# ANALYSIS OF MODEL EQUATIONS FOR STRESS-ENHANCED DIFFUSION IN COAL LAYERS. PART I: EXISTENCE OF A WEAK SOLUTION*

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**Abstract.** This paper is motivated by the study of the sorption processes in the coal. They are modeled by a nonlinear degenerate pseudoparabolic equation for stress-enhanced diffusion of carbon dioxide ($CO_2$) in coal, $\partial_t \phi = \partial_x \left\{ D(\phi) \partial_x \phi + \frac{D(\phi)\phi}{B} \partial_x \left( e^{-m\phi} \partial_t \phi \right) \right\}$, where $B, m$ are positive constants and the diffusion coefficient $D(\phi)$ has a small value when the $CO_2$ volume fraction $\phi$ is $0 \leq \phi < \phi_c$, representative of coal in the glass state and orders of magnitude higher value for $\phi > \phi_c$, when coal is in the rubber-like state. These types of equations arise in a number of cases when nonequilibrium thermodynamics or extended nonequilibrium thermodynamics is used to compute the flux. For this equation, existence of the travelling wave–type solutions was extensively studied. Nevertheless, the existence seems to be known only for a sufficiently short time. We use the corresponding entropy functional in order to get existence, for any time interval, of an appropriate weak solution with square integrable first derivatives and satisfying uniform $L^\infty$-bounds. Due to the degeneracy, we obtain square integrability of the mixed second order derivative only in the region where the concentration $\phi$ is strictly positive. In obtaining the existence result it was crucial to have the regularized entropy as unknown for the approximate problem and not the original unknown (the concentration).

**Key words.** degenerate pseudoparabolic equation, entropy methods, stress-enhanced diffusion

**AMS subject classifications.** 35K70, 35K65, 76R50, 80A17

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**1. Introduction.** One of the promising methods for reducing the discharge of the "greenhouse gas" carbon dioxide ($CO_2$) into the atmosphere is its sequestration in unminable coal seams. A typical procedure is the injection of $CO_2$ via deviated wells drilled inside the coal seams. $CO_2$ displaces the methane adsorbed on the internal surface of the coal. A production well gathers the methane as free gas. This process, known as $CO_2$-enhanced coal bed methane production ($CO_2$-ECBM), is a producer of energy and at the same time reduces greenhouse concentrations as about two $CO_2$ molecules displace one molecule of methane. Worldwide application of ECBM can reduce greenhouse gas emissions by a few percent. Coal has an extensive fracturing system called the cleat system. In fact, it is possible to discern a number of cleat

systems at different scales. In the end, the matrix blocks between the smallest cleat systems typically have diameters of a few tens of microns [13].

The matrix blocks have a polymeric structure (dehydrated cellulose [32]), which provides the adsorption sites for the gases. At low temperatures or low sorption concentration the coal structure behaves like a rigid glassy polymer, in which movement is difficult. At high temperatures or high sorption concentrations, the glassy structure is converted to the less rigid and open rubber-like (swollen) structure [29], [30]. As coal is less dense in the rubber-like state, a conversion from the glassy state to the rubber-like state exhibits swelling. Therefore modeling of diffusion is not only relevant for modeling transport into the matrix blocks, but also for the modeling of swelling, which affects the permeability of the coal seam.

Ritger and Peppas [29], [30] distinguish between transport by Fickean diffusion and a process that occurs on the interface between the glass state and the rubber-like state. Ritger and Peppas state that the conversion process from the glass to the rubber state is controlled by a rate-limiting relaxation phenomenon (see also [2]). Thomas and Windle [31] (see also [16], [17], [19]), however, suggested in their classic paper that the diffusion transport was enhanced by stress gradients that resulted from the accommodation of large molecules in the small cavities providing the adsorption sites. For this, Alfrey, Gurnee, and Lloyd [1] coined the term superdiffusion or case II diffusion. At a critical concentration of the penetrants the glassy polymer is transformed to a rubber state, where the diffusion coefficient is of the order of a factor 1000 larger than in the glassy state.

This paper is the first of a series in which the model equations for case II diffusion [31], [16], [17], [19] will be analyzed. Our longtime interest is to investigate the one-dimensional sorption rate behavior, i.e., whether the equations indeed lead to a rate faster than the square root of time. In this paper, we establish existence of a weak solution for all times.

Nonlinear diffusion equations with a pseudoparabolic regularizing term being the Laplacian of the time derivative are considered in [25] and [26]. Global existence of a strong solution is proved by writing the problem as a linear elliptic operator, acting on the time derivative, equal to the nonlinear diffusion term. Then the linear elliptic operator, acting on the time derivative, is inverted, and the standard geometric theory of nonlinear parabolic equations (see, e.g., [15]) is applicable.

In our situation the physical model leads to a degenerate nonlinear second order elliptic operator, acting on the time derivative, in place of the Laplacian. The invertibility of this nonlinear elliptic operator is not clear anymore and depends on the solution itself. The same type of equation can occur in models that use classical irreversible thermodynamics (CIT) or extended irreversible thermodynamics (EIT). An important example is the model of the two-phase flow through porous media introduced in [14], where the capillary pressure relation is extended with a dynamic term, which contains the time derivative of the saturation. We also refer the reader to [5] for the modeling. This application to multiphase and unsaturated flows through porous media motivated a number of recent papers. In [18] one finds a detailed study of possible travelling wave solutions and in particular of the behavior of such travelling waves near fronts where the concentration is zero. Further studies of the travelling waves are in [8] and [7]. The small- and waiting-time behavior of the equations is studied in [20]. Study of the viscosity limit for the linear relaxation model of the dynamic term is in [10]. Nevertheless, the study of existence of a solution to the nonlinear model from [14] is undertaken only in [4] and [5], where the nondegeneracy is supposed and existence is local in time. Another existence result, also local in

time, is in the paper [9] by Düll, where a related pseudoparabolic equation modeling solvent uptake in polymeric solids is studied. Düll proved the short-time existence of a solution for the problem in $\mathbb{R}$, supposing nonnegative compactly supported initial datum. Contrary to our approach, the problem was written as a system containing a linear elliptic equation and an evolution equation. With such an approach, we did not manage to get estimates as good as those with the entropy approach undertaken in this paper. For studies of travelling waves and sharp fronts in case II diffusion models, we refer the reader to [17] and [33].

We consider the evolution problem

$$(1) \qquad \partial_t \phi = \partial_x \left\{ D(\phi) \partial_x \phi + \frac{D(\phi)\phi}{B} \partial_x \left( e^{-m\phi} \partial_t \phi \right) \right\} \quad \text{in } (0,L) \times (0,T),$$

$$(2) \qquad D(\phi)\partial_x\phi + \frac{D(\phi)\phi}{B}\partial_x\left(e^{-m\phi}\partial_t\phi\right) = 0 \quad \text{on } \{x = L\} \times (0,T),$$

$$(3) \qquad \phi(0,t) = \phi_g(t) \quad \text{on } (0,T), \quad \phi = 0 \quad \text{on } (0,L) \times \{0\}.$$

Our goal is to obtain a global existence of a weak solution, for any time interval, as was obtained in [3] for a degenerate pseudoparabolic regularization of a nonlinear forward-backward heat equation. Our PDE allows a natural generalization of the classic Kullback entropy, and its integrand is given by

$$(4) \qquad \mathcal{E}(\varphi) = \int_0^\varphi \frac{\varphi - \xi}{\xi} \left( e^{-m\xi} \frac{1}{D(\xi)} - \frac{1}{D(0)} \right) d\xi + \frac{1}{D(0)}(\varphi \log \varphi - \varphi).$$

As in [24], we will use $\mathcal{E}'(\varphi)$ as a test function, with the hope of obtaining a convenient a priori estimate. Formal calculation gives the equality

$$\partial_t \int_0^L \left\{ \mathcal{E}(\phi) - \varphi \mathcal{E}'(\varphi_g) + \frac{1}{2B}(e^{-m\phi}\partial_x\phi)^2 \right\} dx$$

$$(5) \qquad + \int_0^L \left( \frac{1}{\phi} e^{-m\phi}(\partial_x\phi)^2 + \phi \partial_t \mathcal{E}'(\varphi_g) \right) dx = 0.$$

The presence of the initial and boundary conditions leads to unbounded nonintegrable $\mathcal{E}'$. The equality (5) cannot be used directly, and we do not get the entropy estimates as in [12]. We had to obtain an additional estimate for the time derivative, and our calculations are more complicated than in the literature.

Existence is proved by showing that the "energy" of the system remains bounded during the time evolution of the system. The "energy" equation is derived from the differential equation by multiplying with an appropriate test function and integrating over the domain. The choice of the test function depends strongly on the choice of the nonlinearities. With an appropriate approximation, this can also be the basis of a numerical scheme that leads to an implicit first order nonlinear system of ODEs. The implicit dependence on the time derivative makes its solvability nontrivial. Solvability of our system of ODEs depends strongly on the initial conditions. The fact that the "energy" is bounded means that the numerical scheme is stable. If convergence can be proved, it shows that at least one solution exists.

As already stated, in this case an appropriate test function is $\Phi(\phi)$, where

$$\Phi'(\xi) = \frac{e^{-m\xi}}{\xi D(\xi)},$$

is, however, singular for $\xi = 0$. Another problem of the test function is that for large values of $\xi$, $\Phi'$ is exponentially small. In order to prove existence we need $\Phi$ that is bijective from $\mathbb{R}$ to $\mathbb{R}$. Concerning the diffusion coefficient $D$, we extend it by setting $D(\xi) = D(-\xi)$ for $\xi < 0$.

We introduce $\Phi_\delta$ by

$$(6) \qquad \Phi'_\delta := \frac{e^{-m\min\{|\xi|,1/\delta\}}}{(|\xi| + \delta)\, D(\xi)}, \quad \delta > 0,\ \xi \in \mathbb{R},$$

and

$$(7) \qquad \Phi_\delta(\phi) := \begin{cases} \displaystyle\int_0^\phi \frac{e^{-m\min\{\xi,1/\delta\}}}{(\xi+\delta)D(\xi)}\, d\xi, & \phi > 0, \\[2mm] \displaystyle -\int_\phi^0 \frac{e^{-m\min\{-\xi,1/\delta\}}}{(-\xi+\delta)D(-\xi)}\, d\xi, & \phi < 0. \end{cases}$$

Obviously, $\Phi_\delta$ is odd and strictly increasing on $\mathbb{R}_+$.

In order to obtain an existence result for problem (1)–(3) we study the following regularized problem in $Q_T = (0, L) \times (0, T)$:

$$(8) \qquad \partial_t \phi = \partial_x \left\{ D(\phi)\, \partial_x \phi + \frac{D(\phi)(|\phi| + \delta)}{B}\, \partial_x \left( e^{-m\min\{|\phi|,1/\delta\}}\partial_t \phi \right) \right\}$$

with boundary condition at $x = L$,

$$(9) \qquad D(\phi)\, \partial_x \phi + \frac{D(\phi)(|\phi| + \delta)}{B}\, \partial_x \left( e^{-m\min\{|\phi|,1/\delta\}}\partial_t \phi \right) \Big|_{x=L} = 0,$$

and boundary and initial conditions (3).

We start by introducing a variational solution for the problem (8), (9), and (3).

DEFINITION 1. *Let*

$$(10) \qquad \mathcal{V} := \{z \in C^\infty[0, L],\ z|_{x=0} = 0\} \quad and \quad \mathcal{H} := \{C^\infty[0, T],\ h(T) = 0\}.$$

*Then the variational formulation corresponding to the problem (3), (8), and (9) is*

$$-\int_0^T \int_0^L \phi(x, t)\, g(x)\, \partial_t h(t)\, dx\, dt + \int_0^T \int_0^L D(\phi)\, \partial_x \phi(x, t)\, \partial_x g(x) h(t)\, dx\, dt$$

$$(11) \qquad + \int_0^T \int_0^L \frac{D(\phi)(|\phi| + \delta)}{B}\, \partial_x g(x)\, h(t)\, \partial_x \left( e^{-m\min\{|\phi|,1/\delta\}}\partial_t \phi \right) dx\, dt = 0$$

*for all $g \in \mathcal{V}$ and for all $h \in \mathcal{H}$, and at the boundary $x = 0$ we have*

$$(12) \qquad \phi - \phi_g = 0.$$

Our goal is to prove existence for (11)–(12). In order to have the entropy estimate, we should formulate the approximate problem in terms of it. Otherwise it *would not be possible to use it as a test function* for the approximate problem, which is finite dimensional. Getting a priori estimates without this approach is not clear.

Let $z := \Phi_\delta(\phi)$, $\phi = \Phi_\delta^{-1}(z)$, $z|_{x=0} = \Phi_\delta(\phi_g(t))$. We reformulate the problem (3), (8), and (9) in terms of $z$:

$$\frac{1}{\Phi'_\delta\left(\Phi^{-1}_\delta(z)\right)}\partial_t z = \partial_x\left\{\frac{D\left(\Phi^{-1}_\delta(z)\right)}{\Phi'_\delta\left(\Phi^{-1}_\delta(z)\right)}\partial_x z\right.$$

$$(13) \quad + \frac{D\left(\Phi^{-1}_\delta(z)\right)\left(\left|\Phi^{-1}_\delta(z)\right|+\delta\right)}{B}\partial_x\left(D\left(\Phi^{-1}_\delta(z)\right)\left(\left|\Phi^{-1}_\delta(z)\right|+\delta\right)\partial_t z\right)\right\} \quad \text{in } Q_T.$$

Moreover we can express the boundary and initial conditions in $z$ as

$$(14) \qquad z(0,t) = \Phi_\delta\left(\phi_g(t)\right) \quad \text{on } (0,T), \quad z(x, t=0) = \Phi_\delta(0) = 0 \quad \text{on } (0,L),$$

$$(15) \qquad \frac{1}{\Phi'_\delta\left(\Phi^{-1}_\delta(z)\right)}\partial_x z + \frac{\left(\left|\Phi^{-1}_\delta(z)\right|+\delta\right)}{B}\partial_x\left(D\left(\Phi^{-1}_\delta(z)\right)\left(\left|\Phi^{-1}_\delta(z)\right|+\delta\right)\partial_t z\right)$$

$$= 0 \quad \text{at } x = L.$$

Our paper is organized as follows: section 2 describes the physical model, first proposed in [31]. We repeat the derivations from [16], [17], [19] for reasons of easy reference and unified notation.

In section 3 we introduce the discretization of the problem (13)–(15). We get the Cauchy problem for an implicit first order system of ODEs. Next the solvability of the discretized problem is proved, and the uniform $L^2$ a priori estimates for the first derivatives and the mixed second derivative are obtained for a small time interval $(0, T_0)$. They imply the short-time existence for the regularized problem.

We continue with section 4 where we use the entropy to establish the existence of a solution for the regularized problem for all times. Next, we establish $L^\infty$-bounds independent of the regularization parameter.

The last section 5 concerns the existence for the original problem. Using the entropy, the estimates for the time derivative, and the $L^\infty$-bounds again, we are able to pass to the limit when the regularization parameter tends to zero and prove the existence of at least one solution for the original problem.

**2. Model equation for stress-induced diffusion.** Consider a coal particle between the fractured cleat systems in coal. The matrix block can be considered as a small (30 $\mu$m diameter) cubical particle consisting of glassy coal. The coal face is exposed to the penetrant, in our case $CO_2$. The coal face of the particle and the mechanism of the sorption process is shown schematically in Figure 1. The coal originates from a cellulose-like polymeric structure [32], with the chemical formula $C_n(H_2O)_m$, from which part of the hydrogen and oxygen have disappeared during the coalification process, which took millions of years. The remaining structure behaves like a glassy polymer, which contains holes (sites) that can accommodate $CO_2$, $CH_4$, etc. In other words, sorption of gases by coal is more a dissolution process than is adsorption of gases at a coal surface. The holes receiving the $CO_2$ are originally too small to accommodate the molecule and need to expand. Consequently, the expanded hole exerts a stress on the neighboring molecules constituting the polymeric coal. Therefore the penetration of $CO_2$ will lead to both a stress gradient and a concentration gradient. The concentration will be expressed as a volume fraction $\phi$, i.e., $\phi = c/\Omega$, where $c$ is the molecular concentration and $\Omega$ is the molecular volume. As the $CO_2$ likes to move toward a region of smaller stress, the transport of the molecule will be caused by both a concentration gradient and a stress gradient. When the stresses become too high, a deformation occurs, in which the glassy polymeric structure is converted to a rubber-like (swollen) structure, which is much more open. Consequently the diffusion coefficient in the rubber-like structure is much higher (more
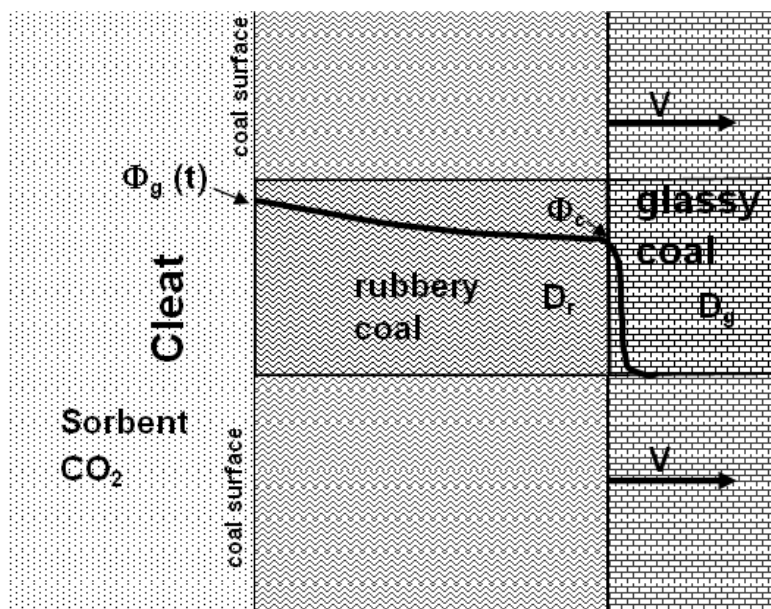
Fig. 1. *A coal face exposed to a sorbent ($CO_2$). To the far right is the virgin coal, which behaves as a glassy polymer. As the sorbent penetrates in the coal, a reorientation of the polymeric coal structure occurs, and the coal becomes rubber-like. The diffusion coefficient in the rubber-like structure is much higher ($> 1000\times$) than in the glassy structure. The rubber-like structure has also a lower density leading to swelling.*

than 1000 times) than the diffusion coefficient in the glassy structure. The stresses are considered to depend on the $CO_2$ concentration in the coal, and conversion to the rubber-like structure occurs instantaneously when a certain critical concentration is exceeded.

These ideas were formulated for the first time by Thomas and Windle [31], and the derivation of the model equations will be explained below.

**2.1. Derivation of model equations.** The salient features of the Thomas and Windle model [31] are well summarized by Hui et al. [16], [17]. We summarize the derivation here with the help of the article by Hui et al. and the book of Landau and Lifshitz [21]; i.e., the molar (diffusive) flux $J$ is not only driven by the volume fraction ($\phi$) (concentration) gradient, but also by the stress ($P_{xx}$) gradient, i.e.,

$$(16) \qquad J = -D\left(\frac{\partial \phi}{\partial x} + \frac{\Omega \phi}{kT}\frac{\partial P_{xx}}{\partial x}\right),$$

where $k$ is the Boltzmann constant. As opposed to the equation in [21], which contains a scalar pressure gradient, the idea here is extended in [19] with the use of the stress gradient $\partial_x P_{xx}$. Hui (see [16], [17]) interprets $P$ as the osmotic pressure. Note that $J$ is the flux of a volume fraction and behaves as a velocity. The diffusion coefficient depends on the concentration. Below a critical volume fraction $\phi_c$, a diffusion coefficient $D_g > 0$ characteristic of a glassy state is used, and above $\phi_c$ the diffusion coefficient $D_r > 0$ characteristic of the rubber (swollen) state is used. It can be expected that $D_r/D_g \gg 1$. In the model an abrupt change of the diffusion coefficients at $\phi_c$ is used, but $D_r$ and $D_g$ are considered constant for $\phi > \phi_c$ and $\phi < \phi_c$, respectively:

$$(17) \qquad D\left(\xi\right) := \begin{cases} D_g, & 0 \leq \xi < \phi_c - \kappa, \\ D_g + (D_r - D_g)(\xi - \phi_c + \kappa)/(2\kappa), & \phi_c - \kappa \leq \xi \leq \phi_c + \kappa, \\ D_r, & \phi_c + \kappa < \xi < +\infty, \end{cases}$$

where $\kappa > 0$ is a small parameter. Extended nonequilibrium thermodynamics [19] suggests that vice versa also the stress $(P_{xx})$ is related to the volumetric flux gradient as

$$(18) \qquad P_{xx} = -\eta_l \frac{\partial J}{\partial x} = \eta_l \frac{\partial \phi}{\partial t},$$

where the second equality follows from a mass conservation law that assumes incompressible flow,

$$(19) \qquad \frac{\partial \phi}{\partial t} + \frac{\partial J}{\partial x} = 0.$$

With $\eta_l$ we denote the elongational velocity [6], i.e., the resistance of movement due to a velocity gradient $\frac{\partial J}{\partial x}$ in the direction of flow. Elongational viscosity is caused by a resistance force of a fluid to accelerate. Hence, the force is proportional to the component of the gradient of the velocity in the flow direction. Elongational viscosity $\eta_l$ is always larger than the shear viscosity $\eta_s$, e.g., in Newtonian fluids $\eta_l = 3\eta_s$. In this case the "fluid $CO_2$" is moving in the coal medium. The resistance to flow is largely determined by the coal-$CO_2$ interaction and not, as in the usual definition of viscosity, as $CO_2$-$CO_2$ interaction. Hence here we deal with an apparent or pseudoviscosity. With increasing $CO_2$ concentration the coal becomes more rubber-like, i.e., it acquires a more open structure, and the apparent viscosity decreases with increasing concentration (see (20)). The elongational viscosity $\eta_l$ is supposed to depend on the volume fraction of the penetrant as

$$(20) \qquad \eta_l = \eta_o \exp\left(-m\phi\right),$$

where $m$ is a material constant and $\eta_0$ is the volumetric viscosity of the unswollen coal sample.

Substituting expression (16) for the flux into the mass balance equation (19), where we also use (18), we obtain

$$(21) \qquad \partial_t \phi = \partial_x \left\{ D\left(\phi\right) \partial_x \phi + \frac{D\left(\phi\right)\phi}{B} \partial_x \left(e^{-m\phi} \partial_t \phi\right) \right\},$$

where the constant $B = k_B T / (\eta_o \Omega)$. This equation is defined in $Q_T = (0, L) \times (0, T)$.

As initial condition we have that the concentration is

$$(22) \qquad \phi\left(x, t = 0\right) = 0 \quad \text{on } (0, T).$$

The boundary condition at $x = 0$ must be derived from thermodynamic arguments. The final equilibrium concentration is reached when the coal has swollen to make the stress $P_{xx} = P_{xx}^0$ equal to zero. In this case the volume fraction of $CO_2$ in the coal is in equilibrium with the $CO_2$ in the fluid phase outside the coal. Also the $CO_2$ in the stressed coal is in equilibrium with the $CO_2$ in the fluid phase. The change in chemical potential is $d\mu = \Omega\, dP_{xx} + k_B T\, d\ln\phi$. Equating the chemical potential in the unstressed and stressed state leads to

$$(23) \qquad \Omega P_{xx} + k_B T \ln\phi = \Omega P_{xx}^0 + k_B T \ln\phi_o,$$

where $\phi_o$ is the volume fraction at the coal boundary that would be in equilibrium with the $CO_2$ in the gas phase if the coal has relaxed to the rubber state with $P_{xx}^0 = 0$.

Substituting (18) and (20) into (23) leads to

$$(24) \qquad t = -\phi_o \frac{\eta_0 \Omega}{k_B T} \int_0^{\phi/\phi_o} \frac{\exp(-m\phi_o y)}{\ln y} \, dy,$$

where we use the initial condition that $\phi = 0$ at $t = 0$. Singularity of the integrand at $y = 1$ guarantees that $\phi$ remains bounded by $\phi_o$ for all times.

At $x = L$ we have the boundary condition on $(0, T)$,

$$(25) \qquad D(\phi) \left( \partial_x \phi + \frac{1}{B} \phi \partial_x \left( \exp(-m\phi) \partial_t \phi \right) \right)_{x=L} = 0.$$

In summary, we have one initial condition equation (22), one boundary condition at $x = L$, viz. (25), and the implicit boundary condition equation (24), which specifies $\phi(x = 0, t)$ as

$$(26) \qquad \phi(0, t) = \phi_g(t).$$

$\phi_g$ satisfies the conditions

$$(27) \qquad 0 \leq \phi_g \leq A_0, \quad \phi_g(0) = 0.$$

*Remark* 2. Equations like (21) can occur in many transport problems in which the flux is calculated using CIT or EIT. A well-known example for CIT in porous media flow is that the deviation of the capillary pressure $P_c$ from its equilibrium value at a given oil saturation $S_o$, i.e., $P_c^o = P_c^o(S_o)$, is a driving force leading to a rate of change of the saturation (scalar flux). This leads [14], [27], [28], [22] to $\partial_t S_o = L(P_c - P_c^o)$ and to the transport equation for counter current imbibition:

$$\varphi \partial_t S_o = \partial_x (\Lambda(S_o) \partial_x P_c)$$
$$= \partial_x (\Lambda(S_o) \partial_x P_c^o(S_o)) + \partial_x \left( \Lambda(S_o) \partial_x \frac{1}{L(S_o)} \partial_t S_o \right).$$

EIT [19] differs from CIT as it characterizes a system not only by its local thermodynamic variables (pressure, temperature, and concentration) but also by its gradients. The explanation in [19] is difficult to follow by nonspecialists as many thermodynamic relations are considered to be known by the reader. In isothermal systems and in the absence of other applied fields, e.g., electric fields, the volumetric flux $J$ is, according to EIT, given by the following system of equations:

$$(28) \qquad \tau_1 \partial_t J + J = -D \left( \frac{\partial \phi}{\partial x} + \frac{\Omega \phi}{kT} \frac{\partial P_{xx}}{\partial x} \right),$$

$$(29) \qquad \tau_2 \partial_t P_{xx} + P_{xx} = -\eta_l \frac{\partial J}{\partial x}.$$

Reference [19] uses a mass flux instead of a volumetric flux and therefore uses a factor $v_1$, being the partial volume per unit mass. Here $\tau_1, \tau_2$ are time constants, which are small with respect to $L^2/D$. The first terms on the left-hand sides of (28) and (29) appear only in EIT and not in CIT. The first terms on the left are of interest for

short-time behavior and are omitted from the model discussed here. Another example from EIT is the Taylor dispersion problem (see equation 10.34 in [19]), where there is an "xxt" derivative in the concentration, apart from many other terms. Hence, EIT or CIT can lead to transport equations of the form of (21).

**3. Short-time existence for the regularized problem.** In this section, we first introduce an approximate problem corresponding to (13)–(15). It is a first order system of ODEs for expansion coefficients, with implicit dependence on the time derivatives. First we prove the solvability on some interval $(0, T_N)$, where $N$ is the parameter describing discretization in space. Then, we use the entropy to prove the solvability on interval $(0, T_0)$, where $T_0$ does not depend on $N$. Finally, we pass to the limit $N \to +\infty$ and prove that the problem (13)–(15) itself has a solution on $(0, T_0)$.

Let $V := \{g \in H^1(0, L) \mid g(0) = 0\}$ be the closure of $\mathcal{V}$ in $H^1(0, L)$, and let $\{\alpha_j\}_{j \in \mathbb{N}}$ be a $C^\infty$-basis for $V$. We set $V_N := \text{span}\{\alpha_1, \ldots, \alpha_N\}$ and introduce the following coefficients:

$$d_1(z) := \frac{1}{\left(\Phi'_\delta\left(\Phi_\delta^{-1}(z)\right)\right)}, \quad d_2(z) := \frac{D\left(\Phi_\delta^{-1}(z)\right)}{\left(\Phi'_\delta\left(\Phi_\delta^{-1}(z)\right)\right)}, \quad \text{and}$$

(30)
$$d(z) := D\left(\Phi_\delta^{-1}(z)\right)\left(\left|\Phi_\delta^{-1}(z)\right| + \delta\right).$$

The coefficients $d_1$ and $d_2$ are continuous, nonnegative, and bounded functions of $z$. $d$ is a continuous function of $z$, bounded away from zero.

We start study of the initial boundary problem (13)–(15) by constructing an approximate solution for every $N$. It is defined as follows.

APPROXIMATE PROBLEM 3. *For $q \in (2, +\infty)$, find $z_N = \sum_{j=1}^{N} c_j(t)\alpha_j(x) + \Phi_\delta(\phi_g(t)) \in W^{1,q}([0, T]; V_N)$ such that*

$$\int_0^L \partial_t z_N d_1(z_N)\alpha_k \, dx + \int_0^L d_2(z_N)\partial_x z_N \partial_x \alpha_k \, dx$$

(31)
$$+ \int_0^L \frac{1}{B}d(z_N)\partial_x(d(z_N)\partial_t z_N)\partial_x \alpha_k \, dx = 0 \text{ for } k = 1, \ldots, N, \quad \text{and}$$

(32)
$$z_N|_{t=0} = P_N\left(z|_{t=0} - \Phi_\delta(\phi_g(0))\right) = 0,$$

*where $P : V \to V_N$ is the projector $P_N(f)(x) := \sum_{j=1}^{N} \alpha_j(x)(f, \alpha_j)_V$.*

Let the vector valued function $\mathbf{F}$ be given by $F_\kappa(t, \mathbf{c}, \partial_t \mathbf{c}) = $ left-hand side of (31) and let $\mathbf{c}$ be the column vector consisting of elements $(c_1(t) \ldots c_N(t))$; then (31), (32) are equivalent to the following Cauchy problem in $\mathbb{R}^N$:

(33)
$$\begin{cases} \mathbf{F}(t, \mathbf{c}, \partial_t \mathbf{c}) = 0, \\ \mathbf{c}|_{t=0} = 0. \end{cases}$$

The Cauchy problem (33) is difficult to solve, since the dependence of $\mathbf{F}$ on $\partial_t \mathbf{c}$ is implicit. It is crucial to reduce it to an ordinary Cauchy problem of the form $\partial_t \mathbf{c} = \varrho(t, \mathbf{c})$.

We note that

$$F_k := \sum_{j=1}^{N} \left\{\int_0^L d_1(z_N)\alpha_k\alpha_j \, dx + \int_0^L \frac{1}{B}d(z_N)\partial_x(d(z_N)\alpha_j)\partial_x\alpha_k \, dx\right\}\frac{dc_j}{dt}$$

(34)
$$+ \sum_{j=1}^{N} \left\{\int_0^L d_2(z_N)\partial_x\alpha_j\partial_x\alpha_k \, dx\right\}c_j + \int_0^L d_1(z_N)\alpha_k\partial_t\Phi_\delta(\phi_g(t)) \, dx.$$

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Then, after introducing the matrices $\mathcal{A}(\mathbf{c})$ and $\mathcal{B}(\mathbf{c})$ and the vector $\mathbf{f}(\mathbf{c})$ by

$$(35) \qquad \mathcal{A}_{kj}(\mathbf{c}) := \int_0^L d_1(z_N)\,\alpha_k\alpha_j\,dx + \int_0^L \frac{1}{B} d(z_N)\,\partial_x(d(z_N)\,\alpha_j)\,\partial_x\alpha_k\,dx,$$

(36)

$$\mathcal{B}_{kj}(\mathbf{c}) := \int_0^L d_2(z_N)\,\partial_x\alpha_j\partial_x\alpha_k\,dx, \quad \text{and} \quad f_k(\mathbf{c}) = \int_0^L d_1(z_N)\,\alpha_k\partial_t\Phi_\delta(\phi_g(t))\,dx,$$

$1 \le k, j \le N$, we see that the problem (31)–(32) is equivalent to the following Cauchy problem:

$$\text{Find } \mathbf{c} \in W^{1,q}(0,T)^N \text{ such that}$$

$$(37) \qquad \mathcal{A}(\mathbf{c})\frac{d\mathbf{c}}{dt} = -\mathcal{B}(\mathbf{c})\mathbf{c} - \mathbf{f}(\mathbf{c}) \text{ a.e. in } (0,T); \quad \mathbf{c}|_{t=0} = 0.$$

PROPOSITION 4. *There is a $T_N > 0$ such that the problem (31)–(32) has a unique solution $z_N \in W^{1,q}(0,T_N;V_N)$ for all $q < +\infty$.*

*Proof.* It is enough to prove that the Cauchy problem (37) has a solution.

Obviously, $\mathcal{A}$, $\mathcal{B}$, and $\mathbf{f}$ are smooth functions of $\mathbf{c}$. Because of the singularity of $\partial_t\varphi_g$ at $t = 0$, $\mathbf{f}(\mathbf{c}) \in L^q(0,T)$ for all $q < +\infty$, but it is not bounded. Hence, the only property to check is the invertibility of the matrix $\mathcal{A}$. Let $\mathbf{b}$ be an arbitrary vector from $\mathbb{R}^N$ and let $b_\alpha(x) = \mathbf{b} \cdot \alpha(x) = \sum_{j=1}^N b_j\alpha_j(x)$. Then we have

$$(\mathcal{A}\mathbf{b}) \cdot \mathbf{b} = \sum_{k,j=1}^N \mathcal{A}_{k,j}b_kb_j = \int_0^L d_1(z_N)(b_\alpha)^2\,dx + \frac{1}{B}\int_0^L d(z_N)\partial_x b_\alpha\partial_x(d(z_N)b_\alpha)\,dx$$

$$= \int_0^L d_1(z_N)(b_\alpha)^2\,dx + \frac{1}{B}\int_0^L (d(z_N)\partial_x b_\alpha)^2\,dx$$

$$+ \frac{1}{B}\int_0^L d(z_N)\partial_x b_\alpha b_\alpha d'(z_N)\partial_x z_N\,dx$$

$$(38) \qquad \ge \int_0^L \left\{d_1(z_N) - \frac{1}{4B}(d'(z_N))^2(\partial_x z_N)^2\right\}(b_\alpha)^2\,dx.$$

Since $\partial_x z_N(x,0) = 0$ and functions $\{\alpha_j\}_{j\in\mathbb{N}}$ are linearly independent, the matrix $\mathcal{A}$ is by (38) invertible in a neighborhood of $t = 0$. Then by the classical theory, the problem (37) has a unique solution on some interval $(0, T_N)$. $\square$

Next, we want to prove that the existence interval does not depend on $N$.

PROPOSITION 5. *There is a constant $C$, independent of $N$, such that*

$$(39) \qquad \|\partial_x z_N\|_{L^\infty(0,T_N;L^2(0,L))} \le C.$$

*Consequently, the vector valued function $\mathbf{c}$ remains bounded at $t = T_N$.*

*Proof.* In (31) we can replace $\alpha_k$ by $z_N - \Phi_\delta(\phi_g)$. Then after using that $\partial_x(d(z_N)\partial_t z_N) = \partial_t(d(z_N)\partial_x z_N)$, we get

$$(40) \qquad \int_0^L d_1(z_N)z_N\partial_t z_N\,dx + \int_0^L d_2(z_N)(\partial_x z_N)^2\,dx$$

$$+ \int_0^L \frac{1}{B}\partial_t(d(z_N)\partial_x z_N)d(z_N)\partial_x z_N\,dx$$

$$= \int_0^L d_1(z_N) \Phi_\delta(\phi_g) \partial_t z_N \, dx = \partial_t \int_0^L \Phi_\delta(\phi_g)(t) \int_0^{z_N} d_1(\xi) \, d\xi \, dx$$

$$- \partial_t \Phi_\delta(\phi_g)(t) \int_0^L \int_0^{z_N} d_1(\xi) \, d\xi \, dx.$$

Integrating over $t$ leads to

$$(41) \qquad \int_0^L \left( \int_0^{z_N(x,t)} d_1(\xi) \xi \, d\xi \right) dx + \int_0^t \int_0^L d_2(z_N)(\partial_x z_N)^2 \, dx \, d\tau$$

$$+ \frac{1}{2B} \int_0^L d(z_N)^2 (\partial_x z_N)^2 \, dx$$

$$= \int_0^L \left( \int_0^{z_N(x,t)} d_1(\xi) \, d\xi \right) dx \, \Phi_\delta(\phi_g)(t) - \int_0^t \partial_\tau \Phi_\delta(\phi_g)(\tau) \left( \int_0^L \int_0^{z_N} d_1(\xi) \, d\xi \, dx \right) d\tau.$$

We easily find out that

$$(42) \qquad \int_0^z d_1(\xi) \xi \, d\xi = \int_0^{\Phi_\delta^{-1}(z)} \Phi_\delta(\eta) \, d\eta \quad \text{and} \quad \int_0^z d_1(\xi) \, d\xi = \Phi_\delta^{-1}(z).$$

The growth of the terms in (42) indicates that it will be possible to control the two terms on the right-hand side of (41) by the first term on the left-hand side of (41).

Let $M_\phi := \max_{0 \le t \le T} |\Phi_\delta(\phi_g(t))|$. Then by the definition of $\Phi_\delta(\varphi)$, we have $C_0(\delta) \log(1 + \varphi/\delta) \le \bar{\Phi}_\delta(\varphi)$ for all $\varphi \ge 0$. Hence $\int_0^z d_1(\xi) \xi \, d\xi \ge C_0(\delta)((|\Phi_\delta^{-1}(z)| + \delta) \log(1+|\Phi_\delta^{-1}(z)|/\delta) - |\Phi_\delta^{-1}(z)|)$, and there is a large enough constant $C_\varphi = C_\varphi(M_\phi, \delta)$ such that $g(z) = C_0(\delta)((|\Phi_\delta^{-1}(z)| + \delta) \log(1 + |\Phi_\delta^{-1}(z)|/\delta) - |\Phi_\delta^{-1}(z)|) - M_\phi |\Phi_\delta^{-1}(z)| + C_\varphi > |\Phi_\delta^{-1}(z)|$ for all $z$. The equality (41) now implies

$$\int_0^L g(z_N(x,t)) \, dx + \int_0^t \int_0^L d_2(z_N)(\partial_x z_N)^2 \, dx \, d\tau + \frac{1}{2B} \int_0^L d(z_N)^2 (\partial_x z_N(t))^2 \, dx$$

$$(43) \qquad \le C_\varphi(M_\phi, \delta) L + \int_0^t |\partial_\tau \Phi_\delta(\phi_g)(\tau)| \left( \int_0^L g(z_N(x,\tau)) \, dx \right) d\tau.$$

Since $\partial_\tau \Phi_\delta(\phi_g) \in L^1(0, T)$, we apply Gronwall's inequality, and estimate (39) follows. Hence **c** remains bounded at $t = T_N$. $\quad \square$

Nevertheless, since the matrix $\mathcal{A}$ could degenerate, some components of $\frac{\partial \mathbf{c}}{\partial t}$ could blow up at $t = T_N$. In order to exclude this possibility and to prove that the maximal solution for (33) exists on $[0, T]$, we need an estimate for the time derivatives. Furthermore, if we want to pass to the limit $N \to +\infty$ in (31), estimate (39) is not sufficient. Our strategy is to obtain an estimate, uniform with respect to $N$, for $\partial_{xt} z_N$ in $L^2(Q_T)$.

THEOREM 6. *There exists $T_0 > 0$, independent of $N$, such that*

$$(44) \qquad\qquad \|\partial_x z_N\|_{L^\infty(0, T_0; L^2(0, L))} \le C,$$

$$(45) \qquad\qquad \|\partial_t z_N\|_{L^2(0, T_0; L^2(0, L))} \le C,$$

$$(46) \qquad\qquad \|\partial_{xt} z_N\|_{L^2(0, T_0; L^2(0, L))} \le C,$$

$$(47) \qquad\qquad \left\| \partial_{xt} \int_0^{z_N} d(\xi) \, d\xi \right\|_{L^2((0, T_0) \times (0, L))} \le C,$$

*with constants independent of $N$. Consequently, the maximal solution for* (33) *exists on* $[0, T_0]$.

*Proof.* We replace $\alpha_k$ in (31) by $\partial_t z_N - \partial_t \Phi_\delta (\phi_g)$. This yields

$$\int_0^L d_1(z_N)(\partial_t z_N)^2 \, dx + \int_0^L d_2(z_N) \partial_x z_N \partial_{xt} z_N \, dx$$

$$(48) \quad + \frac{1}{B} \int_0^L d(z_N) \partial_t (d(z_N) \partial_x z_N) \partial_{xt} z_N \, dx = \int_0^L d_1(z_N) \partial_t z_N \partial_t \Phi_\delta(\phi_g) \, dx.$$

In the estimates which follow we will use the fact that integrability of higher order derivatives implies continuity and boundedness in $x$ or in $t$. We recall that for one-dimensional Sobolev embeddings Morrey's theorem applies and $H^1(0, t)$ (respectively, $H^1(0, L)$) is continuously embedded into the Hölder space $C^{0,1/2}[0, t]$ (respectively, into $C^{0,1/2}[0, L]$). See, e.g., [11] for more details. In our particular situation, we use the explicit dependence of the embedding constant on the length of the time interval and we prefer to derive the estimates directly.

First, as $\partial_x z_N \in L^2\left(0, L; H^1(0, t)\right)$ and $\partial_x z_N|_{\tau=0} = 0$, we have for a.e. $x \in (0, L)$ and for every $\tau \in (0, t)$

$$(49) \quad |\partial_x z_N(x, \tau)| = \left| \int_0^\tau \partial_\xi \partial_x z_N(x, \xi) \, d\xi \right| \le \sqrt{\tau} \sqrt{\int_0^\tau |\partial_\xi \partial_x z_N(x, \xi)|^2 \, d\xi}.$$

Next, as $\partial_\tau z_N \in L^2\left(0, t; H^1(0, L)\right)$ and $\partial_\tau z_N|_{\tau=0} = \partial_\tau \Phi(\phi_g)$, we have for a.e. $\tau \in (0, t)$ and for every $x \in (0, L)$

$$|\partial_\tau z_N(x, \tau)| \le |\partial_\tau \Phi(\phi_g(\tau))| + \left| \int_0^x \partial_\xi \partial_\tau z_N(\xi, \tau) \, d\xi \right|$$

$$(50) \qquad \le |\partial_\tau \Phi(\phi_g(\tau))| + \sqrt{L} \sqrt{\int_0^L |\partial_\xi \partial_\tau z_N(\xi, \tau)|^2 \, d\xi}.$$

Estimates (49)–(50) imply

$$\int_0^L \int_0^t |\partial_\tau z_N(x, \tau)|^2 |\partial_x z_N(x, \tau)|^2 \, dx \, d\tau$$

$$\le 2 \int_0^L \int_0^t \tau \left( \int_0^t |\partial_\xi \partial_x z_N(x, \xi)|^2 \, d\xi \right) \left( |\partial_\tau \Phi(\phi_g(\tau))|^2 + L \int_0^L |\partial_\xi \partial_\tau z_N(\xi, \tau)|^2 \, d\xi \right) d\tau \, dx$$

$$\le 2Lt \|\partial_{x\tau} z_N\|^4_{L^2((0,t)\times(0,L))} + 2 \|\partial_{x\tau} z_N\|^2_{L^2((0,t)\times(0,L))} \int_0^t \tau |\Phi'_\delta(\phi_g)|^2 |\partial_\tau \phi_g|^2 \, d\tau$$

$$(51) \quad \le \left( 2\sqrt{L}\sqrt{t} \|\partial_{x\tau} z_N\|^2_{L^2((0,t)\times(0,L))} + \frac{1}{\sqrt{2L}} \int_0^t \tau^{1/2} |\Phi'_\delta(\phi_g)|^2 |\partial_\tau \phi_g|^2 \, d\tau \right)^2.$$

Now we integrate (48) with respect to time, over $(0, t)$, and estimate the obtained terms. The second term is estimated as follows:

$$\left| \int_0^t \int_0^L d_2(z_N) \partial_x z_N \partial_{x\tau} z_N \, dx \, d\tau \right| \le C \int_0^t \|\partial_{x\tau} z_N(\tau)\|_{L^2(0,L)} \|\partial_x z_N(\tau)\|_{L^2(0,L)} \, d\tau$$

$$(52)$$

$$\le C \sqrt{\int_0^t \|\partial_{x\tau} z_N(\tau)\|^2_{L^2(0,L)} \, d\tau} \sqrt{\int_0^t \int_0^L (\partial_x z_N)^2 \, dx \, d\tau} \le C \|\partial_{x\tau} z_N\|_{L^2((0,t)\times(0,L))},$$

where we have used the estimate (39). We rewrite the third term of (48), omitting the $1/B$ factor, as

$$\int_0^t \int_0^L d(z_N)\partial_\tau\big(d(z_N)\partial_x z_N\big)\partial_{x\tau}z_N \, dx \, d\tau = \int_0^t \int_0^L d(z_N)\left(\partial_{x\tau}z_N\right)^2 dx \, d\tau$$

$$(53) \qquad\qquad + \int_0^t \int_0^L d(z_N)d'(z_N)\partial_\tau z_N \partial_x z_N \partial_{x\tau}z_N \, dx \, d\tau.$$

The last term in (53) is cubic in derivatives of $z_N$. Our idea is to use the estimate (51), showing that for small times it enters with a small coefficient and then controlling it using other terms. Using the estimate (51), we find out that it satisfies the following inequality:

$$\left|\int_0^t \int_0^L d(z_N)\partial_{x\tau}z_N \, d'(z_N)\partial_\tau z_N \partial_x z_N \, dx \, d\tau\right| \le C \left\|\partial_{x\tau}z_N\right\|_{L^2(Q_t)} \left\|\partial_\tau z_N \partial_x z_N\right\|_{L^2(Q_t)}$$

$$(54) \qquad\qquad \le C\sqrt{t}\left(\left\|\partial_{x\tau}z_N\right\|^3_{L^2(Q_t)} + \left\|\tau^{1/6}\Phi'_\delta(\phi_g)\partial_\tau\phi_g\right\|^3_{L^3(0,t)}\right),$$

where $Q_t = (0,t) \times (0,L)$. Let $X^2(t) := \int_0^t \int_0^L |\partial_{x\tau}z_N|^2 dx \, d\tau$. Since $\partial_\tau\Phi_g(\phi_g) \in L^2(0,t)$, estimates (39), (52), (53), and (54) imply

$$(55) \qquad \left\|\sqrt{d_1(z_N)}\partial_\tau z_N\right\|^2_{L^2((0,t)\times(0,L))} + X^2(t) - C_1\sqrt{t}X^3(t) \le C_o,$$

where $C_o$ depends on $\left\|\partial_\tau\Phi_\delta(\phi_g)\right\|_{L^2(0,t)}$ and on the constant from estimate (39). We note that the last term on the left-hand side corresponds to the lower bound for the cubic term, corresponding to the stress gradient part of the diffusive flux. Inequality (55) is satisfied for $t = 0$. The function $\varrho(X) = X^2 - C_1\sqrt{t}X^3$ has its maximum on $(0,+\infty)$ in the point $X_o = 3/\left(2C_1\sqrt{t}\right)$. If $C_o < \varrho(X_o)$, then inequality (55) gives an estimate for $X(t)$. We note that $C_o < \varrho(X_o)$ if $t < \frac{4}{27C_1^2 C_o}$. Hence for $T \le \frac{4}{27C_1^2 C_o} = T_0$ we have estimates (44)–(46).

From (44)–(46) it follows that $\partial_x z_N \partial_t z_N \in L^2\left(0,T_0; L^2(0,L)\right) \le C$, and we have (47) as well. ☐

The estimates (44)–(47) allow us to pass to the limit $N \to +\infty$. Using classical compactness and weak compactness arguments and due to the a priori estimates (44)–(47), we can extract a subsequence of $z_N$, denoted by the same subscripts, which converges to an element $z \in H^1((0,T_0) \times (0,L))$, $\partial_{xt}z \in L^2((0,T_0) \times (0,L))$, in the following sense:

$$(56) \qquad z_N \to z \quad \text{strongly in } L^2((0,T_0) \times (0,L)) \quad \text{and a.e. on } (0,T_0) \times (0,L),$$

$$(57) \qquad\qquad \partial_x z_N \rightharpoonup \partial_x z \quad \text{weakly in } L^2((0,T_0) \times (0,L)),$$

$$(58) \qquad\qquad \partial_t z_N \rightharpoonup \partial_t z \quad \text{weakly in } L^2((0,T_0) \times (0,L)),$$

$$(59) \qquad\qquad \partial_{xt}z_N \rightharpoonup \partial_{xt}z \quad \text{weakly in } L^2((0,T_0) \times (0,L)),$$

$$(60) \qquad \partial_{xt}\int_0^{z_N} d(\xi)\,d\xi \rightharpoonup \partial_{xt}\int_0^z d(\xi)\,d\xi \quad \text{weakly in } L^2((0,T_0) \times (0,L)).$$

Now passing to the limit $N \to \infty$ in (31) does not pose problems, and we conclude that $z$ satisfies (13)–(15).

We summarize the results in the following theorem.

THEOREM 7. *Let $\phi_g \in H^1(0,T)$. Then there exists $T_0 > 0$ such that problem (13)–(15) has at least one variational solution $z \in H^1((0,T_0) \times (0,L))$, $\partial_{xt}z \in L^2((0,T_0) \times (0,L))$.*

COROLLARY 8. *Let $\phi_g \in H^1(0,T)$. Then there exists $T_0 > 0$ such that the variational formulation (11)–(12) of the problem (8), (9), and (3) has at least one solution $\phi = \Phi_\delta^{-1}(z) \in H^1((0,T_0) \times (0,L))$, $\partial_{xt}\phi \in L^2((0,T_0) \times (0,L))$.*

**4. Existence of the regularized problem.** In this section, we first use the regularized entropy function to prove the global existence for the problem (8), (9), and (3) (i.e., for the regularized problem). Then we establish the $L^\infty$-bounds for the solution, independent of the regularization parameter.

Let us prove that any solution $\phi$ for the problem (8), (9), and (3), constructed in Corollary 8, could be extended from $(0,T_0)$ to arbitrary time interval $(0,T)$. First we test (11) by $\Phi_\delta(\phi) - \Phi_\delta(\phi_g(t))$. We have

$$\int_0^t \int_0^L \partial_\tau \phi \Phi_\delta(\phi) \, dx \, d\tau + \int_0^t \int_0^L D(\phi) \partial_x \phi \partial_x \Phi_\delta(\phi) \, dx \, d\tau$$

$$+ \int_0^t \int_0^L \frac{D(\phi)(|\phi| + \delta)}{B} \partial_x \left( e^{-m \min\{|\phi|, 1/\delta\}} \partial_\tau \phi \right) \partial_x \Phi_\delta(\phi) \, dx \, d\tau$$

$$= \int_0^t \int_0^L \partial_\tau \phi \Phi_\delta(\phi_g) \, dx \, d\tau,$$

and it follows that

$$\int_0^L \left( \int_0^{\phi(t)} \Phi_\delta(\xi) \, d\xi \right) dx + \int_0^t \int_0^L \frac{1}{2B} \partial_\tau \left( e^{-m \min\{|\phi|, 1/\delta\}} \partial_x \phi \right)^2 dx \, d\tau$$

$$+ \int_0^t \int_0^L D(\phi) \Phi_\delta'(\phi) (\partial_x \phi)^2 \, dx \, d\tau = \int_0^L \phi(t) \Phi_\delta(\phi_g(t)) \, dx$$

$$- \int_0^t \int_0^L \phi \partial_\tau \Phi_\delta(\phi_g) \, dx \, d\tau$$

and we get as in the proof of Proposition 5

(61) $$\|\partial_x \phi\|_{L^\infty(0,t;L^2(0,L))} \le C.$$

This estimate implies the boundedness of $\phi$. We note that $C$ does not depend on the smoothing of $D$ at $\phi = \phi_c$.

Next we test (11) by

$$e^{-m \min\{|\phi|, 1/\delta\}} \partial_t \phi - e^{-m \min\{|\phi_g|, 1/\delta\}} \partial_t \phi_g$$

and get

$$\int_0^t \int_0^L (\partial_\tau \phi)^2 \, e^{-m \min\{|\phi|, 1/\delta\}} \, dx \, d\tau$$

$$+ \int_0^t \int_0^L D(\phi) \partial_x \phi \partial_x \left( e^{-m \min\{|\phi|, 1/\delta\}} \partial_\tau \phi \right) dx \, d\tau$$

$$+ \int_0^t \int_0^L \frac{D(\phi)(|\phi| + \delta)}{2B} \left( \partial_{x\tau} \int_0^\phi e^{-m \min\{|\xi|, 1/\delta\}} \, d\xi \right)^2 dx \, d\tau$$

$$= \int_0^t \int_0^L \partial_\tau \phi \, e^{-m \min\{|\phi_g|, 1/\delta\}} \partial_\tau \phi_g \, dx \, d\tau.$$

Then, as in the proof of Proposition 5, by estimating the second and the fourth terms and after using (61), we conclude that

(62) $$\|\partial_\tau \phi\|_{L^2((0,t) \times (0,L))} \leq C,$$

(63) $$\left\| \partial_{x\tau} \int_0^\phi e^{-m \min\{|\xi|, 1/\delta\}} \, d\xi \right\|_{L^2((0,t) \times (0,L))} \leq C,$$

and from this it follows that

(64) $$\|\partial_{x\tau} \phi\|_{L^2((0,t) \times (0,L))} \leq C.$$

Therefore, we arrive at the following theorem.

THEOREM 9. *Let $\phi_g \in H^1(0, T)$. Then for all $T > 0$ there exists a weak solution $\phi \in H^1((0,T) \times (0,L))$, $\partial_{xt}\phi \in L^2((0,T) \times (0,L))$ for the variational formulation (11)–(12) of the problem (8), (9), and (3).*

We conclude this section by establishing uniform $L^\infty$-bounds for $\phi$. We have the following proposition.

PROPOSITION 10. *Let $\phi_g \in H^1(0,T)$ and $\phi_g \geq 0$. Then any weak solution $\phi$ of the problem (8), (9), and (3), obtained in Theorem 9, satisfies $\phi(x,t) \geq 0$ a.e. on $Q_T$.*

*Proof.* Let $a_- = -\inf\{a, 0\}$ and $a_+ = \sup\{a, 0\}$. Then $a = a_+ - a_-$ and $\Phi_\delta((\phi_g)_-) = \Phi_\delta(0) = 0$. We test (11) by $\Phi_\delta(\phi_-)$. Note that $\Phi_\delta(\phi_-)|_{x=0} = 0$ and $\Phi_\delta(\phi_-) \geq 0$. Then we have

$$\int_0^t \int_0^L (\partial_\tau \phi) \, \Phi_\delta(\phi_-) \, dx \, d\tau + \int_0^t \int_0^L D(\phi) \, \partial_x \phi \partial_x \Phi_\delta(\phi_-) \, dx \, d\tau$$

$$+ \int_0^t \int_0^L \frac{D(\phi)(|\phi| + \delta)}{B} \partial_x \left( e^{-m \min\{|\phi|, 1/\delta\}} \partial_\tau \phi \right) \Phi_\delta'(\phi_-) \, \partial_x \phi_- \, dx \, d\tau = 0.$$

Since $\phi_-|_{t=0} = 0$, $\phi_+ \phi_- = 0$, and $|\phi| \phi_- = \phi_-^2$, we get

$$\int_0^L \left( \int_0^{\phi_-(x,t)} \Phi_\delta(\xi) \, d\xi \right) dx + \int_0^t \int_0^L D(\phi_-) \, \Phi_\delta'(\phi_-) \, (\partial_x \phi_-)^2 \, dx \, d\tau$$

$$+ \int_0^L \frac{D(\phi)(|\phi| + \delta)}{2B} \left( e^{-m \min\{|\phi|, 1/\delta\}} \partial_x \phi_- \right)^2 (t) \, dx = 0.$$

It follows that $\partial_x \phi_- = 0$ and $\phi_-|_{x=0} = 0$. Therefore $\phi_- = 0$, and consequently $\phi = \phi_+ \geq 0$. □

In the uniform bounds which follow, we use, for given positive constants $m$ and $\delta$, the function

(65) $$G(z) := \int_0^z \exp\{-m \min\{\xi, 1/\delta\}\} \, d\xi, \quad z \geq 0.$$

Then we have the following bounds.

PROPOSITION 11. *Let $\phi_g \in H^1(0,T)$, $\phi_g \geq 0$ and $\partial_t \phi_g \geq 0$ a.e. on $(0,T)$. Then any weak solution $\phi$ of the problem (8), (9), and (3), obtained in Theorem 9, satisfies $\phi_g(t) \geq \phi(x,t)$ a.e. on $Q_T$.*

*Proof.* Let $G$ be given by (65). We test (11) by $(G(\phi) - G(\phi_g))_+$. Note that $(G(\phi) - G(\phi_g))_+\big|_{x=0} = 0$. Then we have

$$\int_0^t \int_0^L \partial_\tau \phi \left(G(\phi) - G(\phi_g)\right)_+ dx\, d\tau + \int_0^t \int_0^L D(\phi)\, \partial_x \phi \partial_x \left(G(\phi) - G(\phi_g)\right)_+ dx\, d\tau$$

(66)

$$+ \int_0^t \int_0^L \frac{D(\phi)(|\phi| + \delta)}{B} \partial_x \left(e^{-m\min\{|\phi|, 1/\delta\}} \partial_\tau \phi\right) \partial_x \left(G(\phi) - G(\phi_g)\right)_+ dx\, d\tau = 0.$$

Note that
(67)

$$\partial_\tau \phi \left(G(\phi) - G(\phi_g)\right)_+ = \partial_\tau \left(\int_0^\phi \left(G(\xi) - G(\phi_g)\right)_+ d\xi\right) + G'(\phi_g) \partial_\tau \phi_g \left(\phi - \phi_g\right)_+$$

and

$$\frac{D(\phi)(|\phi| + \delta)}{B} \partial_\tau \partial_x G(\phi) \partial_x \left(G(\phi) - G(\phi_g)\right)_+ = \partial_\tau \left(\frac{D(\phi)(|\phi| + \delta)}{2B}(\partial_x \left(G(\phi)\right.\right.$$

(68) $$\left.\left. - G(\phi_g))_+\right)^2\right) - \left(\partial_x \left(G(\phi) - G(\phi_g)\right)_+\right)^2 \partial_\tau \left(\frac{D(\phi)(|\phi| + \delta)}{2B}\right).$$

Then using the monotonicity of $\phi_g$ and $G$ we obtain from (66), (67), and (68) the following inequality:

$$\int_0^L \left(\int_0^{\phi(x,t)} \left(G(\xi) - G(\phi_g)\right)_+ d\xi\right) dx + \int_0^t \int_0^L \frac{D(\phi)}{G'(\phi)} \left(\partial_x \left(G(\phi) - G(\phi_g)\right)_+\right)^2 dx\, d\tau$$

$$+ \int_0^L \frac{D(\phi)(|\phi| + \delta)}{2B}(\partial_x \left(G(\phi) - G(\phi_g)\right)_+)^2 dx$$

$$\leq \int_0^t \int_0^L \left(\partial_x \left(G(\phi) - G(\phi_g)\right)_+\right)^2 \partial_\tau \left(\frac{D(\phi)(|\phi| + \delta)}{2B}\right) dx\, d\tau.$$

Since $\partial_\tau \left(\frac{D(\phi)(|\phi| + \delta)}{2B}\right) \in L^2(0, T; L^\infty(0, L))$, we apply Gronwall's lemma and conclude that $(G(\phi) - G(\phi_g))_+ = 0$, from which it follows that $G(\phi) \leq G(\phi_g)$. Inversion of this equation leads to $\phi(x,t) \leq \phi_g(t)$ a.e. on $Q_T$. □

PROPOSITION 12. *Let $\phi_g \in H^1(0, T)$, and let us suppose in addition that there are constants $A_0 > 0$, $\alpha > 0$, and $C_0 > 0$ such that*

(69) $$A_0 \geq \phi_g(t) \geq C_0 t^\alpha \quad \forall t \in [0, T].$$

*Then any weak solution $\phi$ of the problem (8), (9), and (3), obtained in Theorem 9, satisfies $A_0 \geq \phi(x,t) \geq C_0 t^\alpha$ a.e. on $Q_T$.*

*Proof.* The proof follows the lines of Proposition 11. It is enough to prove the lower bound. We test (11) by $(G(C_0 t^\alpha) - G(\phi))_-$. Note that $(G(C_0 t^\alpha) - G(\phi))_-\big|_{x=0} = 0$. Then as in the proof of Proposition 11 we have

(70)
$$\int_0^t \int_0^L \partial_\tau \phi \left(G(C_0 \tau^\alpha) - G(\phi)\right)_- dx\, d\tau$$

$$+ \int_0^t \int_0^L D(\phi)\, \partial_x \phi \partial_x \left(G(C_0 \tau^\alpha) - G(\phi)\right)_- dx\, d\tau$$

$$+ \int_0^t \int_0^L \frac{D(\phi)(|\phi| + \delta)}{B} \partial_x \left(e^{-m\min\{|\phi|, 1/\delta\}} \partial_\tau \phi\right) \partial_x \left(G(C_0 \tau^\alpha) - G(\phi)\right)_- dx\, d\tau = 0.$$

Note that

$$
(71) \qquad G(\phi) = G(C_0 t^\alpha) - (G(C_0 t^\alpha) - G(\phi))_+ + (G(C_0 t^\alpha) - G(\phi))_- ,
$$

$$
\partial_\tau \phi \left( G(C_0 \tau^\alpha) - G(\phi) \right)_- = \frac{\partial_\tau G(C_0 \tau^\alpha)}{G'(\phi)} \left( G(C_0 \tau^\alpha) - G(\phi) \right)_-
$$

$$
(72) \qquad + \frac{1}{2G'(\phi)} \partial_\tau \left( G(C_0 \tau^\alpha) - G(\phi) \right)_-^2 \geq \frac{1}{2G'(\phi)} \partial_\tau \left( G(C_0 \tau^\alpha) - G(\phi) \right)_-^2 ,
$$

and

$$
\frac{D(\phi)(|\phi| + \delta)}{B} \partial_\tau \partial_x G(\phi) \, \partial_x \left( G(C_0 \tau^\alpha) - G(\phi) \right)_-
$$

$$
= \partial_\tau \left( \frac{D(\phi)(|\phi| + \delta)}{2B} (\partial_x \left( G(C_0 \tau^\alpha) - G(\phi) \right)_-)^2 \right)
$$

$$
(73) \qquad - (\partial_x \left( G(C_0 \tau^\alpha) - G(\phi) \right)_-)^2 \partial_\tau \left( \frac{D(\phi)(|\phi| + \delta)}{2B} \right).
$$

Then using the monotonicity of $G$ we obtain from (70), (72), and (73) the following inequality:

$$
\int_0^L \frac{(G(C_0 t^\alpha) - G(\phi))_-^2}{2G'(\phi)} \, dx + \int_0^t \int_0^L \frac{D(\phi)}{G'(\phi)} \left( \partial_x \left( G(C_0 \tau^\alpha) - G(\phi) \right)_- \right)^2 dx \, d\tau
$$

$$
+ \int_0^L \frac{D(\phi)(|\phi| + \delta)}{2B} (\partial_x \left( G(C_0 t^\alpha) - G(\phi) \right)_-)^2 \, dx
$$

$$
\leq \int_0^t \int_0^L (\partial_x \left( G(C_0 \tau^\alpha) - G(\phi) \right)_-)^2 \partial_\tau \frac{1}{2G'(\phi)} \, dx \, d\tau
$$

$$
+ \int_0^t \int_0^L (G(C_0 \tau^\alpha) - G(\phi))_-^2 \, \partial_\tau \left( \frac{D(\phi)(|\phi| + \delta)}{2B} \right) dx \, d\tau.
$$

Since $\partial_t \left( \frac{D(\phi)(|\phi|+\delta)}{2B} \right)$ and $\partial_t \frac{1}{2G'(\phi)}$ are elements of $L^2(0, T; L^\infty(0, L))$, we apply Gronwall's lemma to the function

$$
\int_0^t \| \partial_\tau \phi(\tau) \|_{L^\infty(0,L)} \left\| (G(C_0 \tau^\alpha) - G(\phi(\tau)))_- \right\|_{H^1(0,L)}^2 \, d\tau
$$

and conclude that $\left( G(C_0 t^\alpha) - G(\phi) \right)_- = 0$, from which it follows that $G(\phi) \geq G(C_0 t^\alpha)$. Inversion of $G$ leads to $\phi(x, t) \geq C_0 t^\alpha$ a.e. on $Q_T$. $\qquad \square$

THEOREM 13. *Let $\phi_g \in H^1(0, T)$, $A_0 = \max_{0 \leq t \leq T} \phi_g(t)$, $A_0 \geq \phi_g \geq C_0 t^\alpha$, and $\alpha > 1$. Then there exists a weak solution $\phi$, $C_0 t^\alpha \leq \phi(x, t) \leq A_0$, $\partial_{xt}\phi \in L^2((0,T) \times (0,L))$, $\phi \in H^1((0,T) \times (0,L))$, for the problem (8), (9), and (3).*

*Remark* 14.
- By choosing $\delta < 1/A_0$, we can replace $e^{-m \min\{|\phi|, 1/\delta\}}$ by $e^{-m\phi}$ and $|\phi| + \delta$ by $\phi + \delta$.
- In addition to the assumptions of Theorem 13 let us suppose that $\partial_t \phi_g \geq 0$. Then there exists a weak solution $\phi$, $C_0 t^\alpha \leq \phi(x, t) \leq \phi_g(t)$, $\partial_{xt}\phi \in L^2((0,T) \times (0,L))$, $\phi \in H^1((0,T) \times (0,L))$, for the problem (8), (9), and (3).

**5. Existence for the original problem.** It remains to pass to the limit $\delta \to 0$. This limit will give us the solvability of the starting problem (1)–(3).

After Theorem 13, we are free to replace the nonlinearity $\exp\{-m\phi\}$ by $h(\xi) = e^{-m}\min\{\xi, A_0\}$, $\xi \geq 0$. We have existence for the system (11)–(12); i.e., for every $g \in L^2(0, T; V)$, $V = \{g \in H^1(0, L) \mid g(0) = 0\}$, we have

(74)
$$\int_0^T \int_0^L \partial_t \phi_\delta \, g \, dx \, dt + \int_0^T \int_0^L D(\phi_\delta) \left\{ \partial_x \phi_\delta + \frac{(\phi_\delta + \delta)}{B} \partial_x (h(\phi_\delta) \partial_t \phi_\delta) \right\} \partial_x g \, dx \, dt = 0,$$

(75)
$$\phi_\delta|_{x=0} = \phi_g(t) \qquad \text{and} \qquad \phi_\delta|_{t=0} = 0,$$

and we want to pass to the limit $\delta \to 0$.

Let

(76)
$$\Psi'_\delta(\xi) := \frac{h(\xi)}{D(\xi)(\xi + \delta)}, \quad \xi \geq 0,$$

and

(77)
$$\Psi_\delta(\phi) := \int_0^\phi \frac{1}{\xi + \delta} \left( \frac{h(\xi)}{D(\xi)} - \frac{h(0)}{D(0)} \right) d\xi + \frac{h(0)}{D(0)} \log(\phi + \delta) \quad \text{for} \quad \phi \geq 0.$$

It should be noted that $\Psi_\delta(0) = \frac{h(0)}{D(0)} \log \delta < 0$, which would cause some complications.

THEOREM 15. *Let $\alpha > 0$, $C_0$, and $A_0$ be positive constants and let*

(78)
$$\phi_g \in H^1(0, T), \quad C_0 t^\alpha \leq \phi_g \leq A_0 \quad \text{and} \quad \log \phi_g \in L^2(0, T).$$

*Then problem (1)–(3) has at least one weak solution $\phi \in H^1((0, T) \times (0, L))$ such that $\sqrt{\phi} \, \partial_x \left( e^{-m\phi} \partial_t \phi \right) \in L^2((0, T) \times (0, L))$ and $C_0 t^\alpha \leq \phi \leq A_0$.*

*Proof.*

*Step* 1 (a priori estimates uniform in $\delta$). We test (74) by $\Psi_\delta(\phi_\delta) - \Psi_\delta(\phi_g)$ and get

$$\int_0^t \int_0^L \partial_t \phi_\delta \, \Psi_\delta(\phi_\delta) \, dx \, d\tau + \int_0^t \int_0^L \frac{h(\phi_\delta)}{\phi_\delta + \delta} (\partial_x \phi_\delta)^2 \, dx \, d\tau$$
$$+ \frac{1}{B} \int_0^t \int_0^L D(\phi_\delta)(\phi_\delta + \delta) \partial_t (h(\phi_\delta) \partial_x \phi_\delta) \frac{h(\phi_\delta) \partial_x \phi_\delta}{D(\phi_\delta)(\phi_\delta + \delta)} \, dx \, d\tau$$
$$= \int_0^t \int_0^L \partial_t \phi_\delta \, \Psi_\delta(\phi_g) \, dx \, d\tau.$$

This yields

$$\int_0^L \left( \int_0^{\phi_\delta(t)} \Psi_\delta(\xi) \, d\xi + \frac{1}{2B} (h(\phi_\delta) \partial_x \phi_\delta)^2 \right) dx + \int_0^t \int_0^L \frac{h(\phi_\delta)}{\phi_\delta + \delta} (\partial_x \phi_\delta)^2 \, dx \, d\tau$$

(79)
$$= \int_0^t \int_0^L \partial_t \phi_\delta \, \Psi_\delta(\phi_g) \, dx \, d\tau.$$

In order to get a useful estimate we should find a bound for the first term on the left-hand side of (79). First we note that $\int_0^{\phi_\delta} \int_0^\xi \frac{1}{\eta + \delta} \left( \frac{h(\eta)}{D(\eta)} - \frac{h(0)}{D(0)} \right) d\eta \, d\xi$ defines a continuous function of $\phi_\delta$. Since $\phi_\delta$ takes values between 0 and $A_0$, it is bounded independently of $\delta$. Hence

(80)
$$\left| \int_0^L \int_0^{\phi_\delta(t)} \Psi_\delta(\xi)\, d\xi\, dx \right| \leq \int_0^L \frac{h(0)}{D(0)} \left| \{\phi_\delta + \delta\} \log\{\phi_\delta + \delta\} - \phi_\delta - \delta \log \delta \right| dx + C.$$

Next $(\phi_\delta(t) + \delta) \log(\phi_\delta(t) + \delta) - \phi(t) - \delta \log \delta$ takes value zero at $t = 0$. It is a continuous function of $\phi_\delta$. Obviously $|(\phi(t) + \delta) \log(\phi(t) + \delta) - \phi(t) - \delta \log \delta| \leq \max\{1 - \delta + \delta \log \delta, (A_0 + \delta) \log(A_0 + \delta) - A_0 - \delta \log \delta\}$, and it is uniformly bounded with respect to $\delta$.

With (80), (79) leads to

(81)
$$\int_0^t \int_0^L \frac{h(\phi_\delta)}{\phi_\delta + \delta} (\partial_x \phi_\delta)^2\, dx\, d\tau \leq C + \left| \int_0^t \int_0^L \partial_t \phi_\delta\, \Psi_\delta(\phi_g)\, dx\, d\tau \right|.$$

Next we test (74) by $h(\phi_\delta)\partial_t \phi_\delta - h(\phi_g)\partial_t \phi_g$ and get

$$\int_0^t \int_0^L h(\phi_\delta)(\partial_\tau \phi_\delta)^2\, dx\, d\tau + \int_0^t \int_0^L D(\phi_\delta)\partial_x \phi_\delta \partial_x (h(\phi_\delta)\partial_\tau \phi_\delta)\, dx\, d\tau$$
$$+ \frac{1}{B} \int_0^t \int_0^L D(\phi_\delta)(\phi_\delta + \delta)(\partial_x(h(\phi_\delta)\partial_\tau \phi_\delta))^2\, dx\, d\tau = \int_0^t \int_0^L \partial_t \phi_\delta\, h(\phi_g)\partial_\tau \phi_g\, dx\, d\tau,$$

and from this

$$\int_0^t \int_0^L h(\phi_\delta)(\partial_\tau \phi_\delta)^2\, dx\, d\tau + \frac{1}{B} \int_0^t \int_0^L D(\phi_\delta)(\phi_\delta + \delta)(\partial_x(h(\phi_\delta)\partial_\tau \phi_\delta))^2\, dx\, d\tau$$

(82)
$$\leq B \int_0^t \int_0^L \frac{D(\phi_\delta)}{\phi_\delta + \delta} (\partial_x \phi_\delta)^2\, dx\, d\tau + \int_0^t \int_0^L \frac{h^2(\phi_g)}{h(\phi_\delta)} (\partial_\tau \phi_g)^2\, dx\, d\tau.$$

Let $h_{\min} = e^{-mA_0}$. Then inserting (81) into (82) yields

$$\int_0^t \int_0^L h(\phi_\delta)(\partial_\tau \phi_\delta)^2\, dx\, d\tau + \frac{1}{B} \int_0^t \int_0^L D(\phi_\delta)(\phi_\delta + \delta)(\partial_x(h(\phi_\delta)\partial_\tau \phi_\delta))^2\, dx\, d\tau$$
$$\leq C + \frac{BD_r}{h_{\min}} \left| \int_0^t \int_0^L \partial_\tau \phi_\delta \Psi_\delta(\phi_g)\, dx\, d\tau \right| + \int_0^t \int_0^L \frac{h^2(\phi_g)}{h(\phi_\delta)} (\partial_\tau \phi_g)^2\, dx\, d\tau \leq C$$
$$+ \frac{1}{2} \int_0^t \int_0^L h(\phi_\delta)(\partial_\tau \phi_\delta)^2\, dx\, d\tau + \frac{B^2(D_r)^2}{2h_{\min}^3} \|\Psi_\delta(\phi_g)\|^2_{L^2((0,t)\times(0,L))}$$
$$+ \frac{1}{h_{\min}} \int_0^t \int_0^L (\partial_\tau \phi_g)^2\, dx\, d\tau.$$

*Step* 2 (weak and strong compactness). From the above a priori estimate and assumptions (78) on $\phi_g$, we conclude that

(83)
$$\|\partial_t \phi_\delta\|_{L^2((0,T)\times(0,L))} + \left\| \frac{1}{\sqrt{\phi_\delta + \delta}} \partial_x \phi_\delta \right\|_{L^2((0,T)\times(0,L))} \leq C,$$

(84)
$$\left\| \sqrt{\phi_\delta + \delta}\, \partial_x(h(\phi_\delta)\partial_t \phi_\delta) \right\|_{L^2((0,T)\times(0,L))} \leq C.$$

Hence there are a $\phi \in H^1\left((0,T) \times (0,L)\right)$ and a subsequence $\{\phi_\delta\}$, denoted by the same subscripts, such that

$$(85) \qquad \phi_\delta \to \phi \quad \text{strongly in } L^2\left((0,T) \times (0,L)\right) \text{ and a.e. on} \quad (0,T) \times (0,L),$$

$$(86) \qquad \partial_t \phi_\delta \rightharpoonup \partial_t \phi \quad \text{weakly in} \quad L^2\left((0,T) \times (0,L)\right),$$

$$(87) \qquad \partial_x \phi_\delta \rightharpoonup \partial_x \phi \quad \text{weakly in} \quad L^2\left((0,T) \times (0,L)\right).$$

With the part of the flux containing the second order operator, the situation is more complicated. Obviously, there is $F \in L^2\left((0,T) \times (0,L)\right)$ such that

$$(88) \qquad \sqrt{\phi_\delta + \delta}\, \partial_{xt} \int_0^{\phi_\delta} h\left(\xi\right) d\xi \rightharpoonup F \quad \text{weakly in} \quad L^2\left((0,T) \times (0,L)\right).$$

Using the lower bound $\phi_\delta \geq C_0 t^\alpha$, we get from the estimate (84) and convergence (85) that

$$(89) \qquad \partial_{xt} \int_0^{\phi_\delta} h\left(\xi\right) d\xi \rightharpoonup \partial_{xt} \int_0^{\phi} h\left(\xi\right) d\xi \quad \text{weakly in} \quad L^2\left((0,T) \times (0,L)\right).$$

The convergences (85) and (89) imply that $F$ in (88) is given by $F = \sqrt{\phi}\, \partial_{xt} \int_0^\phi h\left(\xi\right) d\xi$.

*Step* 3 (passing to the limit). Consequently for every $g \in L^2(0,T;V)$ we have

$$(90) \qquad \int_0^T \int_0^L \partial_t \phi_\delta\, g\, dx\, dt \to \int_0^T \int_0^L \partial_t \phi\, g\, dx\, dt \quad \text{for } \delta \to 0,$$

$$(91) \qquad \int_0^T \int_0^L D\left(\phi_\delta\right) \partial_x \phi_\delta\, \partial_x g\, dx\, dt \to \int_0^T \int_0^L D\left(\phi\right) \partial_x \phi\, \partial_x g\, dx\, dt \quad \text{for } \delta \to 0,$$

$$(92) \qquad \begin{aligned} &\int_0^T \int_0^L \frac{D\left(\phi_\delta\right)}{B}\left(\phi_\delta + \delta\right) \partial_x \left(h\left(\phi_\delta\right) \partial_t \phi_\delta\right) \partial_x g\, dx\, dt \\ &\qquad \to \int_0^T \int_0^L \frac{D\left(\phi\right)}{B} \phi\, \partial_x \left(h\left(\phi\right) \partial_t \phi\right) \partial_x g\, dx\, dt \quad \text{for } \delta \to 0. \end{aligned}$$

Furthermore, for every $\zeta \in C^\infty\left([0,L] \times [0,T]\right)$, such that $\zeta(L,t) = 0$ on $[0,T]$, we have

$$\int_0^T \phi|_{x=0}\zeta|_{x=0}\, dt = \int_0^T \phi_g(t)\zeta|_{x=0}\, dt - \int_0^T \int_0^L \partial_x\left((\phi - \phi_\delta)\zeta\right) dx\, dt,$$

and, using the convergences (85) and (87), we obtain that the trace of $\phi$ at $x = 0$ satisfies the boundary condition (26). Hence, we conclude that $\phi$ satisfies the system (1)–(3). $\square$

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