Overview of Turbulence Models for External Aerodynamics

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Cover: Aerospace Design Studio, 66.5 x 45.5 cm, by:
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Tel. +31 (0)71 512 67 25

Printed in The Netherlands

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1. Introduction

As the Reynolds number for the flow around the aerofoils and fuselage of aircraft is usually very large, most of the boundary-layer flow along the surface will be turbulent. It is not possible to compute all the details of the turbulent motion within a reasonable turn-around time (say several hours); therefore for all practical computations the use of *turbulence models* is mandatory.

Up to now most designers and developers of aerofoils compute turbulent flows with older generation models, like the algebraic models of Cebeci-Smith, Baldwin-Lomax, or Johnson-King. More recently some industries and research institutes (like NLR) have also applied two-equation models. All these models assume the existence of a single turbulent velocity and length scale, which is only approximately right for so-called equilibrium flows, such as attached boundary layers in a zero or moderate streamwise pressure gradient. Indeed those models have shown to be rather inaccurate for non-equilibrium flows, like separating boundary layers occurring in high-lift configurations; examples of such configurations are aerofoils under high angle of attack, and multi-element aerofoils. Therefore all these models are only accurate in a limited number of flow types, and they do not meet the high-accuracy requirements for a wide range of configurations. As a result of this, experiments in wind tunnels still serve as the major source of design information.

However, the new generation of turbulence models, *i.e.* Differential Reynolds-Stress Models (DRSM), are expected to deliver much higher accuracy than the mentioned algebraic and two-equation models, and such models can thus upgrade the role of computations in the design and development process. The price of the improved accuracy in a wide range of configurations is an increased computational effort, as multiple differential equations have to be solved for the turbulence. But, provided the numerical methods used are appropriate, an acceptable turn-around time on a modern work station seems to be obtainable.

The present report gives an overview of the above-mentioned types of turbulence models. The models were implemented in in a boundary-layer code that is used at the low-speed aerodynamics laboratory of our faculty. The code was applied to different types of boundary layers, both without and with (favourable or adverse) streamwise pressure gradient. We also used the models to derive the scalings of the boundary-layer at very large Reynolds numbers. The results of the application of the turbulence models are not included in this report, but they are published in three journal papers:

- Application of Turbulence Models to Attached Boundary Layers Relevant to Aeronautics, to appear in *Applied Scientific Research* (1997).

- Scalings of the Turbulent Boundary Layer Along a Flat Plate According to Different Turbulence Models, to appear in *International Journal on Heat and Fluid Flow* (1997).

- Scaling of Equilibrium Boundary Layers Under Adverse Pressure Gradient According to Different Turbulence Models, Submitted to *AIAA Journal* (1997).

The present report is meant to provide the details of turbulence models, and may also be used as a text for undergraduate students in a course on boundary layers. Other valuable papers on turbulence models for external aerodynamics are provided by, for example, Bushnell (1991) and Hanjalić (1994).

2. Equations for the Reynolds stresses

Differential equations for the turbulent Reynolds stresses $-\overline{u_i' u_j'}$ can directly be derived from the Navier-Stokes equations. These equations can be used as a starting point for turbulence modelling.

To derive the Reynolds-stress equations, we use the tensor notation, with

$$x_1 = x, \quad x_2 = y, \quad x_3 = z; \tag{1}$$

$$u_1 = u, \quad u_2 = v, \quad u_3 = w.$$

The Einstein summation convention will be used, which says that if an index appears twice, a summation should be performed for all values that the index can have. For example

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \tag{2}$$

or,

$$u_k \frac{\partial u_i}{\partial x_k} = u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + u_3 \frac{\partial u_i}{\partial x_3}. \tag{3}$$

By using the Einstein convention, the continuity equation for incompressible flows can shortly be written as

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{4}$$

With the tensor notation the incompressible Navier-Stokes equation in the direction x_i (with $i = 1, 2, 3$) can be written as

$$\frac{\partial}{\partial t}(u_i + u_i') + (u_k + u_k')\frac{\partial}{\partial x_k}(u_i + u_i') = -\frac{1}{\rho}\frac{\partial}{\partial x_i}(p + p') + \nu \frac{\partial^2}{\partial x_k \partial x_k}(u_i + u_i'). \tag{5}$$

Here the instantaneous velocity and pressure are split up in a time-averaged contribution (denoted *without* a prime) and a fluctuating quantity (denoted *with* a prime).

If this equation is averaged in time, all terms that are linear in the fluctuations will disappear. For a stationary time-averaged flow, the time averaging leads to

$$u_k \frac{\partial u_i}{\partial x_k} + \overline{u_k' \frac{\partial u_i'}{\partial x_k}} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k}, \tag{6}$$

or to

$$u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} - \overline{u_k' \frac{\partial u_i'}{\partial x_k}}. \tag{7}$$

Because

$$\frac{\partial}{\partial x_k}(u_i' u_k') = u_i' \frac{\partial u_k'}{\partial x_k} + u_k' \frac{\partial u_i'}{\partial x_k}, \tag{8}$$

and because, due to the continuity equation,

$$\frac{\partial u_k'}{\partial x_k} = 0, \tag{9}$$

we can rewrite equation (7) as

$$u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} - \frac{\partial}{\partial x_k}\overline{u_i' u_k'}. \tag{10}$$

This is the Reynolds-Averaged Navier-Stokes equation (RANS). This equation describes how the time-averaged velocity field is influenced by the Reynolds stresses $\overline{u_i' u_k'}$.

A transport equation for the Reynolds stresses can be derived as well. Thereto equation (5) is multiplied by u_j', which gives

$$u_j' \frac{\partial}{\partial t}(u_i + u_i') + u_j'(u_k + u_k')\frac{\partial}{\partial x_k}(u_i + u_i') = -u_j'\frac{1}{\rho}\frac{\partial}{\partial x_i}(p+p') + \nu u_j' \frac{\partial^2}{\partial x_k \partial x_k}(u_i + u_i'), \tag{11}$$

(with $i = 1,2,3$ and $j = 1,2,3$). Then equation (5) in the j-direction is multiplied with u_i' (that is, i and j are just interchanged in the equation), which gives

$$u_i' \frac{\partial}{\partial t}(u_j + u_j') + u_i'(u_k + u_k')\frac{\partial}{\partial x_k}(u_j + u_j') = -u_i'\frac{1}{\rho}\frac{\partial}{\partial x_j}(p+p') + \nu u_i' \frac{\partial^2}{\partial x_k \partial x_k}(u_j + u_j'). \tag{12}$$

Adding equations (11) and (12) gives, after time averaging

$$\underbrace{\overline{u_j' \frac{\partial u_i'}{\partial t}} + \overline{u_i' \frac{\partial u_j'}{\partial t}}}_{1} + \underbrace{\overline{u_k u_j' \frac{\partial u_i'}{\partial x_k}} + \overline{u_k u_i' \frac{\partial u_j'}{\partial x_k}}}_{2} + \underbrace{\overline{u_j' u_k' \frac{\partial u_i}{\partial x_k}} + \overline{u_i' u_k' \frac{\partial u_j}{\partial x_k}}}_{3} + \underbrace{\overline{u_j' u_k' \frac{\partial u_i'}{\partial x_k}} + \overline{u_i' u_k' \frac{\partial u_j'}{\partial x_k}}}_{4} = \tag{13}$$

$$\underbrace{-\frac{1}{\rho}\overline{u_j' \frac{\partial p'}{\partial x_i}} - \frac{1}{\rho}\overline{u_i' \frac{\partial p'}{\partial x_j}}}_{5} + \underbrace{\nu \overline{u_j' \frac{\partial^2 u_i'}{\partial x_k \partial x_k}} + \nu \overline{u_i' \frac{\partial^2 u_j'}{\partial x_k \partial x_k}}}_{6}.$$

The different terms denote the following physical processes:

1. unsteady term

2. convection

3. production

4. turbulent diffusion

5. redistribution of energy through the pressure

6. molecular diffusion and dissipation

When the time-averaged flow is assumed to be stationary, term 1 vanishes:

$$\frac{\partial}{\partial t}\overline{u_i'u_j'} = 0.$$ (14)

Using the continuity equation, term 4 can be rewritten as

$$\frac{\partial}{\partial x_k}\overline{u_i'u_j'u_k'} - \overline{u_i'u_j'\frac{\partial u_k'}{\partial x_k}} = \frac{\partial}{\partial x_k}\overline{u_i'u_j'u_k'}.$$ (15)

The quantity $\overline{u_i'u_j'u_k'}$ is a so-called triple correlation. Using the new expressions for the terms 1 and 4, equation (13) leads to the following equation for the Reynolds stress $\overline{u_i'u_j'}$

$$u_k\frac{\partial}{\partial x_k}\overline{u_i'u_j'} = d_{ij} + P_{ij} + \Phi_{ij} - \epsilon_{ij},$$ (16)

where d_{ij} is the diffusion of the Reynolds stresses, P_{ij} is the production of energy in the Reynolds stresses due to velocity gradients, Φ_{ij} is the pressure-strain correlation causing a redistribution of energy among the different components of the Reynolds-stress tensor, and ϵ_{ij} is the dissipation rate of the energy in the Reynolds stresses. The different components are defined as

$$d_{ij} = \frac{\partial}{\partial x_k}\left(\nu\frac{\partial \overline{u_i'u_j'}}{\partial x_k} - \overline{u_i'u_j'u_k'} - \frac{1}{\rho}\overline{p'u_i'}\delta_{jk} - \frac{1}{\rho}\overline{p'u_j'}\delta_{ik}\right),$$

$$P_{ij} = -\left(\overline{u_i'u_k'}\frac{\partial u_j}{\partial x_k} + \overline{u_j'u_k'}\frac{\partial u_i}{\partial x_k}\right),$$

$$\Phi_{ij} = \frac{1}{\rho}\overline{p'\left(\frac{\partial u_j'}{\partial x_i} + \frac{\partial u_i'}{\partial x_j}\right)},$$

$$\epsilon_{ij} = 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}.$$

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This shows that the total diffusion d_{ij} can be split up in molecular diffusion $\dfrac{\partial}{\partial x_k} \nu \dfrac{\partial \overline{u_i' u_j'}}{\partial x_k}$, turbulent diffusion $-\dfrac{\partial}{\partial x_k} \overline{u_i' u_j' u_k'}$, and pressure diffusion $-\dfrac{1}{\rho}(\overline{p' u_i'} \delta_{jk} + \overline{p' u_j'} \delta_{ik})$.

The equation for the turbulent kinetic energy k $(= \tfrac{1}{2} \overline{u_i' u_i'} = \tfrac{1}{2}(\overline{u'^2 + v'^2 + w'^2}))$ can be found by taking $j = i$ in (16). Using again the Einstein convention (that is taking the sum of the contributions due to $i = 1, 2, 3$), and dividing the result by 2, gives

$$u_k \frac{\partial k}{\partial x_k} = d + P_k - \epsilon, \tag{17}$$

with

$$d = \frac{\partial}{\partial x_k} \left(\nu \frac{\partial k}{\partial x_k} - \frac{1}{2} \overline{u_i' u_i' u_k'} - \frac{1}{\rho} \overline{p' u_k'} \right),$$

$$P_k = -\overline{u_i' u_k'} \frac{\partial u_i}{\partial x_k},$$

$$\epsilon = \nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}}.$$

Note that the contribution of Φ_{ij} vanishes, as these terms serve to redistribute the energy among the different components $(\Phi_{ii} = \dfrac{2}{\rho} \overline{p' \dfrac{\partial u_i'}{\partial x_i}} = 0$, see eq. (9)).

If a flow is considered that is two-dimensional after time averaging, we have

$$u_3 = 0, \quad \overline{u_1' u_3'} = 0, \quad \overline{u_2' u_3'} = 0. \tag{18}$$

Furthermore, all derivatives with respect to x_3 are zero for all time-averaged quantities. Under the assumption of two-dimensionality, equation (17) reduces to

$$u_1 \frac{\partial k}{\partial x_1} + u_2 \frac{\partial k}{\partial x_2} = d + P_k - \epsilon, \tag{19}$$

with

$$d = \nu \frac{\partial^2 k}{\partial x_1^2} + \nu \frac{\partial^2 k}{\partial x_2^2} - \frac{\partial}{\partial x_1} \left(\frac{1}{2} \overline{u_1'(u_1'^2 + u_2'^2 + u_3'^2)} \right) -$$

$$\frac{\partial}{\partial x_2} \left(\frac{1}{2} \overline{u_2'(u_1'^2 + u_2'^2 + u_3'^2)} \right) - \frac{1}{\rho} \frac{\partial}{\partial x_1} \overline{p' u_1'} - \frac{1}{\rho} \frac{\partial}{\partial x_2} \overline{p' u_2'}$$

$$P_k = -\overline{u_1'^2} \frac{\partial u_1}{\partial x_1} - \overline{u_1' u_2'} \frac{\partial u_1}{\partial x_2} - \overline{u_1' u_2'} \frac{\partial u_2}{\partial x_1} - \overline{u_2'^2} \frac{\partial u_2}{\partial x_2},$$

$$\epsilon/\nu = \overline{\left(\frac{\partial u_1'}{\partial x_1}\right)^2} + \overline{\left(\frac{\partial u_1'}{\partial x_2}\right)^2} + \overline{\left(\frac{\partial u_1'}{\partial x_3}\right)^2} + \overline{\left(\frac{\partial u_2'}{\partial x_1}\right)^2} + \overline{\left(\frac{\partial u_2'}{\partial x_2}\right)^2} + \overline{\left(\frac{\partial u_2'}{\partial x_3}\right)^2} +$$
$$\overline{\left(\frac{\partial u_3'}{\partial x_1}\right)^2} + \overline{\left(\frac{\partial u_3'}{\partial x_2}\right)^2} + \overline{\left(\frac{\partial u_3'}{\partial x_3}\right)^2}.$$

If the boundary-layer simplifications are made, which are

$$\frac{\partial}{\partial x_2} \gg \frac{\partial}{\partial x_1}, \quad u_2 \ll u_1,$$

equation (19) reduces to

$$u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} = d + P_k - \epsilon, \tag{20}$$

with

$$d = \frac{\partial}{\partial y}\left(\nu\frac{\partial k}{\partial y} - \frac{1}{\rho}\overline{p'v'} - \frac{1}{2}\overline{v'(u'^2 + v'^2 + w'^2)}\right),$$

$$P_k = -\overline{u'v'}\frac{\partial u}{\partial y}.$$

(using the substitution $u_1 = u$, $u_2 = v$, $x_1 = x$, $x_2 = y$, *et cetera*). This is the boundary-layer equation for the turbulent kinetic energy. It is a convection-diffusion equation, with a source and sink term:

- the turbulent kinetic energy is convected by the velocity vector (u, v), through the advection term $u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y}$;

- the turbulent kinetic energy is diffused by the term d, which consists of molecular diffusion, pressure diffusion, and turbulence diffusion;

- the turbulent kinetic energy is generated by the source term P, which extracts energy from the mean gradient velocity field;

- the turbulent kinetic energy is dissipated into heat by the sink term ϵ, which is the so-called turbulent dissipation rate.

3. Classification of turbulence models

In the past decades a variety of turbulence models has been developed. The most simple models are computationally very cheap, but can only be applied to those flow types for which they were originally developed. More complex models require more computational effort, but they have the advantage that they are applicable to a wider class of turbulent flows, because they incorporate more of the fundamental flow physics.

A possible classification of existing turbulence models is given below:

- integral methods

- Eddy-Viscosity Models (EVM);

  - algebraic models
  - half-equation models
  - one-equation models
  - two-equation models

- Differential Reynolds-Stress Models (DRSM)

- Large-Eddy Simulation (LES)

- Direct Numerical Simulation (DNS)

The most simple models are provided by integral methods. These models do not explicitly consider the variations in turbulence across the boundary-layer thickness, but instead they only make use of integral quantities, represented by shape factors, at each streamwise boundary-layer station. In this way partial differential equations (dependent on x and y) for the flow and turbulence are replaced by ordinary differential equations for the integral parameters. If suitable initial conditions are known at the first station x_o, these ordinary differential equations (depending on x) can easily be integrated with the help of a numerical integration scheme.

Within the integral methods the precise modelling of turbulence is not very clear. The methods heavily rely on empirical relations between the integral parameters. Due to their simplicity, integral methods are still very much used in aerodynamic design codes. In this way, a complete airfoil polar can be computed very quickly on a simple computer.

Eddy-viscosity models relate the Reynolds-stresses $-\overline{u_i' u_j'}$ to the time-averaged gradient velocity field by introducing an eddy viscosity (or: turbulent viscosity) ν_t

$$-\overline{u_i' u_j'} = \nu_t \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij}. \tag{21}$$

The last term $-\frac{2}{3} k \delta_{ij}$ has to be introduced to obtain a consistent expression when the trace of the Reynolds-stress tensor is taken. Due to the continuity equation, taking $j = i$ in (21) now leads to the correct expression

$$-\overline{u_i' u_i'} = -2k. \tag{22}$$

If boundary-layer simplifications are made, equation (21) reduces to

$$-\overline{u'v'} = \nu_t \frac{\partial u}{\partial y}, \quad \text{and} \quad \overline{u'^2} = \overline{v'^2} = \overline{w'^2} = \frac{2}{3} k. \tag{23}$$

Within the eddy-viscosity concept it is assumed that there is an analogy between turbulent transport and molecular transport. It is emphasized, however, that in

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contrast to the molecular viscosity ν, the turbulent viscosity ν_t is *not* a fluid property. Therefore ν_t can vary strongly through the flow.

It should also be noted that as long as $\nu_t(x,y)$ in equation (21) is not further specified yet, no modelling for the Reynolds stresses has been made (except for the implicit assumption that ν_t is a scalar and not a tensor). Of course, the turbulent viscosity has been introduced having in mind that the modelling of the turbulent viscosity is easier than the direct modelling of the Reynolds stress itself.

Considering the dimensions of ν_t shows that

$$\nu_t = f(x,y) \times V \times L, \tag{24}$$

where V is a characteristic turbulent velocity scale, and L is a characteristic turbulent length scale. The function f is dimensionless. These quantities can be modelled in different ways.

Algebraic turbulence models introduce algebraic expressions for V and L, whereas f is taken as a constant. A well-known example of an algebraic model is the mixing length model, which has

$$f = 1, \quad L = l_m, \quad V = l_m \left| \frac{\partial u}{\partial y} \right|. \tag{25}$$

For boundary layers the mixing length is taken as $l_m = \kappa y$ in the inner layer. Other well-known algebraic models are the Cebeci & Smith model and the Baldwin & Lomax model, which will be discussed in sections 4.1 and 4.2.

The Johnson & King model (see section 4.3) is sometimes referred to as a half-equation model. In addition to algebraic relations, this model solves an ordinary differential equation for a quantity related to the turbulent velocity scale. As the complexity of this equation is in between a partial differential equation and an algebraic equation this model is called a half-equation model.

One-equation models solve, besides algebraic expressions, only one partial differential equation for the turbulence. A well-known one-equation model is the $k - l$ model of Bradshaw *et al.* (see section 5), which solves a partial differential equation for the turbulent kinetic energy k $(=V^2)$ and an algebraic equation for the turbulent length scale l $(=L)$.

Two-equation models solve two partial differential equations for the turbulent quantities. The most well-known two-equation model is the $k - \epsilon$ model, which solves a differential equation for k and for ϵ. Analysing the dimensions gives that $V = k^{1/2}$ and $L = k^{3/2}/\epsilon$. Substituting these expressions into (24) gives

$$\nu_t = c_\mu \frac{k^2}{\epsilon}. \tag{26}$$

Thus the function $f(x,y)$ has been replaced by the constant c_μ (which commonly has the value 0.09).

Another two-equation model, which seems to have become quite popular for aeronautical applications, is the $k - \omega$ model. This model solves partial differential equations for the turbulent kinetic energy and for the specific dissipation rate ω, which is proportional to ϵ/k. Analysing the dimensions gives $V = k^{1/2}$ and $L = k^{1/2}/\omega$, which leads to

$$\nu_t = \alpha \frac{k}{\omega}. \tag{27}$$

Thus the function f has been replaced by the constant α. Other two-equation models replace the differential equation for ϵ or ω by an equation for the length scale ($k - l$ model) or for the time scale ($k - \tau$ model).

It is noted that all the mentioned two-equation models can be transformed into each other, because all models can actually be interpreted as representing the single velocity scale and the single length scale appearing in the the turbulent viscosity (24). For example the $k - \epsilon$ model can be transformed in a $k - \omega$ model by using $\epsilon = k\omega$. However, turbulence modellers argue, that some choices of the variables lead to simpler models than others.

The two-equation models have the following restrictions:

- linear relation between the Reynolds stresses and the gradient velocity field;

- scalar character of the turbulent viscosity;

- only one velocity scale and one length scale are considered;

- anisotropy is not explicitly taken into account (that is, $\overline{u'^2} = \overline{v'^2} = \overline{w'^2} = \frac{2}{3}k$ if boundary-layer simplifications are applied).

These restrictions can lead to an inaccurate prediction of flows that are more complex than just simple shear layers, such as:

- flows with abrupt changes in the velocity field;

- flow along strongly curved surfaces;

- stagnant flow regions;

- 3D flows;

- low-frequency unsteadiness.

An example of a complex flow is turbulent boundary-layer separation. The separation shows strong curvature and has strong velocity gradients. Such complex flow types can more accurately be predicted by differential Reynolds-stress models, which solve a partial differential equation for each component of the Reynolds-stress tensor. For example, to calculate a 3D flow 6 equations are solved (e.g. $\overline{u'^2}$, $\overline{v'^2}$, $\overline{w'^2}$, $\overline{u'v'}$, $\overline{v'w'}$, and $\overline{v'w'}$, respectively). In addition, the model also solves a differential equation for

the dissipation rate. The reason that differential Reynolds-stress models are most suited for complex flows is that the production of turbulence energy P does not have to be modelled. Therefore the production due to simple shear, curvature, 3D effects and rotation is represented exactly.

For flows that contain relatively large-scale spatial structures and low-frequency unsteadiness, like near wake flows, flow buffeting and dynamic stall, the use of a Large-Eddy Simulation seems to be a promising approach. In contrast to the earlier mentioned models, which all try to the model the time-averaged equations, the LES only models the small-scale structures, whereas the large-scale structures are calculated from an unsteady time integration. As the small-scale turbulence is almost isotropic, the so-called subgrid models in LES are expected to be not very crucial. LES was originally developed for meteorological applications, like the earth's atmospheric boundary layer, which typically consists of large-scale structures. More recent research on LES is also devoted to aeronautical boundary layers.

The most exact approach of turbulence is the Direct Numerical Simulation. The 3D unsteady Navier-Stokes equations are solved without applying a turbulence model. Very small numerical time steps and fine spatial grids have to be used to represent even the smallest turbulent scales (*i.e.* the Kolmogorov scales), where turbulent dissipation takes place. Therefore a typical DNS takes several hundreds of hours CPU time on a supercomputer. Due to the large computer time, DNS is restricted to simple configurations (like channel flow, pipe flow, Couette flow, and boundary layers) at somewhat low Reynolds numbers. We cannot expect that DNS can be used for predicting the flow around a complete aircraft, or even on a wing, within a few decades or so. For such flow types, the further development of turbulence models remains important. DNS for details of the turbulent flow, however, are important, as in addition to experiments, the results can be used to validate turbulence models.

4. Algebraic models

4.1. Cebeci & Smith model (1974)

Cebeci & Smith have formulated an algebraic model that for a long time has been one of the most popular turbulence models in aeronautics. The boundary layer is split up in an *inner layer* and an *outer layer*. In the inner layer, which covers the viscous sublayer, the buffer layer, and the inertial sublayer, the turbulent viscosity $\nu_{t,i}$ is modelled according to

$$\nu_{t,i} = l^2 \left| \frac{\partial u}{\partial y} \right| \gamma. \tag{28}$$

This actually is Prandtl's mixing length model, with

$$l = \kappa y [1 - \exp(-y^+/A^+)]. \tag{29}$$

Here $y^+ = \dfrac{y u_\tau}{\nu}$ (and $u_\tau = \sqrt{\tau_w/\rho}$, $\tau_w = \mu \left(\dfrac{\partial u}{\partial y}\right)_w$). Close to the wall, the Van Driest damping function is applied. The function $A^+ (= \dfrac{A u_\tau}{\nu})$ only depends on the magnitude of the streamwise pressure gradient according to

$$A^+ = \frac{26}{\sqrt{1 - 11.8\, p^+}}, \tag{30}$$

where p^+ is the dimensionless pressure gradient $\dfrac{\nu U}{u_\tau^3} \dfrac{dU}{dx}$ (here U is the velocity at the edge of the boundary layer).

In equation (28), γ is the intermittency factor, which accounts for the experimental finding that the turbulence becomes intermittent when the outer edge of the boundary layer is approached. Hence the boundary layer is turbulent only during the fraction γ of the time. Klebanoff has experimentally found that

$$\gamma = \left[1 + 5.5 \left(\frac{y}{\delta}\right)^6\right]^{-1}, \tag{31}$$

in which δ is the y-position where $\dfrac{u}{U} = 0.995$. In the inner layer this gives $\gamma \approx 1$.

In the outer layer the turbulent viscosity is modelled according to

$$\nu_{t,o} = \alpha\, U \delta^* \gamma. \tag{32}$$

This shows that in the outer layer the characteristic velocity is assumed to be proportional to the outer edge velocity U whereas the characteristic length scale is proportional to the displacement thickness δ^* $(= \displaystyle\int_0^\infty \left(1 - \dfrac{u}{U}\right) dy)$. The function α is only dependent on the local Reynolds number based on the momentum thickness θ $(= \displaystyle\int_0^\infty \dfrac{u}{U}\left(1 - \dfrac{u}{U}\right) dy)$, according to

$$\alpha = 0.0168 \frac{1.55}{1 + \Pi}, \tag{33}$$

with

$$\Pi = 0.55(1 - \exp(-0.243\sqrt{z_1} - 0.298 z_1)),$$

$$z_1 = \max\left[1, \frac{Re_\theta}{425} - 1\right].$$

The switch between the inner turbulent viscosity and the outer turbulent viscosity $\nu_{t,o}$ is at the y position where both values are equal, i.e. $\nu_{t,i} = \nu_{t,o}$.

4.2. Baldwin & Lomax model (1978)

A disadvantage of the Cebeci & Smith model is that it contains typical boundary-layer parameters, like δ, δ^* and θ. The computation of these parameters is straightforward when the boundary-layer equations are considered, but is less clear when the model is implemented in a general RANS code. The latter approach is often used for the computation of multi-element airfoils and/or for large scale separation regions occurring at airfoils at high incidence. Therefore, Baldwin & Lomax have modified the Cebeci & Smith model in such a way that the typical boundary-layer parameters are avoided. In the case that boundary layers are considered, however, the results with the Baldwin & Lomax model are almost similar to the results with the Cebeci & Smith model. The Baldwin & Lomax model can be applied both to boundary layers along walls and to wake flows, whereas the presented formulation of the Cebeci & Smith model is not applicable to wake flows.

Considering the Baldwin & Lomax model, the turbulent viscosity in the inner layer is the same as in the Cebeci & Smith, except for the interchange of $\dfrac{\partial u}{\partial y}$ by the modulus of the local vorticity ω:

$$\nu_{t,i} = l^2 |\omega|, \tag{34}$$

$$|\omega| = \sqrt{\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)^2}.$$

The turbulent viscosity in the outer layer is modelled as

$$\nu_{t,o} = \alpha\, C_1 F_{\text{wake}} \gamma. \tag{35}$$

Here $\alpha = 0.0168$ and $C_1 = 1.6$. The function F_{wake} is defined as

$$F_{\text{wake}} = \min\left[y_{\max} F_{\max},\, C_2 y_{\max} U_{\text{diff}}^2 / F_{\max}\right],$$

with $C_2 = 0.25$ and

$$F(y) = y|\omega|[1 - exp(-y^+/A^+)].$$

The exponential term is omitted if wake flows are considered. The quantity $F_{\max}$ denotes the maximum value of F, and $y_{\max}$ is the value of y at which the maximum occurs. Further, U_{diff} is the difference between the maximum and minimum total velocity $(\sqrt{u^2 + v^2 + w^2})$ in the profile at a fixed x station. The minimum total velocity is zero for the boundary layer along a fixed wall, but can be nonzero if the model is applied to wake flows. The intermittency is modelled as

$$\gamma = \left[1 + 5.5\left(\frac{C_3 y}{y_{\max}}\right)^6\right]^{-1}, \tag{36}$$

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with $C_3 = 0.3$.

The switch between the inner turbulent viscosity and the outer viscosity takes place at the y position where both turbulent viscosities are equal. Comparing the Baldwin & Lomax model with the Cebeci & Smith model, for the case that boundary layer simplifications are applied, shows that the main difference concerns the treatment of the velocity and length scale in the outer layer.

The constants in the Baldwin & Lomax model have been tuned for giving good predictions for zero-pressure gradient boundary layers at transonic speeds. In the past years the Baldwin & Lomax model has quite often been used to compute airfoils at operational flow conditions.

4.3. Johnson & King model (1984)

The algebraic models of Cebeci & Smith and of Baldwin & Lomax turned out to work reasonably well for attached boundary layers under weakly favourable or adverse pressure gradients. For boundary layers in a strong adverse pressure gradients, however, a too early separation was predicted. The reason for this seems to be that algebraic models are fully determined by local processes, whereas the gradual streamwise evolution of turbulence is not explicitly included in the model. As a result of this, the algebraic model reacts too strongly on changes in the outer-edge velocity, leading to a too early separation. To account more properly for the so-called *memory* or *history* effects of turbulence, a more differential-type of modelling should be included. Therefore Johnson & King modified existing algebraic models by the inclusion of an ordinary differential equation for the maximum turbulent shear stress.

For the whole boundary layer the Johnson & King model evaluates the turbulent viscosity according to

$$\nu_t = \nu_{t,o} \left[1 - exp(-\nu_{t,i}/\nu_{t,o}) \right]. \tag{37}$$

This actually is a blending function between $\nu_{t,i}$, meant to model the inner layer, and $\nu_{t,o}$, meant to model the outer layer.

The turbulent viscosity for the inner layer is modelled as

$$\nu_{t,i} = D^2 \kappa y \sqrt{(-\overline{u'v'})_{\max}}, \tag{38}$$

in which Van Driest's damping function is used, $D = 1 - exp(-y^+/A^+)$ (here $A^+ = 15$ is used). Equation (38) shows that the maximum of the turbulent shear stress is used to model the characteristic turbulent velocity scale in the inner layer. Due to the the logarithmic wall function, for zero pressure gradient boundary layers the turbulent shear stress is constant in the inertial sublayer, implying that there $(-\overline{u'v'})_{\max}$ in the model can be replaced by $-\overline{u'v'}$ as well. The latter expression is used in the algebraic models of Cebeci & Smith and Baldwin & Lomax. Indeed later on it will become clear that the Johnson & King model gives the same result for the inner layer in zero pressure gradient flows, *i.e.* $\sqrt{(-\overline{u'v'})_{\max}} = \kappa y |\partial u/\partial y|$.

The turbulent viscosity in the outer layer is modelled as

$$\nu_{t,o} = \alpha U \delta^* \gamma \beta(x), \tag{39}$$

with $\alpha = 0.0168$ and γ is Klebanoff's intermittency function, $\gamma = [1 + 5.5(y/\delta)^6]^{-1}$. The value of the x-dependent function β will follow from an ordinary differential equation for $(-\overline{u'v'})_{\max}$. Actually, the function β will be chosen such that the following relationship holds:

$$\nu_{t,m} = \frac{(-\overline{u'v'})_{\max}}{\left(\dfrac{\partial u}{\partial y}\right)_m}. \tag{40}$$

Here ν_t is given by equation (37). The subscript m denotes that the quantity is evaluated at the position where $-\overline{u'v'}$ has its maximum.

To complete the model an equation for the turbulent viscosity scale $V_m = \sqrt{(-\overline{u'v'})_{\max}}$ is needed. Thereto the differential equation for the turbulent kinetic energy (20), as derived in section 2, is applied along the path s of maximum turbulent shear stress:

$$u_m \frac{\partial k_m}{\partial x} = d_m + P_m - \epsilon_m, \tag{41}$$

with

$$d_m = -\left(\frac{\partial}{\partial y}\left(\frac{1}{\rho}\overline{p'v'} + \frac{1}{2}\overline{q^2 v'}\right)\right)_m,$$

$$P_m = (-\overline{u'v'})_{\max}\left(\frac{\partial u}{\partial y}\right)_m.$$

Because s is approximately the same as x, the latter coordinate has been used in (41). The characteristic turbulent length scale is modelled as $L_m = V_m^{3/2}/\epsilon_m$, with $V_m = (-\overline{u'v'})_{\max}^{1/2}$. Furthermore the experimental finding is applied that the turbulent shear stress is almost proportional to the turbulent kinetic energy, i.e. $(-\overline{u'v'})_{\max}/k_m = a_1 = $ constant (here a_1 is set to 0.25). Substitution of these expressions into (41) gives

$$V_m + \frac{L_m u_m}{a_1 V_m^2} \frac{dV_m^2}{dx} = L_m \left(\frac{\partial u}{\partial y}\right)_m - \frac{L_m}{V_m^2} D_m^*. \tag{42}$$

The turbulent length scale is modelled as $L_m = 0.4 y_m$, if $y_m/\delta \leq 0.225$, and as $L_m = 0.09\delta$ if $y_m/\delta > 0.225$. The diffusion term in (42), as denoted by D_m^*, also needs further modelling. If the turbulence in the boundary layer is in 'local equilibrium', as is the case for the zero pressure gradient boundary layer, the diffusion and convection are negligible in (42), and the production of turbulence is balanced by dissipation.

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Thus, for the equilibrium case (denoted by the subscript 'eq'), the Johnson & King model reduces to

$$\nu_{t,\mathrm{eq}} = \nu_{t,o} \left[1 - \exp(-\nu_{t,i,\mathrm{eq}}/\nu_{t,o,\mathrm{eq}})\right],$$

$$\nu_{t,i,\mathrm{eq}} = D^2 \kappa y \sqrt{(-\overline{u'v'})_{\max}}, \qquad (43)$$

$$\nu_{t,o,\mathrm{eq}} = 0.0168 U \delta^* \gamma.$$

The diffusion is modelled as

$$D_m^* = \frac{C_{\mathrm{dif}} V_m^3}{a_1 \delta [0.7 - (y/\delta)_m]} \left|1 - \left(\frac{\nu_{t,o}}{\nu_{t,o,\mathrm{eq}}}\right)^{1/2}\right|, \qquad (44)$$

with $C_{\mathrm{dif}} = 0.50$. The diffusion vanishes for the equilibrium case.

5. One-equation models

A one-equation model that has frequently been used for external aerodynamics is the model due to Bradshaw *et al.* (1967), this model solves besides the continuity equation and the momentum equations, also a differential equation for the turbulent kinetic energy, as derived in section 2. This gives, under boundary-layer simplifications

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad (45)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} - \frac{\partial}{\partial y}\overline{u'v'}, \qquad (46)$$

$$u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} = d + P - \epsilon, \qquad (47)$$

with

$$d = -\frac{\partial}{\partial y}\left(\frac{1}{\rho}\overline{p'v'} + \frac{1}{2}\overline{v'(u'^2 + v'^2 + w'^2)}\right),$$

$$P_k = -\overline{u'v'}\frac{\partial u}{\partial y}.$$

In this formulation only high-Reynolds-number turbulence is considered, implying that contributions in which the molecular viscosity ν appear have been neglected. As a consequence, this formulation is not valid very close to the wall, and wall functions have to be used as boundary conditions.

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The Reynolds-stress $\tau = -\rho\overline{u'v'}$ is rewritten as the kinematic Reynolds shear stress according to

$$\hat{\tau} = \frac{\tau}{\rho} = -\overline{u'v'}. \tag{48}$$

Comparison of dimensions shows that the dissipation rate of turbulent kinetic energy can be written as

$$\epsilon = \frac{\hat{\tau}^{3/2}}{L}, \tag{49}$$

where L is a characteristic turbulent length scale. In the one-equation model of Bradshaw *et al.*, the length scale L is modelled by an algebraic equation. In the special case that the production of kinetic energy equals its dissipation, *i.e.* $P_k = \epsilon$, it follows that

$$-\overline{u'v'}\frac{\partial u}{\partial y} = \hat{\tau}\frac{\partial u}{\partial y} = \epsilon = \frac{\hat{\tau}^{3/2}}{L}, \tag{50}$$

which gives

$$\hat{\tau} = L^2 \left(\frac{\partial u}{\partial y}\right)^2. \tag{51}$$

Hence for this specific case the turbulent length scale L is equal to the mixing length l.

In the model the turbulent diffusion is assumed to be determined by the large eddies, which cover the full boundary-layer thickness. The characteristic velocity fluctuations are determined by the local Reynolds-shear stress $\hat{\tau}$. The diffusion is characterized by typical gradients in $\hat{\tau}$, which can be represented by $\hat{\tau}_{\max}$, defined as the maximum of $\hat{\tau}$ for $0.25 \leq y/\delta \leq 1$. In this way the model assumes that $\overline{q^2 v'}$ is proportional to $\hat{\tau}\sqrt{\hat{\tau}_{\max}}$, with $q^2 = \overline{u'^2} + \overline{v'^2} + \overline{w'^2}$. The same approximation is made for the pressure diffusion. This all ends up in the following modeled terms

$$\frac{\overline{p'v'} + \frac{1}{2}\overline{q^2 v'}}{\hat{\tau}\sqrt{\hat{\tau}_{\max}}} = G, \tag{52}$$

$$\frac{\hat{\tau}}{\overline{q^2}} = a_1. \tag{53}$$

In this model L, G and a_1 are determined through calibration with experiments; $L/\delta = f_1(y/\delta)$ and $G = \sqrt{\hat{\tau}_{\max}/U^2}f_2(y/\delta)$ (where U is the local outer-edge velocity and δ is the local y where $u/U = 0.995$). The functions f_1 and f_2 are assumed to have universal shapes. The structure parameter is taken as $a_1 = 0.15$.

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Substitution of the modeled terms into (45) to (47) gives the following one-equation model:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{54}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial \hat{\tau}}{dy}, \tag{55}$$

$$u\frac{\partial}{\partial x}\left(\frac{\hat{\tau}}{2a_1}\right) + v\frac{\partial}{\partial y}\left(\frac{\hat{\tau}}{2a_1}\right) = d + P - \epsilon, \tag{56}$$

with

$$d = -\frac{\partial}{\partial y}\left(G\hat{\tau}\sqrt{\hat{\tau}_{\max}}\right),$$

$$P = \hat{\tau}\frac{\partial u}{\partial y},$$

$$\epsilon = \frac{\hat{\tau}^{3/2}}{L}.$$

6. Two-equation models

6.1. $k - \epsilon$ models

In the $k-\epsilon$ model, two differential equations are solved for the turbulent kinetic energy (k) and the dissipation rate of turbulent kinetic energy (ϵ), respectively. One of the first important publications on the $k - \epsilon$ model was due to Jones & Launder (1972). Through the years the model has become very popular for internal engineering flows, like flows in pipe systems. The application of $k - \epsilon$ models to external aerodynamics, such as boundary layers with streamwise pressure gradient, has only recently gained more interest.

Starting point of the modelling are the exact equations for the turbulent kinetic energy and the dissipation rate of turbulent kinetic energy, both of which can be derived from the Navier-Stokes equations; in section 2 only the derivation of the exact equation for k has been given. However, the equation for ϵ is much more complicated than the equation for k. As a consequence the modelled k equation shows a much closer link with the exact equation than the modelled ϵ equation.

Applying boundary-layer simplifications, the continuity equation and the differential equations for u, k and ϵ read

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{57}$$

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$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{\partial}{\partial y}\left[(\nu + \nu_t)\frac{\partial u}{\partial y}\right],$$ (58)

$$u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] + P_k - \epsilon + D,$$ (59)

$$u\frac{\partial \epsilon}{\partial x} + v\frac{\partial \epsilon}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon}\right)\frac{\partial \epsilon}{\partial y}\right] + (c_{\epsilon 1} f_1 P_k - c_{\epsilon 2} f_2 \epsilon)\frac{\epsilon}{k} + E,$$ (60)

with

$$P_k = \nu_t\left(\frac{\partial u}{\partial y}\right)^2, \quad \nu_t = c_\mu f_\mu \frac{k^2}{\epsilon}.$$

The model contains so-called *high-Reynolds-number* constants (namely c_μ, σ_k, σ_ϵ, $c_{\epsilon 1}$ and $c_{\epsilon 2}$), and *low-Reynolds-number* functions (namely f_μ, f_1, f_2, D and E). Here the relevant Reynolds number is based on the local turbulence quantities; possible definitions are $Re_t = k^2/\nu\epsilon$, or $Re_k = y\sqrt{k}/\nu$.

First the high-Reynolds-number $k - \epsilon$ model will be discussed. The high-Re model is found from (59) and (60) by setting the low-Re functions to $D = E = 0$, and $f_\mu = f_1 = f_2 = 1$. The high-Re model can be applied in those regions of the flow were the turbulence-based Reynolds number is sufficiently high. For example, high-Re turbulence is found in the part of the boundary layer that is both sufficiently far downstream of the transition region and not too close to the fixed wall (*i.e.* in the inertial sublayer or in the defect layer).

Some of the high-Re constants can be derived with the help of the wall functions for k and ϵ. In the inertial sublayer the production of turbulent kinetic energy is assumed to be balanced by its dissipation. This means that both convection and diffusion can be neglected in (59), giving $P_k = \epsilon$, or

$$\nu_t\left(\frac{\partial u}{\partial y}\right)^2 = \epsilon.$$ (61)

In the inertial sublayer the convection can also be neglected in the momentum equation (58), which simplifies to

$$\frac{\partial}{\partial y}(\nu + \nu_t)\frac{\partial u}{\partial y} = 0.$$ (62)

This can directly be integrated to

$$(\nu + \nu_t)\frac{\partial u}{\partial y} = u_\tau^2,$$ (63)

with $u_\tau = \sqrt{\tau_w/\rho}$. Because $\nu_t >> \nu$ in the inertial sublayer, (62) reduces to

$$\nu_t\frac{\partial u}{\partial y} = u_\tau^2.$$ (64)

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Differentiation of the logarithmic wall function for the velocity ($u^+ = \frac{1}{\kappa}\ln y^+ + C$, with $u^+ = u/u_\tau$) gives

$$\frac{\partial u}{\partial y} = \frac{1}{\kappa}\frac{u_\tau^2}{\nu}\frac{1}{y^+}. \tag{65}$$

From equations (61), (64) and (65) it follows that

$$\epsilon = \frac{u_\tau^3}{\kappa y} \tag{66}$$

Non-dimensionalization with ν and u_τ gives the dissipation rate in wall units (as denoted with the $+$ subscript):

$$\epsilon^+ = \frac{1}{\kappa y^+}, \tag{67}$$

with $\epsilon^+ = \epsilon\nu/u_\tau^4$. Equation (67) is the wall function for the turbulent dissipation rate, and is only valid in the inertial sublayer. In the viscous sublayer the wall function gives $\epsilon^+ \to \infty$, which supports that the expression has lost its validity.

Using $\nu_t = c_\mu k^2/\epsilon$, also wall functions for the other turbulent quantities can be derived, giving

$$k^+ = \frac{1}{\sqrt{c_\mu}}, \tag{68}$$

$$\nu_t^+ = \kappa y^+, \tag{69}$$

$$-\overline{u'v'}^+ = 1, \tag{70}$$

with $k^+ = k/u_\tau^2$, $\nu_t^+ = \nu_t/\nu$, $-\overline{u'v'}^+ = -\overline{u'v'}/u_\tau^2$.

The here derived wall functions can be used to provide boundary conditions for the differential equations for k and ϵ. This means that equations (67) and (68) are used as Dirichlet boundary conditions at the first inner computational grid point from the wall. One has to take care that the first inner grid point is chosen sufficiently far away from the wall for the wall functions to hold. As a rule $y_1^+ > 11.5$ is used. An advantage of using wall functions is that the application of many computational grid points for solving the steep gradients in the viscous sublayer and the inertial sublayer can be avoided. A disadvantage is that the predictions, for example for the wall-shear stress, cannot be expected to be accurate in cases where the validity of the wall functions is questionable, such as for boundary layers with separation.

Applying the wall functions for k and $-\overline{u'v'}$ gives

$$\frac{-\overline{u'v'}}{k} = \sqrt{c_\mu}. \tag{71}$$

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Indeed experiments have shown that the turbulent shear stress is proportional to the turbulent kinetic energy in part of the boundary layer, and that the proportionality constant (often denoted as the *structural parameter*) is equal to about 0.3. This constant is used to fix the value of c_μ at 0.09.

The value for the model constant $c_{\epsilon 2}$ can be found by considering the behaviour of the $k - \epsilon$ equations close to the outer edge of the boundary layer. Due to the asymptotic character of the boundary layer equations, all normal gradients must vanish at the outer edge. This defines the proper boundary conditions for k and ϵ at the outer edge, namely the homogeneous boundary conditions $\dfrac{\partial k}{\partial y} = 0$, $\dfrac{\partial \epsilon}{\partial y} = 0$.

Therefore the boundary-layer equations (45) and (46) at the outer edge simplify to

$$U_e \frac{\partial k_e}{\partial x} = -\epsilon_e, \tag{72}$$

$$U_e \frac{\partial \epsilon_e}{\partial x} = -c_{\epsilon 2} \frac{\epsilon_e^2}{k_e}. \tag{73}$$

Here the subscript e is used to denote values at the outer edge. This system consists of two ordinary differential equations for k and ϵ. At some initial station x_0, the value of k and ϵ at the outer edge have to be specified, denoted as $k_{e,0}$ and $\epsilon_{e,0}$. For a constant free stream velocity, an exact solution for k exists:

$$k_e = \left[\frac{(c_{\epsilon 2} - 1)\epsilon_{e,0}}{k_{e,0}^{c_{\epsilon 2}} U_e} (x - x_0) + k_{e,0}^{1-c_{\epsilon 2}} \right]^{\frac{-1}{c_{\epsilon 2}-1}}. \tag{74}$$

For large x, this expression shows that k should decay as x^{-n}, with $n = \frac{1}{c_{\epsilon 2}-1}$. Experiments for the decay of isotropic grid turbulence give $n \approx 1.25$, which fixes the model constant c_{ϵ_2} at 1.8. Most $k - \epsilon$ models use a slightly larger value, namely $c_{\epsilon 2} = 1.92$.

The constants σ_k and σ_ϵ are the so-called turbulent Prandtl numbers for the kinetic energy and the turbulent energy dissipation rate, respectively. The turbulent Prandtl number denotes that the turbulent diffusion of k and ϵ is not necessarily the same as the turbulent diffusion of streamwise momentum. Most $k - \epsilon$ models apply $\sigma_k = 1$ and $\sigma_\epsilon = 1.3$. It is noted that the term *turbulent* Prandtl number is introduced in analogy with the (molecular) Prandtl number, which is defined as the ratio between the diffusion and thermal diffusivity.

Increasing computational sources have given further impetus to abandon the wall functions, and to replace the high-Re $k - \epsilon$ model by an appropriate low-Re $k - \epsilon$ model. Now the boundary layer is computed whole the way up to the wall, which requires a very fine computational grid in the viscous sublayer and in the buffer layer. Through the years a variety of formulations of these low-Re functions has been proposed. As a guideline in the derivation of these functions, Taylor expansions close to the wall should be used; an overview is given by Patel *et al.* (1985). The main consideration are as follows:

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- *limit for small y.* Close to the wall the velocity perturbation can be expanded according to

$$
\begin{aligned}
u' &= a_1 y + b_1 y^2 + ..., \\
v' &= \qquad\quad b_2 y^2 + ..., \\
w' &= a_3 y + b_3 y^2 + ...
\end{aligned}
\tag{75}
$$

With these expansions the quantities $\overline{u'v'}$, $k = \frac{1}{2}\overline{u'_i u'_i}$, $\epsilon = \nu\,\overline{\dfrac{\partial u'_i}{\partial x_j}\dfrac{\partial u'_i}{\partial x_j}}$, and $\nu_t = -\overline{u'v'}/\dfrac{\partial u}{\partial y}$ become

$$
\begin{aligned}
\overline{u'v'} &= \overline{a_1 b_2}\, y^3 + ... , \\
k &= A y^2 + ... , \\
\epsilon &= 2A\nu + ... , \\
\nu_t &= -\frac{\overline{a_1 b_2}\, y^3}{\left(\dfrac{\partial u}{\partial y}\right)_{\text{wall}}} + ... ,
\end{aligned}
\tag{76}
$$

with $A = \overline{a_1^2 + a_3^2}$.

- *D function and boundary condition for k and ϵ.* From the series expansions (76), the proper boundary conditions for k and ϵ at the wall are found to be:

$k = 0$ and $\epsilon = 2A\nu$, with $A = \lim\limits_{y\to 0} k/y^2$. Here A can be determined as $\left(\dfrac{\partial \sqrt{k}}{\partial y}\right)_w^2$ or as $\dfrac{1}{2}\left(\dfrac{\partial^2 k}{\partial y^2}\right)_w$.

All developed low-Re $k - \epsilon$ indeed apply the zero boundary condition for k. However, not all models apply the nonzero value for ϵ; instead such models apply $\epsilon = 0$ at the wall, under the introduction of an additional term D in the k-equation; then the effective dissipation rate is $\epsilon - D$. The limiting behaviour of D close to the wall can be found by substituting the series expansion (76) in the k equation, which gives

$$
\nu\frac{\partial^2 k}{\partial y^2} - \epsilon + D = O(y^3),
\tag{77}
$$

with $\dfrac{\partial^2 k}{\partial y^2} = 2A + ...$ With $\epsilon = 0$ at the wall, (77) shows that $\lim\limits_{y\to 0} D = 2A$.

Possible choices for D are $D = -2\nu\dfrac{k}{y^2}$ and $D = -2\nu\left(\dfrac{\partial \sqrt{k}}{\partial y}\right)^2$.

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- f_2 *and E functions.* Close to the wall the ϵ equation (60) reduces to

$$\nu\frac{\partial^2\epsilon}{\partial y^2} - c_{\epsilon 2}f_2\frac{\epsilon^2}{k} + E = O(y).\tag{78}$$

The choice for f_2 and E should be such that $\dfrac{\partial^2\epsilon}{\partial y^2} = O(1)$ for small y. For example, if $E = 0$ and the nonzero boundary condition for ϵ is prescribed (implying that ϵ is $O(1)$ for small y), it follows that f_2 must be chosen such that it decays as y^2 close to the wall. Some models introduce a nonzero E term, but its physical meaning is not very clear.

Most models choose f_2 such that the decay of isotropic grid turbulence is modelled in agreement with experiments. As already mentioned in the previous description of high-Re models, experiments show that k decays as x^{-n}, with $n \approx 1.25$. However if the turbulence has decayed to sufficiently small levels, the decay rate n increases to about 2.5. Thus the limiting behaviour of f_2 follows from

$$f_2 = \frac{n+1}{n}\frac{1}{c_{\epsilon 2}},\tag{79}$$

with $n \approx 1.25$ for $Re_t \to \infty$ (which means that $f_2 = 1$, using the common value for $c_{\epsilon 2}$), and $n \approx 2.5$ for $Re_t \to 0$ (which means that $f_2 = 1.4/c_{\epsilon 2}$).

- f_μ *function.* This function should be chosen such that the limiting behaviour $\nu_t = O(y^3)$ in eq. (76) is reproduced. Indeed most low-Re models give a power of 3 or 4 in ν_t.

Among existing low-Re models, the model of Launder & Sharma (1974) seems to be superior for a large number of test cases. This model takes $k = 0$ and $\epsilon = 0$ as boundary conditions at the wall, and it determines the low-Re functions as:

$$
\begin{aligned}
f_\mu &= \exp\left(\frac{-3.4}{(1 + Re_t/50)^2}\right),\\
f_1 &= 1.0,\\
f_2 &= 1 - 0.3\,\exp(-Re_t^2),\\
D &= -2\nu\left(\frac{\partial\sqrt{k}}{\partial y}\right)^2,\\
E &= 2\nu\nu_t\left(\frac{\partial^2 u}{\partial y^2}\right)^2.
\end{aligned}\tag{80}
$$

The high-Re constants have the common values: $c_\mu = 0.09$, $c_{\epsilon 1} = 1.44$, $c_{\epsilon 2} = 1.92$, $\sigma_k = 1.0$, and $\sigma_\epsilon = 1.3$.

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6.2. $k - \omega$ models

In this type of models the equation for the dissipation rate in the $k - \epsilon$ model is replaced by an equation for the reciprocal turbulent time scale ω. Considered is the low-Reynolds-number $k - \omega$ model of Wilcox (1993):

$$u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] + P_k - \epsilon, \tag{81}$$

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial y}\right] + \alpha\frac{\omega}{k}P_k - \beta\omega^2, \tag{82}$$

with

$$P_k = \nu_t\left(\frac{\partial u}{\partial y}\right)^2, \quad \nu_t = \alpha^*\frac{k}{\omega}, \quad \epsilon = \beta^* k\omega.$$

The low-Reynolds-number functions are defined by

$$\alpha^* = \frac{\alpha_0^* + Re_t/Re_k}{1 + Re_t/Re_k}, \quad \alpha_0^* = \beta/3, \quad Re_k = 6,$$

$$\alpha = \frac{5}{9}\frac{\alpha_0 + Re_t/Re_\omega}{1 + Re_t/Re_\omega}\frac{1}{\alpha^*}, \quad \alpha_0 = 0.1, \quad Re_\omega = 2.7,$$

$$\beta^* = 0.09\frac{5/18 + (Re_t/Re_\beta)^4}{1 + (Re_t/Re_\beta)^4}, \quad Re_\beta = 8.$$

The other coefficients in the model are $\beta = 3/40$, $\sigma_k = \sigma_\omega = 2$. The turbulence-based Reynolds number Re_t is defined here as $\dfrac{k}{\omega\nu}$. The wall boundary conditions are set to: $k = 0$, $\omega = \lim\limits_{y\to 0}\dfrac{6\nu}{\beta y^2}$.

This low-Reynolds-number model can be transformed into a high-Reynolds-number model by taking the limit $Re_t \to \infty$, which gives $\alpha^* = 1$, $\alpha = 5/9$, $\beta^* = 0.09$.

7. Differential Reynolds-Stress Model (DRSM)

For the flow around airfoils, the merits of second-moment closure over eddy-viscosity based closures become obvious in a number of regions:

- *Stagnant flow regions.* We consider an oncoming flow that hits the perpendicular plane $x = 0$. This flow type resembles the stagnation zone on an airfoil or a jet impinging on a plate. Here the normal velocity gradients (*i.e.* $\dfrac{\partial u_1}{\partial x_1}$

and $\frac{\partial u_2}{\partial x_2}$), or the so-called irrotational strain rates, are dominant compared to the cross velocity gradients (*i.e.* $\frac{\partial u_1}{\partial x_2}$ and $\frac{\partial u_2}{\partial x_1}$), or so-called rotational strain rates. The exact production of kinetic energy (see eq. (19)) is given by (no boundary-layer simplifications are made)

$$P_k = -\overline{u_1'^2}\frac{\partial u_1}{\partial x_1} - \overline{u_2'^2}\frac{\partial u_2}{\partial x_2} - \overline{u_1'u_2'}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right). \tag{83}$$

For incompressible flow, due to the divergence-freedom of the velocity field, the first two terms partly balance each other. This shows that the irrotational strains contribute little to the production of the turbulent kinetic energy, but they mainly help to change the shape of turbulent eddies from oncoming 3D structures to almost 2D pancake-like structures. In contrast to the second-moment closure (where the production term is described exactly), the eddy-viscosity model (21) does not recognize this mechanism, as it models the turbulent production as

$$P_k = 2\nu_t\left(\frac{\partial u_1}{\partial x_1}\right)^2 + 2\nu_t\left(\frac{\partial u_2}{x_2}\right)^2 + \nu_t\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)^2. \tag{84}$$

Hence in the eddy-viscosity model both normal strains increase the turbulent kinetic energy. This leads to an excessive overprediction of the turbulence level. As the reattachment zone of a turbulent separation bubble in a boundary layer will also be influenced by such a mechanism, eddy-viscosity models tend to predict a too early reattachment, as compared to differential Reynolds-stress models.

- *streamline curvature*. Using stability analysis, one can show that a laminar boundary-layer flow is stabilized if the wall has a convex shape, whereas it is destabilized if the wall has a concave shape. The latter shape causes streamwise Görtler vortices in the flow, which are due to a centrifugal instability. This fundamental instability mechanism is also reflected by the exact expression of the turbulent kinetic energy source.

  In contrast to this, the representation of the energy source by the eddy-viscosity model only contains positive terms, showing that this model cannot distinguish between the sign of the curvature. Therefore the eddy-viscosity model cannot be expected to work well if the boundary layer has strong curvature, like in the region close to separation.

- *3D flows*. The eddy-viscosity model (21) can be rewritten as

$$a_{ij} = -\frac{\nu_t}{k}S_{ij}, \tag{85}$$

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with

$$a_{ij} = \frac{\overline{u'_i u'_j}}{k} - \frac{2}{3}\delta_{ij},$$

$$S_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}.$$

Here a_{ij} is the anisotropy tensor, and S_{ij} is the rate of strain tensor. As ν_t is a scalar, the eddy-viscosity model assumes that the anisotropy tensor is aligned with the rate of strain tensor. This is not true for strongly 3D flows, which occur, for example in the boundary layer along a swept wing. The turbulence in such a flow is the result of a combined effect of (i) simple shear strain (as also found in a 2D zero pressure gradient boundary layer), (ii) additional (irrotational) strains due to a favourable or adverse pressure gradient in the direction along the wall, perpendicular to the leading edge, and (iii) additional (irrotational) strains due to spanwise pressure gradients.

The specific differential Reynolds-stress model considered here has been proposed by Hanjalić, Jakirlić, and Hadžić (1995) (see also Jakirlić *et al.*, 1994); the model will be denoted as the HJH model. This model is mainly based on high-Reynolds-number expressions as proposed earlier by different authors, and on a modification of the low-Reynolds-number terms of Launder & Shima (1989).

The different terms, as appearing in the transport equation (16) for the Reynolds stress $\overline{u'_i u'_j}$ are described as follows.

- The triple correlation is approximated with the Generalized Gradient Diffusion Hypothesis

$$-\overline{u'_i u'_j u'_k} = C_s \frac{k}{\epsilon} \left(\overline{u'_k u'_l} \frac{\partial \overline{u'_i u'_j}}{\partial x_l} \right), \tag{86}$$

with $C_s = 0.22$. Thus the unknown triple correlation is modelled with the help of known second-order correlations.

The pressure diffusion is not explicitly modelled. Using (86) the model for the diffusion d_{ij} in (16) is

$$d_{ij} = \frac{\partial}{\partial x_k} \left[\left(\nu + C_s \frac{k}{\epsilon} \overline{u'_k u'_l} \right) \frac{\partial \overline{u'_i u'_j}}{\partial x_l} \right], \quad \text{with } C_s = 0.22 \tag{87}$$

- The term P_{ij} only contains second-order correlations between the fluctuating velocity components. No modelling is needed here.

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- The term Φ_{ij} consists of second-order correlations between the fluctuating pressure and velocity components; these correlations need to be modelled. The term serves to redistribute turbulent kinetic energy among different normal components. Although the pressure fluctuation has an elliptic character, most existing models approximate the pressure-strain correlation by a single-point closure, which consists of the following contributions

$$\Phi_{ij} = \Phi_{ij,1} + \Phi_{ij,2} + \Phi_{ij,1}^w + \Phi_{ij,2}^w. \tag{88}$$

The first term, $\Phi_{ij,1}$, is modelled by using Rotta's return-to-isotropy hypothesis

$$\Phi_{ij,1} = -C_1 \epsilon a_{ij}, \tag{89}$$

where a_{ij} is the anisotropy tensor (85). The hypothesis states that anisotropic turbulence has the tendency to return to an isotropic state; the redistribution of energy, by the pressure-strain force, works in that way. The constant C_1 is the so-called Rotta constant (actually it is made a function in the HJJ model).

The second term in (88), $\Phi_{ij,2}$, is called the rapid-distortion part. It can be modelled by using the isotropization-of-production hypothesis:

$$\Phi_{ij,2} = -C_2\left(P_{ij} - \frac{2}{3}P_k\delta_{ij}\right), \tag{90}$$

with $P_k = \frac{1}{2}P_{ii}$.

Close to the wall, additional wall terms are needed, which are often denoted as pressure wall reflection terms:

$$\Phi_{ij,1}^w = C_1^w \frac{\epsilon}{k} f_w\left(\overline{u_k' u_m'} n_k n_m \delta_{ij} - \frac{3}{2}\overline{u_i' u_k'} n_k n_j - \frac{3}{2}\overline{u_k' u_j'} n_k n_i\right), \tag{91}$$

$$\Phi_{ij,2}^w = C_2^w f_w\left(\Phi_{km,2} n_k n_m \delta_{ij} - \frac{3}{2}\Phi_{ik,2} n_k n_j - \frac{3}{2}\Phi_{kj,2} n_k n_i\right). \tag{92}$$

Here n_i are the components of a unit vector normal to the solid wall. The function f_w in the HJJ model is chosen as

$$f_w = \min\left\{\frac{k^{3/2}}{2.5\epsilon x_n}; 1.4\right\}, \tag{93}$$

where x_n is the distance normal to the wall.

In the original differential Reynolds-stress model, C_1, C_2, C_1^w, and C_2^w were all constants, but Launder & Shima (1989) and Jakirlić et al. (1994) replaced these constants by functions, which depend on the local turbulence-based Reynolds

number, $Re_t = \dfrac{k^2}{\nu\epsilon}$, and on the invariant parameter of the stress anisotropy tensor, A. Here A is defined as

$$A = 1 - \frac{9}{8}(A_2 - A_3), \tag{94}$$

with $A_2 = a_{ij}a_{ji}$ and $A_3 = a_{ij}a_{jk}a_{ki}$. For isotropic turbulence we have $a_{ij} = 0$, which gives $A = 1$. When a wall is approached, the normal stress component perpendicular to the wall decreases much faster than the other two normal stresses, which means that the limit of 2D turbulence is obtained. It can be shown that $A = 0$ for 2D turbulence. Hence the model feels that a wall is approached, not only by a decrease of Re_t, but also by a decrease of A.

The HJJ model also introduces the invariant of the dissipation anisotropy tensor:

$$E = 1 - \frac{9}{8}(E_2 - E_3), \tag{95}$$

with $E_2 = e_{ij}e_{ji}$, $E_3 = e_{ij}e_{jk}e_{ki}$, and $e_{ij} = \frac{\epsilon_{ij}}{\epsilon} - \frac{2}{3}\delta_{ij}$. The HJJ model takes the following functions for the pressure-strain correlation

$$
\begin{aligned}
C_1 &= C + \sqrt{A}E^2, & C_2 &= 0.8\sqrt{A}, \\
C_1^w &= \max\{1 - 0.7C; 0.3\}, & C_2^w &= \min\{A; 0.3\},
\end{aligned} \tag{96}
$$

with

$$C = 2.5AF^{1/4}f, \quad F = \min\{0.6; A_2\}, \quad f = \min\left\{\left(\frac{Re_t}{150}\right)^{3/2}; 1\right\}.$$

- The dissipation tensor ϵ_{ij} is modelled as

$$\epsilon_{ij} = f_s\epsilon_{ij}^* + (1 - f_s)\frac{2}{3}\delta_{ij}\epsilon, \tag{97}$$

with

$$\epsilon_{ij}^* = \frac{\epsilon}{k}\left[\frac{\overline{u_i'u_j'} + f_d(\overline{u_i'u_k'}n_jn_k + \overline{u_j'u_k'}n_in_k + \overline{u_k'u_l'}n_kn_ln_in_j)}{1 + \frac{3}{2}\frac{\overline{u_p'u_q'}}{k}n_pn_qf_d}\right].$$

Here the functions f_s and f_d are chosen as

$$f_s = 1 - \sqrt{A}E^2, \quad f_d = \frac{1}{1 + 0.1Re_t}. \tag{98}$$

For isotropic, fully turbulent flow, we have $f_s = 1$ and $f_d = 0$, and (97) reduces to the isotropic form $\epsilon_{ij} = \frac{2}{3}\delta_{ij}\epsilon$.

Equation (97) denotes an algebraic relation between the dissipation tensor ϵ_{ij} (which contains 6 different components in a 3D flow) and the scalar dissipation ϵ. The latter quantity will be described by a differential equation. With other words, although the differential Reynolds-stress model has introduced 6 differential equations for the second-order velocity correlations (which can be interpreted as representing 6 different turbulent velocity scales), there is still only one differential equation that determines the turbulent length scale. Attempts to develop multiple (length or time) scale models do exist, but such models have not been used very much so far.

The HJJ model solves the following differential equation for the scalar dissipation rate ϵ

$$u_k \frac{\partial \epsilon}{\partial x_k} = \frac{\partial}{\partial x_k}\left[\left(\nu + C_\epsilon \frac{k}{\epsilon}\overline{u'_k u'_l}\right)\frac{\partial \epsilon}{\partial x_l}\right] - C_{\epsilon 1} f_{\epsilon 1} \frac{\epsilon}{k}\overline{u'_i u'_j}\frac{\partial u_i}{\partial x_j} - C_{\epsilon 2} f_{\epsilon 2}\frac{\epsilon \tilde{\epsilon}}{k} \tag{99}$$

$$+ C_{\epsilon 3} f_\mu \nu \frac{k}{\epsilon}\overline{u'_j u'_k}\frac{\partial^2 u_i}{\partial x_j \partial x_l}\frac{\partial^2 u_i}{\partial x_k x_l} + S_{\epsilon 4} + S_l.$$

Here $\tilde{\epsilon} = \epsilon - 2\nu \dfrac{\partial \sqrt{k}}{\partial x_j}\dfrac{\partial \sqrt{k}}{\partial x_j}$; at the wall the boundary condition $\tilde{\epsilon} = 0$ is prescribed. Of course, for the differential equations for the Reynolds stresses, the boundary condition $\overline{u'_i u'_j} = 0$ is prescribed at the wall.

The different constants and functions in (99) are chosen as

$$C_\epsilon = 0.18, \quad C_{\epsilon 1} = 1.44, \quad C_{\epsilon 2} = 1.92, \quad C_{\epsilon 3} = 0.25, \tag{100}$$

$$f_\mu = 1, \quad f_{\epsilon 1} = 1, \quad f_{\epsilon 2} = 1 - \frac{C_{\epsilon 2} - 1.4}{C_{\epsilon 2}}\exp\left[-\left(\frac{Re_t}{6}\right)^2\right].$$

The function S_l is a modification of a term originally proposed by Yap (1987), and often referred to as the Yap-correction. The term is meant to suppress the growth of the turbulent length scale $l \ (= k^{3/2}/\epsilon)$ in region with separation and/or reattachment. In the HJJ model the term is taken as

$$S_l = \max\left\{\left[\left(\frac{1}{C_l}\frac{\partial l}{\partial x_n}\right)^2 - 1\right]\left(\frac{1}{C_l}\frac{\partial l}{\partial x_n}\right)^2 ; 0\right\}\frac{\epsilon \tilde{\epsilon}}{k}A, \tag{101}$$

with x_n is the coordinate normal to the wall.

The term $S_{\epsilon 4}$ is very important for aeronautical applications, as it turns out to be crucial for the the correct prediction of the influence of the streamwise pressure gradient on the turbulent boundary layer. Physically, a favourable pressure gradient will

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decrease the turbulence level, and may even lead to relaminarization of the turbulent boundary layer, whereas an adverse pressure gradient will enhance the turbulence.

To derive S_{ϵ_4}, we first consider the source dissipation due to straining as represented by $-C_{\epsilon 1}\frac{\epsilon}{k}\overline{u_i'u_j'}\frac{\partial u_i}{\partial x_j}$. For a 2D time-averaged flow, this term reads

$$C_{\epsilon 1}\frac{\epsilon}{k}\left(-\overline{u'v'}\frac{\partial u}{\partial y}-\overline{u'v'}\frac{\partial v}{\partial x}-\overline{u'^2}\frac{\partial u}{\partial x}-\overline{v'^2}\frac{\partial v}{\partial y}\right). \tag{102}$$

The first two contributions denote the effects of shear strains (or: rotational strains) and the last two terms are due to normal strains (or: irrotational strains). Hanjalić & Launder (1980) have suggested that in boundary layers with nonzero streamwise pressure gradient the effect of irrotational strains on the dissipation rate might be different; with other words the constant $C_{\epsilon 1}$ in (102) should have a different value when combined with the first two and last two contributions, respectively. Hanjalić & Launder have accounted for this effect by adding the term $S_{\epsilon,4} = C_{\epsilon 4}\frac{\epsilon}{k}(\overline{v'^2}-\overline{u'^2})\frac{\partial u}{\partial x}$ to eq. (99). In the HJJ model the new constant is taken as $C_{\epsilon 4} = 1.16$

A disadvantage of the formulation of this extra term is that it is not coordinate invariant. For a 2D time-averaged flow, this formulation is similar to the formulation of Jakirlić et al. (1994). For an adverse pressure gradient boundary layer (with $\frac{\partial u}{\partial x} < 0$ and $\overline{u'^2} > \overline{v'^2}$), the extra term is positive, and will thus increase the dissipation rate ϵ, implying a decrease of the turbulent length scale. This indeed is the desired effect, as experiments have shown that the increase of the turbulent length scale due to an adverse pressure gradient is far less than would be predicted by the standard ϵ equation.

In this section the differential Reynolds-stress model was presented in the tensor notation. Many terms will disappear if the time-averaged flow is two-dimensional, and if the boundary-layer simplifications are applied. The latter simplifications are

$$\frac{\partial}{\partial y} >> \frac{\partial}{\partial x}, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial x} = -\rho U\frac{dU}{dx}, \tag{103}$$

where U is the free-stream velocity. These simplifications are justified for high Reynolds numbers and for weak streamwise pressure gradients. The resulting equations are parabolic, which can be solved with far less computational effort than the original elliptic Reynolds-Averaged Navier-Stokes formulation.

The two-dimensional boundary-layer equations for the differential Reynolds-stress model are:

Conservation of mass:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{104}$$

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Conservation of momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + U\frac{dU}{dx} - \frac{\partial \overline{u'v'}}{\partial y}. \tag{105}$$

Equations for the Reynolds normal stresses:

$$u\frac{\partial \overline{u'^2}}{\partial x} + v\frac{\partial \overline{u'^2}}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + C_s\frac{k}{\epsilon}\overline{v'^2}\right)\frac{\partial \overline{u'^2}}{\partial y}\right] + P_{11} + \Phi_{11} - \epsilon_{11}, \tag{106}$$

$$u\frac{\partial \overline{v'^2}}{\partial x} + v\frac{\partial \overline{v'^2}}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + C_s\frac{k}{\epsilon}\overline{v'^2}\right)\frac{\partial \overline{v'^2}}{\partial y}\right] + P_{22} + \Phi_{22} - \epsilon_{22}, \tag{107}$$

$$u\frac{\partial \overline{w'^2}}{\partial x} + v\frac{\partial \overline{w'^2}}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + C_s\frac{k}{\epsilon}\overline{v'^2}\right)\frac{\partial \overline{w'^2}}{\partial y}\right] + P_{33} + \Phi_{33} - \epsilon_{33}, \tag{108}$$

with

$$P_{11} = 2P_k, \quad P_{22} = 0, \quad P_{33} = 0,$$

$$P_k = -\overline{u'v'}\frac{\partial u}{\partial y},$$

$$\Phi_{11} = -C_1\epsilon\left(\frac{\overline{u'^2}}{k} - \frac{2}{3}\right) + C_1^w f_w\frac{\epsilon}{k}\overline{v'^2} - \frac{2}{3}C_2 P_k(2 - C_2^w f_w),$$

$$\Phi_{22} = -C_1\epsilon\left(\frac{\overline{v'^2}}{k} - \frac{2}{3}\right) - 2C_1^w f_w\frac{\epsilon}{k}\overline{v'^2} + \frac{2}{3}C_2 P_k(1 - 2C_2^w f_w),$$

$$\Phi_{33} = -C_1\epsilon\left(\frac{\overline{w'^2}}{k} - \frac{2}{3}\right) + C_1^w f_w\frac{\epsilon}{k}\overline{v'^2} + \frac{2}{3}C_2 P_k(1 + C_2^w f_w),$$

$$\epsilon_{11} = \epsilon\left[\frac{2}{3}(1 - f_s) + f_s\frac{\overline{u'^2}}{k}\frac{1}{1 + \frac{3}{2}\frac{\overline{v'^2}}{k}f_d}\right],$$

$$\epsilon_{22} = \epsilon\left[\frac{2}{3}(1 - f_s) + f_s\frac{\overline{v'^2}}{k}\frac{1 + 3f_d}{1 + \frac{3}{2}\frac{\overline{v'^2}}{k}f_d}\right],$$

$$\epsilon_{33} = \epsilon\left[\frac{2}{3}(1 - f_s) + f_s\frac{\overline{w'^2}}{k}\frac{1}{1 + \frac{3}{2}\frac{\overline{v'^2}}{k}f_d}\right].$$

Equation for the turbulent kinetic energy:
This equation is found by taking the sum of the equations for the three normal stresses, and dividing the result by 2. This gives

$$u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + C_s\frac{k}{\epsilon}\overline{v'^2}\right)\frac{\partial k}{\partial y}\right] + P_k - \epsilon. \tag{109}$$

Equation for the Reynolds shear stress:

$$u\frac{\partial \overline{u'v'}}{\partial x} + v\frac{\partial \overline{u'v'}}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + C_s\frac{k}{\epsilon}\overline{v'^2}\right)\frac{\partial \overline{u'v'}}{\partial y}\right] + P_{12} + \Phi_{12} - \epsilon_{12}, \tag{110}$$

with

$$P_{12} = -\overline{v'^2}\frac{\partial u}{\partial y},$$

$$\Phi_{12} = -(C_1 + \frac{3}{2}C_1^w f_w)\frac{\epsilon}{k}\overline{u'v'} + (1 - \frac{3}{2}c_2^w f_w)C_2\overline{v'^2}\frac{\partial u}{\partial y},$$

$$\epsilon_{12} = \epsilon f_s\frac{\overline{u'v'}}{k}\frac{1+f_d}{1+\frac{3}{2}\frac{\overline{v'^2}}{k}f_d}.$$

Equation for the dissipation rate of turbulent kinetic energy:

$$u\frac{\partial \epsilon}{\partial x} + v\frac{\partial \epsilon}{\partial y} = \frac{\partial}{\partial y}\left\{\left(\nu + C_\epsilon\frac{k}{\epsilon}\overline{v'^2}\right)\right\}\frac{\partial \epsilon}{\partial y} + C_{\epsilon 1}f_{\epsilon 1}\frac{\epsilon}{k}P_k - C_{\epsilon 2}f_{\epsilon 2}\frac{\epsilon\tilde{\epsilon}}{k} + \tag{111}$$

$$+C_{\epsilon 3}f_\mu\overline{v'^2}\left(\frac{\partial^2 u}{\partial y^2}\right)^2 + S_{\epsilon 4} + S_l,$$

with

$$S_l = \max\left\{\left[\left(\frac{1}{C_l}\frac{\partial l}{\partial y}\right)^2 - 1\right]\left(\frac{1}{C_l}\frac{\partial l}{\partial y}\right)^2;0\right\}\frac{\epsilon\tilde{\epsilon}}{k}A,$$

$$S_{\epsilon 4} = C_{\epsilon 4}\frac{\epsilon}{k}(\overline{v'^2} - \overline{u'^2})\frac{\partial u}{\partial x}.$$

The last term is not consistent with the boundary-layer simplifications, in which x derivatives are neglected, and consequently $S_{\epsilon 4}$ would vanish. However, this term is kept here, as it turns out to give a significant contribution to the ϵ equation once the steamwise pressure gradients are no longer very weak. If the x-derivative is kept in the $S_{\epsilon 4}$ term, it also must be kept in the other source term of the ϵ equation, which reads

$$C_{\epsilon 1}\frac{\epsilon}{k}\left(-\overline{u'v'}\frac{\partial u}{\partial y} - \overline{u'v'}\frac{\partial v}{\partial x} + (\overline{v'^2} - \overline{u'^2})\frac{\partial u}{\partial x}\right). \tag{112}$$

The second term can be neglected compared to the first term, but the third term is kept. Within the context of the boundary-layer equation (111), this can also be realized by adding the constant $C_{\epsilon 1}$ to the original $C_{\epsilon 4}$ value. This means that $C_{\epsilon 4} = 1.16$ in the elliptic formulation (99), whereas $C_{\epsilon 4} = 1.16 + 1.44 = 2.6$ in the boundary-layer formulation (111).

The wall boundary conditions for the boundary-layer equations are: $u = v = \overline{u'^2} = \overline{v'^2} = \overline{w'^2} = k = 0$ and $\epsilon = 2\nu \left(\dfrac{\partial \sqrt{k}}{\partial y} \right)_w^2$.

Closure

This report has given a detailed overview of different classes of turbulence models that can be used to compute boundary-layers that are relevant to aeronautics. The present author has applied and compared most of these models to several testcases. The results are published in the journal papers mentioned in the introduction.

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An overview is given on the background of different turbulence models that can be used to compute boundary layers in external aerodynamics, such as for aircraft. The overview includes algebraic models (Cebeci-Smith, Baldwin-Lomax), a half-equation model (Johnson-King), two-equation models (K-E-K-), and a differential Reynolds-stress model. The models were compared for boundary layer without and with streamwise pressure gradient. The models were also used to study the large-Reynolds number scalings (wall function and defect layer). The comparisson of models and the scaling analysis are described in three seperate journal contributions.

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