# Topological Extensions of Holomorphic Functional Calculus for Sectorial and Half-plane Type Operators

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The undersigned hereby certify that they have read and recommend to the Faculty of Electrical Engineering, Mathematics and Computer Science (EEMCS) for acceptance a thesis entitled

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# Preface

From ancient times, the humankind has the desire to understand nature. People always try to explain the observations, analyse the past and predict the future. Seers and prophets had always a prominent position in human societies. With the passage of time their place was taken by scientists who could 'see' the future as far as nature allowed them to see.

One of the people who contributed to the birth of science was the ancient Greek Socrates (Σωκρàτης). According to Aristotle, two innovations can one attribute to Socrates: the general definition of notions (*ορìζεσθαι καθòλου*) and the inductive reasoning (*επακτικοì λòγοι*)(cf. [1, 1078b]).

The first one, to define a notion, is an important and difficult task. Consider asking a hundred people what is freedom, justice, beauty, love or friendship. I am almost sure that a hundred different responses will be observed. But the definition of a notion is unique, Socrates taught us. To define a notion, you have to describe it in such a manner, that you can not add, remove or change any word. The second one, the inductive reasoning, is more important. It is worth to examine how Socrates build step by step his syllogisms. With inductive reasoning he introduced the scientific method into human history. Science in the sense that you innovate and build a system of applied information, such that their structure has the pattern, the accuracy and the necessity of the organic order there exists in nature.

The scientific field of mathematics serves ideally these two innovations. In mathematics the definition of a notion is exact and unique, and with rational steps one can induce reasonable results. This privilege brought mathematics into the core of sciences that help the humankind to understand nature. These two difficult tasks are also in the heart of my personal concerns and became the reasons of taking decisions, among several options, for my studies.

This sequence of decisions brought me here, to present this report for the Master Thesis as the final part of the Master Applied Mathematics at Delft University of Technology, with specialization in Computational Science and Engineering. The research was performed under the supervision of Dr. Markus Haase. As he said, the deep motivation to do this project is the belief that nature is well-behaved.

## Topic of the Thesis

The main object of this project, as the title indicates, is to construct topological extensions of holomorphic functional calculus for sectorial and half-plane type operators. The theory of functional calculus deals with the idea of 'inserting operators into functions'. The holomorphic (or Dunford-Riesz) functional calculus is based on a class of functions which contained in the class of bounded holomorphic functions and can be represented by the Cauchy-integral formula. A topological extension means that the primary calculus is extended to a wider class of functions and the definition of the operator is obtained as the limit of a sequence of operators defined by the primary calculus. Sectorial and half-plane type operators are operators that have some spectral conditions.

In the case of sectorial operators, the topologically extended holomorphic functional calculus serves several benefits. The so-called Hirsch calculus for non-negative operators can be incorporated into this holomorphic approach. The resolvent of the logarithm can be defined without the requirement of the injectivity of the operator. Furthermore, this approach can be used in order to obtain convergence rates for the Euler's approximations for bounded semigroups.

In the case of half-plane type operators, the main benefit, the topologically extended holomorphic functional calculus serves, is that the so-called Hille-Phillips calculus for generators of bounded strongly continuous semigroups can be incorporated into this holomorphic approach.

## Acknowledgements

I would like to thank my supervisor Markus Haase for his feedback, trust and the opportunity to do this project, in which I learned how to manage mathematical notions and definitions and how to structure my thinking in order to prove mathematical facts. Moreover, I came to realize the creativity, the beauty but also the difficulties of exploring the limits of human knowledge and producing a small leaf of a tree in the forest of the mathematical world.

Furthermore I want to thank the other two members of the thesis committee, Prof. Dr. J.M.A.M. van Neerven and Dr. W.T. van Horssen for accepting the invitation to examine my project. Special thanks to Mrs.Dori Steeneken for her support regarding institutional issues. I also would like to thank my parents for the materially and mentally support, and my house-mates and friends for the pleasant every day living environment at Delft.

Panos Konstantopoulos
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# Chapter 1

# Introduction

As we mentioned in the preface, the humankind has the desire to predict the future. Evolution equations rule the dynamic behaviour of deterministic systems. Consider, for instance, the initial value problem

$$\begin{cases} u'(t) = Au(t), & t > 0, \\ u(0) = x \end{cases}$$

where $u(\cdot) : \mathbb{R}_+ \to X$ is the unknown, $(X, \|\cdot\|_X)$ is a Banach space, $A$ is an (in general unbounded) operator on $X$, $u'(t) = \frac{\mathrm{d}}{\mathrm{d}t}u(t)$ and $x$ is the given initial value. In the complex-valued case, where $A = \alpha \in \mathbb{C}$, the exponential function $(t \mapsto e^{t\alpha}) : \mathbb{R}_+ \to \mathbb{C}$ determines the evolution of the solution as $u(t) = e^{t\alpha}x$. One may wonder if there is a meaning in the expression $e^{tA}$ in order to describe the evolution of a system in the general case of an operator $A$. It seems as $e^{tA}$ is obtained by somehow inserting the operator $A$ into the complex-valued function $e^{t\alpha}$. The theory of functional calculus deals with the idea of 'inserting operators into functions', in order to provide meaning in expressions like that.

If there exists a unique solution $u$ of the system, then one can define the mapping $T(\cdot) : \mathbb{R}_+ \to \mathcal{L}(X)$, with $T(t)x = u(t)$, where $\mathcal{L}(X)$ is the space of linear bounded operators on $X$. For this mapping, we have $T(0) = I$ (the identity operator), and due to uniqueness, $T(t+s) = T(t)T(s)$ for $t, s \geqslant 0$. This mapping is what we call *semigroup* and it can be completely determined by the solution $u$. The semigroup is related with the operator $A$, which is called the *generator* of the semigroup.

The well-posedness of evolution equations is one of the main study subjects of scientists. One may ask what properties an operator $A$ has to have in order to generate a semigroup that solves the problem. There are several generation theorems, like the Hille-Yosida theorem or the Lumer-Phillips theorem [11, Section II.3], which give necessary and sufficient conditions for a linear operator to generate a semigroup.

In terms of functional calculus, the question is what properties an operator $A$ has to have in order to be insertable into $e^{t\alpha}$, $\alpha \in \mathbb{C}$, and so to give a meaning in the expression $e^{tA}$ in order to describe the solution of an evolution equation. In Section 1.4, we will describe an application, the homogeneous heat equation, where $A = \Delta$ is the Laplacian operator.

The common pattern to construct a functional calculus is: suppose that you are given an operator $A$ and a function $f$ for which you would like to define $f(A)$. Take some representation of $f$ in terms of other functions $g$ of which you already know $g(A)$, for whatever reason. Then insert $A$ into the known parts and hope that the formulas still make sense. Since a calculus is for calculation, it is reasonable to require that the mapping $\Phi : f \mapsto f(A)$ is a homomorphism of algebras.

Holomorphic functional calculus is based on the idea of Dunford and Riesz to use the Cauchy-integral formula (cf. [10])

$$f(\alpha) = \frac{1}{2\pi i}\int_\Gamma f(z)\frac{1}{z-\alpha}\mathrm{d}z$$

in order to construct a functional calculus by defining

$$f(A) = \frac{1}{2\pi i}\int_\Gamma f(A)\,\mathrm{R}(z,A)\mathrm{d}z,$$

where $\Gamma$ is a positively oriented path surrounding the spectrum of $A$ and $\mathrm{R}(z,A) = (z-A)^{-1}$ is the resolvent of the operator $A$. The Cauchy formula requires a spectral condition on the operator $A$ and $f$ to be holomorphic on a neighbourhood of the spectrum of $A$. Here, the known part is the resolvent of the operator $\mathrm{R}(z,A)$.

There are, also, other functional calculi based on different representations. For example the Hille-Phillips calculus (cf. [15]) is based on the Laplace transform

$$f(z) = \int_{\mathbb{R}_+} e^{-zt}\mu(\mathrm{d}t)$$

of a finite complex measure $\mu$ on $\mathbb{R}_+$. Here, the requirement is that $-A$ generates a bounded $C_0-$semigroup (strongly continuous), so the known part is $e^{-tA}$.

The Hirsch functional calculus (cf. [6]) is based on a representation

$$f(z) = \alpha + \int_{\mathbb{R}_+} \frac{z}{1+tz}\mu(\mathrm{d}t),$$

where $\mu$ is a suitable complex measure on $\mathbb{R}_+$ and $\alpha \in \mathbb{C}$ constant. Here, the requirement is that the operator $A$ is non-negative, so the known part is $(1+tA)^{-1}$.

One motivation to do this project is the belief that there has to exist a unified theory of functional calculus in which the definition of $f(A)$ coincides with each divided definitions. Here, we chose the holomorphic functional calculus and we tried, in case of sectorial operators, to incorporate the definition of $f(A)$, according to Hirsch calculus, into the holomorphic approach. In case of half-plane type operators we tried to incorporate the definition of $f(A)$ for semigroup generators, according to Hille-Phillips calculus, into the holomorphic approach.

The purpose to construct a functional calculus is that if one wants to have a property for the operator $f(A)$, it is sufficient to look at the properties of the function $f$. For example in [17], Vitse proves some estimations for $\|f(A)\|$. One may ask whether we can have $\|f(A)\| \leqslant C\|f\|_{\mathcal{F}}$ for some norm on function algebra $\mathcal{F}$ and $C$ constant. The same one might wonder for the convergence rates of Euler's approximations of bounded holomorphic semigroups in [4]. This leads to the question if a functional calculus can be constructed based on representations

$$\int_{\mathbb{R}_+} (zt)^k e^{-tz} \mu(\mathrm{d}t), \qquad k \in \mathbb{N},$$

in case $-A$ generates a bounded holomorphic semigroup.

So, the second motivation is to prove convergence rates for the Euler's approximations of bounded holomorphic semigroups looking on estimations of functions and then having the functional calculus to estimate directly the operator.

The construction of a functional calculus can be seen as composed of two steps. The first step consists in constructing a functional calculus based on an 'elementary' class of functions. In the second step, this 'elementary' functional calculus can be extended to a larger class of functions. One has two options to consider the extension, by algebraic means or by topological means. The algebraically extended functional calculus is available and completely described in [12]. In this project we describe a topologically extended functional calculus. This is desirable in two cases. In case of sectorial operators, the Hirsch calculus can be covered by a topological extension of the holomorhpic (Dunford-Riezs) calculus. And, in case of half-plane type operators, the Hille-Phillips calculus for semigroup generators can be covered by a topological extension of the holomorphic (Dunford-Riesz) calculus.

## 1.1 An Abstract Framework for Algebraic Extensions

In this section it will be described abstractly how to extend a basic functional calculus to a wider class of functions by algebraic means. We will not give extensive proofs, since we will not follow this route in the remaining text but only a description how algebraic extension works. A detailed description of this can be found in [12, Chapter 1].

Suppose an elementary class of functions $\mathcal{E}$ is given and we have constructed a method

$$\Phi : (e \mapsto \Phi(e)) : \mathcal{E} \to \mathcal{L}(X)$$

of 'inserting' the operator into functions from $\mathcal{E}$, so $\mathcal{E}$ is an algebra (without **1** in general) and $\Phi$ is a homomorphism of algebras. We say that the functional calculus is *proper* if the set $\{e \in \mathcal{E} : \Phi(e) \text{ is injective}\}$ is not empty. Now, suppose we have a larger algebra $\mathcal{F}$ which contains **1**, where we would like to extend the method $\Phi$. Let us call the triple $(\mathcal{E}, \mathcal{F}, \Phi)$ an *abstract functional calculus.* Take $f \in \mathcal{F}$. If there is $e \in \mathcal{E}$ such that $ef \in \mathcal{E}$ and $\Phi(e)$ is injective, we call $f$ *regularizable* by $\mathcal{E}$ and $e$ a regularizer for $f$. In this case

$$\mathcal{F}_r = \{f \in \mathcal{F} : \exists e \in \mathcal{E} \text{ s.t. } \Phi(e) \text{ is injective and } ef \in \mathcal{E}\}$$

is a subalgebra of $\mathcal{F}$ which contains $\mathcal{E}$. For $f \in \mathcal{F}_r$ we define

$$\Phi(f) := \Phi(e)^{-1}\Phi(ef)$$

where $e$ is a regularizer for $f$. This would clearly yield a closed operator and one has to make sure that this definition is independent of the regularizer $e$.

**Lemma 1.1** *Let $(\mathcal{E}, \mathcal{F}, \Phi)$ be a proper abstract functional calculus over the Banach space $X$. Then the method*

$$\Phi : (f \mapsto \Phi(f)) : \mathcal{F}_r \to \{\textit{closed operators on } X\},$$

*with $\Phi(f) = \Phi(e)^{-1}\Phi(ef)$, is well-defined and extends the original mapping $\Phi : \mathcal{E} \to \mathcal{L}(X)$.*

**Proof.** Let $h \in \mathcal{E}$ be another regularizer for $f$ and define $A := \Phi(e)^{-1}\Phi(ef)$ and $B := \Phi(h)^{-1}\Phi(hf)$. Since the primary mapping $\Phi$ is a homomorphism we have $\Phi(e)\Phi(h) = \Phi(eh) = \Phi(he) = \Phi(h)\Phi(e)$, so $\Phi(e)^{-1}\Phi(h)^{-1} = \Phi(h)^{-1}\Phi(e)^{-1}$. We obtain

$$\begin{aligned} A = \Phi(e)^{-1}\Phi(ef) &= \Phi(e)^{-1}\Phi(h)^{-1}\Phi(h)\Phi(ef) = \Phi(h)^{-1}\Phi(e)^{-1}\Phi(hef) \\ &= \Phi(h)^{-1}\Phi(e)^{-1}\Phi(e)\Phi(hf) = \Phi(h)^{-1}\Phi(hf) = B. \end{aligned}$$

Hence the definition is independent of the regularizer.

To see that the new $\Phi$ extends the old one, let $e, f \in \mathcal{E}$ with $\Phi(e)$ injective. Then, $\Phi(e)^{-1}\Phi(ef) = \Phi(e)^{-1}\Phi(e)\Phi(f) = \Phi(f)$. Hence, the map on $\mathcal{F}_r$ is indeed an extension of the original. □

We call the original mapping $\Phi : \mathcal{E} \to \mathcal{L}(X)$ *primary functional calculus* and the extension, *extended functional calculus.* Let us collect some basic properties.

**Proposition 1.2** *Let $(\mathcal{E}, \mathcal{F}, \Phi)$ be a proper abstract functional calculus over the Banach space $X$. Then the following assertions hold.*

(i) *If $T \in \mathcal{L}(X)$ commutes with each $\Phi(e)$, $e \in \mathcal{E}$, then it commutes with each $\Phi(f)$, $f \in \mathcal{F}_r$.*

(ii) *One has $\mathbf{1} \in \mathcal{F}_r$ and $\Phi(\mathbf{1}) = I$.*

(iii) *Given $f, g \in \mathcal{F}_r$ one has*

$$\begin{aligned} \Phi(f) + \Phi(g) &\subset \Phi(f+g), \\ \Phi(f)\Phi(g) &\subset \Phi(fg), \end{aligned}$$

*with $\mathcal{D}(\Phi(f)\Phi(g)) = \mathcal{D}((\Phi(fg)) \cap \mathcal{D}(\Phi(g))$.*

(iv) *If $f, g \in \mathcal{F}_r$ such that $fg = \mathbf{1}$, then $\Phi(f)$ is injective with $\Phi(f)^{-1} = \Phi(g)$.*

(v) *Let $f \in \mathcal{F}_r$ and $F$ be a subspace of $\mathcal{D}(\Phi(f))$. Assume that there is a sequence $(e_n)_n \subseteq \mathcal{E}$ such that $\Phi(e_n) \to I$ strongly as $n \to \infty$ and $\mathcal{R}(\Phi(e_n)) \subseteq F$ for all $n \in \mathbb{N}$. Then $F$ is a core for $\Phi(f)$.*

For a proof of the above proposition one can look at [12, Proposition 1.2.2]. In general one cannot expect equality in (iii) of the above proposition. However, if we define

$$\mathcal{F}_b = \{f \in \mathcal{F}_r : \Phi(f) \in \mathcal{L}(X)\},$$

we obtain the following

**Corollary 1.3** *Let $(\mathcal{E}, \mathcal{F}, \Phi)$ be a proper abstract functional calculus over the Banach space $X$. Then*

(i) *For $f \in \mathcal{F}_r$, $g \in \mathcal{F}_b$ the identities*

$$\begin{aligned} \Phi(f) + \Phi(g) &= \Phi(f+g), \\ \Phi(f)\Phi(g) &= \Phi(fg), \end{aligned}$$

*hold.*

(ii) *The set $\mathcal{F}_b$ is a subagebra with $\mathbf{1}$ of $\mathcal{F}$ and the map*

$$\Phi : (f \mapsto \Phi(f)) : \mathcal{F}_b \to \mathcal{L}(X)$$

*is a homomorphism of algebras.*

(iii) *If $f \in \mathcal{F}_b$ is such that $\Phi(f)$ is injective, then*

$$\Phi(f)^{-1}\Phi(g)\Phi(f) = \Phi(g)$$

*for all $g \in \mathcal{F}_r$.*

For a proof of the above corollary one can look at [12, Corollary 1.2.3]. Having this, one can improve statement (iv) of Proposition 1.2.

**Corollary 1.4** *Let $(\mathcal{E}, \mathcal{F}, \Phi)$ be a proper abstract functional calculus over the Banach space $X$. Suppose $f \in \mathcal{F}_r$, $g \in \mathcal{F}$ such that $fg = \mathbf{1}$. Then*

$$g \in \mathcal{F}_r \Leftrightarrow \Phi(f) \text{ is injective},$$

*and $\Phi(f)^{-1} = \Phi(g)$.*

For a proof of the above corollary see [12, Corollary 1.2.4]. These abstract results can be applied in specific examples. For instance, the holomorphic functional calculus for sectorial operators [12, Chapter 2] or for half-plane type operators [3].

## 1.2 An Abstract Framework for Topological Extensions

Now let us describe abstractly how to extend a basic functional calculus to a wider class of functions by topological means. Here we will be more precise, and we will prove results which we will apply later to specific cases.

As before, suppose we have an elementary class of functions $\mathcal{E}$ and we have constructed a primary functional calculus

$$\Phi : (e \mapsto \Phi(e)) : \mathcal{E} \to \mathcal{L}(X).$$

We would like to extend the definition of $\Phi$ to a larger algebra $\mathcal{F}$. One could think of extending the primary calculus to a larger one by topological means, according to the following pattern:

$$\text{If } (e_n)_{n\in\mathbb{N}} \subseteq \mathcal{E},\ f \in \mathcal{A},\ e_n \to f \text{ and } \Phi(e_n) \to T, \text{ then define } \Phi(f) := T,$$

where $\mathcal{A}$ is a subalgebra $\mathcal{A} \subseteq \mathcal{F}$ with $\mathcal{E} \subseteq \mathcal{A}$, equipped with a reasonable convegrent notion. Moreover, we fix a reasonable topology on $\mathcal{L}(X)$ (e.g. the strong operator topology) in order to give meaning to the convergence $\Phi(e_n) \to T$. As we will see in Theorem 1.5, the previous pattern to be reasonable the following is required:

$$(e_n)_n \subseteq \mathcal{E},\ e_n \to 0 \text{ and } \Phi(e_n) \to T \ \Rightarrow\ T = 0. \tag{1.1}$$

Given this, one can extend the original abstract functional calculus towards

$$\mathcal{A} := \{f \in \mathcal{F} : \exists (e_n)_n \subseteq \mathcal{E},\ e_n \to f,\ \lim_n \Phi(e_n) \text{ exists}\}.$$

$\mathcal{A}$ is a subalgebra of $\mathcal{F}$ which contains $\mathcal{E}$. For $f \in \mathcal{A}$ we define

$$\Phi(f) = \lim_{n\to\infty} \Phi(e_n),$$

where $(e_n)_n \subseteq \mathcal{E}$ with $\lim_{n\to\infty} e_n = f$. The following theorem shows that the above definition is independent of the sequence $e_n$.

**Theorem 1.5** *Let* $(\mathcal{E}, \mathcal{F}, \Phi)$ *be an abstract functional calculus over the Banach space* $X$*. If the requirement* (1.1) *is available then the method*

$$\Phi : (f \mapsto \Phi(f)) : \mathcal{A} \to \mathcal{L}(X),$$

*with* $\Phi(f) = \lim_n \Phi(e_n)$*, is well-defined and extends the original mapping* $\Phi : \mathcal{E} \to \mathcal{L}(X)$*.*

**Proof.** Let $(e_n)_n, (h_n)_n \subseteq \mathcal{E}$ be sequences such that $\lim_n e_n = f$, $\lim_n h_n = f$ and define $A := \lim_n \Phi(e_n)$ and $B := \lim_n \Phi(h_n)$. For the sequence $(e_n - h_n)_n \subseteq \mathcal{E}$ we have $\lim_n (e_n - h_n) = 0$ and since the original $\Phi$ is a homomorphism we obtain

$$\lim_n \Phi(e_n - h_n) = \lim_n \Phi(e_n) - \lim_n \Phi(h_n) = A - B$$

exists, so by the requirement (1.1) $A - B = \lim_n \Phi(e_n - h_n) = 0$.

To see that the new $\Phi$ extends the original one, let $f \in \mathcal{E}$ and $(e_n)_n \subseteq \mathcal{E}$ with $e_n \to f$ and $A = \lim_n \Phi(e_n)$. Then, the sequence $(e_n - f)_n \subseteq \mathcal{E}$ with $e_n - f \to 0$. So, $\Phi(e_n - f) = \Phi(e_n) - \Phi(f) \to A - \Phi(f)$. But from the requirement (1.1), $\Phi(e_n - f) \to 0$, hence $\Phi(f) = A = \lim_n \Phi(e_n)$. □

Hence, in order to extend a primary functional calculus on a larger algebra $\mathcal{A}$ it suffices to fix a reasonable topology on $\mathcal{L}(X)$, to fix a reasonable convergence notion on $\mathcal{A}$ and to prove the requirement (1.1). Saying reasonable topology on $\mathcal{L}(X)$ we mean that since we would like $\Phi$ to be a homomorphism of algebras it is reasonable to use a topology in which we have continuity of algebraic operations. For instance, the multiplication in the weak operator topology is not continuous (only separately continuous), but in the strong operator topology it is sequentially continuous. The same holds for the convergence notion on the algebra $\mathcal{A}$. So, fixing a topology on $\mathcal{L}(X)$ and a convergence notion on $\mathcal{A}$ such that algebraic operations are continuous, and having the requirement (1.1), we obtain the following result.

**Proposition 1.6** *Let* $(\mathcal{E}, \mathcal{F}, \Phi)$ *be an abstract functional calculus over the Banach space* $X$. *If we fix the strong operator topology on* $\mathcal{L}(X)$ *and a convergence notion on* $\mathcal{A}$ *such that algebraic operations are (sequentially) continuous, and the requirement* (1.1) *is available, then for* $f, g \in \mathcal{A}$ *the identities*

$$\Phi(f) + \Phi(g) = \Phi(f+g),$$
$$\Phi(f)\Phi(g) = \Phi(fg),$$

*hold. So* $\Phi$ *is a homomorphism of algebras.*

**Proof.** Let $f, g \in \mathcal{A}$. So there exist $(e_n)_n, (h_n)_n \subseteq \mathcal{E}$ such that $e_n \to f$, $h_n \to g$ and $\Phi(f) = \lim_n \Phi(e_n)$, $\Phi(g) = \lim_n \Phi(h_n)$. For the linearity note that $(e_n + h_n)_n \subseteq \mathcal{E}$ with $e_n + h_n \to f + g$ and since the original $\Phi$ is a homomorphism of algebras, $\lim_n \Phi(e_n + h_n)$ exists, so $f + g \in \mathcal{A}$ and by definition $\Phi(f+g) = \lim_n \Phi(e_n + h_n)$. From the linearity of the original $\Phi$ we obtain

$$\Phi(f+g) = \lim_n \Phi(e_n + h_n) = \lim_n \Phi(e_n) + \lim_n \Phi(h_n) = \Phi(f) + \Phi(g).$$

For the multiplicative property, we have that $(e_n h_n)_n \subseteq \mathcal{E}$ with $e_n h_n \to fg$. From the Banach-Steinhaus theorem (cf. [14, Theorem 15.6]) $\Phi(e_n), \Phi(h_n)$ are uniformly bounded in $n$ and $\Phi(f), \Phi(g)$ are bounded operators. Hence for $x \in X$ we obtain

$$\begin{aligned}
&\|(\Phi(e_n h_n) - \Phi(f)\Phi(g))x\|_X = \|(\Phi(e_n)\Phi(h_n) - \Phi(f)\Phi(g))x\|_X \\
&\leqslant \|\Phi(e_n)(\Phi(h_n) - \Phi(g))x\|_X + \|\Phi(g)(\Phi(e_n) - \Phi(f))x\|_X \\
&\leqslant \|\Phi(e_n)\|\|(\Phi(h_n) - \Phi(g))x\|_X + \|\Phi(g)\|\|(\Phi(e_n) - \Phi(f))x\|_X \to 0.
\end{aligned}$$

Hence $fg \in \mathcal{A}$ and

$$\Phi(fg) = \Phi(f)\Phi(g).$$

□

Proposition 1.6 is the alpha and the omega in our procedure, since we will apply this result in the two cases we study, for sectorial and half-plane type operators, in order to prove that an extension of the holomorphic functional calculus by topological means is indeed an algebra homomorphism.

Let us conclude this section with the relation between the algebraic and the topological extension of a given primary functional calculus. We note the following result about compatibility of functional calculi.

**Proposition 1.7** *Let* $(\mathcal{E}, \mathcal{F}, \Phi)$ *be a proper abstract functional calculus over the Banach space* $X$. *If* $f \in \mathcal{F}_b \cap \mathcal{A}$ *and* $\Phi(f)$ *is defined by either calculus, then both definitions lead to the same operator.*

**Proof.** Let $f \in \mathcal{F}_b \cap \mathcal{A}$. Since $f \in \mathcal{A}$, there exists $(e_n)_n \subseteq \mathcal{E}$ such that $e_n \to f$ and $A := \lim_n \Phi(e_n)$. Furthermore, since $f \in \mathcal{F}_b$, there exists $e \in \mathcal{E}$ such that $ef \in \mathcal{E}$, $\Phi(e)$ is injective and $B := \Phi(e)^{-1}\Phi(ef)$. Note that the sequence $(e(e_n - f))_n \subseteq \mathcal{E}$ with $e(e_n - f) \to 0$ and $\lim_n \Phi(e(e_n - f)) = \Phi(e)A - \Phi(ef)$ exists. So, by the requirement (1.1) $\Phi(e)A - \Phi(ef) = 0$, and since $\Phi(e)$ is injective we conclude $A = \Phi(e)^{-1}\Phi(ef) = B$. □

## 1.3 Euler's Approximations

It is known that

$$e^{-zt} = \lim_{n\to\infty} \left(1 - \frac{tz}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{tz}{n}\right)^{-n}$$

for $z \in \mathbb{C}_+$, $t \geqslant 0$. One may wonder if this approximation holds whenever we insert an operator $A$ instead of $z$. A known result is that whenever $-A$ generates a $C_0$−semigroup $(T(t))_{t\geqslant 0}$ on a Banach space $X$, we have that for each $x \in X$

$$T(t)x = \lim_{n\to\infty} \left(1 + \frac{tA}{n}\right)^{-n} x = (n/t\,\mathrm{R}(n/t, -A))^n x$$

strongly and uniformly for $t$ in compact intervals (cf. [11, Corollary 5.5] , [3, Corollary 3.4]). In the case of a bounded holomorphic semigroup one knows convergence in operator norm with a convergence rate $O(1/n)$ (cf. [5]). In [4], Bentkus proved a convergence rate $O(1/(n-1))$ for the Euler's approximations of bounded holomorphic semigroups. Example 2.19 is inspired by this, where we provide a different approach, in terms of functional calculus, in order to obtain a convergence rate for the Euler's approximations of bounded holomorphic semigroups.

Euler's approximations of bounded semigroups play an important role in numerical analysis. In [12, Section 9.2] the rational approximations schemes are described. Let us illustrate a procedure from there, in order to see the importance of Euler's approximations in numerical analysis. Consider the initial value problem

$$\begin{cases} u'(t) = F(u(t)), & t > 0, \\ u(0) = x \end{cases}$$

in a Banach space $X$. An exact solution is a differentiable curve $(u(t))_{t\geqslant 0}$ starting at $x$ and satisfying the above equation, i.e. at any time $t > 0$ the velocity of $u$ at $t$ coincides with the value of the vector field $F$ at the point $u(t)$. In simulating the dynamical system numerically one works with piecewise affine curves instead of exact solutions. Given a $t > 0$ one might think of dividing the interval $[0, t]$ into $n \in \mathbb{N}$ equal parts, each of length $h := t_{k+1} - t_k = t/n$, and consider a continuous curve $u : [0, t] \to X$ starting at $u(0) = x$ and being affine on each interval $[t_{j-1}, t_j]$. The vector $\delta u_k = (u_{k+1} - u_k)/h$, where $u_k = u(t_k)$, is the constant velocity vector of $u$ on $[t_k, t_{k+1}]$. In order $u$ to be considered as an approximate solution this velocity vector should have something to do with the vector field $F$. One choice is to require

$$\delta u_k = (u_{k+1} - u_k)/h = F(u_k).$$

This method is called the forward Euler method. Another choice is to require $\delta u_k = F(u_{k+1})$. This is called the backward Euler method.

Consider the case $F(u(t)) = -Au(t)$ where $A$ is a sectorial operator of angle $\omega < \pi/2$. The forward Euler method leads to

$$u_{k+1} = (1 - hA)u_k,$$

for $k \leqslant n - 1$. Hence $u_n = (1 - hA)^n x$. Obviously, if $A$ is unbounded, this has some drawbacks, since the approximation $u$ will only be defined if the initial value is sufficient smooth (is contained in $\mathcal{D}(A^N)$ for some (large) $N \in \mathbb{N}$). But, using the backward Euler method we obtain $u_{k+1} - u_k = -hAu_{k+1}$, or

$$u_{k+1} = (1 + hA)^{-1} u_k.$$

Note that, since we assume $A$ is sectorial, $(1+hA)^{-1} \in \mathcal{L}(X)$. Hence, the backward Euler method leads us to consider

$$u_n = \left(1 + \frac{t}{n}A\right)^{-n} x$$

as an approximation of the exact solution $e^{-tA}x$. This makes clear the necessity of Euler's approximation. One may ask for convergence $(1+(t/n)A)^{-n}x \to e^{-tA}x$, or for convergence rates depending on the smoothness of $x$.

## 1.4 The Laplace Operator

In this section, we will describe an example of a holomorphic functional calculus for sectorial operators and how it is related with the solution of an evolution equation. Consider, the negative Laplace operator $-\Delta$ on $L^2(\mathbb{R})$. It will be shown that $-\Delta$ is sectorial (according to Definition 2.1). In order to find the resolvent set and to estimate the resolvent of the operator it suffices to examine if the equation

$$\lambda u + \Delta u = f \tag{1.2}$$

has a unique solution $u \in H^2$(the second Sobolev space), $f \in L^2$. Taking the Fourier transform, which is defined as

$$\tilde{u}(s) = \mathcal{F}(u)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-isx} u(x) \mathrm{d}x, \qquad (s \in \mathbb{R})$$

equation (1.2) yields

$$(\lambda - |s|^2)\tilde{u} = \tilde{f}. \tag{1.3}$$

Note that if $\lambda \in \mathbb{R}_+$, $\tilde{u}$ may not be in $L^2$. But, for $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ we obtain the unique solution in the Fourier domain

$$\tilde{u}(s) = \frac{\tilde{f}(s)}{\lambda - |s|^2}$$

for $s \in \mathbb{R}$. From equation (1.3), using Plancherel identity we have

$$\begin{aligned}
\|f\|_{L^2}^2 = \|\tilde{f}\|_{L^2}^2 &= \int_{\mathbb{R}} \left|\lambda - |s|^2\right|^2 |\tilde{u}(s)|^2 \mathrm{d}s = \int_{\mathbb{R}} (|\lambda|^2 - 2\operatorname{Re}\lambda |s|^2 + |s|^4)|\tilde{u}(s)|^2 \mathrm{d}s \\
&= |\lambda|^2 \|\tilde{u}\|_{L^2}^2 - 2\operatorname{Re}\lambda \int_{\mathbb{R}} |s|^2 |\tilde{u}(s)|^2 \mathrm{d}s + \int_{\mathbb{R}} |s|^4 |\tilde{u}(s)|^2 \mathrm{d}s \\
&= |\lambda|^2 \|\tilde{u}\|_{L^2}^2 - 2\operatorname{Re}\lambda \int_{\mathbb{R}} |is\tilde{u}(s)|^2 \mathrm{d}s + \int_{\mathbb{R}} |(is)^2 \tilde{u}(s)|^2 \mathrm{d}s \\
&= |\lambda|^2 \|\tilde{u}\|_{L^2}^2 - 2\operatorname{Re}\lambda \|\tilde{u'}\|_{L^2}^2 + \|\tilde{u''}\|_{L^2}^2 = |\lambda|^2 \|u\|_{L^2}^2 - 2\operatorname{Re}\lambda \|u'\|_{L^2}^2 + \|u''\|_{L^2}^2.
\end{aligned}$$

For $\operatorname{Re}\lambda < 0$, we have that $\|f\|_{L^2} \geqslant |\lambda| \|u\|_{L^2}$. Since equation (1.2) has a unique solution the operator $\lambda + \Delta$ is invertible for $\lambda \in \mathbb{C}_-$, so $\sigma(-\Delta) = [0, \infty) \subset \overline{S_{\pi/2}}$ and for $u = (\lambda + \Delta)^{-1} f = \mathrm{R}(\lambda, -\Delta) f$ we obtain

$$\|\mathrm{R}(\lambda, -\Delta)\| \leqslant \frac{1}{|\lambda|}.$$

Hence, the negative Laplace operator $-\Delta$ is sectorial of angle $\pi/2$. By the Hille-Yosida generation theorem, $\Delta$ generates a bounded $C_0-$semigroup. Hence, the homogeneous heat equation in one direction is well-posed. Let us describe this semigroup. Consider the homogeneous heat equation in one direction

$$\begin{cases} w'(t) = \Delta w(t), & t > 0, \\ w(0) = x \end{cases}$$

where $w(\cdot) : \mathbb{R}_+ \to L^2(\mathbb{R})$ is the unknown and $x \in L^2(\mathbb{R})$ the initial value. Applying the Fourier transform (in the spatial variable) we obtain

$$\begin{cases} \tilde{w}'(t) = -|s|^2 \tilde{w}(t), & t > 0, \\ \tilde{w}(0) = \tilde{x} \end{cases}$$

hence $\tilde{w}(t) = e^{-t|s|^2}\tilde{x}$ is the unique solution in the Fourier domain. Consequently, taking the inverse Fourier transform we have

$$w(t) = \mathcal{F}^{-1}(e^{-t|s|^2}\tilde{x}) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}(e^{-t|s|^2}) * x.$$

One can compute

$$\mathcal{F}^{-1}(e^{-t|s|^2})(y) = \frac{1}{\sqrt{2t}} e^{|y|^2/4t} =: G_t(y), \qquad (y \in \mathbb{R}),$$

so we can write the solution as

$$w(t) = \frac{1}{\sqrt{2\pi}} G_t * x = \frac{1}{\sqrt{4t\pi}} \int_{\mathbb{R}} e^{|y-s|^2/4t} x(s)\ \mathrm{d}s.$$

The semigroup $T(\cdot) : \mathbb{R}_+ \to \mathcal{L}(L^2(\mathbb{R}))$ with

$$T(t)x = \frac{1}{\sqrt{2\pi}} G_t * x, \qquad (t \in \mathbb{R}_+),\ (x \in L^2(\mathbb{R})),$$

is the so-called Gauss-Weierstrass semigroup. In terms of functional calculus, $\Delta$ is 'insertable' in $e^{\lambda z}$ with $\operatorname{Re}\lambda > 0$ and for $x \in L^2$, $e^{\lambda\Delta}x = (1/\sqrt{2\pi})G_\lambda * x$. In conclusion, the solution of the homogeneous heat equation in one direction is described as

$$w(t) = e^{t\Delta}x = T(t)x = (1/\sqrt{2\pi})G_t * x.$$

# Chapter 2

# An Extension for Sectorial Operators

In this chapter a topological extension of the holomorphic functional calculus for sectorial operators is constructed and some applications are presented. Let us fix some notation. For a domain $\Omega \subseteq \mathbb{C}$ we denote $\mathcal{H}^\infty(\Omega)$ the space of all bounded holomorphic functions $f : \Omega \to \mathbb{C}$ with the supremum norm $\|f\|_\infty$. For $0 < \omega \leqslant \pi$, let $S_\omega := \{z \in \mathbb{C}^* : |\arg z| < \omega\}$ be the open sector symmetric about the positive real axis with opening angle $\omega$. For $\omega = 0$ we define $S_0 := (0, \infty)$. For an operator $A$ on a Banach space $X$, we denote $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\sigma(A)$, $\rho(A)$ its domain, range, kernel, spectrum and resolvent set respectively.

We begin constructing a holomorphic functional calculus for sectorial operators based on an elementary function class. Then we mention a functional calculus on a larger class. We do not present main results of this calculus, since one can find a detailed study of this in [12, Chapter 2]. Afterwards, we extend the holomorphic functional calculus on a larger function class by topological means. This extension allows us to present some applications. The Hirsch calculus is covered by the topologically extended holomorphic functional calculus and the definition of the resolvent of the logarithm in case $A$ is not injective is obtained. The chapter is concluded with holomorphic semigroups which can be accessible via this calculus and as an application we observe convergence rates for the Euler's approximations of bounded holomorphic semigroups.

## 2.1 Elementary Functional Calculus

**Definition 2.1** Let $0 \leqslant \omega < \pi$. An operator $A$ on a Banach space $X$ is called sectorial of angle $\omega$ (in short $A \in \mathrm{Sect}(\omega)$ ) if

1. $\sigma(A) \subset \overline{S_\omega}$ and

2. for each $\omega < \omega' < \pi$, there exists a constant $M(A,\omega')$ such that

$$\| \mathrm{R}(\lambda, A)\| \leqslant \frac{M(A,\omega')}{|\lambda|}$$

for all $\lambda \notin \overline{S_{\omega'}}$.

Sectorial operators play an important role in elliptic differential equations (cf. [9], [12, Chapter 8]). An elementary holomorphic functional calculus for sectorial operators will be constructed based on the following class of functions

$$\mathcal{E}_0(S_\phi) := \left\{ e \in \mathcal{H}^\infty(S_\phi) : \int_{\partial S_\theta} |e(z)| \frac{|\mathrm{d}z|}{|z|} < \infty \ , \ \forall\ 0 \leqslant \theta < \phi \right\}.$$

Let us begin with a representation formula for these functions.

**Lemma 2.2** *If $e \in \mathcal{E}_0(S_\phi)$ then for all $0 < \omega < \phi$ and $\alpha \in S_\omega$, $e(\alpha)$ can be represented by the formula:*

$$e(\alpha) = \frac{1}{2\pi i} \int_{\partial S_\omega} \frac{e(z)}{\log z - \log \alpha} \frac{\mathrm{d}z}{z}.$$

**Proof.** Suppose $e \in \mathcal{E}_0(S_\phi)$. Let $0 < \omega < \phi$ and $r < 1$, $R > 1$. Consider the positively oriented closed curve $C$ (Figure 2.1) which consists of four parts :

$$\begin{aligned}
\gamma_1 &= \{-te^{i\omega} : \ -R \leqslant t \leqslant -r\}, \\
\gamma_2 &= \{re^{-it} : \ -\omega \leqslant t \leqslant \omega\}, \\
\gamma_3 &= \{te^{-i\omega} : \ r \leqslant t \leqslant R\}, \\
\gamma_4 &= \{Re^{it} : \ -\omega \leqslant t \leqslant \omega\}.
\end{aligned}$$

For any point $\alpha$ inside $C$, define

$$f(z) := \frac{e(z)}{z} \frac{z-\alpha}{\log z - \log \alpha}, \qquad (z \in C).$$

$f$ is holomorphic inside $C$ since $\lim_{z\to\alpha}(z-\alpha)(\log z - \log \alpha)^{-1} = \alpha$. Hence by Cauchy's integral formula

$$e(\alpha) = f(\alpha) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-\alpha} \mathrm{d}z = \frac{1}{2\pi i} \oint_C \frac{e(z)}{\log z - \log \alpha} \frac{\mathrm{d}z}{z}.$$

On the other hand,

$$\frac{1}{2\pi i} \oint_C \frac{e(z)}{\log z - \log \alpha} \frac{\mathrm{d}z}{z} = \frac{1}{2\pi i} \sum_{n=1}^{4} \int_{\gamma_n} \frac{e(z)}{\log z - \log \alpha} \frac{\mathrm{d}z}{z}.$$

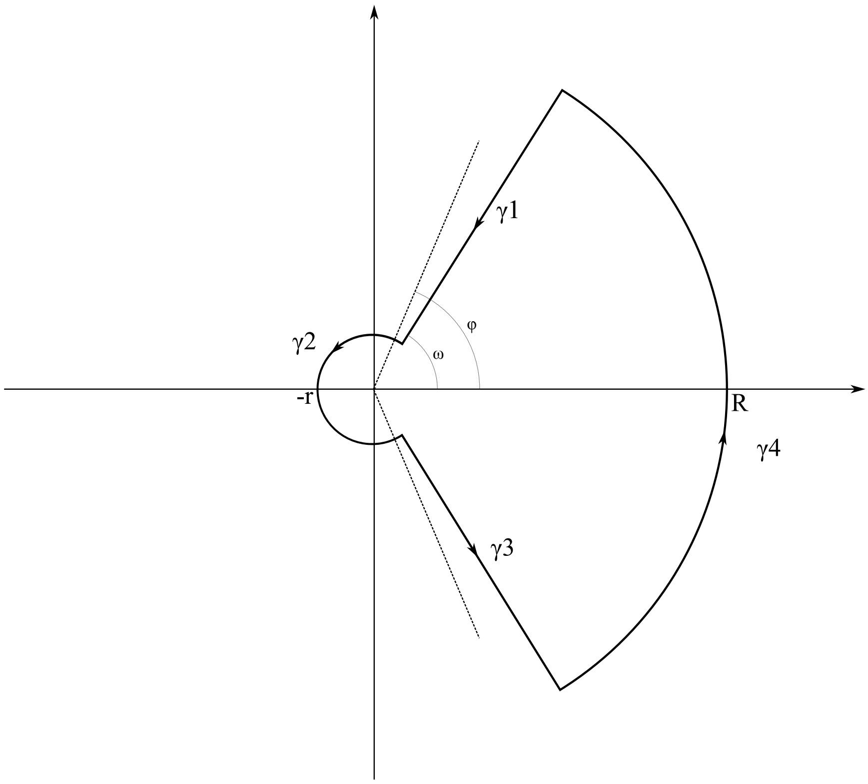


Figure 2.1: Positively oriented closed curve $C$

Since,

$$\frac{|e(z)|}{|\log z - \log \alpha|}\frac{1}{|z|} \leqslant \frac{\|e\|_\infty}{|\ln|z| - \ln|\alpha||}\frac{1}{|z|},$$

an estimation for the integral on $\gamma_2$ is

$$\left|\int_{\gamma_2} \frac{e(z)}{\log z - \log \alpha}\frac{\mathrm{d}z}{z}\right| \leqslant \frac{\|e\|_\infty}{(\ln|\alpha| - \ln r)r}2\omega r$$

and for the integral on $\gamma_4$

$$\left|\int_{\gamma_4} \frac{e(z)}{\log z - \log \alpha}\frac{\mathrm{d}z}{z}\right| \leqslant \frac{\|e\|_\infty}{(\ln R - \ln|\alpha|)R}2\omega R.$$

Now letting $r \to 0$ and $R \to \infty$ the integrals on $\gamma_2$ and $\gamma_4$ tend to 0 and the other two constitute the desired. So,

$$e(\alpha) = \frac{1}{2\pi i}\int_{\partial S_\omega} \frac{e(z)}{\log z - \log \alpha}\frac{\mathrm{d}z}{z}.$$

□

The following lemma shows that a function $e \in \mathcal{E}_0(S_\phi)$ decays at 0 and $\infty$.

**Lemma 2.3** *If $e \in \mathcal{E}_0(S_\phi)$ then, for all $0 < \omega < \phi$, we have*

$$\lim_{\substack{z\to 0\\ z\in S_\omega}} e(z) = 0 \qquad \textit{and} \qquad \lim_{\substack{z\to \infty\\ z\in S_\omega}} e(z) = 0.$$

**Proof.** Suppose $e \in \mathcal{E}_0(S_\phi)$ and let $\omega$ with $0 < \omega < \phi$ . From Lemma 2.2

$$e(\alpha) = \frac{1}{2\pi i}\int_\Gamma \frac{e(z)}{\log z - \log \alpha}\frac{\mathrm{d}z}{z} \qquad (\alpha \in S_\omega),$$

where $\Gamma$ is the positively oriented boundary of the sector $S_{\omega'}$ with $\omega < \omega' < \phi$ (Cauchy's theorem shows that $e(\alpha)$ does not depend on the particular choice of $\omega'$). There exists a constant $M$ such that

$$\frac{1}{|\log z - \log \alpha|} \leqslant M$$

uniformly in $\alpha \in S_\omega$, $z \in \Gamma$. (Otherwise there have to exist sequences $z_n, \alpha_n$ with $|\log z_n - \log \alpha_n|^{-1} \to \infty$, which means $|\log \frac{z_n}{\alpha_n}| \to 0$, which means $\frac{z_n}{\alpha_n} \to 1$ but this can not happen because $\omega' > \omega$.) Hence

$$\frac{|e(z)|}{|\log z - \log \alpha||z|} \leqslant M\frac{|e(z)|}{|z|} \qquad (z \in \partial S_{\omega'}).$$

By Lebesgue's dominated convergence theorem,

$$\lim_{\alpha\to\infty} e(\alpha) = \frac{1}{2\pi i}\int_\Gamma \lim_{\alpha\to\infty}\frac{e(z)}{\log z - \log \alpha}\frac{\mathrm{d}z}{z} = 0$$

and

$$\lim_{\alpha\to 0} e(\alpha) = \frac{1}{2\pi i}\int_\Gamma \lim_{\alpha\to 0}\frac{e(z)}{\log z - \log \alpha}\frac{\mathrm{d}z}{z} = 0.$$

□

From the previous lemmas one can show that functions in $\mathcal{E}_0(S_\phi)$ can be represented via Cauchy's integral formula.

**Lemma 2.4** *If $e \in \mathcal{E}_0(S_\phi)$ then, for all $0 < \omega < \phi$ and $\alpha \in S_\omega$,*

$$e(\alpha) = \frac{1}{2\pi i}\int_{\partial S_\omega} \frac{e(z)}{z-\alpha}\mathrm{d}z.$$

*Furthermore,*

$$\int_{\partial S_\omega} \frac{e(z)}{z}\mathrm{d}z = 0.$$

**Proof.** Fix $e \in \mathcal{E}_0(S_\phi)$. Let $0 < \omega < \phi$ and $r < 1 < R$. Consider, as in the proof of Lemma 2.2, the positively oriented closed curve $C$ which consists of four parts:

$$\begin{aligned}
\gamma_1 &= \{-te^{i\omega} : \quad -R \leqslant t \leqslant -r\},\\
\gamma_2 &= \{re^{-it} : \quad -\omega \leqslant t \leqslant \omega\},\\
\gamma_3 &= \{te^{-i\omega} : \quad r \leqslant t \leqslant R\},\\
\gamma_4 &= \{Re^{it} : \quad -\omega \leqslant t \leqslant \omega\}.
\end{aligned}$$

For any point $\alpha$ inside $C$, by Cauchy's integral formula we have

$$e(\alpha) = \frac{1}{2\pi i} \oint_C \frac{e(z)}{z-\alpha} \mathrm{d}z.$$

On the other hand,

$$\frac{1}{2\pi i} \oint_C \frac{e(z)}{z-\alpha} \mathrm{d}z = \frac{1}{2\pi i} \sum_{n=1}^{4} \int_{\gamma_n} \frac{e(z)}{z-\alpha} \mathrm{d}z.$$

By the second triangle inequality, an estimation for the integral on $\gamma_2$ is

$$\left| \int_{\gamma_2} \frac{e(z)}{z-\alpha} \mathrm{d}z \right| \leqslant \frac{\|e\|_\infty}{|\alpha| - r} 2\omega r.$$

Now for the integral on $\gamma_4$, note that for $\alpha$ inside $C$, there exists a constant $M$ such that

$$\frac{R}{R - |\alpha|} \leqslant M.$$

(Otherwise, it has to exist a sequence $\alpha_n$ with $(1-(|\alpha_n|/R))^{-1} \to \infty$, which means $|\alpha_n|/R \to 1$ but this can not happen because $|\alpha_n| < R$.) So, using the triangle inequality we have that

$$\frac{|e(Re^{it})|}{|Re^{it} - \alpha|} |iRe^{it}| \leqslant \frac{\|e\|_\infty}{R - |\alpha|} R \leqslant M \|e\|_\infty.$$

Hence, by Lebesgue's dominated convergence theorem,

$$\lim_{R\to\infty} \int_{\gamma_4} \frac{e(z)}{z-\alpha} \mathrm{d}z = \lim_{R\to\infty} \int_{-\omega}^{\omega} \frac{e(Re^{it})}{Re^{it} - \alpha} iRe^{it} \mathrm{d}t = \int_{-\omega}^{\omega} \lim_{R\to\infty} \frac{e(Re^{it})}{Re^{it} - \alpha} iRe^{it} \mathrm{d}t = 0$$

since from Lemma 2.3 $\lim_{z\to\infty} e(z) = 0$. Now letting $r \to 0$ and $R \to \infty$ the integrals on $\gamma_2$ and $\gamma_4$ tend to 0 and the other two constitute the desired. So,

$$e(\alpha) = \frac{1}{2\pi i} \int_{\partial S_\omega} \frac{e(z)}{z-\alpha} \mathrm{d}z.$$

Furthermore, since Cauchy's theorem shows that $e(\alpha)$ does not depend on the particular choice of $\omega$, we write

$$e(\alpha) = \frac{1}{2\pi i}\int_\Gamma \frac{e(z)}{z-\alpha}\mathrm{d}z = \frac{1}{2\pi i}\int_\Gamma e(z)\frac{z}{z-\alpha}\frac{\mathrm{d}z}{z},$$

where $\Gamma$ is the positively oriented boundary of the sector $S_{\omega'}$ with $\omega < \omega' < \phi$. There exists a constant $M$ such that $\frac{|z|}{|z-\alpha|} \leqslant M$ uniformly in $z \in \partial S_{\omega'}$, $\alpha \in S_\omega$. (Otherwise there have to exist sequences $z_n, \alpha_n$ with

$$\frac{|z_n|}{|z_n - \alpha_n|} = \frac{1}{\left|1 - \frac{\alpha_n}{z_n}\right|} \to \infty.$$

But that means $\frac{\alpha_n}{z_n} \to 1$ which can not happen since $\omega' > \omega$.) Hence

$$\frac{|e(z)|}{|z-\alpha|} = \frac{|e(z)|}{|z|}\frac{|z|}{|z-\alpha|} \leqslant M\frac{|e(z)|}{|z|}.$$

By Lebesgue's dominated convergence theorem

$$\lim_{\alpha\to 0} e(\alpha) = \frac{1}{2\pi i}\int_\Gamma \lim_{\alpha\to 0} e(z)\frac{z}{z-\alpha}\frac{\mathrm{d}z}{z} = \frac{1}{2\pi i}\int_\Gamma \frac{e(z)}{z}\mathrm{d}z.$$

But, from Lemma 2.3, $\lim_{\alpha\to 0} e(\alpha) = 0$, hence $\int_\Gamma \frac{e(z)}{z}\mathrm{d}z = 0$. □

A functional calculus for sectorial operators can be based on the class of functions $\mathcal{E}_0(S_\phi)$. Given an operator $A$ on a Banach space $X$ with $A \in \mathrm{Sect}(\omega)$, $\omega < \phi < \pi$ we define the mapping

$$\Phi : (e \mapsto e(A)) : \mathcal{E}_0(S_\phi) \to \mathcal{L}(X)$$

by means of a Cauchy integral

$$\Phi(e) := e(A) := \frac{1}{2\pi i}\int_\Gamma e(z)\,\mathrm{R}(z,A)\mathrm{d}z,$$

where $\Gamma$ is the positively oriented boundary of a sector $S_{\omega'}$ with $\omega < \omega' < \phi$ arbitrary and $e \in \mathcal{E}_0(S_\phi)$. Cauchy's integral theorem shows that $e(A)$ is independent of $\omega'$.

**Proposition 2.5** *Let $e \in \mathcal{E}_0(S_\phi)$ and $A \in Sect(\omega)$, with $\omega < \phi < \pi$. Then the mapping $\Phi : \mathcal{E}_0(S_\phi) \to \mathcal{L}(X)$ has the following properties.*

(i) *$\Phi$ is a homomorphism of algebras.*

(ii) *If $T \in \mathcal{L}(X)$ commutes with the resolvent of $A$, it also commutes with $\Phi(e)$.*

(iii) *We have*

$$\mathrm{R}(\lambda, A)\Phi(e) = \left(\frac{e(z)}{\lambda - z}\right)(A), \qquad \text{for } \lambda \notin \overline{S_\phi}.$$

(iv) $\mathcal{R}(e(A)) \subseteq \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$.

**Proof.**

(i) Obviously, $\Phi$ is linear. Suppose $e, h \in \mathcal{E}_0(S_\phi)$. Using Fubini's theorem and the resolvent identity $\mathrm{R}(z, A) - \mathrm{R}(w, A) = (w - z)\,\mathrm{R}(z, A)\,\mathrm{R}(w, A)$ we obtain

$$\begin{aligned}
\Phi(e)\Phi(h) &= \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\Gamma'} e(z)h(w)\,\mathrm{R}(z, A)\,\mathrm{R}(w, A)\ \mathrm{d}w\ \mathrm{d}z \\
&= \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\Gamma'} \frac{e(z)h(w)}{w - z}(\mathrm{R}(z, A) - \mathrm{R}(w, A))\ \mathrm{d}w\ \mathrm{d}z \\
&= \frac{1}{2\pi i}\int_\Gamma e(z)\,\mathrm{R}(z, A)\left(\frac{1}{2\pi i}\int_{\Gamma'}\frac{h(w)}{w - z}\mathrm{d}w\right)\mathrm{d}z \\
&- \frac{1}{2\pi i}\int_{\Gamma'} h(w)\,\mathrm{R}(w, A)\left(\frac{1}{2\pi i}\int_\Gamma\frac{e(z)}{w - z}\mathrm{d}z\right)\mathrm{d}w.
\end{aligned}$$

Since the integrals are independent of the choice of the path we may choose $\Gamma$, $\Gamma'$ such that $\Gamma$ lies to the left of $\Gamma'$ and $\Gamma \cap \Gamma' = \emptyset$. Then, Lemma 2.4 implies

$$\frac{1}{2\pi i}\int_\Gamma \frac{e(z)}{w - z}\mathrm{d}z = 0 \qquad \text{and} \qquad \frac{1}{2\pi i}\int_{\Gamma'}\frac{h(w)}{w - z}\mathrm{d}w = h(z).$$

Hence we obtain

$$\Phi(e)\Phi(h) = \frac{1}{2\pi i}\int_\Gamma e(z)h(z)\,\mathrm{R}(z, A)\mathrm{d}z = \Phi(eh).$$

(ii) Let $T \in \mathcal{L}(X)$ then for all $x \in X$

$$\begin{aligned}
T\Phi(e)x &= T\frac{1}{2\pi i}\int_\Gamma e(z)\,\mathrm{R}(z, A)x\mathrm{d}z = \frac{1}{2\pi i}\int_\Gamma e(z)T\,\mathrm{R}(z, A)x\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_\Gamma e(z)\,\mathrm{R}(z, A)Tx\mathrm{d}z = \Phi(e)Tx
\end{aligned}$$

(iii) Define $g := (\lambda - z)^{-1}e$. By the resolvent identity $A\,\mathrm{R}(z, A) = z\,\mathrm{R}(z, A) - I$ we obtain

$$\begin{aligned}(\lambda - A)g(A) &= \frac{1}{2\pi i}\int_\Gamma g(z)(\lambda - A)\,\mathrm{R}(z, A)\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_\Gamma g(z)[(\lambda - z)\,\mathrm{R}(z, A) + I]\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_\Gamma e(z)\,\mathrm{R}(z, A)\mathrm{d}z + \frac{1}{2\pi i}\int_\Gamma g(z)\mathrm{d}z = e(A).\end{aligned}$$

Note that the second integral equals 0 by Lemma 2.4. Hence we have $((\lambda - z)^{-1}e(z))(A) = \mathrm{R}(z, A)e(A)$.

(iv) Let $y \in \mathcal{R}(e(A))$ then there exists an $x \in X$ such that

$$y = e(A)x = \frac{1}{2\pi i}\int_\Gamma e(z)\,\mathrm{R}(z, A)x\ \mathrm{d}z.$$

Since $\mathrm{R}(z, A)x \in \mathcal{D}(A)$, the integral is in $\overline{\mathcal{D}(A)}$. Now, let $r > 0$ and consider the path $\Gamma_r = \{z \in \Gamma : r^{-1} \leqslant |z| \leqslant r\}$. Then

$$\frac{1}{2\pi i}\int_{\Gamma_r} e(z)\,\mathrm{R}(z, A)x\ \mathrm{d}z \to e(A)x \qquad \text{as} \qquad r \to \infty.$$

On the other hand, using the resolvent identity we obtain

$$\frac{1}{2\pi i}\int_{\Gamma_r} e(z)\,\mathrm{R}(z, A)x\ \mathrm{d}z = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{e(z)}{z}\mathrm{d}z\ x + \frac{1}{2\pi i}\int_{\Gamma_r} \frac{e(z)}{z} A\,\mathrm{R}(z, A)x\ \mathrm{d}z$$

where as $r$ tends to infinity the first integral tends to 0 and the other belongs in $\overline{\mathcal{R}(A)}$ since $A\,\mathrm{R}(z, A)x \in \mathcal{R}(A)$. Hence $y = e(A)x \in \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$. □

## 2.2 Extended Functional Calculus

In this section the definition $f(A)$ will be extended from $f \in \mathcal{E}_0(S_\phi)$ to a larger class of functions. Enlarge $\mathcal{E}_0(S_\phi)$ to

$$\mathcal{E}(S_\phi) = \mathcal{E}_0(S_\phi) \oplus \mathbb{C}\frac{1}{1+z} \oplus \mathbb{C}\mathbf{1}.$$

Let us begin with a useful characterization of these functions.

**Lemma 2.6** *Let $\phi \in (0, \pi]$. For $f : S_\phi \to \mathbb{C}$ the following assertions are equivalent*

(i) $f \in \mathcal{E}(S_\phi)$.

(ii) $f \in \mathcal{H}^\infty(\mathcal{S}_\phi)$ *and there exist* $d, d' \in \mathbb{C}$ *such that*

$$\lim_{z\to\infty} f(z) = d, \qquad \lim_{z\to 0} f(z) = d' \qquad (z \in S_\alpha)$$

*and for all* $0 \leqslant \alpha < \phi$

$$\int_{\substack{\partial S_\alpha \\ |z|\geqslant R}} |f(z) - d| \frac{|\mathrm{d}z|}{|z|} < \infty, \qquad \int_{\substack{\partial S_\alpha \\ |z|\leqslant r}} |f(z) - d'| \frac{|\mathrm{d}z|}{|z|} < \infty,$$

*with* $r < 1$, $R > 1$.

**Proof.**

(i) $\Rightarrow$ (ii) Let $f \in \mathcal{E}(S_\phi)$ then $f(z) = e(z) + a(1+z)^{-1} + b$ for an $e \in \mathcal{E}_0(S_\phi)$ and $a, b \in \mathbb{C}$. So clearly $f \in \mathcal{H}^\infty(S_\phi)$ and

$$\lim_{z\to\infty} f(z) = b =: d, \qquad \lim_{z\to 0} f(z) = a + b =: d'.$$

For $R > 1$, we obtain

$$\int_{\substack{\partial S_\alpha \\ |z|\geqslant R}} |f(z) - d| \frac{|\mathrm{d}z|}{|z|} = \int_{\substack{\partial S_\alpha \\ |z|\geqslant R}} \left| e(z) + \frac{a}{1+z} \right| \frac{|\mathrm{d}z|}{|z|} \leqslant \int_{\substack{\partial S_\alpha \\ |z|\geqslant R}} |e(z)| \frac{|\mathrm{d}z|}{|z|}$$
$$+ |a| \int_{\substack{\partial S_\alpha \\ |z|\geqslant R}} \frac{|\mathrm{d}z|}{|1+z||z|} \leqslant \int_{\substack{\partial S_\alpha \\ |z|\geqslant R}} |e(z)| \frac{|\mathrm{d}z|}{|z|} + |a| \int_{\substack{\partial S_\alpha \\ |z|\geqslant R}} \frac{|\mathrm{d}z|}{(|z|-1)|z|}$$
$$= \int_{\substack{\partial S_\alpha \\ |z|\geqslant R}} |e(z)| \frac{|\mathrm{d}z|}{|z|} + |a| \ln\left(\frac{1-R}{R}\right) < \infty.$$

For $r < 1$, we obtain

$$\int_{\substack{\partial S_\alpha \\ |z|\leqslant r}} |f(z) - d'| \frac{|\mathrm{d}z|}{|z|} = \int_{\substack{\partial S_\alpha \\ |z|\leqslant r}} \left| e(z) - \frac{az}{1+z} \right| \frac{|\mathrm{d}z|}{|z|} \leqslant \int_{\substack{\partial S_\alpha \\ |z|\leqslant r}} |e(z)| \frac{|\mathrm{d}z|}{|z|}$$
$$+ |a| \int_{\substack{\partial S_\alpha \\ |z|\leqslant r}} \frac{|\mathrm{d}z|}{|1+z|} \leqslant \int_{\substack{\partial S_\alpha \\ |z|\leqslant r}} |e(z)| \frac{|\mathrm{d}z|}{|z|} + |a| \int_{\substack{\partial S_\alpha \\ |z|\leqslant r}} \frac{|\mathrm{d}z|}{1-|z|}$$
$$= \int_{\substack{\partial S_\alpha \\ |z|\leqslant r}} |e(z)| \frac{|\mathrm{d}z|}{|z|} - |a| \ln(1-r) < \infty.$$

(ii) $\Rightarrow$ (i) Let $f$ be as in (ii). Then the function $g(z) = f(z) - d - (d' - d)(1+z)^{-1}$ is contained in $\mathcal{E}_0(S_\phi)$ whence $f \in \mathcal{E}(S_\phi)$. $\square$

Given an operator $A \in \mathrm{Sect}(\omega)$ ($\omega < \phi < \pi$) on a Banach space $X$, the class of functions $\mathcal{E}(S_\phi)$ yields an algebra homomorphism (c.f. [12, Section 2.3])

$$\Phi : (f \mapsto f(A)) : \mathcal{E}(S_\phi) \to \mathcal{L}(X),$$

by defining

$$f(A) = e(A) + \alpha(1 + A)^{-1} + b,$$

with $e \in \mathcal{E}_0(S_\phi)$ and $\alpha, b \in \mathbb{C}$. Let us present the following result.

**Lemma 2.7** *Let $\phi \in (0, \pi]$, and $f \in \mathcal{E}(S_\phi)$. Then there exists a constant $C_f$ such that*

$$\sup_{t>0} \|f(tA)\| \leqslant C_f M_{A,\phi},$$

*for each $A \in \mathrm{Sect}(\omega)$ with $0 < \omega < \phi$.*

**Proof.** For $z \in S_\phi$, write $f(z) = e(z) + \alpha(1 + z)^{-1} + b$, with $e \in \mathcal{E}_0(S_\phi)$ and $\alpha, b \in \mathbb{C}$. For $t > 0$ we have

$$\begin{aligned} e(tA) &= \frac{1}{2i\pi} \int_\Gamma e(z)\,\mathrm{R}(z, tA)\mathrm{d}z = \frac{1}{2i\pi} \int_\Gamma e(z)\frac{1}{t}\,\mathrm{R}(\frac{z}{t}, A)\mathrm{d}z \\ &= \frac{1}{2i\pi} \int_\Gamma e(tw)\,\mathrm{R}(w, A)\mathrm{d}w = e(tw)(A). \end{aligned}$$

Using the sectoriality we obtain

$$\begin{aligned} \|e(tA)\| &\leqslant \frac{1}{2\pi} \int_\Gamma |e(tz)|\|\,\mathrm{R}(z, A)\||\mathrm{d}z| \leqslant \frac{M_{A,\phi}}{2\pi} \int_\Gamma |e(tz)|\frac{|\mathrm{d}z|}{|z|} \\ &= \frac{M_{A,\phi}}{2\pi} \int_\Gamma |e(w)|\frac{|\mathrm{d}w|}{|w|} =: \frac{M_{A,\phi}}{2\pi} C_e, \end{aligned}$$

for all $t > 0$. Furthermore, using the non-negativity of $A$ (cf. [6, Section 1.2]) we have

$$\|(1 + tA)^{-1}\| = |1/t|\|((1/t) + A)^{-1}\| \leqslant M_A,$$

for all $t > 0$, with $1 \leqslant M_A \leqslant M_{A,\phi}$. Hence, we obtain

$$\begin{aligned}\|f(tA)\| &\leqslant \|e(tA)\| + |\alpha|\|(1+tA)^{-1}\| + |b| \leqslant \frac{C_e}{2\pi}M_{A,\phi} + |\alpha|M_A + |b| \\ &\leqslant \left(\frac{C_e}{2\pi} + |\alpha| + |b|\right) M_{A,\phi} =: C_f M_{A,\phi},\end{aligned}$$

for all $t > 0$. □

## 2.3 Topological Extension

If $A \in \mathrm{Sect}(\omega)$, one wants to define $f(A)$ for more general functions of $\mathcal{H}^\infty(S_\phi)$, $\omega < \phi < \pi$. As we said in the Introduction this can be done by algebraic or by topological means. In [12, Chapter 2] an extension procedure is described by algebraic means, in which the space of regularizable functions yields a holomorphic functional calculus. However, an algebraically extended functional calculus takes into account properties of the operator. There are many examples, like the resolvent of the logarithm, in which we need the injectivity of the operator in order to define $f(A)$ by the algebraically extended calculus.

Here, an extension of the holomorphic functional calculus for sectorial operators by topological means is described in order to cover cases of non-injective operators that do not fit in the algebraic extension. This could be considered as a special case of the abstract framework we described in Section 1.2. According to that, we have to find an algebra $\mathcal{E}_{\mathrm{top}}(S_\phi)$ with $\mathcal{E}_{\mathrm{top}}(S_\phi) \subseteq H^\infty(S_\phi)$ and $\mathcal{E}(S_\phi) \subseteq \mathcal{E}_{\mathrm{top}}(S_\phi)$ equipped with a reasonable notion of convergence. It will be shown that the boundedly pointwise convergence on $S_\phi \cup \{0\}$ is a possible convergence notion to use. Moreover, recall that the following pattern (look at (1.1)) is required:

$$(f_n)_n \subseteq \mathcal{E}(S_\phi),\ f_n \to 0 \text{ and } f_n(A) \to T \ \Rightarrow\ T = 0,$$

where the topology on $\mathcal{L}(X)$ is the strong operator topology. We obtain the following convergence lemma.

**Lemma 2.8** *If $(f_n)_n \subseteq \mathcal{H}^\infty(S_\phi)$ is a sequence such that $f_n \to 0$ pointwise on $S_\phi$, then*

$$\|(ef_n)(A)\| \to 0 \qquad (e \in \mathcal{E}_0(S_\phi))\,.$$

**Proof.** Fix $e \in \mathcal{E}_0(S_\phi)$. Then, $ef_n \in \mathcal{E}_0(S_\phi)$. By Vitali's theorem (cf. [2, Theorem 2.1]) $f_n \to 0$ uniformly on compact subsets of $S_\phi$. Let $r > 1$ and $K_r = \{z \in \mathbb{C} : r^{-1} \leqslant |z| \leqslant r\}$. Then, using the sectoriality $\|z\,\mathrm{R}(z,A)\| \leqslant M$, we obtain

$$\begin{aligned}\|(ef_n)(A)\| &= \left\| \frac{1}{2\pi i}\int_\Gamma e(z) f_n(z)\,\mathrm{R}(z,A)\mathrm{d}z \right\| \leqslant \frac{M}{2\pi}\int_\Gamma |e(z)||f_n(z)|\frac{|\mathrm{d}z|}{|z|} \\ &\leqslant \frac{M}{2\pi}\left( \sup_{z\in K_r} |f_n(z)| \int_{\Gamma\cap K_r} |e(z)|\frac{|\mathrm{d}z|}{|z|} + \|f_n\|_\infty \int_{\Gamma\setminus K_r} |e(z)|\frac{|\mathrm{d}z|}{|z|}\right)\end{aligned}$$

Letting $n\to\infty$ we have

$$\overline{\lim_{n\to\infty}}\|(ef_n)(A)\| \leqslant \frac{MC}{2\pi}\int_{\Gamma\setminus K_r} |e(z)|\frac{|\mathrm{d}z|}{|z|}$$

where $C=\sup_n \|f_n\|_\infty<\infty$. Now letting $r\to\infty$ we have the claim. □

As a corollary of this lemma, we obtain the requirement we need.

**Corollary 2.9** *Let $(f_n)_n \subseteq \mathcal{E}(S_\phi)$ be a sequence such that $f_n\to 0$ boundedly and pointwise on $S_\phi\cup\{0\}$ and suppose that $f_n(A)\to T\in\mathcal{L}(X)$ in the strong operator topology. Then $T=0$.*

**Proof.** Suppose at first that $(f_n)_n\subseteq \mathcal{E}_0(S_\phi)$. Let $e\in\mathcal{E}_0(S_\phi)$. By Lemma 2.8

$$f_n(A)e(A) = e(A)f_n(A) = (ef_n)(A)\to 0$$

in norm, hence also strongly. On the other hand, $f_n(A)e(A)\to Te(A)$ strongly since multiplication is continuous for the strong topology. So, $Te(A)=e(A)T=0$. Let $y=Tx\in\mathcal{R}(T)$. Then, since $z(1+z)^{-2}\in\mathcal{E}_0(S_\phi)$,

$$A(1+A)^{-2}y = A(1+A)^{-2}Tx = 0.$$

So, $y\in\mathcal{N}(A(1+A)^{-2})=\mathcal{N}(A)$, i.e., $\mathcal{R}(T)\subseteq\mathcal{N}(A)$. By Proposition 2.5 $\mathcal{R}(f_n(A))\subseteq\overline{\mathcal{R}(A)}$. So,

$$y = Tx = \lim_{n\to\infty} f_n(A)x \in \overline{\mathcal{R}(A)}.$$

Hence, $\mathcal{R}(T)\subseteq\overline{\mathcal{R}(A)}$, and we arrive at $\mathcal{R}(T)\subseteq\overline{\mathcal{R}(A)}\cap\mathcal{N}(A)$. However, $\overline{\mathcal{R}(A)}\cap\mathcal{N}(A)=0$ for every sectorial operator (cf. [12, Proposition 2.1.1]), hence $T=0$. Now suppose $(f_n)_n\subseteq\mathcal{E}(S_\phi)$. Then we can write

$$f_n = e_n + \frac{c_n}{1+z} + d_n = e_n + \frac{c_n+d_n}{1+z} + \frac{d_n z}{1+z}$$

for some $(e_n)_n \subseteq \mathcal{E}_0(S_\phi)$ and $(c_n)_n, (d_n)_n \subseteq \mathbb{C}$. We have $\lim_{z\to 0} f_n(z) = c_n + d_n$, so the hypothesis implies $c_n+d_n \to 0$. Hence, $e_n(A)+d_nA(1+A)^{-1} \to T$. Multiplying the operator $(1 + A)^{-1}$ we obtain that

$$(1 + A)^{-1}e_n(A) + d_nA(1 + A)^{-2} \to (1 + A)^{-1}T$$

but $(1 + z)^{-1}e_n + d_n(1 + z)^{-2}z \in \mathcal{E}_0(S_\phi)$. Hence, we are in the first case and it follows that $(1 + A)^{-1}T = 0$, which yields $T = 0$. □

As a consequence of this result we can enlarge the functional calculus topologically towards

$$\mathcal{E}_{\text{top}}(S_\phi) = \left\{ f \in \mathcal{H}^\infty(S_\phi) \,\middle|\, \begin{array}{l} \exists (f_n)_n \subseteq \mathcal{E}(S_\phi) \text{ with } f_n \to f \text{ boundedly and} \\ \text{pointwise on } S_\phi \cup \{0\}, \ \lim f_n(A) \text{ exists strongly.} \end{array} \right\}$$

Given an operator $A \in \mathrm{Sect}(\omega)$, $\omega < \phi < \pi$, we define the mapping

$$\Phi : (f \mapsto f(A)) : \mathcal{E}_{\text{top}}(S_\phi) \to \mathcal{L}(X)$$

setting

$$f(A) = \lim_{n\to\infty} f_n(A),$$

in the strong operator topology, where $(f_n)_n \subseteq \mathcal{E}(S_\phi)$ with $f_n \to f$ boundedly and pointwise on $S_\phi \cup \{0\}$.

**Theorem 2.10** *Suppose $A \in Sect(\omega)$, $\omega < \phi < \pi$. The mapping $\Phi : \mathcal{E}_{top}(S_\phi) \to \mathcal{L}(X)$ is an algebra homomorphism.*

This is just a special case of Proposition 1.6. Indeed, the assumptions are fulfilled since, the boundedly and pointwise convergence notion on $S_\phi \cup \{0\}$ delivers the continuity of multiplication for functions in $\mathcal{E}_{\text{top}}(S_\phi)$ and so does the strong operator topology on $\mathcal{L}(X)$. Furthermore, the requirement (1.1) is available from Corollary 2.9.

Now, let us illustrate a valued application which shows that functions with a particular representation are in the topologically extended function class. We define the set of complex Borel measures on $[0, \infty)$ as

$$\mathbf{M}(\mathbb{R}_+) := \left\{ \mu : \int_{\mathbb{R}_+} |\mu|(\mathrm{d}t) < \infty \right\},$$

where $|\mu|$ stands for the total variation of $\mu$. The following theorem is a very useful tool that will be used afterwards.

**Theorem 2.11** *Let $A \in Sect(\omega)$ and $g \in \mathcal{E}(S_\phi)$ with $\omega < \phi$. Moreover, let $\mu \in \mathbf{M}(\mathbb{R}_+)$ and $f : S_\phi \to \mathbb{C}$ be defined as*

$$f(z) = \int_{\mathbb{R}_+} g(tz)\mu(\mathrm{d}t) \qquad (z \in S_\phi).$$

*Then $f \in \mathcal{E}_{top}(S_\phi)$ and*

$$f(A) = \int_{\mathbb{R}_+} g(tA)\mu(\mathrm{d}t).$$

**Proof.** Let $g \in \mathcal{E}(S_\phi)$. By Lemma 2.6 there exists a constant $M_\phi$ such that $|g(tz)| \leqslant M_\phi$ uniformly in $z \in S_\phi, t \in \mathbb{R}_+$. So, $f \in \mathcal{H}^\infty(S_\phi)$ since

$$|f(z)| \leqslant \int_{\mathbb{R}_+} |g(tz)||\mu|(\mathrm{d}t) \leqslant M_\phi \int_{\mathbb{R}_+} |\mu|(\mathrm{d}t) < \infty.$$

Without loss of generality, assume $\mu(\{0\}) = 0$. Consider the sequence

$$f_n(z) = \int_{(1/n,n)} g(tz)\mu(\mathrm{d}t) \qquad (z \in S_\phi).$$

In case $\mu(\{0\}) \neq 0$ one has to add a constant in the above form. It will be shown that $(f_n)_{n\in\mathbb{N}} \in \mathcal{E}(S_\phi)$. Indeed, from Lemma 2.6 there exist $d, d' \in \mathbb{C}$ such that $\lim_{z\to 0} g(z) = d$, $\lim_{z\to\infty} g(z) = d'$ and for $r < 1$, $R > 1$

$$\int_{\substack{\partial S_\alpha \\ |z|\leqslant r}} |g(z) - d|\frac{|\mathrm{d}z|}{|z|} = C < \infty, \qquad \int_{\substack{\partial S_\alpha \\ |z|\geqslant R}} |g(z) - d'|\frac{|\mathrm{d}z|}{|z|} = C' < \infty,$$

for $0 \leqslant \alpha < \phi$. By Lebesgue's dominated convergence theorem

$$\lim_{z\to 0} f_n(z) = \int_{(1/n,n)} \lim_{z\to 0} g(tz)\mu(\mathrm{d}t) = d \int_{(1/n,n)} \mu(\mathrm{d}t) =: d_n < \infty$$

and

$$\lim_{z\to\infty} f_n(z) = \int_{(1/n,n)} \lim_{z\to\infty} g(tz)\mu(\mathrm{d}t) = d' \int_{(1/n,n)} \mu(\mathrm{d}t) =: d'_n < \infty.$$

Furthermore, for $|z| \leqslant r < 1$ and $0 \leqslant \alpha < \phi$ we have

$$\begin{aligned}
&\int_{\substack{\partial S_\alpha \\ |z|\leqslant r}} |f_n(z) - d_n|\frac{|\mathrm{d}z|}{|z|} = \int_{\substack{\partial S_\alpha \\ |z|\leqslant r}} \left|\int_{(1/n,n)} (g(tz) - d)\mu(\mathrm{d}t)\right| \frac{|\mathrm{d}z|}{|z|} \\
&\leqslant \int_{\substack{\partial S_\alpha \\ |z|\leqslant r}} \int_{(1/n,n)} |g(tz) - d||\mu|(\mathrm{d}t)\frac{|\mathrm{d}z|}{|z|} = \int_{(1/n,n)} \int_{\substack{\partial S_\alpha \\ |z|\leqslant r}} |g(w) - d|\frac{|\mathrm{d}w|}{|w|}|\mu|(\mathrm{d}t) \\
&\leqslant C \int_{(1/n,n)} |\mu|(\mathrm{d}t) < \infty,
\end{aligned}$$

and for $|z| \geqslant R > 1$ we obtain

$$\begin{aligned}
&\int_{\substack{\partial S_\alpha \\ |z|\geqslant R}} |f_n(z) - d_n'| \frac{|\mathrm{d}z|}{|z|} = \int_{\substack{\partial S_\alpha \\ |z|\geqslant R}} \left| \int_{(1/n,n)} (g(tz) - d')\mu(\mathrm{d}t) \right| \frac{|\mathrm{d}z|}{|z|} \\
&\leqslant \int_{\substack{\partial S_\alpha \\ |z|\geqslant R}} \int_{(1/n,n)} |g(tz) - d'||\mu|(\mathrm{d}t) \frac{|\mathrm{d}z|}{|z|} = \int_{(1/n,n)} \int_{\substack{\partial S_\alpha \\ |z|\geqslant R}} |g(w) - d'| \frac{|\mathrm{d}w|}{|w|} |\mu|(\mathrm{d}t) \\
&\leqslant C' \int_{(1/n,n)} |\mu|(\mathrm{d}t) < \infty.
\end{aligned}$$

Hence, by Lemma 2.6 $f_n \in \mathcal{E}(S_\phi)$. The sequence $e_n(z) = f_n(z) - (d_n - d_n')(1 + z)^{-1} - d_n' \in \mathcal{E}_0(S_\phi)$. By the elementary functional calculus we obtain

$$\begin{aligned}
e_n(A) &= \frac{1}{2\pi i} \int_\Gamma e_n(z) \, \mathrm{R}(z, A) \, \mathrm{d}z \\
&= \frac{1}{2\pi i} \int_\Gamma \int_{(1/n,n)} \left( g(tz) - (d - d')(1 + z)^{-1} - d' \right) \mu(\mathrm{d}t) \, \mathrm{R}(z, A) \, \mathrm{d}z \\
&= \int_{(1/n,n)} \frac{1}{2\pi i} \int_\Gamma \left( g(tz) - (d - d')(1 + z)^{-1} - d' \right) \mathrm{R}(z, A) \, \mathrm{d}z \, \mu(\mathrm{d}t) \\
&= \int_{(1/n,n)} g(tA) - (d - d')(1 + A)^{-1} - d' \, \mu(\mathrm{d}t)
\end{aligned}$$

since $g(tz) - (d - d')(1 + z)^{-1} - d' \in \mathcal{E}_0(S_\phi)$. Hence we obtain

$$f_n(A) = e_n(A) + (d_n - d_n')(1 + A)^{-1} + d_n' = \int_{(1/n,n)} g(tA)\mu(\mathrm{d}t).$$

Moreover, $f_n \to f$ pointwise and boundedly on $S_\phi \cup \{0\}$. Since $g \in \mathcal{E}(S_\phi)$, from Lemma 2.7 there exists a constant $C_g$ such that $\|g(tA)\| \leqslant C_g M_{A,\phi}$ for all $t \in \mathbb{R}_+$. Hence, since $\mu(\{0\}) = 0$, we have

$$\begin{aligned}
&\left\| f_n(A) - \int_{\mathbb{R}_+} g(tA)\mu(\mathrm{d}t) \right\| = \left\| \int_{(1/n,n)} g(tA)\mu(\mathrm{d}t) - \int_{(0,\infty)} g(tA)\mu(\mathrm{d}t) \right\| \\
&\leqslant \left( \int_{(0,1/n)} + \int_{(n,\infty)} \right) \|g(tA)\||\mu|(\mathrm{d}t) \leqslant C_g M_{A,\phi} \left( \int_{(0,1/n)} + \int_{(n,\infty)} \right) |\mu|(\mathrm{d}t).
\end{aligned}$$

Letting $n \to \infty$ we have $|\mu|((0, 1/n)), |\mu|((n, \infty)) \to 0$ so the last integrals tend to zero, hence

$$f_n(A) \to \int_{\mathbb{R}_+} g(tA)\mu(\mathrm{d}t)$$

in the operator norm, so also strongly. In conclusion, $f \in \mathcal{E}_{\mathrm{top}}(S_\phi)$ and

$$f(A) = \int_{\mathbb{R}_+} g(tA)\mu(\mathrm{d}t)$$

according to the definition of $\mathcal{E}_{\mathrm{top}}(S_\phi)$. □

## 2.4 Algebraic versus Topological Extension

In this section we describe the relation of algebraic and topological extension of the holomorphic functional calculus for sectorial operators. The compatibility between them could be considered as a special case of Proposition 1.7. In the algebraic extension, as described in [12, Chapter 2], for a sectorial operator $A \in \mathrm{Sect}(\omega)$, the wider class of functions is the class of regularizable functions

$$\mathcal{E}_{\mathrm{alg}}(S_\phi) = \{f \in \mathcal{H}^\infty(S_\phi) : \exists e \in \mathcal{E}_0(S_\phi) \text{ s.t. } e(A) \text{ is injective and } ef \in \mathcal{E}_0(S_\phi)\}\,.$$

For $f \in \mathcal{E}_{\mathrm{alg}}(S_\phi)$, the operator $f(A)$ is defined by

$$f(A) = e(A)^{-1}(ef)(A)$$

with $e \in \mathcal{E}_0(S_\phi)$ being a regularizer for $f$. The algebraic extension procedure takes into account properties of the given operator. If $A$ is injective all of $\mathcal{H}^\infty(S_\phi)$ is already regularized by $\mathcal{E}_0(S_\phi)$. Namely, $z(1+z)^{-2}$ regularizes every function in $\mathcal{H}^\infty(S_\phi)$. If $A$ is not injective, as we will see later, there are functions in $\mathcal{E}_{\mathrm{top}}(S_\phi)$ that give rise to a functional calculus, but they are not regularizable. As it is shown in the following lemma, if a function $f$ is regularizable and in $\mathcal{E}_{\mathrm{top}}(S_\phi)$ the definition of $f(A)$ by either calculus, leads to the same operator.

**Lemma 2.12** *Let $A \in \mathrm{Sect}(\omega)$, $\omega < \phi < \pi$. If $f \in \mathcal{E}_{alg}(S_\phi) \cap \mathcal{E}_{top}(S_\phi)$ then the definition of $f(A)$ by either functional calculus leads to the same operator.*

This is just a special case of Proposition 1.7. The advantages of the topologically extended functional calculus appear if $A$ is not injective when we can cover other calculi as we will see in the next section. If $A$ is injective, we do not gain anything new by the topological extension.

## 2.5 Hirsch Functional Calculus

In this section it will be shown that the so-called Hirsch functional calculus, as presented in [6, Chapter 4], is covered by the topologically extended functional calculus. Let us begin with a basic concept. Consider the set of functions

$$\mathcal{T} := \left\{ f : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C} :\ f(z) = \int_{\mathbb{R}_+} \frac{1}{1+zt}\mu(\mathrm{d}t),\ \mu \in \mathbf{M}(\mathbb{R}_+) \right\}.$$

**Proposition 2.13** *Let $A \in Sect(\omega)$. If $f \in \mathcal{T}$ then $f \in \mathcal{E}_{top}(S_\phi)$, $\omega < \phi$, and*

$$f(A) = \int_{\mathbb{R}_+} (1 + tA)^{-1} \mu(\mathrm{d}t).$$

**Proof.** Let $f \in \mathcal{T}$ and $0 < \phi < \pi$. Define the function $g : S_\phi \to \mathbb{C}$ by $g(z) := (1 + z)^{-1}$. Then, $g \in \mathcal{E}(S_\phi)$. Let $A \in \mathrm{Sect}(\omega)$ with $\omega < \phi$. Then from Theorem 2.11

$$f(z) = \int_{\mathbb{R}_+} g(tz)\mu(\mathrm{d}t) = \int_{\mathbb{R}_+} \frac{1}{1 + zt}\mu(\mathrm{d}t), \qquad (z \in S_\phi)$$

is in $\mathcal{E}_{\mathrm{top}}(S_\phi)$ and

$$f(A) = \int_{\mathbb{R}_+} g(tA)\mu(\mathrm{d}t) = \int_{\mathbb{R}_+} (1 + tA)^{-1}\mu(\mathrm{d}t).$$

□

Using the above result let us see how the definition of $f(A)$ from the so-called Hirsch calculus can be incorporated into the holomorphic approach.

**Example 2.14** Let $\mu \in \mathbf{M}(\mathbb{R}_+)$. Consider functions of the form

$$f(z) = \int_{\mathbb{R}_+} \frac{z(t + 1)}{1 + zt}\mu(\mathrm{d}t).$$

In [6, Section 4.2], the Hirsch functional calculus for non-negative operators $A$ is defined as

$$f(A) = A \int_{[0,1]} (1 + tA)^{-1}(t + 1)\mu(\mathrm{d}t) + \int_{(1,\infty)} A(1 + tA)^{-1}(t + 1)\mu(\mathrm{d}t)$$

on the domain

$$\mathcal{D}(f(A)) = \left\{ x \in X : \int_{[0,1]} (1 + tA)^{-1}(t + 1)\mu(\mathrm{d}t)x \in \mathcal{D}(A) \right\}.$$

Here, using a regularization argument it will be shown that the Hirsch calculus is accessible via the topologically extended functional calculus. First of all, note that every non-negative operator is sectorial and vice versa (cf. [6, Section 1.2]. Using the identity $z(1 + zt)^{-1} = \frac{1}{t} - \frac{1}{t}(1 + zt)^{-1}$ we can write $f$ as

$$f(z) = z \int_{[0,1]} \frac{1}{1 + zt}(t + 1)\mu(\mathrm{d}t) + \int_{(1,\infty)} \frac{t + 1}{t}\mu(\mathrm{d}t) - \int_{(1,\infty)} \frac{1}{1 + zt}\frac{t + 1}{t}\mu(\mathrm{d}t).$$

Define the measures

$$\mu_1(\mathrm{d}t) := \begin{cases} (t+1)\mu(\mathrm{d}t), & t \in [0,1], \\ 0, & t \in (1,\infty) \end{cases},$$

and

$$\mu_2(\mathrm{d}t) := \begin{cases} 0, & t \in [0,1], \\ \frac{(t+1)}{t}\mu(\mathrm{d}t), & t \in (1,\infty). \end{cases}$$

It is clear that $\mu_1, \mu_2 \in \mathbf{M}(\mathbb{R}_+)$. Setting

$$c := \int_{\mathbb{R}_+} \mu_2(\mathrm{d}t), \quad f_1(z) := \int_{\mathbb{R}_+} \frac{1}{1+zt}\mu_1(\mathrm{d}t), \quad f_2(z) := \int_{\mathbb{R}_+} \frac{1}{1+zt}\mu_2(\mathrm{d}t)$$

we have

$$f(z) = c + zf_1(z) - f_2(z).$$

Note that $f_1, f_2 \in \mathcal{T}$. In order to find the operator $(zf_1(z))(A)$ we will use a regularization method. First of all we have

$$z(1+z)^{-1}(A) = \left(\mathbf{1} - \frac{1}{1+z}\right)(A) = I - (1+A)^{-1}$$

The function $(1+z)^{-1}$ regularizes $zf_1(z)$ since $(1+z)^{-1}(A) = (1+A)^{-1}$ is injective and

$$\begin{aligned}(zf_1(z))(A) &= (1+A)\big((1+z)^{-1}zf_1(z)\big)(A) = (1+A)\big((1+z)^{-1}z\big)(A)f_1(A) \\ &= (1+A)(I-(1+A)^{-1})f_1(A) = Af_1(A).\end{aligned}$$

Hence, from Proposition 2.13, we conclude

$$f(A) = c + Af_1(A) - f_2(A)$$

which is the same result as is defined in [6, Chapter 4].

In that way we see that the Hirsch functional calculus for non-negative operators can be covered by an algebraic extension of the topologically extended holomorphic functional calculus for sectorial operators. Another example that can be covered by this concept is the resolvent of the logarithm. Logarithms play a fundamental role in the theory of sectorial operators.

**Example 2.15** Let $A$ be a sectorial operator and $\lambda \in \mathbb{C}$ with $|\operatorname{Im}\lambda| > \pi$, so $\lambda \in \rho(\log A)$. Consider the function $f : \mathbb{C}\setminus\mathbb{R}_- \to \mathbb{C}$ given by $f(z) = (\lambda - \log z)^{-1}$. Then, for all $z \in \mathbb{C}\setminus\mathbb{R}_-$, $f$ can be represented as

$$\frac{1}{\lambda - \log z} = \int_0^\infty \frac{-1}{(t+z)}\frac{\mathrm{d}t}{(\lambda-\log t)^2+\pi^2} = \int_0^\infty \frac{1}{1+zt}\frac{-\mathrm{d}t}{t((\lambda+\log t)^2+\pi^2)}.$$

To see this, let $0 < \omega < \pi$, $r < 1$ and $R > 1$. Consider the positively oriented closed curve $C$ which consists of four parts:

$$\begin{aligned}
\gamma_1 &= \{-te^{i\omega}, \quad -R \leqslant t \leqslant -r\},\\
\gamma_2 &= \{re^{-it}, \quad -\omega \leqslant t \leqslant \omega\},\\
\gamma_3 &= \{te^{-i\omega}, \quad r \leqslant t \leqslant R\},\\
\gamma_4 &= \{Re^{it}, \quad -\omega \leqslant t \leqslant \omega\}.
\end{aligned}$$

For any point $\alpha$ inside $C$, by Cauchy's integral formula we have

$$f(\alpha) = \frac{1}{2\pi i}\oint_C \frac{1}{(z-\alpha)(\lambda - \log z)}\mathrm{d}z = \frac{1}{2\pi i}\sum_{n=1}^{4}\int_{\gamma_n}\frac{1}{(z-\alpha)(\lambda-\log z)}\mathrm{d}z.$$

An estimation for the integral on $\gamma_2$ is

$$\left|\int_{\gamma_2}\frac{\mathrm{d}z}{(z-\alpha)(\lambda-\log z)}\right| \leqslant \int_{\gamma_2}\frac{\mathrm{d}z}{(|\alpha|-|z|)|\operatorname{Re}\lambda - \ln|z||} = \frac{2\omega r}{(|\alpha|-r)|\operatorname{Re}\lambda - \ln r|}$$

and for the integral on $\gamma_4$

$$\left|\int_{\gamma_4}\frac{\mathrm{d}z}{(z-\alpha)(\lambda-\log z)}\right| \leqslant \int_{\gamma_4}\frac{\mathrm{d}z}{(|z|-|\alpha|)|\operatorname{Re}\lambda - \ln|z||} = \frac{2\omega R}{(R-|\alpha|)|\operatorname{Re}\lambda - \ln R|}.$$

Inserting the parametric equations in the integral on $\gamma_1$ we obtain

$$\begin{aligned}
\int_{\gamma_1}\frac{1}{(z-\alpha)(\lambda-\log z)}\mathrm{d}z &= \int_{-R}^{-r}\frac{-e^{i\omega}\mathrm{d}t}{(-te^{i\omega}-\alpha)(\lambda-\log(-t)-i\omega)}\\
&= \int_r^R \frac{-e^{i\omega}\mathrm{d}s}{(se^{i\omega}-\alpha)(\lambda-\log s - i\omega)}
\end{aligned}$$

and on $\gamma_3$

$$\int_{\gamma_3}\frac{1}{(z-\alpha)(\lambda-\log z)}\mathrm{d}z = \int_r^R \frac{e^{-i\omega}\mathrm{d}t}{(te^{-i\omega}-\alpha)(\lambda-\log t + i\omega)}$$

Now, letting $r \to 0$, $R \to \infty$ and $\omega \to \pi$, the integrals on $\gamma_2$ and $\gamma_4$ tend to 0, so we obtain

$$\begin{aligned} f(\alpha) &= \frac{1}{2\pi i}\int_0^\infty \frac{1}{(-t-\alpha)(\lambda-\log t-i\pi)}+\frac{-1}{(-t-\alpha)(\lambda-\log t+i\pi)}\mathrm{d}t \\ &= \frac{1}{2\pi i}\int_0^\infty \frac{2\pi i}{-(t+\alpha)\left((\lambda-\log t)^2+\pi^2\right)}\mathrm{d}t = \int_0^\infty \frac{-\mathrm{d}t}{(t+\alpha)\left((\lambda-\log t)^2+\pi^2\right)} \\ &= \int_0^\infty \frac{1}{1+\alpha t}\frac{-\mathrm{d}t}{t((\lambda+\log t)^2+\pi^2)}. \end{aligned}$$

Now, let us define the measure

$$\mu(\mathrm{d}t) := \frac{-1}{t((\lambda+\log t)^2+\pi^2)}\mathrm{d}t, \qquad t>0$$

and $\mu(\{0\})=0$. It will be shown that $\mu\in\mathbf{M}(\mathbb{R}_+)$. We have

$$\int_{(0,\infty)}|\mu|(\mathrm{d}t) = \int_{(0,\infty)}\frac{1}{|(\lambda+\log t)^2+\pi^2|}\frac{\mathrm{d}t}{t} = \int_{\mathbb{R}}\frac{\mathrm{d}t}{|(\lambda+t)^2+\pi^2|}.$$

We change the variable $s=t+\operatorname{Re}\lambda$ and compute

$$\begin{aligned} |(i\operatorname{Im}\lambda+s)^2+\pi^2| &= \sqrt{\left(s^2-((\operatorname{Im}\lambda)^2-\pi^2)\right)^2+4s^2(\operatorname{Im}\lambda)^2} \\ &\geqslant \sqrt{\left(s^2-((\operatorname{Im}\lambda)^2-\pi^2)\right)^2+4s^2\big((\operatorname{Im}\lambda)^2-\pi^2\big)} \\ &= \sqrt{\left(s^2+((\operatorname{Im}\lambda)^2-\pi^2)\right)^2} = s^2+((\operatorname{Im}\lambda)^2-\pi^2). \end{aligned}$$

So we have

$$\int_{(0,\infty)}|\mu|(\mathrm{d}t) = \int_{\mathbb{R}}\frac{\mathrm{d}s}{|(i\operatorname{Im}\lambda+s)^2+\pi^2|} \leqslant \int_{\mathbb{R}}\frac{\mathrm{d}s}{s^2+((\operatorname{Im}\lambda)^2-\pi^2)} < \infty,$$

for all $\lambda\in\mathbb{C}$ with $|\operatorname{Im}\lambda|>\pi$. Hence, $f\in\mathcal{T}$ and by Proposition 2.13

$$\left(\frac{1}{\lambda-\log z}\right)(A) = \int_{\mathbb{R}_+}(1+tA)^{-1}\mu(\mathrm{d}t) = \int_{\mathbb{R}_+}\frac{-1}{(\lambda-\log t)^2+\pi^2}(t+A)^{-1}\mathrm{d}t.$$

The above example is also important because it shows the advantage of the topologically extended calculus in the case $A$ is not injective. In [12, section 3.5] the resolvent of the logarithm is defined as above but only for injective sectorial operators. If $A$ is not injective, the logarithm of the operator is not defined by the algebraic extension of holomorphic functional calculus, since $(\lambda-\log z)^{-1}$ is not regularizable. The following example is another application which shows the advantage of the topologically extended calculus in the case $A$ is not injective.

**Example 2.16** Consider a measure $\mu \in \mathbf{M}(\mathbb{R}_+)$, with $\mu(\{0\}) = 0$, and the function $f : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$ with

$$f(z) = \int_{(0,\infty)} \frac{t}{t+z} \mu(\mathrm{d}t).$$

We have that $f \in \mathcal{T}$. Indeed, if we change the variable $s = 1/t$ we have

$$f(z) = \int_{(0,\infty)} \frac{t}{t+z} \mu(\mathrm{d}t) = \int_{(0,\infty)} -\frac{1}{1+sz} \mu(\mathrm{d}(1/s)) = \int_{(0,\infty)} \frac{1}{1+sz} \nu(\mathrm{d}s),$$

where

$$\nu(\mathrm{d}s) := \begin{cases} -\mu(\mathrm{d}(1/s)), & s \in (0,\infty), \\ 0, & s = 0 \end{cases} .$$

Since $\mu \in \mathbf{M}(\mathbb{R}_+)$, it is clear that $\nu \in \mathbf{M}(\mathbb{R}_+)$, so $f \in \mathcal{T}$, and for a sectorial operator $A$, by Proposition 2.13

$$f(A) = \int_{(0,\infty)} (1+sA)^{-1} \nu(\mathrm{d}s) = \int_{(0,\infty)} t(t+A)^{-1} \mu(\mathrm{d}t).$$

This is also the definition by Hirsch calculus for non-negative operators (cf. [6, Section 4.2]). It is not possible, in general, to cover the Hirsch calculus with the algebraic extended functional calculus, because if $A$ is not injective, $f$ is not regularizable. For example, take a sequence $(a_n)_{n \geqslant 2}$ with

$$\alpha_n \geqslant 0, \quad \alpha := \sum_{n \geqslant 2} \alpha_n < \infty, \quad \sum_{n \geqslant 2} \alpha_n \log(n+1) = \infty$$

and define $\mu := \sum_{n \geqslant 2} \alpha_n \delta_{1/n}$. Then $\mu$ is a positive measure with total mass $\alpha$. We have

$$\begin{aligned}
\int_0^1 |f(x) - f(0)| \frac{\mathrm{d}x}{x} &= \int_0^1 \left| \int_{(0,\infty)} \frac{t}{t+x} - 1 \; \mu(\mathrm{d}t) \right| \frac{\mathrm{d}x}{x} \\
&= \int_0^1 \left| \sum_{n \geqslant 2} \alpha_n \frac{-x}{1/n + x} \right| \frac{\mathrm{d}x}{x} = \sum_{n \geqslant 2} \alpha_n \int_0^1 \frac{1}{1/n + x} \mathrm{d}x \\
&= \sum_{n \geqslant 2} \alpha_n \left( \log\left(\frac{1}{n} + 1\right) - \log \frac{1}{n} \right) = \sum_{n \geqslant 2} \alpha_n \log(n+1) = \infty.
\end{aligned}$$

Hence, Lemma 2.6 implies that $f$ is not contained $\mathcal{E}(S_\phi)$, i.e. $f(A)$ is not defined by the algebraic extension of holomorphic functional calculus in case $A$ is not injective. But, we have seen that $f(A)$ is defined by the topologically extended calculus.

## 2.6 Holomorphic Semigroups

In this section it will be shown that convergence rates for the Euler's approximation of a bounded holomorphic semigroup can be obtained using the topologically extended functional calculus. Bounded holomorphic semigroups are already accessible by the holomorphic functional calculus for sectorial operators raised from $\mathcal{E}(S_\phi)$, $\phi < \pi/2$ (cf. [12, Section 3.4.]). The idea is, using the Laplace transform of Borel measures, to obtain convergence rates for the Euler's approximation of functions and then having the functional calculus the estimation for the operator comes immediately. Suppose $-A$ generates a bounded holomorphic semigroup. Then, $A \in \mathrm{Sect}(\omega)$ with $0 \leqslant \omega < \pi/2$ (c.f. [11, Theorem 4.6]). Consider a complex Borel measure $\mu \in \mathbf{M}(\mathbb{R}_+)$ and its Laplace transform

$$\mathcal{L}_\mu(z) := \int_{\mathbb{R}_+} e^{-tz}\mu(\mathrm{d}t), \qquad \mathrm{Re}\, z > 0.$$

Let us define the set of functions

$$\mathcal{G}_k = \{f : \mathbb{C}_+ \to \mathbb{C} : f(z) = \int_{\mathbb{R}_+} (tz)^k e^{-tz}\mu(\mathrm{d}t),\ \mu \in \mathbf{M}(\mathbb{R}_+)\}, \qquad (k \in \mathbb{N}_0).$$

Note that for $k = 0$, $f = \mathcal{L}_\mu$. The following result states that $\mathcal{G}_k \subseteq \mathcal{E}_{\mathrm{top}}(S_\phi)$, $\phi < \pi/2$ for all $k \in \mathbb{N}_0$.

**Proposition 2.17** *Let $A \in Sect(\omega)$, $\omega < \pi/2$. For all $k \in \mathbb{N}_0$, if $f \in \mathcal{G}_k$, then $f \in \mathcal{E}_{top}(S_\phi)$, with $\omega < \phi < \pi/2$, and*

$$f(A) = \int_{\mathbb{R}_+} (tA)^k e^{-tA}\mu(\mathrm{d}t).$$

**Proof.** Let $A \in \mathrm{Sect}(\omega)$, $\omega < \pi/2$. We would like to apply Theorem 2.11. For $\omega < \phi < \pi/2$ and for each $k \in \mathbb{N}_0$, define $g : S_\phi \to \mathbb{C}$ with $g(z) := z^k e^{-z}$. Let $k \in \mathbb{N}$. Then $g \in \mathcal{E}_0(S_\phi)$. Indeed, first of all $\lim_{z\to 0} z^k e^{-z} = 0$ and $\lim_{z\to\infty} z^k e^{-z} = 0$. Moreover, by holomorphy, there exist constants $C_1$, $C_2$ and $s \geqslant 1$ such that $|z^k e^{-z}| \leqslant C_1|z|^s$ for $z$ close to zero and $|z^k e^{-z}| \leqslant C_2|z|^{-s}$ for $z$ large. So, we have

$$\begin{aligned}\int_{\partial S_\alpha} |g(z)|\frac{|\mathrm{d}z|}{|z|} &= \int_{\substack{\partial S_\alpha\\ |z|\leqslant 1}} |z^k e^{-z}|\frac{|\mathrm{d}z|}{|z|} + \int_{\substack{\partial S_\alpha\\ |z|\geqslant 1}} |z^k e^{-z}|\frac{|\mathrm{d}z|}{|z|}\\ &\leqslant C_1 \int_{\substack{\partial S_\alpha\\ |z|\leqslant 1}} |z|^{s-1}|\mathrm{d}z| + C_2 \int_{\substack{\partial S_\alpha\\ |z|\geqslant 1}} |z|^{-(s+1)}|\mathrm{d}z| < \infty,\end{aligned}$$

for $0 \leqslant \alpha < \phi$. Hence by definition $g \in \mathcal{E}_0(S_\phi)$. Now, for $k = 0$ an easy application of Lemma 2.6 shows that $e^{-z} \in \mathcal{E}(S_\phi)$. So, for all $k \in \mathbb{N}_0$, if $f \in \mathcal{G}$, from Theorem 2.11 we have that

$$f(z) = \int_{\mathbb{R}_+} g(tz)\mu(\mathrm{d}t) = \int_{\mathbb{R}_+} (tz)^k e^{-tz}\mu(\mathrm{d}t)$$

is in $\mathcal{E}_{\mathrm{top}}(S_\phi)$ and

$$f(A) = \int_{\mathbb{R}_+} (tA)^k e^{-tA}\mu(\mathrm{d}t).$$

□

**Comments 2.18** The argument in case $k = 0$ is superfluous, since it is easy to see (from Lemma 2.6) that $\mathcal{L}_\mu \in \mathcal{E}(S_\phi)$, $\phi < \pi/2$, and one can compute

$$\mathcal{L}_\mu(A) = \int_{\mathbb{R}_+} e^{-tA}\mu(\mathrm{d}t),$$

whenever $A \in \mathrm{Sect}(\omega)$, $\omega < \pi/2$. But, in order to fit all the cases under the same argument we proved that $\mathcal{L}_\mu \in \mathcal{E}_{\mathrm{top}}(S_\phi) \supseteq \mathcal{E}(S_\phi)$.

This result is a very useful tool. For $k = 0$, if we consider the Dirac measure $\delta_{t_0} \in \mathbf{M}(\mathbb{R}_+)$ at a point $t_0 \geqslant 0$, then

$$\int_{\mathbb{R}_+} e^{-tA}\delta_{t_0}(\mathrm{d}t) = e^{-t_0 A}$$

is the semigroup operator. If we consider the measure $\mu = e^{-\lambda t}\mathrm{d}t$ with $\mathrm{Re}\,\lambda > 0$ then

$$\int_{\mathbb{R}_+} e^{-tA}e^{-\lambda t}\mathrm{d}t = \mathrm{R}(\lambda, -A)$$

is the resolvent of the generator. Let us see now how Proposition 2.17 will be applied in order to observe convergence rates for the Euler's approximations of bounded holomorphic semigroups.

**Example 2.19 (Euler's approximations)** It is known that

$$e^{-zt} = \lim_{n\to\infty}\left(1 - \frac{tz}{n}\right)^n = \lim_{n\to\infty}\left(1 + \frac{tz}{n}\right)^{-n} = \lim_{n\to\infty}\left(\frac{n}{t}\left(\frac{n}{t} + z\right)^{-1}\right)^n$$

for $z \in \mathbb{C}_+$, $t \geqslant 0$. We will obtain a convergence rate $O(1/(n-1))$ for this approximation, in operator norm, if we insert an operator $A$ such that $-A$ generates a bounded holomorphic semigroup. For $t = 0$ we have nothing to show, so we consider $t > 0$. To do this it will be shown that the difference $(n/t(n/t+z)^{-1})^n - e^{-tz}$ is in $\mathcal{E}_{\mathrm{top}}(S_\phi)$, $\phi < \pi/2$, and a convergence rate $O(1/(n-1))$ will be obtained. We begin with some basic results. For $\lambda > 0$, we have

$$\lambda \frac{1}{\lambda + z} = \int_{\mathbb{R}_+} e^{-zt} \lambda e^{-\lambda t} \mathrm{d}t = \mathcal{L}_{\mu_\lambda}(z) \qquad (z \in \mathbb{C}_+),$$

with $\mu_\lambda := \lambda e^{-\lambda t}\mathrm{d}t \in \mathbf{M}(\mathbb{R}_+)$. Furthermore, using the convolution of measures we obtain

$$\begin{aligned}\left(\lambda \frac{1}{\lambda + z}\right)^n &= \int_{\mathbb{R}_+} \cdots \int_{\mathbb{R}_+} e^{-z(t_1+\cdots+t_n)} \lambda^n e^{-\lambda(t_1+\cdots+t_n)} \mathrm{d}t_1 \cdots \mathrm{d}t_n \\ &= \int_{\mathbb{R}_+} e^{-zt} (\lambda e^{-\lambda t}\mathrm{d}t * \cdots * \lambda e^{-\lambda t}\mathrm{d}t)\end{aligned}$$

Let us denote

$$\mu_{\lambda,n}(\mathrm{d}t) := \lambda e^{-\lambda t}\mathrm{d}t * \cdots * \lambda e^{-\lambda t}\mathrm{d}t = \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda e^{-\lambda t}\mathrm{d}t.$$

Note that

$$\int_{\mathbb{R}_+} \mu_{\lambda,n}(\mathrm{d}t) = \int_{\mathbb{R}_+} \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda e^{-\lambda t}\mathrm{d}t = 1$$

and

$$\int_{\mathbb{R}_+} t\mu_{\lambda,n}(\mathrm{d}t) = \frac{n}{\lambda}, \qquad \int_{\mathbb{R}_+} t^2 \mu_{\lambda,n}(\mathrm{d}t) = \frac{n}{\lambda^2}.$$

If we take $\lambda = n/s$, we have

$$\int_{\mathbb{R}_+} t\mu_{n/s,n}(\mathrm{d}t) = s, \qquad \int_{\mathbb{R}_+} t^2 \mu_{n/s,n}(\mathrm{d}t) = \frac{s^2}{n}.$$

Furthermore,

$$\left(\frac{n}{t}\left(\frac{n}{t} + z\right)^{-1}\right)^n = \int_{\mathbb{R}_+} e^{-zs} \mu_{n/t,n}(\mathrm{d}s) = \mathcal{L}_{\mu_{n/t,n}}(z).$$

On the other hand,

$$e^{-zt} = \int_{\mathbb{R}_+} e^{-zs} \delta_t(\mathrm{d}s) = \mathcal{L}_{\delta_t}(z).$$

Hence, we conclude that $\mathcal{L}_{\delta_t}(z) = \lim_{n\to\infty} \mathcal{L}_{\mu_{n/t,n}}(z)$, for all $z \in \mathbb{C}_+$, $t > 0$. If $-A$ generates a bounded holomorphic semigroup then $A \in \mathrm{Sect}(\omega)$ with $\omega < \pi/2$ and $\left(n/t\left((n/t)+z\right)^{-1}\right)^n, e^{-tz} \in \mathcal{G}_0$. So, by Proposition 2.17 we have

$$(n/t\, \mathrm{R}(n/t, -A))^n = \left(\frac{n}{t}\left(\frac{n}{t} + A\right)^{-1}\right)^n = \mathcal{L}_{\mu_{n/t,n}}(A), \quad \text{and} \quad e^{-tA} = \mathcal{L}_{\delta_t}(A).$$

Let us define the difference $\Delta_{t,n}(z) := (n/t(n/t+z)^{-1})^n - e^{-tz}$. By Taylor we can write

$$e^{-sz} - e^{-tz} = -ze^{-tz}(s-t) + \int_t^s (s-r)z^2 e^{-rz}\mathrm{d}r,$$

and we compute

$$\begin{aligned}
\Delta_{t,n}(z) &= \int_{\mathbb{R}_+} e^{-sz}\mu_{n/t,n}(\mathrm{d}s) - e^{-tz} = \int_{\mathbb{R}_+} (e^{-sz} - e^{-tz})\mu_{n/t,n}(\mathrm{d}s) \\
&= -ze^{-tz}\int_{\mathbb{R}_+} s\ \mu_{n/t,n}(\mathrm{d}s) + ze^{-tz}t + \int_{\mathbb{R}_+}\int_t^s (s-r)z^2 e^{-rz}\mathrm{d}r\ \mu_{n/t,n}(\mathrm{d}s) \\
&= -ze^{-tz}t + ze^{-tz}t + \int_{\mathbb{R}_+}\int_t^s (s-r)z^2 e^{-rz}\mathrm{d}r\ \mu_{n/t,n}(\mathrm{d}s) \\
&= \int_{\mathbb{R}_+}\int_t^s (s-r)z^2 e^{-rz}\mathrm{d}r\ \mu_{n/t,n}(\mathrm{d}s).
\end{aligned}$$

We change the order of integration in order to integrate with respect to $s$ first. To do this let us define the set

$$A(t,s) = \begin{cases} [t,s], & \text{if } t \leqslant s, \\ [s,t], & \text{if } t \geqslant s. \end{cases}$$

We have

$$\begin{aligned}
\Delta_{t,n}(z) &= \int_0^\infty \int_0^\infty \mathbf{1}_{A(t,s)}(r)|s-r|z^2 e^{-rz}\mathrm{d}r\ \mu_{n/t,n}(\mathrm{d}s) \\
&= \int_0^\infty (rz)^2 e^{-rz} \int_0^\infty \mathbf{1}_{A(t,s)}(r)\frac{|s-r|}{r^2}\mu_{n/t,n}(\mathrm{d}s)\ \mathrm{d}r
\end{aligned}$$

We would like to show that the positive measure

$$\nu_{t,n}(\mathrm{d}r) := \int_0^\infty \mathbf{1}_{A(t,s)}(r)\frac{|s-r|}{r^2}\mu_{n/t,n}(\mathrm{d}s)\ \mathrm{d}r,$$

is in $\mathbf{M}(\mathbb{R}_+)$. We obtain

$$\begin{aligned}
&\int_0^\infty \nu_{t,n}(\mathrm{d}r) = \int_0^\infty\int_0^\infty \mathbf{1}_{A(t,s)}(r)\frac{|s-r|}{r^2}\mu_{n/t,n}(\mathrm{d}s)\ \mathrm{d}r \\
&= \int_0^\infty\int_0^\infty \mathbf{1}_{A(t,s)}(r)\frac{|s-r|}{r^2}\mathrm{d}r\ \mu_{n/t,n}(\mathrm{d}s) = \int_0^\infty\int_t^s \frac{s-r}{r^2}\mathrm{d}r\ \mu_{n/t,n}(\mathrm{d}s).
\end{aligned}$$

Changing the variable $\tau = \frac{r-t}{s-t}$ we obtain an estimation for the inner integral,

$$\begin{aligned}\int_t^s \frac{s-r}{r^2}\mathrm{d}r &= \int_0^1 \frac{(s-t)(1+\tau)}{(t+\tau(s-t))^2}(s-t)\mathrm{d}\tau \leqslant (s-t)^2\int_0^1 \frac{\mathrm{d}\tau}{(t+\tau(s-t))^2}\\ &= (s-t)\left(\frac{1}{t}-\frac{1}{s}\right).\end{aligned}$$

Hence, we have

$$\int_0^\infty \nu_{t,n}(\mathrm{d}r) \leqslant \int_0^\infty \left(\frac{s}{t}-2+\frac{t}{s}\right)\mu_{n/t,n}(\mathrm{d}s) = \int_0^\infty \frac{t}{s}\mu_{n/t,n}(\mathrm{d}s) - 1.$$

We compute,

$$\begin{aligned}\int_0^\infty \frac{t}{s}\mu_{n/t,n}(\mathrm{d}s) &= \int_0^\infty \frac{t}{s}\frac{\left(\frac{n}{t}s\right)^{n-1}}{(n-1)!}\frac{n}{t}e^{-\frac{n}{t}s}\mathrm{d}s = \int_0^\infty \frac{n}{n-1}\frac{\left(\frac{n}{t}s\right)^{n-2}}{(n-2)!}\frac{n}{t}e^{-\frac{n}{t}s}\\ &= \frac{n}{n-1} = 1+\frac{1}{n-1}.\end{aligned}$$

Finally, we obtain

$$\int_0^\infty \nu_{t,n}(\mathrm{d}r) \leqslant 1+\frac{1}{n-1}-1 = \frac{1}{n-1}.$$

So, $\nu_{t,n} \in \mathbf{M}(\mathbb{R}_+)$ and we write

$$\Delta_{t,n}(z) = \int_0^\infty (rz)^2 e^{-rz}\nu_{t,n}(\mathrm{d}r).$$

In other words $\Delta_{t,n} \in \mathcal{G}_2$. If $-A$ generates a bounded holomorphic semigroup then $A \in \mathrm{Sect}(\omega)$ with $\omega < \pi/2$, so from Proposition 2.17, $\Delta_{t,n}(z) \in \mathcal{E}_{\mathrm{top}}(S_\phi)$, with $\omega < \phi < \pi/2$ and

$$\Delta_{t,n}(A) = \int_0^\infty (rA)^2 e^{-rA}\nu_{t,n}(\mathrm{d}r).$$

Since $e^{-rA}$ is a bounded holomorphic semigroup we have bounds on the derivatives, $\sup_{r>0}\|(rA)^2e^{-rA}\| =: M < \infty$, hence we estimate

$$\|\Delta_{t,n}(A)\| \leqslant M\left(\int_{\mathbb{R}_+}|\nu_{t,n}|(\mathrm{d}r)\right) \leqslant M\left(\frac{1}{n-1}\right).$$

# Chapter 3

# An Extension for Half-plane Type Operators

In this chapter, the holomorphic functional calculus for half-plane type operators is described, inspired by [3]. A topological extension is developed for functions which are the Laplace transform of bounded Borel measures, in order to cover the Hille-Phillips calculus for semigroup generators (cf. [12, Section 3.3]). The purpose we are doing this is that the Laplace transform of a bounded Borel measure is not defined in sectors of angle $\phi > \pi/2$. In case of a bounded $C_0-$semigroup with generator $-A$, the operator $A \in \mathrm{Sect}(\pi/2)$ and in order to define a functional calculus we need a class of function defined on a sector of angle $\phi > \pi/2$. So it is clear that the holomorphic functional calculus for sectorial operators does not work if we want to access a bounded $C_0-$semigroup. Here, the Laplace transform of a bounded Borel measure will be defined on half-planes larger than $\overline{\mathbb{C}_+}$ so a class of functions will be developed, defined in such half-plane, and so a construction of a functional calculus can be allowed in order to access bounded $C_0-$semigroups.

## 3.1 Holomorphic Functional Calculus on a Half-plane

For $\omega \in [-\infty, \infty]$ define the right half-plane $R_\omega := \{z \in \mathbb{C} : \mathrm{Re}\, z > \omega\}$.

**Definition 3.1** We say that an operator $A$ on a Banach space $X$ is of half-plane type $\omega \in (-\infty, \infty]$ (in short $A \in \mathrm{HP}(\omega)$) if $\sigma(A) \subseteq \overline{R_\omega}$ and for every $\alpha < \omega$ and for every $z$ with $\mathrm{Re}\, z \leqslant \alpha$ there exists a constant $M_\alpha$ such that

$$\|R(z, A)\| \leqslant M_\alpha.$$

If $A$ is of half-plane type $\omega$ for some $\omega \in \mathbb{R}$, then it is of half-plane type $s_0(A)$, where

$$s_0(A) := \sup\{\alpha : \sup_{\operatorname{Re} z \leqslant \alpha} \|R(z,A)\| < \infty\}.$$

Consider the class of functions

$$\mathcal{E}(R_\phi) = \left\{ f \in \mathcal{H}^\infty(R_\phi) : \int_{\mathbb{R}} |f(r+is)| \mathrm{d}s < \infty, \forall r > \phi \right\}.$$

In order to construct a holomorphic functional calculus based on this class we have to have a Cauchy's integral formula for these functions. The following lemma is a representation formula for the first derivative of functions in $\mathcal{E}(R_\phi)$.

**Lemma 3.2** *If $f \in \mathcal{E}(R_\phi)$ then for all $r > \phi$ and $\alpha \in R_r$, $f'$ can be represented by the formula*

$$f'(\alpha) = \frac{1}{2\pi i} \int_{r+i\mathbb{R}} \frac{f(z)}{(z-\alpha)^2} \mathrm{d}z.$$

**Proof.** Let $f \in \mathcal{E}(R_\phi)$, $r > \phi$ and $K > 0$. Consider the positively oriented closed curve $C$ (Figure 3.1), which consists of two parts

$$\begin{aligned} \gamma_1 &= \{r + Ke^{it} \quad -\pi/2 \leqslant t \leqslant \pi/2\}, \\ \gamma_2 &= \{r - it \qquad -K \leqslant t \leqslant K\}. \end{aligned}$$

Inside $C$, $f'$ is holomorphic and by Cauchy's integral formula for derivatives

$$f'(\alpha) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-\alpha)^2} \mathrm{d}z = \frac{1}{2\pi i} \left( \int_{\gamma_1} + \int_{\gamma_2} \right) \frac{f(z)}{(z-\alpha)^2} \mathrm{d}z.$$

For $z \in \gamma_1$ and $\alpha \in C$ we have $K = |z - r| \geqslant |\alpha - r|$, so by the inverse triangle inequality $|z - \alpha| = |z - r + r - \alpha| \geqslant ||z - r| - |r - \alpha|| = K - |r - \alpha|$. Hence an estimation for the integral on $\gamma_1$ is

$$\left| \int_{\gamma_1} \frac{f(z)}{(z-\alpha)^2} \mathrm{d}z \right| \leqslant \int_{\gamma_1} \frac{|f(z)|}{|z-\alpha|^2} |\mathrm{d}z| \leqslant \frac{\|f\|_\infty}{(K - |r-\alpha|)^2} \int_{\gamma_1} |\mathrm{d}z| = \frac{\|f\|_\infty \pi K}{(K - |r-\alpha|)^2}.$$

Letting $K \to \infty$ the integral on $\gamma_1$ tends to zero and the other constitutes the desired. So,

$$f'(\alpha) = \frac{1}{2\pi i} \int_{r+i\mathbb{R}} \frac{f(z)}{(z-\alpha)^2} \mathrm{d}z,$$

for any $\alpha \in R_r$. □

Having this representation we can prove that $f$ differs from its Cauchy's formula by a constant.

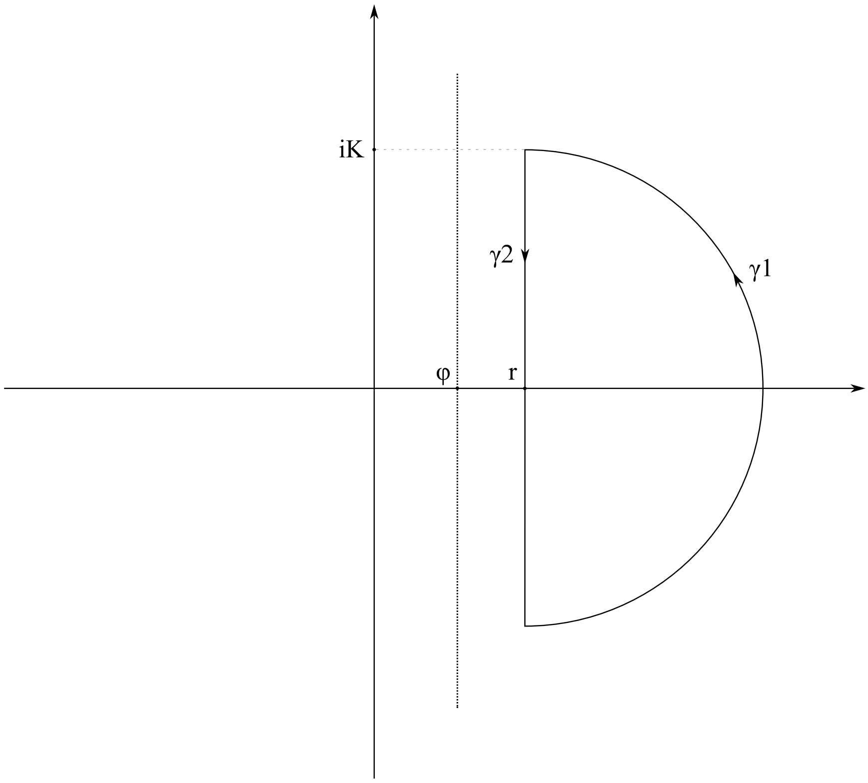


Figure 3.1: Positively oriented closed curve $C$

**Lemma 3.3** *If $f \in \mathcal{E}(R_\phi)$ then for all $r > \phi$ and $\alpha \in R_r$, there is a constant $C$ such that*

$$f(\alpha) = C + \frac{1}{2\pi i}\int_{r+i\mathbb{R}} \frac{f(z)}{z-\alpha}\mathrm{d}z.$$

**Proof.** Let $f \in \mathcal{E}(R_\phi)$ and $r > \phi$. For any point $\alpha \in R_r$ define

$$h(\alpha) = f(\alpha) - \frac{1}{2\pi i}\int_{r+i\mathbb{R}} \frac{f(z)}{z-\alpha}\mathrm{d}z.$$

Then, $h$ is holomorphic and by Lemma 3.2 $h'(\alpha) = 0$ for every $\alpha \in R_r$. Hence $h$ is a constant function, let us say $h(\alpha) := C$ for every $\alpha \in R_r$, and so

$$f(\alpha) = C + \frac{1}{2\pi i}\int_{r+i\mathbb{R}} \frac{f(z)}{z-\alpha}\mathrm{d}z,$$

for every $\alpha \in R_r$. □

The next step is to prove that $C = 0$ and so we have the Cauchy's integral formula for functions in $\mathcal{E}(R_\phi)$.

**Lemma 3.4** *If $f \in \mathcal{E}(R_\phi)$ then for all $r > \phi$ and $\alpha \in R_r$*

$$f(\alpha) = \frac{1}{2\pi i} \int_{r+i\mathbb{R}} \frac{f(z)}{z-\alpha} \mathrm{d}z.$$

*Furthermore $f$ decays at infinity, i.e.*

$$\lim_{\substack{|\alpha|\to\infty \\ \alpha\in R_r}} f(\alpha) = 0$$

**Proof.** Let $f \in \mathcal{E}(R_\phi)$ and $r > \phi$. Furthermore, let $r' \in \mathbb{R}$ with $r' > r$ and $R > 0$. Consider the positively oriented closed curve $K$ (Figure 3.2), which consists of four parts:

$$\begin{aligned}
\gamma_1 &= \{t - iR : & r \leqslant t \leqslant r'\}, \\
\gamma_2 &= \{r' + it : & -R \leqslant t \leqslant R\}, \\
\gamma_3 &= \{-t + iR : & -r' \leqslant t \leqslant -r\}, \\
\gamma_4 &= \{r - it : & -R \leqslant t \leqslant R\}.
\end{aligned}$$

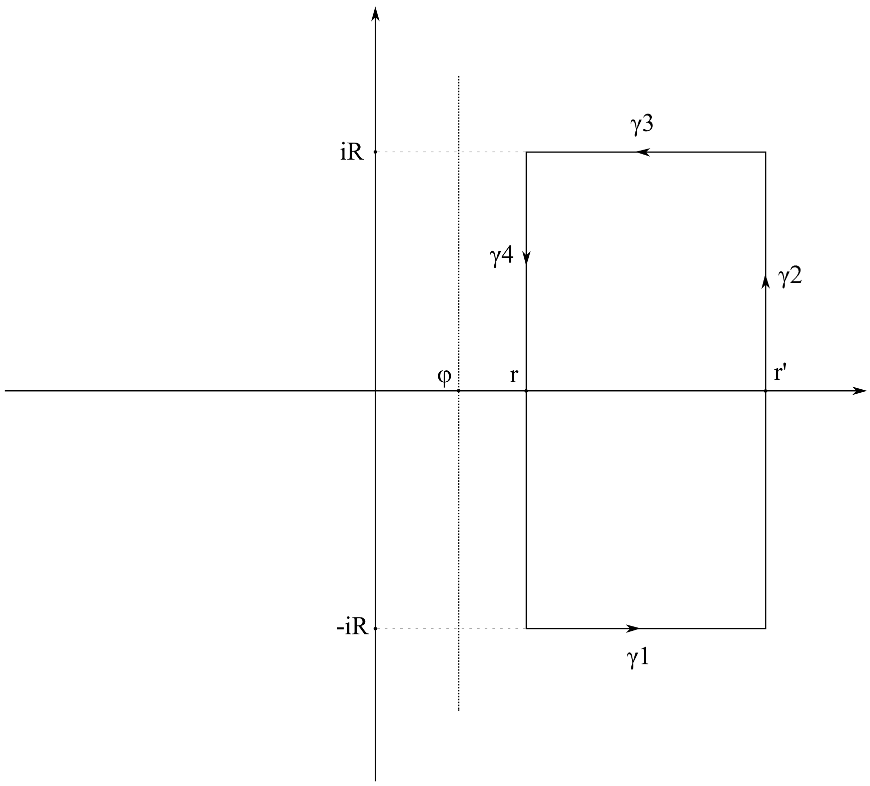


Figure 3.2: Positively oriented closed curve $K$

For any point $\alpha \in K$ by Cauchy's integral formula, we have

$$f(\alpha) = \frac{1}{2\pi i}\oint_K \frac{f(z)}{z-\alpha}\mathrm{d}z = \frac{1}{2\pi i}\sum_{n=1}^{4}\int_{\gamma_n}\frac{f(z)}{z-\alpha}\mathrm{d}z.$$

An estimation for the integral on $\gamma_1$ is

$$\left|\int_{\gamma_1}\frac{f(z)}{z-\alpha}\mathrm{d}z\right| \leqslant \int_r^{r'}\frac{|f(t-iR)|}{|t-iR-\alpha|}\mathrm{d}t \leqslant \int_r^{r'}\frac{\|f\|_\infty}{|\operatorname{Im}(t-iR-\alpha)|}\mathrm{d}t \leqslant \frac{\|f\|_\infty(r'-r)}{|-\operatorname{Im}\alpha - R|}$$

and for the integral on $\gamma_3$

$$\left|\int_{\gamma_3}\frac{f(z)}{z-\alpha}\mathrm{d}z\right| \leqslant \int_{-r'}^{-r}\frac{|f(-t+iR)|}{|-t+iR-\alpha|}\mathrm{d}t \leqslant \int_{-r'}^{-r}\frac{\|f\|_\infty}{|\operatorname{Im}(-t+iR-\alpha)|}\mathrm{d}t \leqslant \frac{\|f\|_\infty(r'-r)}{|R-\operatorname{Im}\alpha|}.$$

Letting $R \to \infty$ the integrals on $\gamma_1$ and $\gamma_3$ tends to 0 so,

$$f(\alpha) = \frac{1}{2\pi i}\left(-\int_{r'+i\mathbb{R}}\frac{f(z)}{z-\alpha}\mathrm{d}z + \int_{r+i\mathbb{R}}\frac{f(z)}{z-\alpha}\mathrm{d}z\right).$$

Hence, from Lemma 3.3 we have that

$$C = -\frac{1}{2\pi i}\int_{r'+i\mathbb{R}}\frac{f(z)}{z-\alpha}\mathrm{d}z,$$

with $\operatorname{Re}\alpha < r'$. Since the above integral is constant we may let $|\alpha| \to \infty$, keeping $\operatorname{Re}\alpha$ fixed, without changing its value. But, there exists a constant $M$ such that

$$\frac{1}{r' - \operatorname{Re}\alpha} \leqslant M.$$

(Otherwise it has to exist a sequence $\alpha_n$ with $(r' - \operatorname{Re}\alpha_n)^{-1} \to \infty$, which means $\frac{\operatorname{Re}\alpha_n}{r'} \to 1$ but this can not happen because $\operatorname{Re}\alpha_n < r'$.) So, since

$$\frac{|f(z)|}{|z-\alpha|} \leqslant \frac{|f(z)|}{|\operatorname{Re}(z-\alpha)|} = \frac{|f(z)|}{r'-\operatorname{Re}\alpha} \leqslant M|f(z)|,$$

by Lebesgue's dominated convergence theorem

$$\lim_{\substack{|\alpha|\to\infty \\ \operatorname{Re}\alpha<r'}}\frac{1}{2\pi i}\int_{r'+i\mathbb{R}}\frac{f(z)}{z-\alpha}\mathrm{d}z = \frac{1}{2\pi i}\int_{r'+i\mathbb{R}}\lim_{\substack{|\alpha|\to\infty \\ \operatorname{Re}\alpha<r'}}\frac{f(z)}{z-\alpha}\mathrm{d}z = 0.$$

Hence, $C = 0$ and we have

$$f(\alpha) = \frac{1}{2\pi i}\int_{r+i\mathbb{R}} \frac{f(z)}{z-\alpha}\mathrm{d}z$$

for all $\alpha \in R_r$. Now, for the second claim, Cauchy's theorem shows that $f(\alpha)$ does not depend on the particular choice of $r$, so we may write

$$f(\alpha) = \frac{1}{2\pi i}\int_{s+i\mathbb{R}} \frac{f(z)}{z-\alpha}\mathrm{d}z \qquad (\alpha \in R_r),$$

with $\phi < s < r$. There exists a constant $N$ such that

$$\frac{1}{|z-\alpha|} \leqslant N$$

uniformly on $R_r$. (Otherwise they have to exist sequences $z_n, \alpha_n$ with $|z_n - \alpha_n|^{-1} \to \infty$, which means $\frac{\alpha_n}{z_n} \to 1$ but this can not happen because $s < r$.) Hence, for $\alpha \in R_r$

$$\frac{|f(z)|}{|z-\alpha|} \leqslant N|f(z)| \qquad (z \in \partial R_s).$$

By Lebesgue's dominated convergence theorem

$$\lim_{\substack{|\alpha|\to\infty \\ \alpha \in R_r}} f(\alpha) = \frac{1}{2\pi i}\int_{s+i\mathbb{R}} \lim_{\substack{|\alpha|\to\infty \\ \alpha \in R_r}} \frac{f(z)}{z-\alpha}\mathrm{d}z = 0.$$

□

A functional calculus for half-plane type operators can be based on the class of functions $\mathcal{E}(R_\phi)$. Given an operator $A$ on a Banach space $X$ with $A \in \mathrm{HP}(\omega)$ and a function $f \in \mathcal{E}(R_\phi)$, with $\phi < \omega < \infty$ we define the mapping

$$\Phi : (f \mapsto f(A)) : \mathcal{E}(R_\phi) \to \mathcal{L}(X)$$

by means of a Cauchy integral

$$\Phi(f) := f(A) := \frac{1}{2\pi i}\int_{r+i\mathbb{R}} f(z)\,\mathrm{R}(z, A)\mathrm{d}z,$$

with $\phi < r < \omega$. Cauchy's integral theorem shows that $f(A)$ is independent of $r$.

**Proposition 3.5** *Let $f \in \mathcal{E}(R_\phi)$ and $A \in \mathrm{HP}(\omega)$, with $\omega > \phi$. Then the mapping $\Phi : \mathcal{E}(R_\phi) \to \mathcal{L}(X)$ has the following properties.*

(i) *$\Phi$ is a homomorphism of algebras.*

(ii) *If $T \in \mathcal{L}(X)$ commutes with the resolvent of $A$, it also commutes with $\Phi(f)$.*

(iii) *We have*

$$\Phi(f)\,\mathrm{R}(\lambda, A) = \left(\frac{f(z)}{\lambda - z}\right)(A), \qquad \textit{for } \operatorname{Re}\lambda < \phi$$

**Proof.**

(i) Suppose $f, g \in \mathcal{E}(R_\phi)$. Directly, $\Phi$ is linear. Using Fubini's theorem and the resolvent identity $\mathrm{R}(z,A) - \mathrm{R}(w,A) = (w-z)\,\mathrm{R}(z,A)\,\mathrm{R}(w,A)$ we obtain

$$\begin{aligned}\Phi(f)\Phi(g) &= \frac{1}{(2\pi i)^2}\int_{r+i\mathbb{R}}\int_{r'+i\mathbb{R}} f(z)g(w)\,\mathrm{R}(z,A)\,\mathrm{R}(w,A)\;\mathrm{d}w\;\mathrm{d}z\\ &= \frac{1}{(2\pi i)^2}\int_{r+i\mathbb{R}}\int_{r'+i\mathbb{R}} \frac{f(z)g(w)}{w-z}(\mathrm{R}(z,A)-\mathrm{R}(w,A))\;\mathrm{d}w\;\mathrm{d}z\\ &= \frac{1}{2\pi i}\int_{r+i\mathbb{R}} f(z)\,\mathrm{R}(z,A)\left(\frac{1}{2\pi i}\int_{r'+i\mathbb{R}}\frac{g(w)}{w-z}\mathrm{d}w\right)\mathrm{d}z\\ &- \frac{1}{2\pi i}\int_{r'+i\mathbb{R}} g(w)\,\mathrm{R}(w,A)\left(\frac{1}{2\pi i}\int_{r+i\mathbb{R}}\frac{f(z)}{w-z}\mathrm{d}z\right)\mathrm{d}w.\end{aligned}$$

Since the integrals are independent of the choice of the path we may choose $r$, $r'$ such that $r < r'$. Then, Lemma 3.4 implies

$$\frac{1}{2\pi i}\int_{r+i\mathbb{R}}\frac{f(z)}{w-z}\mathrm{d}z = 0 \qquad \text{and} \qquad \frac{1}{2\pi i}\int_{r'+i\mathbb{R}}\frac{g(w)}{w-z}\mathrm{d}w = g(z).$$

Hence we obtain

$$\Phi(f)\Phi(g) = \frac{1}{2\pi i}\int_{r+i\mathbb{R}} f(z)g(z)\,\mathrm{R}(z,A)\mathrm{d}z = \Phi(fg).$$

(ii) Let $T \in \mathcal{L}(X)$ then for all $x \in X$

$$\begin{aligned}T\Phi(f)x &= T\frac{1}{2\pi i}\int_{r+i\mathbb{R}} f(z)\,\mathrm{R}(z,A)x\mathrm{d}z = \frac{1}{2\pi i}\int_{r+i\mathbb{R}} f(z)T\,\mathrm{R}(z,A)x\mathrm{d}z\\ &= \frac{1}{2\pi i}\int_{r+i\mathbb{R}} f(z)\,\mathrm{R}(z,A)Tx\mathrm{d}z = \Phi(f)Tx\end{aligned}$$

(iii) By the resolvent identity $\mathrm{R}(z,A) - \mathrm{R}(\lambda,A) = (\lambda - z)\,\mathrm{R}(z,A)\,\mathrm{R}(\lambda,A)$ we obtain

$$\begin{aligned}\Phi(f)\,\mathrm{R}(\lambda,A) &= \frac{1}{2\pi i}\int_{r+i\mathbb{R}} f(z)\,\mathrm{R}(z,A)\,\mathrm{R}(\lambda,A)\mathrm{d}z\\ &= \frac{1}{2\pi i}\int_{r+i\mathbb{R}}\frac{f(z)}{\lambda - z}(\mathrm{R}(z,A)-\mathrm{R}(\lambda,A))\mathrm{d}z\\ &= \frac{1}{2\pi i}\int_{r+i\mathbb{R}}\frac{f(z)}{\lambda - z}\,\mathrm{R}(z,A)\mathrm{d}z - \frac{1}{2\pi i}\,\mathrm{R}(\lambda,A)\int_{r+i\mathbb{R}}\frac{f(z)}{\lambda - z}\mathrm{d}z.\end{aligned}$$

Lemma 3.4 implies that the second integral equals zero, hence we obtain $\Phi(f)\,\mathrm{R}(\lambda,A) = (f(z)(\lambda - z)^{-1})(A)$. □

So, this is an elementary functional calculus for half-plane type operators which is the base of constructing an extension by topological means. The purpose of this extension is to cover the so-called Hille-Phillips calculus, so the bounded strongly continuous semigroups can be accessible via Dunford-Riesz calculus.

## 3.2 Topological Extension

Let us begin this section with a useful result which tells us that a function $f \in \mathcal{E}(R_\phi)$ with $\phi < 0$ is the Laplace transform of a Lebesgue integrable function.

**Lemma 3.6** *Let $\phi < 0$ and $f \in \mathcal{E}(R_\phi)$. For $t \geqslant 0$ define*

$$g(t) := -\frac{1}{2\pi i}\int_{\mathbb{R}} e^{(r+is)t} f(r+is)\mathrm{d}s,$$

*with $\phi < r < 0$ arbitrary. Then, $g \in L^1(0,\infty)$ and $\mathcal{L}[g(t)\mathrm{d}t] = f$ on $R_r$.*

**Proof.** Let $\phi < 0$ and $f \in \mathcal{E}(R_\phi)$. Then, by definition there exists a constant $C_r$ such that $\int_{\mathbb{R}} |f(r+is)|\mathrm{d}s = C_r < \infty$. Cauchy's theorem shows that the definition of $g$ does not depend on the particular choice of $r$. For any $\phi < r < 0$ we have

$$\begin{aligned}\|g\|_{L^1} &= \frac{1}{2\pi}\int_{(0,\infty)} |g(t)|\mathrm{d}t \leqslant \frac{1}{2\pi}\int_{(0,\infty)}\int_{\mathbb{R}} |e^{(r+is)t} f(r+is)|\mathrm{d}s\ \mathrm{d}t \\ &= \frac{1}{2\pi}\int_{(0,\infty)} e^{rt}\int_{\mathbb{R}} |f(r+is)|\mathrm{d}s\ \mathrm{d}t = \frac{C_r}{2\pi}\int_{(0,\infty)} e^{rt}\mathrm{d}t = -\frac{C_r}{2\pi r} < \infty.\end{aligned}$$

So $g \in L^1(0,\infty)$. Furthermore, note that $|g(t)| \leqslant C_r e^{rt}$, for all $t \geqslant 0$. For $z \in R_r$,

$$\begin{aligned}\mathcal{L}[g(t)\mathrm{d}t](z) &= \int_{\mathbb{R}_+} e^{-zt} g(t)\mathrm{d}t = -\frac{1}{2\pi i}\int_{\mathbb{R}_+} e^{-zt}\int_{r+i\mathbb{R}} e^{wt} f(w)\mathrm{d}w\ \mathrm{d}t \\ &= -\frac{1}{2\pi i}\int_{r+i\mathbb{R}} f(w)\mathrm{d}w \int_{\mathbb{R}_+} e^{(w-z)t}\mathrm{d}t = \frac{1}{2\pi i}\int_{r+i\mathbb{R}} \frac{f(w)}{w-z}\mathrm{d}w = f(z),\end{aligned}$$

since $\operatorname{Re} z > r = \operatorname{Re} w$, where the last equation follows from Lemma 3.4. □

Before we continue, let us introduce the notion of weak convergence of measures.

**Definition 3.7** A sequence of measures $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathbf{M}(\mathbb{R}_+)$ converges to $\mu \in \mathbf{M}(\mathbb{R}_+)$ weakly if

$$\int_0^\infty f(t)\mu_n(\mathrm{d}t) \to \int_0^\infty f(t)\mu(\mathrm{d}t)$$

for every $f \in \mathrm{BUC}(\mathbb{R}_+)$ (the space of bounded uniformly continuous functions on $\mathbb{R}_+$).

Now, it is reasonable to introduce another notion of convergence of measures, let us call it 'almost weak convergence of measures'. We will use this convergence on the $\mathcal{H}^\infty(\mathbb{C}_+)$ algebra in order to construct a topological extension.

**Definition 3.8** A sequence of measures $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathbf{M}(\mathbb{R}_+)$ converges to $\mu \in \mathbf{M}(\mathbb{R}_+)$ almost weakly if

$$\int_0^\infty f(s+t)\mu_n(\mathrm{d}t) \to \int_0^\infty f(s+t)\mu(\mathrm{d}t)$$

uniformly in $s \geqslant 0$, for every $f \in \mathrm{BUC}(\mathbb{R}_+)$.

It is clear that the almost weak convergence is 'stronger' than the weak convergence, i.e. if $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathbf{M}(\mathbb{R}_+)$ and $\mu \in \mathbf{M}(\mathbb{R}_+)$ with $\mu_n \to \mu$ almost weakly then $\mu_n \to \mu$ weakly.

The reason we define this notion is that this convergence delivers the continuity of the convolution of measures. And this will be needed afterwards.

**Lemma 3.9** *Let $(\mu_n)_{n\in\mathbb{N}}, (\nu_n)_{n\in\mathbb{N}} \subseteq \mathbf{M}(\mathbb{R}_+)$ such that $\mu_n \to \mu$ , $\nu_n \to \nu$ almost weakly, $\mu, \nu \in \mathbf{M}(\mathbb{R}_+)$. Then, $\mu_n * \nu_n \to \mu * \nu$ almost weakly.*

**Proof.** Let $(\mu_n)_{n\in\mathbb{N}}, (\nu_n)_{n\in\mathbb{N}} \subseteq \mathbf{M}(\mathbb{R}_+)$ such that $\mu_n \to \mu$ , $\nu_n \to \nu$ almost weakly, with $\mu, \nu \in \mathbf{M}(\mathbb{R}_+)$. Let $f \in \mathrm{BUC}(\mathbb{R}_+)$. It suffices to show that

$$\begin{aligned}\int_{\mathbb{R}_+} f(s+t)(\mu_n * \nu_n)(\mathrm{d}t) &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(s+t+r)\mu_n(\mathrm{d}t)\nu_n(\mathrm{d}r)\\ \to \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(s+t+r)\mu(\mathrm{d}t)\nu(\mathrm{d}r) &= \int_{\mathbb{R}_+} f(s+t)(\mu * \nu)(\mathrm{d}t),\end{aligned}$$

uniformly in $s \geqslant 0$. We obtain

$$\begin{aligned}
&\sup_{s\geqslant 0}\left|\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(s+t+r)\mu_n(\mathrm{d}t)\nu_n(\mathrm{d}r)-\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(s+t+r)\mu(\mathrm{d}t)\nu(\mathrm{d}r)\right|\\
&\leqslant \sup_{s\geqslant 0}\left|\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(s+t+r)\mu_n(\mathrm{d}t)\nu_n(\mathrm{d}r)-\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(s+t+r)\mu(\mathrm{d}t)\nu_n(\mathrm{d}r)\right|\\
&+\sup_{s\geqslant 0}\left|\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(s+t+r)\mu(\mathrm{d}t)\nu_n(\mathrm{d}r)-\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(s+t+r)\mu(\mathrm{d}t)\nu(\mathrm{d}r)\right|\\
&\leqslant \sup_{s\geqslant 0}\int_{\mathbb{R}_+}\left|\int_{\mathbb{R}_+} f(s+t+r)(\mu_n-\mu)(\mathrm{d}t)\right||\nu_n|(\mathrm{d}r)\\
&+\sup_{s\geqslant 0}\int_{\mathbb{R}_+}\left|\int_{\mathbb{R}_+} f(s+t+r)(\nu_n-\nu)(\mathrm{d}r)\right||\mu|(\mathrm{d}t)\\
&\leqslant \sup_{s\geqslant 0}\sup_{r\geqslant 0}\left|\int_{\mathbb{R}_+} f(s+t+r)(\mu_n-\mu)(\mathrm{d}t)\right|\int_{\mathbb{R}_+}|\nu_n|(\mathrm{d}r)\\
&+\sup_{s\geqslant 0}\sup_{t\geqslant 0}\left|\int_{\mathbb{R}_+} f(s+t+r)(\nu_n-\nu)(\mathrm{d}r)\right|\int_{\mathbb{R}_+}|\mu|(\mathrm{d}t).
\end{aligned}$$

By the Banach-Steinhauss theorem (cf. [14, Theorem 15.6]), the almost weak convergence implies uniform boundedness of $\|\nu_n\|_{\mathbf{M}(\mathbb{R}_+)}$ (so there exists a constant $N$ such that $\int_{\mathbb{R}_+}|\nu_n|(\mathrm{d}r)\leqslant N$ for all $n\in\mathbb{N}$), and using the almost weak convergence of $\mu_n,\nu_n$, we let $n\to\infty$ and we obtain that

$$\sup_{s\geqslant 0}\left|\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(s+t+r)\mu_n(\mathrm{d}t)\nu_n(\mathrm{d}r)-\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(s+t+r)\mu(\mathrm{d}t)\nu(\mathrm{d}r)\right|\to 0.$$

Hence $\mu_n*\nu_n\to\mu*\nu$ almost weakly. □

The following lemma states that weak convergence of measures implies weak convergence of integrals of vector-valued bounded uniformly continuous functions against these measures. As we will see later, this is needed, since in the case we have a bounded $C_0-$semigroup $T$, the orbit $(T(t)x)_{t\geqslant 0}$ is uniformly continuous.

**Lemma 3.10** *Let $(\mu_n)_{n\in\mathbb{N}}\subseteq\mathbf{M}(\mathbb{R}_+)$ with $\mu_n\to 0$ weakly. Then, for every $f\in BUC(\mathbb{R}_+,X)$, with $X$ a Banach space, we have*

$$\int_{\mathbb{R}_+} f(t)\mu_n(\mathrm{d}t)\to 0$$

*in the weak sense.*

**Proof.** Let $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathbf{M}(\mathbb{R}_+)$ with $\mu_n \to 0$ weakly. Let $f \in \mathrm{BUC}(\mathbb{R}_+, X)$ a bounded uniformly continuous $X-$valued function on $\mathbb{R}_+$. We have that $f$ is strongly measurable and the function $t \mapsto \|f(t)\|_X \in \mathrm{BUC}(\mathbb{R}_+)$ since from the inverse triangle inequality $|\|f(s)\|_X - \|f(t)\|_X| \leqslant \|f(s)-f(t)\|_X$ and $\sup_{t\geqslant 0} \|f(t)\|_X < \infty$. Moreover,

$$\left|\int_{\mathbb{R}_+} \|f(t)\|_X \mu(\mathrm{d}t)\right| \leqslant \int_{\mathbb{R}_+} \|f(t)\|_X |\mu|(\mathrm{d}t) \leqslant \sup_{t\geqslant 0} \|f(t)\|_X \int_{\mathbb{R}_+} |\mu|(\mathrm{d}t) < \infty.$$

We conclude (c.f. [16, Appendix F]) that the vector valued integral of $f$ against $\mu$ is defined as a Bochner integral and

$$\int_{\mathbb{R}_+} f(t)\mu(\mathrm{d}t) = \lim_{k\to\infty} \int_{\mathbb{R}_+} f_k(t)\mu(\mathrm{d}t)$$

where $(f_k)_{k\in\mathbb{N}}$ is any sequence of $X-$valued simple functions such that $f_k(t) \to f(t)$ pointwise on $X$ for each $t \geqslant 0$ and

$$\lim_{k\to\infty} \int_{\mathbb{R}_+} \|f_k(t) - f(t)\|_X |\mu|(\mathrm{d}t) = 0.$$

Now, let $\lambda \in X^*$ be a bounded linear functional on $X$. For each $t \geqslant 0$, since the functions $f_k(t)$ are simple, we have that $\lambda(f_k(t))$ are also simple and $\lambda(f_k(t)) \to \lambda(f(t))$. It is clear from the definition of integrals for $X-$valued simple functions that

$$\lambda\left(\int_{\mathbb{R}_+} f_k(t)\mu_n(\mathrm{d}t)\right) = \int_{\mathbb{R}_+} \lambda(f_k(t))\mu_n(\mathrm{d}t).$$

Hence, using the continuity of $\lambda$ we obtain

$$\begin{aligned}\lambda\left(\int_{\mathbb{R}_+} f(t)\mu_n(\mathrm{d}t)\right) &= \lambda\left(\lim_{k\to\infty}\int_{\mathbb{R}_+} f_k(t)\mu_n(\mathrm{d}t)\right) = \lim_{k\to\infty} \lambda\left(\int_{\mathbb{R}_+} f_k(t)\mu_n(\mathrm{d}t)\right)\\ &= \lim_{k\to\infty}\int_{\mathbb{R}_+} \lambda(f_k(t))\mu_n(\mathrm{d}t) = \int_{\mathbb{R}_+} \lambda(f(t))\mu_n(\mathrm{d}t).\end{aligned}$$

Note that since $f \in \mathrm{BUC}(\mathbb{R}_+, X)$, $\lambda(f(t)) \in \mathrm{BUC}(\mathbb{R}_+)$ and since $\mu_n \to 0$ weakly we obtain that

$$\lim_{n\to\infty} \lambda\left(\int_{\mathbb{R}_+} f(t)\mu_n(\mathrm{d}t)\right) = \lim_{n\to\infty}\int_{\mathbb{R}_+} \lambda(f(t))\mu_n(\mathrm{d}t) = 0.$$

Since $\lambda$ is an arbitrary functional, we conclude that

$$\lim_{n\to\infty}\int_{\mathbb{R}_+} f(t)\mu_n(\mathrm{d}t) = 0,$$

in the weak sense for each $t \geqslant 0$. □

Coming back to our procedure, the following lemma yields a useful formula for functions in $\mathcal{E}(R_\phi)$ with $-1 < \phi < 0$.

**Lemma 3.11** *Let $A \in \mathrm{HP}(0)$ and $f \in \mathcal{E}(R_\phi)$ with $-1 < \phi < 0$. Then the function $f(z)(1+z)^{-2} \in \mathcal{E}(R_\phi)$ and*

$$\frac{f(z)}{(1+z)^2}(A) = \int_{\mathbb{R}_+} \frac{e^{-tz}}{(1+z)^2}(A)\left(\mathcal{L}^{-1}f\right)(t)\mathrm{d}t.$$

**Proof.** Let $A \in \mathrm{HP}(0)$. We begin the proof showing that $(e^{-tz}(1+z)^{-2})(A)$ is defined by the elementary functional calculus. For $t \geqslant 0$ the function $e^{-tz}(1+z)^{-2} \in \mathcal{E}(R_\phi)$ for any $-1 < \phi < 0$. Indeed, fix $-1 < \phi < 0$, then for $z \in R_\phi$ we have

$$\left|\frac{e^{-tz}}{(1+z)^2}\right| \leqslant \frac{e^{-t\,\mathrm{Re}\,z}}{|1+\mathrm{Re}\,z|^2} \leqslant \frac{e^{-t\phi}}{(1+\phi)^2} < \infty,$$

and for $\phi < r < 0$ we obtain

$$\int_{\mathbb{R}}\left|\frac{e^{-t(r+is)}}{(1+r+is)^2}\right|\mathrm{d}s = e^{-tr}\int_{\mathbb{R}}\frac{1}{(1+r)^2+s^2}\mathrm{d}s = \frac{\pi e^{-tr}}{1+r} < \infty.$$

Hence, $(e^{-tz}(1+z)^{-2})(A)$ is defined by the elementary functional calculus for half-plane type 0 operators and using the resolvent boundedness we compute

$$\begin{aligned}\left\|\frac{e^{-tz}}{(1+z)^2}(A)\right\| &\leqslant \frac{1}{2\pi}\int_{r+i\mathbb{R}}\left|\frac{e^{-tz}}{(1+z)^2}\right|\|\,\mathrm{R}(z,A)\||\mathrm{d}z| \leqslant \frac{e^{-tr}M_r}{2\pi}\int_{r+i\mathbb{R}}\frac{1}{|1+z|^2}|\mathrm{d}z|\\ &= \frac{M_r}{2\pi(1+r)}e^{-tr} =: N_r e^{-tr}.\end{aligned}$$

Now let $f \in \mathcal{E}(R_\phi)$ with $-1 < \phi < 0$. Then, it is clear (as above) that $f(z)(1+z)^{-2} \in \mathcal{E}(R_\phi)$. Using Lemma 3.6, for $\phi < r < 0$ we compute

$$\begin{aligned}\frac{f(z)}{(1+z)^2}(A) &= \frac{1}{2\pi i}\int_{r+i\mathbb{R}}\frac{f(z)}{(1+z)^2}\,\mathrm{R}(z,A)\mathrm{d}z\\ &= \frac{1}{2\pi i}\int_{r+i\mathbb{R}}\frac{1}{(1+z)^2}\left(\int_{\mathbb{R}_+} e^{-tz}(\mathcal{L}^{-1}f)(t)\mathrm{d}t\right)\mathrm{R}(z,A)\mathrm{d}z\\ &= \int_{\mathbb{R}_+}\frac{1}{2\pi i}\int_{r+i\mathbb{R}}\frac{e^{-tz}}{(1+z)^2}\,\mathrm{R}(z,A)\mathrm{d}z\;(\mathcal{L}^{-1}f)(t)\mathrm{d}t\\ &= \int_{\mathbb{R}_+}\frac{e^{-tz}}{(1+z)^2}(A)(\mathcal{L}^{-1}f)(t)\mathrm{d}t,\end{aligned}$$

where Fubini's theorem is applicable since $|e^{-tz}(\mathcal{L}^{-1}f)(t)| \leqslant e^{-tr}C_r e^{tr} = C_r < \infty$ (look at the proof of Lemma 3.6). □

Now, we are ready to extend the elementary class of functions by topological means to a larger class according to the general pattern (as in Section 1.2) in order to cover the Hille-Phillips functional calculus for semigroup generators. Note that if an operator $-A$ generates a bounded $C_0$−semigroup then $A \in \mathrm{HP}(0)$. The following theorem is needed in order to define the topological extension.

**Theorem 3.12** *Let $-A$ generate a bounded $C_0$−semigroup and $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}(R_{\phi_n})$, with $-1 < \phi_n < 0$, a sequence such that $\mathcal{L}^{-1}f_n \to 0$ as $n \to \infty$ almost weakly on $\mathbf{M}(\mathbb{R}_+)$ and $f_n(A) \to T \in \mathcal{L}(X)$ strongly. Then $T = 0$.*

**Proof.** Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}(R_{\phi_n})$, with $-1 < \phi_n < 0$. For each $n$, from Lemma 3.6 there exists a function $g_n$ with $g_n(t) := (\mathcal{L}^{-1}f_n)(t)$ for $t \geqslant 0$. Now let $-A$ generates a bounded $C_0$−semigroup $(e^{-tA})_{t\geqslant 0}$, then $A \in \mathrm{HP}(0)$. On the one hand, for each $x \in X$, from Lemma 3.11

$$\frac{f_n(z)}{(1+z)^2}(A)x = \int_{\mathbb{R}_+} \frac{e^{-tz}}{(1+z)^2}(A)x \left(\mathcal{L}^{-1}f_n\right)(t)\mathrm{d}t.$$

Since $e^{-tA}$ is a bounded $C_0$−semigroup we have that $\sup_{t\geqslant 0}\|e^{-tA}\| =: M < \infty$ and for each $x \in X$ the orbit $(e^{-tA}x)_{t\geqslant 0}$ is uniformly continuous. Hence we obtain that $(e^{-tz}(1+z)^{-2})(A)x \in \mathrm{BUC}(\mathbb{R}_+, X)$ since

$$\sup_{t\geqslant 0}\left\|\frac{e^{-tz}}{(1+z)^2}(A)\right\| = \sup_{t\geqslant 0}\left\|e^{-tA}\,\mathrm{R}(-1,A)^2\right\| =: N_{\phi_n} < \infty,$$

and

$$\|e^{-tA}\,\mathrm{R}(-1,A)^2x - e^{-sA}\,\mathrm{R}(-1,A)^2x\|_X \leqslant N_{\phi_n}\|e^{-(t-s)A}x - x\|_X, \qquad (t \geqslant s \geqslant 0).$$

So, by Lemma 3.10 we have

$$\lim_{n\to\infty}\frac{f_n(z)}{(1+z)^2}(A)x = \lim_{n\to\infty}\int_{\mathbb{R}_+}\frac{e^{-tz}}{(1+z)^2}(A)x\left(\mathcal{L}^{-1}f_n\right)(t)\mathrm{d}t = 0$$

in the weak sense. On the other hand from hypothesis we have

$$(f_n(z)(1+z)^{-2})(A) = f_n(A)\,\mathrm{R}(-1,A)^2 = \mathrm{R}(-1,A)^2 f_n(A) \to \mathrm{R}(-1,A)^2 T,$$

in the strong operator topology. So we conclude that $\mathrm{R}(-1,A)^2T = 0$ which yields $T = 0$. □

As a consequence of this result we can enlarge the functional calculus for semigroup generators $-A$ topologically towards

$$\mathcal{E}_{\text{top}}(\overline{\mathbb{C}_+}) = \left\{ f \in \mathcal{H}^\infty(\mathbb{C}_+) \;\middle|\; \begin{array}{l} \exists \mu \in \mathbf{M}(\mathbb{R}_+) \text{ such that } \mathcal{L}_\mu = f, \\ \exists (f_n)_n \subseteq \mathcal{E}(R_{\phi_n}),\ -1 < \phi_n < 0,\ \text{with } \mathcal{L}^{-1} f_n \to \mu \\ \text{almost weakly and } \lim f_n(A) \text{ exists strongly.} \end{array} \right\}$$

Given an operator $A$ such that $-A$ generate a bounded $C_0-$semigroup (so $A \in \text{HP}(0)$), we define the mapping

$$\Phi : (f \mapsto f(A)) : \mathcal{E}_{\text{top}}(\overline{\mathbb{C}_+}) \to \mathcal{L}(X)$$

setting

$$f(A) = \lim_{n\to\infty} f_n(A),$$

in the strong operator topology, where $(f_n)_n \subseteq \mathcal{E}(R_{\phi_n})$, $-1 < \phi_n < 0$ with $\mathcal{L}^{-1} f_n \to \mu$ almost weakly and $\mathcal{L}_\mu = f$.

**Theorem 3.13** *Let $-A$ generate a bounded $C_0-$ semigroup. The mapping $\Phi : (f \mapsto f(A)) : \mathcal{E}_{top}(\overline{\mathbb{C}_+}) \to \mathcal{L}(X)$ is an algebra homomorphism.*

This is just a special case of Proposition 1.6. To see that the almost week convergence of measures delivers the continuity of multiplication note that, for $f, g \in \mathcal{E}_{\text{top}}(\overline{\mathbb{C}_+})$, there exist $\mu, \nu \in \mathbf{M}(\mathbb{R}_+)$ such that $\mathcal{L}_\mu = f$, $\mathcal{L}_\nu = g$ and we obtain

$$\begin{aligned} (fg)(z) = f(z)g(z) = \mathcal{L}_\mu(z)\mathcal{L}_\nu(z) &= \int_{\mathbb{R}_+} e^{-sz}\mu(\mathrm{d}s) \int_{\mathbb{R}_+} e^{-tz}\nu(\mathrm{d}t) \\ &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} e^{-(s+t)z}\mu(\mathrm{d}s)\nu(\mathrm{d}t) = \int_{\mathbb{R}_+} e^{-rz}(\mu * \nu)(\mathrm{d}r). \end{aligned}$$

So, $fg = \mathcal{L}_{\mu*\nu}$ with $(\mu * \nu) \in \mathbf{M}(\mathbb{R}_+)$. Furthermore there exist $(f_n)_n, (g_n)_n \subseteq \mathcal{E}(R_{\phi_n})$ with $-1 < \phi_n < 0$ and $\mathcal{L}^{-1} f_n \to \mu$, $\mathcal{L}^{-1} g_n \to \nu$ almost weakly. From Lemma 3.9 we obtain

$$\mathcal{L}^{-1} f_n g_n = \mathcal{L}^{-1} f_n * \mathcal{L}^{-1} g_n \to \mu * \nu,$$

almost weakly. Furthermore the requirement (1.1) is available from Theorem 3.12. Hence, indeed the assumptions of Proposition 1.6 are fulfilled.

## 3.3 Hille-Phillips Calculus for Semigroup Generators

In this section it will be shown that the Hille-Phillips calculus for generators of bounded $C_0-$semigroups can be covered by the topologically extended holomorphic functional calculus for half plane type operators. Let $\mu \in \mathbf{M}(\mathbb{R}_+)$, and consider its Laplace transform

$$f(z) = \mathcal{L}_\mu(z) = \int_{\mathbb{R}_+} e^{-sz}\mu(\mathrm{d}s), \qquad (\operatorname{Re} z > 0).$$

Suppose that $-A$ generates a bounded $C_0-$semigroup $e^{-tA}$. Then, the mapping $\Phi : \mathbf{M}(\mathbb{R}) \to \mathcal{L}(X)$ with

$$\Phi(\mu) = \mathcal{L}_\mu(A) = f(A) = \int_{\mathbb{R}_+} e^{-sA}\mu(\mathrm{d}s)$$

is called the Hille-Phillips calculus for $A$. (cf. [15], or [12, Section 3.3]). Here, it will be shown that the Hille-Phillips Calculus for semigroup generators can be covered by an algebraic extension of the topologically extended holomorphic calculus for half plane type operators. Let us define the set of functions that are Laplace transforms of measures

$$\mathcal{F} = \{f : \mathbb{C}_+ \to \mathbb{C} : f(z) = \mathcal{L}_\mu(z), \mu \in \mathbf{M}(\mathbb{R}_+)\}.$$

The following result will be useful.

**Proposition 3.14** *Let $-A$ generate a bounded $C_0-$semigroup. If $f \in \mathcal{F}$ then $f(z)(1+z)^{-2} \in \mathcal{E}_{top}(\overline{\mathbb{C}_+})$.*

**Proof.** Let $-A$ generate a bounded $C_0-$semigroup. Then $A \in \mathrm{HP}(0)$. Let $f \in \mathcal{F}$, so there exists a $\mu \in \mathbf{M}(\mathbb{R}_+)$ such that $f(z) = \mathcal{L}_\mu(z)$ for $\operatorname{Re} z > 0$. Let us define $g(z) := f(z)(1+z)^{-2} = (\mathcal{L}_\mu(z))(1+z)^{-2}$, for $\operatorname{Re} z > 0$. Note that

$$\frac{1}{(1+z)^2} = \mathcal{L}_\nu(z), \qquad (\operatorname{Re} z > 0)$$

where $\nu(\mathrm{d}t) = te^{-t}\mathrm{d}t \in \mathbf{M}(\mathbb{R}_+)$. Hence, for $\operatorname{Re} z > 0$ we may write $g(z) = \mathcal{L}_\mu(z)\mathcal{L}_\nu(z) = \mathcal{L}_{\mu*\nu}(z)$, where $\mu * \nu \in \mathbf{M}(\mathbb{R}_+)$. For $n \in \mathbb{N}$ define the sequence of functions $g_n : R_{-1/n} \to \mathbb{C}$ with

$$g_n(z) := g(z + 1/n) = \frac{\mathcal{L}_\mu(z+1/n)}{(1+1/n+z)^2}.$$

Note that

$$\mathcal{L}_\mu(z+1/n) = \int_{\mathbb{R}_+} e^{-zt} e^{(-1/n)t}\mu(\mathrm{d}t) := \int_{\mathbb{R}_+} e^{-zt}\mu_n(\mathrm{d}t) = \mathcal{L}_{\mu_n}(z),$$

where $\mu_n(\mathrm{d}t) = e^{-(1/n)t}\mu(\mathrm{d}t) \in \mathbf{M}(\mathbb{R}_+)$. Furthermore, note that

$$\frac{1}{(1+1/n+z)^2} = \mathcal{L}_{\nu_n}(z), \qquad (\operatorname{Re} z > -1/n)$$

with $\nu_n(\mathrm{d}t) = te^{-(1+1/n)t}\mathrm{d}t = e^{-(1/n)t}\nu(\mathrm{d}t) \in \mathbf{M}(\mathbb{R}_+)$. So, for $\operatorname{Re} z > -1/n$, we may write $g_n(z) = \mathcal{L}_{\mu_n}(z)\mathcal{L}_{\nu_n}(z) = \mathcal{L}_{\mu_n * \nu_n}(z)$. Moreover $g_n \in \mathcal{E}(R_{-1/n})$. Indeed, for $-1/n < r < 0$

$$\begin{aligned}\int_{r+i\mathbb{R}} |g_n(z)|\mathrm{d}z &\leqslant \int_{r+i\mathbb{R}} \frac{1}{|1+1/n+z|^2}\int_{\mathbb{R}_+} e^{-(1/n+\operatorname{Re} z)t}|\mu|(\mathrm{d}t)\ \mathrm{d}z \\ &\leqslant \|\mu\|_{\mathbf{M}(\mathbb{R}_+)} \int_{r+i\mathbb{R}} \frac{1}{|1+1/n+z|^2}\mathrm{d}z < \infty.\end{aligned}$$

After that it will be shown that $\mathcal{L}^{-1}g_n \to \mu * \nu$ in the almost weak sense. It is easy to see that the measures $\mu_n, \nu_n$ converge in variation. Indeed, by Lebesgue's dominated convergence theorem we have

$$\lim_{n\to\infty} \int_{\mathbb{R}_+} |\mu_n|(\mathrm{d}t) = \lim_{n\to\infty} \int_{\mathbb{R}_+} e^{-(1/n)t}|\mu|(\mathrm{d}t) = \int_{\mathbb{R}_+} |\mu|(\mathrm{d}t),$$

so $\mu_n \to \mu$ in variation, hence also almost weakly. In the same way $\nu_n \to \nu$ in variation, so also in the almost weak sense. Hence, using Lemma 3.9 we obtain

$$\mathcal{L}^{-1}g_n = \mu_n * \nu_n \to \mu * \nu = \mathcal{L}^{-1}g,$$

almost weakly. It remains to show that $\lim_{n\to\infty} g_n(A)$ exists strongly. From Lemma 3.11 we write $g_n(A)$ as

$$g_n(A) = \int_{\mathbb{R}_+} \frac{e^{-(z+1/n)t}}{(1+(1/n)+z)^2}(A)(\mathcal{L}^{-1}f)(t)(\mathrm{d}t) = \int_{\mathbb{R}_+} e^{-tA}\,\mathrm{R}(-1-1/n, A)^2\mu_n(\mathrm{d}t).$$

Since $A$ is a half-plane type 0 operator, for $-1-1/n < -1$ there exists a constant $M_{-1}$ such that $\|\,\mathrm{R}(-1-1/n, A)\| \leqslant M_{-1}$ for all $n \in \mathbb{N}$. Furthermore, $e^{-tA}$ is a bounded $C_0-$ semigroup. Hence, since $\mu_n \to \mu$ in variation, by Lebesgue's dominated convergence theorem

$$\begin{aligned}&\left\|\int_{\mathbb{R}_+} e^{-tA}\,\mathrm{R}(-1-1/n, A)^2\mu_n(\mathrm{d}t) - \int_{\mathbb{R}_+} e^{-tA}\,\mathrm{R}(-1, A)^2\mu(\mathrm{d}t)\right\| \\ &\leqslant \int_{\mathbb{R}_+} \|e^{-tA}\|\|\,\mathrm{R}(-1-1/n, A)^2 - \mathrm{R}(-1, A)^2\||\mu_n - \mu|(\mathrm{d}t) \to 0\end{aligned}$$

as $n \to \infty$. So, $g_n(A)$ converges in the strong operator topology. We conclude that $g \in \mathcal{E}_{\mathrm{top}}(\overline{\mathbb{C}_+})$ and by definition $g(A) = \mathcal{L}_{\mu * \nu}(A)$. So

$$g(A) = \int_{\mathbb{R}_+} e^{-tA}\,\mathrm{R}(-1, A)^2 \mu(\mathrm{d}t) = \int_{\mathbb{R}_+} e^{-tA} (\mu * \nu)(\mathrm{d}t).$$

□

**Comments 3.15** Note that, in the above proof, the strong convergence of measures (in variation) is used. So, it seems unnecessary to define and use the 'almost weak' notion of convergence for measures in order to define the topologically extended function class. One could use the strong convergence of measures from the beginning as well. But, it is a matter of preference to prove facts with the weakest assumptions that could be possible.

Proposition 3.14 is important since, we will use $(1+z)^{-2}$ as a regularizer, in order to show that $f$ is in the algebraic extension of $\mathcal{E}_{\mathrm{top}}(\overline{\mathbb{C}_+})$ whenever $f \in \mathcal{F}$.

**Theorem 3.16** *Let $-A$ generate a bounded $C_0-$semigroup. If $f \in \mathcal{F}$ then $f$ is in the algebraic extension of $\mathcal{E}_{top}(\overline{\mathbb{C}_+})$.*

**Proof.** Let $-A$ generate a bounded $C_0-$semigroup and $f \in \mathcal{F}$, so there exists a $\mu \in \mathbf{M}(\mathbb{R}_+)$ such that $f(z) = \mathcal{L}_\mu(z)$ for $\operatorname{Re} z > 0$. Note that $(1+z)^{-2}(A) = \mathrm{R}(-1, A)^2$ is injective and, from Proposition 3.14, $f(z)(1+z)^{-2} \in \mathcal{E}_{\mathrm{top}}(\overline{\mathbb{C}_+})$. Hence, $f$ is regularizable by $\mathcal{E}_{\mathrm{top}}(\overline{\mathbb{C}_+})$ and $(1+z)^{-2}$ is a regularizer for $f$. So we define $f(A) = (1+A)^2((f(z)(1+z)^{-2})(A))$. We compute

$$f(A) = (1+A)^2 \int_{\mathbb{R}_+} e^{-tA} R(-1, A)^2 \mu(\mathrm{d}t) = \int_{\mathbb{R}_+} e^{-tA} \mu(\mathrm{d}t) = \mathcal{L}_\mu(A).$$

□

In this way the Hille-Phillips functional calculus for generators of bounded $C_0-$semigroups is accessible via holomorphic functional calculus. If we consider the Dirac measure $\delta_{t_0} \in \mathbf{M}(\mathbb{R}_+)$ at a point $t_0 \geqslant 0$, then

$$f(A) = \int_{\mathbb{R}_+} e^{-tA} \delta_{t_0}(\mathrm{d}t) = e^{-t_0 A}$$

yields the semigroup operator. If we consider the measure $\mu = e^{-\lambda t}\mathrm{d}t$ with $\operatorname{Re} \lambda > 0$ then

$$f(A) = \int_{\mathbb{R}_+} e^{-tA} e^{-\lambda t} \mathrm{d}t = \mathrm{R}(\lambda, -A)$$

is the resolvent of the generator.

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