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Exact Solution of the Zakharov-Shabat Scattering Problem for Doubly-Truncated Multisoliton Potentials

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Abstract

Recent studies have revealed that multisoliton solutions of the nonlinear Schrödinger equation, as carriers of information, offer a promising solution to the problem of non-linear signal distortions in fiber optic channels. In any nonlinear Fourier transform based transmission methodology seeking to modulate the discrete spectrum of the multisolitons, choice of an appropriate windowing function is an important design issue on account of the unbounded support of such signals. Here, we consider the rectangle function as the windowing function for the multisolitonic signal and provide a recipe for computing the exact solution of the associated Zakharov-Shabat (ZS) scattering problem for the windowed/doubly-truncated multisoliton potential. The idea consists in expressing the Jost solution of the doubly-truncated multisoliton potential in terms of the Jost solution of the original potential. The proposed method allows us to avoid prohibitive numerical computations normally required in order to accurately quantify the effect of time-domain windowing on the nonlinear Fourier spectrum of the multisolitonic signals. Further, the method devised in this work also applies to general type of signals admissible as ZS scattering potential, and, may prove to be a useful tool in the theoretical analysis of such systems.

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Notations

The set of non-zero positive real numbers ($\mathbb{R}$) is denoted by $\mathbb{R}_+$. For any complex number ζ, $\text{Re}(\zeta)$ and $\text{Im}(\zeta)$ refer to the real and the imaginary parts of ζ, respectively. The complex conjugate of ζ is denoted by ζ^*. The upper-half (lower-half) of complex plane ($\mathbb{C}$) is denoted by $\mathbb{C}_+$ ($\mathbb{C}_-$) and its closure by $\overline{\mathbb{C}}_+$ ($\overline{\mathbb{C}}_-$). The Pauli's spin matrices are denoted by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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For the sake of uniformity of notation, we set $\sigma_0 = \text{diag}(1, 1)$. The support of a function $f : \Omega \rightarrow \mathbb{R}$ in Ω is defined as $\text{supp} f = \overline{\{x \in \Omega | f(x) \neq 0\}}$. The Lebesgue spaces of complex-valued functions defined in $\mathbb{R}$ are denoted by L^p for $1 \leq p \leq \infty$ with their corresponding norm denoted by $\| \cdot \|_{\mathsf{L}^p}$ or $\| \cdot \|_p$.

1. Introduction

In optical fiber communication, the propagation of optical field in a loss-less single mode fiber under Kerr-type focusing nonlinearity is governed by the nonlinear Schrödinger equation (NSE) [1, 2] which, in its standard form, reads as

$$i\partial_x q = \partial_t^2 q + 2|q|^2 q, \quad (t, x) \in \mathbb{R} \times \mathbb{R}_+, \tag{1}$$

where $q(t, x)$ is a complex valued function associated with the slowly varying envelope of the electric field, t is the retarded time and x is position along the fiber. This equation also provides a satisfactory description of optical pulse propagation in the guiding-center or path-averaged formulation [3–5] when more general scenarios such as presence of fiber losses, lumped or distributed periodic amplification are included in the mathematical model of the physical channel.

The initial value problem (IVP) corresponding to the NSE was first solved by Zakharov and Shabat [6], which is known to be one of the first successful implementations of the *inverse scattering transform* (IST) method. *Multisolitons* or, more precisely, *K-soliton* solutions were obtained as a special case of this theory. The IST method was later extended to a wider class of nonlinear evolution equations known as the Ablowitz-Kaup-Newell-Segur (AKNS) class of integrable equations [7, 8]. In this pioneering work, IST was, for the first time, presented as a way of Fourier analysis for nonlinear problems prompting researchers to coin the term *nonlinear Fourier transform* (NFT) for IST. In this terminology, any subset of the scattering data that qualifies as the "primordial" scattering data [8] is referred to as the *nonlinear Fourier spectrum*.

The fact that the energy content of *K*-soliton solutions is not dispersed away as it propagates along the fiber makes them promising as carriers of information in optical communication. These ideas were first explored by Hasegawa and Nyu [9] who proposed encoding information in the eigenvalues of the *K*-soliton solutions in a framework which they described as the *eigenvalue communication*. With the recent breakthroughs in coherent optical communication [10, 11] and growing need for increased channel capacity [12–14], these ideas have been recently revived. We refer the reader to the comprehensive review article [15] and the references therein for an overview of NFT-based optical communication methodologies and its potential advantage over the conventional ones.

In this article, we focus on a particular aspect of the NFT-based transmission methodologies which seek to modulate the discrete part of the nonlinear Fourier spectrum using *K*-solitons as information carriers. Given that the support of the *K*-soliton solutions is infinite, it is mandatory to employ a windowing function [16]. The windowing function must be such that it does not considerably alter the nonlinear Fourier spectrum of the original signal. In this work, we consider the simplest of the windowing functions,

the *rectangle function*. It is shown that the resulting scattering problem for the "windowed" or the doubly-truncated K-soliton solution is exactly solvable. The idea is to express the Jost solutions of the windowed potential in terms of the Jost solutions of the original potential. Such an approach has already appeared in the work of Lamb [17] where the scattering problem for a potential truncated from one side is solved exactly using the Jost solutions of the original potential. In particular, the observation that truncated K-soliton solution has rational reflection coefficient has been used to devise exact techniques for IST [18–20]. Adapting Lamb's approach, it is further shown that, in the case of truncation from both sides, one can set up a Riemann-Hilbert (RH) problem to obtain the Jost solutions of the doubly-truncated potential. It must be noted that this method applies to general potentials; however, for K-soliton solutions, the evaluation of certain integrals become a trivial task and the solution of the RH problem can be obtained in a closed form. In particular, the method of *Darboux transformation* (DT) for computing K-soliton solutions provide an adequate representation of the Jost solutions in terms of the so called *Darboux matrix* which, as a function of the spectral parameter, has a rational structure facilitating the solution of the RH problem. This representation further enables us to obtain precise estimates for the effective temporal support as well as spectral width of the K-soliton pulses. The rational structure of the aforementioned Darboux matrix has also been recently exploited to develop fast numerical algorithms for DT [21] and IST [22].

2. Direct Scattering: Doubly-Truncated Potential

The NFT of a given complex-valued signal $q(t)$ is introduced via the associated *Zakharov-Shabat scattering problem* (or ZS problem in short) [6] which can be stated as follows: Let $\zeta \in \mathbb{R}$ and $\mathbf{v} = (v_1, v_2)^{\mathsf{T}} \in \mathbb{C}^2$, then

$$\mathbf{v}_t = -i\zeta\sigma_3\mathbf{v} + U\mathbf{v}, \tag{2}$$

where the matrix elements of U are $U_{11} = U_{22} = 0$ and $U_{12} = q(t) = -U_{21}^* = -r^*(t)$. Here, $q(t)$ is identified as the *scattering potential*. Henceforth, we closely follow the formalism developed in [8, 23]. We assume that the Jost solutions of the *first kind*, denoted by $\boldsymbol{\psi}(t; \zeta)$ and $\overline{\boldsymbol{\psi}}(t; \zeta)$, which are the linearly independent solutions of (2), are known. These solutions are characterized by the following asymptotic behavior as $t \to \infty$: $\boldsymbol{\psi}(t; \zeta)e^{-i\zeta t} \to (0, 1)^{\mathsf{T}}$ and $\overline{\boldsymbol{\psi}}(t; \zeta)e^{i\zeta t} \to (1, 0)^{\mathsf{T}}$. We also assume that the Jost solutions of the *second kind*, denoted by $\boldsymbol{\phi}(t, \zeta)$ and $\overline{\boldsymbol{\phi}}(t, \zeta)$, which are also linearly independent solutions of (2) are known. These solutions are characterized by the following asymptotic behavior as $t \to -\infty$: $\boldsymbol{\phi}(t; \zeta)e^{i\zeta t} \to (1, 0)^{\mathsf{T}}$ and $\overline{\boldsymbol{\phi}}(t; \zeta)e^{-i\zeta t} \to (0, -1)^{\mathsf{T}}$. The scattering coefficients corresponding to $q(t)$ can be written in terms of the Jost solutions by using the Wronskian relations [8]

$$a(\zeta) = \mathscr{W}(\boldsymbol{\phi}, \boldsymbol{\psi}), \quad b(\zeta) = \mathscr{W}(\overline{\boldsymbol{\psi}}, \boldsymbol{\phi}),$$
$$\overline{a}(\zeta) = \mathscr{W}(\overline{\boldsymbol{\phi}}, \overline{\boldsymbol{\psi}}), \quad \overline{b}(\zeta) = \mathscr{W}(\overline{\boldsymbol{\phi}}, \boldsymbol{\psi}). \tag{3}$$

Furthermore, the symmetry properties,

$$\overline{\boldsymbol{\psi}}(t; \zeta) = i\sigma_2\boldsymbol{\psi}^*(t; \zeta^*), \quad \overline{\boldsymbol{\phi}}(t; \zeta) = i\sigma_2\boldsymbol{\phi}^*(t; \zeta^*), \tag{4}$$

yield the relations $\overline{a}(\zeta) = a^*(\zeta^*)$ and $\overline{b}(\zeta) = b^*(\zeta^*)$. Here, we assume that the nonlinear Fourier spectrum of the signal $q(t)$ is as follows: The discrete spectrum consists of the eigenvalues $\zeta_k \in \mathbb{C}_+$ and the norming constants b_k. For convenience, let the discrete spectrum be denoted by the set

$$\mathfrak{S}_K = \{(\zeta_k, b_k) \in \mathbb{C}^2, \; k = 1, 2, \ldots, K\}. \tag{5}$$

The continuous spectrum, also referred to as the *reflection coefficient*, is given by $\rho(\xi) = b(\xi)/a(\xi)$ for $\xi \in \mathbb{R}$.

In this article, we consider windowing using the rectangle function supported in $[-T_-, T_+]$ where $T_-, T_+ > 0$. Define the left-sided signal $q^{(-)}(t; T_+) = q(t)\theta(T_+ - t)$, where $\theta(t)$ is the Heaviside step function so that the windowed signal is $q^{(\sqcap)}(t; T_-, T_+) = q^{(-)}(t; T_+)\theta(t + T_-)$. Here, our objective is to solve the ZS problem corresponding to the windowed potential $q^{(\sqcap)}(t; T_-, T_+)$. To this end, we first derive the Jost solutions for the left-sided signal $q^{(-)}(t; T_+)$ supported in $(-\infty, T_+]$. Starting from the Jost solution of the second kind, it is straightforward to verify that (for $\zeta \in \overline{\mathbb{C}}_+$)

$$\boldsymbol{\phi}^{(-)}(t; \zeta) = \begin{cases} \boldsymbol{\phi}(t; \zeta), & t \leq T_+, \\ e^{-i\sigma_3\zeta(t-T_+)}\boldsymbol{\phi}(T_+; \zeta), & t > T_+. \end{cases} \tag{6}$$

For $t \geq T_+$, the potential is identically zero so that $\boldsymbol{\psi}^{(-)}(t; \zeta) = (0, 1)^\intercal e^{i\zeta t}$, $\zeta \in \overline{\mathbb{C}}_+$. Now, using the Wronskian relations (3), the scattering coefficients, for $\zeta \in \overline{\mathbb{C}}_+$, work out to be

$$a^{(-)}(\zeta) = \phi_1(T_+; \zeta)e^{i\zeta T_+}, \quad b^{(-)}(\zeta) = \phi_2(T_+; \zeta)e^{-i\zeta T_+}, \quad \zeta \in \overline{\mathbb{C}}_+. \tag{7}$$

Next, our aim is to obtain $\boldsymbol{\psi}^{(-)}(t; \zeta)$ for $t < T_+$ for $\zeta \in \mathbb{C}_+$. On the real axis, i.e. $\xi \in \mathbb{R}$, one can obtain $\boldsymbol{\psi}^{(-)}(t; \xi)$ using the linear independence of $\boldsymbol{\phi}^{(-)}(t; \xi)$ and $\overline{\boldsymbol{\phi}^{(-)}}(t; \xi)$:

$$\boldsymbol{\psi}^{(-)}(t; \xi) = -a^{(-)}(\xi)\overline{\boldsymbol{\phi}^{(-)}}(t; \xi) + \overline{b^{(-)}}(\xi)\boldsymbol{\phi}^{(-)}(t; \xi). \tag{8}$$

Evidently, the expression in the right-hand side of the equation above cannot be analytically continued into the upper-half of the complex plane. In order to circumvent this limitation, we adopt a different approach in the following. Let us consider the relation

$$\boldsymbol{\phi}^{(-)}(t; \zeta) = a^{(-)}(\zeta)\overline{\boldsymbol{\psi}^{(-)}}(t; \zeta) + b^{(-)}(\zeta)\boldsymbol{\psi}^{(-)}(t; \zeta), \tag{9}$$

for $\zeta \in \mathbb{R}$. Let $a^{(-)}(\zeta)$ have K' simple zeros in $\mathbb{C}_+$ denoted by $\zeta_k^{(-)}, k = 1, 2, \ldots, K'$. Following [6], we set up a Riemann-Hilbert (RH) problem for the sectionally analytic function $\boldsymbol{F}(\zeta) \equiv \boldsymbol{F}(\zeta; t)$ with simple poles in $\mathbb{C}_+$ defined by

$$\boldsymbol{F}(\zeta; t) = \begin{cases} [a^{(-)}(\zeta)]^{-1}\boldsymbol{\phi}^{(-)}(t; \zeta)e^{i\zeta t}, & \zeta \in \mathbb{C}_+, \\ \overline{\boldsymbol{\psi}^{(-)}}(t; \zeta)e^{i\zeta t}, & \zeta \in \mathbb{C}_-, \end{cases} \tag{10}$$

with the jump condition given by

$$\boldsymbol{F}(\xi + i0) - \boldsymbol{F}(\xi - i0) = \rho^{(-)}(\xi)\boldsymbol{\psi}^{(-)}(t; \xi)e^{i\xi t}, \quad \xi \in \mathbb{R}, \tag{11}$$

4

where $\rho^{(-)}(\xi) = b^{(-)}(\xi)/a^{(-)}(\xi)$. The solution of the RH problem can be stated as

$$F(\zeta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{k=1}^{K'} \frac{e^{i\zeta_k t}}{\left(\zeta - \zeta_k^{(-)}\right)} \frac{\phi^{(-)}(t;\zeta_k^{(-)})}{\dot{a}^{(-)}(\zeta_k^{(-)})} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \rho^{(-)}(\xi)\psi^{(-)}(t;\xi)e^{i\xi t}\frac{d\xi}{\xi - \zeta}. \quad (12)$$

Note that for $t < T_+$ and $\xi \in \mathbb{R}$, we have

$$\rho^{(-)}(\xi)\psi^{(-)}(t;\xi)e^{i\xi t} = -b^{(-)}(\xi)\overline{\phi^{(-)}}(t;\xi)e^{i\xi t} + \frac{|b^{(-)}(\xi)|^2}{a^{(-)}(\xi)}\phi^{(-)}(t;\xi)e^{i\xi t}. \quad (13)$$

The right-hand side of the above equation is known for all $\xi \in \mathbb{R}$; therefore, $F(\zeta)$ can be obtained explicitly provided that the integral in (12) can be computed exactly. This yields $\psi^{(-)}(t;\zeta)$ for $t < T_+$ and $\zeta \in \mathbb{C}_+$, using the symmetry properties stated in (4).

Next, the windowed potential $q^{(\sqcap)}(t;T_-,T_+)$ is obtained as a result of truncation of $q^{(-)}(t;T_+)$ from left. Let the Jost solutions for this potential be $\psi^{(\sqcap)}(t;\zeta)$ (first kind) and $\phi^{(\sqcap)}(t;\zeta)$ (second kind). Let the corresponding scattering coefficients be denoted by $a^{(\sqcap)}(\zeta)$ and $b^{(\sqcap)}(\zeta)$. In the following, our aim would be to obtain an expression for these Jost solutions in terms of the Jost solutions of $q^{(-)}(t;T_+)$. Evidently,

$$\psi^{(\sqcap)} = \begin{cases} e^{-i\sigma_3\zeta(t+T_-)}\psi^{(-)}(-T_-;\zeta), & t < -T_- \\ \psi^{(-)}(t;\zeta), & t \geq -T_-, \end{cases} \quad (14)$$

and $\phi^{(\sqcap)}(t;\zeta) = (1,0)^{\mathsf{T}}e^{-i\zeta t}$ for $t < -T_-$ where $\zeta \in \overline{\mathbb{C}}_+$. Now using the Wronskian relations, we have

$$\begin{aligned} a^{(\sqcap)}(\zeta) &= \psi_2^{(\sqcap)}(-T_-;\zeta)e^{i\zeta T_-} = F_1^*(\zeta^*;-T_-), \\ \overline{b^{(\sqcap)}}(\zeta) &= \psi_1^{(\sqcap)}(-T_-;\zeta)e^{-i\zeta T_-} = -F_2^*(\zeta^*;-T_-)e^{-2i\zeta T_-}, \end{aligned} \quad (15)$$

for $\zeta \in \overline{\mathbb{C}}_+$. This implies $b^{(\sqcap)}(\zeta) = -F_2(\zeta;-T_-)e^{2i\zeta T_-}$ for $\zeta \in \overline{\mathbb{C}}_-$. Note that the functional form obtained for the scattering coefficients $a^{(\sqcap)}(\zeta)$ and $b^{(\sqcap)}(\zeta)$ must hold for all $\zeta \in \mathbb{C}$ if they hold true in any of the half-planes.

The expression in (12) appears to indicate that zeros of $a^{(-)}(\zeta)$ are required in order to compute $F(\zeta)$; however, a closer look at this expression shows that it is not the case: Consider, for $t < T_+$,

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{|b^{(-)}(\xi)|^2}{a^{(-)}(\xi)}\phi^{(-)}(t;\xi)e^{i\xi t}\frac{d\xi}{\xi - \zeta}$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[\frac{1}{a^{(-)}(\xi)} - a^{(-)*}(\xi)\right]\phi^{(-)}(t;\xi)e^{i\xi t}\frac{d\xi}{\xi - \zeta}.$$

Given that $1/a^{(-)}(\xi)$ is holomorphic in $\mathbb{C}_+$ with isolated poles at $\zeta_k^{(-)}$, this part of the integrand can be computed easily by completing the contour in $\mathbb{C}_+$ so that

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\phi^{(-)}(t;\xi)}{a^{(-)}(\xi)}\frac{e^{i\xi t}d\xi}{\xi - \zeta} + \sum_{k=1}^{K'}\frac{e^{i\zeta_k t}}{\left(\zeta - \zeta_k^{(-)}\right)}\frac{\phi^{(-)}(t;\zeta_k^{(-)})}{\dot{a}^{(-)}(\zeta_k^{(-)})} = \begin{cases} [a^{(-)}(\zeta)]^{-1}\phi^{(-)}(t;\zeta)e^{i\zeta t}, & \zeta \in \mathbb{C}_+, \\ 0, & \zeta \in \mathbb{C}_-, \end{cases}$$

yielding

$$F(\zeta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[b^{(-)}(\xi)\overline{\phi^{(-)}}(t;\xi) + a^{(-)*}(\xi)\phi^{(-)}(t;\xi)\right]e^{i\xi t}\frac{d\xi}{\xi - \zeta}, \quad \zeta \in \mathbb{C}_-. \quad (16)$$

3. Doubly-Truncated Multisoliton Potential

Having obtained the general recipe above, we now turn to the case of K-soliton potentials. The K-soliton potentials along with their Jost solutions can be computed quite easily using the *Darboux transformation* (DT) [24–26]. In this discussion, we use the DT procedure described in [24]. Let $\mathfrak{S}_K$ be the discrete spectrum of a K-soliton potential. Define the matrix form of the Jost solutions as $v(t; \zeta) = (\boldsymbol{\phi}, \boldsymbol{\psi})$. The seed solution here corresponds to the *null* potential; therefore, $v_0(t; \zeta) = e^{-i\sigma_3 \zeta t}$. The augmented matrix Jost solution $v_K(t; \zeta)$ can be obtained from the seed solution $v_0(t; \zeta)$ using the Darboux matrix as $v_K(t; \zeta) = \mu_K(\zeta) D_K(t; \zeta, \mathfrak{S}_K) v_0(t; \zeta)$ for $\zeta \in \overline{\mathbb{C}}_+$ where the Darboux matrix is written as

$$D_K(t; \zeta, \mathfrak{S}_K) = \sum_{k=0}^{K} D_k^{(K)}(t; \mathfrak{S}_K) \zeta^k, \tag{17}$$

where the coefficient matrices are such that $D_K^{(K)} = \sigma_0$ and

$$D_k^{(K)} = \begin{pmatrix} d_0^{(k,K)} & d_1^{(k,K)} \\ -d_1^{(k,K)*} & d_0^{(k,K)*} \end{pmatrix}, \quad k = 0, 1, \ldots, K-1. \tag{18}$$

Also, let us recall $a_K(\zeta) = \prod_{k=1}^{K} (\zeta - \zeta_k)(\zeta - \zeta_k^*)^{-1}$ and $(\mu_K)^{-1} = \prod_{k=1}^{K} (\zeta - \zeta_k^*)$ [21]. For $\zeta \in \overline{\mathbb{C}}_+$, it is known that [8]

$$v_K e^{i\sigma_3 \zeta t} = \begin{pmatrix} 1 + \frac{1}{2i\zeta} \mathcal{E}^{(-)} & \frac{1}{2i\zeta} q(t) \\ -\frac{1}{2i\zeta} r(t) & 1 + \frac{1}{2i\zeta} \mathcal{E}^{(+)} \end{pmatrix} + \mathcal{O}\left(\frac{1}{\zeta^2}\right), \tag{19}$$

where

$$\mathcal{E}^{(-)}(t) = \int_{-\infty}^{t} |q(s)|^2 ds, \quad \mathcal{E}^{(+)}(t) = \int_{t}^{\infty} |q(s)|^2 ds. \tag{20}$$

This allows us to conclude that $q(t) = 2i d_1^{(K-1,K)}$ and

$$\mathcal{E}^{(-)} = 2i d_0^{(K-1,K)} + 2i \sum_{k=1}^{K} \zeta_k^*, \quad \mathcal{E}^{(+)} = 2i d_0^{(K-1,K)*} + 2i \sum_{k=1}^{K} \zeta_k^*. \tag{21}$$

The energy in the tails (i.e., the part of the signal outside $[-T_-, T_+]$) is

$$\epsilon_{\text{tails}} = \frac{\mathcal{E}_-(-T_-) + \mathcal{E}_+(T_+)}{\|q\|_2^2} \quad \text{where} \quad \|q\|_2^2 = 4 \sum_{k=1}^{K} \text{Im}\, \zeta_k. \tag{22}$$

The Darboux transformation can be implemented as a recursive scheme [26]. Noting that the seed potential is a null potential, its discrete spectrum is empty. Let us define the successive discrete spectra $\emptyset = \mathfrak{S}_0 \subset \mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \ldots \subset \mathfrak{S}_K$ such that $\mathfrak{S}_j = \{(\zeta_j, b_j)\} \cup \mathfrak{S}_{j-1}$ for $j = 1, 2, \ldots, K$ where (ζ_j, b_j) are distinct elements of $\mathfrak{S}_K$. The Darboux matrix of degree $K > 1$ can be factorized into Darboux matrices of degree one as

$$D_K(t; \zeta, \mathfrak{S}_K | \mathfrak{S}_0) = D_1(t; \zeta, \mathfrak{S}_K | \mathfrak{S}_{K-1})$$
$$\times D_1(t; \zeta, \mathfrak{S}_{K-1} | \mathfrak{S}_{K-2}) \times \ldots \times D_1(t; \zeta, \mathfrak{S}_1 | \mathfrak{S}_0), \tag{23}$$
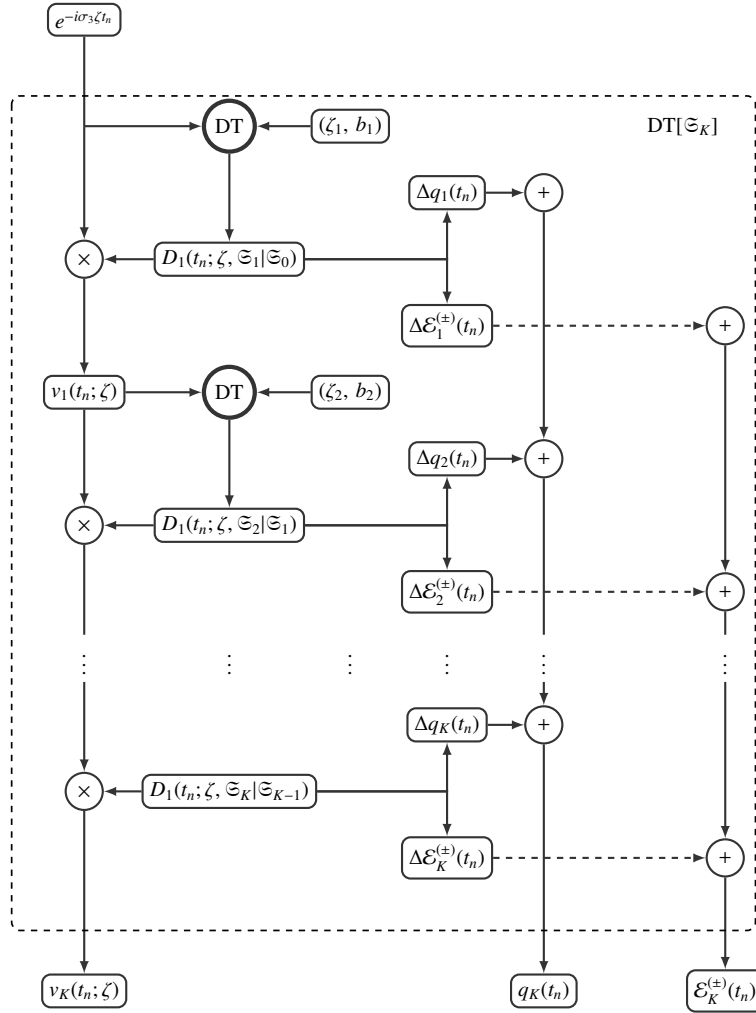
Figure 1: The figure shows the schematic of the Darboux transformation for a given discrete spectrum $\mathfrak{S}_K$ at the grid point t_n. The input is the seed Jost solution, $v_0(t_n; \zeta) = e^{-i\sigma_3 \zeta t_n}$. Here, $\Delta q_j(t_n) = q_j(t_n) - q_{j-1}(t_n)$ and $\Delta \mathcal{E}_j^{(\pm)}(t_n) = \mathcal{E}_j^{(\pm)}(t_n) - \mathcal{E}_{j-1}^{(\pm)}(t_n)$.

where $D_1(t; \zeta, \mathfrak{S}_j | \mathfrak{S}_{j-1})$, $j = 1, \ldots, K$ are the successive Darboux matrices of degree one with the convention that $(\zeta_j, b_j) = \mathfrak{S}_j \cap \mathfrak{S}_{j-1}$ is the bound state being added to the seed potential whose discrete spectra is $\mathfrak{S}_{j-1}$. The resulting scheme is depicted in Fig. 1. Note that the Darboux matrices of degree one can be stated as

$$D_1(t; \zeta, \mathfrak{S}_j | \mathfrak{S}_{j-1}) = \zeta \sigma_0 - \begin{pmatrix} \frac{|\beta_{j-1}|^2 \zeta_j + \zeta_j^*}{1 + |\beta_{j-1}|^2} & \frac{(\zeta_j - \zeta_j^*) \beta_{j-1}}{1 + |\beta_{j-1}|^2} \\ \frac{(\zeta_j - \zeta_j^*) \beta_{j-1}^*}{1 + |\beta_{j-1}|^2} & \frac{\zeta_j + \zeta_j^* |\beta_{j-1}|^2}{1 + |\beta_{j-1}|^2} \end{pmatrix}, \tag{24}$$

and

$$\beta_{j-1}(t;\zeta_j,b_j) = \frac{\phi_1^{(j-1)}(t;\zeta_j) - b_j\psi_1^{(j-1)}(t;\zeta_j)}{\phi_2^{(j-1)}(t;\zeta_j) - b_j\psi_2^{(j-1)}(t;\zeta_j)}, \tag{25}$$

for $(\zeta_j,b_j) \in \mathfrak{S}_K$ and the successive Jost solutions, $v_j = (\boldsymbol{\phi}_j,\boldsymbol{\psi}_j)$, needed in this ratio are computed as

$$v_j(t;\zeta) = \frac{1}{(\zeta - \zeta_j^*)}D_1(t;\zeta,\mathfrak{S}_j|\mathfrak{S}_{j-1})v_{j-1}(t;\zeta). \tag{26}$$

The potential is given by

$$q_j = q_{j-1} - 2i\frac{(\zeta_j - \zeta_j^*)\beta_{j-1}}{1 + |\beta_{j-1}|^2}. \tag{27}$$

and

$$\mathcal{E}_j^{(-)} = \mathcal{E}_{j-1}^{(-)} + \frac{4\operatorname{Im}(\zeta_j)}{1 + |\beta_{j-1}|^{-2}}, \quad \mathcal{E}_j^{(+)} = \mathcal{E}_{j-1}^{(+)} + \frac{4\operatorname{Im}(\zeta_j)}{1 + |\beta_{j-1}|^2}. \tag{28}$$

Finally, let us observe that the computation of the Darboux matrix coefficients can be carried out in $\mathcal{O}\left(K^2\right)$ operations. With N samples of $q(t)$ over $[-T_-, T_+]$, the complexity of computing ϵ_{tails} using the trapezoidal rule (TR) of integration is $\mathcal{O}\left(K^2N\right)$ which can be contrasted with the method proposed here using DT coefficients which affords a complexity of $\mathcal{O}\left(K^2\right)$ yielding an accuracy up to the machine precision[1]. Further, if one attempts to study the effect of propagation of the pulse over M points along the fiber, our method affords a complexity of $\mathcal{O}\left(K^2M\right)$ as opposed to $\mathcal{O}\left(K^2NM\right)$ of any numerical method.

Next, the scattering coefficients corresponding to the truncated K-soliton potential $q^{(-)}(t; T_+)$ work out to be

$$a^{(-)}(\zeta) = \mu_K(\zeta)[D_K(T_+;\zeta,\mathfrak{S}_K)]_{11}, \quad b^{(-)}(\zeta) = \mu_K(\zeta)[D_K(T_+;\zeta,\mathfrak{S}_K)]_{21}e^{-2i\zeta T_+}. \tag{29}$$

In the following, we suppress the dependence on $\mathfrak{S}_K$ for the sake of brevity and proceed to construct the Jost solution $\psi^{(-)}$. Firstly, in order to facilitate the solution of the RH problem introduced above, we intend to compute the terms in (12) exactly by exploiting the rational structure of the Darboux matrix. To this end, let us note that the expression in (13), for $t < T_+$, can be written as

$$\rho^{(-)}(\xi)\boldsymbol{\psi}_K^{(-)}(t;\xi)e^{i\xi t} = \boldsymbol{P}(\xi;t)e^{2i\xi(t-T_+)} + \boldsymbol{Q}(\xi;t), \tag{30}$$

where $\boldsymbol{P}(\xi;t)$ and $\boldsymbol{Q}(\xi;t)$ are vector-valued rational functions of ξ. These functions can be explicitly stated in terms of the Darboux matrix elements as follows:

$$\boldsymbol{P} = -|\mu_K(\xi)|^2[D_K(T_+;\xi)]_{21}\begin{pmatrix} [D_K^*(t;\xi)]_{21} \\ -[D_K^*(t;\xi)]_{11} \end{pmatrix}, \tag{31}$$

$$\boldsymbol{Q} = |\mu_K(\xi)|^2\frac{|[D_K(T_+;\xi)]_{21}|^2}{[D_K(T_+;\xi)]_{11}}\begin{pmatrix} [D_K(t;\xi)]_{11} \\ [D_K(t;\xi)]_{21} \end{pmatrix}. \tag{32}$$

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[1]For a given discrete spectrum and ϵ_{tails}, one can also determine $T_\pm$ using a binary search method where the bracketing interval can be chosen as described in Appendix B.

From these expressions, it follows that the poles of the rational function $\boldsymbol{P}(\xi; t)$ are ζ_k and ζ_k^* while the poles of the rational function $\boldsymbol{Q}(\xi; t)$ are ζ_k, ζ_k^* and zeros of $a^{(-)}(\zeta)$. For the sake of convenience, let us introduce the residues:

$$
\begin{aligned}
\omega_j(\zeta_k) &= \mathrm{Res}\left(Q_j(\xi; t); \zeta_k\right), & \omega_j(\zeta_k^{(-)}) &= \mathrm{Res}\left(Q_j(\xi; t); \zeta_k^{(-)}\right), \\
\pi_j(\zeta_k) &= \mathrm{Res}\left(P_j(\xi; t); \zeta_k\right), & \pi_j(\zeta_k^*) &= \mathrm{Res}\left(P_j(\xi; t); \zeta_k^*\right),
\end{aligned}
\tag{33}
$$

for $j = 1, 2$. For $\zeta \in \mathbb{C}_-$, define

$$
\mathcal{I}_j(\zeta; t) = \lim_{\kappa \to \infty} \frac{1}{2\pi i} \oint_{\Gamma_\kappa} \frac{d\xi}{\xi - \zeta} P_j(\xi; t) e^{2i\xi(t - T_+)},
\tag{34}
$$

$$
\mathcal{J}_j(\zeta; t) = \lim_{\kappa \to \infty} \frac{1}{2\pi i} \oint_{\Gamma_\kappa} \frac{Q_j(\xi; t) d\xi}{\xi - \zeta} + \sum_{k=1}^{K'} \frac{\omega_j(\zeta_k^{(-)})}{\zeta - \zeta_k^{(-)}} = -\sum_{k=1}^{K} \frac{\omega_j(\zeta_k)}{\zeta - \zeta_k},
\tag{35}
$$

where Γ_κ denotes the contour comprising the segment $[-\kappa, \kappa]$ $(\kappa > 0)$ and a semicircular arc with radius κ oriented negatively in $\mathbb{C}_-$ and $j = 1, 2$. Observing,

$$
P_j(\xi; t) = \sum_{k=1}^{K} \left[\frac{\pi_j(\zeta_k)}{\xi - \zeta_k} + \frac{\pi_j(\zeta_k^*)}{\xi - \zeta_k^*} \right],
\tag{36}
$$

the integrals $\mathcal{I}_j$ work out to be

$$
\mathcal{I}_j(\zeta; t) = -\sum_{k=1}^{K} \frac{\pi_j(\zeta_k)}{\zeta - \zeta_k} e^{2i\zeta(t - T_+)} - \sum_{k=1}^{K} \frac{\pi_j(\zeta_k^*)}{\zeta - \zeta_k^*} \left[e^{2i\zeta(t - T_+)} - e^{2i\zeta_k^*(t - T_+)} \right].
\tag{37}
$$

This allows us to write $\boldsymbol{F}(\zeta) = (1, 0)^\mathsf{T} + \boldsymbol{I}(\zeta; t) + \boldsymbol{J}(\zeta; t)$ where we have used the fact that the second term in the right-hand side of (12) is given by

$$
\sum_{k=1}^{K'} \frac{1}{\zeta - \zeta_k^{(-)}} \begin{pmatrix} \frac{[D_K(t; \zeta_k^{(-)})]_{11}}{[\dot{D}_K(T_+; \zeta_k^{(-)})]_{11}} \\ \frac{[D_K(t; \zeta_k^{(-)})]_{21}}{[\dot{D}_K(T_+; \zeta_k^{(-)})]_{11}} \end{pmatrix} = \sum_{k=1}^{K'} \frac{\omega(\zeta_k^{(-)})}{\zeta - \zeta_k^{(-)}},
$$

with $\boldsymbol{\omega} = (\omega_1, \omega_2)^\mathsf{T}$. Next, let us show that the poles of $\boldsymbol{F}(\zeta)$ at ζ_k (as well as at ζ_k^*) are removable. First let us observe that

$$
\begin{pmatrix} \phi_1(t; \zeta) e^{i\zeta t} & \psi_1(t; \zeta) e^{-i\zeta t} \\ \phi_2(t; \zeta) e^{i\zeta t} & \psi_2(t; \zeta) e^{-i\zeta t} \end{pmatrix} = D_K(t; \zeta_k).
$$

Using the symmetry relations for the Darboux matrix, let us also observe that

$$
\begin{aligned}
&-\frac{[D_K^*(t; \zeta_k^*)]_{21}}{[D_K(t; \zeta_k)]_{11}} e^{2i\zeta_k t} + \frac{[D_K^*(T_+; \zeta_k^*)]_{21}}{[D_K(T_+; \zeta_k)]_{11}} e^{2i\zeta_k T_+} \\
&= \frac{[D_K(t; \zeta_k)]_{12}}{[D_K(t; \zeta_k)]_{11}} e^{2i\zeta_k t} - \frac{[D_K(T_+; \zeta_k)]_{12}}{[D_K(T_+; \zeta_k)]_{11}} e^{2i\zeta_k T_+} \\
&= \frac{\psi_1(t; \zeta_k)}{\phi_1(t; \zeta_k)} - \frac{\psi_1(T_+; \zeta_k)}{\phi_1(T_+; \zeta_k)} \equiv 0,
\end{aligned}
$$

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on account of the property of the norming constant b_k. Similarly,

$$\frac{[D_K^*(t;\zeta_k^*)]_{11}}{[D_K(t;\zeta_k)]_{21}}e^{2i\zeta_k t} + \frac{[D_K^*(T_+;\zeta_k^*)]_{21}}{[D_K(T_+;\zeta_k)]_{11}}e^{2i\zeta_k T_+}$$

$$= \frac{[D_K(t;\zeta_k)]_{22}}{[D_K(t;\zeta_k)]_{21}}e^{2i\zeta_k t} - \frac{[D_K(T_+;\zeta_k)]_{12}}{[D_K(T_+;\zeta_k)]_{11}}e^{2i\zeta_k T_+}$$

$$= \frac{\psi_2(t;\zeta_k)}{\phi_2(t;\zeta_k)} - \frac{\psi_1(T_+;\zeta_k)}{\phi_1(T_+;\zeta_k)} \equiv 0.$$

Therefore, $\omega_j(\zeta_k) + \pi_j(\zeta_k)\exp[2i\zeta_k(t - T_+)] = 0$, so that

$$\lim_{\zeta \to \zeta_k} \frac{\left[\omega_j(\zeta_k) + \pi_j(\zeta_k)\exp[2i\zeta(t - T_+)]\right]}{(\zeta - \zeta_k)} = 2i(t - T_+)\pi_j(\zeta_k)\exp[2i\zeta_k(t - T_+)].$$

Consequently,

$$\mathcal{I}(\zeta;t) + \mathcal{J}(\zeta;t) = -\sum_{k=1}^{K} \frac{\pi(\zeta_k;t)}{\zeta - \zeta_k}\left[e^{2i\zeta(t-T_+)} - e^{2i\zeta_k(t-T_+)}\right]$$

$$- \sum_{k=1}^{K} \frac{\pi(\zeta_k^*;t)}{\zeta - \zeta_k^*}\left[e^{2i\zeta(t-T_+)} - e^{2i\zeta_k^*(t-T_+)}\right], \tag{38}$$

turns out to be a vector valued function analytic for all $\zeta \in \mathbb{C}$. Finally, the scattering coefficients for the windowed potential $q^{(\sqcap)}(t; T_-, T_+)$ work out to be

$$a^{(\sqcap)}(\zeta) = 1 + \mathcal{I}_1^*(\zeta^*; -T_-) + \mathcal{J}_1^*(\zeta^*; -T_-),$$
$$b^{(\sqcap)}(\zeta) = -\left[\mathcal{I}_2(\zeta; -T_-) + \mathcal{J}_2(\zeta; -T_-)\right]e^{2i\zeta T_-}, \tag{39}$$

for $\zeta \in \mathbb{C}$. The discrete spectrum can be computed by first computing the zeros of $a^{(\sqcap)}(\zeta)$ (using methods developed for analytic functions [27, 28]) which gives the eigenvalues and evaluating $b^{(\sqcap)}(\zeta)$ at the eigenvalues gives the norming constant.

We conclude this section by demonstrating that the scattering coefficients obtained above are functions of exponential type (See Appendix A): Setting $t = -T_-$ we have, for $|\zeta| > \max_k |\zeta_k|$,

$$\|\mathcal{I}(\zeta;t) + \mathcal{J}(\zeta;t)\| \leq \sum_{k=1}^{K} \frac{\|\pi(\zeta_k; -T_-)\|}{2|\zeta_k|}\left[e^{-4T\,\mathrm{Im}\,\zeta} + e^{4T\,\mathrm{Im}\,\zeta_k}\right]$$

$$+ \sum_{k=1}^{K} \frac{\|\pi(\zeta_k^*; -T_-)\|}{2|\zeta_k|}\left[e^{-4T\,\mathrm{Im}\,\zeta} + e^{-4T\,\mathrm{Im}\,\zeta_k}\right]. \tag{40}$$

From here it is straightforward to conclude that $a^{(\sqcap)}(\zeta)$ and $b^{(\sqcap)}(\zeta)$ satisfy an estimate of the form (A.2).

3.1. Conserved quantities and spectral width

Consider the Fourier spectrum of the multisoliton potential denoted by

$$Q(\xi) = \int q(t)e^{-i\xi t}dt. \tag{41}$$

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For convenience, we introduce the notation

$$\langle \xi^n \rangle = \frac{\frac{1}{2\pi} \int |Q(\xi)|^2 \xi^n d\xi}{\frac{1}{2\pi} \int |Q(\xi)|^2 d\xi} \equiv \frac{\frac{1}{2\pi} \int |Q(\xi)|^2 \xi^n d\xi}{\|q\|_2^2}, \tag{42}$$

for moments in the Fourier domain. Let us observe that the following quantities can be expressed entirely in terms of the eigenvalues:

$$C_0 = \|q\|_2^2 = 4 \sum_k \operatorname{Im} \zeta_k, \tag{43}$$

$$C_1 = - \int \partial_t q(t) q^*(t) dt = 4i \sum_k \operatorname{Im} \zeta_k^2, \tag{44}$$

$$C_2 = \int \left[|q(t)|^4 - \partial_t q^*(t) \partial_t q(t) \right] dt = -\frac{16}{3} \sum_k \operatorname{Im} \zeta_k^3. \tag{45}$$

These quantities do not evolve as the pulse propagates along the fiber. Further, from the first moment

$$\langle \xi \rangle = \frac{\frac{1}{2\pi} \int |Q(\xi)|^2 \xi d\xi}{\|q\|_2^2} = \frac{\int i \partial_t q(t) q^*(t) dt}{\|q\|_2^2} = -\frac{iC_1}{C_0}, \tag{46}$$

and the second moment

$$\begin{aligned} \langle \xi^2 \rangle &= \frac{\frac{1}{2\pi} \int |Q(\xi)|^2 \xi^2 d\xi}{\|q\|_2^2} = \frac{\int [i \partial_t q(t)][i \partial_t q(t)]^* dt}{\|q\|_2^2} \\ &= -\frac{C_2}{C_0} + \frac{1}{C_0} \int |q(t)|^4 dt, \end{aligned} \tag{47}$$

we obtained the variance $\langle \Delta \xi^2 \rangle = \langle \xi^2 \rangle - \langle \xi \rangle^2$ as follows:

$$\langle \Delta \xi^2 \rangle = \frac{1}{C_0} \int |q(t)|^4 dt + \frac{C_1^2}{C_0^2} - \frac{C_2}{C_0} \le \|q\|_\infty^2 + \frac{C_1^2}{C_0^2} - \frac{C_2}{C_0}. \tag{48}$$

This quantity characterizes the width of the Fourier spectrum. Note that the biquadratic integral must be computed numerically. However, $\|q\|_\infty$ can be computed in a straightforward manner: From (27), we have $\|q_j\|_\infty \le \|q_{j-1}\|_\infty + 2 \operatorname{Im}(\zeta_j)$, we have

$$\|q_K\|_\infty \le 2 \sum_{k=1}^{K} \operatorname{Im}(\zeta_j), \tag{49}$$

which yields

$$\langle \Delta \xi^2 \rangle \le \frac{C_0^2}{2} + \frac{C_1^2}{C_0^2} - \frac{C_2}{C_0}. \tag{50}$$

Note that this inequality holds irrespective of how the pulse evolves as it propagates along the fiber.

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Now, turning to the windowed multisolitons and denoting the conserved quantities of the windowed signal by $C_j^{(\sqcap)}$ for $j = 0, 1, 2, \ldots$, we have

$$\langle \Delta \xi^2 \rangle^{(\sqcap)} \le \|q^{(\sqcap)}\|_\infty^2 + \left(\frac{C_1^{(\sqcap)}}{C_0^{(\sqcap)}} \right)^2 - \frac{C_2^{(\sqcap)}}{C_0^{(\sqcap)}}. \tag{51}$$

Note that as the pulse evolves $\|q^{(\sqcap)}\|_\infty$ may not remain bounded by $\|q\|_\infty$ as it does at the initial point, i.e., $x = 0$. The conserved quantities for the windowed potential can be obtained from the asymptotic expansion of $\log[a^{(\sqcap)}(\zeta)]$ as $|\zeta| \to \infty$ while keeping $\zeta \in \mathbb{C}_+$. To this end, let

$$a^{(\sqcap)}(\zeta) \sim 1 + \frac{a_1^{(\sqcap)}}{2i\zeta} + \frac{a_2^{(\sqcap)}}{(2i\zeta)^2} + \frac{a_3^{(\sqcap)}}{(2i\zeta)^3} + \ldots \tag{52}$$

as $|\zeta| \to \infty$ in $\mathbb{C}_+$. The coefficients introduced above can be explicitly stated as

$$(2i)^{-j} a_j^{(\sqcap)} = \sum_{k=1}^{K} \pi_1 (\zeta_k^*)^* e^{4i\zeta_k T} \zeta_k^{j-1} + \sum_{k=1}^{K} \pi_1 (\zeta_k)^* e^{4i\zeta_k^* T} (\zeta_k^*)^{j-1}. \tag{53}$$

Observing that $C_j^{(\sqcap)}$ are defined as

$$\log a^{(\sqcap)}(\zeta) \sim \sum_{j=0}^{\infty} \frac{C_j^{(\sqcap)}}{(2i\zeta)^{j+1}}, \tag{54}$$

the conserved quantities work out to be

$$\begin{aligned} C_0^{(\sqcap)} &= a_1^{(\sqcap)}, \\ C_1^{(\sqcap)} &= a_2^{(\sqcap)} - \frac{(a_1^{(\sqcap)})^2}{2}, \\ C_2^{(\sqcap)} &= a_3^{(\sqcap)} - a_1^{(\sqcap)} a_2^{(\sqcap)} + \frac{(a_1^{(\sqcap)})^3}{3}. \end{aligned} \tag{55}$$

4. Examples

In the following, we treat a simple example of a doubly-truncated 1-soliton in order to demonstrate how to use the recipe provided in this article. Further, we present a general example of a doubly-truncated 6-soliton where the procedure outlined in this article must be implemented numerically. Note that there are no explicit expressions provided in this article for arbitrary K; however, this does not limit our ability to compute them to machine precision thanks to the iterative Darboux transformation procedure[2].

---

[2]The aforementioned iterative scheme and the recipe provided in this article requires some care in implementation in order to avoid arithmetic overflow/underflow of floating point operations. This discussion is not central to the understanding of the main results presented in this article and is, therefore, being omitted.
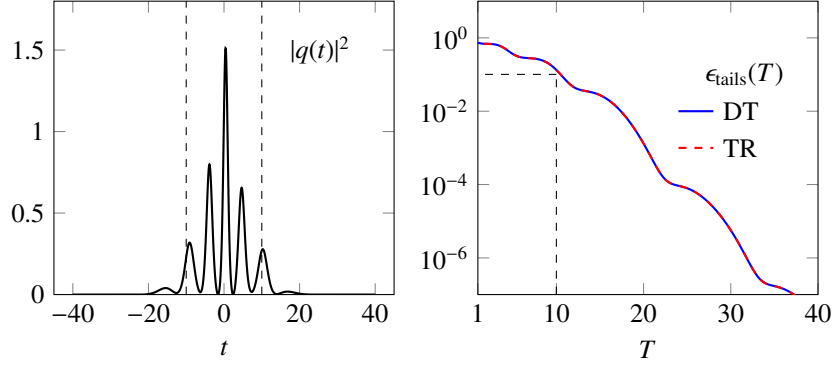
Figure 2: The figure shows the 6-soliton potential (left) corresponding to $\mathfrak{S}_6$ (see Fig. 3) where the dashed lines mark the truncation points ± 10. The fraction of the total energy in the tails $\epsilon_{\text{tails}}(T)$ as a function of the truncation point $\pm T$ is plotted on the right (TR: trapezoidal rule).

4.1. One Soliton

Consider a 1-soliton potential with the discrete spectrum (ζ_1, b_1) where $\zeta = \xi + i\eta_1$. The Darboux matrix can be easily worked out as

$$D_1(t; \zeta) = \zeta \sigma_0 - \begin{pmatrix} \frac{|\beta_0|^2 \zeta_1 + \zeta_1^*}{1+|\beta_0|^2} & \frac{(\zeta_1 - \zeta_1^*)\beta_0}{1+|\beta_0|^2} \\ \frac{(\zeta_1 - \zeta_1^*)\beta_0^*}{1+|\beta_0|^2} & \frac{\zeta_1 + \zeta_1^*|\beta_0|^2}{1+|\beta_0|^2} \end{pmatrix}, \tag{56}$$

where $\beta_0(t; \zeta_1, b_1) = -(1/b_1)e^{-2i\zeta_1 t}$. Let $2T = T_+ + T_-$ and define $Z_+ = 1/\beta_0(T_+)$ and $Z_- = \beta_0(-T_-)$ so that $|Z_\pm| = |b_1|^{\mp 1} e^{-2\eta_1 T_\pm}$. Now

$$P_1(\xi; -T_-) = -\frac{4\eta_1^2 Z_+ Z_-}{\Xi} \frac{1}{(\xi - \zeta_1)(\xi - \zeta_1^*)}$$

where $\Xi = (1 + |Z_+|^2)(1 + |Z_-|^2)$ so that

$$a^{(\sqcap)}(\zeta) = 1 + \frac{2i\eta_1 Z_+^* Z_-^*}{\Xi} \left[\frac{e^{4i\zeta T} - e^{4i\zeta_1^* T}}{\zeta - \zeta_1^*} - \frac{e^{4i\zeta T} - e^{4i\zeta_1 T}}{\zeta - \zeta_1} \right]. \tag{57}$$

The b-coefficient can be computed as follows: Observing

$$P_2(\xi; -T_-) = -\frac{2i\eta_1 Z_+}{\Xi} \frac{[(\xi - \zeta_1) + (\xi - \zeta_1^*)|Z_-|^2]}{(\xi - \zeta_1)(\xi - \zeta_1^*)},$$

we have

$$b^{(\sqcap)}(\zeta) = \frac{2i\eta_1 b_1 |Z_-|^2}{\Xi} \frac{\left(e^{-2i(\zeta - \zeta_1)T_+} - e^{2i(\zeta - \zeta_1)T_-} \right)}{\zeta - \zeta_1} + \frac{2i\eta_1 |Z_+|^2}{b_1^* \Xi} \frac{\left(e^{-2i(\zeta - \zeta_1^*)T_+} - e^{2i(\zeta - \zeta_1^*)T_-} \right)}{\zeta - \zeta_1^*}. \tag{58}$$
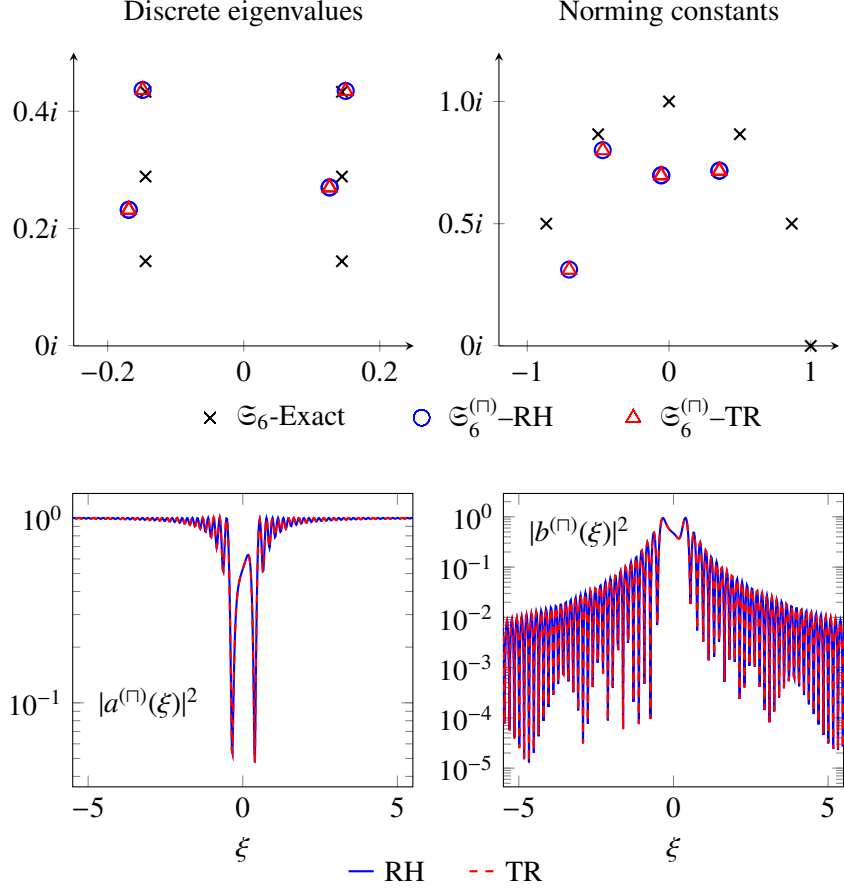
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Figure 3: The top row shows the discrete spectrum $\mathfrak{S}_6$ of a 6-soliton as well as that of its windowed version, i.e., $\mathfrak{S}_6^{(\sqcap)}$. The scattering coefficients of windowed 6-soliton is plotted in the bottom row. A comparison is made between our method (labeled with 'RH') and a numerical method of solving the ZS problem (labeled with 'TR').

Using the asymptotic expansion of $\log[a^{(\sqcap)}(\zeta)]$ as $|\zeta| \to \infty$ in $\mathbb{C}_+$, it is also straightforward to workout

$$\|q^{(\sqcap)}\|_2^2 = 4\eta_1 \frac{1 - |Z_-|^2|Z_+|^2}{(1 + |Z_-|^2)(1 + |Z_+|^2)}. \tag{59}$$

To the leading order in $|Z_\pm|$, the eigenvalue of the windowed 1-soliton is given by

$$\zeta_1^{(\sqcap)} \approx \zeta_1 - 2i\eta_1(|Z_-|^2 + |Z_+|^2), \tag{60}$$

and the norming constant given by

$$b_1^{(\sqcap)} \approx b_1 + 4\eta_1 b_1(T_-|Z_-|^2 - T_+|Z_+|^2). \tag{61}$$

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4.2. 6-Soliton Case

For the second example, we choose a complex vector $\lambda \equiv (\lambda_1, \ldots, \lambda_6) = (\pm 1 + 1i, \pm 1 + 2i, \pm 1 + 3i)$ and $b_k = e^{i(\pi/6)(k-1)}$, $k = 1, \ldots, 6$. The eigenvalues are then taken to be $\zeta_k = \lambda_k/\kappa$ where $\kappa = 2(\sum_{k=1}^{6} \operatorname{Im} \lambda_k)^{1/2}$. The windowed potential is given by $q^{(\sqcap)}(t; T) = \theta(T^2 - t^2)q(t)$ where $T = 10$ (see Fig. 2). The energy in the tails is $\epsilon_{\text{tail}}(10) \approx 10\%$. For the sake of comparison, we solve the scattering problem for $q^{(\sqcap)}(t; T)$ using the (exponential) trapezoidal rule (TR) proposed in [21] with 2^{14} samples. The method proposed in this article is labeled with 'RH' in order to signify the fact that an RH problem is solved exactly to obtain the scattering coefficients. The discrete spectrum for the windowed potential '$\mathfrak{S}_6^{(\sqcap)}$–RH' as depicted in Fig. 3 is determined numerically from the functional form of $a^{(\sqcap)}(\zeta)$ and $b^{(\sqcap)}(\zeta)$ obtained above while '$\mathfrak{S}_6^{(\sqcap)}$–TR' is computed numerically as in [21]. Evidently, results in Fig. 3 confirm the validity of the method proposed in this article.

5. Conclusion

To conclude, we have discussed a method to solve the Zakharov-Shabat scattering problem for the doubly-truncated scattering potential in terms of the Jost solutions of the original potential using the standard techniques of Riemann-Hilbert problems. Exploiting the rational structure of the Darboux matrix, it was possible to obtain the scattering coefficients for the doubly-truncated multisoliton potentials. Significance of this result lies in the fact that on account of the unbounded support of multisolitons, windowing is a practical necessity and an important design issue in optical communication which can now be addressed quite efficiently. Furthermore, it is interesting to note that the results presented in this article may enable us to determine optimal values for various design parameters for K-soliton solutions via the solution of a nonlinear optimization problem. These aspects will be explored in a future publication.

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Appendix A. Scattering coefficients of compactly supported potentials

It is well know that Jost solutions and, consequently, the scattering coefficients for compactly supported potentials are entire functions of the spectral parameter ζ, i.e. analytic for all $\zeta \in \mathbb{C}$ [8, 23]. Let us consider the scattering coefficients in the following. It turns out that, for compactly supported potentials, the scattering coefficients are analytic functions of exponential type (see [29] for properties of such functions) in $\mathbb{C}$. A simple proof of this statement is provided below. Now, given that a doubly-truncated scattering potential is compactly supported, the method proposed in this article must yield scattering coefficients that are of exponential type in $\mathbb{C}$. This is verified in Sec. 3.

Introducing the "local" scattering coefficients $a(t;\zeta)$ and $b(t;\zeta)$ such that $\boldsymbol{\phi}(t;\zeta) = (a(t;\zeta)e^{-i\zeta t}, b(t;\zeta)e^{i\zeta t})^\intercal$, the scattering problem in (2) reads as

$$\partial_t a(t;\zeta) = q(t)b(t;\zeta)e^{2i\zeta t}, \quad \partial_t b(t;\zeta) = r(t)a(t;\zeta)e^{-2i\zeta t}. \tag{A.1}$$

Let $\Omega = [-T_-, T_+]$ where $T_\pm \geq 0$. The initial conditions for the Jost solution $\boldsymbol{\phi}$ are: $a(-T_-;\zeta) = 1$ and $b(-T_-;\zeta) = 0$. The scattering coefficients can be directly obtained from these functions as $a(\zeta) = a(T_+;\zeta)$ and $b(\zeta) = b(T_+;\zeta)$. The following estimate establishes that $a(\zeta)$ and $b(\zeta)$ are of exponential type in $\mathbb{C}$:

Theorem 1. *Let $q \in \mathsf{L}^1$ with support in Ω and set $\kappa = \|q\|_{\mathsf{L}^1(\Omega)}$. Let $f(\zeta)$ denote either $[a(\zeta) - 1]e^{-2i\zeta T_+}$ or $b(\zeta)$; then the estimate*

$$|f(\zeta)| \leq \begin{cases} Ce^{2T_+ \operatorname{Im}\zeta}, & \zeta \in \overline{\mathbb{C}}_+, \\ Ce^{-2T_- \operatorname{Im}\zeta}, & \zeta \in \mathbb{C}_-. \end{cases} \tag{A.2}$$

holds for $C = \|\boldsymbol{D}\| \cosh\kappa$ where $\boldsymbol{D} = (\kappa^2/2, \kappa)^\intercal$.

Proof. Let us define for convenience the modified Jost solution

$$\widetilde{\boldsymbol{P}}(t;\zeta) = \boldsymbol{\phi}(t;\zeta)e^{i\zeta t} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a(t;\zeta) - 1 \\ b(t;\zeta)e^{2i\zeta t} \end{pmatrix}, \tag{A.3}$$

so that $\widetilde{\boldsymbol{P}}(T_+;\zeta)e^{-2i\zeta T_+} = ([a(\zeta)-1]e^{-2i\zeta T_+}, b(\zeta))^\intercal$. The system of equations in (A.1) can be transformed into a set of Volterra integral equations of the second kind for $\widetilde{\boldsymbol{P}}(t;\zeta)$:

$$\widetilde{\boldsymbol{P}}(t;\zeta) = \boldsymbol{\Phi}(t;\zeta) + \int_\Omega \mathcal{K}(t,y;\zeta)\widetilde{\boldsymbol{P}}(y;\zeta)dy, \tag{A.4}$$

where $\boldsymbol{\Phi}(t;\zeta) = (\Phi_1, \Phi_2)^\intercal \in \mathbb{C}^2$ with

$$\Phi_1(t;\zeta) = \int_{-T_-}^t q(z)\Phi_2(z;\zeta)dz,$$
$$\Phi_2(t;\zeta) = \int_{-T_-}^t r(y)e^{2i\zeta(t-y)}dy, \tag{A.5}$$

and the Volterra kernel $\mathcal{K}(x,y;\zeta) = \operatorname{diag}(\mathcal{K}_1, \mathcal{K}_2) \in \mathbb{C}^{2\times2}$ is such that

$$\mathcal{K}_1(x,y;\zeta) = r(y)\int_y^x q(z)e^{2i\zeta(z-y)}dz,$$
$$\mathcal{K}_2(x,y;\zeta) = q(y)\int_y^x r(z)e^{2i\zeta(x-z)}dz, \tag{A.6}$$

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with $\mathcal{K}(x, y; \zeta) = 0$ for $y > x$. Now, the proof can be obtained using the same method as in [8]. For fixed $\zeta \in \overline{\mathbb{C}}_+$, let $\mathcal{K}$ denote the Volterra integral operator in (A.4) corresponding to the kernel $\mathcal{K}(x, y; \zeta)$ such that

$$\mathcal{K}[\widetilde{\boldsymbol{P}}](t; \zeta) = \int_\Omega \mathcal{K}(t, y; \zeta)\widetilde{\boldsymbol{P}}(y; \zeta)dy = \int_{-T_-}^t dz \int_{-T_-}^z dy \begin{pmatrix} q(z)r(y)e^{2i\zeta(z-y)}\widetilde{P}_1(y; \zeta) \\ q(y)r(z)e^{2i\zeta(t-z)}\widetilde{P}_2(y; \zeta) \end{pmatrix}. \tag{A.7}$$

Consider the $\mathsf{L}^\infty(\Omega)$-norm [30, Chap. 9] of $\mathcal{K}$ given by

$$\|\mathcal{K}\|_{\mathsf{L}^\infty(\Omega)} = \operatorname*{ess\,sup}_{t \in \Omega} \int_\Omega \|\mathcal{K}(t, y; \zeta)\|dy, \tag{A.8}$$

so that $\|\mathcal{K}\|_{\mathsf{L}^\infty(\Omega)} \le \kappa^2/2$ [8]. The resolvent $\mathcal{R}$ of this operator exists and is given by the Neumann series $\mathcal{R} = \sum_{n=1}^\infty \mathcal{K}_n$ where $\mathcal{K}_n = \mathcal{K} \circ \mathcal{K}_{n-1}$ with $\mathcal{K}_1 = \mathcal{K}$. It can also be shown using the methods in [8] that $\|\mathcal{K}_n\|_{\mathsf{L}^\infty(\Omega)} \le \kappa^{2n}/(2n)!$, yielding the estimate $\|\mathcal{R}\|_{\mathsf{L}^\infty(\Omega)} \le [\cosh(\kappa) - 1]$. Therefore, for any $\boldsymbol{\Phi}(t; \zeta) \in \mathsf{L}^\infty(\Omega; \mathbb{C}^2)$, the relationship $\widetilde{\boldsymbol{P}}(t; \zeta) = \boldsymbol{\Phi}(t; \zeta) + \mathcal{R}[\boldsymbol{\Phi}](t; \zeta)$ implies, for $\zeta \in \overline{\mathbb{C}}_+$,

$$\|\widetilde{\boldsymbol{P}}(t; \zeta)\|_{\mathsf{L}^\infty(\Omega)} \le \cosh(\kappa)\|\boldsymbol{\Phi}(t; \zeta)\|_{\mathsf{L}^\infty(\Omega)}. \tag{A.9}$$

The result for $\overline{\mathbb{C}}_+$ in (A.2) follows from the observation that, for $\zeta \in \overline{\mathbb{C}}_+$, $\|\boldsymbol{\Phi}(t; \zeta)\|_{\mathsf{L}^\infty(\Omega)} \le \|\boldsymbol{D}\|$ where $\boldsymbol{D} = (\kappa^2/2, \kappa)^\intercal$. Therefore, C can be chosen to be $\|\boldsymbol{D}\|\cosh \kappa$. For the case $\mathbb{C}_-$ of (A.2), we consider $\widetilde{\boldsymbol{P}}_-(t; \zeta) = \widetilde{\boldsymbol{P}}(t; \zeta)e^{-2i\zeta t}$ so that $\widetilde{\boldsymbol{P}}_-(T_+; \zeta) = ([a(\zeta) - 1]e^{-2i\zeta T_+}, b(\zeta))^\intercal$. The Volterra integral equations then reads as $\widetilde{\boldsymbol{P}}_-(t; \zeta)$:

$$\widetilde{\boldsymbol{P}}_-(t; \zeta) = \boldsymbol{\Phi}_-(t; \zeta) + \int_\Omega \mathcal{K}_-(t, y; \zeta)\widetilde{\boldsymbol{P}}_-(y; \zeta)dy, \tag{A.10}$$

where $\boldsymbol{\Phi}_-(t; \zeta) = \boldsymbol{\Phi}(t; \zeta)e^{-2i\zeta t} \in \mathbb{C}^2$ and the Volterra kernel $\mathcal{K}_-(x, y; \zeta) = \operatorname{diag}(\mathcal{K}_1^{(-)}, \mathcal{K}_2^{(-)}) \in \mathbb{C}^{2 \times 2}$ is such that

$$\begin{aligned} \mathcal{K}_1^{(-)}(x, y; \zeta) &= r(y) \int_y^x q(z)e^{-2i\zeta(x-z)}dz, \\ \mathcal{K}_2^{(-)}(x, y; \zeta) &= q(y) \int_y^x r(z)e^{-2i\zeta(z-y)}dz, \end{aligned} \tag{A.11}$$

with $\mathcal{K}_-(x, y; \zeta) = 0$ for $y > x$. Using the approach outlined above, it is possible to show that, for $\zeta \in \mathbb{C}_-$, $\|\widetilde{\boldsymbol{P}}_-(t; \zeta)\|_{\mathsf{L}^\infty(\Omega)} \le \cosh(\kappa)\|\boldsymbol{\Phi}_-(t; \zeta)\|_{\mathsf{L}^\infty(\Omega)}$. The result for the case $\zeta \in \mathbb{C}_-$ in (A.2) then follows from the observation that $\|\boldsymbol{\Phi}_-(t; \zeta)\|_{\mathsf{L}^\infty(\Omega)} \le \|\boldsymbol{D}\|e^{-2\operatorname{Im}(\zeta)T_-}$ for $\zeta \in \mathbb{C}_-$. $\qquad\qquad\square$

Appendix B. Some bounds on the energy in the tails

The unknown Darboux matrix coefficients introduced in the article can also be obtained as the solution of a linear system [21, 24]. From $\boldsymbol{\phi}_K(t; \zeta_k) = b_k\boldsymbol{\psi}_K(t; \zeta_k)$, we have

$$D_K(t; \zeta_k, \mathfrak{S}_K)[\boldsymbol{\phi}_0(t; \zeta_k) - b_k\boldsymbol{\psi}_0(t; \zeta_k)] = 0. \tag{B.1}$$

Note that $\boldsymbol{\phi}_0(t;\zeta_k) - b_k(t)\boldsymbol{\psi}_0(t;\zeta_k) \neq 0$ on account of the fact that ζ_k is not an eigenvalue of the seed potential. Define the Vandermonde matrix $\mathcal{V} = \{V_{jk}\}_{K \times K}$ where

$$V_{jk} = \zeta_j^k, \quad j = 1, 2, \ldots, K, \; k = 0, 1, \ldots, K - 1,$$

and the diagonal matrix $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_K)$. Let the vectors

$$\boldsymbol{f} = \begin{pmatrix} \zeta_1^K \\ \zeta_2^K \\ \vdots \\ \zeta_K^K \end{pmatrix}, \quad \boldsymbol{g} = \Gamma \boldsymbol{f} = \begin{pmatrix} \zeta_1^K \gamma_1 \\ \zeta_2^K \gamma_2 \\ \vdots \\ \zeta_K^K \gamma_K \end{pmatrix}, \tag{B.2}$$

where

$$\gamma_k = \frac{\phi_2^{(0)}(t;\zeta_k) - b_k \psi_2^{(0)}(t;\zeta_k)}{\phi_1^{(0)}(t;\zeta_k) - b_k \psi_1^{(0)}(t;\zeta_k)} = -b_k e^{2i\zeta_k t}. \tag{B.3}$$

The unknown Darboux coefficients can be put into the vector form

$$\boldsymbol{D}_0 = \begin{pmatrix} d_0^{(0,K)} \\ d_0^{(1,K)} \\ \vdots \\ d_0^{(K-1,K)} \end{pmatrix}, \quad \boldsymbol{D}_1 = \begin{pmatrix} d_1^{(0,K)} \\ d_1^{(1,K)} \\ \vdots \\ d_1^{(K-1,K)} \end{pmatrix}. \tag{B.4}$$

The $2K$ linear system of equations (B.1) can be written as

$$-\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{g}^* \end{pmatrix} = \begin{pmatrix} \mathcal{V} & \Gamma \mathcal{V} \\ \Gamma^* \mathcal{V}^* & -\mathcal{V}^* \end{pmatrix} \begin{pmatrix} \boldsymbol{D}_0 \\ \boldsymbol{D}_1 \end{pmatrix}. \tag{B.5}$$

Let $\eta_{\min} = \min_k(\mathrm{Im}\,\zeta_k)$ and

$$\varpi_+ = \max_k(|b_k|), \quad \varpi_- = \max_k\left(|b_k|^{-1}\right). \tag{B.6}$$

Consider the case $t = T_+ > 0$. Denoting the spectral norm of matrices by $\|\cdot\|_s$, we have

$$\|\Gamma\|_s = \max_k\left(|b_k|e^{-2\,\mathrm{Im}\,\zeta_k T_+}\right) \leq \varpi_+ e^{-2\eta_{\min}T_+}, \tag{B.7}$$

so that $\|\Gamma\|_s \to 0$ as $T_+ \to \infty$. Define $\boldsymbol{D}_0(\infty) = -\mathcal{V}^{-1}\boldsymbol{f}$. Putting $\boldsymbol{D}_0 = \boldsymbol{D}_0(\infty) + \widetilde{\boldsymbol{D}}_0$, we have

$$\widetilde{\boldsymbol{D}}_0 = -M\boldsymbol{D}_1,$$
$$\boldsymbol{D}_1 = (1 + M^*M)^{-1}\,M^*\left[\boldsymbol{D}_0(\infty) - \boldsymbol{D}_0^*(\infty)\right],$$

where $M = \mathcal{V}^{-1}\Gamma\mathcal{V}$. Let $\kappa(\mathcal{V}) = \|\mathcal{V}^{-1}\|_s\|\mathcal{V}\|_s$ denote the condition number of $\mathcal{V}$, then $\|M\|_s \leq \kappa\|\Gamma\|_s$. If T_+ is such that $\kappa\|\Gamma\|_s < 1$, then

$$\|\widetilde{\boldsymbol{D}}_0\|_2 \leq \kappa\|\Gamma\|_s\|\boldsymbol{D}_1\|_2,$$
$$\|\boldsymbol{D}_1\|_2 \leq \frac{2\kappa\|\Gamma\|_s}{(1 - \kappa^2\|\Gamma\|_s^2)}\|\,\mathrm{Im}[\boldsymbol{D}_0(\infty)]\|_2.$$

Note that $d_0^{(K-1,K)}(\infty) = -\sum_{k=1}^{K} \zeta_k$, therefore,

$$\mathcal{E}_K^{(+)}(T_+) \le \frac{2\kappa^2 \|\Gamma\|_s^2}{(1 - \kappa^2 \|\Gamma\|_s^2)} \| \operatorname{Im}[\boldsymbol{D}_0(\infty)] \|_2. \tag{B.8}$$

Next we consider the case $t = -T_- < 0$. The linear system (B.5) can be written as

$$-\begin{pmatrix} \Gamma^{-1} \boldsymbol{f} \\ \boldsymbol{f}^* \end{pmatrix} = \begin{pmatrix} \Gamma^{-1} \mathcal{V} & \mathcal{V} \\ \mathcal{V}^* & -(\Gamma^{-1})^* \mathcal{V}^* \end{pmatrix} \begin{pmatrix} \boldsymbol{D}_0 \\ \boldsymbol{D}_1 \end{pmatrix}. \tag{B.9}$$

Observing

$$\|\Gamma^{-1}\|_s = \max_k \left(\frac{e^{-2 \operatorname{Im} \zeta_k T_-}}{|b_k|} \right) \le \varpi_- e^{-2\eta_{\min} T_-},$$

implies $\|\Gamma^{-1}\|_s \to 0$ as $T_- \to \infty$. Define $\boldsymbol{D}_0(-\infty) = -[\mathcal{V}^{-1} \boldsymbol{f}]^*$. As before, putting $\boldsymbol{D}_0 = \boldsymbol{D}_0(-\infty) + \widetilde{\boldsymbol{D}}_0$, we have

$$\widetilde{\boldsymbol{D}}_0 = (M^{-1})^* \boldsymbol{D}_1,$$

$$\boldsymbol{D}_1 = -\left(1 + M^{-1}(M^{-1})^*\right)^{-1} M \left[\boldsymbol{D}_0(-\infty) - \boldsymbol{D}_0^*(-\infty) \right].$$

If T_- is such that $\kappa \|\Gamma^{-1}\|_s < 1$, then

$$\|\widetilde{\boldsymbol{D}}_0\|_2 \le \kappa \|\Gamma^{-1}\|_s \|\boldsymbol{D}_1\|_2,$$

$$\|\boldsymbol{D}_1\|_2 \le \frac{2\kappa \|\Gamma^{-1}\|_s}{(1 - \kappa^2 \|\Gamma^{-1}\|_s^2)} \| \operatorname{Im}[\boldsymbol{D}_0(-\infty)] \|_2.$$

Note that $d_0^{(K-1,K)}(-\infty) = -\sum_{k=1}^{K} \zeta_k^*$, therefore,

$$\mathcal{E}_K^{(-)}(-T_-) \le \frac{2\kappa^2 \|\Gamma^{-1}\|_s^2}{(1 - \kappa^2 \|\Gamma^{-1}\|_s^2)} \| \operatorname{Im}[\boldsymbol{D}_0(-\infty)] \|_2. \tag{B.10}$$

Define

$$T_+^{(0)} = \log(\kappa \varpi_+)/2\eta_{\min}, \quad T_-^{(0)} = \log(\kappa \varpi_-)/2\eta_{\min}, \tag{B.11}$$

then, $T_+ > T_+^{(0)}$ and $T_- > T_-^{(0)}$ ensures that the estimates (B.8) and (B.10) hold, respectively. Now, the effect of propagation can also be incorporated by plugging in the x-dependence of the norming constant in (B.7). For fixed $x \in [0, L]$, we seek $T_+ \in \mathbb{R}$ such that $\|\Gamma\|_s < \kappa^{-1}$, where

$$\|\Gamma\|_s = \max_k \left(|b_k| e^{-2\eta_k(T_+ - 4\xi_k x)} \right).$$

Let $k_{\min}$ be such that $\operatorname{Im}(\zeta_{k_{\min}}) = \eta_{\min}$, then putting $\xi_{\min} = \operatorname{Re}(\zeta_{k_{\min}})$, we have

$$\|\Gamma\|_s \le \varpi_+ e^{-2\eta_{\min}(T_+ - 4\xi_{\min} x)}.$$

Therefore, choosing $T_+ - 4\xi_{\min} x > T_+^{(0)}$ ensures that the estimate (B.8) holds. Using similar arguments as above, it follows that choosing $T_- + 4\xi_{\min} x > T_-^{(0)}$ ensures the validity of estimate (B.10).

Now, in the design of K-soliton pulses, if the tolerance for the fraction of total energy in the tails is ϵ_{tails}, then the domain $[-T_-, T_+]$ must be chosen such that

$$\frac{\mathcal{E}_K^{(+)}(T_+) + \mathcal{E}_K^{(-)}(-T_-)}{4 \sum_{k=1}^{K} \text{Im}\,\zeta_k} \leq \epsilon_{\text{tails}}.$$

If we choose to satisfy the equality above, then one has to solve a nonlinear equation for T_+, T_-. For the sake of simplicity, we let $T = \max(T_-, T_+)$. The inequalities obtained above can be used to compute an upper bound for T as follows: set $X = \exp(-4\eta_{\min} T_{\max})$, then

$$\frac{2\kappa^2 \varpi_+^2 X}{1 - \kappa^2 \varpi_+^2 X} + \frac{2\kappa^2 \varpi_-^2 X}{1 - \kappa^2 \varpi_-^2 X} = \frac{4\epsilon_{\text{tails}} \sum_k \text{Im}\,\zeta_k}{\|\,\text{Im}[\mathcal{V}^{-1}f]\|_2}. \tag{B.12}$$

A good estimate for $T_{\max}$ is

$$T_{\max} = \frac{1}{4\eta_{\min}} \log\left[\frac{\|\,\text{Im}[\mathcal{V}^{-1}f]\|_2 \kappa^2 (\varpi_+^2 + \varpi_-^2)}{2\epsilon_{\text{tails}} \sum_k \text{Im}\,\zeta_k} \right], \tag{B.13}$$

provided that the right-hand side of the equation above is positive. Finally, a search algorithm such as the *bisection method* can be used to obtain the true value of T by choosing a bracketing interval of the form $[0, T_{\max}]$.