pi}\int_{-1}^{+1}\vartheta(\xi)d\xi\int_0^\infty \sin\lambda(x-\xi)\,\frac{(\lambda^2+\alpha^2)^{\frac{1}{2}}}{\lambda}\,e^{(\lambda^2+\alpha^2)^{\frac{1}{2}}y}\,d\lambda,$$
$$(y < 0) \qquad (7.2.4)$$

where we have used the fact that $\vartheta(\xi)$ is a symmetric function. We investigate the limit $y \to 0$ of (7.2.4). The left hand side tends to

$$\lim_{y\to 0}\int_0^x \frac{\partial\vartheta}{\partial y}(\xi,y)d\xi = x, \quad |x| < 1. \qquad (7.2.5)$$

Now remains to consider

$$\lim_{y\to 0}\int_0^\infty \sin\lambda x\,\frac{(\lambda^2+\alpha^2)^{\frac{1}{2}}}{\lambda}\,e^{(\lambda^2+\alpha^2)^{\frac{1}{2}}y}\,d\lambda =$$

$$= \lim_{y\to 0}\int_0^\infty \sin\lambda x\left\{\frac{\lambda}{(\lambda^2+\alpha^2)^{\frac{1}{2}}} + \frac{\alpha^2}{\lambda(\lambda^2+\alpha^2)^{\frac{1}{2}}}\right\}e^{\lambda y}\,d\lambda =$$

$$= \alpha\,K_1(\alpha x) + \alpha\int_0^{\alpha x} K_0(|\eta|)d\eta, \quad x > 0, \qquad (7.2.6)$$

where $K_0(x)$ and $K_1(x)$ are the modified Besselfunctions of the second kind. Because the kernel of the integral equation (7.2.4) is an odd function with respect to x, we have to define the values of $K_1(\alpha x)$ in (7.2.6) for $x < 0$ by

$$K_1(-\alpha x) = -K_1(\alpha x), \quad x > 0. \qquad (7.2.7)$$

We write (7.2.6) in the following form

$$\frac{1}{x} + \left\{\alpha\,K_1(\alpha x) - \frac{1}{x} + \alpha\int_0^{\alpha x} K_0(|\eta|)d\eta\right\} = \frac{1}{x} + k(x), \qquad (7.2.8)$$

where we have introduced the abbreviation $k(x)$. From the series expansion of $K_1(\alpha x)$ in the neighbourhood of $x = 0$ we see that $k(x)$ is a bounded function of x, which tends to zero uniformly on each finite interval when $\alpha \to 0$.

In this way (7.2.4) can be written as the following singular integral equation

$$x = \frac{1}{\pi}\oint_{-1}^{+1}\frac{\vartheta(\xi)}{(x-\xi)}d\xi + \frac{1}{\pi}\int_{-1}^{+1}\vartheta(\xi)\,k(x-\xi)d\xi, \qquad (7.2.9)$$

where the first integral must be taken in the sense of Cauchy.

The first integral in (7.2.9) is independent of α, while the second integral is zero for $\alpha = 0$, hence this one represents the influence of the secondary flow.

7.3 THE CASES $\alpha = 0$ AND $\alpha \to 0$.

First we consider $\alpha = 0$ and hence $k(x) \equiv 0$. Herewith (7.2.9) changes into

$$x = \frac{1}{\pi} \oint_{-1}^{+1} \frac{\vartheta(\xi)}{(x-\xi)} d\xi, \qquad \alpha = 0. \tag{7.3.1}$$

The solution of this simple singular integral equation is given in para. 1.3,

$$\vartheta(x) = \frac{\vartheta_0}{\pi(1-x^2)^{\frac{1}{2}}} - \frac{1}{\pi(1-x^2)^{\frac{1}{2}}} \int_{-1}^{+1} \frac{\xi(1-\xi^2)^{\frac{1}{2}}}{(x-\xi)} d\xi, \tag{7.3.2}$$

where ϑ_0 is some arbitrary number. Using the known integral

$$\int_0^{\pi} \frac{\cos n t}{\cos t - \cos s} dt = \pi \frac{\sin n s}{\sin s} \tag{7.3.3}$$

we find

$$\vartheta(x) = \frac{\vartheta_0}{\pi(1-x^2)^{\frac{1}{2}}} + \frac{(1-2x^2)}{2(1-x^2)^{\frac{1}{2}}}. \tag{7.3.4}$$

Now we determine the unknown constant $\vartheta_0 = \int_{-1}^{+1} \vartheta(x)dx$ by requiring that the square root singularities of $\vartheta(x)$ for $x = \pm 1$ vanish. This yields $\vartheta_0 = \pi/2$ and (7.3.4) changes into

$$\vartheta(x) = (1-x^2)^{\frac{1}{2}}, \qquad \alpha = 0. \tag{7.3.5}$$

Next we consider the case $\alpha \to 0$ and reduce (7.2.9), by the method described in para. 1.3, to a Fredholm integral equation with a quadratic integrable kernel.
Equation (7.2.9) can then be written as

$$(1-x^2)^{\frac{1}{4}} = f(x) - \frac{1}{\pi^2} \int_{-1}^{+1} f(\eta)\, K(x,\eta) d\eta, \tag{7.3.6}$$

where

$$f(x) = \vartheta(x)\, (1-x^2)^{\frac{1}{4}} \tag{7.3.7}$$

and

$$K(x,\eta) = \frac{1}{(1-x^2)^{\frac{1}{4}}(1-\eta^2)^{\frac{1}{4}}} \int_{-1}^{+1} \frac{k(\xi-\eta)(1-\xi^2)^{\frac{1}{2}}}{(x-\xi)} d\xi. \tag{7.3.8}$$

Now it is allowed (ref. 16), for values of α which are small enough, to expand the solution of (7.3.6) into a Neumann series. From this series we shall calculate the first two terms.

Before doing this we determine from the function k(x) (7.2.8) the first term of the asymptotic expansion for $\alpha \to 0$. This can be done by using the known expansions of $K_0(x)$ and $K_1(x)$ in the neighbourhood of x = 0. We obtain

$$k(x) \approx \frac{\alpha^2 x}{2}\{-\ln\alpha + \frac{3}{2} + \ln 2 - \gamma - \ln|x|\} =$$

$$= \frac{\alpha^2 x}{2}\{C(\alpha) - \ln|x|\}, \qquad (7.3.9)$$

where γ = 0,5772157 is Eulers constant. Substitution of (7.3.9) into (7.3.6) and determination of the first two terms of the Neumann series yields

$$\vartheta(x) = \frac{\vartheta_0^*}{(1-x^2)^{\frac{1}{2}}} + (1-x^2)^{\frac{1}{2}} +$$

$$+ \frac{\alpha^2}{2\pi^2(1-x^2)^{\frac{1}{2}}}\int_{-1}^{+1}\frac{(1-\eta^2)^{\frac{1}{2}}(\xi-\eta)\{C(\alpha) - \ln|\xi-\eta|\}(1-\xi^2)^{\frac{1}{2}}\,d\xi\,d\eta}{(x-\xi)}. \qquad (7.3.10)$$

The integrals in this expression can be calculated with the aid of (7.3.3) and

$$\int_0^1 (1-\eta^2)^{\frac{1}{2}}\ln\eta\,d\eta = \frac{-\pi}{4}\left(\tfrac{1}{2} + \ln 2\right). \qquad (7.3.11)$$

We arrive after a straightforward calculation at the result

$$\vartheta(x) = (1-x^2)^{\frac{1}{2}} + \frac{\alpha^2}{2(1-x^2)^{\frac{1}{2}}}\{-\frac{x^4}{6} + x^2(-\tfrac{1}{2}\ln\alpha + \ln 2 + \frac{7}{12} - \frac{\gamma}{2}) +$$

$$+ (-\frac{5}{12} + \frac{\ln\alpha}{2} - \ln 2 + \frac{\gamma}{2})\} \qquad (7.3.12)$$

where we have taken for ϑ_0^*

$$\vartheta_0^* = -\alpha^2\frac{\pi}{2}\{\frac{3}{16} - \frac{\ln\alpha}{4} + \frac{\ln 2}{2} - \frac{\gamma}{4}\}. \qquad (7.3.13)$$

From (7.3.13) we see that $\vartheta_0 + \vartheta_0^*$, which quantity is proportional to the resultant force under the strip, decreases for increasing values of α. This is to be expected because the three-dimensional theory is less restrictive for the water motion than the two-dimensional one.

7.4 THE CASE $\alpha \to \infty$

The limit case $\alpha \to \infty$ can also be dealt with analytically. We start from the original boundary value problem defined by (7.1.1), (7.1.3) and (7.1.4) and consider the limit $b \to \infty$, which is equivalent to $\alpha \to \infty$. The boundary conditions then become

$$\eta = \varepsilon \cos a\,\bar{z}\, e^{i\omega t}, \quad \bar{x} < 0; \quad P = 0, \quad \bar{x} > 0; \quad \bar{y} = 0. \qquad (7.4.1)$$

We introduce the function $\varphi(\bar{x},\bar{y})$ by

$$\varphi(\bar{x},\bar{y}) \cos a\,\bar{z} = - \frac{i}{\omega\varepsilon}\, \vartheta\,(\bar{x},\bar{y},\bar{z}). \qquad (7.4.2)$$

Herewith and using the high frequency approximation the boundary value problem can be written as

$$\frac{\partial^2 \varphi}{\partial \bar{x}^2} + \frac{\partial^2 \varphi}{\partial \bar{y}^2} - a^2 \varphi = 0 \qquad (7.4.3)$$

$$\frac{\partial \varphi}{\partial \bar{y}} = 1, \quad \bar{x} < 0 \;; \quad \varphi = 0 \;, \quad \bar{x} > 0; \quad \bar{y} = 0. \qquad (7.4.4)$$

Applying a Fourier transformation to (7.4.3) and (7.4.4) we obtain

$$\Phi(\lambda,\bar{y}) = F_-(\lambda)\, e^{(\lambda^2+a^2)^{\frac{1}{2}}\,\bar{y}}, \qquad (7.4.5)$$

where

$$\Phi(\lambda,\bar{y}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\lambda\bar{x}}\, \varphi(\bar{x},\bar{y})\, d\bar{x}\;, \quad \mathrm{Im}\,\lambda < 0 \qquad (7.4.6)$$

and $F_-(\lambda)$ is the, still unknown, transform of $\varphi(\bar{x},0)$ for $\bar{x} < 0$, which is regular in a lower half-plane. By transformation of the first boundary condition of (7.4.4) and substitution into (7.4.5) after differentiation with respect to $\bar{y}$ we obtain the simple Hilbert problem

$$(\lambda^2+a^2)^{\frac{1}{2}}\, F_-(\lambda) = G_+(\lambda) - \frac{i}{\sqrt{2\pi}\,\lambda}, \qquad (7.4.7)$$

valid on a line L below and sufficiently close to the real axis in the λ plane. Here $G_+(\lambda)$ is the transform of $\frac{\partial \varphi}{\partial \bar{y}}(x,0)$ for $\bar{x} > 0$. Solution of (7.4.7) yields, using the inverse transformation

$$\varphi(\bar{x},0) = \frac{-i}{2\pi\sqrt{ia}} \int_{-\infty}^{+\infty} \frac{e^{-i\lambda\bar{x}}}{\lambda(\lambda-ia)^{\frac{1}{2}}} d\lambda, \qquad \bar{x} < 0, \tag{7.4.8}$$

where the square roots are defined by $\sqrt{ia} = \frac{(1+i)}{\sqrt{2}} |\sqrt{a}|$ and by $\text{Re}\ (\lambda-ia)^{\frac{1}{2}} \to +\infty$ for $\text{Re}\ \lambda \to +\infty$ on L. Because $\bar{x} < 0$ we can complete the path of integration in (7.4.8) in the way designed in fig. (7.4.1).

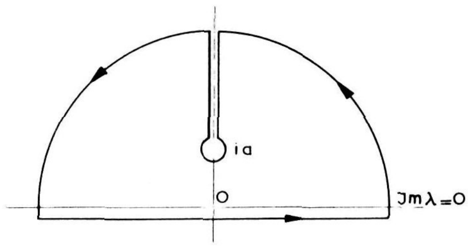

Fig. 7.4.1. The path of integration.

Introducing a new variable of integration $\xi = -i(\lambda-ia)$ we find for (7.4.8)

$$\varphi(\bar{x},0) = \frac{1}{a} - \frac{e^{ax}}{\pi\sqrt{a}} \int_{0}^{\infty} \frac{e^{\xi x}}{(\xi+a)\sqrt{\xi}} d\xi. \tag{7.4.9}$$

By substitution of $\xi = \sigma^2$ into (7.4.9) this integral can be calculated in terms of the error integral,

$$\varphi(\bar{x},0) = \frac{1}{a}\ E(\sqrt{-a\bar{x}}) \tag{7.4.10}$$

where

$$E(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-\xi^2} d\xi. \tag{7.4.11}$$

Assuming a strip situated on the water surface between $\bar{x} = -b$ and $\bar{x} = +b$, b sufficiently large, we find as a first approximation of the dimensionless potential $\vartheta(x,y)$ in the dimensionless coordinates of para. 7.3

$$\vartheta(x,0) = \frac{1}{\alpha}\ E(\sqrt{\alpha(1-|x|)})\ , \quad |x| < 1. \tag{7.4.12}$$

From this we find

$$\vartheta_0 = \int_{-1}^{+1} \vartheta(x,0)\,dx = \frac{2}{\alpha}\int_0^1 E(\sqrt{\alpha(1-x)}\ )\,dx \approx \frac{2}{\alpha}\left[1+\frac{1}{\alpha}\int_0^\infty \{E(\sqrt{y})-1\}dy\right] =$$

$$= \frac{2}{\alpha}\left(1 - \frac{0,5}{\alpha}\right). \qquad (7.4.13)$$

7.5 THE CASE OF FINITE α.

The case of a finite value of α will be treated approximatively. This can be done by approximating $k(x)$, defined in (7.2.8), with the aid of a polynomial

$$k(x) \sim \sum_{n=1}^{N} a_n\, x^{2n-1} \qquad (7.5.1)$$

Here we use the coefficients:

$a_1 = 1,3110\alpha^2$; $a_2 = -0,6543\alpha^4$; $a_3 = 0,24165\alpha^6$; $a_4 = -0,04151\alpha^8$;

$a_5 = 0,002594\alpha^{10}$, which yield an approximation within 1,4% for $|x| < 1$ and $0 \leq \alpha \leq 1,25$. x)

In the following way it can be seen, that the solution of our approximate problem tends to the solution of the exact problem by increasing the accuracy of (7.5.1). Equation (7.2.9) was reduced to a non homogeneous Fredholm integral equation (7.3.6) where both kernel and known functions are quadratic integrable. From this equation we know (ref. 16) that by approximating the kernel with increasing accuracy the approximate solution tends to the exact solution, with possible exception for a domain with measure zero. Since (7.3.6) and (7.2.9) are equivalent this will be true also for our singular integral equation.

We now consider the following approximate equation

$$x = \frac{1}{\pi}\oint_{-1}^{+1} \frac{\vartheta(\xi)}{(x-\xi)}\,d\xi + \frac{1}{\pi}\int_{-1}^{+1} \vartheta(\xi) \sum_{n=1}^{5} a_n (x-\xi)^{2n-1}\,d\xi. \qquad (7.5.2)$$

It is possible to solve this equation exactly. Carrying out the integrations

$$\vartheta_m = \int_{-1}^{+1} \vartheta(\xi)\xi^{m}\,d\xi, \qquad (7.5.3)$$

where the ϑ_m are as yet unknown constants, we can write (7.5.2) as

$$\int_{-1}^{+1} \frac{\vartheta(\xi)}{(x-\xi)}\,d\xi = -\sum_{n=1}^{5} b_n\, x^{2n-1}, \qquad (7.5.4)$$

x) The method as well as the numerical values for this approximation are due to Mr. H.J. Nunnink.

where

$$
\begin{aligned}
b_1 &= a_1\vartheta_0 + 3a_2\vartheta_2 + 5a_3\vartheta_4 + 7a_4\vartheta_6 + 9a_5\vartheta_8 - \pi \\
b_2 &= a_2\vartheta_0 + 10a_3\vartheta_2 + 35a_4\vartheta_4 + 84a_5\vartheta_6 \\
b_3 &= a_3\vartheta_0 + 21a_4\vartheta_2 + 126a_5\vartheta_4 \\
b_4 &= a_4\vartheta_0 + 36a_5\vartheta_2 \\
b_5 &= a_5\vartheta_0 .
\end{aligned}
\tag{7.5.5}
$$

Because $\vartheta(x)$ is symmetric with respect to $x = 0$, the quantities ϑ_n with odd values of n are dropped. By solving (7.5.4) (para 1.3) we can express $\vartheta(x)$ in terms of the unknown constants b_n.

$$\vartheta(x) = \frac{\vartheta_0}{\pi(1-x^2)^{\frac{1}{2}}} + \frac{1}{\pi^2(1-x^2)^{\frac{1}{2}}} \int_{-1}^{+1} \frac{\sum_{n=1}^{5} b_n\, \xi^{2n-1}(1-\xi^2)^{\frac{1}{2}}}{(x-\xi)}\, d\xi . \tag{7.5.6}$$

The integrals occurring in (7.5.6)

$$h_n(x) = \int_{-1}^{+1} \frac{\xi^{2n-1}(1-\xi^2)^{\frac{1}{2}}}{(x-\xi)}\, d\xi , \tag{7.5.7}$$

can easily be calculated by means of the following recursion formula

$$x^2 h_n(x) - h_{n+1}(x) = \pi \frac{(1+2)(1+2.2)\ldots\{1+(n-1)2\}}{(n+1)!\ 2^{n+1}} , \quad n \geq 2. \tag{7.5.8}$$

Herewith we obtain

$$
\begin{aligned}
h_1(x) &= \pi(x^2 - \tfrac{1}{2}) \\
h_2(x) &= \pi(x^4 - \tfrac{1}{2}x^2 - \tfrac{1}{8}) \\
h_3(x) &= \pi(x^6 - \tfrac{1}{2}x^4 - \tfrac{1}{8}x^2 - \tfrac{1}{16}) \\
h_4(x) &= \pi(x^8 - \tfrac{1}{2}x^6 - \tfrac{1}{8}x^4 - \tfrac{1}{16}x^2 - \tfrac{5}{128}) \\
h_5(x) &= \pi(x^{10} - \tfrac{1}{2}x^8 - \tfrac{1}{8}x^6 - \tfrac{1}{16}x^4 - \tfrac{5}{128}x^2 - \tfrac{7}{256}) ,
\end{aligned}
\tag{7.5.9}
$$

and we write (7.5.6) in the form

$$\vartheta(x) = \frac{\vartheta_0}{\pi(1-x^2)^{\frac{1}{2}}} + \frac{1}{\pi^2(1-x^2)^{\frac{1}{2}}} \sum_{n=1}^{5} b_n\, h_n(x). \tag{7.5.10}$$

In order to determine the constants b_n ($\vartheta_0 = b_5/a_5$), we multiply both sides of (7.5.10) by x^{2k} (k=0;1;2;3;4) and integrate with respect to x

$$\vartheta_{2k} = \frac{\vartheta_o}{\pi}\int_{-1}^{+1}\frac{x^{2k}}{(1-x^2)^{\frac{1}{2}}} + \frac{1}{\pi^2}\sum_{n=1}^{5} b_n \int_{-1}^{+1}\frac{h_n(x)}{(1-x^2)^{\frac{1}{2}}}\, x^{2k}\, dx. \quad (7.5.11)$$

Introducing the quantities

$$B(k,n) = \int_{-1}^{+1}\frac{h_n(x)}{(1-x^2)^{\frac{1}{2}}}\, x^{2k}\, dx, \quad (7.5.12)$$

we can write (7.5.11) as

$$\vartheta_{2k} = \frac{\vartheta_o}{\pi}\, A(k) + \frac{1}{\pi^2}\sum_{n=1}^{5} b_n\, B(k,n), \quad (7.5.13)$$

where

$$\begin{aligned}
B(k,1) &= \pi\{A(k+1)-\tfrac{1}{2}A(k)\}\\
B(k,2) &= \pi\{A(k+2)-\tfrac{1}{2}A(k+1)-\tfrac{1}{8}A(k)\}\\
B(k,3) &= \pi\{A(k+3)-\tfrac{1}{2}A(k+2)-\tfrac{1}{8}A(k+1)-\tfrac{1}{16}A(k)\} \quad (7.5.14)\\
B(k,4) &= \pi\{A(k+4)-\tfrac{1}{2}A(k+3)-\tfrac{1}{8}A(k+2)-\tfrac{1}{16}A(k+1)-\tfrac{5}{128}A(k)\}\\
B(k,5) &= \pi\{A(k+5)-\tfrac{1}{2}A(k+4)-\tfrac{1}{8}A(k+3)-\tfrac{1}{16}A(k+2)-\tfrac{5}{128}A(k+1)-\tfrac{7}{256}A(k)\}
\end{aligned}$$

and

$$A(k)=\int_{-1}^{+1}\frac{x^{2k}}{(1-x^2)^{\frac{1}{2}}}\,dx=\frac{\pi(1+2)(1+2.2)\ldots\{1+(k-1)2\}}{k!2^k},\quad A(0)=\pi. \quad (7.5.15)$$

Using (7.5.5) we have obtained four equations (k=1,2,3 and 4) for $\vartheta_o \ldots \vartheta_8$. Because $B(0,n)\equiv 0$ the equation for k=0 yields the identity $\vartheta_o = \vartheta_o$. In order to obtain another relation between the ϑ_{2n} we require again that $\vartheta(x)$ has no singularities for $x=\pm 1$. From (7.5.10) we find

$$\vartheta_o + \frac{1}{\pi}\sum_{n=1}^{5} b_n\, h_n(1) = 0. \quad (7.5.16)$$

Hence we have found five linear equations for the five unknowns $\vartheta_o \ldots \vartheta_8$. Then by (7.5.5) and (7.5.10) the solution of (7.5.2) is determined.

7.6 THE ADDED MASS FOR THE STRIP

We shall now calculate the added mass for the strip. Since in ship building the added mass is only of importance for a whole cross section of the ship, we shall integrate the pressure, which is a function of x, with respect to x. Because we have used the high frequency approximation we may write the resultant force K as

$$K=\int_{-b}^{+b} P(\bar{x},\bar{0},\bar{z},t)d\bar{x}= -\rho \int_{-b}^{+b} \frac{\partial\Theta}{\partial t}d\bar{x}=\rho\omega^2 b^2 \varepsilon \cos a\bar{z} \cos \omega t \int_{-1}^{+1} \vartheta(x,0)dx=$$

$$= \rho \omega^2 b^2 \varepsilon \vartheta_0 \cos a\bar{z} \cos \omega t. \qquad (7.6.1)$$

We seek fictitious mass $m(\bar{z})$ and resistance $\gamma(\bar{z})$ distributions on the strip, which yield the same forces as are exerted by the water. Then we obtain the relation

$$K = \rho \omega^2 b^2 \varepsilon \vartheta_0 \cos a\bar{z} \cos \omega t = -m(\bar{z}) \ddot{\eta} - \gamma(\bar{z})\dot{\eta} \,. \qquad (7.6.2)$$

Substitution of the real part of (7.1.3) into this equation results in

$$m(\bar{z})= b^2\rho \vartheta_0, \qquad \gamma(\bar{z}) \equiv 0. \qquad (7.6.3)$$

The vanishing of the resistance is caused by the fact that at infinitely high frequences no energy can be transported along the water surface.

7.7 THE ADDED MASS FOR A VIBRATING INFINITE CYLINDER.

We now treat the same problem as in the preceeding section but replace the strip by a half cylinder with radius b (fig. 7.7.1)

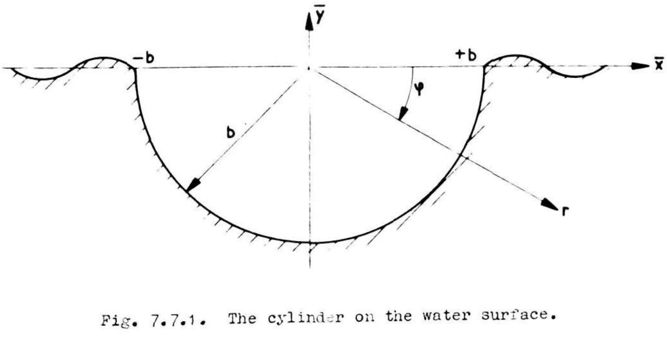

Fig. 7.7.1. The cylinder on the water surface.

Because we have a boundary condition for $\bar{r} = b$ we start with the potential equation on circular cylindrical coordinates $\bar{r}$, φ and $\bar{z}$. Using the high frequency approximation we arrive at the following boundary value problem

$$\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\bar{r} \frac{\partial \Theta}{\partial \bar{r}}\right) + \frac{1}{\bar{r}^2} \frac{\partial^2 \Theta}{\partial \varphi^2} + \frac{\partial^2 \Theta}{\partial \bar{z}^2} = 0, \qquad (7.7.1)$$

$$\eta = \varepsilon \cos a\bar{z}\, e^{i\omega t}, \quad \bar{r} = b; \quad \Theta = 0, \ \varphi = \pi, \ \varphi = 0. \qquad (7.7.2)$$

The first boundary condition can be written as

$$\frac{\partial \Theta}{\partial \bar{r}} = - \dot{\eta} \sin \varphi = -i\omega\, \varepsilon \cos a\bar{z}\, e^{i\omega t} \sin \varphi, \ \bar{r} = b. \qquad (7.7.3)$$

The general solution of (7.7.1) for the case that $\Theta(\bar{r},\bar{z},\varphi,t)$ has the form

$$\Theta = \bar{\vartheta}(\bar{r},\varphi) \cos a\bar{z}\, e^{i\omega t} \qquad (7.7.4)$$

and which remains bounded at infinity is

$$\bar{\vartheta}(\bar{r},\varphi) = \sum_{n=0}^{\infty} K_n(a\bar{r}) \{A_n \cos n\varphi + B_n \sin n\varphi\}. \qquad (7.7.5)$$

Herewith the boundary condition for $\bar{r} = b$ becomes

$$\frac{\partial \bar{\vartheta}}{\partial \bar{r}} (\bar{r},\varphi)\Big|_{\bar{r}=b} = - \frac{a}{2} \sum_{n=0}^{\infty} \{K_{n-1}(ab) + K_{n+1}(ab)\}\{A_n \cos n\varphi + B_n \sin n\varphi\} =$$

$$= -i\omega\varepsilon \sin \varphi. \qquad (7.7.6)$$

Hence

$$\Theta(\bar{r},\bar{z},\varphi,t) = \frac{2\, i\omega\, \varepsilon}{a\{K_0(ab)+K_2(ab)\}} K_1(a\bar{r}) \cos a\bar{z} \sin\varphi\, e^{i\omega t}. \qquad (7.7.7)$$

Integration of the vertical component of the pressure exerted on the cylinder yields the resultant force K,

$$K = \mathrm{Re}\, -\rho \int_0^{\pi} \frac{\partial \Theta}{\partial t} (b,\bar{z},\varphi,t) \sin \varphi\, b\, d\varphi =$$

$$= \frac{\rho\, \omega^2 \varepsilon\, b\, \pi\, K_1(ab)}{a\{K_0(ab)+K_2(ab)\}} \cos a\bar{z} \cos \omega t. \qquad (7.7.8)$$

By formula (7.6.2) we find in this case for the added mass

$$m(\bar{z}) = \frac{\pi \rho b^2}{\alpha} \quad \frac{K_1(\alpha)}{K_0(\alpha)+K_2(\alpha)} , \qquad (7.7.9)$$

where $\alpha = ab$. When α tends to zero we find by expanding $K_0(\alpha)$, $K_1(\alpha)$ and $K_2(\alpha)$ in the neighbourhood of $\alpha = 0$

$$m(\bar{z}) \approx \frac{\pi \rho b^2}{2} \{1+\alpha^2 (\ln\alpha - \ln 2 + \gamma)\}. \qquad (7.7.10)$$

When α tends to infinity we find for the asymptotic behaviour

$$m(\bar{z}) \approx \frac{\pi \rho b^2}{2\alpha} . \qquad (7.7.11)$$

7.8 DISCUSSION OF THE NUMERICAL RESULTS

In fig. 7.8.1 we have drawn the results of the numerical calculations. On the vertical axis is plotted the dimensionless added mass $m/b^2\rho$ and on the horizontal axis the dimensionless quotient α/π of the width of the strip (or cylinder) and the wave length. From the asymptotical expansions about the origin $\alpha = 0$ (7.3.13) and (7.7.10) it is clear that the graph (1) for the strip as well as the graph (3) for the half immersed cylinder start with a horizontal tangent

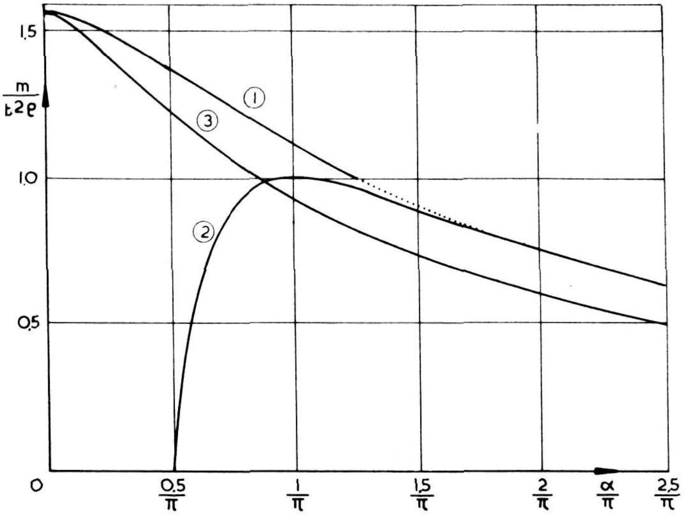

Fig. 7.8.1. Relation between dimensionless added mass, $m/b^2\rho$ and (dimensionless wavelength)$^{-1}$, α/π. (1) strip, (2) asymptotical behaviour strip and (3) half immersed cylinder.

from the point $m/b^2\rho = \pi/2$. The line (1) has been calculated by the method described in para. 7.5, its asymptotic behaviour (2) follows from para.7.4, while (3) represents (7.7.9). Some values are given in the following table:

Table IV, $^{m}/b^{2}\rho$.

$^{\alpha}/\pi$		0	$0,25/\pi$	$0,5/\pi$	$0,75/\pi$	$1/\pi$	$1,25/\pi$	$1,5/\pi$	$2/\pi$	$2,5/\pi$
$^{m}/b^{2}\rho$	strip	$\frac{\pi}{2}$ = 1,57	1,51	1,38	1,24	1,11	1,00	0,908	0,750	0,640
	cylinder	$\frac{\pi}{2}$ = 1,57	1,42	1,23	1,06	0,924	0,816	0,735	0,598	0,505

The value of $m/b^2\rho$ for the strip for $\alpha/\pi = {}^{1,5}/\pi$ has been obtained from the interpolating dotted line and the values for $\alpha/\pi = {}^{2}/\pi$ and ${}^{2,5}/\pi$ come from the asymptotical behaviour (7.4.13).

Apparently the decrease in the added mass, caused by the three-dimensional motion of the water is more pronounced for the cylinder than for the flat strip. This should mean that for ships of shallow draught the influence of the three-dimens ional effects is smaller than is expected from the correction coefficient based on the three-dimensional flow around a vibrating half immersed ellipsoid of revolution.

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S U M M A R Y

In this thesis we have discussed several problems which are related closely to the Hilbert problem, of which a short survey is given in chapter 1.

Chapter 2 deals with the Wiener-Hopf type integral equation by means of the theory of sectionally holomorfic functions. The assumption of a strip of convergence for the Fourier transforms, needed in the classical theory, could be replaced by the assumption of a line of convergence. In chapter 3 an application is made to the shrink-fit problem of an infinite tube shrunk onto a semi infinite rigid shaft. The resulting shrink-fit stresses were calculated. It turned out that for tubes which are sufficiently thin there occurs a region where tube and shaft are out of contact.

In chapter 4 we gave some general considerations about integro-differential equations of the first order and indicated a method to obtain solutions of the equation for the case that the kernel decreases algebraicly while the solution has a sinoidal behaviour at infinity. Next a special integro-differential equation was considered with a derivative of the fourth order, which occurred under the sign of integration. It was asked to obtain the first terms of the asymptotical expansion of a certain physically important quantity with respect to a small parameter b. This expansion possesses a logarithmic branch point at $b = 0$.

The calculations made in the chapters 6 and 7 were concerned with the motion of water with a free surface. First we considered the influence of a fixed dock on incoming waves. We found for instance that, under certain circumstances, when the length of the incoming wave equals the width of the dock only 4,8 % of the energy can pass, the remaining part is reflected. In the last chapter we considered the virtual mass of a vibrating strip on the surface of the water. The aim was to check the use of a vibrating ellipsoid of revolution in ship building, for obtaining a three dimensional correction coefficient. It turned out that there exists a difference of about 20 % between the virtual mass of the strip and a half immersed cylinder, when the wave length of the strip is about two times its width. This means a possible overrating of the three dimensional effects for a ship of shallow draught by using the correction coefficient based on the flow around an ellipsoid of revolution.

S T E L L I N G E N

1. Het vergelijken van het probleem van een planerende cylinder, die zich loodrecht op een beschrijvende voortbeweegt over het wateroppervlak, met een overeenkomstig indrukkingsprobleem uit de elasticiteits theorie, levert op eenvoudige wijze inzicht in de singulariteiten van de druk die het water op de cylinder uitoefent.
   L.N. Sretenski, Bull.Dep.Techn.Sci. U.S.S.R., 7,3,1940.

2. Problemen over de invloed van de aardrotatie op diffractie verschijnselen van getijde golven kunnen,bij bepaalde geometrie van de obstakels, teruggebracht worden tot bekende vraagstukken. Dit geschiedt door invoering van een nieuwe onbekende functie waaruit later de gezochte functie door een eenvoudige integratie is te bepalen.
   J. Crease, J. Fluid Mech. 1, 86, 1956.

3. Een cirkelvormige plaat waarin stationaire thermo-elastische spanningen optreden, die ontstaan door gelijkmatige warmte toevoer aan de rand en een even grote warmte onttrekking in het midden, kan niet in een rotatie symmetrische vorm uitknikken.

4. De theorie van Reissner toegepast op een dunwandige buis (bv. dikten tot een derde van de binnen diameter) levert voor het krimp-spannings probleem van hoofdstuk 3 van dit proefschrift een qualitatief goed beeld van de krimpspanningen. Op een afstand van ongeveer twee maal de wanddikte van het eindvlak van de kern wijken de resultaten minder dan 3 % af van de in hoofdstuk 3 berekende waarden.

5. Bij bepaalde harmonische excitaties van een harmonische oscillator met een aan één zijde begrensde amplitude is het niet mogelijk de beweging te beschrijven met trillings vormen die nul of één keer per periode van de beweging aantikken.
   J.A. Sparenberg, Appl.sci.Res. A6, 53, 1956.

6. De lineaire differentiaal-vergelijkingen,die het resultaat zijn van de methode van de equivalente linearisatie voor het verkrijgen van een benaderde periodieke oplossing van quasilineaire differentiaal vergelijkingen,zijn ook toepasbaar op sterk niet lineaire differentiaal vergelijkingen mits de invloed van het niet lineaire gedeelte voor de beschouwde oplossing klein genoeg is.

7. De door Ales Tondel gegeven conclusie over gevaarlijke gebieden voor de omwentelingssnelheid van een rotor in verband met het niet lineaire elastische gedrag van de kussenblokken, volgt niet uit zijn mathematische analyse van het probleem.
Ales Tondel, Revue de Mécanique Appliquée, 2, 128, 1957.

8. Verschillende physische problemen kunnen op eenvoudige wijze benaderend behandeld worden met de methode beschreven in para. 7.5 van dit proefschrift.

9. De methode van Mac Camy voor het bepalen van de oplossing van een bepaald type integraal vergelijkingen dient aangevuld te worden met een benaderings methode die in het inwendige van het definitie gebied nauwkeuriger resultaten levert.
Mac Camy, Techn. Rep. No 2, Dep. Math. University of California, 1955.

10. De resultaten gegeven door Chi-Chang Chao voor de sommering van reeksen van Fourier van een bepaald type kunnen ook verkregen worden door een eenvoudige toepassing van de Fourier reeksen voor sin ax en cos ax, $-\pi < x < +\pi$.
Chi-Chang Chao, Quart.J.Mech.App.Math. 4, 508, 1956.

11. Het is wenselijk dat op het college gewone differentiaalvergelijkingen aan de studenten inzicht wordt gegeven over de instabiliteits gebieden die optreden bij differentiaal vergelijkingen van de tweede orde met periodieke coefficienten.

12. Het verdient aanbeveling dat het vermenigvuldigen van proefschriften door instellingen van hoger onderwijs kan worden verzorgd.