A Perturbation Method for Delay Differential Equations

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by

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to obtain the degree of Master of Science

at the Delft University of Technology,

to be defended publicly on tuesday July 16, 2019 at 13:30

Student number:	4351347	
Project duration:	December 10, 2018 – July 16, 2019	
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Abstract

In this thesis we construct a perturbation method for delay differential equations (DDEs) based on the method of multiple scales for ordinary differential equations (ODEs) and ordinary difference equations (OΔEs). The method works for nonlinear DDEs, which are linear DDEs in the unperturbed case. The validity of the method is proven under certain conditions, such as a Lipschitz condition on the perturbation, and we illustrate how the method can be applied by working out several examples. We consider a delayed version of Mathieu's equation, which is especially useful, because it can be used when one linearizes a nonlinear oscillator around a period soluction. We also consider a quadratic perturbation. For these examples we have to analyse the relationship between the solutions of the characteristic equation. There already exists a perturbation method for DDEs, for which one solves a corresponding ODE, and uses this solution as an approximation. This method is only applicable when the influence of the delay is small, and is not always accurate due to the different natures of DDEs and ODEs. We study an example for which this method can be used, and show when it fails to give an accurate approximation. We then show how to use our perturbation method for this example, to obtain an accurate approximation.

Dankwoord

Voordat het verslag begint wil ik nog een paar mensen bedanken.

Om te beginnen wil ik Wim bedanken voor de goede samenwerking tijdens dit project. Ik kon altijd bij je binnen lopen om wat te vragen en je wist mij altijd wel verder te helpen met een slimme opmerking. Ook vind ik het erg leuk dat je mij hebt betrokken bij het begeleiden van het Bachelor eindproject van Vincent Li.

Vervolgens wil ik nog mijn vriendengroep bedanken: Carlos, Marco, Massimo, Mike, Rick en Roel. Jullie zijn een van de grote redenen waarom mijn studietijd zo leuk was. In het begin was het vooral poolen, maar in het laatste jaar moet ik dan ook veel aan het presidenten in de pauzes denken. Verder vond ik het fijn dat we tijdens het afstuderen met elkaar over onze projecten konden praten, en misschien zelfs iets er aan toe kunnen voegen. Nog een extra bedankje aan Mike en Carlos voor het lezen van gedeeltes van dit verslag.

Ik wil natuurlijk ook nog mijn ouders en zus bedanken. We kunnen wel met zekerheid stellen dat er een wiskundeknobbeltje zit in onze familie wat mij geholpen heeft tijdens deze studie. Ook al hebben jullie allemaal een wiskunde achtergrond werd het misschien toch wel een beetje vaag wat ik aan het doen was in de Master. Maar jullie bleven altijd geïnteresseerd luisteren als ik even moest vertellen hoe leuk een vak of mijn eindproject wel niet was. Maar het is natuurlijk niet alleen de wiskunde waarvoor ik jullie wil bedanken, ook gewoon het altijd lief en grappig zijn heeft mij gevormd tot wie ik ben en ik weet dat ik altijd bij jullie terecht kan.

Tot slot wil ik jou nog bedanken Darcy. Ten eerste heb je de mooie voorkant van dit verslag gemaakt, en je hebt naar mij geluisterd als ik weer eens over mijn project moest praten. Maar nog veel belangrijker is dat je er altijd voor mij bent en, hoe cliché het ook klinkt, mijn grootste steun en toeverlaat bent.

Contents

Introduction

In many fields of science we try to model the dynamics of a system, just to name a few: population dynamics, heating and cooling, mechanics and electrical circuits. Many of these models are based on differential equations, which relate the change of a quantity to the value of a quantity. For simple models we can find analytical solutions of the governing equations. However, as models get more complicated, we sometimes cannot find the exact solution anymore. When this occurs, usually people find an approximation of the solution numerically. Although these approximations can be very accurate, one cannot always be certain whether the obtained answer is correct. Furthermore, one does not gain insights in the physics of the models. Another way to obtain approximations for these unsolvable problems is the use of asymptotics and perturbation methods. These methods are usable when a relatively small parameter is present in the problem, which occurs often in the field of applied mathematics. We usually denote this small parameter with ε, which we will come across many times in this thesis. The problems that we shall consider always satisfy the following fact: if we set $\varepsilon = 0$, then we get a simpler problem that we have to solve.

In this thesis we shall consider the method of multiple scales. The name is quite self-explanatory, this method is used when there are multiple length or time scales in a problem. Suppose we wish to describe how far the sea reaches upon the beach. If we consider the waves and the tides, then we see that waves work on a timescale of seconds and tides on a timescale of hours. We can use this fact to obtain approximations for problems. Holmes [11] provides a short history on the method of multiple scales. Halfway through the nineteenth century people started using multiple scales to solve very specific problems. One of these was Stokes [24], who used it to describe the flow of fluid around an elliptic cylinder. However, these people did not study the ideas underlying the method of multiple scales. This was done by Poincaré [21], using the work of Lindstedt [19], when studying the motion of planets. Later work on the method, to the point that we shall present it in this thesis, was done by Kuzmak [16] and Cole and Kevorkian [4].

In this thesis we will consider a special class of differential equations, delay differential equations. For these type of equations, the change of a quantity is not only determined by the value of the quantity at the current time, but also at previous time(s). After the First World War, automatic control systems came into use. As there is some time between detecting and reacting, delay differential equations are necessary to create proper models for these systems. In these times, the main research focused on oscillations and instabilities caused by the delays. Erneux [7] gives a good overview of the applicability of delay differential equations in different scientific fields. For the mathematical research on delay differential equations, important work has been done by Bellman and Danskin [2], Bellman and Cooke [1] and Krasovskii [15]. Building upon this, Hale [9] made a fundamental contribution to this research field. In this thesis we will use the work of Hale and Verduyn Lunel [10].

One of the examples given by Erneux [7] is that of two cars following each other. Let x_1 and x_2 be the positions of the leading and following car, respectively. We will use that initially these two cars are driving with a certain velocity v_0 and that there is a constant distance L between the two cars. Suddenly the front car starts to break for some time and then accelerates back to the initial velocity v_0. If the following car would react immediately, then there is no problem and the distance between the two cars stays L. However, this is never the case. There is a certain time before the following car reacts to the change in velocity between the two cars. We will use the following delay differential equation for x_2,

$$\frac{d^2x_2(t)}{dt^2} = \alpha \left(\frac{dx_1(t-T)}{dt} - \frac{dx_2(t-T)}{dt} \right).$$

In simple terms, this equation describes how much the following car accelerates or decelerates depending on the difference in velocity between the two cars. However, due to a delay in reaction, we use the difference in velocity some time T ago. We also have a constant α which represents how intense the following car reacts on a difference in velocity. For a high α, the following car will break and accelerate hard, for a low α the following car will break and accelerate slowly. Figure I.1 shows a plot of the distance between the two cars for different delays. We see that if it takes longer before the following car reacts, then the distance between the two cars becomes a lot smaller and there are also more oscillations. For $T = 3$, we even have that the distance becomes practically zero, so for even greater T we will have a car crash. Although this is a very simple model, it does show that the reaction time is very important. As phone use and drinking ensures a longer reaction time, this model confirms again that these two should not be combined with driving.

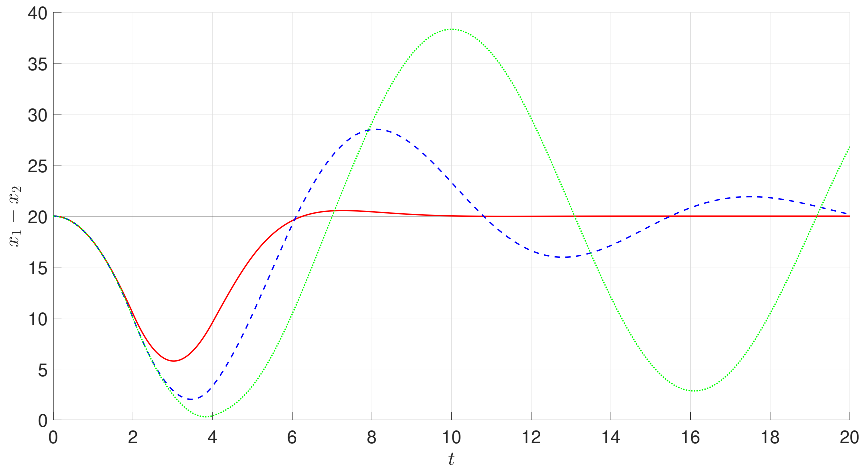


Figure I.1: Plot of the distance between the two cars over time for different delays T. The solid (red) line is for $T = 1$, the dashed (blue) line is for $T = 2$ and the dotted (green) line is for $T = 3$. For the other parameters we have taken $\alpha = 0.5$ and $L = 20$.

The goal of this thesis is to construct a perturbation method for delay differential equations, using the method of multiple scales. To give a clear overview of how the method can be used, we shall consider several examples, each highlighting a different problem that one can come across when using the method.

We start this thesis in Chapter 1 by giving the definition and well-posedness of a retarded functional differential equation. We use the terminology, definitions and theorems of Hale and Verduyn Lunel [10] for this chapter. In Chapter 2, we shall first calculate the solution of a first order delay differential equation, meaning that the highest derivative is of order one. Then we determine the stability for first order delay differential equations. Next, we shall consider how one should calculate the solution for a higher order delay differential equation and perform the actual calculations for a second order delay differential equation. We end this chapter by determining the stability of a second order delay differential equation. For that section we use the work of a bachelor student, whose thesis I have supervised. In Chapter 3 we explain the method of multiple scales for ordinary differential equations and ordinary difference equations and consider illustrating examples for both types. We discuss these types of equations, because we can combine the method of multiple scales for both cases to construct a method of multiple scales for delay differential equations. Such a method already exists, given by, for example, Erneux [7]. However, this method involves solving ordinary differential equations instead of delay differential equations. As a result, the obtained approximations are not always accurate. In this thesis we present a perturbation method which keeps the delay differential equations. We first describe our method for first order delay differential equations in Chapter 4 and then for higher order delay differential equations in Chapter 5. We consider nonlinear delay differential equations with a small parameter ε, which are linear delay differential equations if we set $\varepsilon = 0$. We shall consider examples with different nonlinearities, each resulting in different types of problems that we have to solve when applying our method. Finally, we draw conclusions and give some further remarks on future research in Chapter 6.

Chapter 1

Definition and Well-posedness of a Retarded Functional Differential Equation

In this chapter we give the definition of a retarded functional differential equation, and state three theorems about the well-posedness of such an equation. We closely follow Hale and Verduyn Lunel [10].

1.1 Definition of a Retarded Functional Differential Equation

First, define the following, $r \geq 0$ is a given real number, $C([a,b],\mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. Define the special case $C = C([-r,0],\mathbb{R}^n)$ and define the norm of $\phi \in C$ as $|\phi| = \sup_{-r \geq \theta \geq 0} |\phi(\theta)|$. If for $\sigma \in \mathbb{R}$, $A \geq 0$ and $x \in C([-\sigma - r, \sigma + A], \mathbb{R}^n)$, then for any $t \in [\sigma, \sigma + A]$, we define $x_t \in C$ as $x_t(\theta) = x(t+\theta)$, with $\theta \in [-r, 0]$. Let D be a subset of $\mathbb{R} \times C$, and let $f : D \to \mathbb{R}^n$ be a given function, then we call

$$\dot{x}(t) = f(t, x_t), \tag{1.1}$$

a retarded functional differential equation on D. We denote this as RFDE(f). $x(\sigma,\phi,f)$ is a solution of Eq. (1.1) with initial value ϕ at σ, if there is an $A > 0$, such that $x(\sigma,\phi,f)$ is a solution of Eq. (1.1) on $[\sigma - r, \sigma + A)$ and $x_\sigma(\sigma,\phi,f) = \phi$. If $\sigma \in \mathbb{R}$ and $\phi \in C$ are given, and $f(t,\phi)$ is continuous, then finding a solution of Eq. (1.1) through (σ,ϕ) is equivalent to solving the integral equation

$$x_\sigma = \phi, \tag{1.2}$$

$$x(t) = \phi(0) + \int_\sigma^t f(s, x_s) ds, \qquad t \geq \sigma. \tag{1.3}$$

1.2 Well-posedness

For RFDE(f) to be a well-posed problem, it has to satisfy three criteria. A solution must exist, this solution must be unique, and when changing the problem slightly, the solution should change slightly.

Hale and Verduyn Lunel [10] prove these three criteria under certain circumstances. The procedure is as follows. A mapping T can be defined using the integral equation. Using Schauder's fixed-point theorem, it is proven that a fixed point exists. Then this is a solution of the integral equation, thus a solution exists. Adding a Lipschitz condition on f, ensures that there is a unique fixed point, and thus a unique solution to the integral equation. The continuous dependence is also proven.

Before we state the theorems, the following must be defined. Let V be a subset of $\mathbb{R} \times C$, then $C(V,\mathbb{R}^n)$ is the class of all functions $f :\to \mathbb{R}^n$ that are continuous and define $C^0(V,\mathbb{R}^n) \subseteq C(V,\mathbb{R}^n)$ as the subset of all bounded continuous functions from V to $\mathbb{R}^n$.

Now we can state three theorems, (Chapter 2, Theorems 2.1, 2.2 and 2.3, of [10]), concerning the existence and uniqueness of a solution, and the continuous dependence on initial data and coefficients.

Theorem 1.1 (Existence). *Suppose* Ω *is an open subset in* $\mathbb{R} \times C$ *and* $f^0 \in C(\Omega, \mathbb{R}^n)$. *If* $(\sigma, \phi) \in \Omega$, *then there is a solution of the RFDE(*f^0*) passing through* (σ, ϕ). *More generally, if* $W \subseteq \Omega$ *is compact and* $f^0 \in C(\Omega, \mathbb{R}^n)$ *is given, then there is a neighborhood* $V \subseteq \Omega$ *of* W, *such that* $f^0 \in C^0(V, \mathbb{R}^n)$, *there is a neighborhood* $U \subseteq C^0(V, \mathbb{R}^n)$ *of* f^0 *and an* $\alpha > 0$, *such that for any* $(\sigma, \phi) \in W$, $f \in U$, *there is a solution* $x(\sigma, \phi, f)$ *of the RDFE(*f*) through* (σ, ϕ) *that exists on* $[\sigma - r, \sigma + \alpha]$.

Theorem 1.2 (Uniqueness). *Suppose* Ω *is an open set in* $\mathbb{R} \times C$, $f : \Omega \to \mathbb{R}^n$ *is continuous , and* $f(t, \phi)$ *is Lipschitizan in* ϕ *in each compact set in* Ω. *If* $(\sigma, \phi) \in \Omega$, *then there is a unique solution for RFDE(*f*) through* (σ, ϕ).

Theorem 1.3 (Continuous dependence). *Suppose* $\Omega \subseteq \mathbb{R} \times C$ *is open,* $(\sigma^0, \phi^0) \in \Omega$, $f^0 \in C(\Omega, \mathbb{R}^n)$, *and* x^0 *is a solution of the RFDE(*f^0*) through* (σ^0, ϕ^0) *which exists and is unique on* $[\sigma^0 - r, b]$. *Let* $W^0 \subseteq \Omega$ *be the compact set defined by*

$$W^0 = \{(t, x_t^0) : t \in [\sigma^0, b]\}$$

and let V^0 *be the neighborhood of* W^0 *on which* f^0 *is bounded. If* (σ^k, ϕ^k, f^k), $k = 1, 2, \ldots$ *satisfies* $\sigma^k \to \sigma^0$, $\phi^k \to \phi^0$, *and* $|f^k - f^0|_{V^0} \to 0$ *as* $k \to \infty$, *then there is a* k^0, *such that the RFDE(*f^k*) for* $k \geq k^0$ *is such that each solution* $x^k = x^k(\sigma^k, \phi^k, f^k)$ *through* (σ^k, ϕ^k) *exists on* $[\sigma^k - r, b]$ *and* $x^k \to x^0$ *uniformly on* $[\sigma^0 - r, b]$. *Since all* x^k *may not be defined on* $[\sigma^0 - r, b]$, *by* $x^k \to x^0$ *uniformly on* $[\sigma^0 - r, b]$, *we mean that for any* $\varepsilon > 0$, *there is a* $k_1(\varepsilon)$ *such that* $x^k(t)$, $k \geq k_1(\varepsilon)$, *is defined on* $[\sigma^0 - r + e, b]$, *and* $x^k \to x^0$ *uniformly on* $[\sigma^0 - r + \varepsilon, b]$.

In the next chapters we will consider scalar delay differential equations, DDEs, which are a subset of the RFDEs.

Chapter 2

Solution and Stability

We start this chapter with calculating the solution of a first order DDE. After that we consider the stability of this solution. We continue with the solution for higher order DDEs. A part of this derivation will be for a general higher order DDE, and we work it out entirely for a second order DDE. We end this chapter with a stability analysis of a second order DDE.

2.1 Solution of a First Order Delay Differential Equation

Consider the following DDE,

$$\frac{dy(t)}{dt} + \alpha y(t) + \beta y(t-T) = f(\dots), \tag{2.1}$$

with $T > 0$ the delay and f a nonlinear function of y. Then define $\tau = \frac{t}{T}$ and $\tilde{y}(\tau) = y(t)$. The equation becomes

$$\frac{d\tilde{y}(\tau)}{d\tau} + \alpha T\tilde{y}(\tau) + \beta T\tilde{y}(\tau - 1) = \tilde{f}(\dots),$$

with $\tilde{f}$ a function similar to f. We write $a = \alpha T$, $b = \beta T$, omit the tildes and use t instead of τ, such that we obtain the following equation

$$\frac{dy(t)}{dt} + ay(t) + by(t-1) = f(\dots). \tag{2.2}$$

This equation has one parameter less, so it is easier to analyse.

We also need an initial condition. Note that this initial condition must be prescribed on the interval $[-1, 0]$, due to the delay term. We use

$$y(t) = \phi(t), \qquad \text{for } t \in [-1, 0]. \tag{2.3}$$

We will first derive a solution for a linear DDE and then consider the nonlinear case.

2.1.1 The Linear Case

We consider the following problem,

$$\frac{dy(t)}{dt} + ay(t) + by(t-1) = 0, \tag{2.4}$$

with

$$y(t) = \phi(t), \qquad \text{for } t \in [-1, 0]. \tag{2.5}$$

For linear ordinary differential equations, ODEs, we find the solutions by trying $e^{\lambda t}$. This then results in a characteristic equation for λ. Solving this, and satisfying the initial conditions gives us the solution of a linear ODE. We can do the same for a DDE. The characteristic equation corresponding to Eq. (2.4) is given by

$$\lambda + a + be^{-\lambda} = 0. \tag{2.6}$$

For ODEs the number of solutions for a characteristic equation was the same as the order of the ODE. In this case, there is an infinite number of solutions of the characteristic equation. We will determine these numerically. We can do this by ourselves, but we can also use the Lambert W function to do this. Programs such as Maple and Matlab are able to compute the Lambert W function quickly and accurately, making it easy to use in this case. For higher order DDEs, we will only be able to use the Lambert W function in specific cases, so then we will need to find the solutions of the characteristic equation ourselves.

The Lambert W function, $W(z)$, is discussed in great detail by Corless et al. [5]. It is defined as the inverse of the function $f(z) = ze^z$,

$$z = f^{-1}((ze^z)) = W\left(ze^z\right).$$

In other form this is

$$W(z)e^{W(z)} = z. \tag{2.7}$$

Both z and $W(z)$ can be complex. Like the complex logarithm there is an infinite number of branches of the Lambert W functions, which we denote as $W_k(z)$, with $k = 0, \pm1, \pm2, \ldots$. Figure 2.1 shows a plot of the first branches of the Lambert W function, together with the curves that separate them. When the real parts go to infinity, the imaginary parts go to multiples of π.

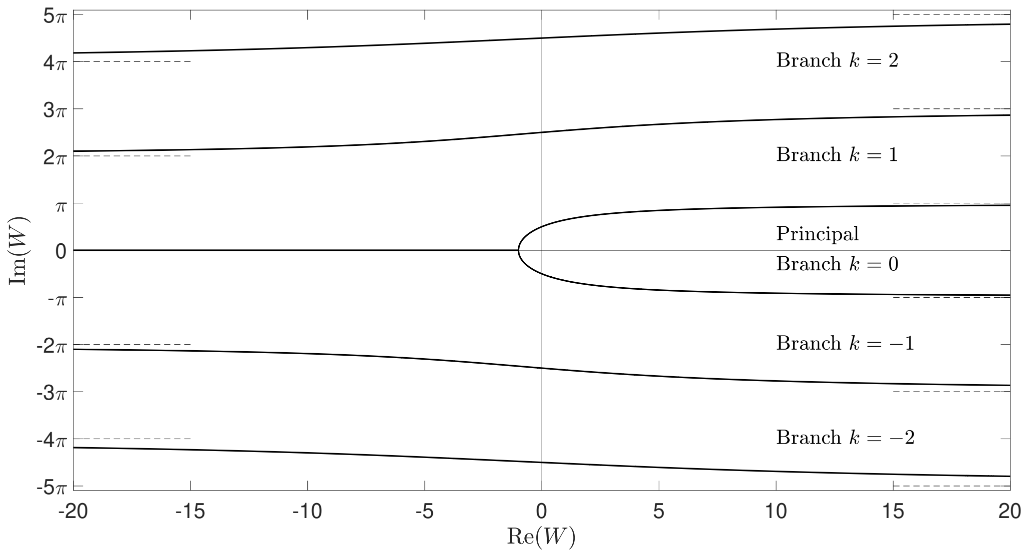


Figure 2.1: Branches of the Lambert W function. Adapted from [5].

We can rewrite the characteristic equation, Eq. (2.6), into the following form,

$$(\lambda + a)e^{\lambda + a} = -be^a,$$

such that we can write down the solutions in terms of the Lambert W function,

$$\lambda_k = W_k\left(-be^a\right) - a, \qquad k = 0, \pm1, \pm2, \ldots. \tag{2.8}$$

In Appendix A we show how to find the solutions of the characteristic equation by ourselves. We calculate a solution curve on which all solutions must lie. We can write down ω as a function of ν for this curve. Consequently, we can write down the characteristic equation as a function of ν, such that we can easily check numerically, for which ν we find solutions. In Appendix A we will show that the solution curve has different shapes for different choices of a and b. We also show how to calculate approximations for the large solutions of the characteristic equation.

Given the solutions of the characteristic equation, λ_k, the solution of the linear DDE, Eq. (2.4), is given by

$$y(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\lambda_k t}. \tag{2.9}$$

We have yet to determine the constants α_k. These will depend on the initial condition, Eq. (2.5). We will use the Laplace transform to determine the constants (this has been done before by for example [27]).

The Laplace transform is defined as

$$\mathcal{L}(f(t))(s) = \int_0^\infty e^{-st} f(t) dt. \tag{2.10}$$

To be able to take the Laplace transform of the DDE, we need to know the Laplace transform of all individual terms. For the derivative, it is well known that

$$\mathcal{L}\left(\frac{dy(t)}{dt}\right) = s\mathcal{L}(y(t)) - y(0).$$

This can be easily derived using partial integration. We can use that $y(0) = \phi(0)$. For the delay term we have

$$\begin{aligned} \mathcal{L}(y(t-1)) &= \int_0^\infty e^{-st} y(t-1) dt \\ &= \int_0^1 e^{-st} y(t-1) dt + \int_1^\infty e^{-st} y(t-1) dt \\ &= \int_0^1 e^{-st} \phi(t-1) dt + e^{-s} \int_0^\infty e^{-st} y(t) dt \\ &= \mathcal{L}(\phi(t-1)) + e^{-s}\mathcal{L}(y(t)). \end{aligned} \tag{2.11}$$

We have extended $\phi(t)$ on the interval $[-1,\infty)$ by setting $\phi(t) = 0$ for $t > 0$. The Laplace transform of the linear DDE, Eq. (2.4), is given by

$$s\mathcal{L}(y(t)) - \phi(0) + a\mathcal{L}(y(t)) + b\left(\mathcal{L}(\phi(t-1)) + e^{-s}\mathcal{L}(y(t))\right) = 0.$$

We solve this for $\mathcal{L}(y(t))$,

$$\mathcal{L}(y(t)) = \left(s + a + be^{-s}\right)^{-1} \left(\phi(0) - b\mathcal{L}(\phi(t-1))\right).$$

We define the fundamental solution $Y(t)$ as

$$\mathcal{L}(Y(t)) = \left(s + a + be^{-s}\right)^{-1}.$$

In Appendix B we show how to calculate $Y(t)$. There are two possible expressions, depending on the multiplicities of the solutions of the characteristic equation. We will continue with the case for which each solution has multiplicity one. For this case, the fundamental solution is given by

$$Y(t) = \sum_{k=-\infty}^{\infty} \frac{1}{1 - be^{-\lambda_k}} e^{\lambda_k t}.$$

We can use this expression, together with the convolution theorem, to calculate the solution of $y(t)$.

Theorem 2.1 (Convolution)**.** *Suppose we have the functions $f(t)$, $g(t)$, $h(t)$, with Laplace transforms $F(s)$, $G(s)$, $H(s)$. If $H(s) = F(s)G(s)$, then*

$$h(t) = (f * g)(t) = \int_0^t f(t-\tau) g(\tau) d\tau = \int_0^t g(t-\tau) f(\tau) d\tau.$$

We use that

$$\mathcal{L}^{-1}\left(\phi(0) - b\mathcal{L}(\phi(t-1))\right) = \phi(0)\delta(t) - b\phi(t-1).$$

For $y(t)$ we find

$$\begin{aligned} y(t) &= \int_0^t Y(t-\tau)\left(\phi(0)\delta(\tau) - b\phi(\tau-1)\right)d\tau \\ &= Y(t)\phi(0) - b\int_{-1}^0 Y(t-\theta-1)\phi(\theta)d\theta \\ &= \sum_{k=-\infty}^{\infty} \alpha_k e^{\lambda_k t}, \end{aligned} \tag{2.12}$$

with

$$\alpha_k = \frac{1}{1-be^{-\lambda_k}}\left(\phi(0) - be^{-\lambda_k}\int_{-1}^0 e^{-\lambda_k\theta}\phi(\theta)d\theta\right). \tag{2.13}$$

We now have the solution to a linear first order DDE. In the next section we shall consider a nonlinear first order DDE.

2.1.2 The Nonlinear Case

Most of the derivation for the linear case can be applied in the nonlinear case. We again take the Laplace transform and solve for $\mathcal{L}(y(t))$. Then we define a fundamental solution $Y(t)$, with $\mathcal{L}(Y(t)) = (s+a+be^{-s})^{-1}$. We consider the case where all solutions of the characteristic equations have multiplicity one. Next, we use $Y(t)$ to find $y(t)$, (see for example [10])

$$y(t) = \int_0^t Y(t-\tau)\left(\phi(0)\delta(\tau) - b\phi(\tau-1) + f(\dots)\right)d\tau.$$

Working this out yields

$$y(t) = \sum_{k=-\infty}^{\infty}\alpha_k e^{\lambda_k t} + \sum_{k=-\infty}^{\infty}\frac{e^{\lambda_k t}}{1-be^{-\lambda_k}}\int_0^t e^{-\lambda_k s}f(\dots)ds, \tag{2.14}$$

Note that this is not an explicit solution if f depends on y. If we have a linear inhomogeneous DDE, which is the case when $f(\dots) = f(t)$, then this does give an explicit solution.

In the next section we consider the stability of a first order DDE.

2.2 Stability of a First Order Delay Differential Equation

To investigate the stability of the just derived solution, we use linearized stability. Assuming that f is of "higher order", the stability of the solution of the nonlinear equation is determined by the stability of the linearized equation. We can determine the stability of the linear equation by considering the signs of the real parts of the solutions of the characteristic equation (see for instance [10, 23]).

Define

$$\mu = \max_{k\in\mathbb{Z}} \mathrm{Re}(\lambda_k). \tag{2.15}$$

Note that the highest value is obtained by the principal branch of the Lambert W function (see [5]). So, $\mu = \mathrm{Re}(\lambda_0)$. We have asymptotic stability if $\mu < 0$, and instability if $\mu > 0$. For $\mu = 0$ the stability depends on the multiplicity of λ_0. The multiplicity is one if $be^a \neq e^{-1}$ and two if $be^a = e^{-1}$. Then $\lambda_{-1} = \lambda_0 = -(a+1)$. To have $\mu = 0$, we must have $a = -1$ and $b = 1$. If the multiplicity is one, then the linear solution is stable, but not asymptotically stable. The stability of the solution of the nonlinear equation is determined by the nonlinear terms. If $a = -1$ and $b = 1$, then the solution is unstable.

We wish to find for which (a, b), we get $\mu = 0$. We must have $\lambda_0 = i\omega$, with $\omega \in (-\pi, \pi)$, since we consider the principal branch. Plugging this in the characteristic equation and splitting it into a real and imaginary part yields

$$a + b\cos\omega = 0$$
$$\omega - b\sin\omega = 0$$

If $\omega = 0$, then we must have $a + b = 0$. If $\omega \in (-\pi, \pi) \setminus \{0\}$, then we must have $(a, b) = \left(\frac{-\omega}{\tan\omega}, \frac{\omega}{\sin\omega}\right)$. We define the curve

$$C = \left\{ \left(\frac{-\omega}{\tan\omega}, \frac{\omega}{\sin\omega} \right) \,\middle|\, \omega \in (-\pi, \pi) \setminus \{0\} \right\}. \tag{2.16}$$

Note that as $\omega \to 0$, the curve approaches $(1, -1)$, so the line $a + b = 0$ and the curve intersect in this point. It can be checked with ease that between these curves $\mu < 0$ and outside these curves $\mu > 0$. We call the area between the curves the stability region, since we have asymptotic stability here. Figure 2.2 shows a plot of the stability region.

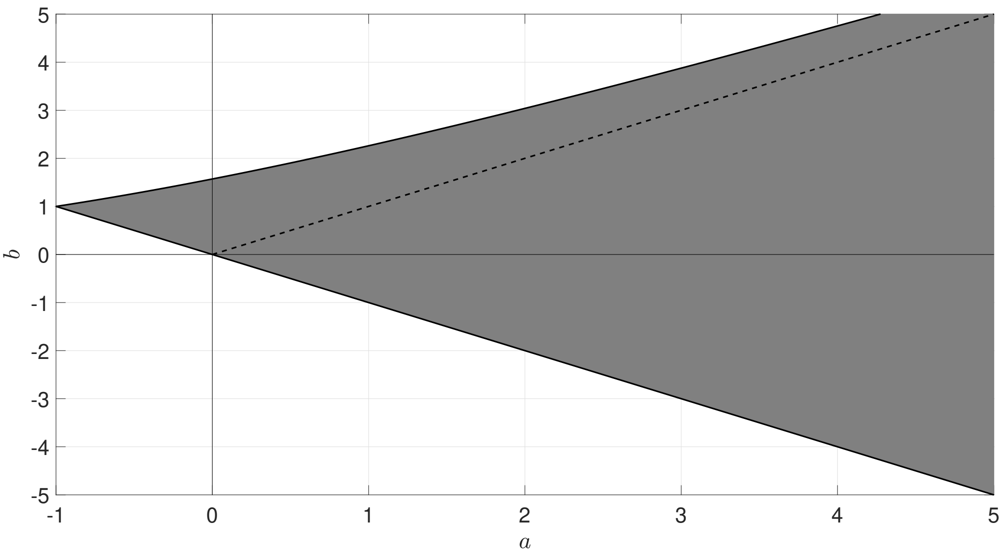


Figure 2.2: Plot of the stability region. In the gray area, $\mu < 0$. On the curves enclosing the region, $\mu = 0$. Outside of the curves, $\mu > 0$. The dashed line is $a - b = 0$.

Recall that the original equation, Eq. (2.1), contained a delay T. By transforming time, we obtained the term $y(t-1)$ instead of $y(t-T)$. Due to this the coefficients changed. Instead of α and β, we got $a = \alpha T$ and $b = \beta T$. We can use the stability region for a and b, to investigate the stability of the original equation, with the parameters α, β and T.

Between the lines $\alpha - \beta = 0$ and $\alpha + \beta = 0$, there is stability regardless of the value for T. Above the line $\alpha - \beta = 0$, there is a critical value for T, such that there is a change in stability. Using the definition of the curve C, we find

$$b^2 - a^2 = \omega^2 \qquad \text{and} \qquad -\frac{a}{b} = \cos\omega.$$

We then find that the critical value for T is given by

$$T^* = \frac{\arccos\left(-\frac{\alpha}{\beta}\right)}{\sqrt{\beta^2 - \alpha^2}}. \tag{2.17}$$

We have the following possibilities.

- We have asymptotic stability if $-\alpha < \beta \leq \alpha$, or when $\beta > \alpha$ and $0 < T < T^*$.
- We have instability if $\beta < -\alpha$, $\beta > \alpha$ and $T > T^*$, or when $\alpha T = -1$ and $\beta T = 1$.
- The stability is determined by the nonlinear terms if $\beta = -\alpha$ or when $\beta > \alpha$ and $T = T^*$. For both cases, we may not have $\alpha T = -1$ and $\beta T = 1$.

In the next section we shall show how to derive the solution of a higher order DDE and work this out for a second order DDE.

2.3 Solution of a Higher Order Delay Differential Equation

Consider the following DDE of order $n > 1$,

$$\frac{d^n y(t)}{dt^n} + \sum_{i=0}^{n-1} a_i \frac{d^i y(t)}{dt^i} + \sum_{i=0}^{n-1} b_i \frac{d^i y(t-1)}{dt^i} = f(\dots), \tag{2.18}$$

with initial conditions

$$\frac{d^i y(t)}{dt^i} = \phi_i(t), \qquad \text{for } t \in [-1, 0], \qquad i = 0, \dots, n-1. \tag{2.19}$$

We introduce the notation

$$d_t^i = \frac{d^i}{dt^i}$$

and the vector

$$\boldsymbol{z}(t) = \begin{pmatrix} y(t) \\ d_t y(t) \\ d_t^2 y(t) \\ \vdots \\ d_t^{n-1} y(t) \end{pmatrix}.$$

We have the following equation for $\boldsymbol{z}(t)$,

$$\frac{d\boldsymbol{z}(t)}{dt} + A\boldsymbol{z}(t) + B\boldsymbol{z}(t-1) = \boldsymbol{f}(\dots), \tag{2.20}$$

with

$$A = \begin{pmatrix} & -1 & & & & \\ & & -1 & & \emptyset & \\ & & & \ddots & & \\ & \emptyset & & & -1 & \\ & & & & & -1 \\ a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} & & & & & \\ & & & \emptyset & & \\ & & & & & \\ b_0 & b_1 & b_2 & \dots & b_{n-2} & b_{n-1} \end{pmatrix}. \tag{2.21}$$

The initial condition for $\boldsymbol{z}(t)$ is

$$\boldsymbol{z}(t) = \boldsymbol{\phi}(t), \qquad \text{for } t \in [-1, 0], \tag{2.22}$$

with

$$\boldsymbol{\phi}(t) = \begin{pmatrix} \phi_0(t) \\ \vdots \\ \phi_{n-1}(t) \end{pmatrix}.$$

We will now consider the characteristic equation corresponding to this system.

2.3.1 The Characteristic Equation

We know that the solutions will be of the form $\boldsymbol{\xi} e^{\lambda t}$, with $\boldsymbol{\xi}$ a constant vector. We can then plug this into the DDE, to obtain a characteristic equation. After rewriting, we obtain the following.

$$\left(\lambda I + A + e^{-\lambda} B\right) \boldsymbol{\xi} = 0.$$

We define the matrix

$$C = \lambda I + A + e^{-\lambda} B. \tag{2.23}$$

For nonzero solutions we must require $\det C = 0$. Working this out yields the characteristic equation. In general we have

$$\lambda^n + \sum_{i=0}^{n-1} a_i \lambda^i + \sum_{i=0}^{n-1} b_i \lambda^i e^{-\lambda} = 0. \tag{2.24}$$

For the second order case we get the following matrix

$$C = \begin{pmatrix} \lambda & -1 \\ a_0 + b_0 e^{-\lambda} & \lambda + a_1 + b_1 e^{-\lambda} \end{pmatrix}$$

and the following characteristic equation

$$\lambda^2 + a_1 \lambda + b_1 \lambda e^{-\lambda} + a_0 + b_0 e^{-\lambda} = 0. \tag{2.25}$$

Note that we could have also obtained these characteristic equations if we had put $e^{\lambda t}$ into the original DDEs.

We will now first consider how to determine the solutions of the characteristic equation numerically. Then we will discuss an interesting characteristic for large solutions of the characteristic equation.

Solution curve

We can find solutions of the characteristic equations numerically, by finding the solution curves. We know that the solutions of the characteristic equation must lie on this curve, making it a lot easier to find the solutions. We have done this for the first order case in Appendix A. We will now do this for the second order case.

To determine the solution curve, we write $\lambda = \nu + i\omega$, with $\nu, \omega \in \mathbb{R}$, and split the characteristic equation into a real and imaginary part. For the second order characteristic equation we obtain

$$\nu^2 - \omega^2 + a_1 \nu + b_1 \left(\nu \cos\omega + \omega \sin\omega\right) e^{-\nu} + a_0 + b_0 e^{-\nu} \cos\omega = 0, \tag{2.26}$$

$$2\nu\omega + a_1 \omega + b_1 \left(-\nu \sin\omega + \omega \cos\omega\right) e^{-\nu} - b_0 e^{-\nu} \sin\omega = 0. \tag{2.27}$$

These can be rewritten into the following form,

$$\begin{aligned} \nu^2 - \omega^2 + a_1 \nu + a_0 &= -\left((b_1 \nu + b_0) \cos\omega + b_1 \omega \sin\omega\right) e^{-\nu}, \\ 2\nu\omega + a_1 \omega &= -\left(-(b_1 \nu + b_0) \sin\omega + b_1 \omega \cos\omega\right) e^{-\nu}. \end{aligned}$$

Squaring and adding these equations yields

$$\left(\nu^2 - \omega^2 + a_1 \nu + a_0\right)^2 + \left(2\nu\omega + a_1 \omega\right)^2 = \left((b_1 \nu + b_0)^2 + b_1^2 \omega^2\right) e^{-2\nu}.$$

We can write this as an equation for ω,

$$\omega^4 + f(\nu)\omega^2 + g(\nu) = 0,$$

with

$$\begin{aligned} f(\nu) &= -2\left(\nu^2 + a_1 \nu + a_0\right) + \left(2\nu + a_1\right)^2 - b_1^2 e^{-2\nu}, \\ g(\nu) &= \left(\nu^2 + a_1 \nu + a_0\right)^2 - (b_1 \nu + b_0)^2 e^{-2\nu}. \end{aligned}$$

Then solve this equation for ω^2,

$$\omega^2 = \frac{-f(\nu) \pm \sqrt{h(\nu)}}{2},$$

with

$$h(\nu) = f(\nu)^2 - 4g(\nu).$$

Since we only wish to consider $\omega \in \mathbb{R}$, we must require $h(\nu) \geq 0$. We will consider an example, for which we set $(a_0, a_1, b_0, b_1) = (1, 1, 1, 1)$. Figure 2.3 shows a plot of $h(\nu)$. The root of h is denoted by ν_1. For $\nu \leq \nu_1$, $h(\nu) \geq 0$.

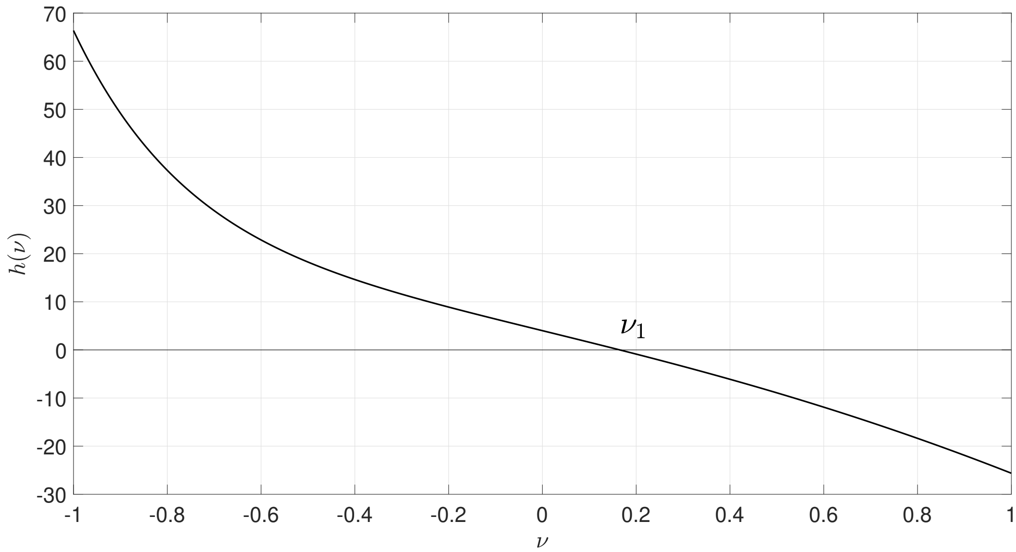


Figure 2.3: Plot of $h(\nu)$ for $(a_0, a_1, b_0, b_1) = (1, 1, 1, 1)$.

Then define the functions

$$k(\nu) = \frac{-f(\nu) + \sqrt{h(\nu)}}{2},$$
$$l(\nu) = \frac{-f(\nu) - \sqrt{h(\nu)}}{2},$$

such that

$$\omega = \pm\sqrt{k(\nu)} \qquad \text{and} \qquad \omega = \pm\sqrt{l(\nu)}.$$

Again, we must check for which intervals $k(\nu) \geq 0$ and $l(\nu) \geq 0$. Figure 2.4 shows a plot of $k(\nu)$ for $\nu \leq \nu_1$. $k(\nu) \geq 0$ for all $\nu \leq \nu_1$, so this is the only restriction on the interval of ν for the solutions $\omega = \pm\sqrt{k(\nu)}$.

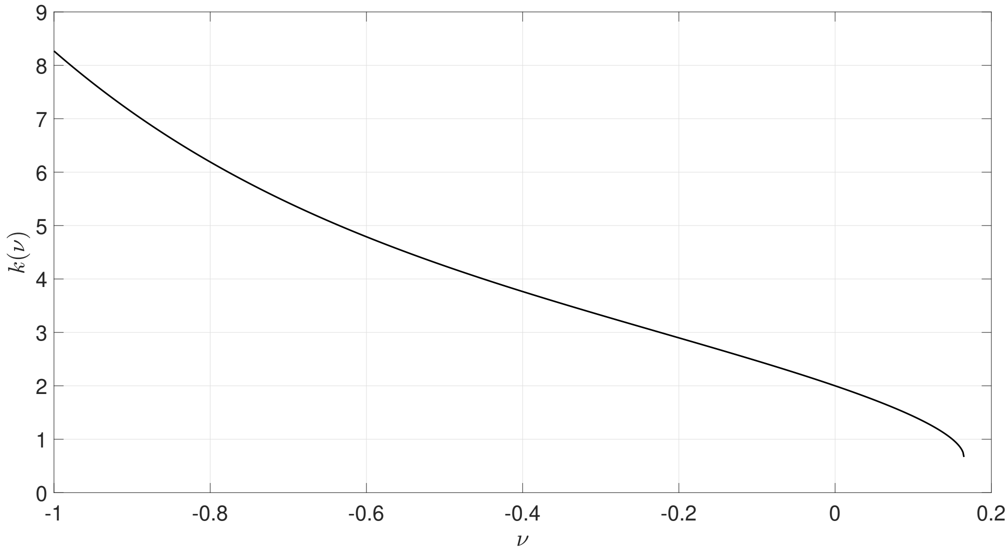


Figure 2.4: Plot of $k(\nu)$ for $(a_0, a_1, b_0, b_1) = (1, 1, 1, 1)$.

Figure 2.5 shows a plot of $l(\nu)$ for $\nu \leq \nu_1$. The roots of $l(\nu)$ are at $\nu = \nu_2$, $\nu = \nu_1$ and $\nu = 0$. We have $l(\nu) \geq 0$ for $\nu \in [\nu_2, \nu_3]$ and $\nu \in [0, \nu_1]$.

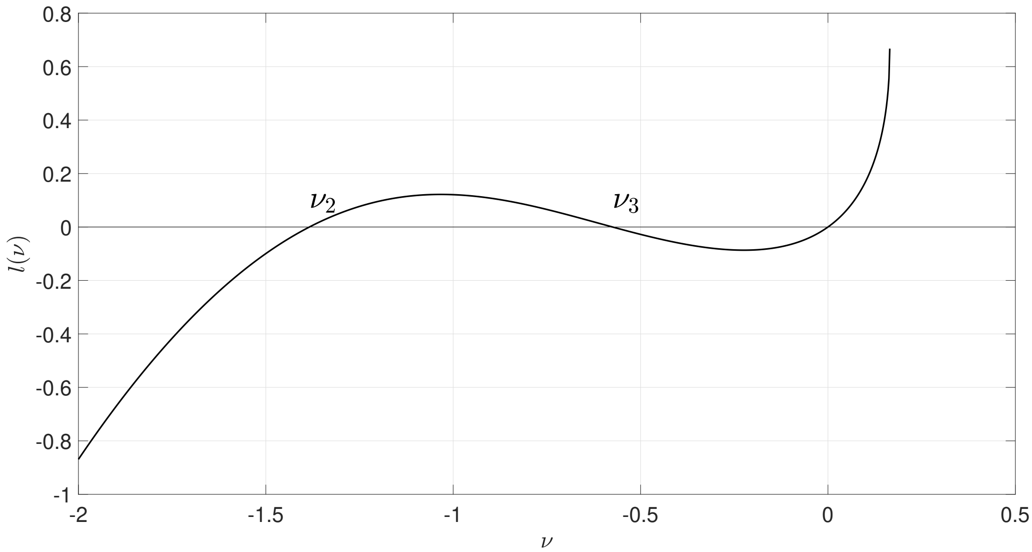


Figure 2.5: Plot of $l(\nu)$ for $(a_0, a_1, b_0, b_1) = (1, 1, 1, 1)$.

We can then use these intervals to plot $\pm\sqrt{k(\nu)}$ and $\pm\sqrt{l(\nu)}$. This is done in Figure 2.6. We call this the solution curve, as all solutions must lie on this curve. We can use this curve to easily calculate the solutions numerically.

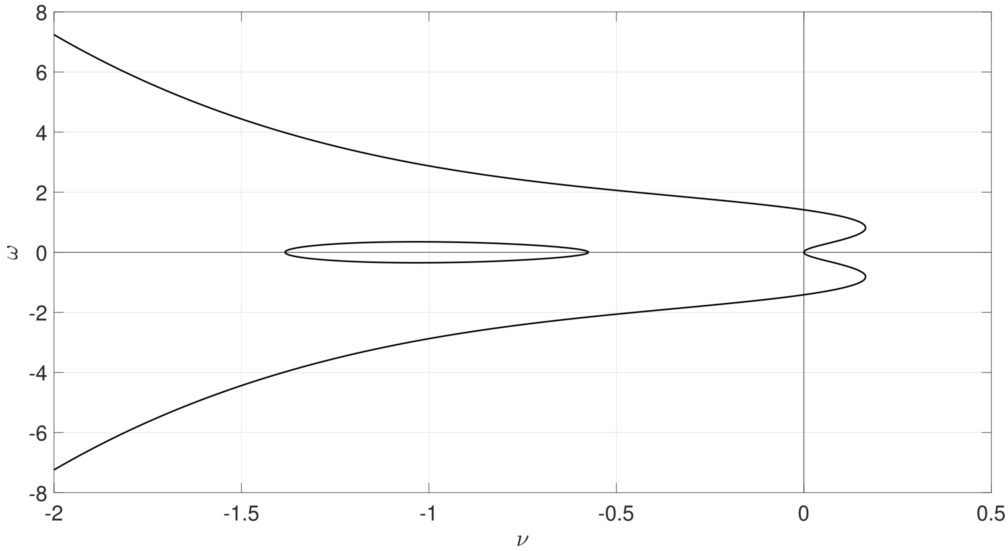


Figure 2.6: Plot of $\pm\sqrt{k(\nu)}$ and $\pm\sqrt{l(\nu)}$ for $(a_0, a_1, b_0, b_1) = (1, 1, 1, 1)$.

We can express ω in terms of ν, such that the real and imaginary characteristic equation, Eqs. (2.26) and (2.27), becomes equations for solely ν. We determine for which ν these equations are satisfied simultaneously. Note that the $\pm$ in $\omega = \pm\sqrt{k(\nu)}$ and $\omega = \pm\sqrt{l(\nu)}$ corresponds to the fact that if λ is a solution of the characteristic equation, then so is its conjugate $\overline{\lambda}$. Consequently, we only have to check for $\omega = \sqrt{k(\nu)}$ and $\omega = \sqrt{l(\nu)}$. Figure 2.7 shows a plot of the characteristic functions, using $\omega = \sqrt{k(\nu)}$. One has to check at which ν, both have a root.

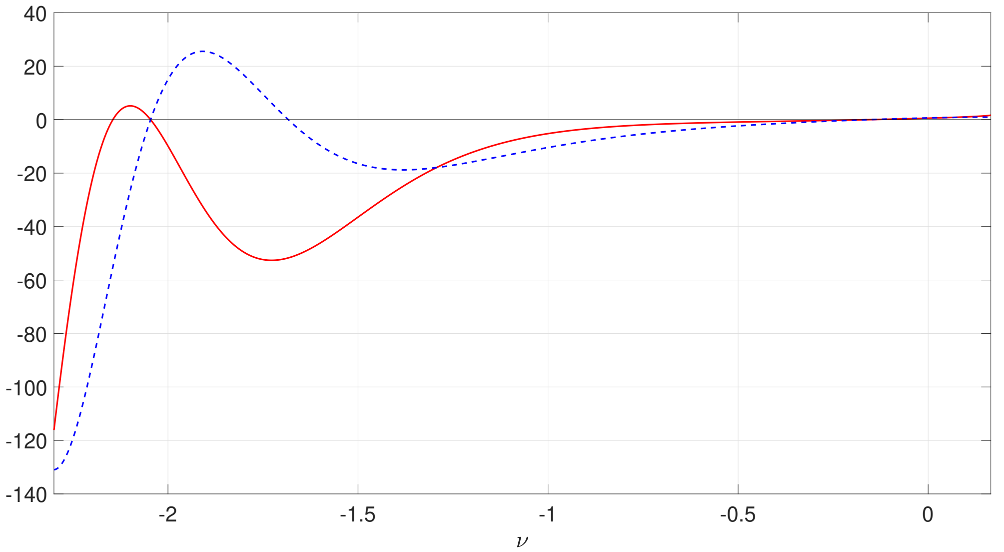


Figure 2.7: Plot of the real, solid (red) line, and imaginary, dashed (blue) line, characteristic functions, using $\omega = \sqrt{k(\nu)}$ and $(a_0, a_1, b_0, b_1) = (1, 1, 1, 1)$.

One can show that we only satisfy the characteristic equation with $\omega = \sqrt{l(\nu)}$ for $\nu = \nu_2$.

Figure 2.8 shows a plot of the solution curve, with the first solutions.

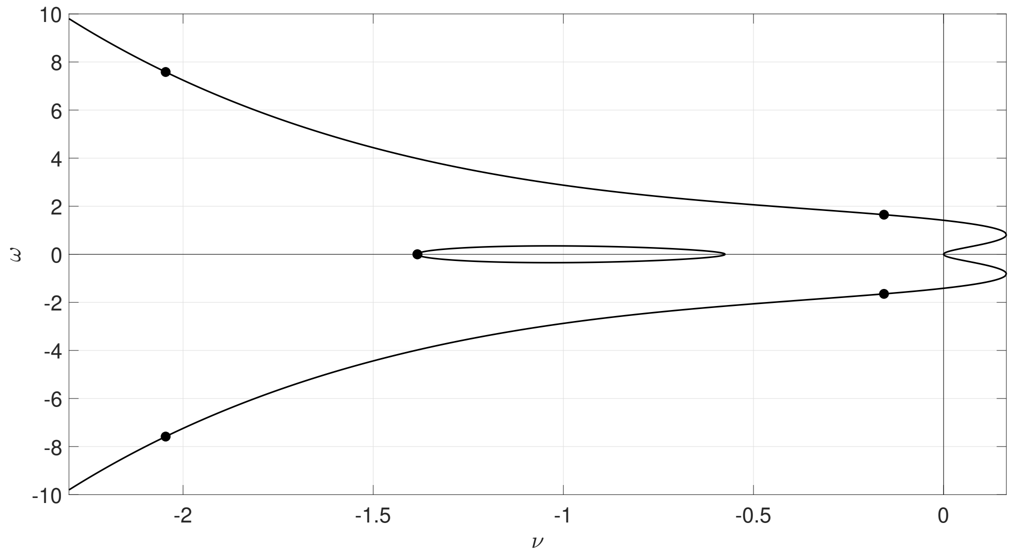


Figure 2.8: Plot of the solution curve, together with the first solutions, for $(a_0, a_1, b_0, b_1) = (1, 1, 1, 1)$.

For a third and fourth order DDE e will have to solve cubic and quartic equations for ω^2, for which we still have analytic solutions. For higher order DDEs, we will not be able to solve the equations for ω^2 analytically, making it harder to find the solutions of the characteristic equations.

We will now consider an interesting feature of the characteristic equation for large solutions.

Large solutions of the characteristic equation

We start by noting that the large solutions will have a negative real part. For these equations we can simplify the characteristic equation. To see this, we rewrite it into the following form,

$$\frac{\lambda^n + \sum_{i=0}^{n-1} a_i \lambda^i}{\sum_{i=0}^{n-1} b_i \lambda^i} = -e^{-\lambda}.$$

Suppose $m < n$ is the highest derivative with a delay that appears in the DDE, so $b_m = 0$ and $b_i = 0$ for $i > m$. Then the previous equation can be rewritten into the following form,

$$\frac{\lambda^{n-m} + \sum_{i=0}^{n-1} a_i \lambda^{i-m}}{\sum_{i=0}^{m} b_i \lambda^{i-m}} = e^{-\lambda}$$

For large λ, we can use that $\lambda^{i-m} \approx 0$ for $i < m$. To find approximations of the large solutions of the characteristic equation, we can solve the following equation

$$\lambda^{n-m} + \sum_{i=m}^{n-1} a_i \lambda^{i-m} + b_m e^{-\lambda} = 0.$$

The characteristic equation has become of lower order, and is easier to analyze. Now assume that $a_i = 0$ for $i = m, \ldots, n-1$. Then the equation can be rewritten into the following form,

$$\frac{\lambda}{n-m} e^{\frac{\lambda}{n-m}} = \frac{(-b_m)^{\frac{1}{n-m}}}{n-m}.$$

Note that there are $n - m$ solutions of $(-b_m)^{\frac{1}{n-m}}$. This means that we will have $n - m$ different first order characteristic equations. The number of solutions is infinite, but intuitively one could say that in this case there are $n - m$ times as many solutions for the characteristic equation, as for a first order characteristic equation.

Now that we are able to find the solutions of the characteristic equations, we can find the solutions of the DDEs. We start with a linear DDE.

2.3.2 The Linear Case

As for the first order case, we use the Laplace transform to derive the solution. We first take the Laplace transform of the DDE, and then solve the resulting equation for $\mathcal{L}(\boldsymbol{z}(t))$. We obtain the following equation,

$$\mathcal{L}(\boldsymbol{z}(t)) = \left(sI + A + Be^{-s}\right)^{-1} \left(\boldsymbol{\phi}(0) - B\mathcal{L}(\boldsymbol{\phi}(t-1))\right). \tag{2.28}$$

Note that $\left(sI + A + Be^{-s}\right)^{-1}$ is the inverse of the matrix C, Eq. (2.23), with s instead of λ. Define the fundamental solution $Z(t)$ as $\mathcal{L}(Z(t)) = C^{-1}$. We must first determine the inverse of the matrix C, and can then perform the inverse Laplace transform. To illustrate this, we will do this for a second order DDE, for which the matrix C is given by

$$C = \begin{pmatrix} s & -1 \\ a_0 + b_0 e^{-s} & s + a_1 + b_1 e^{-s} \end{pmatrix}.$$

The inverse is given by

$$C^{-1} = \frac{1}{\det C} \begin{pmatrix} s + a_1 + b_1 e^{-s} & 1 \\ -a_0 - b_0 e^{-s} & s \end{pmatrix}.$$

We calculate the inverse Laplace transform element-wise, analogous to the first order case, and obtain

$$Z(t) = \sum_{k=-\infty}^{\infty} D_k e^{\lambda_k t},$$

with

$$D_k = \frac{1}{2\lambda_k + a_1 + b_1(1-\lambda_k)e^{-\lambda_k} - b_0 e^{-\lambda_k}} \begin{pmatrix} \lambda_k & 1 \\ -a_0 - b_0 e^{-\lambda_k} & \lambda_k + a_1 + b_1 e^{-\lambda_k} \end{pmatrix}.$$

We can then use $Z(t)$ to determine $\boldsymbol{z}(t)$,

$$\boldsymbol{z}(t) = \int_0^t Z(t-\tau)\left(\boldsymbol{\phi}(0)\delta(\tau) - B\boldsymbol{\phi}(\tau-1)\right) d\tau.$$

Working this out yields

$$\boldsymbol{z}(t) = \sum_{k=-\infty}^{\infty} \boldsymbol{\alpha}_k e^{\lambda_k t}, \tag{2.29}$$

with

$$\boldsymbol{\alpha}_k = D_k \left(\boldsymbol{\phi}(0) - Be^{-\lambda_k} \int_{-1}^{0} e^{-\lambda_k \theta} \boldsymbol{\phi}(\theta) d\theta\right). \tag{2.30}$$

We originally started with a higher order DDE for $y(t)$. The solution for $y(t)$ is the first element of $\boldsymbol{z}(t)$.

Now we shall consider the nonlinear case.

2.3.3 The Nonlinear Case

For the nonlinear case the derivation stays the same, but have to take into account $\boldsymbol{f}$. We find

$$\boldsymbol{z}(t) = \sum_{k=-\infty}^{\infty} \boldsymbol{\alpha}_k e^{\lambda_k t} + \sum_{k=-\infty}^{\infty} D_k e^{\lambda_k t} \int_0^t e^{-\lambda_k s} \boldsymbol{f}(\dots) ds. \tag{2.31}$$

If $\boldsymbol{f}(\dots)$ has a $\boldsymbol{z}$ dependence, then this is not an explicit solution. For the linear inhomogeneous case, this does give the solution for $\boldsymbol{z}$.

We have worked out the solution for a second order DDE. In the next section we will consider the stability of this solution.

2.4 Stability of a Second Order Delay Differential Equation

For this section we use the Bachelor thesis of Vincent Li [17, 18], whom I co-supervised. The focus of this work was on the stability of the following linear second order DDE,

$$\frac{d^2y(t)}{dt^2} + ay(t) + by(t-1) = 0. \tag{2.32}$$

The corresponding characteristic equation is given by

$$\lambda^2 + a + be^{-\lambda} = 0.$$

The stability of a solution is determined by the largest real part of all the solutions of the characteristic equation. Let λ_k, $k = 0, \pm1, \pm2, \dots$ be all the solutions of the characteristic equation and define

$$\mu = \max_{k\in\mathbb{Z}} \mathrm{Re}(\lambda_k).$$

If $\mu > 0$, then the solution of the DDE is unstable, if $\mu < 0$, then the solution is asymptotically stable, and if $\mu = 0$, then we have to continue our investigation. Areas for which $\mu < 0$ and $\mu > 0$ are separated by curves on which $\mu = 0$. We wish to know the curves on which $\mu = 0$. To find these curves, we write $\lambda = i\omega$, and obtain the following equation for ω,

$$-\omega^2 + a + b(\cos\omega - i\sin\omega) = 0.$$

To satisfy the imaginary part of this equation, we must have $b = 0$ or $\omega = n\pi$. If $b = 0$, then we do not have a DDE anymore, but an ODE. To have purely imaginary solutions we must have $a > 0$ and find $\omega = \pm\sqrt{a}$. If $\omega = n\pi$, then we obtain the following restriction for a and b,

$$a + (-1)^n b = n^2\pi^2.$$

These lines enclose triangles around the positive a-axis. Li [17, 18] has proven that these triangles are exactly the regions for which $\mu < 0$. Figure 2.9 shows a plot of a part of the stability region.

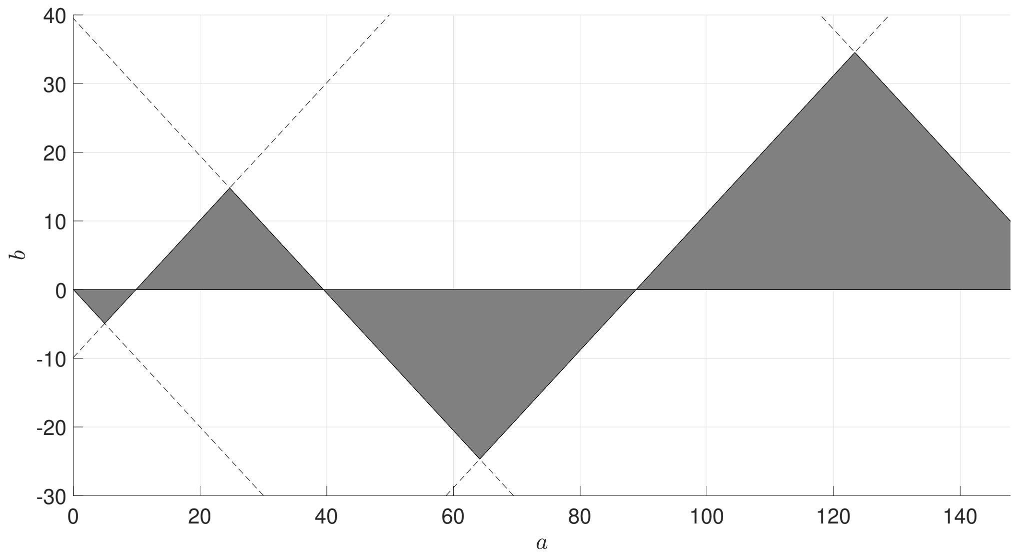


Figure 2.9: Plot of the stability region for the DDE in Eq. (2.32). There exist imaginary solutions on the boundaries of the triangles and the dashed lines.

One of the interesting characteristics of this stability region is the alternating sign of b. For the first triangle $b < 0$, for the second triangle $b > 0$, etcetera. We can see why this occurs by first noting that $\lambda = \lambda(a, b)$ and then taking the derivative of the characteristic equation with respect to b,

$$2\lambda\frac{\partial\lambda}{\partial b} + \left(1 - b\frac{\partial\lambda}{\partial b}\right)e^{-\lambda} = 0.$$

We can solve this for $\frac{\partial \lambda}{\partial b}$,

$$\frac{\partial \lambda}{\partial b} = \frac{e^{-\lambda}}{be^{-\lambda} - 2\lambda}$$

and work this out for $\lambda = i\sqrt{a}$,

$$\left.\frac{\partial \lambda}{\partial b}\right|_{\lambda = i\sqrt{a}} = \frac{\sin\sqrt{a}}{2\sqrt{a}} + i\frac{\cos\sqrt{a}}{2\sqrt{a}}.$$

If $\sin\sqrt{a} > 0$, then if b increases/decreases, the real part of λ increases/decreases as well. So, we find a stability region for $\sin\sqrt{a} > 0$ and $b < 0$. Similarly, we find the stability region for $\sin\sqrt{a} < 0$ and $b > 0$. We also know that the entire regions enclosed by these triangles must be stability regions, because of the continuity of the solutions of the characteristic equation. We find that the sign of b must change for $\sin\sqrt{a} = 0$, so for $a = n^2\pi^2$. This corresponds to the vertices of the triangles that lie on the positive a-axis.

One of the remaining problems from the thesis of Vincent Li, is the stability on the lines on which $\lambda = \pm in\pi$. He has proven that there is stability, but no asymptotic stability, on the lines surrounding the triangles. We expect that there is instability on the remainder of the lines, but have not proven this. To support this claim, we consider the vertices of the triangles that do not lie on the a-axis, in which there are multiple imaginary solutions. Suppose we are at the point where $\pm i\pi$ and $\pm 2i\pi$ are solutions of the characteristic equation. We are then interested in what happens with $\pm 2i\pi$, when we continue on the $\pm i\pi$ line and vice versa. On these lines we can write b as a function of a, such that we find $\lambda = \lambda(a)$. We can then use the characteristic equation to find an expression for $\lambda'(a)$ and numerically integrate this to see how the solutions change along the line. If we do this for $\pm 2i\pi$ along the line on which $\pm i\pi$ are solutions and for $\pm i\pi$ along the line on which $\pm 2i\pi$ are solutions, then we find that the real parts become positive. For the first we let $a \to \infty$ and for the second $a \to -\infty$. We find that $in\pi \to i(n+2)\pi$ as $a \to \infty$ and $in\pi \to i(n-2)\pi$ as $a \to -\infty$. This corresponds to the fact that parallel lines "intersect" in infinity. As this was done numerically, and only for a few cases, this is not a formal proof, but does support our claim of instability on the lines not surrounding the triangles.

In this chapter we have calculated the solutions for a first and second order DDE and analysed the stability of the solutions. We have also given a general method to derive the solution of a higher order DDE. In the next chapter we introduce and discuss the method of multiple scales for ordinary differential equations and ordinary difference equations.

Chapter 3

Method of Multiple Scales

In the next chapter we shall introduce our perturbation method for delay differential equations, which is based on the method of multiple scales for ordinary differential equations, ODEs, and for ordinary difference equations, OΔEs. In this chapter we will show how the method of multiple scales works for these two cases.

3.1 Ordinary Differential Equation

To illustrate the method of multiple scales we consider the following example as discussed by, for example, Holmes [11] and Reiss [22],

$$y'' + \varepsilon y' + y = 0, \tag{3.1}$$

with

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 1. \tag{3.2}$$

ε is a small parameter, $0 < \varepsilon \ll 1$, so this corresponds to an oscillator with weak damping.

A way to solve perturbed equations, is by trying an expansion for the solution,

$$y(t) = y_0(t) + \varepsilon y_1(t) + \dots .$$

We assume the following ordering for this expansion,

$$y_0(t) \gg \varepsilon y_1(t) \gg \dots .$$

$y_0(t)$ is the main part of the approximation and the other terms are corrections. Plugging this expansion into the equation yields

$$(y_0'' + \varepsilon y_1'' + \dots) + \varepsilon(y_0' + \varepsilon y_1' + \dots) + (y_0 + \varepsilon y_1 + \dots) = 0.$$

For each order of ε, we can get an equation. The $\mathcal{O}(1)$ equation is

$$y_0'' + y_0 = 0,$$

which corresponds to the unperturbed problem. The $\mathcal{O}(\varepsilon)$ equation is

$$y_1'' + y_1 = -y_0'.$$

The initial conditions for y_0 are $y_0(0) = 0$ and $y_0'(0) = 1$. The initial conditions for y_n, with $n \geq 1$ are $y_n(0) = 0$ and $y_n'(0) = 0$. We can solve the $\mathcal{O}(1)$ equation for y_0,

$$y_0(t) = \sin t.$$

The right-hand side of the $\mathcal{O}(\varepsilon)$ equation becomes $-\cos t$, such that we find the following solution for y_1,

$$y_1 = -\frac{1}{2} t \sin t.$$

An approximation for the solution $y(t)$ is given by

$$y(t) \sim \sin t - \frac{1}{2}\varepsilon t \sin t. \tag{3.3}$$

We can also solve the equation exactly,

$$y(t) = \frac{1}{\sqrt{1-\frac{\varepsilon^2}{4}}} e^{-\frac{1}{2}\varepsilon t} \sin\left(\sqrt{1-\frac{\varepsilon^2}{4}}t\right) \tag{3.4}$$

Figure 3.1 shows a plot of both the exact solution and the approximation for $\varepsilon = 0.1$. At first the approximation fits the exact solution well, but after some time the amplitude of the approximation becomes much greater than that of the exact solution. This is caused by the second term in the expansion, $-\frac{1}{2}\varepsilon t \sin t$. Note that the ordering that we assume, $y_0(t) \gg \varepsilon y_1(t)$, is violated for $t = \mathcal{O}(\frac{1}{\varepsilon})$. This means that the approximation is only valid for $t = \mathcal{O}(1)$. We call the growth of the second term in the expansion secular growth, and the second term is called a secular term.

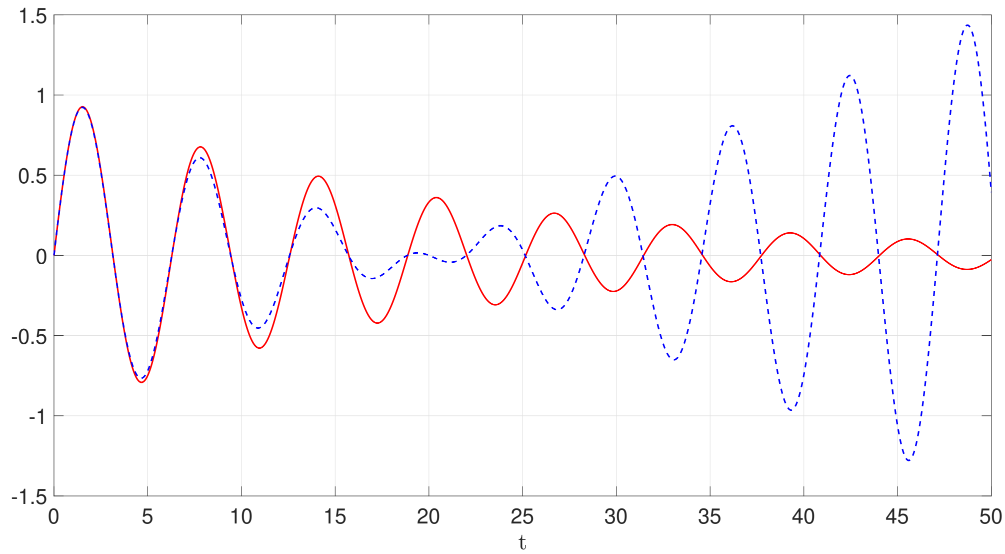


Figure 3.1: Plot of the exact solution, solid (red) line, and the approximation, dashed (blue) line for $\varepsilon = 0.1$.

We can distinguish two timescales if we take a look at the exact solution, Eq. (3.4). The oscillation occurs on a fast timescale, t, and the damping occurs on a slow timescale, εt. This motivates us to use two timescales, t and $\tau = \varepsilon t$. We will try to find a solution which depends on these two timescales, $\tilde{y}(t,\tau) = y(t)$. We assume that the timescales are independent, so what happens on a short timescale is not influenced by what happens on a long timescale and vice versa.

Due to the two timescales, partial derivatives are introduced,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau},$$
$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial^2}{\partial t \partial \tau} + \varepsilon^2 \frac{\partial^2}{\partial \tau^2}.$$

Eqs. (3.1) and (3.2) then become

$$\left(\frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial^2}{\partial t \partial \tau} + \varepsilon^2 \frac{\partial^2}{\partial \tau^2}\right)\tilde{y} + \varepsilon\left(\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}\right)\tilde{y} + \tilde{y} = 0,$$

and

$$\tilde{y} = 0 \qquad \text{and} \qquad \left(\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}\right)\tilde{y} = 1 \qquad \text{for } t = \tau = 0.$$

These are in fact too few conditions to have a unique solution, but this freedom will allow us to avoid secular terms. We will find an approximation for $\tilde{y}$, using an asymptotic expansion,

$$\tilde{y}(t,\tau) = y_0(t,\tau) + \varepsilon y_1(t,\tau) + \dots .$$

We again find an equation for each order of ε. The first two equations are given by

$$\begin{aligned} \frac{\partial^2 y_0}{\partial t^2} + y_0 &= 0, \\ \frac{\partial^2 y_1}{\partial t^2} + y_1 &= -2\frac{\partial^2 y_0}{\partial t \partial \tau} - \frac{\partial y_0}{\partial t}. \end{aligned}$$

For the initial conditions we also collect like orders of ε. The initial conditions for y_0 and y_1 are

$$\begin{aligned} y_0 &= 0 \qquad \text{and} \qquad \frac{\partial y_0}{\partial t} = 1, \\ y_1 &= 0 \qquad \text{and} \qquad \frac{\partial y_1}{\partial t} = -\frac{\partial y_0}{\partial \tau}, \end{aligned}$$

for $(t,\tau) = (0,0)$. Note that in the equation for y_0, τ only appears as a parameter. We shall solve for t, and instead of constants, we get functions of τ,

$$y_0(t,\tau) = A(\tau)\cos t + B(\tau)\sin t.$$

The initial conditions for y_0, can be used to obtain initial conditions for A and B,

$$A(0) = 0 \qquad \text{and} \qquad B(0) = 1. \tag{3.5}$$

The equation for y_1 becomes

$$\frac{\partial^2 y_1}{\partial t^2} + y_1 = \left(2\frac{dA}{d\tau} + A\right)\sin t - \left(2\frac{dB}{d\tau} + B\right)\cos t.$$

$\sin t$ and $\cos t$ will cause secular terms, but we can use our freedom here to avoid this. We must require

$$\begin{aligned} 2\frac{dA}{d\tau} + A &= 0, \\ 2\frac{dB}{d\tau} + B &= 0. \end{aligned}$$

Together with the initial conditions in Eq. (3.5), we can solve for A and B,

$$\begin{aligned} A(\tau) &= 0, \\ B(\tau) &= e^{-\frac{1}{2}\tau}. \end{aligned}$$

We obtain the following approximation,

$$y \sim e^{-\frac{1}{2}\varepsilon t}\sin t. \tag{3.6}$$

Note that for $t = \mathcal{O}(1)$ we can expand the exponential to obtain

$$y \sim \left(1 - \frac{1}{2}\varepsilon t + \dots\right)\sin t,$$

in which we recognize the approximation that we obtained using the asymptotic expansion with only one timescale.

Figure 3.2 shows a plot of the exact solution and the approximation obtained using the method of multiple scales. The plots are indistinguishable.

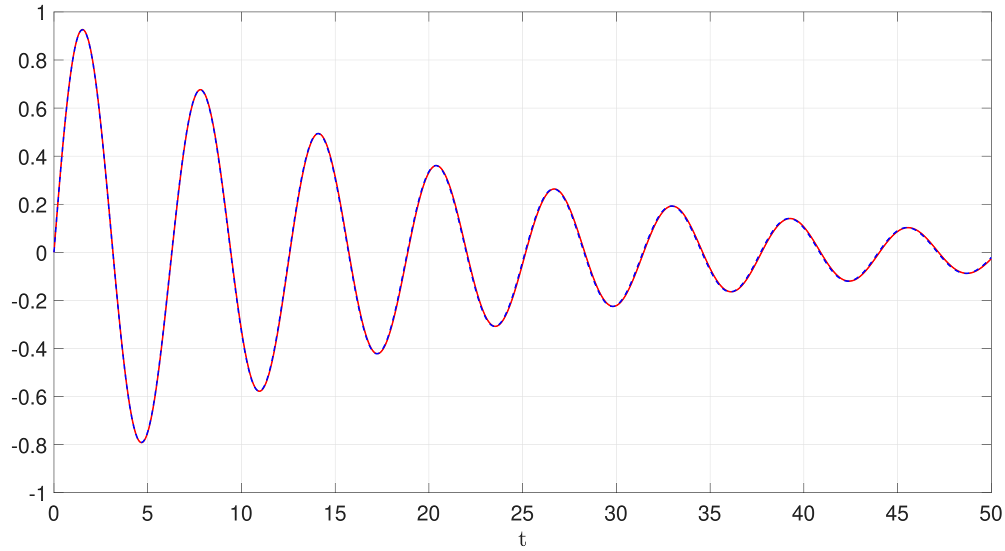


Figure 3.2: Plot of the exact solution, solid (red) line, and the approximation obtained using the method of multiple scales, dashed (blue) line, for $\varepsilon = 0.1$.

Due to the nice nature of this example the error will stay small for all t. In general the approximation obtained using this method is valid for $t = \mathcal{O}(\frac{1}{\varepsilon})$. With this we mean that the error is $\mathcal{O}(\varepsilon)$.

Next we shall consider the method of multiple scales for ordinary difference equations, OΔEs.

3.2 Ordinary Difference Equation

In a difference equation we do not consider a continuous, but a discrete time. So, we will have x_1, x_2, $\ldots$, x_n, x_{n+1}, $\ldots$ and an equation for x_{n+1}, which depends on the previous time(s). One method of multiple scales to solve difference equations, developed by Hoppensteadt and Miranker [12], involves transforming differences to derivatives, such that one has to solve an ODE. Although some results are the same for ODEs and OΔEs, other results are different, so one must be careful using this method. Van Horssen and Ter Brake [26] have developed a method, which keeps the differences throughout the calculations. We shall first introduce the operators that are used in this method, and then consider an illustrating example.

3.2.1 Operators

Three operators are defined, the shift operator E, the difference operator Δ and the identity operator I,

$$Ex_n = x_{n+1}, \qquad \Delta x_n = x_{n+1} - x_n \qquad \text{and} \qquad Ix_n = x_n.$$

We have the following relation between these operators,

$$E - \Delta = I.$$

For ODEs we introduced two timescales, a fast timescale t and a slow timescale εt. Similarly for OΔEs we can introduce n and εn. We will find approximations for $\tilde{x}(n, \varepsilon n) = x(n)$. As a result, the OΔEs will become partial difference equations, PΔEs. We define the partial operators in the following way,

$$\begin{aligned}
E_1 x(n, \varepsilon n) &= x(n+1, \varepsilon n), \\
E_\varepsilon x(n, \varepsilon n) &= x(n, \varepsilon n + \varepsilon), \\
\Delta_1 x(n, \varepsilon n) &= x(n+1, \varepsilon n) - x(n, \varepsilon n), \\
\Delta_\varepsilon x(n, \varepsilon n) &= x(n, \varepsilon n + \varepsilon) - x(n, \varepsilon n).
\end{aligned}$$

The identity operator remains the same. The relations between the partial operators are

$$E_1 - \Delta_1 = I \qquad \text{and} \qquad E_\varepsilon - \Delta_\varepsilon = I.$$

We wish to know the relation between the operators E and Δ, and the partial operators, such that we can rewrite the OΔEs into PΔEs. First,

$$E\tilde{x}(n, \varepsilon n) = \tilde{x}(n+1, \varepsilon n + \varepsilon) = E_1 E_\varepsilon \tilde{x}(n, \varepsilon n),$$

such that

$$E = E_1 E_\varepsilon.$$

We use this, together with the relations between the partial operators, to write the difference operator in terms of the partial difference operators,

$$\begin{aligned} \Delta &= E - I = E_1 E_\varepsilon - I = (\Delta_1 + I)(\Delta_\varepsilon + I) - I \\ &= \Delta_1 + \Delta_\varepsilon + \Delta_1 \Delta_\varepsilon. \end{aligned}$$

It is assumed that

$$\begin{aligned} \Delta_1 x(n, \varepsilon n) &= \mathcal{O}(x(n, \varepsilon n)), \\ \Delta_\varepsilon x(n, \varepsilon n) &= \mathcal{O}(\varepsilon x(n, \varepsilon n)), \end{aligned}$$

which will be important when we get equations by collecting like orders of ε. We will now consider an example, to illustrate the method.

3.2.2 Example

We will consider an example, similar to the example for the ODEs,

$$x_{n+2} + \varepsilon x_{n+1} + x_n = 0, \tag{3.7}$$

with

$$x_0 = 0 \qquad \text{and} \qquad \Delta x_0 = 0, \tag{3.8}$$

and ε a small parameter.

We first rewrite the equation using the operators,

$$\left(\Delta^2 + (2+\varepsilon)\Delta + (2+\varepsilon)I\right) x_n = 0.$$

Then introduce the slow timescale εn and construct an equation for $\tilde{x}(n, \varepsilon n) = x_n$. We use the partial operators to do this,

$$\begin{aligned} &\left((\Delta_1 + \Delta_\varepsilon + \Delta_1\Delta_\varepsilon)^2 + (2+\varepsilon)(\Delta_1 + \Delta_\varepsilon + \Delta_1\Delta_\varepsilon) + (2+\varepsilon)I\right)\tilde{x}(n, \varepsilon n) = 0 \\ \Longleftrightarrow\ &\left(\Delta_1^2 + 2E_1\right)\tilde{x}(n, \varepsilon n) + \left(2E_1^2\Delta_\varepsilon + \varepsilon E_1\right)\tilde{x}(n, \varepsilon n) + \mathcal{O}(\varepsilon^2 \tilde{x}(n, \varepsilon n)) = 0. \end{aligned}$$

We have split the equation in an $\mathcal{O}(\tilde{x}(n, \varepsilon n)$ and a $\mathcal{O}(\varepsilon\tilde{x}(n, \varepsilon n)$ part. We will use an expansion to obtain an approximation for $\tilde{x}(n, \varepsilon n)$, $\tilde{x}(n, \varepsilon n) = x_0(n, \varepsilon n) + \varepsilon x_1(n, \varepsilon n) + \ldots$. We obtain an equation for each order of ε. The first equation is given by

$$(\Delta_1^2 + 2E_1)x_0(n, \varepsilon n) = 0.$$

The initial condition is

$$x_0 = 0 \qquad \text{and} \qquad \Delta_1 x_0 = 1, \qquad \text{for } (n, \varepsilon n) = (0, 0).$$

Note that the subscripts 0 here, differs from the subscript 0 in Eq. (3.8). What we mean with the subscripts should be clear from the context. The solution is given by

$$x_0(n, \varepsilon n) = A(\varepsilon n)\cos\left(\frac{1}{2}n\pi\right) + B(\varepsilon n)\sin\left(\frac{1}{2}n\pi\right).$$

Using the initial conditions for x_0, we get initial conditions for A and B,

$$A(0) = 0 \qquad \text{and} \qquad B(0) = 1. \tag{3.9}$$

The $\mathcal{O}(\varepsilon)$ equation is

$$\begin{aligned}\varepsilon(\Delta_1^2 + 2E_1)x_1(n, \varepsilon n) &= -\left(2E_1^2\Delta_\varepsilon + \varepsilon E_1\right) x_0(n, \varepsilon n) \\ &= \left(2\Delta_\varepsilon A(\varepsilon n) - \varepsilon B(\varepsilon n)\right)\cos\left(\frac{1}{2}n\pi\right) + \left(2\Delta_\varepsilon B(\varepsilon n) + \varepsilon A(\varepsilon n)\right)\sin\left(\frac{1}{2}n\pi\right).\end{aligned}$$

To avoid secular terms, we must require

$$\begin{aligned}2\Delta_\varepsilon A(\varepsilon n) - \varepsilon B(\varepsilon n) &= 0, \\ 2\Delta_\varepsilon B(\varepsilon n) + \varepsilon A(\varepsilon n) &= 0.\end{aligned}$$

Together with initial conditions, Eq. (3.9), the solutions are given by (see [6, 26])

$$\begin{aligned}A(\varepsilon n) &= \left(1 + \frac{\varepsilon^2}{4}\right)^{\frac{n}{2}} \sin(n\mu(\varepsilon)), \\ B(\varepsilon n) &= \left(1 + \frac{\varepsilon^2}{4}\right)^{\frac{n}{2}} \cos(n\mu(\varepsilon)).\end{aligned}$$

$\mu(\varepsilon)$ is defined as

$$\cos(\mu(\varepsilon)) = \frac{1}{\sqrt{1 + \frac{\varepsilon^2}{4}}} \qquad \text{and} \qquad \sin(\mu(\varepsilon)) = \frac{\varepsilon}{2\sqrt{1 + \frac{\varepsilon^2}{4}}}.$$

Using the trigonometric identity for the angle sum, the approximation can be written in the following form,

$$x_0(n, \varepsilon n) = \left(1 + \frac{\varepsilon^2}{4}\right)^{\frac{n}{2}} \sin\left(\frac{1}{2}n\pi + n\mu(\varepsilon)\right). \tag{3.10}$$

The exact solution can be calculated, such that one can see that the error is of $\mathcal{O}(\varepsilon)$, for $n = \mathcal{O}(\frac{1}{\varepsilon})$. This has been done by Van Horssen and Ter Brake [26]. Figure 3.3 shows a plot of the exact solution and the approximation. Secular behaviour is observed, but only for $n = \mathcal{O}(\frac{1}{\varepsilon^2})$.

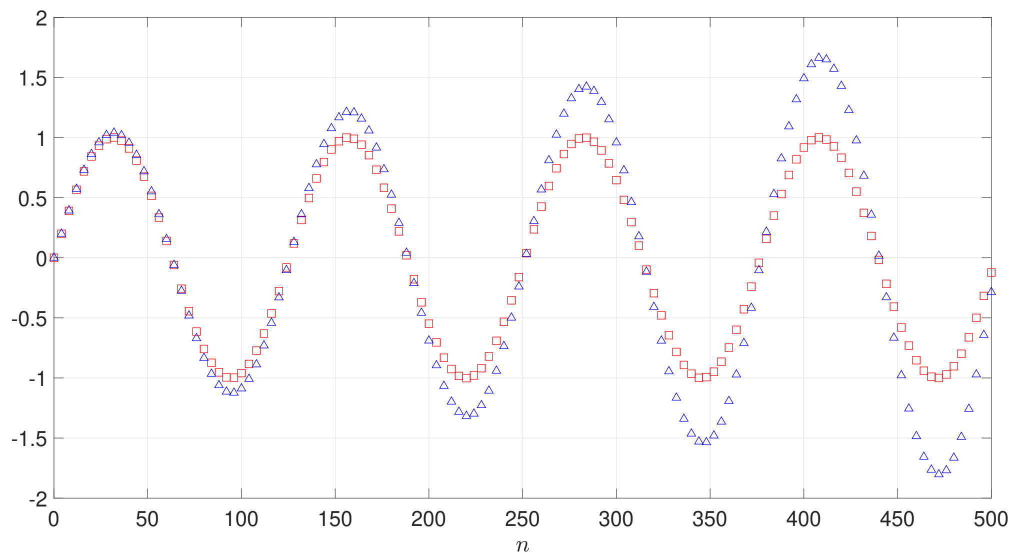


Figure 3.3: Plot of the exact solution of Eqs. (3.7) and (3.8), (blue) triangles, and an approximation obtained using the method of multiple scales, (red) squares, for $\varepsilon = 0.1$. Only every fourth point has been plotted.

Now that we have seen how the method of multiple scales works for ODEs and O∆Es, we shall consider a method of multiple scales for DDEs.

Chapter 4

Method of Multiple Scales for First Order Delay Differential Equations

We will use the method of multiple scales for ODEs and OΔEs, to construct a method of multiple scales for DDEs. In this chapter we shall only consider first order DDEs. Many of the difficulties that arise for first order DDEs will also arise for higher order DDEs. It is then favorable to have solved these problems for the simplest case, such that only slight adjustments are necessary.

Method of multiple scales already exist for DDEs (see for example [7]), but these assume that either the delay is a perturbation on an ODE, for example

$$\frac{dy(t)}{dt} + ay(t) = \varepsilon y(t-1),$$

or that the delay itself is small, for example

$$\frac{dy(t)}{dt} + ay(t) = by(t-\varepsilon).$$

For the first case, the unperturbed equation is an ODE. For the second case, a Taylor expansion can be used to again obtain an ODE. To obtain approximations, the ODEs are solved, instead of the DDEs. The effect of the delay terms are small in these equations, so one might expect this method to work. However, due to the different natures of ODEs and DDEs, the approximations are not always accurate. Furthermore, this method cannot be used anymore when the effect of the delay term becomes bigger. In this chapter we shall construct a method which keeps the DDEs throughout the derivation. We consider (nonlinear) DDEs, which are linear DDEs in the unperturbed case.

We shall first show what kind of functions cause secular terms. Then we present the operators that will be used in this method. These operators will be similar to the operators used in the method of multiple scales for OΔEs. Next, an example is used to illustrate how the method works. After that, the validity of the method is proven under certain conditions. Then we shall consider a delayed version of Mathieu's equation and a quadratic perturbation, for which we need to analyse the characteristic equation. Finally, we will consider a DDE perturbation on an ODE. We will show when the already existing perturbation method fails, and yield an accurate approximation for this case, using our perturbation method.

4.1 Secular Terms and Detuning

In the method of multiple scales for ODEs and OΔEs, the initial conditions did not fully determine a unique solution. This freedom was used to avoid secular terms. It was important to identify which functions would cause secular growth, such that these can be eliminated. In this section, we shall show which functions cause secular growth for DDEs.

Consider

$$\frac{dy(t)}{dt} + ay(t) + by(t-1) = f(t).$$

As seen before, solutions to the homogeneous equation can cause secular growth. We shall investigate that phenomenon here as well. In Section 2.1 the solution to a nonlinear first order DDE was derived. This solution consists of a homogeneous part and a part related to the nonlinear part of the DDE. The nonlinear part of this solution is given by

$$\sum_{k=-\infty}^{\infty} \frac{e^{\lambda_k t}}{1 - be^{-\lambda_k}} \int_0^t e^{-\lambda_k s} f(s) ds,$$

with λ_k the solutions of the characteristic equation corresponding to the linear DDE. Set $f(t) = e^{\lambda_l t}$, with λ_l one of the solutions of the characteristic equation. Then the nonlinear part of the solution becomes

$$\frac{te^{\lambda_l t}}{1 - be^{-\lambda_l}} + \sum_{\substack{k=-\infty \\ k \neq l}}^{\infty} \frac{e^{\lambda_l t} - e^{\lambda_k t}}{(1 - be^{-\lambda_k})(\lambda_l - \lambda_k)},$$

where we have split it up into a $k = l$ and $k \neq l$ part. We see that for $k = l$ we get $te^{\lambda_l t}$, which is a secular term.

Note that we have used the solution for which all solutions of the characteristic equation must have multiplicity one. If we consider the case for which $\lambda_0 = \lambda_{-1}$, then the solution will contain a term of the form $te^{\lambda_0 t}$. Then using $f(t) = e^{\lambda_0 t}$ will result in a term of the form $t^2 e^{\lambda_0 t}$ in the solution, so we still have secular behaviour.

We now consider a source function, close to a solution of the homogeneous equation. This is called detuning. Write $f(t) = e^{\alpha t}$, with $\alpha = \lambda_l + \varepsilon$. ε is small, $0 < \varepsilon \ll 1$. The nonlinear part of the solution is then given by

$$\sum_{k=-\infty}^{\infty} \frac{e^{\alpha t} - e^{\lambda_k t}}{(1 + be^{-\lambda_k})(\alpha - \lambda_k)}.$$

For $k = l$, $\alpha - \lambda_k = \varepsilon$, so we divide by ε. If we use an expansion, then this will violate the ordering of an expansion, so detuning still causes secular behaviour.

We shall now consider the operators used in the method of multiple scales for DDEs.

4.2 Operators

The operators that we consider for DDEs are similar to the operators for OΔEs. We have the shift operator E, the difference operator Δ and identity operator I, now defined as

$$Ex(t) = x(t-T), \qquad \Delta x(t) = x(t) - x(t-T), \qquad \text{and} \qquad Ix(t) = x(t), \tag{4.1}$$

for a delay T. The relation between the operators is

$$E + \Delta = I. \tag{4.2}$$

For most examples that we consider in this thesis we will have to use a fast and a slow timescale, t and $\tau = \varepsilon t$. We will use the following partial operators,

$$E_1 x(t,\tau) = x(t-T,\tau), \tag{4.3}$$
$$E_\varepsilon x(t,\tau) = x(t-T,\tau-\varepsilon T), \tag{4.4}$$
$$\Delta_1 x(t,\tau) = x(t,\tau) - x(t-T,\tau), \tag{4.5}$$
$$\Delta_\varepsilon x(t,\tau) = x(t,\tau) - x(t,\tau-\varepsilon T). \tag{4.6}$$

The relations between the partial operators are

$$E_1 + \Delta_1 = I \qquad \text{and} \qquad E_\varepsilon + \Delta_\varepsilon = I. \tag{4.7}$$

As for the OΔEs, we have

$$E = E_1 E_\varepsilon. \tag{4.8}$$

Then,

$$\begin{aligned} \Delta &= I - E = I - E_1 E_\varepsilon = I - (I - \Delta_1)(I - \Delta_\varepsilon) \\ &= \Delta_1 + \Delta_\varepsilon - \Delta_1 \Delta_\varepsilon. \end{aligned} \tag{4.9}$$

As for OΔEs, we assume that

$$\Delta_1 x(t,\tau) = \mathcal{O}(x(t,\tau)), \tag{4.10}$$

$$\Delta_\varepsilon x(t,\tau) = \mathcal{O}(\varepsilon x(t,\tau)). \tag{4.11}$$

We shall now use these operators for a method of multiple scales for first order DDEs.

4.3 Method of Multiple Scales

We will consider the following types of equations in this chapter,

$$\frac{dy(t)}{dt} + ay(t) + by(t-1) = \varepsilon f(t, y(t), y(t-1)).$$

Usually, f is a function for which we cannot find the exact solution of the equation. ε is a small parameter, so we are considering a small (nonlinear) perturbation on a linear first order DDE. We will use the following initial condition,

$$y(t) = \phi(t), \qquad \text{for } t \in [-1, 0].$$

For most examples that will be discussed in this chapter, we will introduce a slow timescale, $\tau = \varepsilon t$. We will construct an equation for $\tilde{y}(t,\tau) = y(t)$. For the derivative we use

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau},$$

as we have seen for ODEs. For the delay term we will use

$$\begin{aligned} E &= E_1 E_\varepsilon = E_1 (I - \Delta_\varepsilon) \\ &= E_1 - E_1 \Delta_\varepsilon, \end{aligned}$$

whereas we used only partial Δ operators for OΔEs. Both yield the same equations, so one can use either method.

The equation for $\tilde{y}(t,\tau)$ is given by

$$\left(\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}\right) \tilde{y}(t,\tau) + a\tilde{y}(t,\tau) + bE_1 \tilde{y}(t,\tau) - bE_1 \Delta_\varepsilon \tilde{y}(t,\tau) = \varepsilon \tilde{f}(\dots),$$

with $\tilde{f}(\dots)$ a function similar to $f(t, y(t), y(t-1))$. We will use expansions for $\tilde{y}(t,\tau)$ and $\tilde{f}(\dots)$ to obtain an approximation for $\tilde{y}(t,\tau)$,

$$\begin{aligned} \tilde{y}(t,\tau) &= y_0(t,\tau) + \varepsilon y_1(t,\tau) + \dots, \\ \tilde{f}(\dots) &= f_0(\dots) + \varepsilon f_1(\dots) + \dots. \end{aligned}$$

We assume that $f_0(\dots)$ only depends on y_0, $f_1(\dots)$ only on y_0 and y_1, and so on. We can then collect like orders of ε in the equation for $\tilde{y}(t,\tau)$. The $\mathcal{O}(1)$ equation is given by

$$\frac{\partial y_0(t,\tau)}{\partial t} + ay_0(t,\tau) + bE_1 y_0(t,\tau) = 0.$$

Note that this is exactly the equation that we would get if we would set $\varepsilon = 0$ in the original equation. τ only appears as a parameter in this equation, so we can solve it for t. We use the results of Chapter 2, with functions of τ instead of constants. The solution for y_0 is of the following form,

$$y_0(t,\tau) = \sum_{k=-\infty}^{\infty} \alpha_k(\tau)e^{\lambda_k t},$$

with λ_k the solutions of the characteristic equation

$$\lambda + a + be^{-\lambda} = 0.$$

Generally we have the initial condition $\tilde{y}(t,\tau) = \tilde{\phi}(t,\tau)$ for $(t,\tau) \in [-1,0] \times [-\varepsilon, 0]$. Then one should use an expansion for $\tilde{\phi}$ to obtain initial conditions for each y_i. For simplicity, we will assume that $\tilde{\phi} = \mathcal{O}(1)$, such that y_0 will satisfy the initial condition. We shall also assume that $\tilde{\phi}$ has no τ dependence. This is not a strange assumption, as it only means that εt does not occur in the initial condition. The initial condition for y_0 becomes a constant initial condition for α_k,

$$\alpha_k(\tau) = \Phi_k, \qquad \text{for } \tau \in [-\varepsilon, 0],$$

with

$$\Phi_k = \frac{1}{1 - be^{-\lambda_k}} \left(\phi(0) - be^{-\lambda_k} \int_{-1}^{0} e^{-\lambda_k \theta} \phi(\theta) d\theta \right).$$

Note that these are precisely the constants that we find for the exact solution of the unperturbed problem.

The $\mathcal{O}(\varepsilon)$ equation is given by

$$\frac{\partial y_1(t,\tau)}{\partial t} + ay_1(t,\tau) + bE_1 y_1(t,\tau) = -\frac{\partial y_0(t,\tau)}{\partial \tau} + \frac{b}{\varepsilon} E_1 \Delta_\varepsilon y_0(t,\tau) + f_0(\dots).$$

Note that we assume that $\Delta_\varepsilon y_0(t,\tau) = \mathcal{O}(\varepsilon y_0(t,\tau))$, so the term $\frac{b}{\varepsilon} E_1 \Delta_\varepsilon y_0(t,\tau)$ is of the same order as all other terms in this equation. Using our expression for y_0, the right-hand side of the $\mathcal{O}(\varepsilon)$ equation becomes

$$\sum_{k=-\infty}^{\infty} \left(-\frac{d\alpha_k(\tau)}{d\tau} + \frac{b}{\varepsilon} e^{-\lambda_k} \Delta_\varepsilon \alpha_k(\tau) \right) e^{\lambda_k t} + f_0(\dots).$$

We have shown that terms of the form $e^{\lambda t}$, with λ a solution of the characteristic equation, will cause secular terms. So, we need to eliminate these terms on the right-hand side of the $\mathcal{O}(\varepsilon)$ equation. If for some λ_l, $f_0(\dots)$ does not contain $e^{\lambda_l t}$, then to eliminate $e^{\lambda_l t}$ from the right-hand side, we must require

$$\frac{d\alpha_l(\tau)}{d\tau} = \frac{b}{\varepsilon} e^{-\lambda_l} \Delta_\varepsilon \alpha_l(\tau).$$

Note that $\alpha_l(\tau) = \Phi_l$ is a solution which satisfies the equation and the initial condition for α_l. By the existence and uniqueness theorems, as stated in Chapter 1, this must be the unique solution. If $f_0(\dots)$ does contain $e^{\lambda_l t}$, then we get different equations. For different choices of f, we will get very different problems. We will discuss several examples in this chapter.

We start with a simple example, for which we also know the exact solution. We can compare our approximation with this exact solution.

4.4 Linear Example

We shall consider the following problem,

$$\frac{dy(t)}{dt} + ay(t) + by(t-1) = \varepsilon y(t), \tag{4.12}$$

with

$$y(t) = \phi(t), \qquad \text{for } t \in [-1, 0]. \tag{4.13}$$

Note that this is a linear DDE, for which we know the exact solution. We shall first give this exact solution. Then we shall determine an approximation, using the method of multiple scales. Finally, we shall compare our approximation with the exact solution.

4.4.1 Exact solution

The exact solution is given by

$$y(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\lambda_k t}, \tag{4.14}$$

with λ_k the solutions of the characteristic equation,

$$\lambda + a - \varepsilon + be^{-\lambda} = 0, \tag{4.15}$$

and the constants are determined by the initial condition,

$$\alpha_k = \frac{1}{1 - be^{-\lambda_k}} \left(\phi(0) - be^{-\lambda_k} \int_{-1}^{0} e^{-\lambda_k \theta} \phi(\theta) d\theta \right). \tag{4.16}$$

We continue with the approximation.

4.4.2 Approximation

We can copy much of the general derivation. We will use the slow timescale $\tau = \varepsilon t$, obtain an equation for $\tilde{y}(t, \tau)$ and use an expansion. We then get equations for each order of ε. The $\mathcal{O}(1)$ equation is the same as for the general derivation, so we can immediately write down the solution,

$$y_0(t, \tau) = \sum_{k=-\infty}^{\infty} \beta_k(\tau) e^{\mu_k t},$$

with μ_k the solutions of the characteristic equation

$$\mu + a + be^{-\mu} = 0. \tag{4.17}$$

We assume that $\phi(t) = \mathcal{O}(1)$ and has no τ dependence, such that for β_k we get the following initial condition

$$\beta_k(\tau) = \Phi_k, \qquad \text{for } \tau \in [-\varepsilon, 0],$$

with

$$\Phi_k = \frac{1}{1 - be^{-\mu_k}} \left(\phi(0) - be^{-\mu_k} \int_{-1}^{0} e^{-\mu_k \theta} \phi(\theta) d\theta \right). \tag{4.18}$$

Note that this is the same as α_k, but with μ_k instead of λ_k. The $\mathcal{O}(\varepsilon)$ equation is given by

$$\frac{dy_1(t, \tau)}{dt} + ay_1(t, \tau) + bE_1 y_1(t, \tau) = -\frac{\partial y_0(t, \tau)}{\partial \tau} + \frac{b}{\varepsilon} E_1 \Delta_\varepsilon y_0(t, \tau) + y_0(t, \tau).$$

Using our expression for y_0, we can write down the right-hand side of the $\mathcal{O}(\varepsilon)$ equation,

$$\sum_{k=-\infty}^{\infty} \left(-\frac{d\beta_k(\tau)}{d\tau} + \frac{b}{\varepsilon} e^{-\mu_k} \Delta_\varepsilon \beta_k(\tau) + \beta_k(\tau) \right) e^{\mu_k t}.$$

To avoid secular terms, we get the following equation for β_k,

$$\frac{d\beta_k(\tau)}{d\tau} - \left(\frac{b}{\varepsilon} e^{-\mu_k} + 1 \right) \beta_k(\tau) + \frac{b}{\varepsilon} e^{-\mu_k} \beta_k(\tau - \varepsilon) = 0.$$

This is a linear first order DDE, with a delay $T = \varepsilon$ instead of $T = 1$. We can readily solve this,

$$\beta_k(\tau) = \sum_{l=-\infty}^{\infty} \gamma_{kl} e^{\nu_{kl} \tau},$$

with ν_{kl} the solutions of the characteristic equation

$$\nu - \left(\frac{b}{\varepsilon} e^{-\mu_k} + 1 \right) + \frac{b}{\varepsilon} e^{-\mu_k} e^{-\varepsilon \nu} = 0, \tag{4.19}$$

and the constants are given by the following expression,

$$\gamma_{kl} = \frac{\Phi_k}{1 - be^{-\mu_k} e^{-\varepsilon \nu_{kl}}} \left(1 - \frac{b}{\varepsilon} e^{-\mu_k} e^{-\varepsilon \nu_{kl}} \int_{-\varepsilon}^{0} e^{-\nu_{kl}\eta} d\eta \right).$$

Our approximation is given by

$$\begin{aligned} y_0(t,\tau) &= \sum_{k,l=-\infty}^{\infty} \gamma_{kl} e^{\nu_{kl}\tau} e^{\mu_k t} \\ &= \sum_{k,l=-\infty}^{\infty} \gamma_{kl} e^{(\mu_k + \varepsilon \nu_{kl})t}. \end{aligned}$$

We shall now compare this approximation with the exact solution.

4.4.3 Comparison

We shall show that the error of the approximation is $\mathcal{O}(\varepsilon)$ for $t = \mathcal{O}(\frac{1}{\varepsilon})$.

We first consider the relation between the solutions of the different characteristic equations. For ν, we can find solutions of different orders of ε. There are two possible balances, $\mathcal{O}(1)$ and $\mathcal{O}(\frac{1}{\varepsilon})$. For the first balance we expand the exponential in Eq. (4.19), to obtain the following equation,

$$\nu - \left(\frac{b}{\varepsilon} e^{-\mu_k} + 1 \right) + \frac{b}{\varepsilon} e^{-\mu_k} \left(1 - \varepsilon \nu + \dots \right) = 0$$

Note that the two $\frac{b}{\varepsilon} e^{-\mu_k}$ cancel. Then solve the $\mathcal{O}(1)$ equation for ν to obtain the following first order approximation,

$$\nu \sim \left(1 - be^{-\mu_k} \right)^{-1}. \tag{4.20}$$

For the other balance write $\nu = \frac{1}{\varepsilon}\tilde{\nu}$, with $\tilde{\nu} = \mathcal{O}(1)$. The $\mathcal{O}(\frac{1}{\varepsilon})$ equation is given by

$$\tilde{\nu} - be^{-\mu_k} + be^{-\mu_k} e^{-\tilde{\nu}} = 0.$$

This has infinitely many solutions. $\tilde{\nu} = 0$ is a solution of this equation, but we do not consider this, since then we do not have that ν is $\mathcal{O}(\frac{1}{\varepsilon})$. Using the characteristic equation, one can show that

$$\gamma_{kl} = \frac{\Phi_k}{1 - be^{-\mu_k} e^{-\varepsilon \nu_{kl}}} \frac{1}{\nu_{kl}}.$$

For $\nu_{kl} = \mathcal{O}(\frac{1}{\varepsilon})$ we find $\gamma_{kl} = \mathcal{O}(\varepsilon)$ and for $\nu_{kl} = \left(1 - be^{-\mu_k}\right)^{-1}$ we find $\gamma_{kl} = \Phi_k \left(1 + \mathcal{O}(\varepsilon)\right)$. So, for a first order approximation of β_k, we can neglect the $\mathcal{O}(\frac{1}{\varepsilon})$ solutions. We will continue to write $\nu_k = \left(1 - be^{-\mu_k}\right)^{-1}$ and let γ_k be the corresponding coefficient.

Now consider λ. We expand λ as $\lambda = \lambda^0 + \varepsilon\lambda^1 + \dots$ and expand the exponential in Eq. (4.15) to obtain

$$\left(\lambda^0 + \varepsilon \lambda^1 + \dots \right) + a - \varepsilon + be^{-\lambda^0} \left(1 - \varepsilon \lambda^1 + \dots \right) = 0.$$

The $\mathcal{O}(1)$ equation is given by

$$\lambda^0 + a + be^{-\lambda^0} = 0,$$

which is exactly the characteristic equation for μ, Eq. (4.17). The $\mathcal{O}(\varepsilon)$ equation is given by

$$\lambda^1 - 1 - be^{-\lambda^0} \lambda^1 = 0.$$

Solving this for λ^1 yields

$$\lambda^1 = \left(1 - be^{-\lambda^0} \right)^{-1},$$

which is the same expression that we found for ν_k. So, we have

$$\lambda_k = \mu_k + \varepsilon \nu_k + \mathcal{O}(\varepsilon^2).$$

Then,

$$e^{\mu_k t}e^{\nu_k \tau} = e^{(\lambda_k+\mathcal{O}(\varepsilon^2))t} = e^{\lambda_k t}(1+\mathcal{O}(\varepsilon)), \tag{4.21}$$

for $t = \mathcal{O}(\frac{1}{\varepsilon})$.

We must now show that the $\mathcal{O}(1)$ parts of γ_k and α_k are the same. Using that $e^{-\lambda_k} = e^{-\mu_k}(1+\mathcal{O}(\varepsilon))$, one can show that $\alpha_k = \Phi_k(1+\mathcal{O}(\varepsilon))$. We have already seen that $\gamma_k = \Phi_k(1+\mathcal{O}(\varepsilon))$, such that

$$\gamma_k = \alpha_k(1+\mathcal{O}(\varepsilon)). \tag{4.22}$$

Using Eqs. (4.21) and (4.22), we have

$$y_0(t,\tau) = y(t) + \mathcal{O}(\varepsilon),$$

for $t = \mathcal{O}(\frac{1}{\varepsilon})$.

We have shown that the approximation is $\mathcal{O}(\varepsilon)$ accurate on a timescale of $t = \mathcal{O}(\frac{1}{\varepsilon})$ for this example. In the next section we will prove the validity of the method in a more general sense.

4.5 Validity of the Method of Multiple Scales

Consider the equation

$$\frac{dy(t)}{dt} + ay(t) + by(t-1) = \varepsilon f(t, y(t), y(t-1)), \tag{4.23}$$

with

$$y(t) = \phi(t), \qquad \text{for } t \in [-1,0].$$

We have an approximation of the solution, $\tilde{y}$, which satisfies

$$\frac{d\tilde{y}(t)}{dt} + a\tilde{y}(t) + b\tilde{y}(t-1) = \varepsilon f(t, \tilde{y}(t), \tilde{y}(t-1)) + \varepsilon^2 R(t), \tag{4.24}$$

with

$$\tilde{y}(t) = \phi_0(t), \qquad \text{for } t \in [-1,0].$$

The initial conditions are close together, $\phi(t) = \phi_0(t) + \varepsilon\phi_1(t) + \dots$.

We will show that $\tilde{y}(t)$ is $\mathcal{O}(\varepsilon)$ accurate on a timescale of $\mathcal{O}(\frac{1}{\varepsilon})$.

Note that the linear parts of the equations for y and $\tilde{y}$ are the same, which can be seen by setting $\varepsilon = 0$ in both equations. We have the fundamental solution $Y(t)$, with $\mathcal{L}(Y(t))(s) = (s - a - be^{-s})^{-1}$. Then,

$$y(t) = Y(t)\phi(0) + b\int_{-1}^{0} Y(t-\theta-1)\phi(\theta)d\theta + \varepsilon\int_0^t Y(t-s)f(s,y(s),y(s-1))ds, \tag{4.25}$$

$$\tilde{y}(t) = Y(t)\phi_0(0) + b\int_{-1}^{0} Y(t-\theta-1)\phi_0(\theta)d\theta + \varepsilon\int_0^t Y(t-s)f(s,y(s),y(s-1))ds + \varepsilon^2\int_0^t Y(t-s)R(s)ds \tag{4.26}$$

We assume that f satisfies the following Lipschitz conditions: there exist positive constants L_1 and L_2, such that

$$|f(t,y(t),y(t-1)) - f(t,\tilde{y}(t),y(t-1))| \le L_1|y(t) - \tilde{y}(t)|, \tag{4.27}$$

$$|f(t,y(t),y(t-1)) - f(t,y(t),\tilde{y}(t-1))| \le L_2|y(t-1) - \tilde{y}(t-1)|. \tag{4.28}$$

Under these conditions, the following inequality holds,

$$|f(t,y(t),y(t-1)) - f(t,\tilde{y}(t),\tilde{y}(t-1))| \le L_1|y(t) - y\tilde{(t)}| + L_2|y(t-1) - \tilde{y}(t-1)|. \tag{4.29}$$

Recall that for the fundamental solution $Y(t)$ we have an expression with an infinite sum of $e^{\lambda_k t}$, with λ_k the solutions of the characteristic equation. The λ_k with the highest real value will dominate the solution of the unperturbed problem. We call this $e^{\lambda_k t}$ the "largest solution of the unperturbed problem ($\varepsilon = 0$)". $Y(t)$ is bounded by this "largest solution" and we assume that $R(t)$ is as well. We define

$$\mu = \max_k \mathrm{Re}(\lambda_k). \tag{4.30}$$

Recall that this maximal value is found at the principal branch of the Lambert W function, λ_0. Hale and Verduyn Lunel [10] show that if λ_k has multiplicity m, then for $n = 0, \ldots, m-1$, $t^n e^{\lambda_k t}$ are all solutions of the unperturbed equation. If $be^a = e^{-1}$, then λ_0 has multiplicity two, so the bounding function is $te^{\mu t}$. If $be^a \neq e^{-1}$, then the multiplicity of λ_0 is one, so the bounding function is $e^{\mu t}$. We will continue with the latter case, and assume that

$$|Y(t)| \leq M_0 e^{\mu t}, \tag{4.31}$$
$$|R(t)| \leq M_1 e^{\mu t}, \tag{4.32}$$

with M_0 and M_1 positive constants. The derivation for the other case is similar.

Recall that the initial conditions are close together, so we can write

$$|\phi(\theta) - \phi_0(\theta)| \leq \varepsilon M_2, \tag{4.33}$$

with M_2 a positive constant. Note that this only has to hold for $\theta \in [-1, 0]$.

Subtract Eqs. (4.25) and (4.26) and take the absolute value,

$$\begin{aligned} |y(t) - \tilde{y}(t)| \leq & |Y(t)||\phi(0) - \phi_0(0)| + b\int_{-1}^{0} |Y(t-\theta-1)||\phi(\theta) - \phi_0(\theta)|d\theta \\ & + \varepsilon \int_0^t |Y(t-s)||f(s, y(s), y(s-1)) - f(s, \tilde{y}(s), \tilde{y}(s-1))|ds \\ & + \varepsilon^2 \int_0^t |Y(t-s)||R(s)|ds \end{aligned} \tag{4.34}$$

We will consider three parts separately. First, we consider the part involving the initial conditions. Using our assumptions, we find that

$$\begin{aligned} |Y(t)||\phi(0) - \phi_0(0)| + b\int_{-1}^{0} |Y(t-\theta-1)||\phi(\theta) - \phi_0(\theta)|d\theta & \leq \varepsilon M_0 M_2 e^{\mu t} + \varepsilon \frac{bM_0 M_2}{\mu}\left(1 - e^{-\mu}\right) e^{\mu t} \\ & = \varepsilon M_3 e^{\mu t}, \end{aligned} \tag{4.35}$$

with

$$M_3 = M_0 M_2 + \frac{bM_0 M_2}{\mu}\left(1 - e^{-\mu}\right).$$

For the second part, we consider the function f. We use Eqs. (4.29) and (4.31) to find,

$$\begin{aligned} & \int_0^t |Y(t-s)||f(s, y(s), y(s-1)) - f(s, \tilde{y}(s), \tilde{y}(s-1))|ds \\ \leq & \int_0^t |Y(t-s)| \Big(L_1 |y(s) - \tilde{y(s)}| + L_2 |y(s-1) - \tilde{y}(s-1)| \Big) ds \\ = & \int_0^t |Y(t-s)| L_1 |y(s) - \tilde{y(s)}| ds + \int_{-1}^{t-1} |Y(t-s-1)| L_2 |y(s) - \tilde{y}(s)| ds \\ \leq & \int_{-1}^t \left(L_1 |Y(t-s)| + L_2 |Y(t-s-1)| \right) |y(s) - \tilde{y}(s)| ds \\ \leq & \int_{-1}^t M_0 L e^{\mu(t-s)} |y(s) - \tilde{y}(s)| ds, \end{aligned} \tag{4.36}$$

with

$$L = L_1 + L_2 e^{-\mu}.$$

For the third part, we consider R. We use Eqs. (4.31) and (4.32), to find

$$\begin{aligned}\int_0^t |Y(t-s)||R(s)|ds &\leq M_0 M_1 \int_0^t e^{\mu(t-s)} e^{\mu s} ds \\ &= M_0 M_1 t e^{\mu t}.\end{aligned} \tag{4.37}$$

Using Eqs. (4.35) to (4.37), the inequality in Eq. (4.34) becomes

$$|y(t) - \tilde{y}(t)| \leq \varepsilon M_3 e^{\mu t} + \varepsilon^2 M_0 M_1 t e^{\mu t} + \int_{-1}^t \varepsilon M_0 L e^{\mu(t-s)} |y(s) - \tilde{y}(s)| ds, \tag{4.38}$$

which is equivalent to

$$e^{-\mu t} |y(t) - \tilde{y}(t)| \leq \varepsilon M_3 + \varepsilon^2 M_0 M_1 t + \int_{-1}^t \varepsilon M_0 L e^{-\mu s} |y(s) - \tilde{y}(s)| ds \tag{4.39}$$

We are considering $t = \mathcal{O}(\frac{1}{\varepsilon})$, so let $t \leq \frac{K}{\varepsilon}$, with K a positive constant. Then,

$$e^{-\mu t} |y(t) - \tilde{y}(t)| \leq \varepsilon (M_3 + M_0 M_1 K) + \int_{-1}^t \varepsilon M_0 L e^{-\mu s} |y(s) - \tilde{y}(s)| ds. \tag{4.40}$$

Grönwall [8] has shown that a function satisfying an integral inequality is bounded by the solution of the corresponding integral equation. We can use that here to obtain

$$\begin{aligned} e^{-\mu t} |y(t) - \tilde{y}(t)| &\leq \varepsilon (M_3 + M_0 M_1 K) \exp\left(\int_{-1}^t \varepsilon M_0 L ds \right) \\ &= \varepsilon (M_3 + M_0 M_1 K) e^{\varepsilon M_0 L (t+1)}. \end{aligned}$$

The right-hand side is $\mathcal{O}(\varepsilon)$ (for $t = \mathcal{O}(\frac{1}{\varepsilon})$), so

$$|y(t) - \tilde{y}(t)| = \mathcal{O}\left(\varepsilon e^{\mu t}\right). \tag{4.41}$$

We consider the absolute error, $|y(t) - \tilde{y}(t)|$, and the relative error,

$$\frac{|y(t) - \tilde{y}(t)|}{e^{\mu t}}.$$

For $\mu < 0$ the absolute and the relative error both stay small. For $\mu \geq 0$, we only consider the relative error as the exact solution and approximation are unbounded.

We summarize the results of this section in the following theorem.

Theorem 4.1. *Let $y(t)$ satisfy Eq. (4.23) and $\tilde{y}(t)$ Eq. (4.24). Assume f satisfies Eq. (4.29), let μ be defined as in Eq. (4.30), let $Y(t)$ be the fundamental solution, which satisfies Eq. (4.31), assume $R(t)$ satisfies Eq. (4.32), and assume that $|y(t) - \tilde{y}(t)| = \mathcal{O}(\varepsilon)$ for $t \in [-1, 0]$. Then, the absolute error between y and $\tilde{y}$ is $\mathcal{O}(\varepsilon)$ for $\mu < 0$, and the relative error between y and $\tilde{y}$ is $\mathcal{O}(\varepsilon)$ for all μ, on a timescale $t = \mathcal{O}(\frac{1}{\varepsilon})$.*

In this chapter we consider first order DDEs. We have already considered the solution for higher order DDEs in Chapter 2. In the next chapter we shall consider the method of multiple scales for higher order DDEs. We will already briefly touch upon the validity of the method of multiple scales for higher order DDEs.

4.5.1 Validity for higher order delay differential equations

We can rewrite higher order DDEs into the following system of DDEs,

$$\frac{d\boldsymbol{y}(t)}{dt} + A\boldsymbol{y}(t) + B\boldsymbol{y}(t-1) = \varepsilon \boldsymbol{f}(t, \boldsymbol{y}(t), \boldsymbol{y}(t-1)), \tag{4.42}$$

with the initial condition $\boldsymbol{y}(t) = \boldsymbol{\phi}(t)$ for $t \in [-1, 0]$. Using the method of multiple scales, we will find an approximation, $\tilde{\boldsymbol{y}}(t)$, which satisfies

$$\frac{d\tilde{\boldsymbol{y}}(t)}{dt} + A\tilde{\boldsymbol{y}}(t) + B\tilde{\boldsymbol{y}}(t-1) = \varepsilon \boldsymbol{f}(t, \tilde{\boldsymbol{y}}(t), \tilde{\boldsymbol{y}}(t-1)) + \varepsilon^2 \boldsymbol{R}(t), \tag{4.43}$$

with initial condition $\tilde{\boldsymbol{y}}(t) = \boldsymbol{\phi}(t)$ for $t \in [-1, 0]$. As for the first order case, these initial conditions are close together.

This time, we will have the same fundamental matrix $Y(t)$, such that we can write down the integral equations for $\boldsymbol{y}(t)$ and $\tilde{\boldsymbol{y}}(t)$. We can subtract these integral equations and take the norm, $\|\cdot\|$, which we define as

$$\|\boldsymbol{y}(t)\| = \max_i y_i(t),$$

with $y_i(t)$ the elements of $\boldsymbol{y}(t)$. We will get a restriction for each part of the equation and use Grönwall's lemma to obtain a restriction for $\|\boldsymbol{y}(t) - \tilde{\boldsymbol{y}}(t)\|$. This will again be of the form

$$\|\boldsymbol{y}(t) - \tilde{\boldsymbol{y}}(t)\| = \mathcal{O}\left(\varepsilon e^{\mu t}\right), \tag{4.44}$$

with μ the largest real part of all the solutions of the characteristic equation. The restriction holds for $t = \mathcal{O}(\frac{1}{\varepsilon})$. We find that the error is $\mathcal{O}(\varepsilon)$ on a timescale $t = \mathcal{O}(\frac{1}{\varepsilon})$. Whether this is the absolute and/or relative error depends on the sign of μ.

In the remainder of this chapter we shall consider examples for first order DDEs. For the example that we shall consider next, the relation between the solutions of the characteristic equation will need to be investigated.

4.6 Mathieu's Equation

We shall consider an example closely related to Mathieu's equation (see for example [13, 20] for Mathieu's equation for ordinary differential equations and [14] for the stability of a DDE version of Mathieu's equation). Mathieu's equation for ODEs is of the following form,

$$\frac{d^2y(t)}{dt^2} + (\delta + \varepsilon \cos t)y(t) = 0. \tag{4.45}$$

One assumes that $\delta = \mathcal{O}(1)$ and that ε is a small parameter. It can be shown that for certain values of δ, the solution becomes unstable. This phenomenon is called parametric resonance. We will consider the following DDE version of Mathieu's equation,

$$\frac{dy(t)}{dt} + ay(t) + by(t-1) = \varepsilon e^{\gamma t} y(t), \tag{4.46}$$

with the following initial condition,

$$y(t) = \phi(t), \qquad \text{for } t \in [-1, 0]. \tag{4.47}$$

We will set a and b, such that we can calculate all solutions of the characteristic equation. We can then find for which γ, resonance will occur. We could also consider a different forcing, such as $\varepsilon \cos(\gamma t) y(t)$. This will lead to similar behaviour. We will consider this forcing for a second order DDE in Section 5.2.

We introduce the slow timescale $\tau = \varepsilon t$, and construct an equation for $\tilde{y}(t, \tau)$. We do this exactly as before and obtain

$$\left(\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}\right) \tilde{y}(t, \tau) + a\tilde{y}(t, \tau) + bE_1\tilde{y}(t, \tau) - bE_1\Delta_\varepsilon \tilde{y}(t, \tau) = \varepsilon e^{\gamma t}\tilde{y}(t, \tau).$$

Then expand $\tilde{y}$ as $\tilde{y}(t, \tau) = y_0(t, \tau) + \varepsilon y_1(t, \tau) + \dots$, and collect like orders of ε. The $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$ equations are given by

$$\frac{\partial y_0(t, \tau)}{\partial t} + ay_0(t, \tau) + bE_1 y_0(t, \tau) = 0, \tag{4.48}$$

$$\frac{\partial y_1(t, \tau)}{\partial t} + ay_1(t, \tau) + bE_1 y_1(t, \tau) = -\frac{\partial y_0(t, \tau)}{\partial \tau} + \frac{b}{\varepsilon} E_1 \Delta_\varepsilon y_0(t, \tau) + e^{\gamma t} y_0(t, \tau). \tag{4.49}$$

We solve y_0 for t,

$$y_0(t,\tau) = \sum_{k=-\infty}^{\infty} \alpha_k(\tau) e^{\lambda_k t}, \tag{4.50}$$

with λ_k the solutions of the characteristic equation

$$\lambda + a + be^{-\lambda} = 0. \tag{4.51}$$

We assume that $\phi(t)$ is $\mathcal{O}(1)$ and has no τ dependence, such that we have a constant initial condition for α_k, $\alpha_k(\tau) = \Phi_k$ for $\tau \in [-\varepsilon, 0]$, with

$$\Phi_k = \frac{1}{1 - be^{-\lambda_k}} \left(\phi(0) - be^{-\lambda_k} \int_{-1}^{0} e^{-\lambda_k \theta} \phi(\theta) d\theta \right). \tag{4.52}$$

The right-hand side of the $\mathcal{O}(\varepsilon)$ equation then becomes

$$\sum_{k=-\infty}^{\infty} \left[\left(-\frac{d\alpha_k(\tau)}{d\tau} + \frac{b}{\varepsilon} e^{-\lambda_k} \Delta_\varepsilon \alpha_k(\tau) \right) e^{\lambda_k t} + \alpha_k(\tau) e^{(\lambda_k + \gamma)t} \right].$$

As said before, we wish to find γ, such that resonance occurs. For this we must find a solution of the characteristic equation, λ_k, such that $\lambda_k + \gamma$ is again a solution of the characteristic equation.

We must know for which solutions this is possible. To this end, we write $\lambda = \nu + i\omega$, with $\nu, \omega \in \mathbb{R}$, and split the characteristic equation into a real and imaginary part. We will only need the imaginary part of the characteristic equation,

$$\omega - be^{-\nu} \sin \omega = 0.$$

First consider $\omega = n\pi$, with $n \in \mathbb{Z}$. Then we see that this only satisfies the equation if $n = 0$. Second, consider $\omega \neq n\pi$. Then we see that we can determine ν for this value of ω. Consequently, there is only one solution with this imaginary part, so $\lambda + \gamma$ cannot be a solution as well. The only possibility is to have more than one real solutions of the characteristic equation. One can show that there are two distinct real solutions if $be^a \in (0, e^{-1})$. These are λ_{-1} and λ_0, with $\lambda_{-1} < \lambda_0$. We choose $\gamma = \lambda_{-1} - \lambda_0 < 0$, such that we will find a stable solution.

We can then write down the equations for α_k, such that there are no secular terms. For $k \neq -1$, the equations are

$$\frac{d\alpha_k(\tau)}{d\tau} = \frac{b}{\varepsilon} e^{-\lambda_k} \Delta_\varepsilon \alpha_k(\tau). \tag{4.53}$$

Note that $\alpha_k(\tau) = \Phi_k$ is a solution that satisfies the equation and initial condition for α_k. By the existence and uniqueness theorems, as stated in Chapter 1, this must be the unique solution. For $k = -1$, we get the following equation

$$\frac{d\alpha_{-1}(\tau)}{d\tau} = \frac{b}{\varepsilon} e^{-\lambda_{-1}} \Delta_\varepsilon \alpha_{-1}(\tau) + \Phi_0, \tag{4.54}$$

where we have used that $\alpha_0(\tau) = \Phi_0$. The solution is given by

$$\alpha_{-1}(\tau) = \Phi_{-1} + \sum_{l=-\infty}^{\infty} \frac{e^{\mu_l \tau}}{1 - be^{-\lambda_{-1}} e^{-\varepsilon \mu_l}} \int_0^\tau e^{-\mu_l \sigma} \Phi_0 d\sigma,$$

with μ_l the solutions of the characteristic equation

$$\mu - \frac{b}{\varepsilon} e^{-\lambda_{-1}} + \frac{b}{\varepsilon} e^{-\lambda_{-1}} e^{-\varepsilon \mu} = 0.$$

We rewrite the characteristic equation for μ into the following form,

$$\varepsilon \mu - be^{-\lambda_{-1}} = -be^{-\lambda_{-1}} e^{-\varepsilon \mu}.$$

Then use the characteristic equation for λ, Eq. (4.51), to see that $-be^{-\lambda_{-1}} = \lambda_{-1} + a$, such that the characteristic equation for μ becomes

$$\varepsilon \mu + \lambda_{-1} + a = (\lambda_{-1} + a) e^{-\varepsilon \mu}.$$

We can rewrite this into the following form,

$$(\varepsilon\mu + \lambda_{-1} + a)\, e^{\varepsilon\mu+\lambda_{-1}+a} = (\lambda_{-1} + a)\, e^{\lambda_{-1}+a}.$$

We again use the characteristic equation for λ, this time to see that $(\lambda_{-1} + a)\, e^{\lambda_{-1}+a} = -be^a$, such that

$$(\varepsilon\mu + \lambda_{-1} + a)\, e^{\varepsilon\mu+\lambda_{-1}+a} = -be^a.$$

Note that this is the characteristic equation for λ, in rewritten form, thus we must have

$$\lambda_l + a = \varepsilon\mu_l + \lambda_{-1} + a,$$

or equivalently

$$\varepsilon\mu_l = \lambda_l - \lambda_{-1}.$$

It is then easy to see that $\mu_{-1} = 0$. We can calculate the integral in the expression for α_{-1}, to obtain

$$\alpha_{-1}(\tau) = \Phi_{-1} + \frac{\Phi_0}{1 - be^{-\lambda_{-1}}}\tau + \sum_{\substack{l=-\infty \\ l\neq -1}}^{\infty} \frac{\varepsilon\Phi_0}{(1 - be^{-\lambda_l})(\lambda_l - \lambda_{-1})} \left(e^{\mu_l\tau} - 1\right). \tag{4.55}$$

We have also rewritten some terms, using our expression for μ_l. We can finally write down the expression for y_0,

$$y_0(t,\tau) = \sum_{k=-\infty}^{\infty} \Phi_k e^{\lambda_k t} + \frac{\Phi_0}{1 - be^{-\lambda_{-1}}}\tau e^{\lambda_{-1}t} + \sum_{\substack{l=-\infty \\ l\neq -1}}^{\infty} \frac{\varepsilon\Phi_0}{(1 - be^{-\lambda_l})(\lambda_l - \lambda_{-1})} \left(e^{\lambda_l t} - e^{\lambda_{-1}t}\right). \tag{4.56}$$

We can compare this with a numerically calculated solution.

4.6.1 Numerical Comparison

We use a very simple method to calculate a numerical solution. We discretize time, $t_i = i\Delta t$, and will calculate the numerical approximation $\hat{y}_i \approx y(t_i)$. We use the Euler forward approximation for the derivative,

$$\frac{dy(t_i)}{dt} \approx \frac{\hat{y}_{i+1} - \hat{y}_i}{\Delta t}. \tag{4.57}$$

For the delay term we use $y(t-1) \approx \hat{y}_{i-\frac{1}{\Delta t}}$. Eq. (4.46), becomes the following equation for $\hat{y}$

$$\frac{\hat{y}_{i+1} - \hat{y}_i}{\Delta t} + a\hat{y}_i + b\hat{y}_{i-\frac{1}{\Delta t}} = \varepsilon e^{\gamma t_i}\hat{y}_i.$$

We can rewrite this into the following form,

$$\hat{y}_{i+1} = \hat{y}_i + \Delta t\left(-a\hat{y}_i - b\hat{y}_{i-\frac{1}{\Delta t}} + \varepsilon e^{\gamma t_i}\hat{y}_i\right). \tag{4.58}$$

We start with a vector for $t \in [-1, 0]$, for which we have the initial condition. Then we iterate and calculate $\hat{y}_{i+1}$, using the previous values.

Figure 4.1 shows a plot of the numerical solution, together with the approximation. The two plots are close together.

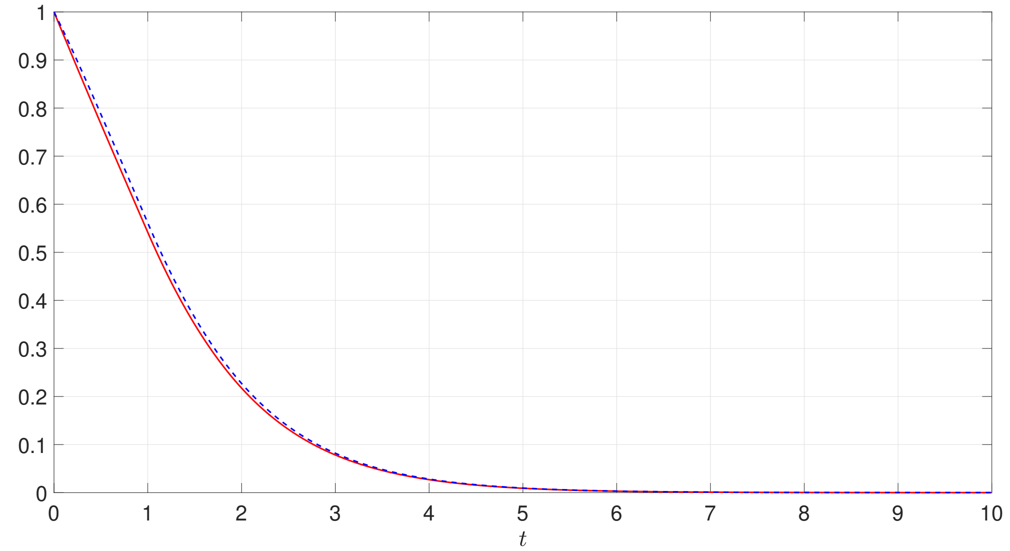


Figure 4.1: Plot of a numerically calculated solution, solid (red) line, and an approximation of the solution, dashed (blue) line, of Eq. (4.46), with initial condition Eq. (4.47) with $\phi(t) = e^t$. For the parameters, $a = \frac{1}{2}$, $b = \frac{1}{5}$, $\varepsilon = 0.1$ and $\gamma = \lambda_{-1} - \lambda_0$ have been used.

As the numerical solution and the approximation, both go to zero, the absolute error will go to zero as well. Instead, we will consider the relative error. Figure 4.2 shows a plot of the relative error. One can clearly distinguish the linear line, which corresponds to the term $\frac{\Phi_0}{1-be^{-\lambda_{-1}}}\tau e^{\lambda_{-1}t}$ in the approximation. Also note that the relative error is $\mathcal{O}(\varepsilon)$ on a timescale $t = \mathcal{O}(\frac{1}{\varepsilon})$.

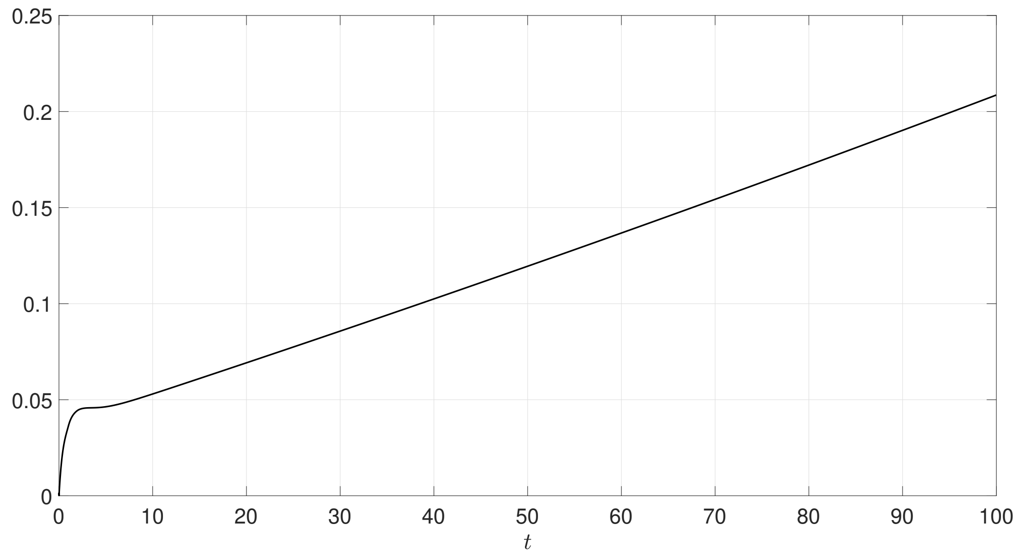


Figure 4.2: Plot of the relative error of the approximation for the solution of the DDE Eq. (4.46), with initial condition Eq. (4.47) with $\phi(t) = e^t$. For the parameters, $a = \frac{1}{2}$, $b = \frac{1}{5}$, $\varepsilon = 0.1$ and $\gamma = \lambda_{-1} - \lambda_0$ have been used.

In this example we had to answer the question whether $\lambda+\gamma$ could again be a solution of the characteristic equation. In the next example, we will have to answer the question whether the sum of two solution of the characteristic equation can be a solution of the characteristic equation.

4.7 Quadratic Perturbation

We shall consider the following DDE with a quadratic perturbation,

$$\frac{dy(t)}{dt} + ay(t) + by(t-1) = \varepsilon y(t)^2, \tag{4.59}$$

with the following initial condition,

$$y(t) = \phi(t), \qquad \text{for } t \in [-1, 0].$$

The beginning of the analysis is as before. We again find,

$$y_0(t, \tau) = \sum_{k=-\infty}^{\infty} \alpha_k(\tau) e^{\lambda_k t},$$

with λ_k the solution of the characteristic equation,

$$\lambda + a + be^{-\lambda} = 0. \tag{4.60}$$

We have a constant initial condition for α_k, $\alpha_k(\tau) = \Phi_k$ for $\tau \in [-\varepsilon, 0]$, with

$$\Phi_k = \frac{1}{1 - be^{-\lambda_k}} \left(\phi(0) - be^{-\lambda_k} \int_{-1}^{0} e^{-\lambda_k \theta} \phi(\theta) d\theta \right). \tag{4.61}$$

The right-hand side of the $\mathcal{O}(\varepsilon)$ is of the following form,

$$\sum_{k=-\infty}^{\infty} \left[\left(-\frac{d\alpha_k(\tau)}{d\tau} + \frac{b}{\varepsilon} e^{-\lambda_k} \Delta_\varepsilon \alpha_k(\tau) \right) e^{\lambda_k t} + \sum_{l=-\infty}^{\infty} \alpha_k(\tau) \alpha_l(\tau) e^{(\lambda_k + \lambda_l)t} \right]. \tag{4.62}$$

We see that we must determine whether $\lambda_k + \lambda_l$ can again be a solution of the characteristic equation. This is discussed in detail in Appendix C. One can tune the parameters a and b, such that the sum of two conjugate solutions is exactly a real solution, either λ_{-1} or λ_0. As λ_{-1} and λ_0 are greater than the real parts of all other solutions, this will only occur for cases for which some of the solutions of the characteristic equations have positive real part. Thus, we can only consider a problem, which is already unstable in the unperturbed case. We will continue with the case $\lambda_1 + \lambda_{-2} = \lambda_0$, and there is no other possibility to have that the sum of two solutions of the characteristic equation is again a solution of the characteristic equation. Then for $k \neq 0$ we have the equations

$$\frac{d\alpha_k(\tau)}{d\tau} = \frac{b}{\varepsilon} e^{-\lambda_k} \Delta_\varepsilon \alpha_k(\tau), \tag{4.63}$$

such that $\alpha_k(\tau) = \Phi_k$. For $k = 0$ we have the following equation,

$$\frac{d\alpha_0(\tau)}{d\tau} = \frac{b}{\varepsilon} e^{-\lambda_0} \Delta_\varepsilon \alpha_0(\tau) + 2\Phi_1 \Phi_{-2}. \tag{4.64}$$

The solution is given by

$$\alpha_0(\tau) = \Phi_0 + \sum_{l=-\infty}^{\infty} \frac{e^{\mu_l \tau}}{1 - be^{-\lambda_0} e^{-\mu_l}} \int_0^{\tau} e^{-\mu_l \sigma} 2\Phi_1 \Phi_{-2} d\sigma, \tag{4.65}$$

with μ_l the solutions of the characteristic equation

$$\mu = \frac{b}{\varepsilon} e^{-\lambda_0} \left(1 - e^{-\varepsilon \mu} \right).$$

As in the previous example, $\varepsilon \mu_l = \lambda_l - \lambda_0$. We can use this to work out the expression for α_0,

$$\alpha_0(\tau) = \Phi_0 + \frac{2\Phi_1 \Phi_{-2}}{1 - be^{-\lambda_0}} \tau + \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} \frac{2\varepsilon \Phi_{-2} \Phi_1}{(1 - be^{-\lambda_l})(\lambda_l - \lambda_0)} \left(e^{\mu_l \tau} - 1 \right). \tag{4.66}$$

The approximation is given by

$$y_0(t,\tau) = \sum_{k=-\infty}^{\infty} \Phi_k e^{\lambda_k t} + \frac{2\Phi_1\Phi_{-2}}{1-be^{-\lambda_0}}\tau e^{\lambda_0 t} + \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \frac{2\varepsilon\Phi_{-2}\Phi_1}{\left(1-be^{-\lambda_l}\right)\left(\lambda_l-\lambda_0\right)}\left(e^{\lambda_l\tau}-e^{\lambda_0 t}\right). \tag{4.67}$$

Finally, we will consider an example for which we will have to introduce a different timescale.

4.8 DDE perturbation on an ODE

In the beginning of this chapter we mentioned that there already exists a method of multiple scales for certain kinds of DDEs, for which we will only solve ODEs. In this section we will consider when this method fails, and in which way we can use our method to obtain a good approximation for these cases. For this we will have to introduce a timescale that we have not seen in the previous examples.

We consider the equation

$$\frac{dy(t)}{dt} + ay(t) = \varepsilon y(t-1), \tag{4.68}$$

with ε a small parameter. We use the following initial condition,

$$y(t) = \phi(t), \qquad \text{for } t \in [-1, 0]. \tag{4.69}$$

We will first state the exact solution. Then we will use the method as described by Erneux [7] to obtain an approximation, and show when this method fails. Finally we will obtain an accurate approximation using our method.

4.8.1 Exact solution

The exact solution is given by

$$y(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\lambda_k t}, \tag{4.70}$$

with λ_k the solutions of

$$\lambda + a = \varepsilon e^{-\lambda}, \tag{4.71}$$

and

$$\alpha_k = \frac{1}{1+\varepsilon e^{-\lambda_k}}\left(\phi(0) + \varepsilon e^{-\lambda_k}\int_{-1}^{0} e^{-\lambda_k\theta}\phi(\theta)d\theta\right). \tag{4.72}$$

We can group the solutions of the characteristic equation according to their dependence on ε. There is one $\mathcal{O}(1)$ solution. To obtain an approximation for this solution, we can expand it, $\lambda = \lambda_0 + \varepsilon\lambda_1 + \ldots$, and collect like orders of ε in the characteristic equation. We obtain

$$\lambda = -a + \varepsilon e^a + \ldots. \tag{4.73}$$

To find the other orders, we first write $\mu = \lambda + a$. The equation for μ is given by

$$\mu = \varepsilon e^a e^{-\mu}.$$

As $\mu \in \mathbb{C}$, we can write $\mu = re^{i\theta}$, with $r \geq 0$ and $\theta \in (-\pi, \pi]$. Note that $\mu = 0$ is not a solution, so we do not have to consider $r = 0$. The equation for μ becomes the following equation,

$$re^{i\theta} = \varepsilon e^a e^{-r\cos\theta}e^{-ir\sin\theta}.$$

We assume that $\varepsilon > 0$, such that

$$r = \varepsilon e^a e^{-r\cos\theta},$$

and

$$\theta = -r\sin\theta + 2k\pi,$$

with $k \in \mathbb{Z}$. We can rewrite these equations into the following form,

$$\begin{aligned} r\cos\theta &= \ln \varepsilon e^a - \ln r, \\ r\sin\theta &= -\theta + 2k\pi. \end{aligned}$$

For the first equation, we are interested in the balance of the right-hand side. For $r \sim \varepsilon$ and $r \sim \frac{1}{\varepsilon}$, $\ln \varepsilon e^a$ and $\ln r$ balance. For r, in between these, $\ln \varepsilon e^a$ is dominant. One can show that $r \sim \varepsilon$ is not an option. We have already seen that there is one $\mathcal{O}(1)$ solution of the characteristic equation. We will now consider a solution with r greater than $\mathcal{O}(1)$, but smaller than $\mathcal{O}(\frac{1}{\varepsilon})$. We neglect θ in the second equation, such that we have the following system,

$$\begin{aligned} r\cos\theta &= \ln \varepsilon e^a, \\ r\sin\theta &= 2k\pi. \end{aligned}$$

Note that $\ln \varepsilon e^a$ is negative, such that the real part of the solution, $r\cos\theta$, will be negative. Squaring both equations and adding them yields the following equation for r,

$$r^2 = \ln^2 \varepsilon + 4k^2\pi^2. \tag{4.74}$$

We can solve this and use a Taylor expansion for the square root in the expression. We then have,

$$r \sim -\ln \varepsilon \left(1 + \frac{1}{2}\left(\frac{2k\pi}{\ln \varepsilon}\right)^2\right). \tag{4.75}$$

So, to a first order, another group of solutions of the characteristic equation is found for $r \sim -\ln \varepsilon$. These are the smallest solutions after the $\mathcal{O}(1)$ solution. We will use this group to construct a new timescale, which we will use to obtain an approximation.

We continue with the approximations. We start with the simple approximations and show for which cases these fail.

4.8.2 Simple approximations

We start with a simple expansion,

$$y(t) \sim y_0(t) + \varepsilon y_1(t) + \dots.$$

The $\mathcal{O}(1)$ equation is an ODE,

$$\frac{dy_0(t)}{dt} + a y_0(t) = 0.$$

Note that we can only satisfy an initial condition at a single point in time, instead of satisfying the initial condition on the interval $t \in [-1, 0]$. We choose to satisfy the initial condition at $t = 0$. If $\phi(0) = \mathcal{O}(1)$, then

$$y_0(t) = \phi(0)e^{-at}.$$

Else we get $y_0(t) = 0$. The $\mathcal{O}(\varepsilon)$ equation is given by

$$\frac{dy_1(t)}{dt} + a y_1(t) = \phi(0)e^a e^{-at}.$$

The solution is given by

$$y_1(t) = \phi(0)e^a t e^{-at}.$$

So, we have the following approximation

$$y(t) \sim \phi(0)\left(1 + \varepsilon e^a t\right)e^{-at}.$$

Note that we have a secular term, so we use the method of multiple scales. We do this as described by Erneux [7]. We introduce the slow timescale $\tau = \varepsilon t$, and derive the following equation for $\tilde{y}(t,\tau) = y(t)$,

$$\left(\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}\right)\tilde{y}(t,\tau) + a\tilde{y}(t,\tau) = \varepsilon \tilde{y}(t-1, \tau - \varepsilon).$$

Whereas we have kept all the delays throughout the analysis, Erneux [7] uses a Taylor expansion for the second input,

$$\tilde{y}(t-1,\tau-\varepsilon) \sim \tilde{y}(t-1,\tau) - \varepsilon\frac{\partial \tilde{y}(t-1,\tau)}{\partial \tau} + \dots.$$

We expand $\tilde{y}$, $\tilde{y}(t,\tau) = y_0(t,\tau) + \varepsilon y_1(t,\tau) + \dots$. We collect the like orders of ε, and obtain the following $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$ equations,

$$\begin{aligned}
&\frac{\partial y_0(t,\tau)}{\partial t} + ay_0(t,\tau) = 0,\\
&\frac{\partial y_1(t,\tau)}{\partial t} + ay_1(t,\tau) = -\frac{\partial y_0(t,\tau)}{\partial \tau} + y_0(t-1,\tau).
\end{aligned}$$

We can solve the equation for y_0 for t,

$$y_0(t,\tau) = A(\tau)e^{-at}.$$

The initial condition for y_0 becomes an initial condition for A. Note that we run into the same problems as for the simple expansion, we cannot satisfy the initial condition on the entire interval. We will use $A(0) = \phi(0)$. Using the expression for y_0, the right-hand side of the $\mathcal{O}(\varepsilon)$ equation becomes

$$\left(-\frac{dA(\tau)}{d\tau} + e^a A(\tau)\right)e^{-at}.$$

To avoid secular terms, we must require

$$A(\tau) = \phi(0)e^{e^a\tau}.$$

We find the following approximation,

$$\begin{aligned}
y_0(t,\tau) &= \phi(0)e^{e^a\tau}e^{-at}\\
&= \phi(0)e^{(-a+\varepsilon e^a)t}.
\end{aligned} \tag{4.76}$$

Note that using the simple expansion, we got a term of the form e^{-at}. Using the method of multiple scales, we got $e^{(-a+\varepsilon e^a)t}$. The exponents correspond to a first and second order approximation of the $\mathcal{O}(1)$ solution of the characteristic equation, respectively. As the real part of all other solutions are large and negative, these will damp out quickly, and one might expect these approximations to hold very well. We will first consider an example for which they do, and then an example for which they do not. This depends on the value of the initial condition at $t = 0$.

We first use $\phi(t) = e^t$, such that $\phi(0) = 1$. Figure 4.3 shows a plot of the exact solution and the two approximations for this initial condition. One can clearly see that the error of both solutions is $\mathcal{O}(\varepsilon)$ for all t. This is also caused by the fact that all solutions go to zero for increasing t. If we would plot the relative error, then we would see differences between the two approximations. In this case, the approximations work as one would like them to.

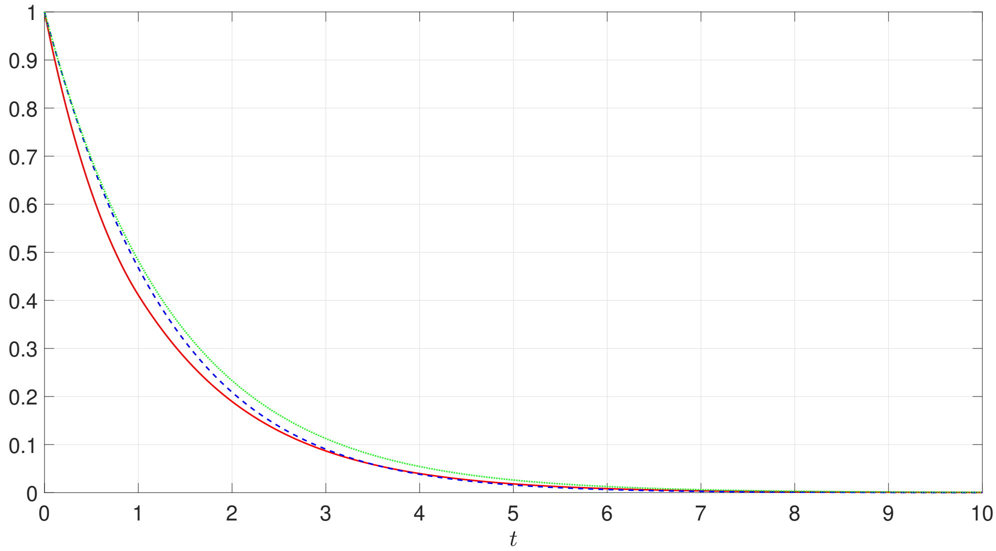


Figure 4.3: Plot of the exact solution, solid (red) line, of Eq. (4.68) with initial condition Eq. (4.69) with $\phi(t) = e^t$. Also, the approximation using a simple expansion, dashed (blue) line, and the approximation using a simple method of multiple scales, dotted (green) line, have been plotted. For the parameters $a = 1$ and $\varepsilon = 0.1$ have been used.

Now consider the case, $\phi(t) = \sin(\pi t)$, such that $\phi(0) = 0$. As a consequence, our approximations will be $y \sim 0$. Figure 4.4 shows a plot of the exact solution for this initial condition. Note that this solution has a small amplitude, so the absolute error of the approximations will be small. However, the approximations give no information about the behaviour of the exact solution. Furthermore, the relative error will always be 1. We conclude that these simple approximations fail, when the initial condition at $t = 0$ is small. We have shown this here with the extreme case $\phi(0) = 0$, but we also find this if we would have $\phi(0) = \mathcal{O}(\varepsilon)$. That this is the case can be seen by investigating the constants in the exact solution, Eq. (4.72). In between the brackets we have the terms $\phi(0)$ and $\varepsilon e^{-\lambda_k} \int_{-1}^{0} e^{-\lambda_k \theta} \phi(\theta) d\theta$. If $\phi(\theta) = \mathcal{O}(1)$ for $\theta \in [-1, 0]$, then the first terms is dominant, so we only make a small error. If $\phi(0) = \mathcal{O}(\varepsilon)$, while $\phi(\theta) = \mathcal{O}(1)$ for $\theta \in [-1, 0)$, then the two terms balance, so we cannot only consider $\phi(0)$. For smaller $\phi(0)$, the second term becomes the dominant term.

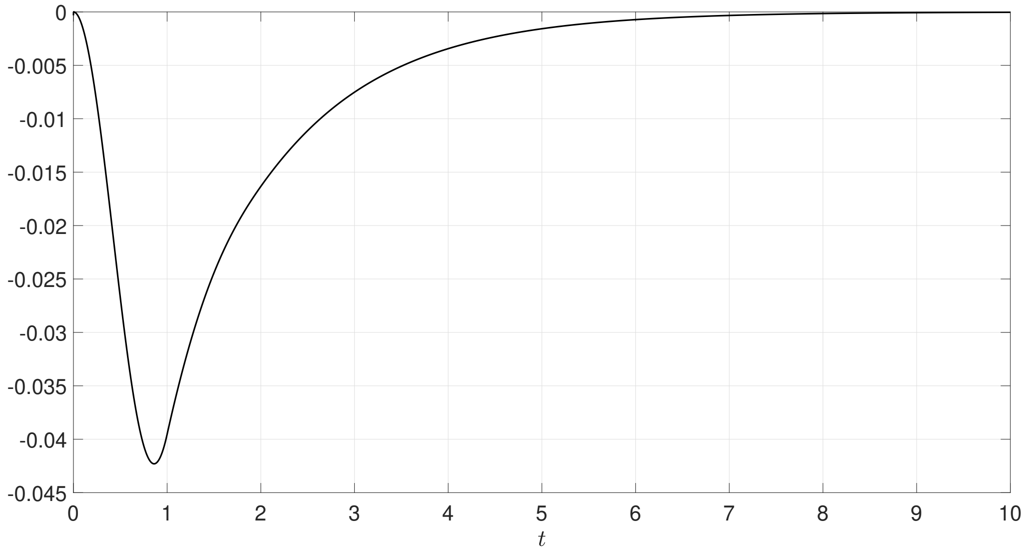


Figure 4.4: Plot of the exact solution of Eq. (4.68) with initial condition Eq. (4.69) with $\phi(t) = \sin(\pi t)$. For the parameters $a = 1$ and $\varepsilon = 0.1$ have been used.

One could state that only considering e^{at} is still a good approximation, but that we need to find the appropriate constant. Figure 4.5 shows a plot of the exact solution, together with $\alpha_0 e^{\lambda_0 t}$. We see that after some time, this is a good approximation. However, in the beginning we really need the other branches of the solution to obtain an accurate approximation.

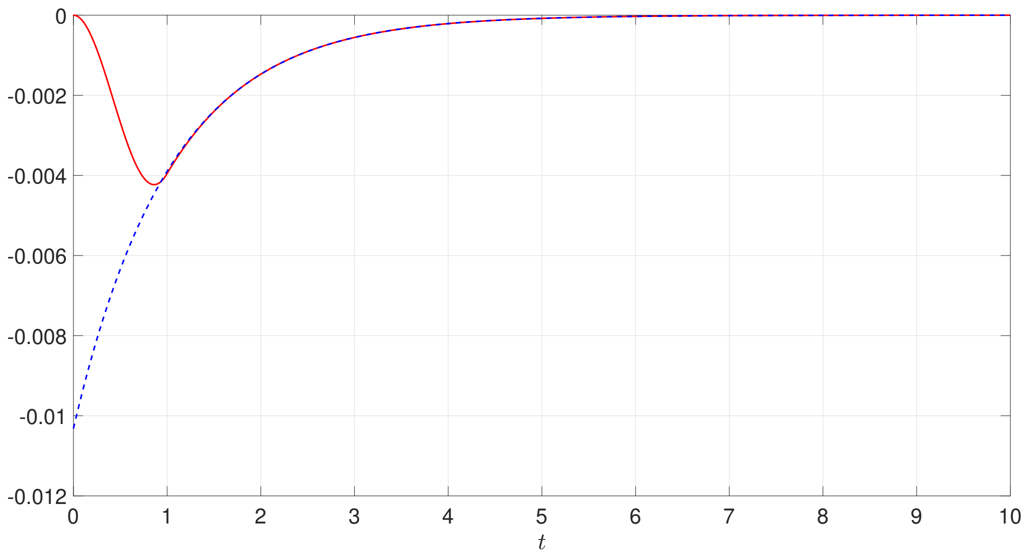


Figure 4.5: Plot of the exact solution, solid (red) line, and the approximation using only one branch of the exact solution, dashed (blue) line. Initial condition $\phi(t) = \sin(\pi t)$ and parameters $a = 1$ and $\varepsilon = 0.1$ have been used.

Now that we have established that these simple approximations fail when the initial condition at $t = 0$ is small, we will consider a better method for these cases. For this we will use the method of multiple scales, but with a fast timescale, instead of a slow timescale.

4.8.3 Approximations with a different timescale

The timescale that we use, depends on the solutions of the characteristic equation. We have previously shown that there is one $\mathcal{O}(1)$ solution. We found that the simple approximations only used this $\mathcal{O}(1)$ solution. We also wish to include the $\mathcal{O}(\ln\varepsilon)$ solutions that we found. Define

$$\delta = -\frac{1}{\ln\varepsilon}, \tag{4.77}$$

and the timescale

$$s = \frac{t}{\delta}. \tag{4.78}$$

Note that this is a fast timescale, whereas we have only considered adding the slow timescale $\tau = \varepsilon t$ before. We will again have to obtain an equation for $\tilde{y}(t,s) = y(t)$. We will need partial operators as before,

$$\begin{aligned}
E_1\tilde{y}(t,s) &= \tilde{y}(t-1,s),\\
\Delta_1\tilde{y}(t,s) &= \tilde{y}(t,s) - \tilde{y}(t-1,s),\\
E_{\frac{1}{\delta}}\tilde{y}(t,s) &= \tilde{y}\left(t, s-\frac{1}{\delta}\right),\\
\Delta_{\frac{1}{\delta}}\tilde{y}(t,s) &= \tilde{y}(t,s) - \tilde{y}\left(t, s-\frac{1}{\delta}\right).
\end{aligned}$$

Previously, we assumed that $\Delta_\varepsilon\tilde{y}(t,\tau) = \mathcal{O}(\varepsilon\tilde{y}(t,\tau))$. This was not a strange assumption, as we only move a little bit in time. It is harder to see what order we have for $\Delta_{\frac{1}{\delta}}$. We found δ by balancing the terms in the characteristic equation, so we will use this balance for the assumption

$$\Delta_{\frac{1}{\delta}}\tilde{y}(t,s) = \mathcal{O}\left(\frac{1}{\varepsilon\delta}\tilde{y}(t,s)\right).$$

The equation for $\tilde{y}(t,s)$ is

$$\left(\frac{\partial}{\partial t} + \frac{1}{\delta}\frac{\partial}{\partial s}\right)\tilde{y}(t,s) + a\tilde{y}(t,s) = \varepsilon E_1\tilde{y}(t,s) - \varepsilon E_1\Delta_{\frac{1}{\delta}}\tilde{y}(t,s).$$

We rewrite this and indicate the orders of each term,

$$\underbrace{\delta\frac{\partial\tilde{y}(t,s)}{\partial t}}_{\mathcal{O}(\delta\tilde{y}(t,s))} + \underbrace{\frac{\partial\tilde{y}(t,s)}{\partial s}}_{\mathcal{O}(\tilde{y}(t,s))} + \underbrace{a\delta\tilde{y}(t,s)}_{\mathcal{O}(\delta\tilde{y}(t,s))} = \underbrace{\varepsilon\delta E_1\tilde{y}(t,s)}_{\mathcal{O}(\varepsilon\delta\tilde{y}(t,s))} - \underbrace{\varepsilon\delta E_1\Delta_{\frac{1}{\delta}}\tilde{y}(t,s)}_{\mathcal{O}(\tilde{y}(t,s))}.$$

We then introduce the expansion $\tilde{y}(t,s) = y_0(t,s) + \delta y_1(t,s) + \ldots$ and obtain the following $\mathcal{O}(1)$ and $\mathcal{O}(\delta)$ equations,

$$\begin{aligned}
\frac{\partial y_0(t,s)}{\partial s} + \varepsilon\delta E_1\Delta_{\frac{1}{\delta}}y_0(t,s) &= 0,\\
\frac{\partial y_1(t,s)}{\partial s} + \varepsilon\delta E_1\Delta_{\frac{1}{\delta}}y_1(t,s) &= -\frac{\partial y_0(t,s)}{\partial t} - ay_0(t,s).
\end{aligned}$$

Note that when we used the slow timescale $\tau = \varepsilon t$, τ would only appear as a parameter in the $\mathcal{O}(1)$ equation. We could then solve this equation for t, and solve for τ by eliminating secular terms in the next equation. In this case, both t and s appear as variables in the $\mathcal{O}(1)$ equation, so we have a partial delay differential equation, PDDE. We will solve this using the method of separation of variables (as done by Van Horssen [25] for partial difference equations, PΔEs). We assume that we can write $y_0(t,s) = T_0(t)S_0(s)$. The $\mathcal{O}(1)$ equation then becomes

$$T_0(t)\frac{dS_0(s)}{ds} + \varepsilon\delta E_1T_0(t)\Delta_{\frac{1}{\delta}}S_0(s) = 0.$$

We can rewrite this equation into the following form,

$$\frac{T_0(t)}{T_0(t-1)} = -\varepsilon\delta \frac{S_0(s) - S_0\left(s - \frac{1}{\delta}\right)}{\frac{dS_0(s)}{ds}}.$$

The left-hand side depends on t and the right-hand side depends on s, so they both must be equal to some separation constant, ζ. We then have two equations,

$$T_0(t) = \zeta T_0(t-1),$$

and

$$\frac{dS_0(s)}{ds} + \frac{\varepsilon\delta}{\zeta} S_0(s) - \frac{\varepsilon\delta}{\zeta} S_0\left(s - \frac{1}{\delta}\right) = 0.$$

We first solve for T_0. We will use $S_0(s)$ to satisfy the initial condition. We find,

$$T_0(t) = \zeta^t.$$

With this expression, the right-hand side of the $\mathcal{O}(\delta)$ equation becomes

$$(-\ln\zeta - a)\, T_0(t) S_0(s).$$

Note that $T_0(t)S_0(s)$ will cause secular behaviour, so we must require

$$\zeta = e^{-a}.$$

The solution for T_0 then becomes

$$T_0(t) = e^{-at}. \tag{4.79}$$

We recognize that this corresponds to the $\mathcal{O}(1)$ solution of the characteristic equation. The equation for S_0 becomes

$$\frac{dS_0(s)}{ds} + \varepsilon\delta e^a S_0(s) - \varepsilon\delta e^a S_0\left(s - \frac{1}{\delta}\right) = 0.$$

To be able to solve this, we need an initial condition for $S_0(s)$. Using the initial condition for $y_0(t, s)$, we must have

$$T_0(t)S_0(s) = \phi(t).$$

We can rewrite this as an initial condition for S_0,

$$S_0(s) = \tilde{\phi}(s), \qquad \text{for } s \in \left[-\frac{1}{\delta}, 0\right],$$

with

$$\tilde{\phi}(s) = e^{a\delta s}\phi(\delta s).$$

The solution for S_0 is then given by

$$S_0(s) = \sum_{k=-\infty}^{\infty} \beta_k e^{\mu_k s}, \tag{4.80}$$

with μ_k the solutions of the characteristic equation

$$\mu + \varepsilon\delta e^a - \varepsilon\delta e^a e^{-\mu/\delta} = 0, \tag{4.81}$$

and the constants are given by

$$\beta_k = \frac{1}{1 + \varepsilon e^a e^{-\mu_k/\delta}} \left(\tilde{\phi}(0) + \varepsilon\delta e^a e^{-\mu_k/\delta} \int_{-\frac{1}{\delta}}^{0} e^{-\mu_k \eta} \tilde{\phi}(\eta) d\eta \right). \tag{4.82}$$

Our approximation is given by

$$\begin{aligned} y_0(t,s) &= \sum_{k=-\infty}^{\infty} \beta_k e^{\mu_k s} e^{-at} \\ &= \sum_{k=-\infty}^{\infty} \beta_k e^{\left(\frac{\mu_k}{\delta}-a\right)t}. \end{aligned} \tag{4.83}$$

We will compare this approximation with the exact solution. We have the characteristic equations, as in Eqs. (4.71) and (4.81). It is easy to show that

$$\frac{\mu_k}{\delta} = \lambda_k + a + \mathcal{O}(\varepsilon). \tag{4.84}$$

Just put this into the characteristic equation for μ and work it out into the following form,

$$\lambda + a + \mathcal{O}(\varepsilon) + \varepsilon e^{a} = \varepsilon e^{-\lambda}\left(1 + \mathcal{O}(\varepsilon)\right). \tag{4.85}$$

The $\mathcal{O}(1)$ part of this equation is the characteristic equation for λ, so this approximation indeed holds. For β_k, we will use $\theta = \delta\eta$ in the integral, together with the approximation for μ_k, and obtain

$$\beta_k = \frac{1}{1+\varepsilon e^{-\lambda_k}}\left(\phi(0) + \varepsilon e^{-\lambda_k}\int_{-1}^{0} e^{-\lambda_k\theta}\phi(\theta)d\theta\right)\left(1+\mathcal{O}(\varepsilon)\right). \tag{4.86}$$

Note that the first part is equal to α_k, Eq. (4.72). We then find

$$y_0(t,s) = \sum_{k=-\infty}^{\infty} \alpha_k e^{(\lambda_k + \mathcal{O}(\varepsilon))t}\left(1+\mathcal{O}(\varepsilon)\right). \tag{4.87}$$

For $t = \mathcal{O}(1)$,

$$y_0(t,s) = y(t)\left(1+\mathcal{O}(\varepsilon)\right), \tag{4.88}$$

so the relative error is $\mathcal{O}(\varepsilon)$.

Figure 4.6 shows a plot of the exact solution and the approximation. The approximation describes the behaviour of the exact solution very well.

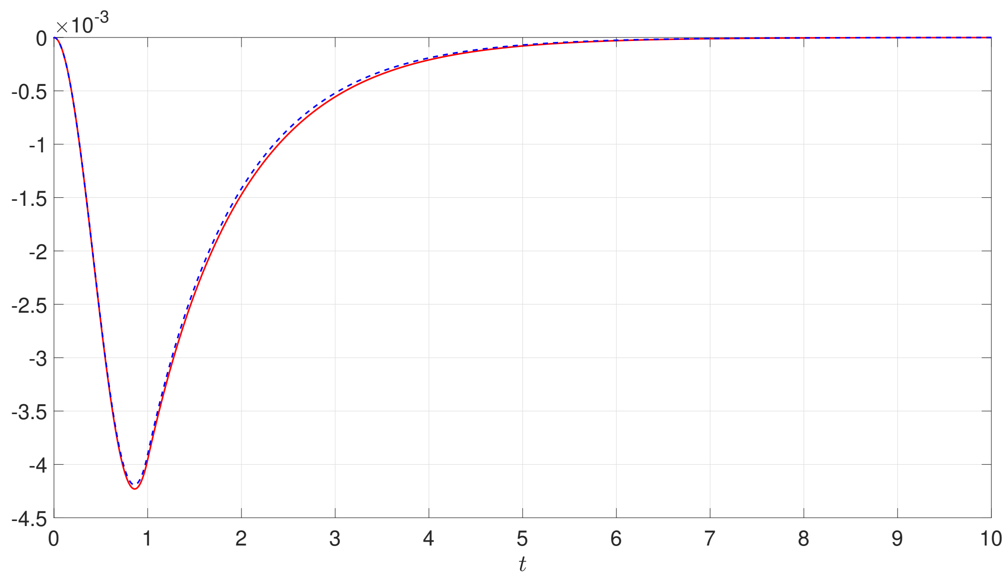


Figure 4.6: Plot of the exact solution, solid (red) line, and an approximation of the solution, dashed (blue) line, of Eq. (4.68) with initial condition Eq. (4.69) with $\phi(t) = \sin(\pi t)$. For the parameters, $a = 1$ and $\varepsilon = 0.01$ have been used.

Our approximation is only valid for $t = \mathcal{O}(1)$. We can also construct an approximation that is valid on a longer timescale.

4.8.4 Three timescales

If we wish to have a valid approximation for $t = \mathcal{O}(\frac{1}{\varepsilon})$, we also have to include the timescale $\tau = \varepsilon t$. We then obtain the following equation for $\tilde{y}(t, s, \tau)$ (omitting the (t, s, τ)),

$$\underbrace{\frac{\partial \tilde{y}}{\partial s}}_{\mathcal{O}(\tilde{y})} + \underbrace{\varepsilon\delta E_1 \Delta_{\frac{1}{\delta}} \tilde{y}}_{\mathcal{O}(\tilde{y})} = -\underbrace{\delta \frac{\partial \tilde{y}}{\partial t}}_{\mathcal{O}(\delta\tilde{y})} - \underbrace{\varepsilon\delta \frac{\partial \tilde{y}}{\partial \tau}}_{\mathcal{O}(\varepsilon\delta\tilde{y})} - \underbrace{a\delta\tilde{y}}_{\mathcal{O}(\delta\tilde{y})} + \underbrace{\varepsilon\delta E_1 \tilde{y}}_{\mathcal{O}(\varepsilon\delta\tilde{y})} - \underbrace{\varepsilon\delta E_1 \Delta_\varepsilon \tilde{y}}_{\mathcal{O}(\varepsilon^2\delta\tilde{y})} + \underbrace{\varepsilon\delta E_1 \Delta_{\frac{1}{\delta}} \Delta_\varepsilon \tilde{y}}_{\mathcal{O}(\varepsilon\tilde{y})}$$

Using the expansion $\tilde{y}(t, s, \tau) = y_0(t, s, \tau) + \delta y_1(t, s, \tau) + \ldots$ yields the same equations as before with τ appearing as a parameter,

$$\begin{aligned} \frac{\partial y_0(t,s,\tau)}{\partial s} + \varepsilon\delta E_1 \Delta_{\frac{1}{\delta}} y_0(t,s,\tau) &= 0, \\ \frac{\partial y_1(t,s,\tau)}{\partial s} + \varepsilon\delta E_1 \Delta_{\frac{1}{\delta}} y_1(t,s,\tau) &= -\frac{\partial y_0(t,s,\tau)}{\partial t} - a y_0(t,s,\tau). \end{aligned}$$

This time we write $y_0(t, s, \tau) = T_0(t)S_0(s)\mathscr{T}_0(\tau)$. The solution will then be of the form,

$$y_0(t, s, \tau) = \sum_{k=-\infty}^{\infty} \beta_k e^{\mu_k s} e^{-at} \mathscr{T}_0(\tau),$$

with μ_k and β_k as for the case with timescales t and s. We have the initial condition $\mathscr{T}_0(\tau) = 1$ for $\tau \in [-\varepsilon, 0]$.

As the right-hand side of the $\mathcal{O}(\delta)$ equation becomes zero, and we have a zero initial condition for y_1, we find $y_1(t, s, \tau) = 0$. Instead we will use an expansion of the form $\tilde{y}(t, s, \tau) = y_0(t, s, \tau) + \varepsilon y_1(t, s, \tau) + \varepsilon\delta y_2(t, s, \tau) + \ldots$. Normally we would now obtain an equation for $\mathscr{T}(\tau)$ by avoiding secular terms in the $\mathcal{O}(\varepsilon)$ equation. However, as we shall see, it is better to avoid secular terms in the $\mathcal{O}(\varepsilon\delta)$ equation. We start by writing both equations down. The $\mathcal{O}(\varepsilon)$ equation is given by

$$\frac{\partial y_1(t,s,\tau)}{\partial s} + \varepsilon\delta E_1 \Delta_{\frac{1}{\delta}} y_1(t,s,\tau) = \delta E_1 \Delta_{\frac{1}{\delta}} \Delta_\varepsilon y_0(t,s,\tau),$$

and the $\mathcal{O}(\varepsilon\delta)$ equation is given by

$$\frac{\partial y_2(t,s,\tau)}{\partial s} + \varepsilon\delta E_1 \Delta_{\frac{1}{\delta}} y_2(t,s,\tau) = -\frac{\partial y_1(t,s,\tau)}{\partial t} - \frac{\partial y_0(t,s,\tau)}{\partial \tau} - a y_1(t,s,\tau) + E_1 y_0(t,s,\tau).$$

We can write down the right-hand side of the $\mathcal{O}(\varepsilon)$ equation, using our expression for y_0, and the fact that $\mu_0 = 0$,

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \delta e^a \left(1 - e^{-\frac{\mu_k}{\delta}}\right) e^{\mu_k s} e^{at} \Delta_\varepsilon \mathscr{T}_0(\tau).$$

Note that we do not get an equation for $k = 0$. This is the branch that dominates the solution for large t, so we need this equation to find a good expression for $\mathscr{T}_0(t)$. Note that the $k = 0$ part for y_1 will be zero, as the right-hand side and the initial condition are zero for this part.

The $k = 0$ part of the right-hand side of the $\mathcal{O}(\varepsilon\delta)$ equation is given by

$$\left(-\frac{d\mathscr{T}(\tau)}{d\tau} + e^a \mathscr{T}(\tau)\right) e^{-at}.$$

To avoid secular behaviour for this branch, we must require

$$\frac{d\mathscr{T}(\tau)}{d\tau} = e^a \mathscr{T}(\tau).$$

We cannot satisfy the initial condition on the interval $\tau \in [-\varepsilon, 0]$, but only on $\tau = 0$. Due to this, we will make an $\mathcal{O}(\varepsilon)$ error on this interval. The solution is given by

$$\mathscr{T}(\tau) = e^{e^a \tau}.$$

The approximation using three timescales is given by

$$\begin{aligned} y_0(t,s,\tau) &= \sum_{k=-\infty}^{\infty} \beta_k e^{\mu_k s} e^{-at} e^{e^a \tau} \\ &= \sum_{k=-\infty}^{\infty} \beta_k e^{\left(\frac{\mu_k}{\delta} - a + \varepsilon e^a\right)t}. \end{aligned} \tag{4.89}$$

That this approximation holds for $t = \mathcal{O}(1)$ is not changed by the slow timescale. For $t = \mathcal{O}(\frac{1}{\varepsilon})$ we get $y_0 \sim \beta_0 e^{(-a+\varepsilon e^a)t}$. We have already seen that $y \sim \alpha_0 e^{\lambda_0 t}$, $\beta_0 = \alpha_0(1+\mathcal{O}(\varepsilon))$ and $\lambda_0 \sim -a + \varepsilon e^a + \mathcal{O}(\varepsilon^2)$, such that

$$y_0(t,s,\tau) = y(t)\left(1 + \mathcal{O}(\varepsilon)\right),$$

for $t = \mathcal{O}(\frac{1}{\varepsilon})$.

Chapter 5

Method of Multiple Scales for a Higher Order Delay Differential Equation

In the previous chapter we considered the method of multiple scales for first order DDEs. We shall now show how the method of multiple scales works for higher order DDEs. Most of the procedure is very similar to that for the first order case. We will use this method to find an approximation for a second order Mathieu's equation. We will also consider a detuned case of this example.

5.1 Method of Multiple Scales

The equations that we will consider are of the following form,

$$\frac{d^n y(t)}{dt^n} + \sum_{i=0}^{n-1} a_i \frac{d^i y(t)}{dt^i} + \sum_{i=0}^{n-1} b_i \frac{d^i y(t-1)}{dt^i} = \varepsilon f(\dots), \tag{5.1}$$

with f a nonlinear function of y. So, we consider a linear higher order DDE, with a nonlinear perturbation. We include the following initial conditions

$$\frac{d^i y(t)}{dt^i} = \phi_i(t), \qquad \text{for } t \in [-1, 0], \qquad i = 0, \dots, n-1. \tag{5.2}$$

In Chapter 2 we transformed this into a system of DDEs. The resulting vector contained derivatives with respect to t, which results in difficulties when we add the slow timescale $\tau = \varepsilon t$. It will be simpler to first introduce the slow timescale and collect the like orders of ε to obtain the different equations. These equations can then be rewritten into a system, which we can solve using the results of Chapter 2.

The delay terms will be handled the same as for the first order case. For the ith derivative we have

$$\frac{d^i}{dt^i} = \sum_{k=0}^{i} \binom{i}{k} \frac{\partial^k}{\partial t^k} \frac{\partial^{i-k}}{\partial \tau^{i-k}}.$$

We then have the following equation for $\tilde{y}(t, \tau) = y(t)$,

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k}{\partial t^k} \frac{\partial^{n-k}}{\partial \tau^{n-k}} \tilde{y}(t,\tau) + \sum_{i=0}^{n-1} a_i \sum_{k=0}^{i} \binom{i}{k} \frac{\partial^k}{\partial t^k} \frac{\partial^{i-k}}{\partial \tau^{i-k}} \tilde{y}(t,\tau) \\
&+ \sum_{i=0}^{n-1} b_i \sum_{k=0}^{i} \binom{i}{k} \frac{\partial^k}{\partial t^k} \frac{\partial^{i-k}}{\partial \tau^{i-k}} E_1 \tilde{y}(t,\tau) - \sum_{i=0}^{n-1} b_i \sum_{k=0}^{i} \binom{i}{k} \frac{\partial^k}{\partial t^k} \frac{\partial^{i-k}}{\partial \tau^{i-k}} E_1 \Delta_\varepsilon \tilde{y}(t,\tau) = \varepsilon \tilde{f}(\dots),
\end{aligned}$$

with $\tilde{f}$ a function similar to f. We will find an approximation for $\tilde{y}(t,\tau)$ by using an expansion, $\tilde{y}(t,\tau) = y_0(t,\tau) + \varepsilon y_1(t,\tau) + \dots$. We then collect the like orders of ε. The $\mathcal{O}(1)$ equation is given by

$$\frac{\partial^n y_0(t,\tau)}{\partial t^n} + \sum_{i=0}^{n-1} a_i \frac{\partial^i y_0(t,\tau)}{\partial t^i} + \sum_{i=0}^{n-1} b_i \frac{\partial^i E_1 y_0(t,\tau)}{\partial t^i} = 0.$$

This is a higher order DDE with τ appearing as a parameter. We define the vector

$$\boldsymbol{z}_0(t,\tau) = \begin{pmatrix} y_0(t,\tau) \\ \partial_t y_0(t,\tau) \\ \vdots \\ \partial_t^{n-1} y_0(t,\tau) \end{pmatrix},$$

with $\partial_t = \dfrac{\partial}{\partial t}$. We have the following equation for $\boldsymbol{z}_0$,

$$\frac{\partial \boldsymbol{z}_0(t,\tau)}{\partial t} + A\boldsymbol{z}_0(t,\tau) + BE_1\boldsymbol{z}_0(t,\tau) = 0,$$

with

$$A = \begin{pmatrix} & -1 & & & & \\ & & -1 & & \emptyset & \\ & & & \ddots & & \\ & \emptyset & & & -1 & \\ & & & & & -1 \\ a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} & & & \emptyset & & \\ & & & & & \\ b_0 & b_1 & b_2 & \dots & b_{n-2} & b_{n-1} \end{pmatrix}.$$

We assume that $\phi_i(t) = \mathcal{O}(1)$ and has no τ dependence, for all $i = 0, \dots, n-1$, such that we find the following initial condition for $\boldsymbol{z}_0(t,\tau)$,

$$\boldsymbol{z}_0(t,\tau) = \boldsymbol{\phi}(t), \qquad \text{for } t \in [-1,0],$$

with

$$\boldsymbol{\phi}(t) = \begin{pmatrix} \phi_0(t,\tau) \\ \vdots \\ \phi_{n-1}(t,\tau) \end{pmatrix}.$$

We solve the equation for $\boldsymbol{z}_0(t,\tau)$ for t. In Chapter 2 we have shown that for a second order DDE the solution is given by

$$\boldsymbol{z}_0(t,\tau) = \sum_{k=-\infty}^{\infty} \boldsymbol{\alpha}_k(\tau) e^{\lambda_k t},$$

with λ_k the solutions of the characteristic equation

$$\lambda^2 + a_1\lambda + b_1\lambda e^{-\lambda} + a_0 + b_0 e^{-\lambda} = 0.$$

The initial condition for $\boldsymbol{z}_0(t,\tau)$ becomes an initial condition for $\boldsymbol{\alpha}_k(t)$. Due to the assumption that there is no τ dependence in the initial condition, we have

$$\boldsymbol{\alpha}_k(\tau) = \boldsymbol{\Phi}_k, \qquad \text{for } \tau \in [-\varepsilon, 0],$$

with

$$\boldsymbol{\Phi}_k = D_k \left(\boldsymbol{\phi}(0) - Be^{-\lambda_k} \int_{-1}^{0} e^{-\lambda_k \theta} \boldsymbol{\phi}(\theta) d\theta \right)$$

and

$$D_k = \frac{1}{2\lambda_k + a_1 + b_1(1-\lambda_k)e^{-\lambda_k} - b_0 e^{-\lambda_k}} \begin{pmatrix} \lambda_k & 1 \\ -a_0 - b_0 e^{-\lambda_k} & \lambda_k + a_1 + b_1 e^{-\lambda_k} \end{pmatrix}.$$

These are exactly the constants which we would find if we solved the unperturbed equation. Using the definition of $\boldsymbol{z}_0(t,\tau)$, we know that $\boldsymbol{\alpha}_k(\tau)$ is of the following form,

$$\boldsymbol{\alpha}_k(\tau) = \begin{pmatrix} \beta_k(\tau) \\ \lambda_k \beta_k(\tau) \end{pmatrix}.$$

We write $\boldsymbol{\Phi}_k = \begin{pmatrix} \Phi_k^0 \\ \Phi_k^1 \end{pmatrix}$, such that we have the following initial condition for β_k,

$$\beta_k(\tau) = \Phi_k^0, \qquad \text{for } \tau \in [-\varepsilon, 0].$$

We can calculate Φ_k^0 explicitly,

$$\Phi_k^0 = \frac{1}{2\lambda_k + a_1 + b_1(1-\lambda_k)e^{-\lambda_k} - b_0 e^{-\lambda_k}} \left(\lambda_k \phi_0(0) + \phi_1(0) - b_0 e^{-\lambda_k} \int_{-1}^{0} e^{-\lambda_k \theta} \phi_0(\theta) d\theta - b_1 e^{-\lambda_k} \int_{-1}^{0} e^{-\lambda_k \theta} \phi_1(\theta) d\theta \right).$$

We have the following solution for $y_0(t,\tau)$,

$$y_0(t,\tau) = \sum_{k=-\infty}^{\infty} \beta_k(\tau) e^{\lambda_k t}.$$

Note that we find this kind of solution for every higher order DDE, but we have only calculated the initial condition for $\beta_k(\tau)$ for a second order DDE.

We continue with the $\mathcal{O}(\varepsilon)$ equation, which is given by

$$\begin{aligned} \frac{\partial^n y_1(t,\tau)}{\partial t^n} + \sum_{i=0}^{n-1} a_i \frac{\partial^i y_1(t,\tau)}{\partial t^i} + \sum_{i=0}^{n-1} b_i \frac{\partial^i E_1 y_1(t,\tau)}{\partial t^i} =& - n \frac{\partial^{n-1}}{\partial t^{n-1}} \frac{\partial}{\partial \tau} y_0(t,\tau) - \sum_{i=1}^{n-1} i a_i \frac{\partial^{i-1}}{\partial t^{i-1}} \frac{\partial}{\partial \tau} y_0(t,\tau) \\ & - \sum_{i=1}^{n-1} i b_i \frac{\partial^{i-1}}{\partial t^{i-1}} \frac{\partial}{\partial \tau} E_1 y_0(t,\tau) + \frac{1}{\varepsilon} \sum_{i=0}^{n-1} b_i \frac{\partial^i E_1 \Delta_\varepsilon y_1(t,\tau)}{\partial t^i} \\ & + f_0(\dots), \end{aligned}$$

where we have assumed that we can write

$$\tilde{f}(\dots) = f_0(\dots) + \varepsilon f_1(\dots) + \dots.$$

We work out the right-hand side of the $\mathcal{O}(\varepsilon)$ equation,

$$\sum_{k=-\infty}^{\infty} \left[-\left(n\lambda_k^{n-1} + \sum_{i=1}^{n-1} i a_i \lambda_k^{i-1} + \sum_{i=1}^{n-1} i b_i \lambda_k^{i-1} e^{-\lambda_k} \right) \frac{d\beta_k(\tau)}{d\tau} + \sum_{i=0}^{n-1} \frac{b_i}{\varepsilon} \lambda_k^i e^{-\lambda_k} \Delta_\varepsilon \beta_k(\tau) \right] e^{\lambda_k t} + f_0(\dots).$$

If $f_0(\dots)$ does not contain the term $e^{\lambda_l t}$ for some λ_l, then we have the following equation for β_l,

$$\left(n\lambda_l^{n-1} + \sum_{i=1}^{n-1} i a_i \lambda_l^{i-1} + \sum_{i=1}^{n-1} i b_i \lambda_l^{i-1} e^{-\lambda_l} \right) \frac{d\beta_l(\tau)}{d\tau} = \sum_{i=0}^{n-1} \frac{b_i}{\varepsilon} \lambda_l^i e^{-\lambda_l} \Delta_\varepsilon \beta_l(\tau).$$

We note that $\beta_l(\tau) = \Phi_l^0$ is a solution which satisfies the equation and the initial condition for β_l. By the existence and uniqueness theorems, as stated in Chapter 1, this must be the unique solution. If $f_0(\dots)$ does contain the term $e^{\lambda_k t}$, then the equation for β_k is different.

Our general discussion ends here. Next, we consider a second order DDE version of Mathieu's equation.

5.2 Second Order Mathieu's Equation

We consider the following problem in this example,

$$\frac{d^2 y(t)}{dt^2} + a y(t) + b y(t-1) = 2\varepsilon \cos(\gamma t) y(t), \tag{5.3}$$

with

$$y(t) = \phi(t) \qquad \text{and} \qquad \frac{dy(t)}{dt} = \psi(t), \qquad \text{for } t \in [-1, 0]. \tag{5.4}$$

This is a DDE version of Mathieu's equation. Extensive research has been done on this equation and the stability of the solution. Mathieu's equation occurs when one linearises around a periodic solution of a nonlinear oscillator, and thus is of practical use. It is then very interesting to see if we can solve a DDE version of this equation.

Note that for the first order DDE version of Mathieu's equation, we considered a perturbation of the form $\varepsilon e^{\gamma t} y(t)$. We then had to answer the question whether $\lambda_k + \gamma$ could again be a solution of the characteristic equation, if λ_k was a solution of the characteristic equation. We can already see that we will have to answer the question whether $\lambda_l \pm i\gamma$ can again be a solution of the characteristic equation for this example.

As seen before, we introduce the timescale $\tau = \varepsilon t$ and write down the equation for $\tilde{y}(t, \tau) = y(t)$,

$$\left(\frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial}{\partial t} \frac{\partial}{\partial \tau} + \varepsilon^2 \frac{\partial^2}{\partial \tau^2} \right) \tilde{y}(t, \tau) + a\tilde{y}(t, \tau) + bE_1 \tilde{y}(t, \tau) - bE_1 \Delta_\varepsilon \tilde{y}(t, \tau) = 2\varepsilon \cos(\gamma t) \tilde{y}(t, \tau).$$

We will use the following expansion, $\tilde{y}(t, \tau) = y_0(t, \tau) + \varepsilon y_1(t, \tau) + \ldots$ and have equations for each order of ε. The $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$ equations are given by

$$\frac{\partial^2 y_0(t, \tau)}{\partial t^2} + a y_0(t, \tau) + bE_1 y_0(t, \tau) = 0,$$
$$\frac{\partial^2 y_1(t, \tau)}{\partial t^2} + a y_1(t, \tau) + bE_1 y_1(t, \tau) = -2 \frac{\partial}{\partial t} \frac{\partial}{\partial \tau} y_0(t, \tau) + \frac{b}{\varepsilon} E_1 \Delta_\varepsilon y_0(t, \tau) + 2\cos(\gamma t) y_0(t, \tau).$$

We will get the following solution of the $\mathcal{O}(1)$ equation,

$$y_0(t, \tau) = \sum_{k=-\infty}^{\infty} \alpha_k(\tau) e^{\lambda_k t},$$

with λ_k the solutions of the characteristic equation

$$\lambda^2 + a + be^{-\lambda} = 0,$$

and the initial conditions for $\alpha_k(\tau)$ are

$$\alpha_k(\tau) = \Phi_k, \qquad \text{for } \tau \in [-\varepsilon, 0],$$

with

$$\Phi_k = \frac{1}{2\lambda_k - be^{-\lambda_k}} \left(\lambda_k \phi(0) + \psi(0) - be^{-\lambda_k} \int_{-1}^{0} e^{-\lambda_k \theta} \phi(\theta) d\theta \right).$$

The right-hand side of the $\mathcal{O}(\varepsilon)$ is of the following form,

$$\sum_{k=-\infty}^{\infty} \left[\left(-2\lambda_k \frac{d\alpha_k(\tau)}{d\tau} + \frac{b}{\varepsilon} e^{-\lambda_k} \Delta_\varepsilon \alpha_k(\tau) \right) e^{\lambda_k t} + \alpha_k(\tau) \left(e^{(\lambda_k + i\gamma)t} + e^{(\lambda_k - i\gamma)t} \right) \right],$$

where we have used that $2\cos(\gamma t) = e^{i\gamma t} + e^{-i\gamma t}$. As stated in the beginning of this example, we must determine whether $\lambda_k \pm i\gamma$ can again be a solution of the characteristic equation. We know that if λ is a solution of the characteristic equation, then so is its conjugate $\overline{\lambda}$. If we take $\gamma = 2\text{Im}(\lambda)$, then $\overline{\lambda} + \gamma = \lambda$. To make the effect of the method of multiple scales most clear, we wish to have purely imaginary solutions of the characteristic equation. To this end, write $\lambda = i\omega$. We use the characteristic equation to obtain the following equation for ω,

$$-\omega^2 + a + b(\cos\omega - i\sin\omega) = 0.$$

To satisfy the imaginary part of this equation, we must require $\omega = n\pi$, with $n \in \mathbb{Z}$. We can use the real part of the equation to obtain a restriction on the parameters a and b,

$$a + (-1)^n b = n^2 \pi^2.$$

We will continue with $a = 2\pi^2$ and $b = \pi^2$, such that $\lambda = \pm i\pi$ are solutions of the characteristic equation. Figure 5.1 shows a plot of the solution curve, together with the actual solutions for λ. $\lambda = \pm i\pi$ are the solutions of the characteristic equation with the largest real part.

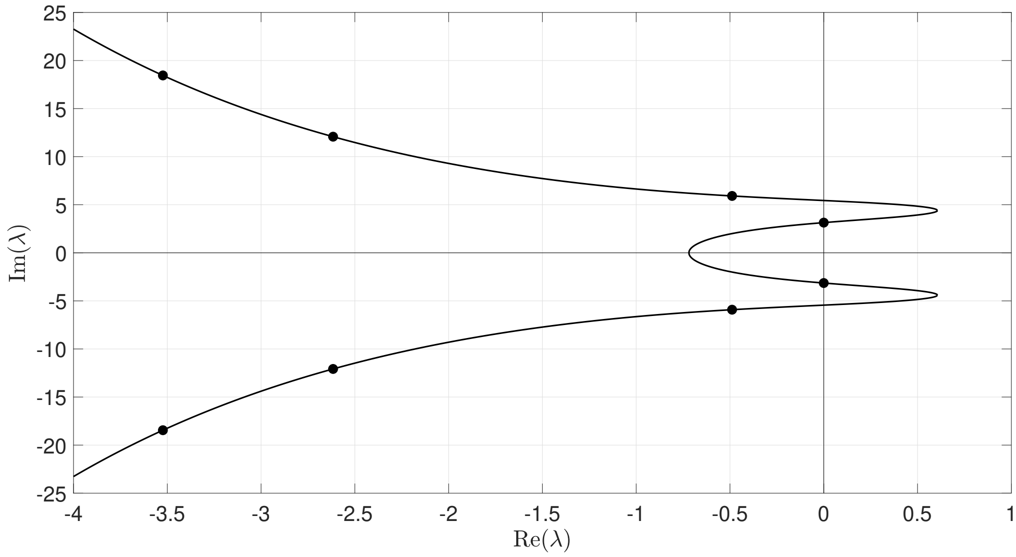


Figure 5.1: Plot of the solution curve, together with the actual solutions for λ. For the parameters $a = 2\pi^2$ and $b = \pi^2$ have been used

We define $\lambda_{-1} = -i\pi$ and $\lambda_0 = i\pi$. Let $\gamma = 2\pi$, such that $\lambda_{-1} + i\gamma = \lambda_0$. For $k \neq -1, 0$ there does not exist a λ_l such that $\lambda_l \pm i\gamma = \lambda_k$. We have seen in the general derivation that then $\alpha_k(\tau) = \Phi_k$ for all τ. For α_{-1} and α_0 we have the following equation,

$$\frac{d\alpha_k(\tau)}{d\tau} = \frac{b}{2\varepsilon\lambda_k e^{\lambda_k}}\Delta_\varepsilon \alpha_k(\tau) + \frac{1}{2\lambda_k}\alpha_l(\tau),$$

with $(k,l) = (-1,0)$ and $(k,l) = (0,-1)$. We define the vector $\boldsymbol{\beta}(\tau) = \begin{pmatrix} \alpha_{-1}(\tau) \\ \alpha_0(\tau) \end{pmatrix}$, for which we have the following equation,

$$\frac{d\boldsymbol{\beta}(\tau)}{d\tau} + \tilde{A}\boldsymbol{\beta}(\tau) + \tilde{B}\boldsymbol{\beta}(\tau - \varepsilon) = 0,$$

with

$$\tilde{A} = -\begin{pmatrix} \frac{b}{2\varepsilon\lambda_{-1}e^{\lambda_{-1}}} & \frac{1}{2\lambda_{-1}} \\ \frac{1}{2\lambda_0} & \frac{b}{2\varepsilon\lambda_0 e^{\lambda_0}} \end{pmatrix} \qquad \text{and} \qquad \tilde{B} = \begin{pmatrix} \frac{b}{2\varepsilon\lambda_{-1}e^{\lambda_{-1}}} & 0 \\ 0 & \frac{b}{2\varepsilon\lambda_0 e^{\lambda_0}} \end{pmatrix}.$$

The initial condition for $\boldsymbol{\beta}$ is given by

$$\boldsymbol{\beta}(\tau) = \boldsymbol{\Phi}, \qquad \text{for } \tau \in [-\varepsilon, 0],$$

with $\boldsymbol{\Phi} = \begin{pmatrix} \Phi_{-1} \\ \Phi_0 \end{pmatrix}$. Define the matrix

$$\tilde{C} = \mu I + \tilde{A} + \tilde{B}e^{-\varepsilon\mu}.$$

We find the characteristic equation by working out $\det \tilde{C} = 0$,

$$\left(\mu - \frac{b}{2\varepsilon\lambda_{-1}e^{\lambda_{-1}}}\left(1 - e^{-\varepsilon\mu}\right)\right)\left(\mu - \frac{b}{2\varepsilon\lambda_0 e^{\lambda_0}}\left(1 - e^{-\varepsilon\mu}\right)\right) - \frac{1}{4\lambda_{-1}\lambda_0} = 0.$$

Then use that $\lambda_{-1} = -i\pi$ and $\lambda_0 = i\pi$, to find

$$\mu^2 + \frac{c^2}{\varepsilon^2}\left(1 - e^{-\varepsilon\mu}\right)^2 - \frac{1}{4\pi^2} = 0,$$

with $c = \frac{b}{2\pi}$. We will consider two balances for μ. If $\mu = \mathcal{O}(1)$, then we can expand the exponential and obtain the following equation,

$$\mu^2 + \frac{c^2}{\varepsilon^2}\left(1 - (1 - \varepsilon\mu + \dots)\right)^2 - \frac{1}{4\pi^2} = 0.$$

We solve the $\mathcal{O}(1)$ equation and obtain

$$\mu \sim \pm\frac{1}{2\pi\sqrt{1+c^2}}.$$

These are approximations of the two real solutions of the characteristic equation. The other balance that we consider is $\mu = \mathcal{O}(\frac{1}{\varepsilon})$. We write $\mu = \frac{\tilde{\mu}}{\varepsilon}$ and obtain the following equation,

$$\frac{\tilde{\mu}^2}{\varepsilon^2} + \frac{c^2}{\varepsilon^2}\left(1 - e^{-\tilde{\mu}}\right)^2 - \frac{1}{4\pi^2} = 0.$$

We consider the $\mathcal{O}(\frac{1}{\varepsilon^2})$ equation,

$$\tilde{\mu}^2 + c^2\left(1 - e^{-\tilde{\mu}}\right)^2 = 0.$$

There are infinitely many solutions of this equation. We highlight three solutions, $\tilde{\mu} = 0, \pm i\pi$. We do not have to consider $\tilde{\mu} = 0$, since then $\mu \neq \mathcal{O}(\frac{1}{\varepsilon})$. For $\tilde{\mu} = \pm i\pi$ we must continue our expansion, to obtain the real part. We will have to use the expansion $\mu \sim \frac{1}{\varepsilon}\left(\mu^0 + \varepsilon^2\mu^1 + \dots\right)$. We obtain an equation for μ^1 and using that $\mu^0 = \pm i\pi$, we obtain

$$\mu^1 = -\frac{1}{4\pi^2\left(\pi^2+4\right)} \mp i\frac{1}{2\pi^3\left(\pi^2+4\right)}.$$

We will use the following approximations for μ,

$$\mu \sim -\frac{\varepsilon}{4\pi^2\left(\pi^2+4\right)} \pm \frac{i\pi}{\varepsilon}.$$

We can find the other $\mathcal{O}(\frac{1}{\varepsilon})$ solutions numerically.

To find the solution for $\boldsymbol{\beta}$, we first need to calculate the inverse of $\tilde{C}$, with s instead of μ,

$$\tilde{C}^{-1} = \frac{1}{\det\tilde{C}}\begin{pmatrix} s - \frac{b}{2\varepsilon\lambda_0 e^{\lambda_0}}\left(1 - e^{-\varepsilon\mu}\right) & \frac{1}{2\lambda_{-1}} \\ \frac{1}{2\lambda_0} & s - \frac{b}{2\varepsilon\lambda_{-1}e^{\lambda_{-1}}}\left(1 - e^{-\varepsilon\mu}\right)\end{pmatrix}.$$

We use this inverse to calculate the fundamental matrix

$$\Gamma(\tau) = \sum_{l=-\infty}^{\infty}\tilde{D}_l e^{\mu_l\tau},$$

with

$$\tilde{D}_l = \frac{1}{2\mu_l + \frac{2c^2}{\varepsilon}\left(1 - e^{-\varepsilon\mu_l}\right)e^{-\varepsilon\mu_l}}\begin{pmatrix} \mu_l - \frac{b}{2\varepsilon\lambda_0 e^{\lambda_0}}\left(1 - e^{-\varepsilon\mu_l}\right) & \frac{1}{2\lambda_{-1}} \\ \frac{1}{2\lambda_0} & \mu_l - \frac{b}{2\varepsilon\lambda_{-1}e^{\lambda_{-1}}}\left(1 - e^{-\varepsilon\mu_l}\right)\end{pmatrix}.$$

We can then use $\Gamma(\tau)$ to calculate the $\boldsymbol{\beta}(\tau)$,

$$\boldsymbol{\beta}(\tau) = \sum_{l=-\infty}^{\infty}\boldsymbol{\xi}_l e^{\mu_l\tau},$$

with

$$\boldsymbol{\xi}_l = \tilde{D}_l\left(\boldsymbol{\Phi} - \tilde{B}\boldsymbol{\Phi}e^{-\varepsilon\mu_l}\int_{-\varepsilon}^{0} e^{-\mu_l\eta}d\eta\right).$$

We write $\boldsymbol{\xi}_l = \begin{pmatrix} \xi_l^{-1} \\ \xi_l^0 \end{pmatrix}$, such that

$$\alpha_{-1}(\tau) = \sum_{l=-\infty}^{\infty} \xi_l^{-1} e^{\mu_l \tau} \qquad \text{and} \qquad \alpha_0(\tau) = \sum_{l=-\infty}^{\infty} \xi_l^{0} e^{\mu_l \tau}.$$

We have the following approximation,

$$y_0(t,\tau) = \sum_{\substack{k=-\infty \\ k\neq -1,0}}^{\infty} \Phi_k e^{\lambda_k t} + \sum_{l=-\infty}^{\infty} \left(\xi_l^{-1} e^{\lambda_{-1} t} + \xi_l^0 e^{\lambda_0 t}\right) e^{\mu_l \tau}.$$

Figure 5.2 shows a plot of a numerical solution and the approximation. The two solutions are indistinguishable. Note that due to resonance, the amplitude of the solutions increases over time.

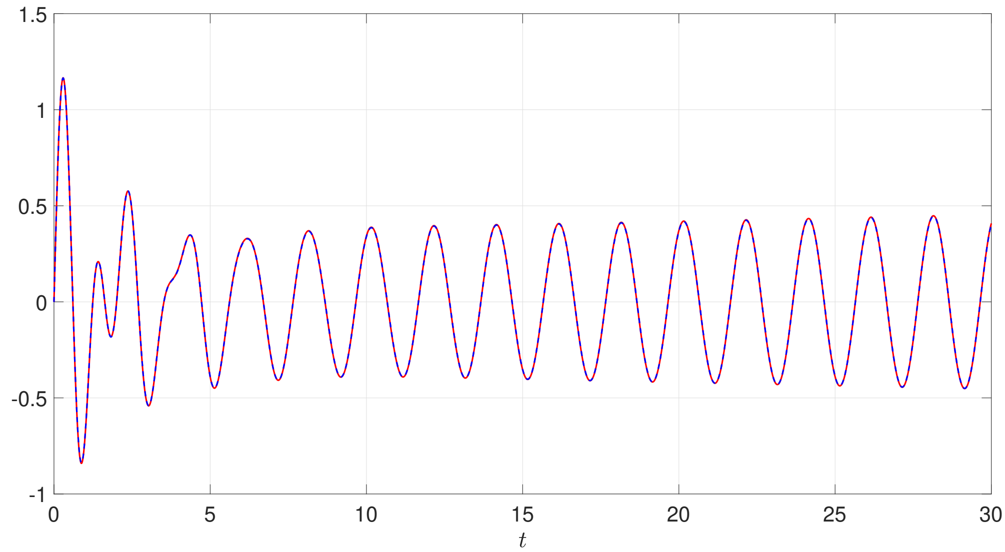


Figure 5.2: Plot of numerical solution, solid (red) line, and the approximation of the solution, dashed (blue) line, of Eq. (5.3) with initial condition Eq. (5.4) with $\phi(t) = \sin(2\pi t)$ and $\psi(t) = 2\pi\cos(2\pi t)$. For the parameters, $a = 2\pi^2$, $b = \pi^2$, $\gamma = 2\pi$ and $\varepsilon = 0.1$ have been used.

In the next section, we shall consider a detuned version of this example.

5.2.1 Detuned version

We shall consider the following variation,

$$\frac{d^2 y(t)}{dt^2} + ay(t) + by(t-1) = 2\varepsilon\cos((\gamma+\varepsilon\delta)t)y(t).$$

We will continue using $a = 2\pi^2$, $b = \pi^2$ and $\gamma = 2\pi$. We never have exactly the resonant frequency, so it is interesting to see what happens when we are close to the resonant frequency.

We will find the following terms, $\lambda_k \pm i(\gamma + \varepsilon\delta)$. Although these will never be exactly solutions of the characteristic equation, $\lambda_{-1} + i(\gamma+\varepsilon\delta)$ and $\lambda_0 - i(\gamma+\varepsilon\delta)$, are close to solutions of the characteristic equation. We have seen in Section 4.1 that these terms will still violate the ordering of a simple expansion, so we still need to use the method of multiple scales. We use the slow timescale $\tau = \varepsilon t$ and have the following equation for $\tilde{y}(t,\tau) = y(t)$,

$$\left(\frac{\partial^2}{\partial t^2} + 2\varepsilon\frac{\partial}{\partial t}\frac{\partial}{\partial \tau} + \varepsilon^2\frac{\partial^2}{\partial \tau^2}\right)\tilde{y}(t,\tau) + a\tilde{y}(t,\tau) + bE_1\tilde{y}(t,\tau) - bE_1\Delta_\varepsilon\tilde{y}(t,\tau) = 2\varepsilon\cos(\gamma t + \delta\tau)\tilde{y}(t,\tau).$$

We can copy a lot of the derivation that we have seen before in this example. We will again use an expansion for $\tilde{y}$, $\tilde{y}(t,\tau) = y_0(t,\tau) + \varepsilon y_1(t,\tau) + \dots$. The $\mathcal{O}(1)$ equations results in the following expression for $y_0(t,\tau)$,

$$y_0(t,\tau) = \sum_{k=-\infty}^{\infty} \alpha_k(\tau) e^{\lambda_k t},$$

with λ_k the solutions of the characteristic equations. For α_k we have the same initial condition as before. The right-hand side of the $\mathcal{O}(\varepsilon)$ equation is now given by

$$\sum_{k=-\infty}^{\infty} \left[\left(-2\lambda_k \frac{d\alpha_k(\tau)}{d\tau} + \frac{b}{\varepsilon} e^{-\lambda_k} \Delta_\varepsilon \alpha_k(\tau) \right) e^{\lambda_k t} + \alpha_k(\tau) \left(e^{i\delta\tau} e^{(\lambda_k + i\gamma)t} + e^{-i\delta\tau} e^{(\lambda_k - i\gamma)t} \right) \right].$$

For $k \neq -1, 0$ we obtain the same equation and again find $\alpha_k(\tau) = \Phi_k$. For α_{-1} and α_0 we have the following equations,

$$\begin{aligned} 2\lambda_{-1} \frac{d\alpha_{-1}(\tau)}{d\tau} &= -\frac{b}{\varepsilon} \Delta_\varepsilon \alpha_{-1}(\tau) + \alpha_0(\tau) e^{-i\delta\tau}, \\ 2\lambda_0 \frac{d\alpha_0(\tau)}{d\tau} &= -\frac{b}{\varepsilon} \Delta_\varepsilon \alpha_0(\tau) + \alpha_{-1}(\tau) e^{i\delta\tau}, \end{aligned}$$

where we have used that $e^{-\lambda_{-1}} = e^{-\lambda_0} = -1$, since $\lambda_{-1} = -i\pi$ and $\lambda_0 = i\pi$. We again have coupled equations. However, these are even harder to solve than before due to the presence of $e^{\pm i\delta\tau}$.

We will solve this problem using the Laplace transform. We first define a few functions, to make the derivation clearer. For the initial conditions we use

$$\alpha_{-1}(\tau) = \psi_{-1}(\tau) \qquad \text{and} \qquad \alpha_0(\tau) = \psi_0(\tau), \qquad \text{for } \tau \in [-\varepsilon, 0],$$

with

$$\psi_{-1}(\tau) = \begin{cases} \Phi_{-1}, & \text{for } \tau \in [-\varepsilon, 0], \\ 0, & \text{else} \end{cases} \qquad \text{and} \qquad \psi_0(\tau) = \begin{cases} \Phi_0, & \text{for } \tau \in [-\varepsilon, 0], \\ 0 & \text{else} \end{cases}.$$

We will need the Laplace transforms of $\psi_{-1}(\tau - \varepsilon)$ and $\psi_0(\tau - \varepsilon)$,

$$\begin{aligned} \Psi_i(\sigma) = \mathcal{L}(\psi_i(\tau - \varepsilon)) &= \int_0^\infty e^{-\varepsilon\sigma} \psi_i(\tau - \varepsilon) d\tau \\ &= \Phi_i \frac{1 - e^{-\varepsilon\sigma}}{\sigma}. \end{aligned}$$

We will use the notation $A_i(\sigma) = \mathcal{L}(\alpha_i(\tau))$. Finally, we will use that

$$\mathcal{L}\left(\alpha_0(\tau) e^{-i\delta\tau}\right) = A_0(\sigma + i\delta) \qquad \text{and} \qquad \mathcal{L}\left(\alpha_{-1}(\tau) e^{i\delta\tau}\right) = A_{-1}(\sigma - i\delta).$$

We can then take the Laplace transform of the equations for α_{-1} and α_0,

$$\begin{aligned} 2\lambda_{-1}\left(\sigma A_{-1}(\sigma) - \Phi_{-1}\right) &= -\frac{b}{\varepsilon}\left(A_{-1}(\tau) - \left(\Psi_{-1}(\sigma) + e^{-\varepsilon\sigma} A_{-1}(\tau)\right)\right) + A_0(\sigma + i\delta), \\ 2\lambda_0\left(\sigma A_0(\sigma) - \Phi_0\right) &= -\frac{b}{\varepsilon}\left(A_0(\tau) - \left(\Psi_0(\sigma) + e^{-\varepsilon\sigma} A_0(\tau)\right)\right) + A_{-1}(\sigma - i\delta). \end{aligned}$$

We can rewrite these equations into the following form,

$$\begin{aligned} \left(2\lambda_{-1}\sigma + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon\sigma}\right)\right) A_{-1}(\sigma) &= 2\lambda_{-1}\Phi_{-1} + \frac{b}{\varepsilon}\Psi_{-1}(\sigma) + A_0(\sigma + i\delta), \\ \left(2\lambda_0\sigma + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon\sigma}\right)\right) A_0(\sigma) &= 2\lambda_0\Phi_0 + \frac{b}{\varepsilon}\Psi_0(\sigma) + A_{-1}(\sigma - i\delta). \end{aligned}$$

Then use that

$$2\lambda_i\Phi_i + \frac{b}{\varepsilon}\Psi_i = \left(2\lambda_i\sigma + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon\sigma}\right)\right)\frac{\Phi_i}{\sigma},$$

such that

$$\left(2\lambda_{-1}\sigma + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon\sigma}\right)\right)\left(A_{-1}(\sigma) - \frac{\Phi_{-1}}{\sigma}\right) = A_0(\sigma + i\delta),$$
$$\left(2\lambda_0\sigma + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon\sigma}\right)\right)\left(A_0(\sigma) - \frac{\Phi_0}{\sigma}\right) = A_{-1}(\sigma - i\delta).$$

We will use $\sigma \to \sigma + i\delta$ in the second equation, solve the resulting equation for $A_0(\sigma + i\delta)$ and use this in the first equation. This results in the following equation for $A_{-1}(\sigma)$,

$$\left(2\lambda_{-1}\sigma + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon\sigma}\right)\right)\left(A_{-1}(\sigma) - \frac{\Phi_{-1}}{\sigma}\right) = \frac{\Phi_0}{\sigma + i\delta} + \frac{A_{-1}(\sigma)}{2\lambda_0(\sigma + i\delta) + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon(\sigma + i\delta)}\right)}.$$

We solve this equation for $A_{-1}(\sigma)$,

$$\begin{aligned} A_{-1}(\sigma) = &\left[\left(2\lambda_{-1}\sigma + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon\sigma}\right)\right)\left(2\lambda_0(\sigma + i\delta) + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon(\sigma + i\delta)}\right)\right) - 1\right]^{-1} \\ &\times \left[\frac{2\lambda_{-1}\sigma + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon\sigma}\right)}{\sigma}\left(2\lambda_0(\sigma + i\delta) + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon(\sigma + i\delta)}\right)\right)\Phi_{-1}\right. \\ &\quad \left. + \frac{2\lambda_0(\sigma + i\delta) + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon(\sigma + i\delta)}\right)}{\sigma + i\delta}\Phi_0\right]. \end{aligned}$$

We will use this to calculate an expression for α_{-1}. We could also calculate an expression for A_0 and use this to calculate α_0. However, we will use the fact that we must have $\alpha_0 = \overline{\alpha_{-1}}$ to have a real valued approximation. Consequently, we only have to calculate α_{-1} and use this to calculate α_0.

We shall now study the characteristic equation for this problem.

Characteristic equation

Using the equation for $A_{-1}(\sigma)$, we see that the characteristic equation for this problem is given by

$$\left(2\lambda_{-1}\sigma + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon\sigma}\right)\right)\left(2\lambda_0(\sigma + i\delta) + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon(\sigma + i\delta)}\right)\right) - 1 = 0.$$

There are two balances for μ, $\mu = \mathcal{O}(1)$ and $\mu = \mathcal{O}(\frac{1}{\varepsilon})$. For $\mu = \mathcal{O}(1)$ we write $\mu = \mu_0 + \varepsilon\mu_1 + \dots$. We can then expand the exponentials and collect like orders of ε in the equation,

$$\begin{aligned} &\left((2\lambda_{-1} + b)\mu_0 + \varepsilon\left((2\lambda_{-1} + b)\mu_1 - \frac{1}{2}b\mu_0^2\right) + \dots\right) \\ &\times\left((2\lambda_0 + b)(\mu_0 + i\delta) + \varepsilon\left((2\lambda_0 + b)\mu_1 - \frac{1}{2}b(\mu_0 + i\delta)^2\right) + \dots\right) - 1 = 0. \end{aligned}$$

The $\mathcal{O}(1)$ equation is given by

$$(2\lambda_{-1} + b)(2\lambda_0 + b)\mu_0(\mu_0 + i\delta) - 1 = 0.$$

The solution is given by

$$\mu_0 = -\frac{i\delta}{2} \pm \frac{1}{2}\sqrt{\frac{4 - \delta^2 z}{z}},$$

with $z = (2\lambda_{-1} + b)(2\lambda_0 + b)$. The $\mathcal{O}(\varepsilon)$ equation is given by

$$(2\lambda_{-1} + b)\mu_0\left((2\lambda_0 + b)\mu_1 - \frac{1}{2}b(\mu_0 + i\delta)^2\right) + (2\lambda_0 + b)(\mu_0 + i\delta)\left((2\lambda_{-1} + b)\mu_1 - \frac{1}{2}b\mu_0^2\right) = 0.$$

We solve this for μ_1 and obtain

$$\mu_1 = \frac{b\left((2\lambda_{-1} + b)\mu_0(\mu_0 + i\delta)^2 + (2\lambda_0 + b)\mu_0^2(\mu_0 + i\delta)\right)}{2(2\lambda_{-1} + b)\left((2\lambda_{-1} + b)\mu_0 + (2\lambda_0 + b)(\mu_0 + i\delta)\right)}.$$

For $\mu = \mathcal{O}(\frac{1}{\varepsilon})$ we write $\mu = \frac{1}{\varepsilon}\left(\mu_0 + \varepsilon\mu_1 + \varepsilon^2\mu_2 + \ldots\right)$. We have the following equation,

$$\frac{1}{\varepsilon^2}\left(\left(2\lambda_{-1}\mu_0 + b\left(1 - e^{-\mu_0}\right)\right) + \varepsilon\left(2\lambda_{-1}\mu_1 + be^{-\mu_0}\mu_1\right) + \varepsilon^2\left(2\lambda_{-1}\mu_2 + be^{-\mu_0}\left(\mu_2 - \frac{1}{2}\mu_1^2\right)\right) + \ldots\right)$$
$$\times\left(\left(2\lambda_0\mu_0 + b\left(1 - e^{-\mu_0}\right)\right) + \varepsilon\left(2\lambda_0(\mu_1 + i\delta) + be^{-\mu_0}(\mu_1 + i\delta)\right) + \varepsilon^2\left(2\lambda_0\mu_2 + be^{-\mu_0}\left(\mu_2 - \frac{1}{2}(\mu_1 + i\delta)^2\right)\right) + \ldots\right)$$
$$- 1 = 0.$$

The $\mathcal{O}(\frac{1}{\varepsilon^2})$ equation is given by

$$\left(2\lambda_{-1}\mu_0 + b\left(1 - e^{-\mu_0}\right)\right)\left(2\lambda_0\mu_0 + b\left(1 - e^{-\mu_0}\right)\right) = 0.$$

So, either $2\lambda_{-1}\mu_0 + b\left(1 - e^{-\mu_0}\right) = 0$ or $2\lambda_0\mu_0 + b\left(1 - e^{-\mu_0}\right) = 0$. If we take the conjugate of the first equation, then we obtain the second equation. So, the solutions of the second equation are conjugates of the solutions of the first equation. If $2\lambda_{-1}\mu_0 + b\left(1 - e^{-\mu_0}\right) = 0$, then the $\mathcal{O}(\frac{1}{\varepsilon})$ equation is given by

$$\left(2\lambda_{-1} + be^{-\mu_0}\right)\left(2\lambda_0\mu_0 + b\left(1 - e^{-\mu_0}\right)\right)\mu_1 = 0.$$

Note that $2\lambda_{-1} + be^{-\mu_0} = 0$ corresponds to μ_0 having a multiplicity higher than one. For our choice of parameters this is not the case, so we have $2\lambda_{-1} + be^{-\mu_0} \neq 0$. We also have $2\lambda_0\mu_0 + b\left(1 - e^{-\mu_0}\right) \neq 0$, since we are considering the other case. We find $\mu_1 = 0$. The $\mathcal{O}(1)$ equation is then given by

$$\left(2\lambda_{-1} + be^{-\mu_0}\right)\left(2\lambda_0\mu_0 + b\left(1 - e^{-\mu_0}\right)\right)\mu_2 - 1 = 0.$$

We solve this equation for μ_2 and obtain,

$$\mu_2 = \frac{1}{\left(2\lambda_{-1} + be^{-\mu_0}\right)\left(2\lambda_0\mu_0 + b\left(1 - e^{-\mu_0}\right)\right)}.$$

If $2\lambda_0\mu_0 + b\left(1 - e^{-\mu_0}\right) = 0$, then the $\mathcal{O}(\frac{1}{\varepsilon})$ equation is given by

$$\left(2\lambda_{-1}\mu_0 + b\left(1 - e^{-\mu_0}\right)\right)\left(2\lambda_0 + be^{-\mu_0}\right)(\mu_1 + i\delta) = 0.$$

We find $\mu_1 = -i\delta$. The $\mathcal{O}(1)$ equation is given by

$$\left(2\lambda_{-1}\mu_0 + b\left(1 - e^{-\mu_0}\right)\right)\left(2\lambda_0 + be^{-\mu_0}\right)\mu_2 - 1 = 0.$$

We solve this equation for μ_2 and obtain,

$$\mu_2 = \frac{1}{\left(2\lambda_{-1}\mu_0 + b\left(1 - e^{-\mu_0}\right)\right)\left(2\lambda_0 + be^{-\mu_0}\right)}.$$

Now that we have found approximations of solutions of the characteristic equation, we can continue our example.

Continuation of the example

We will first define the fundamental solution, $\beta(\tau)$, as

$$\mathcal{L}(\beta(\tau)) = \left[\left(2\lambda_{-1}\sigma + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon\sigma}\right)\right)\left(2\lambda_0(\sigma + i\delta) + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon(\sigma + i\delta)}\right)\right) - 1\right]^{-1}.$$

We can immediately solve this, using the solutions of the characteristic equation. We skip the derivation as this is the same as before. We have the following expression for β,

$$\beta(\tau) = \sum_{l=-\infty}^{\infty} \zeta_l e^{\mu_l \tau},$$

with

$$\zeta_l = \frac{1}{\left(2\lambda_{-1} + be^{-\varepsilon\mu_l}\right)\left(2\lambda_0(\mu_l + i\delta) + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon(\mu_l + i\delta)}\right)\right) + \left(2\lambda_{-1}\mu_l + \frac{b}{\varepsilon}\left(1 - e^{-\varepsilon\mu}\right)\right)\left(2\lambda_0 + be^{-\varepsilon(\mu_l + i\delta)}\right)}.$$

To calculate α_{-1}, we will first need to know the inverse Laplace transforms of

$$\frac{2\lambda_{-1}\sigma + \frac{b}{\varepsilon}\left(1-e^{-\varepsilon\sigma}\right)}{\sigma}\left(2\lambda_0(\sigma+i\delta)+\frac{b}{\varepsilon}\left(1-e^{-\varepsilon(\sigma+i\delta)}\right)\right)\Phi_{-1}$$

and

$$\frac{2\lambda_0(\sigma+i\delta)+\frac{b}{\varepsilon}\left(1-e^{-\varepsilon(\sigma+i\delta)}\right)}{\sigma+i\delta}.$$

We will omit the calculations and only given the results,

$$\begin{aligned}&\mathcal{L}^{-1}\left(\frac{2\lambda_{-1}\sigma + \frac{b}{\varepsilon}\left(1-e^{-\varepsilon\sigma}\right)}{\sigma}\left(2\lambda_0(\sigma+i\delta)+\frac{b}{\varepsilon}\left(1-e^{-\varepsilon(\sigma+i\delta)}\right)\right)\Phi_{-1}\right)\\ =&4\lambda_0\lambda_{-1}D'(\tau)+4\lambda_0\lambda_{-1}i\delta D(\tau)-\frac{2b}{\varepsilon}\left(\lambda_0+e^{-i\varepsilon\delta}\lambda_{-1}\right)D(\tau-\varepsilon)\\ &+\frac{b}{\varepsilon}\left(2\lambda_0 i\delta+\frac{b}{\varepsilon}\right)\left(u(\tau)-u(\tau-\varepsilon)\right)-\frac{b^2}{\varepsilon^2}e^{-i\varepsilon\delta}\left(u(\tau-\varepsilon)-u(\tau-2\varepsilon)\right),\end{aligned}$$

and

$$\mathcal{L}^{-1}\left(\frac{2\lambda_0(\sigma+i\delta)+\frac{b}{\varepsilon}\left(1-e^{-\varepsilon(\sigma+i\delta)}\right)}{\sigma+i\delta}\right)=2\lambda_0 D(\tau)+\frac{b}{\varepsilon}e^{-i\delta\tau}\left(u(\tau)-u(\tau-\varepsilon)\right),$$

with $D(\tau)$ the Dirac delta function and $u(\tau)$ the Heaviside step function. Usually we denote the Dirac delta function with $\delta(\tau)$, but we already use δ as a parameter, so to avoid confusion we use $D(\tau)$.

Now that we have β and the inverse Laplace transforms, we can calculate α_{-1}

$$\begin{aligned}\alpha_{-1}(\tau)=\int_0^\tau \beta(\tau-\sigma)\Bigg\{&\Phi_{-1}\Bigg[4\lambda_0\lambda_{-1}D'(\sigma)+4\lambda_0\lambda_{-1}i\delta D(\sigma)-\frac{2b}{\varepsilon}\left(\lambda_0+e^{-i\varepsilon\delta}\lambda_{-1}\right)D(\sigma-\varepsilon)\\ &+\frac{b}{\varepsilon}\left(2\lambda_0 i\delta+\frac{b}{\varepsilon}\right)\left(u(\sigma)-u(\sigma-\varepsilon)\right)-\frac{b^2}{\varepsilon^2}e^{-i\varepsilon\delta}\left(u(\sigma-\varepsilon)-u(\sigma-2\varepsilon)\right)\Bigg]\\ &+\Phi_0\left[2\lambda_0 D(\sigma)+\frac{b}{\varepsilon}e^{-i\delta\sigma}\left(u(\sigma)-u(\sigma-\varepsilon)\right)\right]\Bigg\}d\sigma.\end{aligned}$$

We can work this out

$$\begin{aligned}\alpha_{-1}(\tau)=&\Phi_{-1}\Bigg[4\lambda_0\lambda_{-1}\beta'(\tau)+4\lambda_0\lambda_{-1}i\delta\beta(\tau)-\frac{2b}{\varepsilon}\left(\lambda_0+e^{-i\varepsilon\delta}\lambda_{-1}\right)\beta(\tau-\varepsilon)\\ &+\frac{b}{\varepsilon}\left(2\lambda_0 i\delta+\frac{b}{\varepsilon}\right)\int_0^\varepsilon\beta(\tau-\sigma)d\sigma-\frac{b^2}{\varepsilon^2}e^{-i\varepsilon\delta}\int_\varepsilon^{2\varepsilon}\beta(\tau-\sigma)d\sigma\Bigg]\\ &+\Phi_0\left[2\lambda_0\beta(\tau)+\frac{b}{\varepsilon}\int_0^\varepsilon\beta(\tau-\sigma)e^{-i\delta\sigma}d\sigma\right]\end{aligned}$$

This can be written in a more compact form,

$$\alpha_{-1}(\tau)=\sum_{l=-\infty}^{\infty}\xi_l e^{\mu_l\tau},$$

with

$$\begin{aligned}\xi_l=\zeta_l\Bigg\{&\Phi_{-1}\Bigg[4\lambda_0\lambda_{-1}\mu_l+4\lambda_0\lambda_{-1}i\delta-\frac{2b}{\varepsilon}\left(\lambda_0+e^{-i\varepsilon\delta}\lambda_{-1}\right)e^{-\varepsilon\mu_l}\\ &+\frac{b}{\varepsilon}\left(2\lambda_0 i\delta+\frac{b}{\varepsilon}\right)\int_0^\varepsilon e^{-\mu_l\sigma}d\sigma-\frac{b^2}{\varepsilon^2}e^{-i\varepsilon\delta}\int_\varepsilon^{2\varepsilon}e^{-\mu_l\sigma}d\sigma\Bigg]\\ &+\Phi_0\left[2\lambda_0+\frac{b}{\varepsilon}\int_0^\varepsilon e^{-(\mu_l+i\delta)\sigma}d\sigma\right]\Bigg\}\end{aligned}$$

We can calculate the integrals,

$$\xi_l = \zeta_l \Bigg\{ \Phi_{-1} \Bigg[4\lambda_0\lambda_{-1}\mu_l + 4\lambda_0\lambda_{-1} i\delta - \frac{2b}{\varepsilon}\left(\lambda_0 + e^{-i\varepsilon\delta}\lambda_{-1}\right) e^{-\varepsilon\mu_l} + \frac{b}{\varepsilon}\left(2\lambda_0 i\delta + \frac{b}{\varepsilon\mu_l}\right)\left(1 - e^{-\varepsilon\mu_l}\right) - \frac{b^2}{\varepsilon^2\mu_l} e^{-i\varepsilon\delta} e^{-\varepsilon\mu_l}\left(1 - e^{-\varepsilon\mu_l}\right) \Bigg] + \Phi_0 \left[2\lambda_0 + \frac{b}{\varepsilon(\mu_l + i\delta)}\left(1 - e^{-\varepsilon(\mu_l + i\delta)}\right) \right] \Bigg\}.$$

For α_0 we use that $\alpha_0 = \overline{\alpha_{-1}}$ to obtain the following expression,

$$\alpha_0(\tau) = \sum_{l=-\infty}^{\infty} \overline{\xi_l} e^{\overline{\mu_l}\tau}.$$

Finally, we can write down our approximation,

$$\begin{aligned} y_0(t,\tau) &= \sum_{\substack{k=-\infty \\ k\neq -1,0}}^{\infty} \Phi_k e^{\lambda_k t} + \alpha_0(\tau) e^{\lambda_0 t} + \alpha_{-1}(\tau) e^{\lambda_{-1} t} \\ &= \sum_{\substack{k=-\infty \\ k\neq -1,0}}^{\infty} \Phi_k e^{\lambda_k t} + \sum_{l=-\infty}^{\infty} \left(\xi_l e^{\mu_l \tau} e^{\lambda_{-1} t} + \overline{\xi_l} e^{\overline{\mu_l}\tau} e^{\lambda_0 t} \right). \end{aligned}$$

We use that $\lambda_{-1} = -i\pi$, $\lambda_0 = i\pi$ and write $\xi_l = \xi_l^R + i\xi_l^I$ and $\mu_l = \mu_l^R + i\mu_l^I$. After some calculations we obtain

$$\begin{aligned} y_0(t,\tau) &= \sum_{\substack{k=-\infty \\ k\neq -1,0}}^{\infty} \Phi_k e^{\lambda_k t} + \sum_{l=-\infty}^{\infty} 2e^{\mu_l^R \tau} \left(\xi_l^R \cos(\pi t - \mu_l^I \tau) + \xi_l^I \sin(\pi t - \mu_l^I \tau) \right) \\ &= \sum_{\substack{k=-\infty \\ k\neq -1,0}}^{\infty} \Phi_k e^{\lambda_k t} + \sum_{l=-\infty}^{\infty} 2e^{\mu_l^R \varepsilon t} \left(\xi_l^R \cos((\pi - \varepsilon\mu_l^I)t) + \xi_l^I \sin((\pi - \varepsilon\mu_l^I)t) \right). \end{aligned}$$

We will consider our approximation for two values of δ. For small δ the frequency is close to the resonance frequency, so we still expect instability. As δ increases, we expect that at some point this turns into stability. To see for which δ this occurs, we investigate the $\mathcal{O}(1)$ solutions of μ,

$$\mu_0 = -\frac{i\delta}{2} \pm \frac{1}{2}\sqrt{\frac{4 - \delta^2 z}{z}},$$

with $z = (2\lambda_{-1} + b)(2\lambda_0 + b)$. We distinguish two cases,

$$0 < \delta < \frac{2}{\sqrt{z}} \qquad \text{and} \qquad \delta \geq \frac{2}{\sqrt{z}}.$$

For the first, we have a μ_0 with a positive real part, which causes instability. As this happens for the term $e^{\mu_0\tau}$, we expect to see this on a timescale $t = \mathcal{O}(\frac{1}{\varepsilon})$. For $\delta \geq \frac{2}{\sqrt{z}}$, there is no real part anymore. So, if we find instability, this will occur on a timescale $t = \mathcal{O}(\frac{1}{\varepsilon^2})$, for which our approximation is not valid anymore.

Figure 5.3 shows the approximation, together with a numerically calculated approximation, for $\delta = \frac{1}{\sqrt{z}}$. It is clear that these both show instability, as expected.

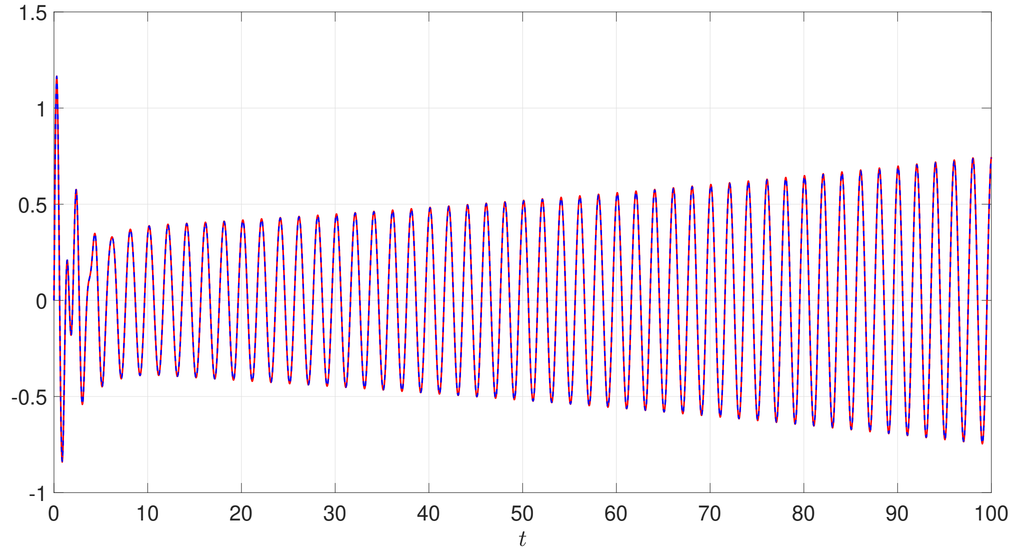


Figure 5.3: Plot of the numerical solution, solid (red) line, and our approximation, dashed (blue) line, for a second order DDE detuned Mathieu equation. We have used $a = 2\pi^2$, $b = \pi^2$, $\gamma = 2\pi$, $\varepsilon = 0.1$, $\delta = \frac{1}{\sqrt{z}}$, $\phi(t) = \sin(2\pi t)$ and $\psi(t) = 2\pi\cos(2\pi t)$.

Figure 5.4 shows a plot of the approximation, together with a numerically calculated approximation, for $\delta = 1 > \frac{2}{\sqrt{z}}$. As expected, there is no instability for $t = \mathcal{O}(\frac{1}{\varepsilon})$. Instead we notice that there are two periods, one on a fast timescale and one on a slow timescale. For the fast timescale this corresponds to the solution $\lambda = \pm i\pi$, resulting in a period of 2. For the slow period this corresponds to the solution $\mu \approx -i\delta$, resulting in a period of $\frac{2\pi}{\varepsilon\delta}$. These periods correspond to the periods that we see in the figure.

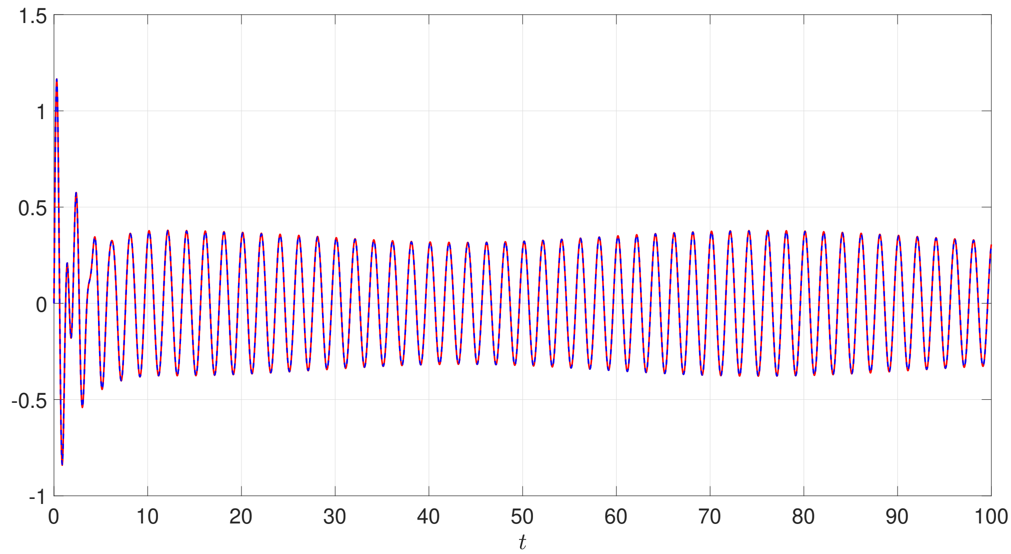


Figure 5.4: Plot of the numerical solution, solid (red) line, and our approximation, dashed (blue) line, for a second order DDE detuned Mathieu equation. We have used $a = 2\pi^2$, $b = \pi^2$, $\gamma = 2\pi$, $\varepsilon = 0.1$, $\delta = 1$, $\phi(t) = \sin(2\pi t)$ and $\psi(t) = 2\pi\cos(2\pi t)$.

Chapter 6

Conclusion and Further Research

The goal of this thesis was to construct a perturbation method for delay differential equations. We have first shown how to calculate the solution of a DDE in Chapter 2. In Chapter 3 the method of multiple scales was explained for ordinary differential equations and ordinary difference equations. The work of these two chapters was combined in Chapter 4 to create the method of multiple scales for DDEs. In this chapter we only considered first order DDEs. In Chapter 5 we considered the method of multiple scales for higher order DDEs. We have proven the validity of the method under certain conditions and illustrated how the method should be applied, by working out examples.

We started with a linear example for which we could also calculate the exact solution, with which we could compare our approximation. We have considered Mathieu's equation for a first order DDE in Chapter 4 and for a second order DDE in Chapter 5. Especially the second order case is of practical use, as it can be used when one linearizes a nonlinear oscillator around a periodic solution. For the first order case we used the perturbation $\varepsilon e^{\gamma t} y(t)$ and had to determine whether there exists solutions of the characteristic equation, λ, such that $\lambda + \gamma$ is also a solution of the characteristic equation. For the second order case we used the perturbation $\varepsilon \cos(\gamma t) y(t)$ and had to determine whether there exist solutions of the characteristic equation, λ, such that $\lambda + i\gamma$ or $\lambda - i\gamma$ is also a solution of the characteristic equation. We have also considered a detuned version for the second order case. If we are still close enough to the resonant frequency, then the solution remains unstable. However, as we get further away, the solution becomes stable. For this case we found a solution with a oscillation on the fast and on the slow timescale. We have also considered a quadratic perturbation, $\varepsilon y(t)^2$, for which we had to determine whether the sum of two solutions of the characteristic equation could again be a solution. Finally, we have also considered a DDE perturbation on an ODE. This is exactly the type of equation for which there already existed a perturbation method. This perturbation method focuses on obtaining an ODE, which one solves to obtain an approximation for the DDE. We have shown that this method does not work if the initial condition at $t = 0$ is small. We considered the extreme case $\phi(0) = 0$, such that this perturbation method yielded the approximation, $y(t) \sim 0$. For this case, the approximation gives no information about the exact solution and the relative error is 1 for all t. We then used our perturbation method to obtain an approximation for this example. For all other examples we consider a fast timescale t and a slow timescale εt. For this example we also had to include an even faster timescale, $(-\ln \varepsilon)t$.

As we have proven the validity of our method, and for each example the approximation showed good agreement with a numerically calculated solution, we conclude that the perturbation method for DDEs, as constructed in this thesis, works well.

An example that we have not considered in this thesis, is that of a cubic perturbation, $\varepsilon y(t)^3$. These also occur often in the field of applied mathematics, just think of the van der Pol-equation. In the field of DDEs, a famous equation with a cubic perturbation is Minorsky's equation. This equation is used for certain stabilization problems in which delays occur. When handling a cubic problem we will have to answer the question whether the sum of three solutions of the characteristic equation can again be a solution of the characteristic equation. We divide this question into three subquestions.

- If we have the solution λ, can 3λ be a solution?

- If we have the distinct solutions λ_k and λ_l, can $2\lambda_k + \lambda_l$ be a solution?
- If we have the distinct solutions λ_k, λ_l and λ_m, can $\lambda_k + \lambda_l + \lambda_m$ be a solution?

These first two can be solved in the same manner that we have solved the quadratic perturbation example. The final question with three distinct solution turns out to be a difficult problem and remains unsolved. The problems that we run into are similar to that of partial differential equations with quadratic and cubic nonlinearities and boundary conditions which result in transcendental equations for the eigenvalues of the corresponding linear problem. One could assume that there are no distinct solutions such that the sum of three solutions is again a solution. With this assumption one can solve a problem with a cubic perturbation, such as Minorsky's equation. However, one cannot be sure of the validity of the obtained approximation. Although it may be very challenging to solve the cubic case, the outcome will be great, as we will be able to obtain approximations for many DDEs that occur naturally in real life.

We have considered a small collection of examples in this thesis, which we thought would illustrate the constructed perturbation method well. Now we can continue to expand the use of this method by considering many more examples and solving the challenges that come with them.

Appendix A

Solution Curve and Approximations for a First Order Characteristic Equation

For a linear first order scalar DDE,

$$\frac{dy(t)}{dt} + ay(t) + by(t-1) = 0, \tag{A.1}$$

we have the following corresponding characteristic equation

$$\lambda + a + be^{-\lambda} = 0. \tag{A.2}$$

The solutions of this equation can be expressed in terms of the Lambert W function. As Maple and Matlab can calculate the Lambert W function easily, this is a useful tool to express the solutions with. However, it does not present much insight in the locations of the solutions of the characteristic equation. We will try to get a sense of the solutions of the characteristic equation, using more elementary functions. We will first derive a solution curve on which all solutions must lie. Such a solution curve can be very useful when determining the solutions of the characteristic equation numerically. For higher order DDEs, we cannot generally express the solutions of the characteristic equation in terms of the Lambert W function, so we will have to use a solution curve to find the solutions numerically. Next, we will derive approximations for the larger solutions of the characteristic equation.

A.1 Solution Curve

We start by writing $\lambda = \nu + i\omega$, with $\nu, \omega \in \mathbb{R}$, and splitting the characteristic equation into a real and imaginary part,

$$\begin{aligned} \nu + a + be^{-\nu}\cos\omega &= 0, \\ \omega - be^{-\nu}\sin\omega &= 0. \end{aligned}$$

Define $\mu = \nu + a$ and $c = -be^{a}$, the equations become

$$\begin{aligned} \mu &= ce^{-\mu}\cos\omega, \\ \omega &= -ce^{-\mu}\sin\omega. \end{aligned}$$

Squaring both equations and adding them yields

$$\mu^2 + \omega^2 = c^2e^{-2\mu}.$$

We define the function

$$f(\mu) = c^2 e^{-2\mu} - \mu^2, \tag{A.3}$$

then we find

$$\omega = \pm\sqrt{f(\mu)}. \tag{A.4}$$

Note that the $\pm$ corresponds to the fact that conjugates are both solutions for the characteristic equation. We are interested in $f(\mu) \geq 0$, such that we find real values for ω. Note that $f(\mu) \to \mp\infty$ for $\mu \to \pm\infty$. We are interested in the roots of $f'(\mu)$. First,

$$f'(\mu) = -2c^2 e^{-2\mu} - 2\mu.$$

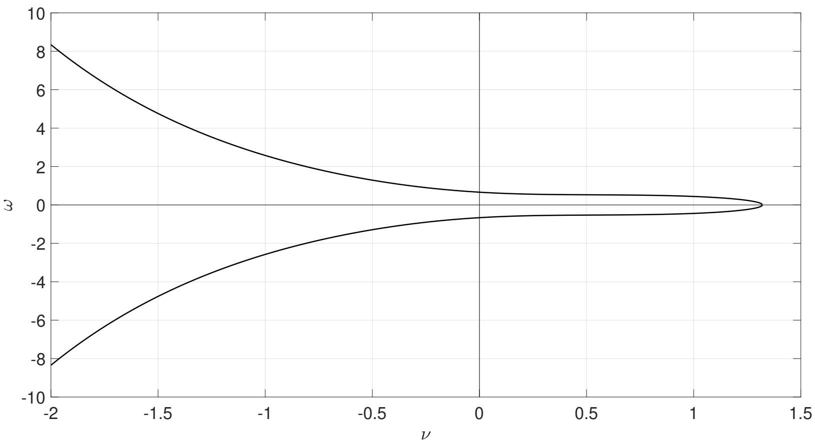
For the roots we have

$$2\mu e^{-2\mu} = -2c^2.$$

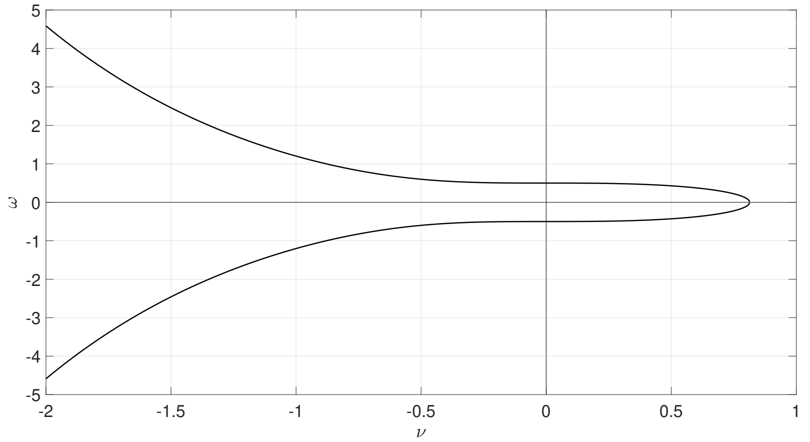
Define $d = -2c^2$ and note that d is negative, since $c = 0$ is not an option. If this would be the case, then we are not considering a DDE anymore.

One can easily see that $2\mu e^{-2\mu} < 0$ for $\mu < 0$ and $2\mu e^{-2\mu} \geq 0$ for $\mu \geq 0$. To have the equality, we must have $\mu < 0$. Also, $2\mu e^{-2\mu} \in (-e^{-1}, 0)$ for $\mu \in (-\frac{1}{2}, 0)$, $2\mu e^{-2\mu} = -e^{-1}$ for $\mu = -\frac{1}{2}$ and $2\mu e^{-2\mu} \in (-e^{-1}, 0)$ for $\mu \in (-\infty, -1)$. Then, if $d < -e^{-1}$, we cannot find any μ to satisfy the equality. Thus, $f'(\mu) < 0$ for all μ then. Then, $f(\nu)$ has exactly one root. We find real values for ω, for μ smaller than that root. If $d = -e^{-1}$, then $\mu = -1$ is the only (double) root of $f'(\mu)$. If $-e^{-1} < d < 0$, then there are two roots of $f'(\mu)$, $\mu_0 < -\frac{1}{2}$ and $-\frac{1}{2} < \mu_1 < 0$. We are interested in the values of f at these points. One can find that always $f(\mu_1) > 0$. Also, there exists a value d^*, such that $f(\mu_0) = 0$. For $-e^{-1} < d < d^*$, $f(\mu_0) > 0$. For $d^* < d < 0$, $f(\mu_0) < 0$.

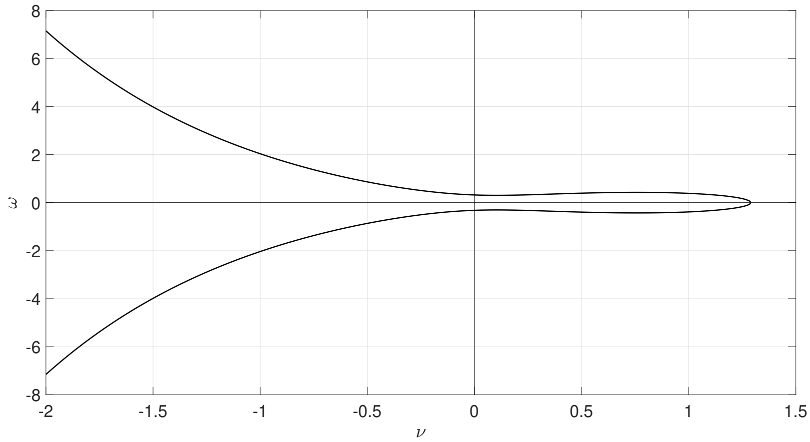
Figure A.1 shows what the curve (ν, ω) looks like for different regions of d.



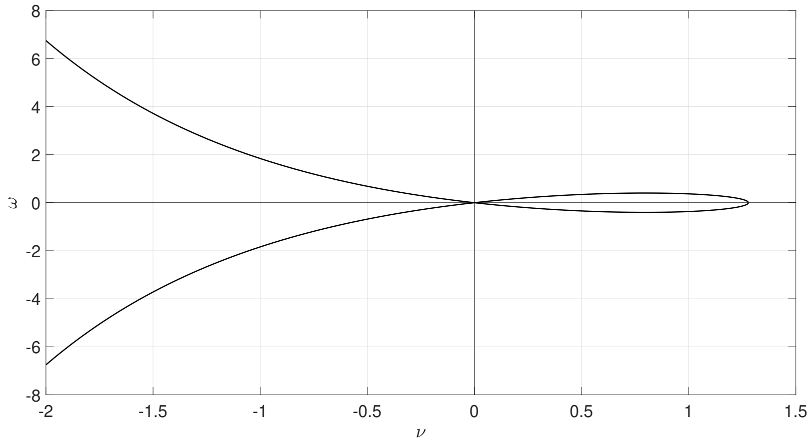
(a) $d < -e^{-1}$.



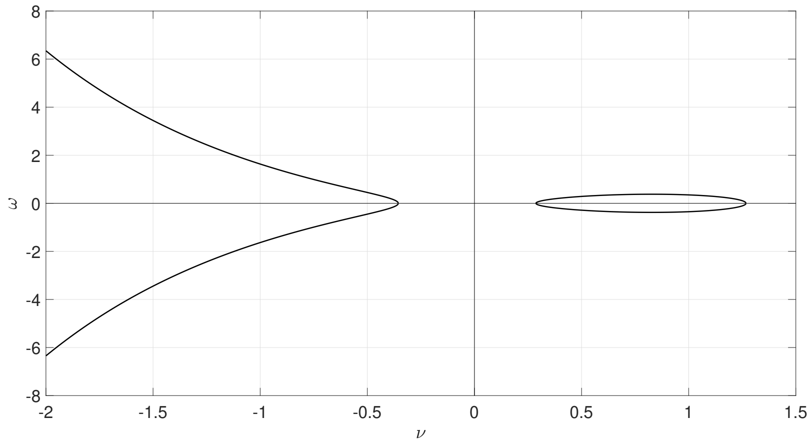
(b) $d = -e^{-1}$.



(c) $-e^{-1} < d < d^{*}$.



(d) $d = d^{*}$.



(e) $d^* < d < 0$.

Figure A.1: Plots of solution curves for different regions of d.

One can now write ω as a function of ν and put this into the characteristic equations. We then have two equations only depending on ν. We can find numerically, when these two equations are satisfied simultaneously.

A.2 Approximations for the Large Solutions of the Characteristic Equation

We will find approximations for the large solutions of the characteristic equation. To this end, we first define $\mu = \lambda + a$ and $c = -be^a$, such that we can rewrite the characteristic equation into the following form,

$$\mu = ce^{-\mu}.$$

$\mu \in \mathbb{C}$, so there exist $r > 0$ and $\theta \in (-\pi, \pi]$, such that $\mu = re^{i\theta}$. We exclude the case $r = 0$, because $\mu = 0$ is only a solution if $c = 0$, in which case we are not considering a DDE. Using this notation, the equation becomes

$$re^{i\theta} = ce^{-r\cos\theta}e^{-ir\sin\theta},$$

where we have written out $e^{i\theta}$. From this equality we immediately obtain

$$r = |c|e^{-r\cos\theta}.$$

If $c > 0$, then

$$\theta = -r\sin\theta + 2k\pi,$$

with $k \in \mathbb{Z}$. If $c < 0$, then

$$\theta = \pi - r\sin\theta + 2k\pi.$$

Define

$$K(k) = \begin{cases} 2k, & c > 0, \\ 2k+1, & c < 0. \end{cases}$$

Then,

$$r\sin\theta = -\theta + K(k)\pi.$$

We can rewrite $r = |c|e^{-r\cos\theta}$, by taking the logarithm,

$$\cos\theta = -\frac{\ln\frac{r}{|c|}}{r}.$$

We will use expansions for r and θ, to obtain approximations. Write, $r = r_0 + r_1 + \dots$ and $\theta = \theta_0 + \theta_1 + \dots$, with $r_0 \gg r_1 \gg \dots$ and $\theta_0 \gg \theta_1 \gg \dots$. Note that for large r, $\cos\theta \sim 0$, so we must have $\theta_0 = \pm\frac{1}{2}\pi$. We will continue with $\theta_0 = \frac{1}{2}\pi$.

We have the following equations,

$$\cos\theta = -\frac{\ln\frac{r}{|c|}}{r},$$
$$r = \frac{-\theta + K(k)\pi}{\sin\theta}.$$

Then, using Taylor expansions around $\theta_0 = \frac{1}{2}\pi$ and r_0, the equations can be rewritten into the following form,

$$(\theta_1+\theta_2+\dots)-\frac{1}{6}(\theta_1+\theta_2+\dots)^3+\dots = \frac{\ln\frac{r_0}{|c|}}{r_0}+\frac{1-\ln\frac{r_0}{|c|}}{r_0^2}(r_1+r_2+\dots)+\frac{-3+2\ln\frac{r_0}{|c|}}{2r_0^3}(r_1+\dots)^2+\dots, \quad \text{(A.5)}$$

and

$$r_0 + r_1 + \dots = \left(K(k) - \frac{1}{2}\right)\pi\left(1 + \frac{1}{2}(\theta_1 + \dots)^2 + \dots\right) - \left((\theta_1 + \dots) + \frac{1}{2}(\theta_1 + \dots)^3 + \dots\right). \quad \text{(A.6)}$$

We will use Eq. (A.5) to find expressions for the terms in the expansion for θ and Eq. (A.6) for r. We will first find r_0, then θ_1, then r_1, etcetera. Every time we must find the dominant term in the equations, and then the next term must be equal to that dominant term. This can be checked most easily with a program like Maple or Matlab. We then find

$$r_0 = \left(K(k) - \frac{1}{2}\right)\pi \quad \text{(A.7)}$$

and

$$\theta_1 = \frac{\ln\frac{r_0}{|c|}}{r_0}. \quad \text{(A.8)}$$

For r_1, we have two options, $\frac{1}{2}\theta_1^2 r_0$ and $-\theta_1$. It turns out that the first is the dominant term, so we have

$$r_1 = \frac{1}{2}\theta_1^2 r_0. \quad \text{(A.9)}$$

We continue this and find

$$\theta_2 = \frac{1-\ln\frac{r_0}{|c|}}{r_0^2}r_1, \qquad r_2 = -\theta_1, \quad \text{(A.10)}$$
$$\theta_3 = \frac{1}{6}\theta_1^3, \qquad r_3 = \theta_1\theta_2 r_0. \quad \text{(A.11)}$$

Then we have the following approximations,

$$r \sim r_0 + r_1 + r_2 + r_3, \quad \text{(A.12)}$$
$$\theta \sim \theta_0 + \theta_1 + \theta_2 + \theta_3. \quad \text{(A.13)}$$

Using the approximations, we can calculate an approximation for μ, and then for λ. Figure A.2 shows a plot of the exact solution for λ, and the approximations. These are the first branches, but not λ_0. For λ_0, the approximation is inaccurate.

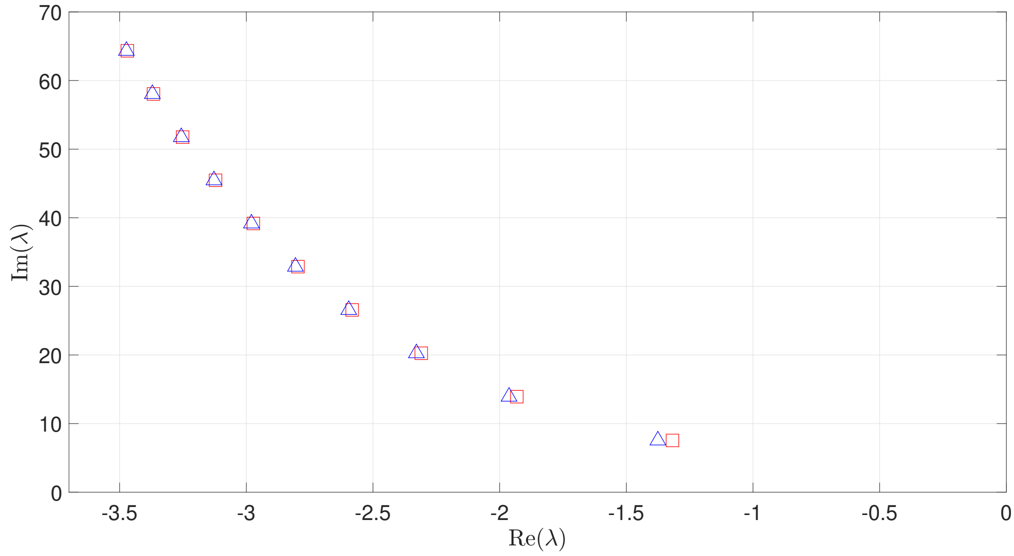


Figure A.2: Plot of the exact solutions, (blue) triangle, and the approximations, (red) square.

Figure A.3 shows a plot of the relative error of the approximations.

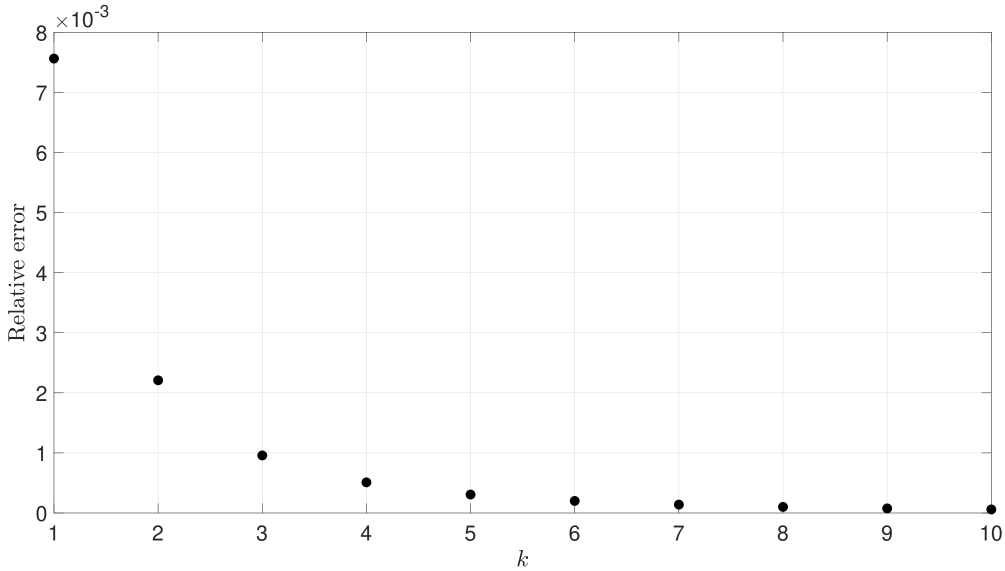


Figure A.3: Plot of the relative error of the approximations.

Appendix B

The Fundamental Solution of a First Order Delay Differential Equation

In Chapter 2 we find the solutions for first and higher order DDEs. In the derivation we take the Laplace transform of the DDE for $y(t)$. This yields an equation for $\mathcal{L}(y(t))$. For a linear first order DDE we find

$$\mathcal{L}(y(t)) = \left(s + a + be^{-s}\right)^{-1} \left(\phi(0) - b\mathcal{L}(\phi(t-1))\right).$$

We then define the fundamental solution, $Y(t)$, as

$$\mathcal{L}(Y(t)) = \left(s + a + be^{-s}\right)^{-1}.$$

In this appendix we will find an expression for $Y(t)$.

We can solve for $Y(t)$ using the inverse Laplace transform,

$$Y(t) = \lim_{\xi\to\infty} \frac{1}{2\pi i} \int_{\gamma - i\xi}^{\gamma + i\xi} \frac{e^{st}}{s + a + be^{-s}} ds. \tag{B.1}$$

γ must be taken such that the real parts of all the poles of $\left(s + a + be^{-s}\right)^{-1} e^{st}$ are smaller than γ. The poles are exactly the solutions of the characteristic equation corresponding to the linear DDE. The largest real part is found at the principal branch of the Lambert W function, λ_0 (see [5]). Let $\gamma > \mathrm{Re}(\lambda_0)$, then this inverse Laplace transform is well-defined. To calculate this integral, we consider the contour, Γ, as in Figure B.1. We have split Γ up into two pieces, the vertical line which we need for the inverse Laplace transform, Γ_1, and an arc, Γ_2.

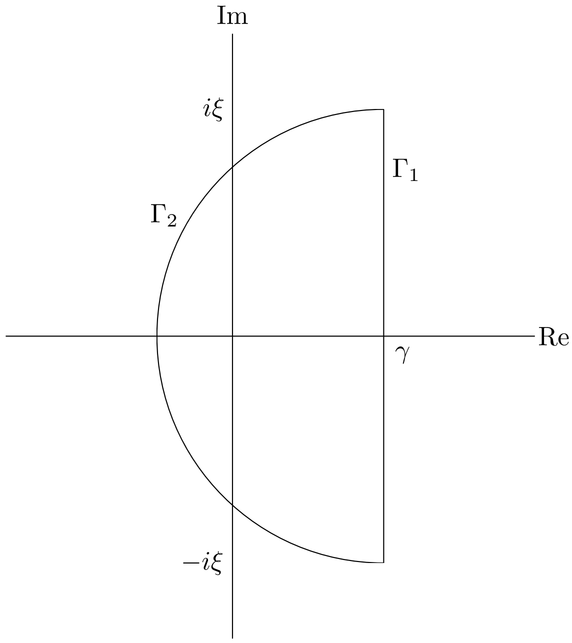


Figure B.1: The contour Γ which is used to calculate the integral in Eq. (B.1)

To evaluate the integral, we use the residue theorem, also known as Cauchy's residue theorem, (see for example [3])

$$\lim_{\xi\to\infty}\oint_\Gamma \frac{e^{st}}{s+a+be^{-s}} = 2\pi i \sum_{k=-\infty}^{\infty} \text{Res}\left(\frac{e^{st}}{s+a+be^{-s}}, \lambda_k\right),$$

with "Res" the residues in the poles. For a pole of multiplicity m, this can be calculated using the following equation

$$\text{Res}\left(\frac{e^{st}}{s+a+be^{-s}}, \lambda_k\right) = \frac{1}{(m-1)!}\lim_{s\to\lambda_k}\frac{d^{m-1}}{ds^{m-1}}\left((s-\lambda_k)^m\frac{e^{st}}{s+a+be^{-s}}\right). \tag{B.2}$$

In our case, the multiplicity is one for all λ_k, except when when $be^a = e^{-1}$. In this case $\lambda_{-1} = \lambda_0 = -(a+1)$. We use L'Hôpital's rule to evaluate the limit. In the case of multiplicity 2,

$$\text{Res}\left(\frac{e^{st}}{s+a+be^{-s}}, -(a+1)\right) = (6t+2)e^{-(a+1)t}, \tag{B.3}$$

and for the other cases,

$$\text{Res}\left(\frac{e^{st}}{s+a+be^{-s}}, \lambda_k\right) = \frac{1}{1-be^{-\lambda_k}}e^{\lambda_k t}. \tag{B.4}$$

The arc in Figure B.1, Γ_2, can be described as $s = \gamma + \xi e^{i\theta}$, with $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Then,

$$\int_{\Gamma_2}\frac{e^{st}}{s+a+be^{-s}}ds = \int_{\pi/2}^{3\pi/2}\frac{e^{(\gamma+\xi e^{i\theta})t}}{\gamma+\xi e^{i\theta}+a+be^{-(\gamma+\xi e^{i\theta})}}i\xi e^{i\theta}d\theta.$$

We will estimate the magnitude of this integral,

$$\left|\int_{\pi/2}^{3\pi/2}\frac{e^{(\gamma+\xi e^{i\theta})t}}{\gamma+\xi e^{i\theta}+a+be^{-(\gamma+\xi e^{i\theta})}}i\xi e^{i\theta}d\theta\right| \le \pi\xi^2\sup_{\Gamma_2}\left|\frac{e^{(\gamma+\xi e^{i\theta})t}}{\gamma+\xi e^{i\theta}+a+be^{-(\gamma+\xi e^{i\theta})}}\right|.$$

Using that the real part of $e^{i\theta}$ is negative for our choice of θ, taking the limit $\xi\to\infty$ causes the numerator to go to zero exponentially and the denominator to infinity exponentially. Thus, the integral over Γ_2 will go to zero for $\xi \to \infty$. This means that the integral over the vertical line, Γ_1, which we wish to know for the inverse Laplace transform, is equal to the contour integral.

If $be^a = e^{-1}$, then

$$Y(t) = (12t + 4)e^{-(a+1)t} + \sum_{\substack{k=-\infty \\ k\neq -1,0}}^{\infty} \frac{1}{1 - be^{-\lambda_k}} e^{\lambda_k t}. \tag{B.5}$$

If this is not the case, then

$$Y(t) = \sum_{k=-\infty}^{\infty} \frac{1}{1 - be^{-\lambda_k}} e^{\lambda_k t}. \tag{B.6}$$

These are the two possible fundamental solutions.

Appendix C

Sum of Solutions of the Characteristic Equation for a First Order Delay Differential Equation

In this chapter we shall consider whether the sum of two solutions of the characteristic equation for a first order DDE, can again be a solution of this characteristic equation. The general first order linear DDE that we consider is

$$\frac{dy(t)}{dt} + ay(t) + by(t-1) = 0.$$

The corresponding characteristic equation is

$$\lambda + a + be^{-\lambda} = 0.$$

We can write $\lambda = \nu + i\omega$, with $\nu, \omega \in \mathbb{R}$, and split the characteristic equation into a real and imaginary part,

$$\nu + a + be^{-\nu}\cos\omega = 0, \tag{C.1}$$

$$\omega - be^{-\nu}\sin\omega = 0. \tag{C.2}$$

We see that to satisfy the first equation for "large" negative ν, $\cos\omega$ must be "small". Thus, we must have $\omega \approx \pm\frac{1}{2}\pi + k\pi$, with $k \in \mathbb{Z}$. As negative and positive ω correspond to conjugate solutions, we will focus only on positive ω. To satisfy the latter equation, we see that we must have $b\sin\omega > 0$. To this end, define

$$K(k) = \begin{cases} 2k, & b > 0, \\ 2k+1, & b < 0, \end{cases}$$

such that

$$\omega \approx \left(K(k) + \frac{1}{2}\right)\pi.$$

Now suppose that we have two solutions of the characteristic equation, λ_k and λ_l. The imaginary part of the sum of these solutions will be approximately $N\pi$, with N an integer. This will cannot be a solution, except when there is a real solution and $N = 0$.

We will show how to determine the parameters a and b, such that this occurs. For simplicity of calculation, we will use the Lambert W function. In Chapter 2 we already showed that the solutions of the characteristic equation are given by

$$\lambda_k = W_k\left(-be^a\right) - a,$$

with $k = 0, \pm1, \pm2, \ldots$ the branches of the Lambert W function. Define $c = -be^a$. For $c \in [-e^{-1}, 0)$, $\lambda_{-1} \in \mathbb{R}$ and for $c \in [-e^{-1}, \infty)$, $\lambda_0 \in \mathbb{R}$. Suppose we wish to have $\lambda_k + \overline{\lambda}_k = \lambda_0$. Then take any $c \in [-e^{-1}, \infty)$ and let

$$a_0(k, c) = 2\text{Re}(W_k(c)) - W_0(c). \tag{C.3}$$

To satisfy the definition of c, we must let

$$b_0(k,c) = -ce^{-a_0(k,c)}. \tag{C.4}$$

Similarly we can define

$$a_{-1}(k,c) = 2\mathrm{Re}(W_k(c)) - W_{-1}(c), \tag{C.5}$$

$$b_{-1}(k,c) = -ce^{-a_{-a}(k,c)}. \tag{C.6}$$

For this case, we must take $c \in [-e^{-1}, 0)$. We can also ensure that $\lambda_k + \overline{\lambda}_k = \lambda_{-1}$ and $\lambda_l + \overline{\lambda}_l = \lambda_0$ at the same time. This will not be possible for any k and l. But suppose we take $k = 2$ and $l = 1$, then it is possible. Figure C.1 shows a plot of $a_{-1}(2,c)$ and $a_0(1,c)$ for $c \in [-e^{-1}, 0)$. One has to calculate where these lines intersect, such that we have $\lambda_2 + \lambda_{-3} = \lambda_{-1}$ and $\lambda_1 + \lambda_{-2} = \lambda_0$ for the same set of parameters, (a, b).

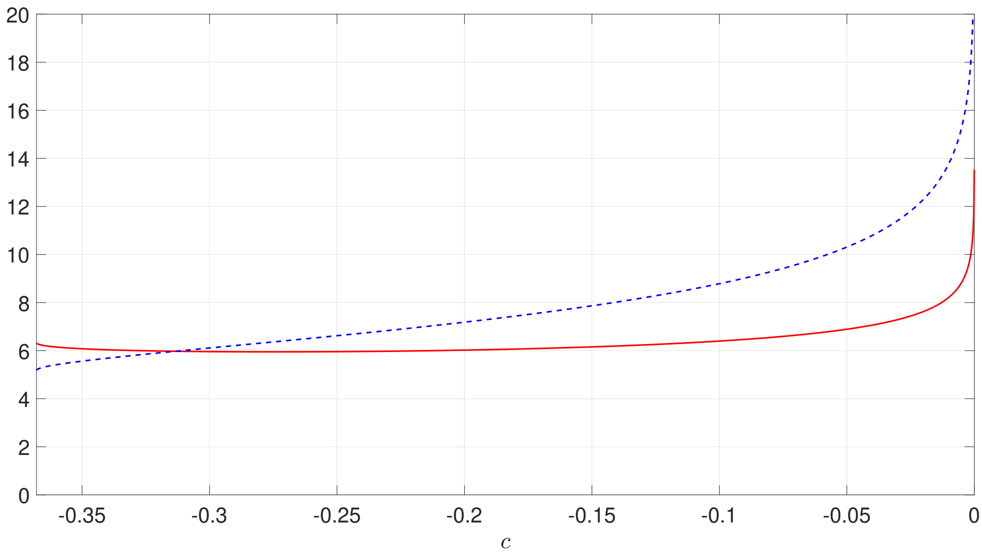


Figure C.1: Plot of $a_{-1}(2,c)$, solid (red) line, and $a_0(1,c)$, dashed (blue) line.

Note that what we have just derived, ensures that the sum of two solutions of the characteristic equation is again a solution of the characteristic equation. Furthermore, we have given a reasoning why this can be the only possibility. However, for this reasoning we have used an approximation for the imaginary part of the solutions. Therefore, we are not certain whether there do not exist other solutions for which the sum is again a solution. We will derive a method to check whether there are more possibilities.

Suppose we have two solutions, λ_k and λ_l and assume that $\lambda_k + \lambda_l$ is also a solution. We split these into a real and imaginary part as before, $\lambda = \nu + i\omega$. Then define $\mu = \nu + a$. We have the following six equations that should be satisfied simultaneously,

$$\begin{aligned} \mu_k &= -be^a e^{-\mu_k} \cos\omega_k, \\ \omega_k &= be^a e^{-\mu_k} \sin\omega_k, \\ \mu_l &= -be^a e^{-\mu_l} \cos\omega_l, \\ \omega_l &= be^a e^{-\mu_l} \sin\omega_l, \\ b\,(\mu_k + \mu_l - a) &= -be^{2a} e^{-(\mu_k+\mu_l)} \cos(\omega_k + \omega_l), \\ b\,(\omega_k + \omega_l) &= -be^{2a} e{-}(\mu_k + \mu_l) \sin(\omega_k + \omega_l). \end{aligned}$$

We rewrite the latter two, using the first four and the following trigonometric identities,

$$\begin{aligned} \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta, \\ \sin(\alpha + \beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha. \end{aligned}$$

We then get

$$\begin{aligned} b\left(\mu_k+\mu_l-a\right) &= -b^2e^{2a}e^{-(\mu_k+\mu_l)}\left(\cos\omega_k\cos\omega_l-\sin\omega_k\sin\omega_l\right) \\ &= -\left(-be^a e^{-\mu_k}\cos\omega_k\right)\left(-be^a e^{-\mu_l}\cos\omega_l\right)+\left(be^a e^{-\mu_l}\sin\omega_k\right)\left(be^a e^{-\mu_l}\sin\omega_l\right) \\ &= -\mu_k\mu_l+\omega_k\omega_l. \end{aligned}$$

Similarly, we obtain

$$b\left(\omega_k+\omega_l\right) = -\omega_k\mu_l-\omega_l\mu_k.$$

We now have two equations and four unknowns, μ_k, ω_k, μ_l and ω_l. We can solve the equations for μ_k and ω_k. The solutions are given by

$$\begin{aligned} \mu_k &= -\frac{b\left(\omega_l^2+(\mu_l-a)(\mu_l+b)\right)}{\omega_l^2+(\mu_l+b)^2}, \\ \omega_k &= -\frac{b(a+b)\omega_l}{\omega_l^2+(\mu_l+b)^2}. \end{aligned}$$

We can also express μ_k in terms of ω_k, using the imaginary part of the characteristic equation,

$$\mu_k = \ln\left(be^a\frac{\sin\omega_k}{\omega_k}\right).$$

We define the functions

$$f(\mu,\omega) = -\frac{b\left(\omega^2+(\mu-a)(\mu+b)\right)}{\omega^2+(\mu+b)^2}, \tag{C.7}$$

$$g(\mu,\omega) = -\frac{b(a+b)\omega}{\omega^2+(\mu+b)^2}, \tag{C.8}$$

$$h(\omega) = \ln\left(be^a\frac{\sin\omega}{\omega}\right). \tag{C.9}$$

We can then write down μ_k as a function of ω_l in two ways. First, write μ_l as a function of ω_l using h. Then calculate μ_k and ω_k, using the functions f and g respectively. We then have the first expression for μ_k. Finally, calculate the second expression, by putting the expression for ω_k in h. We define the functions

$$F(\omega) = f(h(\omega),\omega), \tag{C.10}$$

$$G(\omega) = h(g(h(\omega),\omega)), \tag{C.11}$$

such that $\mu_k = F(\omega_l)$ and $\mu_k = G(\omega_l)$ should hold. We define the function

$$H(\omega) = G(\omega) - F(\omega). \tag{C.12}$$

We seek the roots of $H(\omega)$. Given a root of H, we can calculate the solutions λ_k and λ_l. However, we must still check whether λ_k, λ_l and $\lambda_k+\lambda_l$ are indeed solutions of the characteristic equation.

We will now focus on finding the roots of H. If λ is a solution, then so is $\overline{\lambda}$. So, we only have to check for $\omega \geq 0$. For $\omega = 0$, we have real solutions. The only possibility for the sum of two real solutions to again be a solution, is when $2\lambda_0 = \lambda_{-1}$ or $2\lambda_{-1} = \lambda_0$. We will continue with the case $\omega > 0$.

We can ease our task of finding all roots of H, if we can limit the interval of ω which we have to consider. Note that we have $\omega_k = g(h(\omega_l),\omega_l)$. For large ω, $g(h(\omega),\omega) \sim \frac{1}{\omega}$. As the solutions of the characteristic equation are isolated, we can find a Ω, such that for $\omega > \Omega$, we know that $g(h(\omega),\omega)$ can never be the imaginary part of a solution. Consequently, we only need to consider $\omega \in (0,\Omega)$.

To find the roots of H, we need to know whether there are sign flips. To this end, it is interesting to consider certain limits of $H(\omega)$. It is interesting to know with which sign we start and end, so we consider the limits $\omega \to 0$ and $\omega \to \infty$. For $\omega \to 0$, we have $h(\omega) \to \ln(b) + a$. Then $g(h(\omega), \omega) \to 0$, $h(g(h(\omega), \omega)) \to \ln(b) + a$ and $f(h(\omega), \omega) \to -\frac{b\ln(b)}{\ln(b)+a+b}$, such that

$$H(\omega) \to \frac{b\ln(b)}{\ln(b)+a+b} + \ln(b) + a, \qquad \text{as } \omega \to 0. \tag{C.13}$$

For $\omega \to \infty$, $h(\omega) \to -\infty$. This happens logarithmic, so it goes to infinity "slower" than ω. Then, $g(h(\omega), \omega) \to 0$, $h(g(h(\omega), \omega)) \to \ln(b) + a$ and $f(h(\omega), \omega) \to -b$, such that

$$H(\omega) \to \ln(b) + a + b. \tag{C.14}$$

If $\omega \to 0$ and $\omega \to \infty$ lead to different signs for H, then we know that there will be a sign flip, and thus possibly a root.

We will now set a and b, and see how the function H can be used. We note two things about plotting H. First, H is not real valued for every ω, due to the fact that we can have negative input in a logarithm. We only consider the intervals for which H is real valued. Second, the plot may not be accurate for $\omega \to n\pi$, $n = 1, 2, \ldots$. Then $h(\omega) \to \infty$ analytically, since $\frac{\sin\omega}{\omega} \to 0$. Numerically however, we get $\sin\omega \to \epsilon$, with ϵ the machine precision. Suppose we have $\epsilon = 10^{-16}$, then $\ln(\epsilon) = -16$. So, instead of $-\infty$, we get some finite negative number. This will then continue to influence the behaviour of H in these regions. One should take this into consideration when investigating the plots.

Figure C.2 shows a plot of $H(\omega)$, together with the markers of the imaginary parts of the first solutions. We have used $a = a_0(1, 10)$ and $b = b_0(1, 10)$, such that we know that $\lambda_1 + \lambda_{-2} = \lambda_0$. We find two intersections for $H(\omega)$, and one corresponds to the imaginary part of λ_1. The other intersection does not correspond to a solution, thus we ignore this root of $H(\omega)$. We can then use the functions f, g and h to calculate the two solutions. This yields λ_1 and λ_{-2}, as expected.

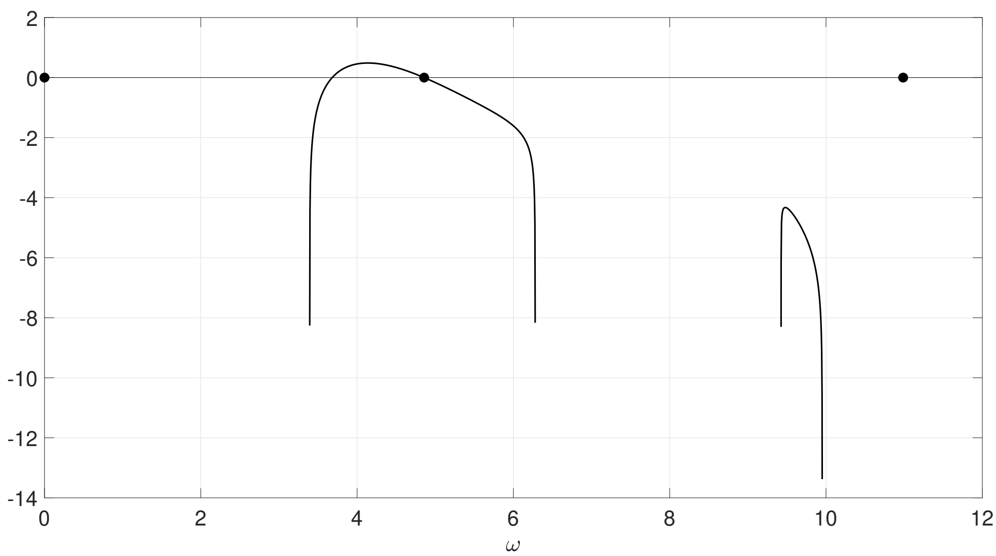


Figure C.2: Plot of $H(\omega)$ using $a = a_0(1, 10)$ and $b = b_0(1, 10)$. The markers are the imaginary parts of the solutions of the characteristic equation for the first branches.

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