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Vroegrijk, Tom

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QH-SINGULARITY OF PARTIALLY ORDERED SPACES

TOM VROEGRIJK

Dedicated to Cheyenne Sels

Abstract. Each partial order generates a transitive quasi-uniformity. In this article we will study the properties of quasi-uniformities that are defined by a partial order and are QH-singular.

1. Introduction

In exercise 17 on page 35 of Isbells book [1] on uniform spaces it is claimed that if $\mathcal{U}$ and $\mathcal{V}$ are distinct uniformities on a set X, the topologies defined by the Hausdorff uniformities on the hyperspace of X are also distinct. In [7] Smith showed that this claim was false. From that point on uniformities on a set X that do generate the same hyperspace topology were called H-equivalent. A uniformity $\mathcal{U}$ for which there is no distinct uniformity $\mathcal{V}$ that is H-equivalent to $\mathcal{U}$ is called H-singular.

After Smiths article [7] several papers on the properties of H-singular uniform spaces appeared (see for example [9] and [10]). Some recent results on this topic can be found in [2] and [6]. With the publications [3] and [5] Cao, Künzi and Reilly started the study of H-singularity in the asymmetric case. With each quasi-uniformity $\mathcal{U}$ on a set X we can associate a quasi-uniform structure on the hyperspace of X called the Hausdorff quasi-uniformity. Here too we can ask ourselves if there exist quasi-uniformities $\mathcal{U}$ for which there is no distinct quasi-uniformity $\mathcal{V}$ such that $\mathcal{U}$ and $\mathcal{V}$ define Hausdorff quasi-uniformities that have the same underlying topology. Such quasi-uniformities will be called QH-singular.

In [8] the author obtained some general results on QH-singularity of quasi-uniform spaces. The purpose of this article is to investigate the properties of QH-singular quasi-uniformities that are defined by a partial order.

2. Preliminaries

Let X be a set and $U, V \subseteq X \times X$ relations on X. For an $x \in X$ we define $U(x)$ as $\{y \in X \mid (x, y) \in U\}$. The relation $V \circ U$ contains all (x, z) for which there is a $y \in X$ such that $y \in U(x)$ and $z \in V(y)$. We will denote $U \circ U$ as U^2 and $U \circ U^n$ as U^{n+1} whenever $n \geq 2$.

A filter $\mathcal{U}$ on $X \times X$ is called a *quasi-uniformity* iff it has the following properties:

(1) $\forall x \in X \, \forall U \in \mathcal{U} : (x, x) \in U$,
(2) $\forall U \in \mathcal{U} \, \exists V \in \mathcal{U} : V^2 \subseteq U$.

The elements of a quasi-uniformity $\mathcal{U}$ will be called *entourages*. The pair $(X, \mathcal{U})$ is a *quasi-uniform space*. A subset $\mathcal{U}' \subseteq \mathcal{U}$ is a *base* for $\mathcal{U}$ iff each $U \in \mathcal{U}$ contains a $U' \in \mathcal{U}'$. A *transitive* quasi-uniformity is a quasi-uniformity with a base that consists of transitive relations. For an extensive monograph on quasi-uniform spaces we refer the reader to [4].

Each quasi-uniformity $\mathcal{U}$ has an underlying topology $\tau(\mathcal{U})$. In this topology the neighbourhoodfilter of a point x is generated by the sets $U(x)$ with $U \in \mathcal{U}$.

The quasi-uniformity $\mathcal{U}^{-1}$ is called the *conjugate of* $\mathcal{U}$ and consists of all entourages U^{-1}, where $U^{-1} = \{(y,x) \mid (x,y) \in \mathcal{U}\}$.

The set of all subsets of X will be denoted as $\mathcal{P}(X)$. For a subset $A \in \mathcal{P}(X)$ and an entourage $U \in \mathcal{U}$ we define $U(A)$ as the union of all $U(x)$ with $x \in A$. For any relation U on X we define

$$U_+ = \{(A,B) \in \mathcal{P}(X) \times \mathcal{P}(X) \mid B \subseteq U(A)\}$$

and

$$U_- = \{(A,B) \in \mathcal{P}(X) \times \mathcal{P}(X) \mid A \subseteq U^{-1}(B)\}.$$

If $(X,\mathcal{U})$ is a quasi-uniform space, then the filter generated by the sets U_- is a quasi-uniformity $\mathcal{U}_H^-$ on $\mathcal{P}(X)$ that we will call the lower Hausdorff quasi-uniformity. Analogously, the sets U_+ generate the upper Hausdorff quasi-uniformity $\mathcal{U}_H^+$ on $\mathcal{P}(X)$. We will denote the intersection $U_- \cap U_+$ as U_H. The Hausdorff quasi-uniformity $\mathcal{U}_H$ on the hyperspace $\mathcal{P}(X)$ is the filter that is generated by the sets U_H.

If $\mathcal{U}$ and $\mathcal{V}$ are two quasi-uniformities on a set X, then we say that $\mathcal{V}$ is *QH-finer* than $\mathcal{U}$ (or that $\mathcal{U}$ is *QH-coarser* than $\mathcal{V}$) iff $\tau(\mathcal{U}_H) \subseteq \tau(\mathcal{V}_H)$. If the topologies $\tau(\mathcal{U}_H)$ and $\tau(\mathcal{V}_H)$ are equal, then we say that $\mathcal{U}$ and $\mathcal{V}$ are *QH-equivalent*. The set of all quasi-uniformities on X that are QH-equivalent with $\mathcal{U}$ is the *QH-equivalence class* of $\mathcal{U}$. A quasi-uniformity $\mathcal{U}$ is called *QH-singular* iff its QH-equivalence class only contains $\mathcal{U}$. We will say that $\mathcal{U}$ is *transitively QH-singular* iff there is no transitive quasi-uniformity $\mathcal{V}$ that is distinct from $\mathcal{U}$ and QH-equivalent with $\mathcal{U}$.

3. QH-singularity of subspaces

Each partial order defines a unique transitive quasi-uniformity. In the preliminaries we defined the QH-equivalence class of a quasi-uniformity. The purpose of this article is to get some insight into the structure of the QH-equivalence class of a quasi-uniformity defined by a partial order and to discover some properties of quasi-uniformities within this equivalence class.

Definition 1. For a partial order $\leq$ on a set X we define $U_{\leq}$ as

$$\{(x,y) \in X \times X \mid x \leq y\}.$$

The filter that consists of all subsets of $X \times X$ that contain $U_{\leq}$ is a transitive quasi-uniformity that we will denote as $\mathcal{U}_{\leq}$.

Proposition 1. *If $\leq$ is a partial order on X, then $\mathcal{U}_{\leq}$ is a the finest element in its QH-equivalence class.*

Proof. Suppose that $\mathcal{V}$ is a quasi-uniformity that is QH-equivalent to $\mathcal{U}$. Take a $V \in \mathcal{V}$ and an $x \in X$. By assumption we have that there is a $U \in \mathcal{U}$ such that $U_H(\{x\}) \subseteq V_H(\{x\})$ and thus $(U_{\leq})_H(\{x\}) \subseteq V_H(\{x\})$. This implies $U_{\leq}(x) \subseteq V(x)$ and because x was chosen arbitrarily we get $U_{\leq} \subseteq V$. Hence we obtain that $\mathcal{V} \subseteq \mathcal{U}_{\leq}$. □

Proposition 2. *If $\mathcal{V}$ is a quasi-uniformity that is coarser than $\mathcal{U}_\leq$, then $\mathcal{V}$ is QH-equivalent with $\mathcal{U}_\leq$ iff for each $A \subseteq X$ there is a $V \in \mathcal{V}$ such that $V(A) \subseteq U_\leq(A)$ and for each $x \in A$ there is a $y \in A$ with the property $V(y) \subseteq U_\leq(x)$.*

Proof. Since $\mathcal{V}$ is a quasi-uniformity that is coarser than $\mathcal{U}_\leq$ we automatically obtain that $\mathcal{V}$ is QH-coarser than $\mathcal{U}_\leq$. This means that both quasi-uniformities are QH-equivalent iff $\mathcal{V}$ is QH-finer than $\mathcal{U}_\leq$. That this is true iff for each $A \subseteq X$ there is a $V \in \mathcal{V}$ such that $V(A) \subseteq U_\leq(A)$ and for each $x \in A$ there is a $y \in A$ with the property $V(y) \subseteq U_\leq(x)$ is a direct consequence of the first corollary of [8]. □

The following results describe how QH-singularity transfers to certain types of subspaces of partially ordered sets. We will use these results in the final section to prove the main theorems of this article.

A subset Y of a partially ordered space $(X, \leq)$ is a *downset* (*upset*) iff $x \in Y$ whenever there is a $y \in Y$ such that $x \leq y$ ($x \geq y$).

Proposition 3. *Let Y be a downset in a partially ordered space $(X, \leq)$. If $(X, \leq)$ is transitively QH-singular, then the partially ordered subspace $(Y, \leq)$ is transitively QH-singular.*

Proof. Suppose that $(Y, \leq)$ is not transitively QH-singular and that $\mathcal{V}$ is a transitive quasi-uniformity on Y that is QH-equivalent with $\mathcal{U}_{\leq_Y}$, where $\leq_Y$ is the restriction of the partial order $\leq$ to Y. Take a transitive $V \in \mathcal{V}$. Define $V^\dagger$ such that $V^\dagger(x)$ is equal to $U_\leq(x)$ whenever $x \notin Y$ and equal to $V(x) \cup U_\leq(x)$ for $x \in Y$. It is easy to verify that $V^\dagger$ is a transitive relation if Y is a downset. Because $V_1^\dagger \cap V_2^\dagger$ equals $(V_1 \cap V_2)^\dagger$ whenever V_1, V_2 are transitive elements of $\mathcal{V}$, the collection of all relations $V^\dagger$ forms a base for a quasi-uniformity. Let $\mathcal{V}^\dagger$ be this quasi-uniformity.

It is clear that $\mathcal{V}^\dagger$ is coarser than $\mathcal{U}_\leq$. Take a subset A of X. Because $\mathcal{V}$ is QH-equivalent with $(\mathcal{U}_{\leq_Y})$ we can use proposition 2 to find a transitive $V \in \mathcal{V}$ that satisfies $V(A \cap Y) \subseteq U_\leq(A \cap Y)$ and for each $x \in A \cap Y$ there is a $y \in A \cap Y$ with the property $V(y) \subseteq U_{\leq_Y}(x)$

To prove that $V^\dagger(A) \subseteq U_\leq(A)$ take an $x \in A$. If x is not an element of $A \cap Y$, then $V^\dagger(x)$ is simply $U_\leq(x)$, so $V^\dagger(x) \subseteq U_\leq(A)$. In case $x \in A \cap Y$ and $y \in V^\dagger(x)$ we know that y is either contained in $V(x)$ or in $U_\leq(x)$. If $y \in V(x)$, then $y \in Y$ and therefore $y \in V(A \cap Y) \subseteq U_\leq(A \cap Y) \subseteq U_\leq(A)$. On the other hand, if y is not contained in $V(x)$, then $y \in U_\leq(x) \subseteq U_\leq(A)$.

Take an $x \in A$. We only need to prove that there is a $y \in A$ such that $V^\dagger(y) \subseteq U_\leq(x)$. If x is not contained in $A \cap Y$, then this is trivially true since $V^\dagger(x) = U_\leq(x)$. Suppose that $x \in A \cap Y$. We know that there must be a $y \in A \cap Y$ such that $V(y) \subseteq U_\leq(x)$. This yields that $x \leq y$ and thus we obtain $V^\dagger(y) = V(y) \cup U_\leq(y) \subseteq U_\leq(x)$. □

In the following three results $(X, \leq)$ will be a partially ordered space, Y will be a subset of X and $\mathcal{V}$ will be a quasi-uniformity on Y. Throughout these propositions we will define $\tilde{\mathcal{V}}$ as the filter on $X \times X$ generated by all relations $\tilde{V}$ where $\tilde{V}(x)$ is equal to $U_\leq(x)$ if $x \notin Y$ and equal to $V(x)$ when $x \in Y$.

Lemma 1. *Let Y be an upset in a partially ordered space $(X, \leq)$ and let $\mathcal{V}$ be a transitive quasi-uniformity on Y that is coarser than $\mathcal{U}_{\leq_Y}$ and that satisfies $z \geq x$ whenever $z \in V(y)$ and $y \geq x$ whenever $V \in \mathcal{V}$, $x \notin Y$ and $y \in Y$. $\tilde{\mathcal{V}}$ is a transitive quasi-uniformity on X.*

Proof. Take a transitive $V \in \mathcal{V}$ and $x, y, z \in X$ such that $z \in \tilde{V}(y)$ and $y \in \tilde{V}(x)$. If x and y are not in Y, then we have $z \geq y \geq x$ and therefore $z \in \tilde{V}(x)$. In the case that $x \in Y$ we automatically obtain $y \in Y$ and thus $z \in V^2(x) \subseteq \tilde{V}(x)$. Finally, if $x \notin Y$ and $y \in Y$, then we have $z \in V(y)$ and $y \geq x$. By assumption this yields $z \geq x$ and thus $z \in \tilde{V}(x)$. □

Proposition 4. *Let Y be an upset in a partially ordered space $(X, \leq)$ and let $\mathcal{V}$ be a transitive quasi-uniformity on Y that is QH-equivalent to $\mathcal{U}_{\leq_Y}$ and that satisfies $z \geq x$ whenever $z \in V(y)$ and $y \geq x$ for some $V \in \mathcal{V}$, $x \notin Y$ and $y \in Y$. The quasi-uniformity $\tilde{\mathcal{V}}$ is QH-equivalent with $\mathcal{U}_{\leq}$.*

Proof. By definition we have that $\tilde{\mathcal{V}}$ is coarser than $\mathcal{U}_{\leq}$. Let A be a subset of X. Proposition 2 tells us that we can find a $V \in \mathcal{V}$ such that $V(A \cap Y) \subseteq U_{\leq_Y}(A \cap Y)$ and for each $x \in A \cap Y$ there is a $y \in A \cap Y$ with the property $V(y) \subseteq U_{\leq_Y}(x)$.

Take an $x \in A$ and a $z \in \tilde{V}(x)$. If $x \in Y$, then we have

$$z \in \tilde{V}(x) = V(x) \subseteq V(A \cap Y) \subseteq U_{\leq_Y}(A \cap Y) \subseteq U_{\leq}(A).$$

For $x \notin Y$ we have that $\tilde{V}(x) = U_{\leq}(x)$ and thus $z \in U_{\leq}(A)$. This proves that $\tilde{V}(A) \subseteq U_{\leq}(A)$.

Finally, we want to show that there is a $y \in A$ such that $\tilde{V}(y) \subseteq U_{\leq}(x)$. In case $x \notin Y$ we can simply choose y to be equal to x, since $\tilde{V}(y) = \tilde{V}(x) = U_{\leq}(x)$. If x is an element of Y, then we know that there is a $y \in A \cap Y$ with the property $V(y) \subseteq U_{\leq_Y}(x)$. This implies $\tilde{V}(y) = V(y) \subseteq U_{\leq}(x)$. □

Proposition 5. *Let Y be a subset of a partially ordered space $(X, \leq)$ such that $x \leq y$ for each $y \in Y$ whenever $x \notin Y$. If $(X, \leq)$ is transitively QH-singular, then $(Y, \leq)$ is transitively QH-singular.*

Proof. Suppose that there exists a transitive quasi-uniformity $\mathcal{V}$ on Y that is QH-equivalent with $\mathcal{U}_{\leq}$. Because $x \leq y$ for each $y \in Y$ whenever $x \notin Y$ we have that Y is an upset. On the other hand, this also implies that $z \geq x$ whenever $V \in \mathcal{V}$, $x \notin Y$, $y \in Y$ and $z \in X$ such that $z \in V(y)$ and $y \geq x$. The previous proposition now yields that $\tilde{\mathcal{V}}$ is a transitive quasi-uniformity that is QH-equivalent with $\mathcal{U}_{\leq}$. □

4. The ordered space ω

That the ordered space ω is not QH-singular was already established in [3]. In this section we will characterise all quasi-uniformities that are in the QH-equivalence class of the quasi-uniformity $\mathcal{U}_\omega$ determined by the order on ω. We will denote $U_{\leq}$ as U_ω if $\leq$ is the order relation on ω.

Proposition 6. *A quasi-uniformity $\mathcal{V}$ on ω is QH-coarser than $\mathcal{U}_\omega$ iff $\tau(\mathcal{V})$ is coarser than $\tau(\mathcal{U}_\omega)$.*

Proof. It follows from the definition that the underlying topology of $\mathcal{V}$ is coarser than $\tau(\mathcal{U})$ whenever $\tau(\mathcal{V}_H) \subseteq \tau((\mathcal{U}_\omega)_H)$. On the other hand, if $\tau(\mathcal{V}) \subseteq \tau(\mathcal{U}_\omega)$, then we have for each $n \in \omega$ and $V \in \mathcal{V}$ that $U_\omega(n) \subseteq V(n)$. This implies $U_\omega \subseteq V$ and thus $\mathcal{V} \subseteq \mathcal{U}_\omega$. The latter yields that $\mathcal{V}$ is QH-coarsers than $\mathcal{U}_\omega$. □

A subset Y of a quasi-uniform space $(X, \mathcal{U})$ will be called *relatively $\mathcal{U}$-precompact* iff for each $U \in \mathcal{U}$ there is a finite set $K \subseteq X$ such that $Y \subseteq U(K)$.

Proposition 7. *Let $\mathcal{V}$ be a quasi-uniformity on ω. The following are equivalent:*

(1) *for each* $A \subseteq \omega$ *there is a* $V \in \mathcal{V}$ *such that for each* $x \in A$ *there is a* $y \in A$ *with the property* $V(y) \subseteq U_\omega(x)$,
(2) *each relatively* $\mathcal{V}^{-1}$*-precompact subset of* ω *is finite.*

Proof. Suppose that there is an infinite relatively $\mathcal{V}^{-1}$-precompact subset A of ω. Take an arbitrary $V \in \mathcal{V}$. By assumption there is an $n \in \omega$ such that $A \subseteq V^{-1}([0,n])$. Choose $x \in A$ such that $n < x$. Because A is infinite such an x must exist. Since $A \subseteq V^{-1}([0,n])$ we now have that for each $y \in A$ the set $V(y)$ intersects with $[0,n]$. This means that there is no $y \in A$ such that $V(y) \subseteq U_\omega(x)$.

To prove the converse we assume that there is an $A \subseteq \omega$ such that for each $V \in \mathcal{V}$ there is an $x \in A$ with the property that $V(y) \nsubseteq U_\omega(x)$ for any $y \in A$. Take $V \in \mathcal{V}$ and choose an $x \in A$ with this property. Whenever V is an element of $\mathcal{V}$ we know that $V(y)$ is not contained in $U_\omega(x)$. Clearly, x cannot be equal to 0, since this would imply that $U_\omega(x)$ equals ω. For any $y \in A$ the set $V(y)$ intersects with $[0, x-1]$ and thus $A \subseteq V^{-1}([0, x-1])$. Because V was arbitrary we have that A is relatively $\mathcal{V}^{-1}$-precompact. □

Proposition 8. *A quasi-uniformity* $\mathcal{V}$ *on* ω *is* QH*-equivalent to* $\mathcal{U}_\omega$ *iff the following conditions hold:*

(1) $\tau(\mathcal{V}) \subseteq \tau(\mathcal{U}_\omega)$,
(2) *for each* $n \in \omega$ *there is a* $V \in \mathcal{V}$ *such that* $V^{-1}([0,n]) = [0,n]$,
(3) *each relatively* $\mathcal{V}^{-1}$*-precompact subset of* ω *is finite.*

Proof. First we will prove the necessity of these conditions. That QH-equivalence of $\mathcal{V}$ and $\mathcal{U}_\omega$ implies conditions (1) and (3) follows from the previous propositions and proposition 2. To prove that the second condition holds let us assume that there is an $n \in \omega$ such that for each $V \in \mathcal{V}$ the set $V^{-1}([0,n])$ is not equal to $[0,n]$. If we define A as $[n+1, +\infty[$, then $V(A)$ intersects with $[0,n]$ for each $V \in \mathcal{V}$. Clearly the set $U_\omega(A)$ is equal to A and thus there is no $V \in \mathcal{V}$ for which $V(A) \subseteq U_\omega(A)$. This contradicts with the assumption that $\mathcal{V}$ on ω is QH-equivalent to $\mathcal{U}_\omega$.

Now suppose that the three stated conditions are true. The first condition yields that $\mathcal{V}_H$ is coarser than $(\mathcal{U}_\omega)$. By proposition 2 this means that in order to prove that $\mathcal{V}$ is QH-equivalent to $\mathcal{U}_\omega$ we still need to show that for each $A \subseteq \omega$ there is a $V \in \mathcal{V}$ such that $V(A) \subseteq U_\omega(A)$. Assume that this is not the case. This means that we can find an $A \subseteq \omega$ such that for each $V \in \mathcal{V}$ we have $V(A) \nsubseteq U_\omega(A)$. The set A does not contain 0, because in this case $U_\omega(A)$ would be equal to ω. Define n as $\min(A) - 1$. Since $V(A)$ hits $[0,n]$ for each $V \in \mathcal{V}$ we obtain that there is no entourage $V \in \mathcal{V}$ for which $V^{-1}([0,n]) \subseteq [0,n]$ □

Proposition 9. *A quasi-uniformity* $\mathcal{V}$ *on* ω *is* QH*-equivalent to* $\mathcal{U}_\omega$ *iff the following conditions hold:*

(1) $\tau(\mathcal{V}) \subseteq \tau(\mathcal{U}_\omega)$,
(2) $\tau(\mathcal{U}_\omega^{-1}) \subseteq \tau(\mathcal{V}^{-1})$,
(3) *each relatively* $\mathcal{V}^{-1}$*-precompact subset of* ω *is finite.*

Proof. Let $\mathcal{V}$ be a quasi-uniformity that is QH-equivalent to $\mathcal{U}_\omega$. It was established in [5] that the conjugates of QH-equivalent quasi-uniformities generate the same topology. It follows from the previous result that $\mathcal{V}$ satisfies conditions (1) and (3).

To prove the converse assume that the quasi-uniformity $\mathcal{V}$ satisfies the three given conditions. Because of the previous result we only need to prove that for each $n \in \omega$

there is a $V \in \mathcal{V}$ such that $V^{-1}([0,n]) = [0,n]$ to show that $\mathcal{V}_H$ and $(\mathcal{U}_\omega)_H$ generate the same topology. From the second condition we obtain that for each $k \in \omega$ there is a $V_k \in \mathcal{V}$ such that $V_k^{-1}(k) \subseteq U_\omega^{-1}(k) = [0,k]$. Take $n \in \omega$ and define V as $V_0 \cap \ldots \cap V_n$. This entourage is clearly an element of $\mathcal{V}$ and $V^{-1}([0,n]) \subseteq [0,n]$. □

Example 1. Define the entourage W_k on ω such that $W_k(n)$ is equal to $U_\omega(n-1)$ whenever n is odd and $n \geq k$ and equal to $U_\omega(n)$ in all other cases. It is an easy exercise to check that these relations are transitive. Because $W_{k'} \subseteq W_k$ whenever $k \leq k'$ we obtain that these entourages also form a base for a transitive quasi-uniformity $\mathcal{W}$.

The quasi-uniformity $\mathcal{W}$ in fact satisfies all the conditions in the previous proposition. First of all it follows directly from the definition that $U_\omega(n) \subseteq W_k(n)$ for all $k, n \in \omega$, so this means $\tau(\mathcal{V}) \subseteq \tau(\mathcal{U}_\omega)$.

Now take an $n \in \omega$ and define k as $n+2$. If $m \leq n$, then $W_k(m)$ equals $U_\omega(m)$ and thus $n \in W_k(m)$. In case $m > n$ we have that $W_k(m) \subseteq U_\omega(n+1)$ and therefore $n \notin W_k(m)$. Hence we obtain that $W_k^{-1}(n) = [0,n] = U_\omega^{-1}(n)$. This yields $\tau(\mathcal{U}_\omega^{-1}) \subseteq \tau(\mathcal{V}^{-1})$.

Finally, let Y be a relatively $\mathcal{W}^{-1}$-precompact subset of ω. By definition we have that $W_0^{-1}(n) \subseteq [0, n+1]$ for all $n \in \omega$. Now let K be a finite subset of ω such that $Y \subseteq W_0^{-1}(K)$. If k_0 is the maximum of K, then $W_0^{-1}(K) \subseteq [0, k_0+1]$ and this means that Y must be finite.

This example suggests that the existence of a totally ordered subspace implies QH-singularity. In the following section we will see that this is not the case, but that there is some sort of upper bound for the size of totally ordered subspaces in QH-singular partially ordered spaces. In fact we will construct quasi-uniformities similar to the one in the previous example to prove the main results of this article.

5. Chains and antichains

A subset of a partially ordered space that is totally ordered is called a *chain*. An *antichain* is a subset of which all distinct elements are incomparable. In this section we investigate the behaviour of chains and antichains in QH-singular partially ordered spaces.

Proposition 10. *A partially ordered set $(X, \leq)$ that is equal to a finite union of antichains is QH-singular.*

Proof. Suppose that X can be written as $A_0 \cup \ldots \cup A_n$ where each A_k is an antichain. Let $\mathcal{V}$ be a quasi-uniformity that is QH-equivalent to $\mathcal{U}_\leq$. We already saw that $\mathcal{V}$ must be coarser than $\mathcal{U}_\leq$. From theorem 2.4 of [3] we obtain that for each $0 \leq k \leq n$ we can find a $V_k \in \mathcal{V}$ such that $V_k(x) \subseteq U_\leq(x)$ whenever $x \in A_k$. If we define V as the intersection of all V_k we obtain an element of $\mathcal{V}$ with the property that $V \subseteq U_\leq$. Hence $\mathcal{V}$ and $\mathcal{U}_\leq$ must be equal. □

Definition 2. We will define the *depth* of an element $x \in X$ as the supremum of all $n \in \omega$ with the property that there exists a chain of length n of which x is the smallest element.

Proposition 11. *Let $(X, \leq)$ be a partially ordered set. If there is an $n \in \omega$ such that $|C| \leq n$ for each chain C in X, then $(X, \leq)$ is QH-singular.*

Proof. Let A_k be the collection of all $x \in X$ with depth equal to k. It is clear that X is equal to $A_1 \cup \ldots \cup A_n$. We will now show that each A_k is in fact an antichain. Take $x, y \in A_k$ with $x < y$. By definition we can find a chain $y_1 < \ldots < y_k$ such that y equals y_1. We now have that the chain $x < y_1 < \ldots < y_k$ consists of $k+1$ elements and x is the smallest element in the chain, but this is impossible since the depth of x is equal to k. By using the previous proposition we obtain that $(X, \leq)$ is QH-singular. □

Definition 3. The supremum of all cardinalities of antichains in X will be called the *width* of X.

Example 2. The space $\omega \times \omega$ with the pointwise ordering (i.e. $(n_1, m_1) \leq (n_2, m_2)$ iff $n_1 \leq n_2$ and $m_1 \leq m_2$) only has finite antichains, but it has countable width.

Suppose the elements $(n_k, m_k)_{k\in\omega}$ form an antichain. Define N as $\{k \in \omega \mid n_k \leq n_0\}$ and M as $\{k \in \omega \mid m_k \leq m_0\}$. Since all elements (n_k, m_k) are incomparable the set $N \cup M$ must be equal to ω. This means that either N or M must be infinite. Let us assume that N is an infinite set. This yields that there is an $n \leq n_0$ such that there is an infinite number of elements (n_k, m_k) that satisfy $n_k = n$. This would of course imply that the elements $(n_k, m_k)_{k\in\omega}$ do not form an antichain. Hence we can conclude that each antichain must be finite.

On the other hand, the subset $A_k = \{(n, m) \in \omega \times \omega \mid n + m = k\}$ is clearly an antichain with $k + 1$ elements, so $\omega \times \omega$ has countable width.

Definition 4. Let β be an ordinal. For a map $\Lambda : \beta \to X$ we define $\lambda_\Lambda(x)$ as $\min\{\gamma \in \beta \mid x \not\leq \Lambda(\gamma)\}$ and $\lambda^*_\Lambda(x)$ as $\min\{\gamma \in \beta \mid x \leq \Lambda(\gamma)\}$.

Definition 5. Let Λ be a map from an ordinal β to X. Define the relation V^α_Λ, with $\alpha \in \beta$, such that $V^\alpha_\Lambda(x)$ is equal to the union of $U_{\leq}(x)$ and the set of all $y \in X$ for which there is an even α' that is greater than or equal to α and satisfies the properties $x \leq \Lambda(\alpha')$ and $\Lambda(\alpha' + 1) \leq y$.

Lemma 2. *If β is an ordinal and $\Lambda : \beta \to X$ is strictly decreasing, then V^α_Λ is a transitive relation.*

Proof. Suppose (x, y) and (y, z) are both elements of the relation V^α_Λ. If either $x \leq y$ or $y \leq z$, then it is easy to see that $z \in V^\alpha_\Lambda(x)$.

Now let us take a look at the situation where $x \not\leq y$ and $y \not\leq z$. This means that we can find an even ordinal $\alpha' \geq \alpha$ such that $x \leq \Lambda(\alpha')$ and $\Lambda(\alpha' + 1) \leq y$ and an even ordinal $\alpha'' \geq \alpha$ such that $y \leq \Lambda(\alpha'')$ and $\Lambda(\alpha''+1) \leq z$. First of all this implies that $x \leq \Lambda(\alpha')$ and $y \leq \Lambda(\alpha'')$. Moreover, we have that $\Lambda(\alpha' + 1) \leq y \leq \Lambda(\alpha'')$. Since both α' and α'' are both even and Λ is strictly decreasing we obtain that $\Lambda(\alpha'+1) \leq \Lambda(\alpha''+1)$. This yields $\Lambda(\alpha'+1) \leq \Lambda(\alpha''+1) \leq z$ and thus $z \in V^\alpha_\Lambda(x)$. □

It follows from the definition that $V^\alpha_\Lambda \supseteq V^{\alpha'}_\Lambda$ whenever $\alpha \leq \alpha'$. This implies that the sets V^α_Λ form a filter basis on $X \times X$ that consists of transitive relations. The filter generated by these sets is therefore a transitive quasi-uniformity.

Definition 6. Define $\mathcal{V}_\Lambda$ as the transitive quasi-uniformity on X generated by the entourages V^α_Λ with $\alpha \in \beta$.

The construction of this quasi-uniformity is based on the quasi-uniformity on ω in example 1. It was in fact this example that led to the ideas behind the main results of this article.

Lemma 3. *Let Λ be a strictly decreasing map from an ordinal β to X and A a subset of X. If A does not contain an antichain A' for which* $\sup\{\lambda_\Lambda(x) \mid x \in A'\}$ *is equal to β, then we can find an $\alpha \in \beta$ such that for each $x \in A$ with $\alpha < \lambda_\Lambda(x)$ there exists a $y \in A$ with $\lambda(y) < \lambda(x)$ that satisfies $x \leq y$.*

Proof. Suppose that for each $\alpha \in \beta$ there is an $x \in A$ with $\alpha < \lambda_\Lambda(x)$ such that for each $y \in A$ with $\lambda_\Lambda(y) < \lambda_\Lambda(x)$ it holds that $x \not\leq y$. Choose an $x_0 \in A$ such that $0 < \lambda_\Lambda(x_0)$ and therefore $x_0 \leq \Lambda(0)$. Assume that for some $\gamma \in \beta$ we have found a family $(x_\alpha)_{\alpha\in\gamma}$ of elements in A such that $x_{\alpha'} \not\leq x_\alpha$ and $\lambda_\Lambda(x_\alpha) < \lambda_\Lambda(x_{\alpha'})$ whenever $\alpha < \alpha'$.

Suppose that $\sup\{\lambda_\Lambda(x_\alpha) \mid \alpha \in \gamma\}$ is not equal to β. Because of our initial assumption we can find an $x_\gamma \in A$ with $\sup\{\lambda_\Lambda(x_\alpha) \mid \alpha \in \gamma\} < \lambda_\Lambda(x_\gamma)$ and such that for each $y \in A$ with $\lambda_\Lambda(y) < \lambda_\Lambda(x_\gamma)$ it holds that $x_\gamma \not\leq y$. This means that $x_\gamma \not\leq x_\alpha$ and $\lambda_\Lambda(x_\alpha) < \lambda_\Lambda(x_\gamma)$ whenever $\alpha < \gamma$.

Using transfinite induction we obtain an indexed family $(x_\alpha)_{\alpha\in\gamma_0}$ in A such that the supremum of all $\lambda_\Lambda(x_\alpha)$ with $\alpha \in \gamma_0$ is equal to β. By construction we have that $x_{\alpha'} \not\leq x_\alpha$ whenever $\alpha < \alpha'$.

Now suppose that $x_\alpha \leq x_{\alpha'}$. This means that for each $\gamma \in \beta$ we have $x_{\alpha'} \not\leq \Lambda(\gamma)$ if $x_\alpha \not\leq \Lambda(\gamma)$ and therefore $\lambda_\Lambda(x_{\alpha'}) \leq \lambda_\Lambda(x_\alpha)$. This contradicts the fact that $\lambda_\Lambda(x_\alpha) < \lambda_\Lambda(x_{\alpha'})$ whenever $\alpha < \alpha'$. Hence we obtain that distinct elements in the family $(x_\alpha)_{\alpha\in\gamma_0}$ are incomparable and that the subset of all elements x_α is an antichain. □

Proposition 12. *Let β be an ordinal and $\Lambda : \beta \to X$ a strictly decreasing function. If X does not contain an antichain Y such that* $\sup\{\lambda_\Lambda(y) \mid y \in Y\}$ *equals β, then $\mathcal{V}_\Lambda$ is QH-equivalent with $\mathcal{U}_\leq$.*

Proof. It is clear that $\mathcal{V}_\Lambda$ is coarser than $\mathcal{U}_\leq$. Now take a subset A of X. In case $\sup\{\lambda_\Lambda(x) \mid x \in A\}$ is strictly smaller than β we have that $V^\alpha_\Lambda(x) = U_\leq(x)$, with α equal to $\sup\{\lambda_\Lambda(x) \mid x \in A\}$, for all $x \in A$. This implies that V^α_Λ satisfies the conditions of proposition 2.

Now suppose that $\sup\{\lambda_\Lambda(x) \mid x \in A\}$ is equal to β. By assumption A cannot contain an antichain A' such that $\sup\{\lambda_\Lambda(x) \mid x \in A'\}$ equals β. Using the previous proposition we obtain that there is an $\alpha \in \beta$ such that for each $x \in A$ with $\alpha < \lambda_\Lambda(x)$ there exists a $y \in A$ with $\lambda_\Lambda(y) < \lambda_\Lambda(x)$ that satisfies $x \leq y$. We will show that V^α_Λ satisfies the conditions of proposition 2.

Take a $y \in V^\alpha_\Lambda(A)$. We want to show that y is an element of $U_\leq(A)$. Choose a $z \in A$ such that $y \in V^\alpha_\Lambda(z)$. If $z \leq y$, then there is nothing left to prove, so we will assume that this is not the case. This means that we can find an even $\alpha' \in \beta$ such that $\Lambda(\alpha'+1) \leq y$ and $\alpha \leq \alpha' < \lambda_\Lambda(z)$. Because $\sup\{\lambda_\Lambda(x) \mid x \in A\} = \beta$ we know that there is an $x \in A$ with the property $\lambda_\Lambda(x) > \alpha' + 1$ and thus $x \leq \Lambda(\alpha'+1)$. This implies that $x \leq y$ and that $y \in U_\leq(A)$.

Let z be an element of A. To complete this proof we need to show that there is a $y \in A$ such that $V^\alpha_\Lambda(y) \subseteq U_\leq(z)$. If $\lambda_\Lambda(z) \leq \alpha$, then $V^\alpha_\Lambda(z) = U_\leq(z)$ so we can simply choose y to be equal to z. In case $\lambda_\Lambda(z) > \alpha$ there must be a $y \in A$ such that $\lambda_\Lambda(y) < \lambda_\Lambda(z)$ and $z \leq y$. Take an element $y' \in V^\alpha_\Lambda(y)$. If $y \leq y'$, then we have $z \leq y \leq y'$ and thus $y' \in U_\leq(z)$. If $y \not\leq y'$, then $\Lambda(\alpha'+1) \leq y'$ for some even α' with the property $\alpha \leq \alpha' < \lambda_\Lambda(y)$. Because $\alpha' < \lambda_\Lambda(y) < \lambda_\Lambda(z)$ we know that $z \leq \Lambda(\alpha'+1)$ and thus $z \leq y'$. □

Theorem 1. *Let $(X, \leq)$ be a QH-singular partially ordered space. If $C \subseteq X$ is a chain, then there is an antichain Y such that $|Y|$ is at least the coinitiality of C.*

Proof. Denote the coinitiality of C as β. If β is finite, then it must be equal to 1 because C is a chain. In this case the proposition is obviously true. If β is infinite, then it is an infinite cardinal and thus a limit ordinal. The quasi-uniformity $\mathcal{V}_\Lambda$ is distinct from $U_{\leq}$. For each $\alpha \in \beta$ we can take an even $\alpha' \in \beta$ that is greater than or equal to α. We now have that $\Lambda(\alpha'+1) \in V_\Lambda^\alpha(\Lambda(\alpha))$, but because Λ is strictly decreasing we know that $\Lambda(\alpha'+1) \notin U_{\leq}(\Lambda(\alpha))$.

Choose a coinitial well-ordered subset C' of C such that $|C'|$ is equal to β. Define $\Lambda : \beta \to X$ as the unique decreasing function that maps β bijectively onto C'. Since $(X, \leq)$ is QH-singular the previous proposition implies that there is an antichain A such that $\sup\{\lambda_\Lambda(y) \mid y \in C'\}$ is equal to β.

Choose a family $(a_i)_{i \in I}$ in A with the property that $\lambda_\Lambda(a_i) \neq \lambda_\Lambda(a_j)$ whenever $i \neq j$ and such that $\sup\{\lambda_\Lambda(a_i) \mid i \in I\} = \beta$. The set $\{\lambda_\Lambda(a_i) \mid i \in I\}$ is by definition cofinal in β. Because β is the coinitiality of C it is a regular cardinal. This means that the cardinal number of $\{\lambda_\Lambda(a_i) \mid i \in I\}$ is β and thus $\beta \leq |A|$. ☐

Using the same techniques as in the previous results we can now prove a similar theorem about the cofinallity of chains in QH-singular partially ordered spaces.

Definition 7. Let Λ be a map from an ordinal β to X. Define the relation W_Λ^α, with $\alpha \in \beta$, such that $W_\Lambda^\alpha(x)$ is equal to the union of $U_{\leq}(x)$ and the set of all $y \in X$ for which there is an even α' that is greater than or equal to α and satisfies the properties $x \leq \Lambda(\alpha'+1)$ and $\Lambda(\alpha) \leq y$.

Lemma 4. *If β is an ordinal and $\Lambda : \beta \to X$ a strictly increasing, then W_Λ^α is a transitive relation.*

Proof. The proof of this result is analogous to that of lemma 2. ☐

Definition 8. Define $\mathcal{W}_\Lambda$ as the transitive quasi-uniformity on X generated by the entourages W_Λ^α with $\alpha \in \beta$.

Lemma 5. *Let β be an ordinal and $\Lambda : \beta \to X$ a strictly increasing function. If $A \subseteq X$ does not contain an antichain A' such that*

$$\sup\{\lambda_\Lambda^*(x) \mid x \in A'\} = \beta,$$

then we can find an $\alpha \in \beta$ such that for each $x \in A$ with $\alpha < \lambda_\Lambda^*(x)$ there exists a $y \in A$ with $\lambda_\Lambda^*(x) < \lambda_\Lambda^*(y)$ that satisfies $x \leq y$.

Proof. The proof of this result is analogous to that of lemma 3. ☐

Proposition 13. *Let β be an ordinal and $\Lambda : \beta \to X$ a strictly increasing function. If X does not contain an antichain Y such that $\sup\{\lambda_\Lambda^*(y) \mid y \in Y\}$ is equal to β, then $\mathcal{W}_\Lambda$ is QH-equivalent with $\mathcal{U}_{\leq}$.*

Proof. The quasi-uniformity $\mathcal{W}_\Lambda$ is clearly coarser than $\mathcal{U}_{\leq}$. Once more we will use proposition 2 to prove that these quasi-uniformities are actually QH-equivalent. Let A be a subset of X. Suppose that the supremum of $\{\lambda_\Lambda^*(x) \mid x \in A\}$ is not equal to β. Choose an $\alpha \in \beta$ such that $\lambda_\Lambda^*(x) < \alpha$ for each $x \in A$. Whenever $\alpha \leq \alpha'+1$ we have $\lambda_\Lambda^*(x) \leq \alpha'$ for each $x \in A$ and thus $U_{\leq}(\Lambda(\alpha')) \subseteq U_{\leq}(x)$. This implies that for each element $x \in A$ the set $W_\Lambda^\alpha(x)$ is equal to $U_{\leq}(x)$.

Let us now assume that the supremum $\{\lambda^*_\Lambda(x) \mid x \in A\}$ is indeed equal to β. Choose an arbitrary $\alpha_1 \in \beta$ for which there is an $x_1 \in A$ such that $x_1 \leq \Lambda(\alpha_1)$ and use the previous proposition to obtain an $\alpha_2 \in \beta$ with the property that for each $x \in A$ with $\alpha_2 < \lambda^*_\Lambda(x)$ there exists a $y \in A$ with $\lambda^*_\Lambda(x) < \lambda^*_\Lambda(y)$ that satisfies $x \leq y$. Define α_0 as the maximum of α_1 and α_2.

To prove that $W^{\alpha_0}_\Lambda(A) \subseteq U_\leq(A)$ take a $y \in A$ and a $z \in W^{\alpha_0}_\Lambda(y)$. If $y \leq z$ there is nothing left to prove, so let us assume that this is not the case. This means that there is an even $\alpha' \in \beta$ such that $\alpha_0 \leq \alpha'$, $\lambda^*_\Lambda(y) \leq \alpha' + 1$ and $\Lambda(\alpha') \leq z$. Because $\alpha_1 \leq \alpha'$ we have $x_1 \leq \Lambda(\alpha_1) \leq \Lambda(\alpha') \leq z$ and therefore we obtain that $z \in U_\leq(A)$.

Finally we need to show that for each $z \in A$ there is a $y \in A$ that satisfies $W^{\alpha_0}_\Lambda(y) \subseteq U_\leq(z)$. Take $z \in A$. If $\lambda^*_\Lambda(z) \leq \alpha_0$, then $z \leq \Lambda(\alpha')$ for each even α' that is greater than α_0 and thus $W^{\alpha_0}_\Lambda(z) = U_\leq(z)$. This means that we can choose y to be equal to z. If $\alpha_0 < \lambda^*_\Lambda(z)$ then we know that there is a $y \in A$ with $\lambda^*_\Lambda(z) < \lambda^*_\Lambda(y)$ and $z \leq y$. If $y \not\leq x$ and $x \in W^{\alpha_0}_\Lambda(y)$, then there is an even α' that is greater than or equal to α_0 such that $\lambda^*_\Lambda(y) \leq \alpha' + 1$ and $\Lambda(\alpha') \leq x$. Since $\lambda^*_\Lambda(z) < \lambda^*_\Lambda(y)$ we know that $\lambda^*_\Lambda(z) \leq \alpha'$ and thus $z \leq \Lambda(\alpha') \leq x$. Hence we can conclude that $W^{\alpha_0}_\Lambda(y) \subseteq U_\leq(z)$. □

Theorem 2. *Let $(X, \leq)$ be a QH-singular partially ordered space. If $C \subseteq X$ is a chain, then there is an antichain Y such that $|Y|$ is at least the cofinality of C.*

Proof. The proof of this result is analogous to the proof of theorem 1. □

Example 3. It follows from the previous theorem that the space $\omega\times\omega$ from example 2 is not QH-singular. It is clear that the set $(n, 0)_{n\in\omega}$ is a countable chain, but we already saw that $\omega \times \omega$ only has finite antichains.

Theorem 3. *If $(X, \leq)$ is a QH-singular partially ordered set, then both the coinitiality and cofinality of each chain in X are less than or equal to the width of X.*

Proof. This follows from theorems 1 and 2. □

Example 4. We will define the partial order relation $\preceq$ on $\omega \times \omega$ such that $(n_1, m_1) \preceq (n_2, m_2)$ iff $n_1 = n_2$ and $m_1 \leq m_2$. The space $\omega \times \omega$ endowed with this particular partial order is not QH-singular. If it were QH-singular, then it would also be transitively QH-singular. This would imply that the subspace $\{(0, m) \mid m \in \omega\}$, which is a downset, would also be transitively QH-singular according to proposition 3. The subspace $\{(0, m) \mid m \in \omega\}$, however, is clearly order isomorphic to the ordinal ω and we already saw in the previous section that the latter is in fact not transitively QH-singular.

The partially ordered space $(\omega \times \omega, \preceq)$ does in fact satisfy the conditions stated in the previous theorem. The subspace $\{(n, 0) \mid n \in \omega\}$ is an antichain, so the width of this space is at least countable. Moreover, it is clear that each chain is contained in a subset $\{(n_0, m) \mid m \in \omega\}$ for some n_0. This means that both the coinitiality and cofinality of each chain are less than or equal to the width of X

Proposition 14. *If $(X, \leq)$ is QH-singular and totally ordered, then $(X, \leq)$ is finite.*

Proof. Since $(X, \leq)$ is totally ordered its width is equal to 1. From the previous proposition we obtain that the coinitiality and cofinality of each chain in X are at most 1. Therefore $(X, \leq)$ cannot contain any infinite increasing or decreasing sequences and must be finite. □

References

[1] J.R. Isbell, *Uniform spaces*, Mathematical Surveys **12**, American Mathematical Society (Providence, R.I. 1964).

[2] A. Bouziad, Uniformities with the same Hausdorff hypertopology, *Top. Appl.* **156** (2009), 1315-1326.

[3] J. Cao, H. Künzi, I. Reilly, Hausdorff quasi-uniformities inducing the same hypertopologies, *Publ. Math. Debrecen* **67** (2005), 27-40.

[4] P. Fletcher, W. Lindgren, *Quasi-uniform spaces*, Lecture Notes in Pure and Applied Mathematics **77** (Marcel Dekker, Inc., 1982).

[5] H. Künzi, Iterations of quasi-uniform hyperspace constructions, *Acta Math. Hungar.* **113** (2006), 213-225.

[6] M. Rosa, P. Vitolo, A question related to the Isbell problem, *Top. Appl.* **160** (2013), 1829-1848.

[7] D. Smith, Hyperspaces of a uniformisable space, *Proc. Camb. Phil. Soc.* **62** (1966), 25-28.

[8] T. Vroegrijk, Proximally well-monotone covers and QH-singularity, *Acta Math. Hung.* **148** (2016), 437-449.

[9] A. Ward, On H-equivalence of uniformities, *Pacific J. Math.* **22** (1967), 189-196.

[10] A. Ward, On H-equivalence of uniformities II, *Pacific J. Math.* **28** (1969), 207-215.