Critical Voltage of a Mesoscopic Superconductor

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We study the influence of a voltage-driven nonequilibrium of quasiparticles on the properties of short mesoscopic superconducting wires. We employ a numerical calculation based upon the Usadel equation. Going beyond linear response, we find a nonthermal energy distribution of the quasiparticles caused by the applied bias voltage. It is demonstrated that this nonequilibrium drives the system from the superconducting state to the normal state, at a current density far below the critical depairing current density.

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The energy distribution function of quasiparticles in a normal metal is under equilibrium conditions given by the Fermi-Dirac distribution f_0. In recent years it has been demonstrated that in a voltage (V)-biased mesoscopic wire (length L) a two-step nonequilibrium distribution develops [1] with additional rounding by quasiparticle scattering due to spin-flip and/or Coulomb interactions [2]. Figure 1(a) shows the distribution, which resembles two shifted Fermi-Dirac functions:

$$f(x, \varepsilon) = (1 - x)f_0(\varepsilon - eV/2) + xf_0(\varepsilon + eV/2) \quad (1)$$

with ε the quasiparticle energy and x the coordinate along the wire. For strong enough relaxation ($L \gg L_\phi$, with L_ϕ the phase coherence length) and/or high temperatures ($k_BT \gg eV$) the distribution returns to a Fermi-Dirac distribution with a local effective temperature.

The questions we address here are how the distribution function is modified when the normal wire is replaced by a superconducting wire [for a typical result see Fig. 1(b)] and how this affects observable properties such as the current-voltage characteristics of the system and the breakdown of the superconducting state. The static nonequilibrium distribution leads to the occurrence of a resistance of the superconductor. Another source of voltage might potentially develop due to phase-slip events, either thermally activated or as quantum phase slips [3,4]. The problem that we study focuses on wires which are wide enough to ignore the contribution of quantum phase slips—but still more narrow than the superconducting phase coherence length ξ_0—to the resistance and are also far enough below the critical temperature T_c to ignore the thermally assisted contribution. Within these constraints we relate the distribution function to observable quantities. To do this, it is convenient to separate the part of f which is symmetric in particle-hole space, f_L (energy mode), from the asymmetric part, f_T (charge mode), since they each have a different spatial and spectral form [Figs. 1(c) and 1(d)]. In particular, we will show that the breakdown is characterized by a voltage rather than by a current; in other words, the system cannot be trivially treated as two resistors modelling the normal current to supercurrent conversion, with a superconducting element characterized by its depairing current in between.

The transport and spectral properties of dirty superconducting systems ($\ell_e \ll \xi_0$, with ℓ_e the elastic mean free path) are described by the quasiclassical Green functions obeying the Usadel equation [5]. For out of equilibrium systems we use the Keldysh technique in Nambu (particle-hole) space, neglecting spin-dependent interactions. We ignore inelastic scattering in the wire and use the time-independent formalism. The Usadel equation (for an s-wave superconductor) then takes the form $\hbar D\nabla(\check{G}\nabla\check{G}) = -i[\check{H}, \check{G}]$, where the check notation ($\check{G}$) denotes a 4×4 matrix, D is the diffusion constant and ∇ is the spatial derivative [6]. The elements of $\check{G}$ and $\check{H}$, when split up in Keldysh space, are 2×2 matrices in Nambu space, denoted by a hat:

$$\check{G} = \begin{pmatrix} \hat{G}^R & \hat{G}^K \\ 0 & \hat{G}^A \end{pmatrix}, \qquad \check{H} = \begin{pmatrix} \hat{H} & 0 \\ 0 & \hat{H} \end{pmatrix} \quad (2)$$

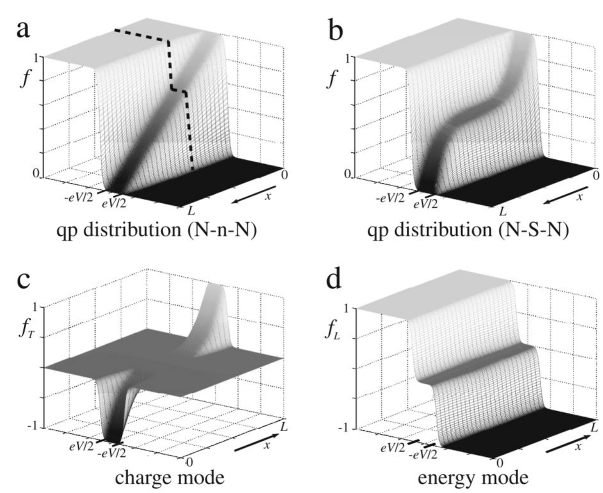

FIG. 1. Quasiparticle distribution function $f(x, \varepsilon)$ as a function of energy ε and position x for a normal wire (a) and a superconducting wire (b) between normal metallic reservoirs for $k_BT \ll eV < \Delta_0$, with (c) and (d) the decomposition of (b) into the charge mode f_T and energy mode f_L.

Here, $\hat{G}^R$ and $\hat{G}^A$ are the retarded and advanced components describing equilibrium properties and $\hat{G}^K$ is the Keldysh component which describes the nonequilibrium properties. Their elements are the quasiclassical (energy-dependent) normal and anomalous Green functions and, for the Keldysh component only, the quasiparticle distribution functions (which take account of the nonequilibrium). For the Hamiltonian $\hat{H}$ we write

$$\hat{H} = \begin{pmatrix} \varepsilon & -\Delta \\ \Delta^* & -\varepsilon \end{pmatrix} \tag{3}$$

where ε is the (eigen)energy and the chosen gauge is such that the pair potential Δ is in equilibrium a real quantity, $\Delta = \Delta^*$. The matrix Green function $\check{G}$ satisfies the normalization condition $\check{G}\check{G} = \check{1}$, leading to $\hat{G}^R\hat{G}^R = \hat{G}^A\hat{G}^A = \hat{1}$ and $\hat{G}^R\hat{G}^K + \hat{G}^K\hat{G}^A = \hat{0}$. If superconducting reservoirs in the system are kept at zero voltage (avoiding ac Josephson effects), $\hat{G}^K$ can be written as $\hat{G}^K = \hat{G}^R\hat{f} - \hat{f}\hat{G}^A$. Here $\hat{f}$ is the diagonal generalized distribution number matrix of the quasiparticles in Nambu space. To relate $\hat{f}$ to observable quantities we decompose it into an even part (or energy or longitudinal mode) and an odd part (or charge or transverse mode) in particle-hole space: $\hat{f} = f_L\tau_0 + f_T\tau_3$, where τ_i are the Pauli matrices in particle-hole space [7]. The full distribution function is retained by $2f(x, \varepsilon) = 1 - f_L(x, \varepsilon) - f_T(x, \varepsilon)$.

The retarded matrix Green function in terms of the position and energy-dependent normal $g(\varepsilon, x)$ and anomalous $F_i(\varepsilon, x)$ Green functions is

$$\hat{G}^R = \begin{pmatrix} g(\varepsilon, x) & F_1(\varepsilon, x) \\ F_2(\varepsilon, x) & -g(\varepsilon, x) \end{pmatrix}. \tag{4}$$

Substituting this in the retarded part of the Usadel equation: $\hbar D\nabla(\hat{G}^R\nabla\hat{G}^R) = -i[\hat{H}, \hat{G}^R]$ and using the normalization condition ($g^2 + F_1F_2 = 1$), we find the retarded Usadel equations:

$$\begin{aligned} \hbar D[g\nabla^2 F_1 - F_1\nabla^2 g] &= -2i\Delta g - 2i\varepsilon F_1, \\ \hbar D[F_1\nabla^2 F_2 - F_2\nabla^2 F_1] &= 2i\Delta F_2 + 2i\Delta^* F_1. \end{aligned} \tag{5}$$

The second equation is essential when calculating the nonequilibrium properties of superconductors. Its left-hand side is proportional to the divergence of the spectral (energy-dependent) supercurrent, which is (compared to the equilibrium case) no longer a conserved quantity. A general relation between the advanced matrix Green function and the retarded matrix Green function is given by $\hat{G}^R = -\tau_3(\hat{G}^A)^\dagger\tau_3$. Using this, the Keldysh matrix Green function $\hat{G}^K$ can be written entirely in terms of g, F_1, F_2, f_L, and f_T:

$$\hat{G}^K = \begin{pmatrix} (g + g^\dagger)f_+ & F_1 f_- - F_2^\dagger f_+ \\ F_2 f_+ - F_1^\dagger f_- & -(g + g^\dagger)f_- \end{pmatrix} \tag{6}$$

where $f_\pm = f_L \pm f_T$. Working out the kinetic part of the Usadel equation $\hbar D\nabla(\hat{G}^R\nabla\hat{G}^K + \hat{G}^K\nabla\hat{G}^A) = -i[\hat{H}, \hat{G}^K]$ we find (combining the diagonal components) the kinetic equations describing the nonequilibrium part:

$$\hbar D\nabla j_{\text{energy}} = 0, \quad \hbar D\nabla j_{\text{charge}} = 2R_L f_L + 2R_T f_T. \tag{7}$$

The various elements in Eq. (7) are given by

$$\begin{aligned} j_{\text{energy}} &= \Pi_L\nabla f_L + \Pi_X\nabla f_T + j_\varepsilon f_T, \\ j_{\text{charge}} &= \Pi_T\nabla f_T - \Pi_X\nabla f_L + j_\varepsilon f_L, \\ \Pi_L &= \frac{1}{4}(2 + 2|g|^2 - |F_1|^2 - |F_2|^2), \\ \Pi_T &= \frac{1}{4}(2 + 2|g|^2 + |F_1|^2 + |F_2|^2), \\ \Pi_X &= \frac{1}{4}(|F_1|^2 - |F_2|^2), \\ j_\varepsilon &= \frac{1}{2}\Re\{F_1\nabla F_2 - F_2\nabla F_1\}, \\ R_L &= -\frac{1}{2}\Im\{\Delta F_2 + \Delta F_1^\dagger\}, \\ R_T &= -\frac{1}{2}\Im\{\Delta F_2 - \Delta F_1^\dagger\}. \end{aligned} \tag{8}$$

Equations (7) are two coupled diffusion equations for f_L and f_T, describing the divergences in the spectral energy current and the spectral charge current. The total charge current is given by $J = \frac{1}{2e\rho}\int j_{\text{charge}} d\varepsilon$ with ρ the resistivity. The terms Π_L and Π_T can be related to an effective diffusion constant for the energy and charge mode, respectively, and Π_X as a "cross-diffusion" between them. j_ε is the spectral supercurrent and R_L and R_T describe the "leakage" of spectral current to different energies, where the total leakage current $\propto \int[R_L f_L + R_T f_T]d\varepsilon$ is zero. In the small signal limit the terms Π_X, j_ε, and R_L are small and can in many cases be neglected (linear approach), effectively decoupling f_L and f_T. In this article we go beyond this limit.

The Usadel equation is supplemented by a self-consistency relation:

$$\hat{H}_{(1,2)} = \frac{N_0 V_{\text{eff}}}{4}\int_{-\hbar\omega_D}^{\hbar\omega_D} \hat{G}^K_{(1,2)} d\varepsilon. \tag{9}$$

Here, N_0 is the normal density of states around the Fermi energy, V_{eff} the effective attractive interaction, and the integral limits are set by the Debye energy $\hbar\omega_D$. The resulting equation for Δ becomes $\Delta = -\frac{1}{4}N_0 V_{\text{eff}} \times \int_{-\hbar\omega_D}^{\hbar\omega_D}[(F_1 - F_2^\dagger)f_L - (F_1 + F_2^\dagger)f_T]d\varepsilon$.

To calculate spectral and transport properties, one needs to know the self-consistent solution of Δ. In most practical cases, this has to be done numerically. A convenient solution scheme is to first find the Green functions of the system by solving the retarded equations for a certain Δ, next to determine the quasiparticle distribution functions by solving the kinetic equations, and then calculate a new Δ using the self-consistency relation. This process has to be repeated until Δ converges. As a starting value for Δ we use the BCS form at zero temperature. A typical solution employs a grid of (on the order of) 10^4 energies, 10^2 spatial coordinates, and 10^3 iterations of Δ. The stability of the solution scheme was tested extensively by inserting different initial values. At all the applied voltages self-consistent steady state solutions are found. To simplify the calculations a parametrization is used that automatically fulfills the normalization condition. It is convenient to take $g = \cosh(\theta)$, $F_1 = \sinh(\theta)e^{i\chi}$, and $F_2 = -\sinh(\theta)e^{-i\chi}$, where θ and χ are position- and energy-dependent (complex) variables. At the interfaces between the superconducting wire and the normal metallic reservoirs we use the following boundary conditions: $\theta = \nabla\chi = 0$ (retarded equation) and $f_{L,T} = \frac{1}{2}(\tanh\frac{\varepsilon+eV}{2k_BT} \pm \tanh\frac{\varepsilon-eV}{2k_BT})$ (kinetic equation), where the latter are the usual reservoir distribution functions.

The transport properties of the NSN system (see inset Fig. 2) can now be calculated with the equations described above. In a previous analysis a finite differential conductance was found at zero bias employing a linear response calculation [8]. With the approach introduced here, the full current-voltage relation can be obtained. The result at several temperatures is displayed in Fig. 2, with the voltage normalized to $\Delta_0(= \Delta_{\mathrm{bulk},T=0})$ and the current density normalized to the critical current density $J_c \approx 0.75\frac{\Delta_0}{\xi_0\rho e}$ [9], with $\xi_0 = \sqrt{\hbar D/\Delta_0}$. At $T = 0$ we observe a linear resistance at low voltages caused by the decay of f_T [Fig. 1(c)], and a critical point (voltage) above which the resistance is equal to the normal state resistance. At higher temperatures ($T = 0.5, 0.75T_c$) a linear approach would only give an adequate approximation in a limited voltage range. We will argue below that the superconductor switches to the normal state by f_L which is controlled by the voltage and cannot be interpreted as a critical current.

In Fig. 3 the electrostatic potential $\phi = \int_0^\infty f_T \Re\{g\} d\epsilon$ along the wire is shown at zero temperature prior to ($eV/\Delta_0 = 0.013, 0.646$) and immediately after ($eV/\Delta_0 = 0.651$) the transition. The potential can be seen to drop to zero over a distance on the order of the coherence length due to the normal current to supercurrent conversion. This mechanism also gives rise to the finite zero bias resistance. The profile hardly changes over the full range of voltages, until the critical value is reached, after which the electrostatic potential drops in a linear fashion, indicating the system is in the normal state. The minimal changes emphasize the limited influence of f_T on the superconducting state (i.e., on Δ).

The current density at which the superconductor switches to the normal state (for $T = 0$) is much smaller than the critical current density in an infinitely long wire ($J/J_c = 1$). This excludes the depairing mechanism as the (main) cause of the transition. Neither is the transition triggered at the weaker superconducting edges as indicated by the shape of the electrostatic potential profile in Fig. 3.

The parameter that determines whether or not the superconducting state exist is Δ, as follows from Eq. (9). The integral in this self-consistency equation sums all pair states (either occupied by a Cooper pair or empty). F_i gives the Cooper pair density of states and f_L and f_T determine which of those states are doubly occupied or doubly empty and which are singly occupied (broken) due to the presence of quasiparticles. In equilibrium at $T = 0$, a switch to the normal state can only be caused by reaching a critical phase gradient, entering Δ via F_i. In the presence of quasiparticles, Δ (and thus potentially the state of the system) is also influenced by the distribution functions. It was noticed above that the charge mode f_T has a very limited influence on Δ. The effect of the energy mode f_L is examined below.

By a small modification of our system to a T-shaped geometry as shown in Fig. 4, we can in a direct way disentangle the effects of f_L and f_T on Δ. This setup can be thought of as the connection of the superconducting wire to the center of a normal wire. In the middle of such a wire f_T is equal to zero, but f_L is not. The result for the pair potential at the edge of the superconducting wire as a function of the voltage of the reservoirs is shown in Fig. 4. Although there is no net current flowing through the superconductor, at a certain voltage the pair potential collapses. The voltage that is necessary to trigger this

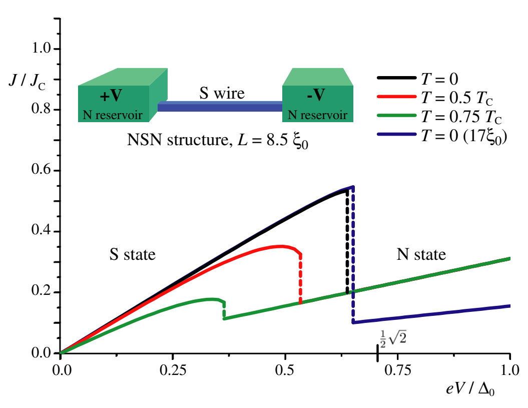

FIG. 2 (color online). The calculated current (J)-voltage (V) relation of a superconducting wire of length $L = 8.5\xi_0$ between normal metallic reservoirs (see inset) at several temperatures, and for a wire of length $17\xi_0$ at $T = 0$. J_c is the critical current density, and Δ_0 the bulk gap energy.

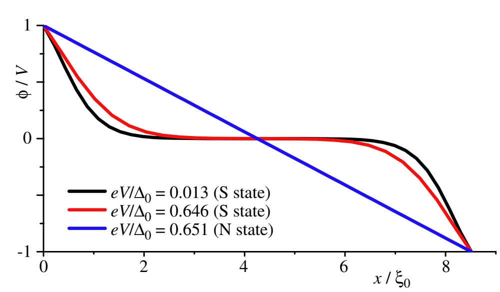

FIG. 3 (color online). The normalized electrostatic potential ϕ as a function of position x along the superconducting wire for bias voltages prior to and immediately after the transition (at $T = 0$).

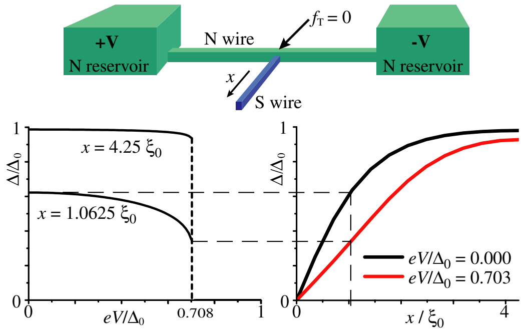

FIG. 4 (color online). Top: T-shaped geometry. Bottom: pair potential Δ in an S wire (of length $L = 4.25\xi_0$). For two different positions along the wire (left) and as a function of position for two different voltages (right). The breakdown voltage is at $eV/\Delta_0 = 0.707$.

transition to the normal state is very close to the transition in Fig. 2 (where we used the two terminal setup). Apparently the influence of f_L is important, since it can cause the superconductor to switch to the normal state irrespective of the value of the supercurrent. Clearly the influence of f_L on the state of the superconductor is larger than the influence of the supercurrent on this same quantity.

Upon approaching the critical voltage, Eq. (9) has multiple solutions and selecting the stable solution is a complicated issue of nonequilibrium thermodynamics [10,11]. For a uniform gap in the case of Fig. 4 (here called *bulk*) we find analytically from Eq. (9) that $\Delta = \Delta_0$ for eV $< \frac{1}{2}\Delta_0$, and $\Delta = 0$ for eV $> \Delta_0$. At intermediate voltages, both solutions exist together with a third solution at $\Delta = \sqrt{2eV\Delta_0 - \Delta_0^2}$. In order to investigate the stability of these solutions we use the approach taken by Bardeen [12] to define the energy difference between the normal and the superconducting state based on comparing potential and kinetic energies of the electron systems and apply it locally. We realize that the validity of this approach remains to be justified. However, using it we find that the numerically calculated energy difference (Fig. 5) for the T-shaped structure gives the same results as the analytical ones for the bulk superconductor. For long wires, the numerical results approach the analytical calculation. This analysis suggests that the bias voltage drives the system towards a first order phase transition [13].

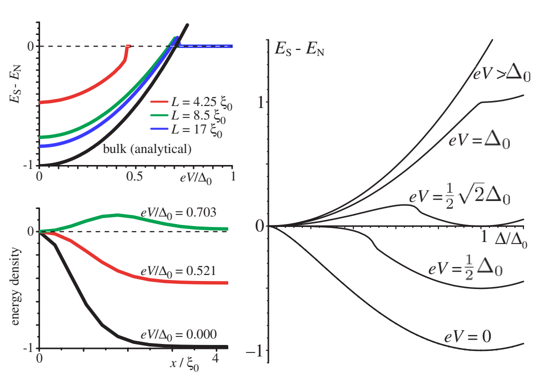

FIG. 5 (color online). Energy difference between the superconducting and normal state. Right: analytical bulk solution showing the bistable voltage range. Left: numerical solutions for (top) increasing wire length as function of voltage and (bottom) as function of position. Energies are normalized to $H_c^2(0)/8\pi$.

In conclusion, we have studied the role of the energy mode f_L of the quasiparticle distribution on the properties of a superconducting nanowire. We find a nonthermal distribution for f_L (caused by an applied bias voltage) which drives the system from the superconducting state to the normal state irrespective of the current. In general, the significant role played by f_L found in these superconducting nanowires stresses the importance of treating f_L and f_T on equal footing [14].

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