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Affine Caps

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Preface

This document is my bachelor thesis. This is the final report of my bachelor project "Affine Caps", for the program Applied Mathematics of the Technical University Delft. I had the chance to choose a project or invent my own project. I choose to do the first, the department of optimization had my preference since I attended the courses "Optimization" and "Combinatorial Optimization". As I like to solve puzzles and play games, this final project about Set seemed perfect for me.
The project was more theoretical in some parts than I wished for, but after all I am glad I choose this project. It showed me that I can do much more than I thought and it confirmed my preference for optimization. It taught me how to write a thesis and how to work in a structured way.
First of all I want to thank my mentor Dion Gijswijt for his time and patience. He gave me the freedom to think of new solutions and helped me with new questions if I did not know what to do. Second I want to thank the optimization department for the seminars, the preparations for my presentations and the feedback on my presentation were always very helpful. Finally I want to thank my friends, for all the laughs, the distractions and the motivational speeches to work hard.

S.B. Vertregt
Delft, July 2015

Abstract

This thesis concerns the proof that twenty points are a 4-cap and gives an upper bound for higher dimensions. First I explain what the card game set is and give a mathematical interpretation for the game. Then I introduce some concepts that will be used in the proof. Then I will explain some lower dimension proofs to make more clear what the main structure is of the proof fo the 4-cap. In chapter 3 I will give the proof of the 4-cap and give the first found upper bound for higher dimensions. In chapter 4 an other upper bound for higher dimension will be proven and we will compare both found bounds for different dimensions.

Contents

Chapter 1

Introduction

The card game Set is a game for quick thinkers. You can only win if you can find combinations faster than your opponents. The main goal is collecting cards, by finding a 'set' before your opponents do, you can collect the three cards of the set.
The game starts with 12 cards face-up on the table. If a set is found by one of the players, these cards are removed by that player and 3 other cards are put face-up on the table, such that there are 12 cards on the table.
If there is no set found at one point in the game, 3 other cards are put face-up on the table, until there is a set found. The cards are not replaced until there are again 12 cards face-up on the table.
This continues until the stack of cards is empty. It is possible there are still cards face-up on the table at the end of the game, that do not contain a set. Every player counts the number of sets, the player with the most sets wins!

1.1 The card game Set

The card game Set is played with a deck of 81 cards, specially designed for this game. On each card is

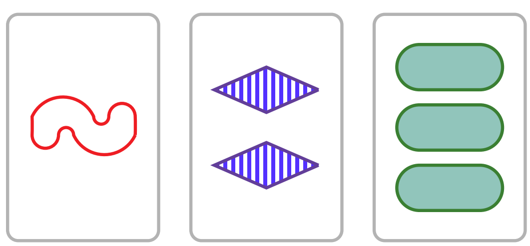


Figure 1.1: All options in characteristics

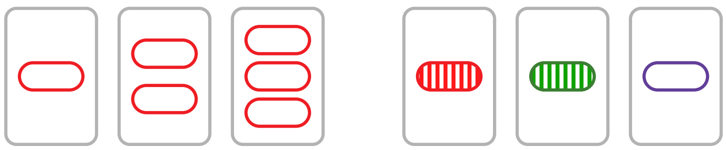
a unique design with 4 characteristics.
The card shows one possible combination of 4 characteristics — number, filling, color, shape — and each characteristic has 3 possible values.

number	:	One, Two, Three
filling	:	Open, striped, Solid
color	:	Red, Purple, Green
shape	:	Diamonds , Ovals, Wiggles

The goal of the game is to collect the highest amount of cards. You can do this by searching for combinations, that satisfy the following rule, in the cards on the table.

Rule (Set Rule)**.** *Three cards are called a* Set *if, with respect to each of the 4 characteristics, the cards are either all the same or all different.*

Example 1. *Figure 1.2a shows an example of a set. All cards have the same color, shading and shape and all cards are different in number of shapes.*
Figure 1.2b shows an example of a combination of three cards that do not form a set. The cards all have the same shape and number. The cards are all different in color, but the filling of two cards are the same (striped) and one is different (Open). Therefore, the filling characteristic does not satisfy the Set rule.



(a) A set (b) Not a set

Figure 1.2

Although finding a set could take seconds, in other cases there will be 12 cards face-up on the table that do not contain a Set. To continue the game, three more cards are drawn. But still with 15 cards on the table it is possible that there is no Set. Therefore the following problem is important for Set.

Problem. *How many cards can be dealt without creating a Set?*

We find that it is possible to draw 20 cards without creating a set(see Figure 1.3), whether this is the maximum number of cards without creating a set is not yet proven.

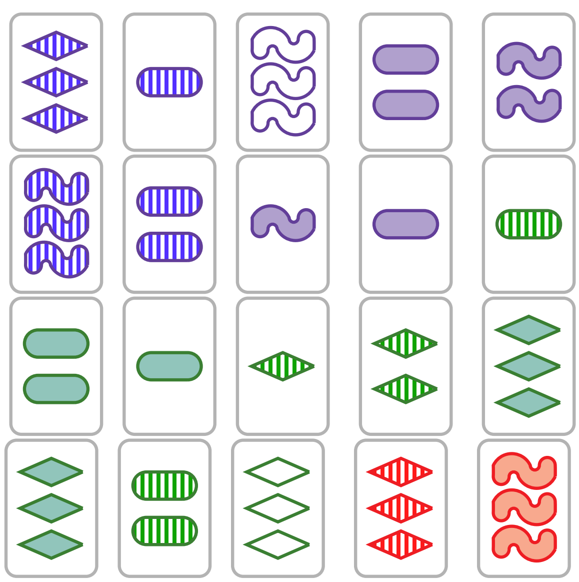


Figure 1.3: Twenty cards without a set

There is no really easy way to show that these cards do not create a set, other than systematically searching for combinations. For instance, we only have two red cards, therefore we can not have an entirely red set. Hence, you can check entirely green, purple and multicoloured sets until you have checked all combinations. To solve this problem systematically, we make a mathematical interpretation of a card.

1.2 Mathematical interpretation of Set

Set can be described in a mathematical manner. The 81 cards are the elements of the vector space $\mathbb{F}_3^4$. The field $\mathbb{F}_3$ has three elements, for each characteristic there are 3 different options.

Definition 1 (Field)**.** *A field is a finite set F, with two maps addition $+ : F \times F \to F$ and multiplication $\cdot : F \times F \to F$ such that the following hold;*

1. *The maps $+$ and $\cdot$ are associative, $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$*
2. *The maps $+$ and $\cdot$ are commutative, $a + b = b + a$ and $a \cdot b = b \cdot a$*
3. *The maps $+$ and $\cdot$ are distributive, $a \cdot (b + c) = a \cdot b + a \cdot c$.*
4. *F has a unit element for addition $(+)$, which we denote with 0, and a unit element for multiplication $(\cdot)$, which we denote with 1, with $1 \neq 0$.*
5. *Every $a \in F$ has a inverse for addition.*
6. *Every $0 \neq a \in F$ has a inverse for multiplication.*

In this case, we work in the field $\mathbb{F}_3$. More general we can define $\mathbb{F}_p$, with p a prime number, as the set of the elements $\{0, 1, \cdots, p - 1\}$, with addition and multiplication modulo p.
Every Set card has 4 characteristics, therefore a card is an element of $\mathbb{F}_3^4$. An element of $\mathbb{F}_3^4$ can be described as a 4-tuple of the form (x_1, x_2, x_3, x_4), each coordinate corresponds to one characteristic and can assume 3 different values: $0, 1, 2$.

Example 2. *The element $(1, 0, 2, 1)$ can correspond to "Two Open Green Ovals"*

We can desribe the set rule also mathematically, three points form a set if and only if $a + b + c = 0$ or three points form a set if and only if a, b and c are collinear.

Definition 2 (Line)**.** *We define a line as the set $\{a + \lambda b \quad | \quad a, b \in \mathbb{F}_3^4, b \neq 0, \lambda \in \mathbb{F}_3\}$.*

Proof. Suppose we have three points x, y and z lie on one line, with the definition of a line, we know we can write the points as;

$$\begin{cases} x = x + 0 \cdot (y - x) \\ y = x + 1 \cdot (y - x) \\ z = x + 2 \cdot (y - x) \end{cases}$$

Adding up all points gives; $x + y + z = x + x + (y - x) + x + 2(y - x) = 3y = 0 \bmod 3$.
Suppose $x + y + z = 0$, then we can rewrite this a $x - y + z = -2y \Leftrightarrow x - y = y - z$. The difference between x and y is the same as the difference between y and z, thus x, y and z lie on a line $\{x + \lambda(y - x) \quad | \quad a, b \in \mathbb{F}_3^4, b \neq 0, \lambda \in \mathbb{F}_3\}$. □

Example 3. *Suppose we have the following 3 points* $a = (1, 0, 2, 1)$ *,* $b = (1, 2, 1, 1)$ *and* $c = (1, 1, 0, 1)$*. We know* a, b *and* c *are a Set, because* $a + b + c = (3, 3, 3, 3) = (0, 0, 0, 0)$ *mod* 3*. Therefore* a, b *and* c *are collinear.*
We can also find the missing point to form a Set. Suppose we have point $a = (1, 0, 2, 1)$ *and* $b = (1, 2, 1, 1)$*. We know that* a *and* b *can be in the same Set.We add up point* a *and* b*;* $a + b = (2, 2, 3, 2)$*, for a set holds* $a + b + c = (0, 0, 0, 0)$*.* $(2, 2, 3, 2) + (1, 1, 0, 1) = (3, 3, 3, 3)$*.therefore using modular 3 arithmetics we can find the third point to form the Set;* $c = (1, 1, 0, 1)$*.*

To define the problem in a mathematical manner we introduce de Cap of a dimension.
Definition 3 (d-Cap)**.** *A d-cap is a subset of* $\mathbb{F}_3^d$ *not containing any lines, therefore the d-cap does not contain 3 collinear points. A maximum d-cap is a largest d-cap possible in* $\mathbb{F}_3^d$*.*

Problem. *How many elements of* $\mathbb{F}_3^4$ *form a maximal 4-cap? Or more general; How many elements of* $\mathbb{F}_3^d$ *form a maximal d-cap?*

In the following chapters we will give a proof that twenty is the maximum size of a 4-cap, we have seen that 20 is a possibility, but what we do not know yet if 21 is also a possibility. We will explain some definitions that we need for the proof. Last we will look at an upper bound for higher dimensions.

Chapter 2

Geometry in a vector space

This chapter is about the geometry and properties of the vector space $\mathbb{F}_3^4$. Since a Set is defined as a line, we also need to know how a plane is defined in a vector space. Also the space is affine and not linear, therefore we have to redefine certain basic properties and definitions.

2.1 Affine spaces

The vector space $\mathbb{F}_3^4$ is called an affine space, this means the vector space is shifted.

Definition 4 (Affine space)**.** *Let $L \subset V$ a linear subspace and $p \in V$, then we call $A = L + p = \{v + p \text{ such that } v \in L\}$ an affine subspace.*

There is no unit element in a affine space, but every element can become the unit element. We use affine transformations, that keep the collinearity between the elements, to make the unit element.
A set is defined as a line through the vector space. Next we want to define planes and subspaces. To define a basis for such a k-dimensional space we also need the following definitions.

Definition 5 (Affine independent)**.** *Vectors $v_1, \cdots, v_n$ are affine independent if $\exists \lambda_1, \cdots, \lambda_n$ such that $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$ and $\lambda_1 + \cdots \lambda_n = 0$ and not all λ_i equal to zero.*
In terms of linear independence, $v_1, \cdots, v_n$ are affine independent if and only if $\begin{bmatrix} v_1 \\ 1 \end{bmatrix}, \cdots, \begin{bmatrix} v_n \\ 1 \end{bmatrix}$ are linear independent.
Definition 6 (Dimension)**.** *The dimension of a linear space L is defined as the number of basis vectors to span the whole linear space. The dimension of a affine space A is defined as the number of basis vectors of the corresponding linear space L.*

2.2 Planes in $\mathbb{F}_3^d$

In chapter 1 we have defined that 3 points, that are a set, lie on one line or $\{a + \lambda b \mid a, b \in \mathbb{F}_3^4, b \neq 0, \lambda \in \mathbb{F}_3\}$. Because of the definition of a set, we know that three points on a line is the maximum, because of uniqueness of the cards. Therefore if we have two of those points, we know what the third point has to be. Thus a basis for a line is 2 points.
If we want to define a plane we need 3 points, suppose we have three points, a, b and c, such that a, b

and c are not collinear. Another definition for a plane is the collection points $\{a+\lambda_1 b+\lambda_2 c \quad | \quad a,b,c \in \mathbb{F}_3^4, b,c \neq 0, \lambda_1, \lambda_2 \in \mathbb{F}_3\}$. Than that plane defined by a, b and c will have following points;

$$\begin{array}{ccc} a & a+b & a+2b \\ a+c & a+b+c & a+2b+c \\ a+2c & a+b+2c & a+2b+2c \end{array}$$

We know that that these points are all the points in the plane because of uniqueness of the points. In terms of vectors we can define that the plane is spanned by the two vectors ab and ac.

2.3 Hyperplanes in $\mathbb{F}_3^d$

We defined what a plane is as basis for larger dimensions, because We are interested in different d-caps, we want to expand this plane more generally. Therefore we are interested in hyperplanes. Instead of looking at our whole space, we decompose our space into three parallel hyperplanes.

Definition 7 (Hyperplane)**.** *A hyperplane is a subset of the vector space such that $\forall x$ in $\mathbb{F}_3^d$ the following holds: $\{x \in \mathbb{F}_3^d : a_1x_1 + a_2x_2 + \cdots + a_dx_d = b\}$. In other words; A hyperplane is the solution set of $\{a \neq 0, x \in \mathbb{F}_3^d : a \cdot x = b\}$. The whole vector space can be decomposed in three parallel hyperplanes with $b = 0$ or $b = 1$ or $b = 2$.*

Therefore a hyperplane is $d-1$ dimensional, because the dimension of a solution set $\{a \neq 0, x \in \mathbb{F}_3^d : a \cdot x = b\}$ is equal to dimension of x minus rank of A. We know that a is a vector, thus the rank of a is 1. As x is an element of $\mathbb{F}_3^d$, x has dimension d, therefore we find dimension $d-1$ for a hyperplane.
Definition 8 (k-flat)**.** *A k-dimensional affine subspace of a vector space is called a k-flat.*

To proof that there a twenty points in the 4-cap, we will count cap points on different lines and planes. Therefore we will need the following proposition;

Proposition 1. *The number of hyperplanes containing a fixed k-flat in $\mathbb{F}_3^d$ is given by* $\frac{3^{d-k}-1}{2}$

Proof. Suppose $\{v_1, \cdots, v_k\}$ are a basis B for the k-flat. We want to expand this basis B to a basis for a hyperplane of dimension $d-1$. Suppose we find the following basis for our hyperplane of dimension $d-1$; $\{v_1, \cdots, v_k, u_{k+1}, \cdots, u_{d-1}\}$. Counting combinations we find;

$$\begin{aligned} &\Rightarrow \frac{(3^d - 3^k)(3^d - 3^{k+1}) \cdots (3^d - 3^{d-1})}{(3^{d-1} - 3^k)(3^{d-1} - 3^{k+1}) \cdots (3^{d-1} - 3^{d-2})} \\ &= 3^{d-1-k} \cdot \frac{(3^d - 3^k)(3^d - 3^{k+1}) \cdots (3^d - 3^{d-1})}{(3^d - 3^{k+1})(3^d - 3^{k+2}) \cdots (3^d - 3^{d-1})} \\ &= 3^{d-1-k} \cdot \frac{3^d - 3^k}{3^d - 3^{d-1}} \\ &= 3^{d-1-k} \cdot 3^{k-d+1} \cdot \frac{3^{d-k} - 1}{2} = \frac{3^{d-k} - 1}{2} \end{aligned}$$

□

2.4 Basic proof of small caps

Now we will use all the definitions to give a basic proof and clarify the structure used in the proof. We will count points of the cap on certain k-flats.

Proposition 2. *A maximum 2-cap has four points.*

We know that $\{(0,0),(0,1),(1,0),(1,1)\}$ is a 2-cap, adding any other point will give a Set. To prove this we can add up all possible combinations of points and show that this is a maximal cap, but this wil not prove that there is no other combination of points that can be of size greater than 4. Therefore we will make a more systematically proof.

Proof. We show by contradiction that no larger cap exists. We have seen that a cap can have four points. Suppose there is a 2-cap with five points, x_1, x_2, x_3, x_4, x_5. We can decompose $\mathbb{F}_3^2$ into three parallel lines. Each line can contain at most 2 points of the cap. We have 5 points, so two lines with 2 points and one line H with one point of the cap. Suppose x_5 is located on H. There are exactly 4 lines through x_5 (see proposition 1), L_1, L_2, L_3, H. We know x_1, x_2, x_3, x_4 are not located on H, so x_1, x_2, x_3, x_4 must be located on L_1, L_2, L_3.
We have four points that are located on three lines. By the pigeon hole principle, we know that there are two points x_j, x_k located on L_i. Therefore L_i contains three points x_j, x_k, x_5. This contradicts that x_1, x_2, x_3, x_4, x_5 can be a 2-cap. □

We can use Proposition 2 to determine the maximum size of 3-cap. We will prove by counting cap points on different planes.

Proposition 3. *A maximum 3-cap has nine points.*

We know that $\{(0,0,0),(2,0,0),(0,2,0),(2,2,0),(1,1,1),(0,1,2),(1,0,2),(1,2,2),(2,1,2)\}$ is a 3-cap, adding any other point will give a set.

Proof. We know 3-cap can be nine points. This proof will proceed by contradiction. Suppose there is a 3-cap with 10 points. We can decompose $\mathbb{F}_3^3$ in three parallel hyperplanes, these hyperplanes are all of dimension two. Therefore we know that no hyperplane contains more than four points of the cap (see Proposition 2). There is a hyperplane H with the least number of points of the cap, this hyperplane must contain two or three points. (four points wil give a total of twelve points in the cap, zero or one point wil give a total of at most 9 points in the cap). There are at least seven points in the cap not contained in H, $x_1, x_2, x_3, x_4, x_5, x_6, x_7$.
Let a and b be two points of the cap on plane H. There are four planes which contain a and b (see proposition 1):

$$\frac{3^{d-k}-1}{2} = \frac{3^2-1}{2} = 4$$

Let the four planes be H, P_1, P_2, P_3. The points $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ are not contained in plane H, therefore $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ have to be contained in P_1, P_2, P_3. With the Pigeon hole principle, we know that one P_i must contain three points x_c, x_d, x_e, therefore P_i must contain a, b, x_c, x_d, x_e and this is in contradiction with Proposition 2. Therefore a maximal 3-cap contains 9 points. □

This method will not work for larger dimensions, because the variation in number of cap points distributed over the different hyperplanes grows significantly with larger dimensions and also the number of different planes through two points grows significantly. Therefore we will prove proposition 3 again with a different method.

Chapter 3

Counting hyperplanes and hyperplane triples

To prove the maximum cap size for larger dimensions we introduce a counting method for hyperplane triples [Davis and Maclagan, 2003]. In the previous chapter we counted cap points on a hyperplane. Now we will count distributions of cap points over parallel hyperplanes. We call these distributions hyperplane triples.

Definition 9 ((unordered) hyperplane triples)**.** *Given a hyperplane decomposition, a hyperplane triple is the distribution of points in the d-cap over the three parallel hyperplanes. In other words, given a d-cap C and hyperplane decomposition H_1, H_2, H_3. The hyperplane triple is defined as follows, $\{|C \cap H_1|, |C \cap H_2|, |C \cap H_3|\}$, where $|C \cap H_i|$ is the size of $C \cap H_i$.*
The hyperplane triple is called unordered if we talk about affine hyperplanes. Then the hyperplane triple $\{4, 4, 2\}$ is equal to the hyperplane triple $\{4, 2, 4\}$. In this case we always look at affine hyperplanes and unordered hyperplane triples.

To give an example of a proof with hyperplane triples, we will prove again that a maximum 3-cap has nine points with this new method.

Proof. This proof will again proceed by contradiction. Suppose there is a 3-cap with 10 points. We can decompose $\mathbb{F}_3^3$ into three parallel hyperplanes, H_1, H_2, H_3. We know from Proposition 2 that a plane can have at most 4 cap points, thus we find the following hyperplane triples; $\{4, 4, 2\}$ or $\{4, 3, 3\}$. We want to count the different ways to decompose $\mathbb{F}_3^3$ into three parallel hyperplanes. Let x_{442} be the number of $\{4, 4, 2\}$ hyperplane triples and x_{433} be the number of $\{4, 3, 3\}$ hyperplane triples. Then the total number of decompositions is $x_{442} + x_{433}$. On the other hand the number of directions perpendicular to one decomposition in three parallel hyperplane is equal to the number lines through the origin. Therefore each non zero point determines a direction. The number of non zero points in $\mathbb{F}_3^3$ is equal to $3^3 - 1$. On each line through the origin are 3 points located, the origin and two others. Thus the number of lines through the origin and number of directions is equal to $26/2 = 13$. Therefore,

$$x_{442} + x_{433} = 13 \tag{3.1}$$

To obtain another equation, we count 2-marked hyperplanes. The 2-marked hyperplanes are pairs of the form $(H, \{x, y\} \subset H \cap C)$, with H a hyperplane and $x \neq y$. With proposition 1 we can check that there are four planes containing two fixed points. Therefore the total number of 2-marked hyperplanes is $4\binom{10}{2} = 180$. Now counting the 2-marked hyperplanes in the hyperplane triples, we find the following equation:

$$\left[\binom{4}{2}+\binom{4}{2}+\binom{2}{2}\right]x_{442}+\left[\binom{4}{2}+\binom{3}{2}+\binom{3}{2}\right]x_{433}=180,$$

$$13x_{442}+12x_{433}=180. \tag{3.2}$$

Solving this system of equations (3.1) and (3.2), we find $x_{442}=24$ and $x_{433}=-11$. The number of hyperplane triples for a given partition must be positive, therefore this is a contradiction. □

3.1 4-caps

Proposition 4. *A maximum 4-cap has twenty points.*

Proof. we have seen in Figure 1.3 that a 4-cap can contain twenty points. It remains to be shown that no 4-cap contains more than twenty points. We proceed by contradiction. Suppose the 4-cap contains twenty-one points. We find the following possible hyperplane triples;

$$\{9,9,3\},\{9,8,4\},\{9,7,5\},\{9,6,6\},\{8,8,5\},\{8,7,6\},\{7,7,7\}$$

let x_{ijk} be the number of $\{i,j,k\}$ hyperplane triples.

Again, we are counting the number of hyperplane triples and the number of planes or directions through the origin and for the different hyperplane triples.
For the first equation, we look at all the ways to decompose $\mathbb{F}_3^4$ into 3 parallel hyperplanes. This is the sum of all the number of hyperplane triples. There is an unique line through the origin perpendicular to three parallel hyperplanes for each set of parallel hyperplanes. Counting these unique lines will give the total sum of decompositions. We use Proposition 1 to find the first equation. Therefore the first equation is

$$x_{993}+x_{984}+x_{975}+x_{966}+x_{885}+x_{876}+x_{777}=40 \tag{3.3}$$

For the second equation, we will count 2-marked hyperplanes. First we count how many different 2-marked hyperplanes there are for each hyperplane triple. We find the following;

$$\left[\binom{9}{2}+\binom{9}{2}+\binom{3}{2}\right]x_{993}+\cdots+\left[\binom{7}{2}+\binom{7}{2}+\binom{7}{2}\right]x_{777}$$

Then we count the total number of 2-marked hyperplanes in $\mathbb{F}_3^4$ with Proposition 1. We find

$$\frac{3^{4-1}}{2}\cdot\binom{21}{2}=13\cdot\binom{21}{2}=2730.$$

This gives us the following equation

$$75x_{993}+70x_{984}+67x_{975}+66x_{966}+66x_{885}+64x_{876}+63x_{777}=2730 \tag{3.4}$$

The third equation we will count 3-marked hyperplanes. First we count how many different 3-marked hyperplanes there are for each hyperplane triple. We find the following;

$$\left[\binom{9}{3}+\binom{9}{3}+\binom{3}{3}\right]x_{993}+\cdots+\left[\binom{7}{3}+\binom{7}{3}+\binom{7}{3}\right]x_{777}$$

Then we count the total number of 3-marked hyperplanes in $\mathbb{F}_3^4$ with proposition 1. We find;

$$\frac{3^{4-2}}{2} \cdot \binom{21}{3} = 4 \cdot \binom{21}{2} = 5320$$

This gives us the last equation:

$$169x_{993} + 144x_{984} + 129x_{975} + 124x_{966} + 122x_{885} + 111x_{876} + 105x_{777} = 5320 \tag{3.5}$$

To conclude the contradiction we need to solve the system of equations:

$$\begin{cases} x_{993} + x_{984} + x_{975} + x_{966} + x_{885} + x_{876} + x_{777} = 40 \\ 75x_{993} + 70x_{984} + 67x_{975} + 66x_{966} + 66x_{885} + 64x_{876} + 63x_{777} = 2730 \\ 169x_{993} + 144x_{984} + 129x_{975} + 124x_{966} + 122x_{885} + 111x_{876} + 105x_{777} = 5320 \end{cases}$$

We found that adding 3 times equation (3.5) to 693 times equation (3.3) and substracting 16 times equation (3.4) gives;

$$5x_{984} + 8x_{975} + 9x_{966} + 3x_{885} + 2x_{876} = 0$$

All the coefficients have to be positive, therefore the only solution is $x_{984} = x_{975} = x_{966} = x_{885} = x_{876} = 0$. When we substract 63 times equation (3.3) from equation (3.4), we find;

$$12x_{993} + 7x_{984} + 4x_{975} + 3x_{966} + 3x_{885} + x_{876} = 210$$

If we use both outcomes we find $12x_{993} = 210$. This contradicts with x_{993} being an integer. Therefore there is no solution to the system of equation that meets the requirements and therefore our hypothesis can not be true. □

3.2 Bound on 5-caps

We have seen in [Edel et al., 2002] that a 5-cap can contain fortyfive points. The proof for the 4-cap is easy to set up for other dimensions. The things you need to know are the dimension, the maximum cap size of the previous dimension and the expected size of the new dimension.

The entire proof is dependent on the equations. In four dimensions 3 equations were enough to give a good upper bound, but this is not the case for higher dimensions. Therefore we need to count 4-marked hyperplanes and that is the main reason this proof can not be generalized for more dimensions. We know that for any three selected points we can define a plane through these points, because the three points are in the cap, they are not on one line and the three points can only be in a plane.
If we select four points randomly, we need to distinguish two cases, namely the case that the four points are in the same two dimensional space and the case that the four points are in a three dimensional space. This makes counting 4-marked hyperplanes to difficult.
Therefore we choose the search for upper bounds with just three equations.

Proof of an upper bound . We proceed by contradiction. Suppose the 5-cap can contain fifty points. We find the following possible hyperplane triples;

$$\{20, 20, 10\}, \{20, 19, 11\}, \{20, 18, 12\}, \{20, 17, 13\}, \{20, 16, 14\}, \{20, 15, 15\}, \{19, 19, 12\},$$
$$\{19, 18, 13\}, \{19, 17, 14\}, \{19, 16, 15\}, \{18, 18, 14\}, \{18, 17, 15\}, \{18, 16, 16\}, \{17, 17, 16\}$$

We use Proposition 1 to find the first equation to count the total number of ways to decompose $\mathbb{F}_3^5$ into three parallel 4 dimensional hyperplanes. The first equation is;

$$\begin{aligned} &x_{202010} + x_{201911} + x_{201812} + x_{201713} + x_{201614} + x_{201515} + x_{191912} \\ &\quad + x_{191813} + x_{191714} + x_{191615} + x_{181814} + x_{181715} x_{181616} + x_{171716} = 121 \end{aligned} \tag{3.6}$$

Then we count the 2-marked hyperplanes, first for each hyperplane triple and again we use Proposition 1 to count the total number of 2-marked hyperplanes. We find;

$$\left[\binom{20}{2} + \binom{20}{2} + \binom{10}{2}\right] x_{202010} + \cdots + \left[\binom{17}{2} + \binom{17}{2} + \binom{16}{2}\right] x_{171716} = 49000$$

$$\begin{aligned} &425x_{202010} + 416x_{201911} + 409x_{201812} + 404x_{201713} + 401x_{201614} \\ &+ 400x_{201515} + 408x_{191912} + 402x_{191813} + 398x_{191714} + 396x_{191615} \\ &\quad + 397x_{181814} + 394x_{181715} + 393x_{181616} + 392x_{171716} = 49000 \end{aligned} \tag{3.7}$$

Lastly we count the 3-marked hyperplanes, for each hyperplane triple and with Proposition 1 to count the total number of 3-marked hyperplanes. We find;

$$\left[\binom{20}{3} + \binom{20}{3} + \binom{10}{3}\right] x_{202010} + \cdots + \left[\binom{17}{3} + \binom{17}{3} + \binom{16}{3}\right] x_{171716} = 254800$$

$$\begin{aligned} &2400x_{202010} + 2274x_{201911} + 2176x_{201812} + 2106x_{201713} + 2064x_{201614} \\ &+ 2050x_{201515} + 2158x_{191912} + 2071x_{191813} + 2013x_{191714} + 1984x_{191615} \\ &\quad + 1996x_{181814} + 1951x_{181715} + 1936x_{181616} + 1920x_{171716} = 49000 \end{aligned} \tag{3.8}$$

□

This gives us a system of fourteen unknown variables and three equations. Still there are two conditions for each variable, namely they have to be positive and integer.
The integer linear program in Appendix A.2 was made to solve these equations. The ILP did not return a solution to 5-cap is fifty points. This concludes our contradiction and now we know that forty nine points is an upper bound to the 5-cap.

3.3 Matlab implementation

The proof is based on a contradiction, therefore you need to know a lower bound for which we know the cap exists. We are searching for the smallest upper bound for which the equation gives a solution. Previous I set up the equations myself and let matlab solve the equations I made. To set up the equations, we need quite a lot of computations and to find such a smallest upper bound, we have to set up at least to 2 sets of equations, but mostly more. As you can imagine, this is time consuming. Therefore I made a function with Matlab (see Appendix C) that calculates the hyperplane triples and solves the ILP. The function gave me the following results;

Dimension	3	4	5	6	7	8
Known bounds for caps	9	20	45	112	-	-
Upper bound for the cap, no integer solution	9	21	50	114	292	773*
Upper bound for the cap, integer solution	9	20	48	114	291	771**

* We used 292 as previous cap size in the calculations

** We used 291 as previous cap size in the calculations

First we made calculations with the integer constraint for the solution, because in higher dimensions the calcutation time significantly increased, we also made calculations without this constraint.

As you can see in the table, the calculated cap sizes are not far apart of each other. Therefore if you use the best known previous cap size value, the non integer solution will have almost the same cap size as with the integer constraint.

Chapter 4

Upper bounds

In [Bierbrauer and Edel, 2002] a more general upper bound for cap sizes is discussed. This bound is dependent on dimension and size of . They do not only calculate the maximal cap size, but they look at the ratio between the total number of elements in the set $\mathbb{F}_q^d$ and the amount of elements of the cap of $\mathbb{F}_q^d$.
Denote by $C_d(q)$ the maximum size of a cap in the affine space $\mathbb{F}_q^d$ and denote $c_d(q) = C_d(q)/q^d$. The number $c_d(q)$ gives the ratio between the number of cap points and the total number of points.
Theorem 1. *Let $q > 2$ be a prime-power. If $d \geq 3$, then*

$$c_d(q) \leq \frac{q^{-d} + c_{d-1}(q)}{1 + c_{d-1}(q)}$$

4.1 Proof of theorem 1

In this case, we only consider $q = 3$. This suffices for the prime-power. Let $k > 3$ and $A \subset \mathbb{F}_3^d$ be a cap. Let $F = |\mathbb{F}_3^d| = 3^d$. And at last, ζ is a complex primitive third root of unity. We look for an upper bound for the cap size $|A|$. Consider the following complex number;

$$S = \sum_{y \in \mathbb{F}_3^d \setminus \{0\}} \sum_{a_1, a_2, a_3 \in A} \zeta^{(\sum_i a_i) y} \tag{4.1}$$

Lemma 1. $S = |A|(F - |A|^2)$

Proof. We consider the following;

$$S = \sum_{y \in \mathbb{F}_3^d \setminus \{0\}} \sum_{a_1, a_2, a_3 \in A} \zeta^{(\sum_i a_i) y} = \sum_{y \in \mathbb{F}_3^d} \sum_{a_1, a_2, a_3 \in A} \zeta^{(\sum_i a_i) y} - |A|^3$$

Because for $y = 0$, we have $\sum_{a_1, a_2, a_3 \in A} 1$ and this is equal to

$$\sum_{a_1 \in A} 1 \cdot \sum_{a_2 \in A} 1 \cdot \sum_{a_3 \in A} 1 = |A| \cdot |A| \cdot |A|$$

Now we consider two cases, first $\sum_i a_i \neq 0$, we find ;

$$\begin{aligned} S + |A|^3 = & \sum_{y \in \mathbb{F}_3^d} \sum_{a_1,a_2,a_3 \in A} \zeta^{(\sum_i a_i)y} = \sum_{a_1,a_2,a_3 \in A} \sum_{y \in \mathbb{F}_3^d} \zeta^{(\sum_i a_i)y} \\ = & \sum_{a_1,a_2,a_3 \in A} \zeta^{a_i} \sum_{y \in \mathbb{F}_3^d} \zeta^{(a(y-e_i)} \end{aligned}$$

Substituting $z := y - e_i$ gives the following;

$$\sum_{a_1,a_2,a_3 \in A} \zeta^{a_i} \sum_{y \in \mathbb{F}_3^d} \zeta^{(a(y-e_i)} = \sum_{a_1,a_2,a_3 \in A} \zeta^{a_i} \sum_{z \in \mathbb{F}_3^d} \zeta^{(az)}$$

Therefore we know that $\sum_{y \in \mathbb{F}_3^d} \zeta^{(\sum_i a_i)y} = 0$.
Second we consider the case $\sum_i a_i = 0$, this can only be the case when $a_1 = a_2 = a_3$, because A is a cap of $\mathbb{F}_3^d$.

$$S + |A|^3 = \sum_{y \in \mathbb{F}_3^d} \sum_{a_1,a_2,a_3 \in A} \zeta^{(\sum_i a_i)y} = \sum_{y \in \mathbb{F}_3^d} \sum_{a_1,a_2,a_3 \in A} 1 = F|A|$$

Therefore we find: $S = |A|(F - |A|^2)$. □

Definition 10. *Let $0 \neq \lambda \in \mathbb{F}_3$ and let $0 \neq y \in \mathbb{F}_3^d$. Consider the complex number $U(\lambda)_y = \sum_{a \in A} \zeta^{(\lambda a);y}$ and $u(\lambda)_y = |U(\lambda)_y|$. Define a real vector $u(\lambda)$ of length $F - 1$ whose coordinates are parameterized by $0 \neq y \in \mathbb{F}_3^d$, the corresponding entry being $u(\lambda)_y$.*

Lemma 2. *Let $0 \neq \lambda \in \mathbb{F}_3$ and $0 \neq y \in \mathbb{F}_3^d$. Then*

$$u(\lambda)_y \leq 3 \cdot C_{d-1}(3) - |A| = c_{d-1}(3) \cdot F - |A|$$

Proof. Define v_c the number of elements $a \in A$, with A a cap, such that $a \cdot y = c$. We know that $u \in \mathbb{F}_3^d$ that satisfy $u \cdot y = c$ form a hyperplane of dimension $d - 1$. It holds that $v_c \leq C_{k-1}$. Thus

$$u(\lambda)_y = |\sum_{c \in \mathbb{F}_3} v_c \zeta^c| = |\sum_{c \in \mathbb{F}_3} (C_{d-1} - v_c)\zeta^c|$$

We can substitute $c_{d-1} - v_c$ for v_c, because the affine space this gives the same result . We can rewrite this further;

$$u(\lambda)_y = |\sum_{c \in \mathbb{F}_3} (C_{d-1} - v_c)\zeta^c| \leq \sum_{c \in \mathbb{F}_3} C_{d-1} - v_c = q \cdot C_{d-1} - |A|$$

□

Lemma 3. *Let $0 \neq \lambda \in \mathbb{F}_3$. Then*

$$\|u(\lambda)\|^2 = |A|(F - |A|).$$

Proof. The definition of the norm for complex vectors is given by;

$$\|u(\lambda)\|^2 = \sum_{y \neq 0} U_y \bar{U_y}$$

We use this definition and find the following;

$$\begin{aligned}
\|u(\lambda)\|^2 + U_0\bar{U}_0 &= \sum_y U_y\bar{U}_y = \sum_y \Big(\sum_{a\in A}\zeta^{ay}\Big)\overline{\Big(\sum_{a\in A}\zeta^{ay}\Big)} \\
&= \sum_y \Big(\sum_{a\in A}\zeta^{ay}\Big)\Big(\sum_{a\in A}\zeta^{-ay}\Big) = \sum_y \Big(\sum_{a,b\in A}\zeta^{(a-b)y}\Big) \\
\|u(\lambda)\|^2 &= \sum_y \Big(\sum_{a,b\in A}\zeta^{(a-b)y}\Big) - U_0\bar{U}_0 \\
&= \sum_y \Big(\sum_{a,b\in A}\zeta^{(a-b)y}\Big) - |A|^2
\end{aligned}$$

There are two cases, first suppose that $(a-b)=0$. Then the sum over a,b is equal to $F\cdot|A|$. Second, suppose $(a-b)\neq 0$. We know from the proof of lemma 1 that the sum over y is equal to zero. Therefore we find;

$$\|u(\lambda)\|^2 = F\cdot|A| - |A|^2 = |A|(F-|A|)$$

□

From Lemma 2 and 3 we find a lower bound for $c_{k-1}-c_k$. We choose $|A|=C_k$. We know that the norm can be written as follows;

$$\|u(\lambda)\|^2 = |u(\lambda)_{y_1}|^2 + |u(\lambda)_{y_2}|^2 + |u(\lambda)_{y_3}|^2 + \dots$$

Now we use Lemma 2 to bound the $|u(\lambda)_{y_i}|$. We find;

$$\|u(\lambda)\|^2 \leq (F-1)|u(\lambda)_y| = (F-1)(Fc_k - |A|)$$

From Lemma 3 we obtain;
Theorem 2. $(c_{k-1}-c_k)^2 \geq {}^{c_k(1-c_k)}/_{F-1}$

We can define S in an other way, thus the next Lemma is an obvious result of definitions.
Lemma 4. *We define* $S=\sum_{y\neq 0}U(\lambda_1)_yU(\lambda_2)_yU(\lambda_3)_y$ *and this we can rewrite as;*

$$S \leq \sum_{y\neq 0} u(\lambda_1)_y u(\lambda_2)_y u(\lambda_3)_y$$

To finish the proof of Theorem 1, we use all proven lemmas. First we start with using Lemma 2 to give a bound for $u(\lambda_1)_y$.

$$|S| \leq \sum_{y\neq 0} u(\lambda_1)_y u(\lambda_2)_y u(\lambda_3)_y \leq F\cdot(c_{d-1}-c_d)\sum_{y\neq 0} u(\lambda_2)_y u(\lambda_3)_y$$

Then we use Lemma 3. The remaining sum can be written as a dot-product and we use Cauchy Schwarz inequality to apply Lemma 3.

$$|S| \leq F\cdot(c_{d-1}-c_d)\sum_{y\neq 0} u(\lambda_2)_y u(\lambda_3)_y \leq F\cdot(c_{d-1}-c_d)\cdot\|u(\lambda)_y\|^2 \leq F\cdot(c_{d-1}-c_d)\cdot F\cdot c_d\cdot(F-F\cdot c_d)$$

Last we use Lemma 1 with $|A| = C_d$.

$$\begin{aligned}
|F \cdot c_d \cdot (F - F^2 \cdot c_d^2)| &\leq F^3 \cdot c_d(c_{d-1} - c_d) \cdot (1 - c_d) \\
|(1 - F \cdot c_d^2)| &\leq F \cdot (c_{d-1} - c_d) \cdot (1 - c_d) \\
F \cdot c_d^2 - 1 &\leq F \cdot (c_{d-1} - c_d) \cdot (1 - c_d) \\
c_d^2 - F^{-1} &\leq c_{d-1} - c_d + c_d^2 - c_{d-1} \cdot c_d \\
c_d &\leq \frac{F^{-1} + c_{d-1}}{1 + c_{d-1}}
\end{aligned}$$

Thus, the proof of theorem 1 is complete.

4.2 Application of the upper bound

We find that Theorem 1 provides us with a much faster and less complicated upperbound, than the proof of Proposition 4 in Chapter 3. Comparison with Proposition 4 gives us;

Dimension	3	4	5	6	7	8
Known bounds	9	20	45	112	-	-
Upper bound, no integer solution	9	21	50	114	292	773
Upper bound, integer solution	9	20	48	114	291	771
Upper bound theorem 1*	9	21	50	126	324	847
Upper bound theorem 1**	9	21	48	114	292	771

*calculated with the previous calculated cap size value
** calculated with the best known previous cap size value

The results show that Theorem 1 is almost just a good as Proposition 4 if we apply the integer constraint. In terms of calculating time Theorem 1 is much faster than Proposition 4 with the integer constraint. The upper bound of theorem 1 will not take more calculating time with higher dimensions, while Proposition 4 will take more time with higher dimensions. If we compare the results of Theorem 1 and Proposition 4 (without the integer constraint), we find that Theorem 1 is even better than Proposition 4, while the calculating time is the same. Thus overall the upper bound of theorem 1 is a better upper bound.

In terms of ratio, we find the following with theorem 1;

Dimension	3	4	5	6	7	8
Previous cap size	4	9	20	45	112	291
Ratio of theorem 1	0.3333	0.2593	0.2013	0.1574	0.1336	0.1176

We can see that the ratio between the cap size and the size of the total space reduces quikly. Thus far it is not know how fast this ratio will tend to zero. The rate at which the total space grows with one extra dimension is significantly larger then the rate at which the d-cap size grows.

Appendix A

Matlab implementation

A.1 ILP 4-cap = 21

```
f = ones(1,7);
intcon = 1:7;
A = [] ;
b = [] ;
Aeq = [1 1 1 1 1 1 1; 75 70 67 66 66 64 63;169 144 129 124 122 111 105];
beq = [40; 2730; 5320];
lb = [0;0;0;0;0;0;0];
ub = [Inf;Inf;Inf;Inf;Inf;Inf;Inf];
intlinprog(f,intcon,A, b, Aeq,beq,lb,ub)
```

A.2 ILP 5-cap = 50

```
f = [1;1;1;1;1;1;1;1;1;1;1;1;1;1];
intcon = 1:14;
A = [] ;
b = [] ;
Aeq = [1 1 1 1 1 1 1 1 1 1 1 1 1 1;
    425 416 409 404 401 400 408 402 398 396 397 394 393 392;
    2400 2274 2176 2106 2064 2050 2158 2071 2013 1984 1996 1951 1936 1920];
beq = [121; 49000; 254800];
lb = [0;0;0;0;0;0;0;0;0;0;0;0;0;0];
ub = [Inf;Inf;Inf;Inf;Inf;Inf;Inf;Inf;Inf;Inf;Inf;Inf;Inf;Inf];
intlinprog(f,intcon,A, b, Aeq,beq,lb,ub)
```

Appendix B

Matlab functions

B.1 Counting triples

```
function [ m ] = triptel( c,e )
m = 0;
for i = e:-1:1
    for j = i:-1:1
        k = c - i - j;
        if k <= j && k>= 0
            m = m+1;
        end
    end
end

end
```

B.2 Combinations

```
function res = kies(n,k)
if k>n || k<0
    res = 0;
else
    res = nchoosek(n,k)

end
```

Appendix C

IPL for solving equations Chapter 3

```
function [ a ] = Cap2(d,c,e)
t = triptel(c,e);
B = zeros(t,3);
m = 0;

for i = e:-1:1
    for j = i:-1:1
        k = c - i - j;
        if k <= j && k>= 0
            m = m+1;
            B(m,:)= [i,j,k];
        end
    end
end

f = ones(1,t);
intcon = [1:t]; % Integer constraint
A = [];
b = [];
Aeq = zeros(3,t);

for i=1:t
    x = B(i,1);
    y = B(i,2);
    z = B(i,3);
    Aeq(1,i)= 1;
    Aeq(2,i) = kies(x,2) + kies(y,2) + kies(z,2);
    Aeq(3,i) = kies(x,3) + kies(y,3) + kies(z,3);
end

g = (3^(d-1)-1)/2;
h = (3^(d-2)-1)/2;
l = (3^(d)-1)/2;
beq = [l,g*nchoosek(c,2),h*nchoosek(c,3)];
```

```
lb = zeros(1,t);
ub = zeros(1,t);

for i = 1:t
    ub(1,i)= inf;
end

a = intlinprog(f,intcon,A, b, Aeq,beq,lb,ub);

end
```

Appendix D

Solving upper bound with inequality in Chapter 4

```
function [ k,e ] = bb(d,c)

k = (3^(-d) + (c/3^(d-1)))/(1 + (c/3^(d-1)));
e = (3^d)*k;

end
```

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