Quantum correlation matrices and Tsirelson's problem

Previous work and three-player considerations

Master Thesis by Gerrit Vos

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August 19, 2019

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To be defended on:	29-08-2019
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Abstract

Tsirelson once claimed that the set of quantum correlations, defined by strategies of non-local two-player games, does not depend on which of two possible models is chosen: the *tensor product model* or the *commuting operator model*. He later came back from this claim, and the resulting conjecture is now known as Tsirelson's problem. The problem has since been proven equivalent to notoriously hard problems in operator theory, such as the Connes' Embedding Problem and the QWEP conjecture. In this master thesis, we look at the finite dimensional case of Tsirelson's problem, working out all the details of an existing proof and giving a new, shorter proof which also extends to the nuclear case. Moreover, we give an overview of the equivalence of Tsirelson's problem and two of Kirchberg's conjectures, including the QWEP conjecture. Finally, we give some results and considerations for the three-player case of Tsirelson's problem. The appendix contains proofs of many related results used throughout the thesis, and also a beginner's introduction to quantum mechanics.

Preface

After some fine years at the Technical University of Delft, the end of my time as a student was getting in sight and there was just one final step ahead before graduation: the master thesis. Since I was interested in the field of Quantum Information Theory and Operator Algebras, I approached Martijn Caspers who graciously accepted to be my supervisor. He presented me with the interesting Tsirelson problem and its range of equivalent problems, and pointed out that little research seemed to have been done into the three-player scenario. Therefore, we hoped, there might potentially be some interesting new results there.

After getting a feel about the topic of correlations matrices and the definition of Tsirelson's problem (Chapters 3, 4, 5), I first studied [25] and delved into the finite dimensional proof until I understood all the details (see Chapter 5.2). As a means of trying to undestand the line of thought, I tried a much simpler approach to see where it went wrong. It turned out that it didn't go wrong and thus we discovered a much simpler proof that seems to be nowhere in any literature - although we suspect experts must be aware of the approach. This approach also turned into a nice proof for the nuclear case, see Theorem 5.4.1. I did some more research in other special cases such as the one in [16].

Next, I studied the papers [7], [9] and a part of [19] about the equivalence between Tsirelson's problem and a conjecture on tensor norms of C^*-algebras by Kirchberg. After this was done and I had written my own version of the proof (most of which was preserved to become Chapter 6 in the current thesis) we decided it was time to delve into the three-player case and see if anything could be said using this equivalence. This didn't last for long however; it turned out that with my current knowledge there was still very little I could say abot the three-player scenario. Therefore we decided I would study other equivalences, and I took Brown&Ozawa's great book [1] to learn more about the QWEP conjecture and Connes' Embedding Problem. As it turned out, it was especially the QWEP conjecture and its related theory that had some interesting links to Tsirelson. Hence I decided to write another piece (Chapter 7) on this topic.

Even with this though, the number of things I could say about the three-player scenario were limited. Mostly, I discovered that some seemingly obvious properties seemed to go wrong. As the thesis progressed this part (Chapter 8) gradually got more structured. Finally, it became a decent overview of the things we do and do not know including some new insights in problems that occur.

I have attempted to keep this report as readable as possible for someone with a similar background as I had when I started this project. I set out with the goal of keeping the thesis self-contained apart from referring to the book of Murphy [15], but I had to quickly give up on perfectly achieving this goal. To achieve that goal at least to some extent, many results are proven in the Appendix in the back.

I would like to very much thank my supervisor Martijn Caspers whose advice and help has been invaluable over the entire course of the project. He was able to help me on when I got stuck countless times and has given much helpful feedback on my thesis. Further, I would like to thank Jan van Neerven and Dion Gijswijt for being a part of my thesis committee and for the critical commentary they will no doubt deliver at the time of my defense. Finally, I would like to thank the whole Analysis department for the great atmosphere and 'gezellige' lunches.

Contents

1 Introduction

This thesis is about correlation matrices. With that, we do not mean the standard type of correlation matrices given by the Pearson correlation of two random variables, although it is similar in spirit. The correlation matrices of this thesis arise from strategies of so-called non-local games and are intimately related to Bell inequalities. A non-local two-player game consists of two (abstract) space-like separated players that are quite unimaginatively named Alice and Bob. They each receive an input x respectively y and have to choose an output a respectively b. We assume there are finitely many inputs and outputs. A two-player strategy is given by a set of conditional probabilities $\mathbb{P}(a, b|x, y)$; the correlation matrix is given by $(\mathbb{P}(a, b|x, y))_{a,b;x,y}$.

Since Alice and Bob are spacelike-separated, there is a limited set of strategies they can carry out. It is not very hard to determine the set of classically obtainable correlation matrices; this turns out to be the polytope with vertices given by deterministic strategies $\mathbb{P}(a|x)\mathbb{P}(b|y)$. When allowing quantum mechanical tricks (particularly entangled states), a larger range of *quantum correlation matrices* becomes obtainable which is not so easily described. We do know that this set is strictly larger; there are several examples of Bell inequality violations proving this, such as the famous CHSH-inequality violation. As we will see in Chapter 4, Bell inequalities are nothing but facets of the classical polytope of correlation matrices. Thus, a violation of a Bell inequality is nothing but a specific quantum correlation matrix that falls outside the classical polytope.

For the main part of the thesis we will focus on quantum correlation matrices. The standard way to define a quantum correlation is by the formula

$$\mathbb{P}(a, b|x, y) = \langle\psi|A_a^x \otimes B_b^y|\psi\rangle$$

where $|\psi\rangle$ is a (possibly entangled) state on a bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where each player has access to one tensor leg. Furthermore, the $\{A_a^x\}_{a,x}$ and $\{B_b^y\}_{b,y}$ are POVMs (or equivalently, as it will turn out, projective measurements) that the players use to measure the state on their part of the Hilbert space. Given input x and y, Alice and Bob will apply the POVMs $\{A_a^x\}_a$ and $\{B_b^y\}_b$ and return the outcome corresponding to the resulting measurement. The model corresponding to this definition of quantum correlations will be called the *tensor product model*.

There is a different, more general model we can use to define quantum correlations: the *commuting operator model*. The physical intuition behind this model is the question: can we really assume that we can break down the universe that nature presents us with into separate parts? We will refrain from going further into this discussion (see e.g. [7] for more), and just state the mathematical alternative. Namely, we have just one 'giant' Hilbert space and Alice and Bob's measurement operators are only assumed to be commuting. This assumption is necessary to assure Alice and Bob can independently measure their states. Quantum correlations are then defined by the formula

$$\mathbb{P}(a, b|x, y) = \langle\psi|A_a^x B_b^y|\psi\rangle .$$

These different definitions of quantum correlations yield potentially different sets. In [28], Tsirelson considered these sets for the first time and claimed, without proof, that they are one and the same. When someone finally asked for a proof, he found a mistake in his would-be proof and instead posted the question as an open problem on the site of TU Braunschweig [29], no less than 13 years later! He did resolve the finite dimensional case, for which he gave a sketch of a proof. We give a very different (and remarkably shorter) proof in Chapter 5.

The original problem was quite recently resolved by Slofstra in [26] in 2016, and the proof was further simplified in [27] and [4]. These papers proved that the set of tensor product correlations is not closed, while it was already known that the set of commuting correlations is closed. The 'weak' version of Tsirelson's problem, which asks whether the closure of the set of tensor product correlations is equal to the commuting correlations, remains open, and is now commonly referred to as 'the' Tsirelson problem, which we will refer to as (T2).

A few special cases are known; for instance, the authors of [16] showed Tsirelson in the case where one player has two inputs and outputs. A proof for the case where one player's operators generate a nuclear C^*-algebra is given in this thesis. The general case, we now know for sure, is really very hard; indeed, it was proven equivalent to the notorious Connes Embedding Problem, a problem that has been open since 1976 [2]. This problem is equivalent to a number of other open problems in C^* and von Neumann algebra theory. In particular, there is an interesting connection to a problem in the theory of tensor norms on C^*-algebras, found by Kirchberg in 1993 [14]. This problem asks whether

$$C^*(\mathbb{F}_2) \otimes_{\min} C^*(\mathbb{F}_2) \stackrel{?}{=} C^*(\mathbb{F}_2) \otimes_{\max} C^*(\mathbb{F}_2).$$

We will refer to this problem as (K2). One implication was found simultaneously by Fritz [7] and Junge, Navascues, Palazuelos, Perez-Garcia, Scholz and Werner [9] in 2012, both of whom showed the converse implication only for a matrix-valued version of Tsirelson's problem. The latter paper is built on the theory of operator systems. The first link to operator systems was found by Werner and Scholz [25] in 2008; ironically, the result achieved there was later proven to be incorrect. Indeed, it is essentialy claimed that the set of tensor product correlations is the same as the closure of the set of finite dimensional correlations, which contradicts the result of [26], [27], [4] mentioned above that the set of tensor product correlations is not closed. Still, the ideas in that paper can be said to be crucial to later developements.

Fritz' proof in [7] also has operator system theory lurking in the background, but it is written down in a more elementary and accessible manner. Hence, it is this proof that will be the basis of our Chapter 6, and we will also frequently refer to results in Fritz' comprehensive appendix of background material.

The full equivalence was proven one year later by Ozawa [19]. He used Fritz' statement (Prop. 3.4) as a starting point for his proof, and proceeds to use several results relating to the Connes Embedding Conjecture for the implication (K2) $\Rightarrow$ (T2). These results are unfortunately outside the scope of this thesis, so we will not give this proof; instead we give the proof for the matrix-valued version of Tsirelson as proven in [7] and [9].

We do look at one other statement equivalent to the Tsirelson problem: the QWEP conjecture, which asks if every C^*-algebras has the Quotient Weak Expectation Property. We will elobarate in Chapter 7 on what this means, how its link to (K2) appears, and what we can deduce about Tsirelson's problem. One conclusion we can draw is that the problem statement of Tsirelson does not depend on whether both players have the same number of inputs and outputs, which is an assumption made in much of the literature. We draw mostly from material from Brown & Ozawa's book [1], much of which originates from Ozawa's 2004 paper [18]. The equivalence between (K2) and QWEP was originally proven by Kirchberg in [14].

In our final normal chapter, we take a look at the three-player version of Tsirelons's problem (T3) and consider what happens to the connections explored in the earlier chapters. It turns out that there are many things we cannot duplicate to the three-player case. Most importantly, the proof of (K2) $\Rightarrow$ (T2) does not carry over to the three-player case since it goes via the Connes Embedding Conjecture, as there is no know corresponding problem for the three-player case. Things that do carry over are the results from [7] and [9]; specifically the implication (K3) $\Rightarrow$ (T3) (where (K3) is the tripartite version of (K2)) and the converse implication for the matrix-valued Tsirelson. We also take a shot at proving (T2) $\Rightarrow$ (T3) via the QWEP conjecture and explain where it goes wrong. Finally, we investigate what happens when considering combinations of the tensor product model and the commuting operator model. Fritz made some claims and observations about this case in [7], but we argue that these claims might have been premature.

At the end, there is an appendix which includes a collection of proofs of results we need throughout the main part of the thesis. It also includes an elementary introduction to quantum mechanics for readers that are unfamiliar with the basic concepts.

2 Terms and definitions

This thesis is built on material from Murphy's book [15] and we will frequently refer to results therein as 'Theorem x.x.x from Murphy' without giving a citation each time. We will assume familiarity with at least the first 3 chapters of that book, including Hilbert spaces, C^*-algebras, the Gelfand-Naimark Theorem and some representation theory.

Let us start with some basic notation. A Hilbert space is generally denoted by $\mathcal{H}$; the space of its bounded linear operators by $\mathbb{B}(\mathcal{H})$. A C^*-algebra is indicated by some calligraphic letter - usually $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$. Operators are indicated by normal capital letters. C^*-algebras will always be assumed to be unital throughout this thesis, so no need for hassles like approximate units or unitisations.

2.1 Completely positive maps

One essential term that is missing from Murphy's book is completely positive maps. We give a short overview here; a more extensive collection of results can be found in [1, 1.5].

A (concrete) operator system in a C^*-algebra $\mathcal{A}$ is a self-adjoint subspace containing the unit. Note that an operator system inherits a notion of positive elements from $\mathcal{A}$. More generally, the matrix space $M_n(E)$ inherits a notion of positive elements from $M_n(\mathcal{A})$. Indeed, as is decribed in Appendix A.2, this can be seen as the defining property of an operator system.

Definition 2.1.1. Let $E \subseteq \mathcal{A}$ be an operator system. A map $\Phi : E \to \mathcal{B}$ is called *completely positive* if, for every $n \in \mathbb{N}$, the map

$$\Phi^{(n)} := \mathbb{1}_{M_n} \otimes \Phi : x \mapsto \begin{pmatrix} \Phi(x_{11}) & \dots & \Phi(x_{1n}) \\ \vdots & \ddots & \vdots \\ \Phi(x_{n1}) & \dots & \Phi(x_{nn}) \end{pmatrix}, \qquad x \in M_n(E)$$

is positive. We denote unital completely positive maps shorthand by ucp maps, and contractive completely positive maps by ccp maps.

Let us state some elementary facts about positive maps in general. By writing self-adjoint elements as linear combination of two positive elements, we find that self-adjoint elements get sent to self-adjoint elements. More generally, if $x \in E$ is any element, then we can write $x = a + ib$ for a, b self-adjoint. Thus, $\Phi(x)^* = \Phi(a)^* - i\Phi(b)^* = \Phi(a) - i\Phi(b) = \Phi(x^*)$.

In most cases, ucp maps will be defined on C^*-algebras. But in some cases, we will first define an ucp map on a dense operator system and then extend to the whole C^*-algebra. This is possible because of the following result.

Proposition 2.1.2. *Let $\Phi : E \to \mathcal{B}$ be a ucp map. Then Φ is contractive.*

Proof. First let $x \in M_n(E)$ be self-adjoint. Note that $\|x\|\mathbb{1} - x \geq 0$; indeed,

$$\big\|(\|x\|\mathbb{1} - x) - \|x\|\mathbb{1}\big\| = \|x\|,$$

so by Lemma 2.2.2 from Murphy $\|x\|\mathbb{1} - x \geq 0$. Because $\Phi^{(n)}$ is positive and unital, we have

$$\Phi^{(n)}(x) \leq \Phi^{(n)}(\|x\|\mathbb{1}) = \|x\|\mathbb{1}.$$

Thus by Theorem 2.2.3 (3) from Murphy, $\|\Phi^{(n)}(x)\| \leq \|x\|$.

Now let $x \in E$ be any element. Let $\tilde{x} := \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in M_2(E)$; this is called a self-adjoint dilation of x. Note that $\|\tilde{x}\| = \|x\|$ (and $\|\Phi^{(2)}(\tilde{x})\| = \|\Phi(x)\|$). Thus, $\|\Phi(x)\| = \|\Phi^{(2)}(\tilde{x})\| \leq \|\tilde{x}\| = \|x\|$ by the above argument for $n = 2$. □

In the case of abelian C^*-algebras, completely positive maps are just the same as positive maps.

Proposition 2.1.3. [22, 3.11] *Let $\mathcal{A}$ be an abelian C^*-algebra and let $\Phi : \mathcal{A} \to \mathcal{B}$ be a positive map. Then Φ is completely positive.*

Proof. As $\mathcal{A}$ is abelian, we can assume that $\mathcal{A} = C(\Omega)$ for a compact Hausdorff space Ω (see 2.1.10 and 1.3.5 in Murphy). An element $f \in \mathcal{A}$ is positive iff $f(x) \geq 0$ for all $x \in \Omega$. Similarly, an element $f \in M_n(\mathcal{A}) \cong M_n(C(\Omega)) \cong C(\Omega, M_n(\mathbb{C}))$ is positive iff $f(x)$ is a positive matrix for all $x \in \Omega$.

Now, let $f \in M_n(\mathcal{A})^+$ and assume that $f = g \cdot A$ for $g \in C(\Omega)^+$ and $A \in M_n(\mathbb{C})^+$. In that case, by positivity of Φ, it is clear that

$$\Phi^{(n)}(f) = \Phi(g) \cdot A \in M_n(\mathbb{C})^+.$$

So, for the set S^+ of positive linear combinations of such elements, it is clear that Φ is completely positive. It remains to show that S^+ is dense in $M_n(\mathcal{A})^+ = C(\Omega, M_n(\mathbb{C}))^+$.

Let $f \in C(X, M_n(\mathbb{C}))^+$ and $\varepsilon > 0$. For $x \in X$, let O_x be an open neighbourhood of x such that for $y \in O_x$, $\|f(x) - f(y)\| < \varepsilon$. Since Ω is compact, there exists a finite number of points $x_1, \ldots, x_k$ such that $\{O_{x_i}\}_{i=1}^k$ is an open cover for X. Define $A_i = f(x_i)$.

Now let $(g_i)_{i=1}^k$ be a partition of unity (i.e. $g_i \geq 0$ and $\sum_{i=1}^k g_i = 1$) such that $\mathrm{supp(g_i) \subseteq O_{x_i}}$. Then for $x \in X$ we have

$$\left\| f(x) - \sum_{i=1}^k g_i(x) A_i \right\| = \left\| \sum_{i=1}^k (f(x) - A_i) g_i(x) \right\| \leq \sum_{i=1}^k \|f(x) - A_i\| g_i(x) < \varepsilon \sum_{i=1}^k g_i(x) = \varepsilon.$$

Since $\sum_{i=1}^k g_i(x) A_i \in S^+$, we are done. □

2.2 Quantum theoretic terms

We briefly state the terms we use in the thesis. For those that have never seen quantum mechanics before, we included an introduction in Appendix B.

A *pure state* on a Hilbert space $\mathcal{H}$ is a unit vector of $\mathcal{H}$. A *mixed state* is a convex combination of pure states. An *entangled state* is a pure or mixed state on a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ that cannot be written as a tensor product of pure/mixed states on the tensor legs.

A *projective measurement* is a set of orthogonal projections $\{P_1, \ldots, P_n\}$ such that $\sum_{i=1}^n P_i = \mathbb{1}_{\mathcal{H}}$. This is a special case of a Positive Operator-Valued Measurement (POVM), which is a set of positive operators $\{A_1, \ldots, A_n\}$ such that $\sum_{i=1}^n A_i = \mathbb{1}_{\mathcal{H}}$. Both have corresponding outcomes $\lambda_1, \ldots, \lambda_n$. The probability of observing outcome λ_i when measuring a pure state $|\psi\rangle$ is $\langle\psi|A_i|\psi\rangle$.

2.3 Tensor products

Note that the direct sum of two finite dimensional spaces has dimension equal to the sum of the dimensions. The tensor product can be seen as a way to multiply the dimensions of two spaces.

Algebraic tensor products

We give an informal definition of the algebraic tensor product; for a more rigorous definition and several results, see [1, Ch. 3.1].

Let V, W be vector spaces. The elementary tensor of two elements $v \in V$ and $w \in W$ is written as $v \otimes w$. The elementary tensors have certain arithmetic rules; in short, these are given by bilinearity of the map $(v, w) \mapsto v \otimes w$. The algebraic tensor product space $V \otimes W$ of V and W is defined as the space of linear combinations of elementary tensors.

Tensor products on Hilbert spaces

To define a tensor product on Banach spaces, we need to define a norm on the algebraic tensor product of the underlying vector spaces, and then take the completion with respect to this norm. For general Banach spaces this is a rather subtle issue, as there can be many different possible tensor norms leading to completely different spaces. However, on Hilbert spaces, there is a unique canonical way to do this. If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, then we define an inner product on their algebraic tensor product by

$$\langle a \otimes b, c \otimes d\rangle_{\mathcal{H}\otimes\mathcal{K}} = \langle a, c\rangle_{\mathcal{H}} \langle b, d\rangle_{\mathcal{K}}.$$

After completion with respect to the corresponding norm, we obtain a Hilbert space that we will also denote as $\mathcal{H} \otimes \mathcal{K}$.

Tensor products on C^*-algebras

One can canonically define a $*$-algebra structure on the algebraic tensor product by pointwise multiplication and involution. The question is how to define a C^*-norm. Like Banach spaces, C^*-algebras are rather complicated in this regard; they can admit several different pre-C^*-norms on their algebraic tensor product (by which we mean that they satisfy all properties of a C^*-norm bar completeness). The extreme (and most often used) cases are the minimal and maximal tensor norms.

Definition 2.3.1 (Minimal tensor norm)**.** *Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras and let $\pi_A : \mathcal{A} \to \mathbb{B}(\mathcal{H}_A)$ and $\pi_B : \mathcal{B} \to \mathbb{B}(\mathcal{H}_B)$ be their universal representations. Then $\pi_A \otimes \pi_B$ defines a $*$-homomorphism $\mathcal{A} \otimes \mathcal{B} \to \mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. The* minimal (or spacial) tensor norm *is given by*

$$\| \cdot \|_{\min} = \|\pi_A \otimes \pi_B(\cdot)\|_{\mathbb{B}(\mathcal{H}_A\otimes\mathcal{H}_B)}.$$

The completion with respect to this norm is denoted by $\mathcal{A} \otimes_{\min} \mathcal{B}$.

Remark 2.3.2. It turns out that instead of the universal representation, any faithful representation suffices to define the same minimal tensor norm. See e.g. [7, B.6] or [1, 3.3.11].

Remark 2.3.3. It is not immediately clear that the above norm is indeed the 'minimal' tensor norm, i.e. that it is always smaller or equal than another pre-C^* norm on the algebraic tensor product. This turns out to be a very deep result by Takesaki - see for example Chapter 6.4 from Murphy.

The representation used for the definition of the minimal tensor norm can be seen as a special case of a representation $\mathcal{A} \otimes \mathcal{B} \to \mathbb{B}(\mathcal{H})$. The maximal tensor norm is defined as a supremum over these kinds of representations; therefore it is not hard to see that the minimal tensor norm is smaller than the maximal tensor norm.

Definition 2.3.4 (Maximal tensor norm)**.** *The* maximal tensor norm *is given by*

$$\| \cdot \|_{\max} = \sup_{\pi} \pi(\cdot),$$

where the supremum is taken over all representations $\pi : \mathcal{A} \otimes \mathcal{B} \to \mathbb{B}(\mathcal{H})$ of the algebraic tensor product. The completion with respect to this norm is denoted by $\mathcal{A} \otimes_{\max} \mathcal{B}$.

The maximal tensor product satisfies the following universal property (see [7, B.9] for a proof):

Proposition 2.3.5 (Universal property of maximal tensor norm)**.** *For any $*$-homomorphism $\pi : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{C}$ to a C^*-algebra $\mathcal{C}$, there exist restrictions $\pi_A : \mathcal{A} \to \mathcal{C}$ and $\pi_B : \mathcal{B} \to \mathcal{C}$ with commuting ranges. Conversely, given any $*$-homomorphism $\pi_A : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ and $\pi_B : \mathcal{B} \to \mathbb{B}(\mathcal{H})$ with commuting ranges, there exists a unique $*$-homomorphism $\pi : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{C}$ that extends π_A and π_B.*

Corollary 2.3.6. *Let $\mathcal{A}$, $\mathcal{B}$ be C^*-algebras. The maximal norm (and thus also the minimal norm) on $\mathcal{A} \otimes \mathcal{B}$ satisfies the following estimate for any $a \in \mathcal{A}, b \in \mathcal{B}$:*

$$\|a \otimes b\|_{\max} \leq \|a\|_{\mathcal{A}} \|b\|_{\mathcal{B}}.$$

Proof. Let $\pi : \mathcal{A} \otimes \mathcal{B} \to \mathbb{B}(\mathcal{H})$ be any representation and let π_A, π_B be the restrictions from the universal property. Then

$$\|\pi(a \otimes b)\| = \|\pi_A(a)\pi_B(b)\| \leq \|\pi_A(a)\| \|\pi_B(b)\| \leq \|a\|_{\mathcal{A}} \|b\|_{\mathcal{B}},$$

using the fact that $*$-homomorphisms on C^*-algebras are contractive (Theorem 2.1.7 from Murphy). Taking the supremum over all representations, we find $\|a \otimes b\|_{\max} \leq \|a\|_{\mathcal{A}} \|b\|_{\mathcal{B}}$. □

Remark 2.3.7. Unlike with the minimal tensor norm, it is not hard to show that the maximal tensor norm is indeed the 'largest' tensor norm one can define. Indeed, if α was another tensor norm on $\mathcal{A} \otimes \mathcal{B}$ and its completion is denoted by $\mathcal{A} \otimes_\alpha \mathcal{B}$, then the canonical $*$-homomorphisms $\pi_A : \mathcal{A} \to \mathcal{A} \otimes_\alpha \mathcal{B}$ and $\pi_B : \mathcal{B} \to \mathcal{A} \otimes_\alpha \mathcal{B}$ have commuting ranges. Therefore, by the universal property, there exists a $*$-homomorphism $\pi : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{A} \otimes_\alpha \mathcal{B}$ which is the identity on $\mathcal{A} \otimes \mathcal{B}$. But since $*$-homomorphisms between C^*-algebras are automatically contractive, it follows that $\|x\|_\alpha \leq \|x\|_{\max}$ for $x \in \mathcal{A} \otimes \mathcal{B}$.

One example of a C^*-algebras where the minimal and maximal norm are different is the space $\mathbb{B}(\mathcal{H})$ - it is known that $\mathbb{B}(\mathcal{H}) \otimes_{\min} \mathbb{B}(\mathcal{H}) \neq \mathbb{B}(\mathcal{H}) \otimes_{\max} \mathbb{B}(\mathcal{H})$ [10] and even that there is an uncountably infinite number of inequivalent C^*-norms on $\mathbb{B}(\mathcal{H}) \otimes \mathbb{B}(\mathcal{H})$ [20]. In the case of nuclear C^*-algebras, we know that this can't happen.

Definition 2.3.8. A C^*-algebra $\mathcal{A}$ is called *nuclear* if for every C^*-algebra $\mathcal{B}$, there is a unique tensor norm on $\mathcal{A} \otimes \mathcal{B}$ - i.e. the minimal and maximal tensor norm coincide.

This is one of several equivalent definitions for nuclear C^*-algebras (see e.g. [1, 2.3]); it is the one that will be most convenient for us.

2.4 Free groups, free products and group C^*-algebras

Free groups, free products and group C^*-algebras are notions that are essential for this thesis but which are not covered in Murphy's book (or any courses I followed). Hence I will introduce them here for readers who aren't familiar with them either.

Free groups

Firstly, the *free group* $\mathbb{F}_n$ of n generators $g_1, \ldots, g_n$ is the group of reduced words in those generators and their inverses. We will explain what this means. Firstly, we refer to the generators g_i as *letters.* A word consists of a sequence of such letters, for example $g_1g_3g_2g_1$. Instead of writing multiple of the same letters adjacently, we write powers: for example $g_1^5g_2^2g_3^3g_1$. These can also be inverses, such as g_1^{-5}.

The group operation is concatenation; i.e. $(g_1g_2) \circ (g_2g_1) = g_1g_2g_2g_1 = g_1g_2^2g_1$. The only relations are taking powers, such as here, and the inverse relation - e.g. $(g_1g_2) \circ (g_2^{-1}g_1) = g_1g_1 = g_1^2$. The right hand sides of both of these words are said to be in *reduced* form. The unit is the 'empty' word, which we simply write as $\mathbb{1}_{\mathbb{F}_n}$.

Free products

The free product of groups is quite similar in definition to free groups. If G, H are two groups, then the *free product* of G and H is written as $G * H$ and defined as all words of alternating elements $g_1h_1g_2h_2\ldots$, $g_i \in G\setminus\{e\}$, $h_i \in H\setminus\{e\}$. Upon concatenating two words, we take group operations of any adjacent elements of the same group until all the letters are again alternating and not equal to the identity (i.e. it is in 'reduced' form). We can concretely write down the free product as

$$G * H = \bigcup_{n=1}^{\infty}(S_1^n \cup S_2^n),$$

where $S_1^n = \{g_1 \ldots g_n : g_1 \in G \setminus \{e\}, g_2 \in H \setminus \{e\}, \ldots\}$ and $S_2^n = \{g_1 \ldots g_n : g_1 \in H \setminus \{e\}, g_2 \in G \setminus \{e\}, \ldots\}$.

The free product of C^*-algebras are more tricky; we do not write down the full details here (see e.g. [17]). If $\mathcal{A}, \mathcal{B}$ are C^*-algebras, we first construct the free $*$-algebra. We start again with words of alternating elements. We define a multiplication by concatenation, multiplying adjacent elements from the same C^*-algebra. We then take linear combinations of these words to create a vector space structure, and define an involution on words by the standard rule $(AB)^* = B^*A^*$ and on the whole space by anti-linear extension. We need to adhere to the standard $*$-algebra relations such as the distributive property; in order to do that, we quotient out all elements like $a_1(b_1 + b_2)a_2 - a_1b_1a_2 - a_1b_2b_1$, etc. Now we would expect to concretely write down the $*$-algebra $\mathcal{A} *_{\mathbb{C}} \mathcal{B}$ as

$$\mathcal{A} *_{\mathbb{C}} \mathcal{B} = \mathbb{C}\mathbb{1} \oplus \bigoplus_{n=1}^{\infty}(W_1^n \oplus W_2^n), \tag{1}$$

where

$$W_1^n = \mathrm{Span}\{\mathrm{a}_1 \ldots \mathrm{a}_\mathrm{n} : \mathrm{a}_1 \in \mathcal{A} \setminus \mathbb{C}\mathbb{1}, \mathrm{a}_2 \in \mathcal{B} \setminus \mathbb{C}\mathbb{1}, \ldots\},$$
$$W_2^n = \mathrm{Span}\{\mathrm{a}_1 \ldots \mathrm{a}_\mathrm{n} : \mathrm{a}_1 \in \mathcal{B} \setminus \mathbb{C}\mathbb{1}, \mathrm{a}_2 \in \mathcal{A} \setminus \mathbb{C}\mathbb{1}, \ldots\}.$$

However, there is a problem here; for example, the word $a_1(b_1+\lambda\mathbb{1})a_2 \in W_1^3$ could be written alternatively as $a_1b_1a_2 + \lambda \cdot a_1a_2 \in W_1^3 \oplus W_1^1$, so elements here are not uniquely defined. To remedy this, we choose states τ_A, τ_B and define subspaces

$$\mathcal{A}^\circ = \{a \in \mathcal{A} : \tau_A(a) = 0\}, \quad \mathcal{B}^\circ = \{b \in \mathcal{B} : \tau_B(b) = 0\}.$$

For example, in the case $\mathcal{A} = \mathcal{B} = \ell_\infty^m$ that will be considered in this thesis, we have the natural choice $\tau(x) = \langle x, (1, \dots, 1)\rangle$; thus, the space $(\ell_\infty^m)^\circ$ will consist of elements whose inner product with the all-ones vector is 0.

Now we define $\mathcal{A} *_\mathbb{C} \mathcal{B}$ through (1) but by redefining

$$W_1^n = \mathrm{Span}\{\mathrm{a_1 \dots a_n : a_1} \in \mathcal{A}^\circ, \mathrm{a_2} \in \mathcal{B}^\circ, \dots\}, \quad \mathrm{W_2^n} = \mathrm{Span}\{\mathrm{a_1 \dots a_n : a_1} \in \mathcal{B}^\circ, \mathrm{a_2} \in \mathcal{A}^\circ, \dots\}.$$

We can replace a letter $a \in \mathcal{A}$ with $(a - \tau_A(a)) + \tau_A(a) \in \mathcal{A}^\circ \oplus \mathbb{C}\mathbb{1}$. With this trick and the distributive property, we can 'reduce' words of letters in $\mathcal{A}$ and $\mathcal{B}$ to elements in (1).

Finally, we need to define a norm to complete over. There are actually multiple ways to do this; we will be using the *universal free product* in this thesis (one can also define a reduced one which depends on a choice of state, see [17, Ch. 7]). The norm is defined somewhat similarly to the maximal tensor norm: we let $\|x\| := \sup \|\pi(x)\|$ where the supremum is taken over all representations $\pi : \mathcal{A} *_\mathbb{C} \mathcal{B} \to \mathbb{B}(\mathcal{H})$. First note that this norm is finite; indeed, if $x = g_1 \dots g_n$ we have for any representation π that $\|\pi(x)\| \le \|\pi(g_1)\| \dots \|\pi(g_n)\| \le \|g_1\| \dots \|g_n\|$ since π is a representation of C^*-algebras (and thus contractive) on the separate letters. Now it is easy to show that this is a seminorm; to show that it is a norm we need to show that a faithful representation exists. This follows from the construction of the reduced free product; we refer again to [17].

The C^*-algebraic free product satisfies the following universal property:

Proposition 2.4.1 (Universal property of C^*-algebraic free product)**.** Let $\pi_A : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ and $\pi_B : \mathcal{B} \to \mathbb{B}(\mathcal{H})$ be $*$-homomorphisms on C^*-algebras. Then there exists a unique $*$-homomorphism $\pi : \mathcal{A} * \mathcal{B} \to \mathbb{B}(\mathcal{H})$ extending them.

Proof. We can define π on words by applying π_A and π_B to the separate letters and linearly extend it to the algebraic free product $\mathcal{A} *_\mathbb{C} \mathcal{B}$. This defines a representation on that algebraic free product; hence it is contractive by definition of the free product norm. Therefore we can extend π to $\mathcal{A} * \mathcal{B}$. □

Group C^*-algebras

The last thing we need to define is the (maximal) group C^*-algebra, where we use elements from a group as the building blocks to construct a C^*-algebra. The construction is rather similar to the free product of C^*-algebras (although less technically involved). If G is a group, we start with the space $\mathbb{C}[G] = \mathrm{span}\{\eta_\mathrm{g} : \mathrm{g} \in \mathrm{G}\}$. Instead of just g, we write η_g for the group elements to distinguish between the C^*-algebra and the original group. The multiplication on this space is defined through the group operation and the standard algebra rules. The involution of a group elements η_g is given by $\eta_g^* = \eta_{g^{-1}}$. This extends to the whole space similarly as with the free product. The norm is defined in a similar fashion through a supremum of representations:

$$\left\| \sum_g \lambda_g \eta_g \right\| := \sup_\pi \left\| \sum_g \lambda_g \pi(g) \right\|,$$

where the supremum is taken over all unitary representations $\pi : G \to \mathcal{A}$ to C^*-algebras $\mathcal{A}$, i.e. representations that map to unitary elements. After completing with respect to this norm we obtain the desired C^*-algebra, which we denote by $C^*(G)$. This C^*-algebra satisfies the following universal property:

Proposition 2.4.2 (Universal property of maximal group C^*-algebra)**.** *Let G be a group. For any unitary representation $\pi : G \to \mathcal{A}$, there exists a $*$-homomorphism $\hat{\pi} : C^*(G) \to \mathcal{A}$ 'extending' π, by which we mean that $\pi(g) = \hat{\pi}(\eta_g)$ for all $g \in G$.*

Proof. By linear extension, it is clear how to define $\hat{\pi}$ on $\mathbb{C}[G]$. By definition of the norm, $\hat{\pi}$ is contractive on $\mathbb{C}[G]$, so we can extend it to $C^*(G)$. $\square$

3 Correlation matrices and Tsirelson's problem

3.1 Introduction

We start by describing a very general fictional game setting. There are 2 players: Alice and Bob, and a 'game master'. The game master gives inputs x and y from finite sets $\mathbf{X} = \{1, \ldots, X\}$ and $\mathbf{Y} = \{1, \ldots, Y\}$ to Alice and Bob respectively. Alice and Bob are not allowed to communicate in any way. They now need to give outputs a and b from finite sets $\mathbf{A} = \{1, \ldots, A\}$ and $\mathbf{B} = \{1, \ldots, B\}$; they win the game only for specific combinations of a, b, x and y which are known in advance. They can coordinate a strategy beforehand.

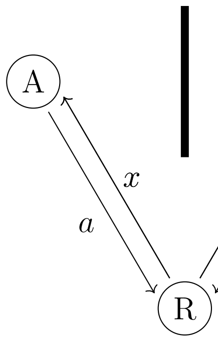


Figure 1: A schematic overview of a non-local two-player game

We denote a (randomized) strategy by a probability measure $\mathbb{P}$, indicating the probability of every pair of outputs (a, b) given any pair of inputs (x, y):

$$(\mathbb{P}(a, b|x, y))_{a\in\mathbf{A},b\in\mathbf{B},x\in\mathbf{X},y\in\mathbf{Y}}.$$

such a matrix of probabilities is called a *correlation matrix*. It reflects the correlation between Alice and Bob's output probabilities.

3.2 Classical case

In the classical deterministic case, all Alice and Bob can do for a strategy is coordinate beforehand which output they will give for every input. Somewhat more generally, they can agree on personal probability distributions over the possible outputs for each input; in other words, personal conditional probability distributions. We call these distributions P and Q for Alice and Bob respectively; so the probability of output a and b given input x and y is given by $P(a|x)Q(b|y)$.

If we define $\mathcal{P}(\mathbf{A}|\mathbf{X})$ as all possible conditional probability distributions over $\mathbf{A}$ and $\mathbf{X}$:

$$\mathcal{P}(\mathbf{A}|\mathbf{X}) = \{(P(a|x))_{x,a} \in \mathbb{R}_+^{AX} : \forall x \in \mathbf{X}, \sum_{a\in\mathbf{A}} P(a|x) = 1\},$$

then all possible strategies are given by

$$\{(P(a|x)Q(b|y))_{a,b;x,y} : P \in \mathcal{P}(\mathbf{A}|\mathbf{X}),\ Q \in \mathcal{P}(\mathbf{B}|\mathbf{Y})\}. \tag{2}$$

At this point there is no correlation between Alice's and Bob's outputs; the probabilities for Alice do not depend on Bob's outcome, and vice versa.

To add some correlation, we can allow Alice and Bob to have shared randomness; for example, there could be a dice roll of which they can both see the result. The resulting set of classical correlation matrices is denoted by $C_c(\mathbf{AB}|\mathbf{XY})$, after [5]. It now also includes probability distributions over the strategies in (2), i.e.

$$C_c(\mathbf{AB}|\mathbf{XY}) = \left\{ \left(\sum_{i \in I} p_i P_i(a|x) Q_i(b|x) \right)_{a,b;x,y} : P_i \in \mathcal{P}(\mathbf{A}|\mathbf{X}),\ Q_i \in \mathcal{P}(\mathbf{B}|\mathbf{Y}),\ p_i \in [0,1],\ \sum_{i \in I} p_i = 1 \right\}.$$

This set can also be seen as the convex hull of (2), with the deterministic strategies as extreme values. We often write simply C_c when the corresponding sets are clear.

3.3 Quantum case with entangled states

If we involve quantum mechanics, we can define even more elaborate strategies. Instead of shared randomness, Alice and Bob now share an entangled (bipartite) state $|\psi\rangle$ over Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Their outputs are decided by some POVM $\{A_a^x\}_{a \in \mathbf{A}}$ respectively $\{B_b^y\}_{b \in \mathbf{B}}$, depending on the inputs (x, y).

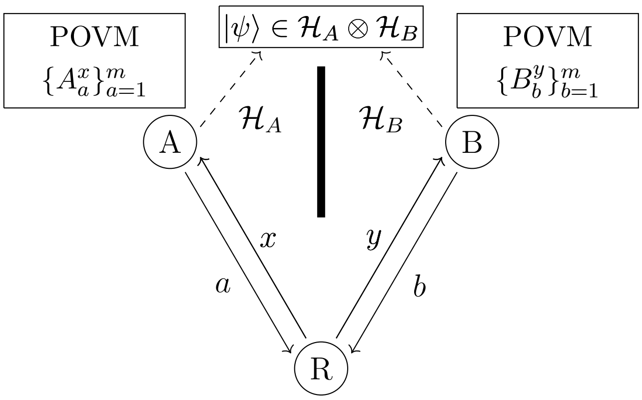


Figure 2: A schematic overview of the tensor product model

From a physical point of view, it usually suffices to consider finite dimensional Hilbert spaces. The resulting set of correlation matrices is simply called the 'quantum correlations' and denoted by C_q:

$$C_q(\mathbf{AB}|\mathbf{XY}) = \Big\{ (\langle\psi| A_a^x \otimes B_b^y |\psi\rangle)_{x,y;a,b} : \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B,\ \dim(\mathcal{H}) < \infty,\ |\psi\rangle \in \mathrm{Ball}(\mathcal{H}), \mathrm{A_a^x} \in \mathbb{B}(\mathcal{H}_\mathrm{A})^+,$$
$$B_b^y \in \mathbb{B}(\mathcal{H}_B)^+,\ \sum_a A_a^x = \sum_b B_b^y = \mathbb{1} \forall (x,y) \in \mathbf{X} \times \mathbf{Y} \Big\}.$$

Again, we write C_q when the corresponding sets are clear.

If we drop the finite dimensional requirement, we get another set of quantum correlations. Following notation from [5], we denote this set by C_{qs} (probably for 'separated quantum correlations'):

$$C_{qs}(\mathbf{AB}|\mathbf{XY}) = \Big\{ (\langle\psi| A_a^x \otimes B_b^y |\psi\rangle)_{x,y;a,b} : \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B,\ |\psi\rangle \in \mathrm{Ball}(\mathcal{H}), \mathrm{A_a^x} \in \mathbb{B}(\mathcal{H}_\mathrm{A})^+,$$
$$B_b^y \in \mathbb{B}(\mathcal{H}_B)^+,\ \sum_a A_a^x = \sum_b B_b^y = \mathbb{1} \forall (x,y) \in \mathbf{X} \times \mathbf{Y} \Big\}.$$

As we shall see in Chapter 5.3, neither of these sets are closed. Therefore, we define the closure $C_{qa} = \overline{C_{qs}}$. I am unsure what the 'a' in 'qa' is named after, perhaps the german word 'abschluss'.

3.4 Quantum case with commuting projections

In the above, we assumed that the Hilbert space was divided in separate tensor legs for Alice and Bob. It is not clear whether this is consistent with physical reality. A perhaps more reasonable model would be to assume one (giant) Hilbert space on which both Alice's and Bob's operators act. All we need then to conserve our experimental setup is for the operators on Alice's side to commute with those on Bob's side; that way, it does not matter who does their measurement first.

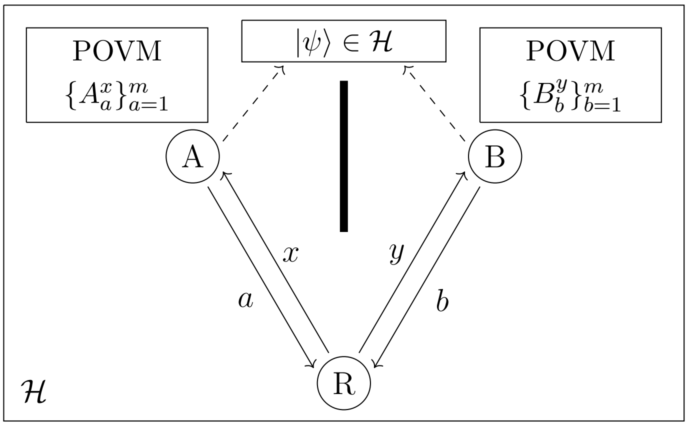


Figure 3: A schematic overview of the commuting operator model

This defines the set of 'commuting quantum correlations', denoted by C_{qc}:

$$C_{qc}(\mathbf{AB}|\mathbf{XY}) = \Big\{(\langle\psi|\, A_a^x \cdot B_b^y \,|\psi\rangle)_{x,y;a,b} : \mathcal{H}, |\psi\rangle \in \mathrm{Ball}(\mathcal{H}), \mathrm{A_a^x}, \mathrm{B_b^y} \in \mathrm{Proj}(\mathcal{H}),$$
$$[A_a^x, B_b^y] = 0, \sum_a A_a^x = \sum_b B_b^y = \mathbb{1}\Big\}..$$

By setting $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\tilde{A}_x^a := A_a^x \otimes \mathbb{1}_{\mathcal{H}_B}$, $\tilde{B}_y^b := B_b^y \otimes \mathbb{1}_{\mathcal{H}_B}$, we see that correlation matrices from C_{qs} are also in C_{qc}; therefore we have the inclusion $C_{qs} \subseteq C_{qc}$.

3.5 Overview

In Chapter 5, we shall show the following inclusion relations of the various correlation sets:

$$C_c \subsetneq C_q \subseteq C_{qs} \subsetneq C_{qa} \subseteq C_{qc}.$$

Strictness of the final inclusion is still open, which will be the main problem in this thesis:

Conjecture 3.5.1 (Tsirelson's Problem or T2)**.** *For all input sets* $\mathbf{X}, \mathbf{Y}$ *and output sets* $\mathbf{A}, \mathbf{B}$*, we have* $C_{qa}(\mathbf{AB}|\mathbf{XY}) = C_{qc}(\mathbf{AB}|\mathbf{XY})$*.*

Remark 3.5.2. The attentive reader might have remarked that we have not defined a finite dimensional version of the set C_{qc}, as we did for C_{qs}. The reason is that we already know these sets would be the same; in other words, the finite dimensional version of Tsirelson's problem is already resolved. A sketch of the proof was given by Tsirelson himself in the original problem statement [29]. We give the full version with all details, as well as our own simpler proof, in Chapter 5.2.

Remark 3.5.3. In the above, we have used pure states to define the various sets of quantum correlations. These sets would remain unchanged if we defined them instead with mixed states. This is because we can 'purify' a mixed state by increasing the dimension of the Hilbert space: if $\rho = \sum_{i=1}^{n} p_i |\varphi_i\rangle\langle\varphi_i|$ is a mixed state on $\mathcal{H}$, then we define $|\psi\rangle \in \mathcal{H} \otimes \mathbb{C}^n$ by $|\psi\rangle = \sum_{i=1}^{n} |\varphi_i\rangle \otimes \sqrt{p_i}\,|e_i\rangle$. By Pythagoras, this is a unit vector, and therefore a valid pure state. By straightforward calculation, we can check that $\rho = (\mathbb{1}_{\mathcal{H}} \otimes \mathrm{Tr}_{\mathbb{C}^{\mathrm{n}}})(|\psi\rangle\langle\psi|)$. This implies that, for $A \in \mathbb{B}(\mathcal{H})$:

$$\langle\psi| A \otimes \mathbb{1}_{\mathbb{C}^n} |\psi\rangle = \mathrm{Tr}(\mathrm{A} \otimes \mathbb{1}_{\mathbb{C}^{\mathrm{n}}} |\psi\rangle\langle\psi|) = \mathrm{Tr}(\mathrm{A}(\mathbb{1}_{\mathcal{H}} \otimes \mathrm{Tr}_{\mathbb{C}^{\mathrm{n}}})(|\psi\rangle\langle\psi|) = \mathrm{Tr}(\mathrm{A}\rho).$$

If $\mathcal{H}$ has a tensor product form $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, then this can be preserved by adding the ancilla to one of the two tensor legs.

4 Classical correlations vs quantum correlations

In this chapter we give some results on the relation between the two 'simplest' correlation sets C_c and C_q. The content is rather stand-alone as we will not use any of these results later on.

4.1 $C_c \subseteq C_q$

First let us check that the expected result $C_c \subseteq C_q$ indeed holds. Let $(\mathbb{P}(a,b|x,y))_{a,b;x,y} \in C_c(\mathbf{AB}|\mathbf{XY})$ be a classical correlation given by $\mathbb{P}(a,b|x,y) = \sum_{i=1}^n p_i P_i(a|x) Q_i(b|y)$. The goal is to find some $A_a^x, B_b^y \in M_n(\mathbb{C})$ and $|\psi\rangle \in \mathrm{Ball}(\mathbb{C}^{n^2})$ such that $\langle \psi | A_a^x \otimes B_b^y | \psi \rangle = \sum_{i=1}^n p_i P_i(a|x) Q_i(b|y)$.

We define measurement operators

$$A_a^x := \begin{pmatrix} P_1(a|x) & 0 & \dots & 0 \\ 0 & P_2(a|x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_n(a|x) \end{pmatrix}, \quad B_b^y := \begin{pmatrix} Q_1(b|y) & 0 & \dots & 0 \\ 0 & Q_2(b|y) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Q_n(b|y) \end{pmatrix}$$

and a state $|\psi\rangle = \sum_{i=1}^n \sqrt{p_i}\, |ii\rangle = \sum_{i=1}^n \sqrt{p_i} e_i \otimes e_i$. Then it holds that

$$\langle \psi | A_a^x \otimes B_b^y | \psi \rangle = \sum_{i=1}^n p_i P_i(a|x) Q_i(b|y).$$

Also, A_a^x and B_b^y are positive matrices summing up to the identity over a respectively b, and ψ is a unit vector. Thus, $\mathbb{P}(a,b|x,y) \in C_q(\mathbf{AB}|\mathbf{XY})$.

4.2 $C_c \neq C_q$: Bell violations

As mentioned in Chapter 3.2, the classical correlations are given by a convex hull, i.e. a polytope. This means that we can describe it by a set of inequalities corresponding to its edges; these are called Bell inequalities (see [21, Ch. 1.2]). Such an inequality is described by

$$\sum_{a,b;x,y} \lambda_{a,b;x,y} \mathbb{P}(a,b|x,y) \leq C \quad \text{for all} \quad (\mathbb{P}(a,b|x,y))_{a,b;x,y} \in C_c(\mathbf{AB}|\mathbf{XY}).$$

Therefore, to prove that $C_c \neq C_q$, it suffices to find a correlation $(\mathbb{P}(a,b|x,y))_{a,b;x,y} \in C_q(\mathbf{AB}|\mathbf{XY})$ that *violates* such a Bell inequality. The most common (and most simple) example is the *CHSH-inequality.*

In the CHSH-scenario, Bob and Alice have only 2 inputs and outputs. Let us start by describing the experiment from the classical point of view. Alice has a particle of which she can measure two properties, given by random variables A_1 and A_2. She chooses which property to measure depending on the input x. Similarly, Bob has random variables B_1 and B_2 of which he measures one depending on the input y. The outputs are given by the sets $\mathbf{A} = \mathbf{B} = \{1, -1\}$.

We consider the random variable $S = A_1B_1 + A_2B_1 + A_1B_2 - A_2B_2 = \sum_{x,y=1}^2 (-1)^{1+\min(x,y)} A_i B_j$. In classical physics, we assume that the values of the random variables are deterministic; they have fixed, although hidden, values. Therefore, the value of either $A_1 + A_2$ or $A_1 - A_2$ is always 0, while the other is ± 2. As such, the value of S is always ± 2. Hence,

$$|\mathbb{E}(S)| = \left| \sum_{s \in \{-2,2\}} \mathbb{P}(S = s) s \right| \leq 2 \sum_{s \in \{-2,2\}} \mathbb{P}(S = s) = 2.$$

Note that $\mathbb{E}(A_x B_y) = \sum_{a,b\in\{-1,1\}}(-1)^{ab}\mathbb{P}(a,b|x,y)$. Hence we can reformulate this as a Bell inequality (or actually 2) as defined above:

$$\sum_{a,b;x,y}(-1)^{1+ab+\min(x,y)}\mathbb{P}(a,b|x,y) \leq 2, \quad \sum_{a,b;x,y}(-1)^{1+ab+\min(x,y)}\mathbb{P}(a,b|x,y) \geq -2.$$

We now describe a specific quantum experiment violating this inequality. Recall the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

also known as the bitflip and the phaseflip operators. We implement the measurements through the observables $A_1 = \sigma_x, A_2 = \sigma_z$ and $B_1 = -1/\sqrt{2}(\sigma_z + \sigma_x), B_2 = 1/\sqrt{2}(\sigma_z - \sigma_x)$ (note that these are observables, not the projections from the definition). The state is given by the well-known EPR-pair $|\psi\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$.

Now we have

$$\mathbb{E}(A_1B_1) = \langle\psi|A_1 \otimes B_1|\psi\rangle = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = -\frac{1}{\sqrt{2}}$$

And similarly

$$\mathbb{E}(A_1B_2) = -\frac{1}{\sqrt{2}}, \quad \mathbb{E}(A_2B_1) = -\frac{1}{\sqrt{2}}, \quad \mathbb{E}(A_2B_2) = \frac{1}{\sqrt{2}}.$$

Hence,

$$\mathbb{E}(S) = -2\sqrt{2} < -2,$$

which violates the Bell inequality. Therefore, the projective measurement operators following from the observables A_1, A_2, B_1, B_2 (i.e. the projections on the eigenspaces) together with the state $|\psi\rangle$ yield a correlation matrix in C_q that is not in C_c.

5 Known quantum correlation results

5.1 Preliminaries: C^*-states vs vector states

In the C^*-algebra theory we will use hereafter, we need the notion of a C^*-algebraic state. This is a positive linear functional $\omega : \mathbb{B}(\mathcal{H}) \to \mathbb{C}$ such that $\omega(\mathbb{1}_{\mathcal{H}}) = 1$. In this case we say that ω is a state *on* $\mathcal{H}$. See Murphy for more details.

Note that a 'pure quantum state' $|\psi\rangle$ as above can be identified with the C^*-algebraic state on $\mathcal{A}$ defined as $A \mapsto \langle\psi|A|\psi\rangle$. More generally, a density matrix ρ can be identified with the state defined by $A \mapsto \mathrm{Tr}(A\rho)$. In the context of a concretely represented C^*-algebra $\mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$, these are called a *(pure) vector state* respectively *mixed vector state*. Recall that we can purify a mixed vector state by Remark 3.5.3.

We see that the mixed vector states are a subset of all C^*-algebraic states. In finite dimensions, the converse inclusion is also true. Indeed, if $\omega : M_n(\mathbb{C}) \to \mathbb{C}$ is a positive linear functional with $\omega(\mathbb{1}_{\mathbb{C}^n}) = 1$, then clearly $\omega(A) = \mathrm{Tr}(A\rho)$ for some $\rho \in M_n(\mathbb{C})$. (One can check that the coefficients of ρ are given by $\rho_{i,j} = \omega(E_{j,i})$, where $E_{i,j}$ is the matrix with a 1 at position (i, j) and zeroes elsewhere.) Filling in $A = \mathbb{1}_{\mathbb{C}^n}$ shows that $\mathrm{Tr}(\rho) = 1$. Also, positivity of ω implies that $\mathrm{Tr(A}\rho) \geq 0$ whenever A is positive definite. In particular, it follows that $\langle\psi|\rho|\psi\rangle = \mathrm{Tr}(|\psi\rangle\langle\psi|\,\rho) \geq 0$, therefore ρ is positive definite. It follows that ρ is a density matrix, and thus ω is a mixed vector state.

In general, the converse is not true; there are C^*-algebraic states which cannot be written as mixed vector states on the same Hilbert space. There are two common tricks we can use to convert a general state ω on a Hilbert space $\mathcal{H}$ to a (mixed) vector state.

First, if we are allowed to change our Hilbert space, we can use the GNS representation corresponding to ω. Then ω will automatically become a (pure!) vector state on the resulting Hilbert space $\mathcal{H}_\omega$. Indeed, if ξ is the equivalence class of the multiplicative unit $(1, \ldots, 1)$ in $\mathcal{H}_\omega$, then $\langle\pi_\omega(a)\xi, \xi\rangle = \omega(a)$. To be precise, we define a corresponding state $\tilde{\omega}$ on $\mathbb{B}(\mathcal{H}_\omega)$ by $\tilde{\omega}(a) = \langle a\xi, \xi\rangle$. Then $\tilde{\omega}$ is a vector state on $\mathcal{H}_\omega$ such that $\omega = \tilde{\omega} \circ \pi_\omega$. Note that the new Hilbert space $\mathcal{H}_\omega$ generally does not carry over any properties from $\mathcal{H}$. In particular, if $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, then $\mathcal{H}_\omega$ does not need to have a similar tensor form.

Second, we can use the fact that mixed vector states on $\mathcal{H}$ are weak-* dense in the set of general states - see Theorem A.1.1. This has the advantage that the representation remains fixed.

5.2 Finite dimensional case

In the finite dimensional case, Tsirelson's problem has already been resolved; hence the definition of the set C_q does not depend on the type of model chosen. A sketch of the proof was already given by Tsirelson himself in the original problem statement [29], and worked out in a bit more detail in [25]. We work out all the details of this proof below. The proof turns out to be of considerable length, and can be skipped at first reading.

We introduce some notation from Murphy (above Lemma 4.1.6). Let p be a projection of a Hilbert space $\mathcal{H}$ into $\mathcal{K}$. If $a \in \mathbb{B}(\mathcal{H})$, we define $a_p = a_K$ to be the compression of $a \in \mathbb{B}(\mathcal{H})$ to $\mathcal{K}$. In other words, $a_p \in \mathcal{B}(\mathcal{K})$ is defined as $a_p : \xi \mapsto pa\xi$. Further, when $\mathcal{A}$ is a $*$-algebra on $\mathcal{H}$ and $p \in \mathcal{A}'$, we define $\mathcal{A}_p = \{a_p : a \in \mathcal{A}\}$. By Lemma 4.1.6 from Murphy, the map $pAp \mapsto A_p,\ u \mapsto u_p$ is a $*$-isomorphism. Moreover, if $p \in \mathcal{A}''$, then $(\mathcal{A}')_p = (\mathcal{A}_p)'$.

Theorem 5.2.1. *Let $\mathcal{H}$ be a finite dimensional Hilbert space and $\mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$ be a von Neumann algebra containing $\mathbb{1}_{\mathcal{H}}$. Then there exist finite dimensional Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ and injective $*$-homomorphisms $\pi_A : \mathcal{A} \to \mathbb{B}(\mathcal{H}_A)$ and $\pi_B : \mathcal{A}' \to \mathbb{B}(\mathcal{H}_B)$ such that the following statement holds: for every state ω on $\mathbb{B}(\mathcal{H})$, there is a state $\tilde{\omega}$ on $\mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that $\tilde{\omega}(\pi_A(a) \otimes \pi_B(b)) = \omega(a \cdot b)$ for all $a \in \mathcal{A}, b \in \mathcal{B}$.*

Proof. We start by describing the so-called central decomposition of $\mathcal{A}$. The center $\mathcal{Z}(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'$ is a unital commutative C^*-algebra, hence by Gelfand-Naimark it is isomorphic to $C(\Omega)$, where Ω is the character space of $\mathcal{H}$. Since $\mathcal{H}$ is finite dimensional, Ω is a finite set of points $\{x_1, \ldots, x_n\}$. Therefore $C(\Omega)$ is linearly spanned by the indicator functions $\mathbb{1}_{x_i}$. Define $p_i \in \mathcal{Z}(\mathcal{A})$ to be the elements satisfying $\hat{p_i} = \mathbb{1}_{x_i}$. Then the p_i are orthogonal central projections summing to $\mathbb{1}_{\mathcal{H}}$.

Defining $\mathcal{A}_i = \mathcal{A}_{p_i}$, it follows from Lemma 4.1.6 of Murphy that $\mathcal{A} = \sum_{i=1}^n p_i\mathcal{A} = \sum_{i=1}^n p_i\mathcal{A}p_i \cong \bigoplus_{i=1}^n \mathcal{A}_i$. Similarly, we can decompose $\mathcal{A}' \cong \bigoplus_{i=1}^n \mathcal{A}_i'$, since it has the same center. Note here that, again by Lemma 4.1.6 from Murphy, we have $\mathcal{A}_i' = (\mathcal{A}')_{p_i} = (\mathcal{A}_i)'$. Also, we define $\mathcal{H}_i = p_i(\mathcal{H})$, which leads to the decomposition $\mathcal{H} \cong \bigoplus_{i=1}^n \mathcal{H}_i$.

Next, we prove that each $\mathcal{A}_i$ is a factor, i.e. $\mathcal{Z}(\mathcal{A}_i) = \mathbb{C}\mathbb{1}_{\mathcal{H}_i} = \mathbb{C}p_i$. Take $i \in \{1, \ldots, n\}$ and $p = p_ib \in \mathcal{Z}(\mathcal{A}_i)$. For $a \in \mathcal{A}$, we have

$$a(p_ib) = (\sum_{k=1}^n p_ka)p_ib = (p_ia)(p_ib) = (p_ib)(p_ia) = (p_ib)a.$$

Therefore $p_ib \in \mathcal{Z}(\mathcal{A})$, so we can write it as a linear combination $p_ib = \sum_{k=1}^n \lambda_ip_k$. But then $p_ib = p_ibp_i = (\sum_{k=1}^n \lambda_kp_k)p_i = \lambda_ip_i$, and therefore $p = \lambda_i\mathbb{1}_{\mathcal{H}_i} \in \mathbb{C}\mathbb{1}_{\mathcal{H}_i}$.

We will need the following Lemma (see Lemma A.4.2 in the Appendix for a proof):

Lemma 5.2.2. *Let $\mathcal{M} \subseteq \mathbb{B}(\mathcal{H})$ be a factor, where $\mathcal{H}$ is a finite dimensional Hilbert space. Then there exists a unitary $u \in \mathbb{B}(\mathcal{H})$ such that $u\mathcal{M}u^* = \mathbb{1}_{M_d(\mathbb{C})} \otimes M_q(\mathbb{C})$ for some $d, q \in \mathbb{N}$ (and thus $\mathcal{H} \cong \mathbb{C}^{dq} \cong \mathbb{C}^d \otimes C^q$).*

Since the $\mathcal{A}_i'$ are also factors, we can by the Lemma find $n_i, m_i \in \mathbb{N}$ such that $u\mathcal{A}_i'u^* = \mathbb{1}_{M_{n_i}(\mathbb{C})} \otimes M_{m_i}(\mathbb{C})$, acting on the Hilbert space $\mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}$. Then (since $u\mathcal{A}_iu^* = u\mathcal{A}_i''u^* = (u\mathcal{A}_i'u^*)'$ we have $u\mathcal{A}_iu^* = M_{n_i}(\mathbb{C}) \otimes \mathbb{1}_{M_{m_i}(\mathbb{C})}$ (we could have done this the other way around, but this way will be more convenient for us). For notational convenience, let us now assume without loss of generality that both unitary equivalences are in fact equalities. It follows that we can write

$$\mathcal{A} = \bigoplus_{i=1}^n M_{n_i}(\mathbb{C}) \otimes \mathbb{1}_{M_{m_i}(\mathbb{C})}, \quad \mathcal{A}' = \bigoplus_{i=1}^n \mathbb{1}_{M_{n_i}(\mathbb{C})} \otimes M_{m_i}(\mathbb{C}).$$

So to illustrate, for elements $a \in \mathcal{A}$ and $b \in \mathcal{A}'$ we can write

$$a = \bigoplus_{i=1}^n a_i = \bigoplus_{i=1}^n \tilde{a}_i \otimes \mathbb{1}_{M_{m_i}}(\mathbb{C}), \qquad ab = \bigoplus_{i=1}^n \tilde{a}_i \otimes \tilde{b}_i. \tag{3}$$

Now define $\mathcal{H}_A = \bigoplus_{i=1}^n \mathbb{C}^{n_i}$ and $\mathcal{H}_B = \bigoplus_{i=1}^n \mathbb{C}^{m_i}$. Then we can define a projection $p_{\mathcal{H}} : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}$ by mapping a double direct sum to the direct sum of its diagonal entries, i.e.

$\bigoplus_{i=1}^n \bigoplus_{j=1}^n a_i \otimes b_j \mapsto \bigoplus_{i=1}^n a_i \otimes b_i$.

Using notation from (3), we define injective $*$-homomorphisms $\pi_A : \mathcal{A} \to \mathbb{B}(\mathcal{H}_A)$ and $\pi_B : \mathcal{A}' \to \mathbb{B}(\mathcal{H}_B)$ by

$$\pi_A(a) = \bigoplus_{i=1}^n \tilde{a}_i, \qquad \pi_B(b) = \bigoplus_{i=1}^n \tilde{b}_i.$$

To prove the final statement, let ω be a state on $\mathbb{B}(\mathcal{H})$. Let $u_{\mathcal{H}} : \mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathbb{B}(\mathcal{H})$ be the restriction mapping defined by $x \mapsto p_{\mathcal{H}} x p_{\mathcal{H}}$. We define our new state $\tilde{\omega}$ on $\mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ as

$$\tilde{\omega} = \omega \circ u_{\mathcal{H}} : \quad \tilde{\omega}(x) = \omega(p_{\mathcal{H}} x p_{\mathcal{H}}).$$

Since $u_{\mathcal{H}}$ is contractive and $\tilde{\omega}$ is unital, Corollary 3.3.4 from Murphy tells us that this linear functional is indeed a state. Finally, for $a \in \mathcal{A}$, $b \in \mathcal{A}'$:

$$\begin{aligned}\tilde{\omega}(a \otimes b) &= \tilde{\omega}\left(\bigoplus_{i=1}^n \tilde{a}_i \otimes \bigoplus_{j=1}^n \tilde{b}_j\right) = \tilde{\omega}\left(\bigoplus_{i=1}^n \bigoplus_{j=1}^n \tilde{a}_i \otimes \tilde{b}_j\right) \\ &= \omega\left(\bigoplus_{i=1}^n \tilde{a}_i \otimes \tilde{b}_i\right) \overset{(3)}{=} \omega(a \cdot b).\end{aligned}$$

$\square$

Corollary 5.2.3. *Let $P(a,b|x,y) = \omega(A_a^x B_b^y)$ be a correlation given by a state ω on $\mathbb{B}(\mathcal{H})$, and measurement operators $A_a^x, B_b^y \in \mathbb{B}(\mathcal{H})$ with $[A_a^x, B_b^y] = 0$. Then there exist POVMs $\tilde{A}_a^x \in \mathbb{B}(\mathcal{H}_A), \tilde{B}_b^y \in \mathbb{B}(\mathcal{H}_B)$ and a vector state $\tilde{\omega}$ on $\mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that $\tilde{\omega}(\tilde{A}_a^x \otimes \tilde{B}_b^y) = P(a,b|x,y)$.*

Proof. This follows from the previous proposition by defining $\mathcal{A}$ to be the von Neumann algebra generated by the A_a^x, noting that $B_b^y \in \mathcal{A}'$ and defining $\tilde{A}_a^x = \pi_A(A_a^x), \tilde{B}_b^y = \pi_B(B_b^y)$. Furthermore, as mentioned in Chapter 5.1, every state on a finite dimensional Hilbert space is automatically a mixed vector state, which we can purify by Remark 3.5.3 by increasing the dimension of the Hilbert space if necessary. $\square$

Note that $\dim(\mathcal{H}_A \otimes \mathcal{H}_B) \geq \dim(\mathcal{H})$, even without the expansion needed for purification of the vector state; indeed, $\dim(\mathcal{H}) = \sum_{i=1}^n n_i m_i \leq \sum_{i=1}^n n_i \sum_{j=1}^n m_j = \dim(\mathcal{H}_A \otimes \mathcal{H}_B)$.

The above proof is Tsirelson's original idea with all details worked out, using the structure of finite dimensional C^*-algebras to create a clever splitting of the Hilbert space. A different proof, using the notion of 'quansality', was given by [16].

The elaborate splitting for the previous proof only really helps you in defining the new state because there is a natural restriction mapping $\mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathbb{B}(\mathcal{H})$. What if instead, we just define $\mathcal{H}_A = \mathcal{H}_B = \mathcal{H}$? It turns out that it is not much harder to prove that the natural candidate for the new state $\tilde{\omega}(a \otimes b) = \omega(ab)$ is indeed a valid state. This much shorter proof does not seem to be written down anywhere in the literature (although we are sure it must be known to the experts). We take $\mathcal{A}$ and $\mathcal{B}$ to be the C^*-algebras generated by Alice and Bob's operators (we do not need von Neumann algebra theory anymore).

Theorem 5.2.4. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{B}(\mathcal{H})$ be C^*-algebras with $\mathcal{H}$ finite dimensional and $\mathcal{B} \subseteq \mathcal{A}'$. Let ω be a state on $\mathbb{B}(\mathcal{H})$. Then there exists a vector state $\tilde{\omega}$ on $\mathbb{B}(\mathcal{H} \otimes \mathcal{H})$ such that $\tilde{\omega}(a \otimes b) = \omega(a \cdot b)$ for all $a \in \mathcal{A}, b \in \mathcal{B}$.*

Proof. Note that, since the Hilbert spaces are finite dimensional, the unital $*$-algebra $\mathcal{A} \otimes \mathcal{B}$ with the inherited norm from $\mathbb{B}(\mathcal{H} \otimes \mathcal{H})$ becomes a C^*-algebra (in infinite dimensions this would require a norm completion). Therefore the map $\varphi : \mathcal{A} \otimes \mathcal{B} \to \mathbb{B}(\mathcal{H})$ given by $a \otimes b \mapsto a \cdot b$ is a linear map between C^*-algebras. Using commutativity, it can be routinely checked that φ is in fact a $*$-homomorphism, which by Theorem 2.1.7 from Murphy means that φ is contractive.

Now the map $\tilde{\omega} \in (\mathcal{A} \otimes \mathcal{B})^*$ defined by $\tilde{\omega} = \omega \circ \varphi$ is clearly a linear functional with $\tilde{\omega}(\mathbb{1} \otimes \mathbb{1}) = 1$. This implies that $||\tilde{\omega}|| \geq 1$, but also $||\tilde{\omega}|| \leq \|\varphi\|\|\omega\| \leq 1$, so $\|\tilde{\omega}\| = 1$. By Corollary 3.3.4 from Murphy, $\tilde{\omega}$ is positive and therefore a state. By Theorem 3.3.8 from Murphy, $\tilde{\omega}$ can be extended to $\mathbb{B}(\mathcal{H} \otimes \mathcal{H})$. As remarked in Chapter 5.1, $\tilde{\omega}$ is automatically a mixed vector state since it works on a finite dimensional Hilbert space. By Remark 3.5.3, we can assume without loss of generality that $\tilde{\omega}$ is a pure vector state. Finally, we have by definition that $\tilde{\omega}(a \otimes b) = \omega(a \cdot b)$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. □

5.3 Other quantum inclusion results

$C_{qs} \neq C_{qa}$

The original problem that Tsirelson posed actually asked multiple questions: firstly, if $C_{qs} = C_{qc}$, and if not, than maybe $C_{qa} = C_{qc}$? The latter became known as the weak Tsirelson problem. Even stronger, one might ask whether $C_q = C_{qc}$, which was titled the strong Tsirelson problem, with the original case becoming the intermediate Tsirelson problem.

The strong and intermediate Tsirelson problem were resolved by Slofstra in 2016 [26], who showed the non-closedness of C_{qs} which means that $C_{qs} \subsetneq C_{qa} \subseteq C_{qc}$. The proof was later simplified by Slofstra [27] with a constructive example of input sizes 184 and 235, and output sizes 8 and 2. The proof was even further simplified by Dykema, Paulsen and Prakash [4], who showed that a violating quantum strategy can be found even in the relatively simple case of input sizes 5 and output sizes 2. These results are outside the scope of this thesis and will not be discussed further.

With that, only the weak Tsirelson problem remains open, which is now commonly referred to as 'the' Tsirelson problem.

$C_{qa} \subseteq C_{qc}$

We already know that $C_{qs} \subseteq C_{qc}$, so to prove that $C_{qa} \subseteq C_{qc}$ we only need to show that C_{qc} is closed. We postpone this proof until Chapter 6; see Corollary 6.2.6. This proves closedness only for the case where both players have the same number of inputs and ouputs, but in Corollary 7.2.6 we will see that the general case also holds.

5.4 Special cases for Tsirelson's problem

Theorem 5.2.4 used a simple proof to show the finite dimensional case of Tsirelson. A similar proof works when the algebra generated by Alice's operators is nuclear:

Theorem 5.4.1. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{B}(\mathcal{H})$ be C^*-algebras such that $\mathcal{B} \subseteq \mathcal{A}'$. Moreover, assume that $\mathcal{A}$ is nuclear. Let ω be a state on $\mathbb{B}(\mathcal{H})$. Then there exists a state $\tilde{\omega}$ on $\mathbb{B}(\mathcal{H} \otimes \mathcal{H})$ such that $\tilde{\omega}(a \otimes b) = \omega(a \cdot b)$ for all $a \in \mathcal{A}, b \in \mathcal{B}$.*

Proof. The only part in the previous proof where we used finite dimensionality was where we claimed that $\mathcal{A} \otimes \mathcal{B}$ was a C^*-subalgebra of $\mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. In infinite dimensions, we need a norm completion to make $\mathcal{A} \otimes \mathcal{B}$ into a C^*-algebra. such as $\mathcal{A} \otimes_{\min} \mathcal{B}$. In order to define a state on $\mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, we need to extend the map $\varphi : a \otimes b \mapsto ab$ that we used before to $\mathcal{A} \otimes_{\min} \mathcal{B}$.

By nuclearity, we have $\mathcal{A} \otimes_{\min} \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$. Theorem 6.3.7 from Murphy now tells us exactly that there is a unique $*$-homomorphism $\varphi : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathbb{B}(\mathcal{H})$ satisfying $\varphi(a \otimes b) = a \cdot b$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. This gives us the extension of φ we need. Now we can define $\tilde{\omega} \in (\mathcal{A} \otimes_{\min} \mathcal{B})^*$ by $\tilde{\omega} := \omega \circ \varphi$ and similarly prove that $\tilde{\omega}$ is a state that can be extended to a state on $\mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. $\square$

Corollary 5.4.2. *Let $P(a, b|x, y) \in C_{qc}$ be a correlation matrix given by a Hilbert space $\mathcal{H}$, a state ω on $\mathcal{H}$ and measurement operators $\{A_a^x\}, \{B_b^y\} \in \mathbb{B}(\mathcal{H})$ such that the C^*-algebra $\mathcal{A}$ generated by $\{A_a^x\}$ is nuclear. Then $P(a, b|x, y) \in C_{qa}$; more specifically, there are vector states ω_n such that $\omega_n(A_a^x \otimes B_b^y) \to P(a, b|x, y)$ for all a, b, x, y.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be the C^*-algebras generated by Alice's and Bob's operators. By the previous Proposition, there exists a state $\tilde{\omega}$ such that $\tilde{\omega}(A_a^x \otimes B_b^y) = \omega(A_a^x B_b^y)$. However, we can no longer assume that $\tilde{\omega}$ is a vector state. Therefore, we use the weak-* density of vector states (Theorem A.1.1): let ω_n be vector states so that $\omega_n \to \tilde{\omega}$ pointwise. Then in particular, $\omega_n(A_a^x \otimes B_b^y) \to P(a, b|x, y)$ for all a, b, x, y. $\square$

Note that the nuclear version implies the finite dimensional version: indeed, by Theorem 6.3.9 from Murphy, every finite dimensional C^*-algebra is nuclear.

Another known case is when the number of inputs and outputs for Alice is 2, see [16]. The case where both players have 2 inputs and outputs is the well-known CSHS-scenario.

6 Tsirelson's problem and Kirchberg's conjecture

In this section we focus on the equivalence to Kirchberg's conjecture. It is important to note that this equivalence holds only if we assume Alice and Bob have the same number of inputs and outputs. We prove one implication and a matrix-valued version of the other implication, similar to the one [9] and Fritz [7] have shown it. The full equivalence was later shown by Ozawa [19] but his proof is outside the scope of this project. Our proof runs along the same lines as Fritz' proof.

Kirchberg's conjecture (which we will abbreviate by (K2)) has many equivalent forms, some of which are stated in Appendix A.5. The formulation we need for now is the following one:

Conjecture 6.0.1 (K2). *For some (or equivalently, for all) $k, m \geq 2$, $(k,m) \neq (2,2)$, we have*

$$*_{x=1}^{k} \ell_{\infty}^{m} \otimes_{\min} *_{x=1}^{k} \ell_{\infty}^{m} = *_{x=1}^{k} \ell_{\infty}^{m} \otimes_{\max} *_{x=1}^{k} \ell_{\infty}^{m}.$$

The sufficient condition is proven in Appendix A.5. Recognise how the CHSH-scenario, for which Tsirelson is known to be true (cf. Chapter 5.4), is left out here. This is due to the fact that $\mathbb{Z}_2 * \mathbb{Z}_2$ does not contain $\mathbb{F}_2$ as a subgroup, while other groups of that form do (see Lemma A.5.6).

Any C^*-algebra containing one or more copies of $*_{x=1}^{k} \ell_{\infty}^{m}$ with minimal or maximal tensor products between them will be called a *FJO-algebra*, after the three papers [7], [9] and [19] that established the equivalence between Kirchberg's en Tsirelson's problems.

In the following, $\{A_a^x\}$ respectively $\{B_b^y\}$ denote sets of POVMs that can be thought of as Alice's respectively Bob's set of observables. We will assume in this Chapter that Alice and Bob have the same number of inputs and the same number of outputs: $x, y \in \{1, \dots, k\}$ and $a, b \in \{1, \dots, m\}$. We define $\Gamma := (k, m)$ as a 'choice of integers'. The value of Γ defines a 'version' of Tsirelson's problem, and we denote the related sets of quantum correlations by $C_{qa}(\Gamma)$ and $C_{qc}(\Gamma)$.

6.1 Relation to ℓ_{∞}^{m} and the free product

We start by considering only one player. The first step in the proof establishes a link between the 'simple' space ℓ_{∞}^{m} of functions on m discrete points with the supremum norm, and $\mathbb{B}(\mathcal{H})$, the space containing the operators from the quantum measurements. Indeed, since the operators from an m-outcome POVM $\{A_1^x, \dots, A_m^x\}$ add up to 1, it makes sense to link them to the standard basis $e_1, \dots, e_m \in \ell_{\infty}^{m}$ (note here that ℓ_{∞}^{m} is just $\mathbb{C}^m$ with the supremum norm). Through linear extension, we can define a linear map $\Phi_x : \ell_{\infty}^{m} \to \mathbb{B}(\mathcal{H})$ satisfying $\Phi_x(e_a) = A_a^x$. By construction, it is clear that this map is unital.

With pointwise multiplication and conjugation, the space ℓ_{∞}^{m} becomes a C^*-algebra. This gives us a notion of positive elements - namely, those elements with real, positive coefficients. In other words, an element $a = \sum_{i=1}^{m} a_i e_i \in \ell_{\infty}^{m}$ is positive iff $a_i \in \mathbb{R}_{\geq 0}$ for $i = 1, \dots, m$. Thus, since the A_a^x are all positive, $\Phi_x(a) = \sum_{i=1}^{m} a_i A_a^x$ is positive whenever a is positive. This means that Φ_x is a positive map. In fact, since ℓ_{∞}^{m} is a commutative C^*-algebra, this means that Φ_x is completely positive. We conclude that Φ_x is a unital completely positive map (in short: ucp map).

The above shows how we can define a ucp map between C^*-algebras from a quantum measurement. Conversely, note that any ucp map $\Phi_x : \ell_{\infty}^{m} \to \mathbb{B}(\mathcal{H})$ defines a quantum measurement.

Indeed, if we define $A_a^x := \Phi_x(e_a)$, then we find $\sum_{i=1}^m A_a^x = \Phi_x(\sum_{i=1}^n e_a) = \Phi_x(\mathbb{1}_{\ell_\infty^m}) = \mathbb{1}_\mathcal{H}$ by unitality. Also, the A_a^x are positive since the e_a are. Therefore, the operators $A_1^x, \ldots, A_m^x$ constitute a valid POVM.

The next goal is to define some ucp map $\Phi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ defined on some C^*-algebra $\mathcal{A}$, which defines all measurements $\{A_a^x\}$ simultaneously. Somehow we need to 'combine' the Φ_x above to a single map. The following proposition states that this can be done via the free product. For a proof we refer to [8, Cor. 3.8].

Proposition 6.1.1. *Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras. For ucp maps $\Phi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ and $\Psi : \mathcal{B} \to \mathbb{B}(\mathcal{H})$, there is a ucp map $\tilde{\Phi} : \mathcal{A} * \mathcal{B} \to \mathbb{B}(\mathcal{H})$ extending Φ and Ψ (in the sense of the embeddings $\mathcal{A} \hookrightarrow \mathcal{A} * \mathcal{B}$ and $\mathcal{B} \hookrightarrow \mathcal{A} * \mathcal{B}$).*

Repeatedly applying this proposition shows how we can define a single ucp map $\Phi : *_{x=1}^k \ell_\infty^m \to \mathbb{B}(\mathcal{H})$ extending the maps Φ_x. Conversely, any ucp map

$$\Phi : *_{x=1}^k \ell_\infty^m \to \mathbb{B}(\mathcal{H}), \quad \Phi(e_a^x) = A_a^x$$

defines a set of k measurements, one for each copy of ℓ_∞^m. The operators are given by $A_a^x := \Phi(e_a^x)$, where e_a^x is the a'th orthonormal basis vector in the x'th copy of ℓ_∞^m.

We can conclude the following:

Proposition 6.1.2. *Sets of k m-outcome POVMs on Hilbert spaces $\mathcal{H}$ correspond to ucp maps $\Phi : *_{x=1}^k \ell_\infty^m \to \mathbb{B}(\mathcal{H})$, i.e.*

$$\{(A_a^x)_{x,a} \mid A_a^x \in \mathbb{B}(\mathcal{H})^+, \sum_{a=1}^m A_a^x = \mathbb{1}_\mathcal{H}\} = \{(\Phi(e_a^x))_{x,a} \mid \Phi : *_{x=1}^k \ell_\infty^m \to \mathbb{B}(\mathcal{H}) \ \textit{ucp}\}.$$

A similar correspondence holds for projective measurements and $*$-representations, which we will later use to show that it suffices to consider projective measurements:

Proposition 6.1.3. *Sets of k m-outcome projective measurements on Hilbert spaces $\mathcal{H}$ correspond to $*$-homomorphisms $\pi : *_{x=1}^k \ell_\infty^m \to \mathbb{B}(\mathcal{H})$, i.e.*

$$\begin{aligned} &\{(A_a^x)_{x,a} \mid A_a^x \in \mathrm{Proj}(\mathbb{B}(\mathcal{H})), A_a^x \ \textit{pairwise orthogonal}, \sum_{a=1}^m A_a^x = \mathbb{1}_\mathcal{H}\} \\ &= \{(\pi(e_a^x))_{x,a} \mid \pi : *_{x=1}^k \ell_\infty^m \to \mathbb{B}(\mathcal{H}) * -\textit{homomorphism}\}. \end{aligned}$$

Proof. Let $\pi : *_{x=1}^k \ell_\infty^m \to \mathbb{B}(\mathcal{H})$ be a $*$-homomorphism. Define $A_a^x := \pi(e_a^x)$. By the properties of $*$-homomorphisms, we have $(A_a^x)^2 = \pi(e_a^x)^2 = \pi((e_a^x)^2) = \pi(e_a^x) = A_a^x$ and $(A_a^x)^* = \pi(e_a^x)^* = \pi(e_a^x) = A_a^x$, so the A_a^x are projections. Similarly, we have $A_{a_1}^x A_{a_2}^x = \pi(e_{a_1}^x e_{a_2}^x) = \pi(0) = 0$ whenever $a_1 \neq a_2$, so the projections $\{A_a^x\}_{a=1}^m$ are orthogonal. Finally, $\sum_{a=1}^m A_a^x = \pi\left(\sum_{a=1}^m e_a^x\right) = \pi((1, \ldots, 1)) = \mathbb{1}_\mathcal{H}$.

Conversely, let $(A_a^x)_{x,a}$ be a projective measurement. Define maps $\pi_x : \ell_\infty^m \to \mathbb{B}(\mathcal{H})$ as given before. We already know they are ucp, and since the images of the e_a^x are orthogonal projections they are $*$-homomorphisms. Now, by the universal property of free products, there is a unique $*$-homomorphism $\pi : *_{x=1}^k \ell_\infty^m \to \mathbb{B}(\mathcal{H})$ extending the π_x. $\square$

6.2 Link to tensor norms and Kirchberg implies Tsirelson

We now know what the link is to free products. Next, we will see where the minimal and maximal tensor products enter the equation. The essential step is in the following propositions, which showcase the link to operator system theory.

Proposition 6.2.1. *Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras. If $\Phi_A : \mathcal{A} \to \mathbb{B}(\mathcal{H}_A)$ and $\Phi_B : \mathcal{B} \to \mathbb{B}(\mathcal{H}_B)$ are ucp maps, then the map defined by*

$$\Phi_A \otimes_{\min} \Phi_B : \mathcal{A} \otimes_{\min} \mathcal{B} \to \mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B), \qquad a \otimes b \mapsto \Phi_A(a) \otimes \Phi_B(b)$$

is also ucp.

Proof. This is a direct corollary of Theorem A.2.7. □

Proposition 6.2.2. *Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras. If $\Phi_A : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ and $\Phi_B : \mathcal{B} \to \mathbb{B}(\mathcal{H})$ are ucp maps with commuting ranges, then the map defined by*

$$\Phi_A \otimes_{\max} \Phi_B : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathbb{B}(\mathcal{H}), \qquad a \otimes b \mapsto \Phi_A(a)\Phi_B(b)$$

is well-defined and ucp.

See Theorem A.2.8 for the proof.

The tensor product model respectively commuting operator model become visible in these propositions. They allow us to prove the following result, which states that the respective sets of quantum correlations are equal in some sense to the set of states on the respective tensor products of C^*-algebras. This proposition gives a natural framework in which to study Tsirelson's problem, and also immediately proves that Kirchberg's conjecture implies Tsirelson's problem.

Recall that $\Gamma := (k, m)$ represents the number of inputs and outputs. For ease of notation, we define

$$\mathcal{G}_{\min}(\Gamma) := *_{x=1}^{k} \ell_\infty^m \otimes_{\min} *_{x=1}^{k} \ell_\infty^m, \qquad \mathcal{G}_{\max}(\Gamma) := *_{x=1}^{k} \ell_\infty^m \otimes_{\max} *_{x=1}^{k} \ell_\infty^m$$

as a shortcut for the FJO-algebras.

Proposition 6.2.3. *We have*

$$C_{qa}(\Gamma) = \{[\omega(e_a^x \otimes e_b^y)]_{a,b;x,y} : \omega \in S(\mathcal{G}_{\min}(\Gamma))\} \tag{4}$$

and

$$C_{qc}(\Gamma) = \{[\omega(e_a^x \otimes e_b^y)]_{a,b;x,y} : \omega \in S(\mathcal{G}_{\max}(\Gamma))\}. \tag{5}$$

Here $S(A)$ denotes the state space of the C^*-algebra A.

Proof. We essentially follow Fritz' proof from [7, 3.4]. Let $P(a, b|x, y) \in C_{qs}(\Gamma)$ be a quantum correlation in the tensor product model. By Proposition 6.1.2, there exist ucp maps Φ_A, Φ_B describing Alice's and Bob's measurements respectively. Let $\mathcal{H}_{\mathcal{A}}$, $\mathcal{H}_{\mathcal{B}}$ be their corresponding Hilbert spaces and $\omega \in S(\mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B))$ their shared (vector) state. The unital linear functional

$$\tilde{\omega}(\lambda_1 \otimes \lambda_2) := \omega(\Phi_A(\lambda_1) \otimes \Phi_B(\lambda_2)), \quad \lambda_1, \lambda_2 \in *_{x=1}^{k} \ell_\infty^m$$

satisfies $\tilde{\omega}(e_a^x \otimes e_b^y) = P(a,b|x,y)$ by construction. To show that $\tilde{\omega}$ is a state on $\mathcal{G}_{\min}(\Gamma)$, it remains to check that $\tilde{\omega}$ is positive. By proposition 6.2.1, the map $\Phi_A \otimes_{\min} \Phi_B : \mathcal{G}_{\min}(\Gamma) \to \mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ defined by $\Phi_A \otimes_{\min} \Phi_B(\lambda_1 \otimes \lambda_2) = \Phi_A(\lambda_1) \otimes \Phi_B(\lambda_2)$ is ucp. So $\tilde{\omega}$ is the composition of two positive maps and therefore positive.

Now let $P(a,b|x,y) \in C_{qa}(\Gamma)$, i.e. it is a limit of quantum correlations from the tensor product model. Each of these quantum correlations gives rise to a state $\tilde{\omega}_n$ as argued above. Now we define $\tilde{\omega}(x) := \lim_U \tilde{\omega}_n(x)$ to be the pointwise ultralimit with respect to some non-principal ultrafilter U. (If the reader is unfamiliar with ultrafilters and ultralimits, the state $\tilde{\omega}$ can also be defined through a Hahn-Banach argument, by defining it first on the closed linear span of the elements $\Phi_A(e_a^x) \otimes \Phi_B(e_b^y)$.)

Since the ultralimit of a converging sequence is the same as the usual limit, we have $\tilde{\omega}(e_a^x \otimes e_b^y) = P(a,b|x,y)$. Also, a pointwise ultralimit of linear maps is still linear (since an ultralimit preserves sums). Finally, $\|\tilde{\omega}\| \leq 1$ since $\|\tilde{\omega}_n\| = 1$, and also $\tilde{\omega}(\mathbb{1} \otimes \mathbb{1}) = 1$ so $\|\tilde{\omega}\| = 1$. Now by Corollary 3.3.4 of Murphy, $\tilde{\omega}$ is positive. Therefore $\tilde{\omega}$ defines a state satisfying $\tilde{\omega}(e_a^x \otimes e_b^y) = P(a,b|x,y)$, which shows one inclusion of (4).

For the converse inclusion, fix some faithful representation $\pi : *_{x=1}^k \ell_\infty^m \to \mathbb{B}(\mathcal{H})$. Then $\mathcal{G}_{\min}(\Gamma) \hookrightarrow \mathbb{B}(\mathcal{H} \otimes \mathcal{H})$ by definition of the $\otimes_{\min}$ norm. We define the measurement operators $A_x^a := \pi(e_a^x)$, $B_y^b := \pi(e_b^y)$. Note that the Hilbert space and these operators can be defined independently of the given state.

Now if $P(a,b|x,y)$ is given by a state ω on $\mathcal{G}_{\min}(\Gamma)$, all that remains to do is converting the C^*-algebraic state to a vector state. One could try using the GNS-representation, but that would destroy the tensor form of the Hilbert space $\mathcal{H} \otimes \mathcal{H}$ (see Chapter 5.1). Instead, we use the second trick from that chapter (see Theorem A.1.1), namely the fact that for a concretely represented C^*-algebra, mixed vector states are weak-* dense in the state space. In other words, there are vector states ω_n on $\mathcal{G}_{\min}(\Gamma)$ such that $\omega_n(x) \to \omega(x)$ pointwise. So, if $\varepsilon > 0$, we can find some ω_n such that

$$|P(a,b;x,y) - \omega_n(e_a^x \otimes e_b^y)| = |\omega(e_a^x \otimes e_b^y) - \omega_n(e_a^x \otimes e_b^y)| < \varepsilon \quad \forall a,b,x,y.$$

Since the supremum norm and Euclidean norm are equivalent on $\mathbb{R}^{m^2k^2}$, this means that $[\omega(e_a^x \otimes e_b^y)]_{a,b;x,y} \in C_{qa}(\Gamma)$.

We now turn our attention to (5). The proof runs along similar lines as in the tensor product case. Let $P(a,b|x,y) \in C_{qc}(\Gamma)$ and Φ_A, Φ_B the ucp maps following from Proposition 6.1.2, with corresponding Hilbert space $\mathcal{H}$ and vector state ω on $\mathbb{B}(\mathcal{H})$. This time, we define the linear functional $\tilde{\omega}$ on $\mathcal{G}_{\max}(\Gamma)$ as

$$\tilde{\omega}(\lambda_1 \otimes \lambda_2) := \omega(\Phi_A(\lambda_1)\Phi_B(\lambda_2)), \quad \lambda_1, \lambda_2 \in *_{x=1}^k \ell_\infty^m$$

which clearly satisfies $P(a,b|x,y) = \tilde{\omega}(e_a^x \otimes e_b^y)$. By Proposition 6.2.2 the map $\Phi_A \otimes_{\max} \Phi_B : \mathcal{G}_{\max}(\Gamma) \to \mathbb{B}(\mathcal{H})$ defined by $\Phi_A \otimes_{\max} \Phi_B(\lambda_1 \otimes \lambda_2) = \Phi_A(\lambda_1)\Phi_B(\lambda_2)$ is ucp, which implies that $\tilde{\omega}$ is indeed a state.

For the converse, let ω be a state on $\mathcal{G}_{\max}(\Gamma)$. This time, we can simply use the GNS-representation of ω to define a vector state $\tilde{\omega}$ on $\mathcal{H}_\omega$ satisfying $\omega = \tilde{\omega} \circ \pi_\omega$ (see Chapter 5.1). We define the operators $A_a^x := \pi_\omega(e_a^x \otimes \mathbb{1})$, $B_b^y := \pi_\omega(\mathbb{1} \otimes e_b^y)$. Then the operators commute pairwise and $\tilde{\omega}(A_a^x B_b^y) = \omega(e_a^x \otimes e_b^y) = P(a,b|x,y)$, as required. $\square$

Remark 6.2.4. Note that both times we constructed a quantum correlation, we were able to do so via a $*$-representation (an arbitrary faithful representation for the tensor model and the GNS representation for the commuting operator model). By Proposition 6.1.3, the resulting measurements are actually projective measurements. In other words, we pass from a general quantum correlation to a FJO-algebra and a state thereon, where the measurements (defined through some representation) are automatically projective. This means that to generate either set of quantum correlations, it suffices to consider projective measurements.

Remark 6.2.5. In fact, the proof exhibits that in the tensor product model, there is a universal quantum system, in the sense of a Hilbert space and fixed (projective) measurements generating all tensor product quantum correlations as the state varies over the state space. This is also true for the commuting operator model. To see this, replace the GNS representation in the proof by the universal representation (i.e. the direct sum over all states of the corresponding GNS representations). Each state now becomes a vector state (it only takes the argument from its own component).

Finally we prove the corollary that we referred to in Chapter 5.3.

Corollary 6.2.6. *$C_{qc}(\Gamma)$ is closed*

Proof. By the Banach-Alaoglu Theorem, the state space of a C^*-algebra is weak-$*$ compact. The space $C_{qc}(\Gamma)$ is obtained from $S(\mathcal{G}_{\min}(\Gamma))$ as the image of the map $\omega \mapsto [\omega(e_a^x \otimes e_b^y)]$. Evaluation on each of the elements $e_a^x \otimes e_b^y$ is a continuous mapping, so the map $\omega \mapsto [\omega(e_a^x \otimes e_b^y)]$ is also continuous. Therefore the image $C_{qc}(\Gamma)$ is also compact and therefore closed. $\square$

6.3 Matrix-Tsirelson implies Kirchberg

The converse implication has been proven in its actual form by [19] after [7] and [9] proved a variant with a stronger matrix-valued version of Tsirelson. Unfortunately, the proof by Ozawa uses advanced techniques linked to the original Connes Embedding Problem, which falls outside the scope of this thesis. Instead, we present here the matrix-valued proof, again mostly following the proof of Fritz [7].

We define the matrix-valued Tsirelson problem directly in the form as in (4) and (5). One can interpret this in terms of non-local games by using *steering* (see [7]) or in the form of the original definition in Chapter 3 by using isometries instead of vector states (see [19]).

To adapt the situation of (4) to the matrix-valued situation, we consider states on the matrix algebras $M_n(\mathcal{G}_{\min}(\Gamma))$ and $M_n(\mathcal{G}_{\max}(\Gamma))$ on the min- and max tensor product respectively. We thus define the matrix-valued sets of correlation matrices as follows:

$$C_{qa}^n(\Gamma) := \{[\omega(E_{i,j} \otimes e_a^x \otimes e_b^y)]_{a,b;x,y;i,j} : \omega \in S(M_n(\mathcal{G}_{\min}(\Gamma)))\} \tag{6}$$

and

$$C_{qc}^n(\Gamma) := \{[\omega(E_{i,j} \otimes e_a^x \otimes e_b^y)]_{a,b;x,y;i,j} : \omega \in S(M_n(\mathcal{G}_{\max}(\Gamma)))\}. \tag{7}$$

Note that these sets can be considered as subsets of $\mathbb{R}^{m^2k^2n^2}$.

We will now prove that Kirchberg's conjecture is fully equivalent with this matrix-valued version. The essential trick, which necessitates the matrix version, is known as Pisier's trick, see Theorem A.3.3. This allows one to prove norm equality only on a certain generating subset of the C^*-algebra, at the cost of having to prove it for all matrix algebras. Deeper down, the matrix requirement comes from the Arveson extension theorem which requires a completely positive map.

Theorem 6.3.1. *Let $\Gamma = (k, m)$ be fixed. The following statements are equivalent:*

(i) We have

$$\mathcal{G}_{\min}(\Gamma) = \mathcal{G}_{\max}(\Gamma).$$

(ii) The matrix-valued Tsirelson problem (T2M): for all n, we have $C^n_{qa}(\Gamma) = C^n_{qc}(\Gamma)$.

Proof. The direction $(i) \Rightarrow (ii)$ is (still) clear from the definitions (6) and (7). For the converse, let n and $\Gamma = (k, m)$ be arbitrary and define $S = \mathrm{Span}\{\mathrm{e}^{\mathrm{x}}_{\mathrm{a}} \otimes \mathrm{e}^{\mathrm{y}}_{\mathrm{b}}\}$. Now by Pisier's trick (Theorem A.3.3) it suffices to show that $\|x\|_{\min} = \|x\|_{\max}$ for all $x \in M_n(S)^{\mathrm{sa}}$. Here the $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ norms on $M_n(S)$ are induced by the respective norms on $M_n(\mathcal{G}_{\min}(\Gamma))$ and $M_n(\mathcal{G}_{\max}(\Gamma))$.

Thus, let $x \in M_n(S)^{\mathrm{sa}}$. Since $M_n(S)$ can be identified with $M_n(\mathbb{C}) \otimes S$, we can write x in the form

$$x = \sum_{i,j,a,b,x,y} \lambda_{i,j} E_{i,j} \otimes e^x_a \otimes e^y_b.$$

By the assumption, it follows that for each $\omega \in S(M_n(\mathcal{G}_{\min}(\Gamma)))$ there exists a $\tilde{\omega} \in S(M_n(\mathcal{G}_{\max}(\Gamma)))$ such that $\omega(x) = \tilde{\omega}(x)$, and vice versa. Finally, we use the fact that for self-adjoint $a \in \mathcal{A}$, $\|a\| = \sup_{\omega \in S(\mathcal{A})} |\omega(a)|$ to conclude:

$$\|x\|_{\min} = \sup_{\omega \in S(M_n(\mathcal{G}_{\min}(\Gamma)))} |\omega(x)| = \sup_{\omega \in S(M_n(\mathcal{G}_{\max}(\Gamma)))} |\omega(x)| = \|x\|_{\max}.$$

□

Remark 6.3.2. Recall that for (K2) it is sufficient to show the statement for one $\Gamma \neq (2, 2)$ in order to show it for all $\Gamma \neq (2, 2)$. Therefore, this result shows that if we can show the matrix-valued Tsirelson for just one $\Gamma \neq (2, 2)$, it follows for all Γ. This is not the case (or at least, not known) for the original, 1-dimensional Tsirelson; see [19, Thm. 36]. In this paper, the Connes Embedding Conjecture, which is equivalent to (K2), is deduced from the assumption that (T2) holds for all Γ. In other words, there is no known 1-1 equivalence between a given Tsirelson scenario and its corresponding FJO algebra.

7 Relations to QWEP conjecture

In this section we will show that Kirchberg's QWEP conjecture implies (K2). In the first subsection, we will prove that QWEP implies the following related statement:

Conjecture 7.0.1. *If $\mathbb{F}_k$ is a countable free group, then*

$$C^*(\mathbb{F}_k) \otimes_{\min} C^*(\mathbb{F}_k) = C^*(\mathbb{F}_k) \otimes_{\max} C^*(\mathbb{F}_k).$$

In the second subsection we will show how the implication QWEP $\Rightarrow$ (K2) follows from this. In Appendix A.5, we will prove that Conjecture 7.0.1 is equivalent to (K2), using some results from this chapter.
We will introduce necessary concepts such as the Lifting Property (LP), the Weak Expectation Property (WEP) and Quotient Weak Expectation Property (QWEP), and show some related results from Brown&Ozawa's book [1]. The link between FJO-algebras and these terms will help us say some things about the 3-player case in chapter 8.

7.1 Related theory

The fundamental result behind this theory is the following one by Kirchberg:

Theorem 7.1.1. *For any free group $\mathbb{F}$ and any Hilbert space $\mathcal{H}$, we have*

$$C^*(\mathbb{F}) \otimes_{\max} \mathbb{B}(\mathcal{H}) = C^*(\mathbb{F}) \otimes_{\min} \mathbb{B}(\mathcal{H}).$$

The proof, although interesting, is not very relevant to this section so we included it in the appendix (Theorem A.6.4).

Now, let us see what this means for us if we want to prove that $C^*(\mathbb{F}) \otimes_{\max} \mathcal{A} = C^*(\mathbb{F}) \otimes_{\min} \mathcal{A}$ for some C^*-algebra $\mathcal{A}$. If $\mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$ is a faithful representation of $\mathcal{A}$ and $C^*(\mathbb{F}) \subseteq \mathcal{B}(\mathcal{K})$ is another representation, then it follows that both $C^*(\mathbb{F}) \otimes_{\min} \mathcal{A}$ and $C^*(\mathbb{F}) \otimes_{\min} \mathbb{B}(\mathcal{H})$ inherit their norm from $\mathcal{B}(\mathcal{K} \otimes \mathcal{H})$, and therefore the inclusion $C^*(\mathbb{F}) \otimes_{\min} \mathcal{A} \subseteq C^*(\mathbb{F}) \otimes_{\min} \mathbb{B}(\mathcal{H})$ is clear. Combined with Theorem 7.1.1, all that remains to prove is that $C^*(\mathbb{F}) \otimes_{\max} \mathcal{A} \subseteq C^*(\mathbb{F}) \otimes_{\max} \mathbb{B}(\mathcal{H})$.

The problem here is that on the algebraic tensor product $C^*(\mathbb{F}) \otimes \mathbb{B}(\mathcal{H})$, there might not be as many representations as on the 'smaller' algebraic tensor product $C^*(\mathbb{F}) \otimes \mathcal{A}$, and therefore the maximal norm might be different. Thus the question of whether this inclusion holds is not trivial. [1, Prop. 3.6.6] gives two equivalent formulations for this property.

Theorem 7.1.2. *Let $\mathcal{A} \subseteq \mathcal{B}$ be an inclusion of C^*-algebras and $\mathcal{A}''$ (resp. $\mathcal{B}''$) be the double commutant in the universal representation. Then the following are equivalent:*

1. *For every C^*-algebra $\mathcal{C}$, there is a natural inclusion*

$$\mathcal{A} \otimes_{\max} \mathcal{C} \subseteq \mathcal{B} \otimes_{\max} \mathcal{C}.$$

2. *There exists a ccp map $\varphi : \mathcal{B} \to \mathcal{A}''$ such that $\varphi|_{\mathcal{A}} = \mathbb{1}_{\mathcal{A}}$.*
3. *For every *-homomorphism $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ there exists a ccp map $\varphi : \mathcal{B} \to \pi(\mathcal{A})''$ such that $\varphi|_{\mathcal{A}} = \pi$.*

We first give a Lemma about extensions of maps on $\mathcal{A}$ to $\mathcal{A}''$.

Lemma 7.1.3. *Let $\mathcal{A}$ be a C^*-algebra and $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ be a representation. Then there is an extension $\tilde{\pi} : \mathcal{A}'' \to \mathbb{B}(\mathcal{H})$ such that $\tilde{\pi}(\mathcal{A}'') = \pi(\mathcal{A})''$.*

Proof. By Theorem 5.1.3 from Murphy, the representation (H, π) (following notation from Murphy) can be written as a direct sum of cyclic representations, i.e. $(\mathcal{H}, \pi) = \bigoplus_{x \in \Lambda}(\mathcal{H}_\xi, \pi_\xi)$, where $\mathcal{H}_\xi = [\pi(\mathcal{A})\xi]$. By Theorem 5.1.7 from Murphy, a cyclic representation with cyclic vector ξ is unitarily equivalent to the GNS representation corresponding to the state $\tau_\xi : a \mapsto \langle \pi(a)(\xi), \xi \rangle$. Thus,

$$\bigoplus_{\xi \in \Lambda}(\mathcal{H}_\xi, \pi_\xi) \overset{\Psi}{\cong} \bigoplus_{\xi \in \Lambda}(\mathcal{H}_{\tau_\xi}, \pi_{\tau_\xi}).$$

As mentioned, $\mathcal{A}''$ is the double commutant of $\mathcal{A}$ within the universal representation $(\mathcal{H}_U, \pi_U) = \bigoplus_{\tau \in \mathcal{S}(\mathcal{A})}(\mathcal{H}_\tau, \pi_\tau)$. Now we can define a map $\tilde{\pi} : \mathcal{A}'' \to \mathbb{B}(\mathcal{H})$ by sending an element $x = (x_\tau) \in \bigoplus_{\tau \in \mathcal{S}(\mathcal{A})} \mathbb{B}(\mathcal{H}_\tau)$ to $\Psi^{-1}((x_{\tau_\xi})_{\xi \in \Lambda}) \in \mathbb{B}(\mathcal{H})$. This is an extension of π by uniqueness of GNS representations. Since the commutant must leave every non-trivial H_τ invariant, we have (for general direct sums of representations)

$$\left(\bigoplus_\lambda \pi_\lambda(\mathcal{A})\right)'' = \bigoplus_\lambda \pi_\lambda(\mathcal{A})''.$$

This implies that

$$\tilde{\pi}(\mathcal{A}'') = \tilde{\pi}\left(\bigoplus_{\tau \in \mathcal{S}(\mathcal{A})} \pi_\tau(\mathcal{A})''\right) = \Psi^{-1}\left(\bigoplus_{\xi \in \Lambda} \pi_{\tau_\xi}(\mathcal{A})''\right) = \bigoplus_{\xi \in \Lambda} \pi_\xi(\mathcal{A})'' = \pi(\mathcal{A})''$$

where we used that unitary equivalences preserve commutants. □

Proof of Thm.7.1.2. 3. ⇒ 2. is clear. For 2. ⇒ 3, let $\varphi : \mathcal{B} \to \mathcal{A}''$ be the ccp map given by condition 2. Let $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ be a *-homomorphism. By Lemma 7.1.3 we can extend π to $\mathcal{A}''$ and its range will be contained within (even equal to) $\pi(\mathcal{A})''$. Therefore $\pi \circ \varphi$ yields the map $\mathcal{B} \to \pi(\mathcal{A})''$ we need.

For 1. ⇒ 3. we refer to [1, Prop. 3.6.6.] We only prove 3. ⇒ 1, since that is the one we will need later on. Let $\mathcal{C}$ be a C^*-algebra and let $\pi : \mathcal{A} \otimes_{\max} \mathcal{C} \to \mathbb{B}(\mathcal{H})$ be a faithful representation. Note that the inclusion maps $\mathcal{A} \to \mathcal{B} \otimes_{\max} \mathcal{C}$ and $\mathcal{C} \to \mathcal{B} \otimes_{\max} \mathcal{C}$ yield an extension $\mathcal{A} \otimes_{\max} \mathcal{C} \to \mathcal{B} \otimes_{\max} \mathcal{C}$ by the universal property. Our goal will be to prove that this map is injective, which we will accomplish by constructing a commutative diagram that factors the representation π through $\mathcal{B} \otimes_{\max} \mathcal{C}$. It will look as follows:

$$\begin{array}{ccc}
\mathcal{B} \otimes_{\max} \mathcal{C} & \xrightarrow{\varphi \otimes_{\max} \pi_C} & \pi_A(\mathcal{A})'' \otimes_{\max} \pi_C(\mathcal{C}) \\
\uparrow & & \downarrow \\
\mathcal{A} \otimes_{\max} \mathcal{C} & \xrightarrow{\quad\quad \pi \quad\quad} & \mathbb{B}(\mathcal{H})
\end{array}$$

Then using that the bottom arrow is injective, it follows that the left map has to be injective also.

By universality, the representation π has restrictions π_A and π_C with commuting ranges. Note that $\pi_A(\mathcal{A})''$ and $\pi_C(\mathcal{C})$ still commute. Therefore, by the universal property, the inclusions $\pi_A(\mathcal{A})'', \pi_C(\mathcal{C}) \subseteq \mathbb{B}(\mathcal{H})$ induce a *-homomorphism which is the right-most arrow in the diagram:

$$\pi_A(\mathcal{A})'' \otimes_{\max} \pi(\mathcal{C}) \to \mathbb{B}(\mathcal{H}).$$

By assumption, there exists a ccp extension $\varphi : \mathcal{B} \to \pi_A(\mathcal{A})''$ of the map π_A. Now by Theorem A.2.7, the map $\varphi \otimes_{\max} \pi_C : \mathcal{B} \otimes_{\max} \mathcal{C} \to \pi_A(\mathcal{A})'' \otimes_{\max} \pi_C(\mathcal{C})$ is well-defined (and ccp). This completes the diagram, and it commutes because φ is an extension of π_A. □

Definition 7.1.4. When a C^* algebra $\mathcal{A}$ satisfies the properties from Theorem 7.1.2 with $\mathcal{B} = \mathbb{B}(\mathcal{H}_U)$, its universal representation, we say that $\mathcal{A}$ has the *Weak Expectation Property*, or WEP in short.

The following corollary, which is of central importance not only for the current chapter but also for the three-player case, is now immediate

Corollary 7.1.5. *If $\mathcal{A}$ is a C^*-algebra and $\mathbb{F}$ is any free group, then $C^*(\mathbb{F}) \otimes_{\max} \mathcal{A} = C^*(\mathbb{F}) \otimes_{\min} \mathcal{A}$ if and only if $\mathcal{A}$ has the WEP.*

Proof. For the if statement, the final inclusion $C^*(\mathbb{F}) \otimes_{\max} \mathcal{A} \subseteq C^*(\mathbb{F}) \otimes_{\max} \mathbb{B}(\mathcal{H})$ that we needed follows from Theorem 7.1.2 (as soon as we take $\mathcal{H}$ to be the universal representation). Conversely, if the tensor products are the same, then we have

$$C^*(\mathbb{F}) \otimes_{\max} \mathcal{A} = C^*(\mathbb{F}) \otimes_{\min} \mathcal{A} \subseteq C^*(\mathbb{F}) \otimes_{\min} \mathbb{B}(\mathcal{H}) \overset{7.1.1}{=} C^*(\mathbb{F}) \otimes_{\max} \mathbb{B}(\mathcal{H}).$$

□

This corollary shows that in order to prove Conjecture 7.0.1, it is enough to show that $C^*(\mathbb{F})$ has the WEP. We will not need the converse of this corollary (and it needs the direction of Theorem 7.1.2 we have not proven)

Definition 7.1.6. A C^*-algebra $\mathcal{A}$ is said to be QWEP (or to have the Quotient Weak Expectation Property) if it is the quotient of a C^*-algebra with the WEP.

It seems grammatticaly strange that a C^*-algebra 'is' QWEP and not 'has' the QWEP. However, this seems to be the convention in the literature and we adhere to it.

Conjecture 7.1.7 (QWEP conjecture). *Every C^*-algebra is QWEP*

This is one of the many equivalent statements to Tsirelson's problem, and it will help us to say a little about the three-player case Tsirelson. Before we can prove that QWEP implies Conjecture 7.0.1, we need to introduce one more property.

Definition 7.1.8. Let A be a C^*-algebra, $J \subseteq \mathcal{B}$ be a closed two-sided ideal in a C^*-algebra $\mathcal{B}$, and $\pi : \mathcal{B} \to \mathcal{B}/J$ be the quotient map. A u.c.p. map $\varphi : \mathcal{A} \to \mathcal{B}/J$ is called *liftable* if there exists a u.c.p. map $\psi : \mathcal{A} \to \mathcal{B}$ such that $\pi \circ \psi = \varphi$. $\mathcal{A}$ is said to have the *lifting property* (LP) if every u.c.p. map from $\mathcal{A}$ to a quotient C^*-algebra $\mathcal{B}/J$ is liftable.

Theorem 7.1.9. *For any countable free group $\mathbb{F}_k$, the group C^*-algebra $C^*(\mathbb{F}_k)$ has the LP.*

Again we refer to the appendix for the proof (Theorem A.6.1).

We can now prove that the QWEP conjecture implies Conjecture 7.0.1, which we do via the following proposition

Proposition 7.1.10. *If a C^*-algebra $\mathcal{A}$ is QWEP and has the LP, then it has the WEP. In particular, the QWEP conjecture implies that the LP implies the WEP.*

Proof. The proof that will follow is summarised in the following diagram.

$$
\begin{tikzcd}
\mathcal{A} \arrow[rr, hook] \arrow[dr] \arrow[dd, "\psi"'] & & \mathcal{A}'' \\
& \mathbb{B}(\mathcal{H}_A) \arrow[ur, dashed, "\Phi"] \arrow[dr, "\bar{\psi}"] \arrow[dd, dotted, "\tilde{\psi}"] & \\
\mathcal{B} \arrow[rr, hook] \arrow[dr] & & \mathcal{B}'' \arrow[uu, "\pi"'] \\
& \mathbb{B}(\mathcal{H}_B) \arrow[ur, "\varphi"] &
\end{tikzcd}
$$

Let $\mathcal{B}$ be a C^*-algebra with the WEP such that $\mathcal{A}$ is a quotient of $\mathcal{B}$. Let π be the quotient mapping. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be the universal representations of $\mathcal{A}$ and $\mathcal{B}$ respectively, and $\mathcal{A}''$ and $\mathcal{B}''$ the double commutants within this representation. Because $\mathcal{B}$ has the WEP, there exists a ccp map $\varphi : \mathbb{B}(\mathcal{H}_B) \to \mathcal{B}''$ such that the lower triangle commutes. ①

Since $\mathcal{A}$ has the LP, there exists a ccp lifting ψ of the identity. Note that the extension of π to $\mathcal{B}''$ makes the outer square commute. ②

By the Arveson extension theorem (A.1.4), the map $\psi : \mathcal{A} \to \mathcal{B} \subseteq \mathbb{B}(\mathcal{H}_B)$ has a ccp extension $\tilde{\psi} : \mathcal{B}(\mathcal{H}_A) \to \mathbb{B}(\mathcal{H}_B)$. In other words, the left parallelogram commutes. ③

Now we can define a ccp map $\bar{\psi} : \mathbb{B}(\mathcal{H}_A) \to \mathcal{B}''$ by setting $\bar{\psi} = \varphi \circ \tilde{\psi}$. Using this definition and ① and ③, we find that $\bar{\psi}$ is an extension of ψ, i.e. the triangle $\mathcal{A} - \mathcal{B} - \mathcal{B}''$ commutes. ④

Finally, we define the ccp map $\Phi = \pi \circ \bar{\psi}$. Combining this definition with ④ and ②, we find that the upper triangle commutes, i.e. $\Phi|_{\mathcal{A}} = \mathbb{1}_A$ as required. □

Proof that QWEP implies Conjecture 7.0.1. Let $\mathbb{F}_k$ be a countable free group. By Proposition 7.1.10 the QWEP conjecture implies that, since $C^*(\mathbb{F}_k)$ has the LP by Theorem 7.1.9, it also has the WEP. By Corollary 7.1.5, this implies Conjecture 7.0.1. □

7.2 QWEP implies (K2)

In this section we will prove that QWEP also implies (K2) as stated in Chapter 6, namely $*_{x=1}^{k} \ell_\infty^m \otimes_{\min} *_{x=1}^{k} \ell_\infty^m = *_{x=1}^{k} \ell_\infty^m \otimes_{\max} *_{x=1}^{k} \ell_\infty^m$ for some/all $k, m \geq 2$, $(k, m) \neq (2, 2)$.

We begin by proving a well-known Lemma, which states that free group C^*-algebras are universal in a sense.

Lemma 7.2.1. *Let $\mathcal{A}$ be a C^*-algebra. Then $\mathcal{A} \cong C^*(\mathbb{F})/J$ for some free group $\mathbb{F}$ and some ideal $J \subseteq C^*(\mathbb{F})$. Here $\mathbb{F}$ can be taken with countable generating set if $\mathcal{A}$ is separable.*

Proof. Note that $\mathcal{A}$ is generated by its set of unitaries $\mathcal{U}(\mathcal{A})$. Let $\mathbb{F}$ be a free group whose set of generators has the same cardinality as $\mathcal{U}(\mathcal{A})$, i.e. there exists a bijection from the set of generators of $\mathbb{F}$ to $\mathcal{U}(\mathcal{A})$. By the universal property of the maximal group C^*-algebra, there is a canonical *-homomorphism $\varphi : C^*(\mathbb{F}) \to \mathcal{A}$. This map is surjective because the range contains the unitaries and must be a C^*-algebra. Thus, the map $C^*(\mathbb{F})/\ker(\varphi) \to \mathcal{A}$ is a $*$-isomorphism.

If $\mathcal{A}$ is separable, then we can find a countable set of unitaries which is dense in $\mathcal{U}(\mathcal{A})$, and thus also generates $\mathcal{A}$. Now we map the generators of some countable free group $\mathbb{F}_k$ to these unitarie.

By the same argument, this gives a surjective $*$-homomorphism $\varphi : C^*(\mathbb{F}_k) \to \mathcal{A}$, after which we quotient out the kernel. $\square$

Next, we prove another corollary of Theorem 7.1.1 in the spirit of corollary 7.1.5.

Corollary 7.2.2. *For C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, if $\mathcal{A}$ has the LP and $\mathcal{B}$ has the WEP, then $\mathcal{A} \otimes_{\max} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$.*

Proof. Let $\mathcal{B} \subseteq \mathbb{B}(\mathcal{H})$ be a faithful representation and $A \cong C^*(\mathbb{F})/J$, with π the quotient map. Since $\mathcal{A}$ has the LP, there exists a lifting $\psi : \mathcal{A} \to C^*(\mathbb{F})$ satisfying $\psi|_{\mathcal{A}} = \mathbb{1}_{\mathcal{A}}$. Now the map

$$\mathcal{A} \otimes_{\min} \mathbb{B}(\mathcal{H}) \xrightarrow{\psi\otimes\mathbb{1}} C^*(\mathbb{F}) \otimes_{\min} \mathbb{B}(\mathcal{H}) = C^*(\mathbb{F}) \otimes_{\max} \mathbb{B}(\mathcal{H}) \xrightarrow{\pi\otimes\mathbb{1}} \mathcal{A} \otimes_{\max} \mathbb{B}(\mathcal{H})$$

is a contraction. Since the minimum norm is obviously smaller than the maximum norm, this means that both are equal, i.e. $\mathcal{A} \otimes_{\min} \mathbb{B}(\mathcal{H}) = \mathcal{A} \otimes_{\max} \mathbb{B}(\mathcal{H})$.

Finally, since B has the WEP, we have $\mathcal{A} \otimes_{\max} \mathcal{B} \subseteq \mathcal{A} \otimes_{\max} \mathbb{B}(\mathcal{H})$ (see Theorem 7.1.2). Therefore, by the same argument as below Theorem 7.1.1, we have $\mathcal{A} \otimes_{\min} \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$. $\square$

This shows that we need to prove that $*_{x=1}^k \ell_\infty^m$ has the LP and the WEP. Assuming the QWEP conjecture, Proposition 7.1.10 implies that it suffices to prove the LP. This is the contents of our next statement.

Lemma 7.2.3. *Let $\mathcal{A}$ be a separable C^*-algebra, and write $\mathcal{A} \cong C^*(\mathbb{F})/J$ by Lemma 7.2.1. Then $\mathcal{A}$ has the LP if the identity map $\mathbb{1}_A : \mathcal{A} \to \mathbb{C}^*(\mathbb{F})/J$ has the lifting property.*

Proof. Assume wlog that $\mathcal{A} = C^*(\mathbb{F})/J$. Let $q : C^*(\mathbb{F}) \to \mathcal{A}$ be the quotient map and let $\psi : \mathcal{A} \to C^*(\mathbb{F})$ be a ucp lifting.

Now let $\varphi : \mathcal{A} \to \mathcal{B}/I$ be some ucp map to a quotient of a C^*-algebra $\mathcal{B}$. Then $\varphi \circ q$ is a ucp map from $C^*(\mathbb{F})$ into B/J. By Theorem 7.1.9, there exists a ucp lifting $\tilde{\psi} : C^*(\mathbb{F}) \to B$. Now the map $\tilde{\psi} \circ \psi$ gives a ucp lifting from $\mathcal{A}$ into $\mathcal{B}$, by commutativity of the following diagram

$$\begin{tikzcd} & C^*(\mathbb{F}) \arrow[r, "\tilde{\psi}"] \arrow[d, "q"] & \mathcal{B} \arrow[d] \\ \mathcal{A} \arrow[ur, "\psi"] \arrow[r, equal] & \mathcal{A} \arrow[r, "\varphi"] & \mathcal{B}/J \end{tikzcd}$$

$\square$

Theorem 7.2.4. *$*_{x=1}^k \ell_\infty^m$ has the LP*

Proof. [24, Thm. 7] has shown that the LP is preserved under free products. Therefore it suffices to prove that ℓ_∞^m has the LP. Here we use that $\ell_\infty^m \cong C^*(\mathbb{Z}_m) = \mathbb{C}[\mathbb{Z}_m]$ via the discrete Fourier transform - we give the details in the Lemma below. The canonical projection $p : \mathbb{Z} \to \mathbb{Z}_m$ induces a surjective *-homomorphism $C^*(p) : C^*(\mathbb{F}_1) \to C^*(\mathbb{Z}_m)$ (where $C^*(\mathbb{F}_1) = C^*(\mathbb{Z}) = \mathbb{C}[\mathbb{Z}]$). Quotienting out by the kernel of this map gives an isomorphism $C^*(\mathbb{Z}_m) \cong C^*(\mathbb{F}_1)/J$. We can now construct a lifting by simply choosing for each group element $\delta_g \in C^*(\mathbb{Z}_m)$ a positive original in $C^*(\mathbb{F}_1)$ and expanding linearly. This map is positive and thus completely positive since $C^*(\mathbb{Z}_m)$ is abelian (see Proposition 2.1.3).

$$\begin{tikzcd} & C^*(\mathbb{F}_1) \arrow[d] \\ C^*(\mathbb{Z}_m) \arrow[ur] \arrow[r, equal] & C^*(\mathbb{F}_1)/J \end{tikzcd}$$

By the previous Lemma, $C^*(\mathbb{Z}_m)$ and thus ℓ_∞^m has the LP. $\square$

Lemma 7.2.5. *The discrete Fourier transform gives a *-isomorphism $\ell_\infty^m \cong C^*(\mathbb{Z}_m)$.*

Proof. Let u be the generator of $\mathbb{Z}_m$, and let $\omega_m := e^{2\pi i/m}$. The discrete Fourier transform sends u to the vector $(\omega_m, \omega_m^2, \ldots, 1)$. This extends to a homomorphism $u^n \mapsto (\omega_m^n, \omega_m^{2n}, \ldots, 1)$ defined on $\mathbb{Z}_m$. One can check that these vectors form an orthonormal basis for $\mathbb{C}^m$. Hence, by linear extension, we get a *-isomorphism $\mathbb{C}[\mathbb{Z}_m] \to \mathbb{C}^m$. $\square$

As a result, we can now prove that the simplicity assumption we made at the beginning of chapter 6 - that both players have the same number of inputs and outputs - is legal. This assumption is made in pretty much all of the literature about this topic so it is good to have this straightened out. In fact, we can even assume the more general case that each input has its own number of outputs.

Corollary 7.2.6. *The following conjectures are equivalent:*

i) (K2): $*_{x=1}^k \ell_\infty^m \otimes_{\min} *_{x=1}^k \ell_\infty^m = *_{x=1}^k \ell_\infty^m \otimes_{\max} *_{x=1}^k \ell_\infty^m$ for all $k, m \in \mathbb{N}$ or equivalently, for some $k, m \geq 2$ with $(k, m) \neq (2, 2)$.

ii) $*_{x=1}^k \ell_\infty^{m_x} \otimes_{\min} *_{y=1}^l \ell_\infty^{n_y} = *_{x=1}^k \ell_\infty^{m_x} \otimes_{\max} *_{y=1}^l \ell_\infty^{n_y}$ for all $k, l \in \mathbb{N}$ and $m_1, \ldots, m_k, n_1, \ldots, n_l \in \mathbb{N}$. Equivalently, it is enough to prove the equality for one choice of integers $k, l, m_1, \ldots, m_k, n_1, \ldots, n_l \in \mathbb{N}$ such that either $k \geq 3$ or $m_1, m_2 > 2$, and either $l \geq 3$ or $n_1, n_2 > 2$.

As a result, Tsirelson's problem is equivalent whether or not we consider the same number of inputs and ouputs for both players.

Proof. ii) $\Rightarrow$ i) is trivial. The argument of Theorem 7.2.4 uses only that free products of spaces with the LP have the LP, and thus it also shows that a space of the form $*_{x=1}^k \ell_\infty^{m_x}$ has the LP. Therefore, ii) is implied by QWEP in exactly the same way as i) was. The fact that i) implies QWEP is proven in [18, 3.19, 6.2], or in [1, 13.3.1]. Statement ii) is equivalent to (T2) with different numbers of inputs and outputs by variants of Propositions 6.1.2 and 6.2.3 with completely analogous proofs.

Finally, the sufficient condition for statement (ii) follows from a variant of Lemma A.5.6; in the case $k = 2, m_1, m_2 > 2$ we can use the ping-pong lemma on the subgroup $\mathbb{Z}_{m_1} * \mathbb{Z}_{m_2}$. $\square$

8 Three-player case

Papers such as [23] have shown how a multipartite setting can yield some surprising results in that they are quite opposite to what holds in the twopartite setting. In that paper, it is shown that tripartite quantum states can allow for unbounded violations of tripartite Bell inequalities. Therefore it might be interesting to consider the tripartite version of Tsirelson's problem. Not much seems to have been written about this topic - all we could find were short remarks in for example [7] and [16].

8.1 Introduction of three-player Tsirelson

We now introduce a third player Charlie to the fictional game setting, who receives an input $c \in \{1, \dots, k\}$ and has a set of measurements $\{C_c^z\}$ to determine an output $z \in \{1, \dots, m\}$. We can define 'extreme' three-player correlation sets similarly to the two player case

$$C_{qa}^{(3)}(\Gamma) = \text{closure}\Big(\Big\{ \langle\psi|\mathrm{A_a^x} \otimes \mathrm{B_b^y} \otimes \mathrm{C_c^z}|\psi\rangle : \mathrm{A_a^x} \in \mathbb{B}(\mathcal{H}_\mathrm{A})^+, \mathrm{B_b^y} \in \mathbb{B}(\mathcal{H}_\mathrm{B})^+, \mathrm{C_c^z} \in \mathbb{B}(\mathcal{H}_\mathrm{C})^+, ...\Big\}\Big),$$
$$C_{qc}^{(3)}(\Gamma) = \Big\{ \langle\psi|A_a^x B_b^y C_c^z|\psi\rangle : A_a^x, B_b^y, C_c^z \in \mathbb{B}(\mathcal{H})^+ \text{ pairwise commuting }, ...\Big\}$$

where due to lack of space we omitted the following requirements: $|\psi\rangle$ is a state on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ respectively $\mathcal{H}$, the A_a^x, B_b^y, C_c^z are POVMs (i.e. $\sum_a A_a^x = \mathbb{1}$, etc), and $x, y, z \in \{1, \dots, k\}$, $a, b, c \in \{1, \dots, m\}$.

This leads to the following definition of the three-player Tsirelson problem:

Conjecture 8.1.1 (T3). *For all $\Gamma = (k, m)$, the equality $C_{qa}^{(3)}(\Gamma) = C_{qc}^{(3)}(\Gamma)$ holds.*

As we will see, a lot is still unclear about the three-player case. Let us first talk about some easy cases that we do know.

8.2 Known cases: finite dimensions & nuclearity

In finite dimensions, (T3) is easily proven. In fact, the proof of Theorem 5.2.4 still works in exactly the same way, but now with 3 tensored spaces instead of 2.

Next, we prove (T3) in the nuclear case. It turns out that two players need to have a set of operators that generates a nuclear C^*-algebra.

Proposition 8.2.1. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \mathbb{B}(\mathcal{H})$ be three C^*-algebras that are all in each other's commutant. Moreover, assume that $\mathcal{A}$ and $\mathcal{B}$ are nuclear. Let ω be a state on $\mathbb{B}(\mathcal{H})$. Then there exists a state $\tilde{\omega}$ on $\mathbb{B}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$ such that $\tilde{\omega}(a \otimes b \otimes c) = \omega(a \cdot b \cdot c)$ for all $a \in \mathcal{A}, b \in \mathcal{B}, c \in \mathcal{C}$.*

Proof. Define the $*$-homomorphism $\varphi : \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \to \mathbb{B}(\mathcal{H})$ by

$$\varphi(a \otimes b \otimes c) = abc.$$

Like before, we will be looking to extend this function to $\mathcal{A} \otimes_{\min} \mathcal{B} \otimes_{\min} \mathcal{C}$. Here we refer to Appendix A.7 where we look at associativity of triple tensor products:

$$\mathcal{A} \otimes_{\min} \mathcal{B} \otimes_{\min} \mathcal{C} = \mathcal{A} \otimes_{\min} (\mathcal{B} \otimes_{\min} \mathcal{C}) = \mathcal{A} \otimes_{\min} (\mathcal{B} \otimes_{\max} \mathcal{C}) = \mathcal{A} \otimes_{\max} (\mathcal{B} \otimes_{\max} \mathcal{C}) = \mathcal{A} \otimes_{\max} \mathcal{B} \otimes_{\max} \mathcal{C},$$

using nuclearity of $\mathcal{B}$ respectively $\mathcal{A}$ in the middle two steps. We finish the proof as in 5.4.1 with an application of the universal property for the triple maximal tensor product (the proof of [7, B.9] extends analogously to the triple case). □

Corollary 8.2.2. *Let $P(a,b,c|x,y,z) \in C_{qc}$ be a correlation matrix, such that the measurement operators of Alice and Bob generate nuclear C^*-algebras. Then $P(a,b,c|x,y,z) \in C_{qa}$.*

Proof. Let $\omega \in \mathbb{B}(\mathcal{H})$ be the state defining $P(a,b,c|x,y,z)$ and A_a^x, B_b^y, C_c^z the respective measurement operators. With $\mathcal{A}, \mathcal{B}, \mathcal{C}$ as the C^*-algebras generated by the respective measurement operator sets, Proposition 8.2.1 implies that there exists a state $\tilde{\omega} \in \mathbb{B}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$ such that $\tilde{\omega}(A_a^x \otimes B_b^y \otimes C_c^z) = P(a,b,c|x,y,z)$. By Theorem A.1.1, $\tilde{\omega}$ can be approximated by vector states ω_n; therefore, $\omega_n(A_a^x \otimes B_b^y \otimes C_c^z) \to P(a,b,c|x,y,z)$. In other words, $P(a,b,c|x,y,z) \in C_{qa}$. □

8.3 Characterisation in terms of FJO-algebras

We can extend the proof of Proposition 6.2.3 to the three-player case completely analogously:

Proposition 8.3.1. *We have*

$$C_{qa}^{(3)}(\Gamma) = \left\{ [\omega(e_a^x \otimes e_b^y \otimes e_c^z)]_{a,b,c;x,y,z} : \omega \in S\left(\bigotimes_{\min}^{3} *_{x=1}^{k} \ell_\infty^m \right) \right\} \tag{8}$$

and

$$C_{qc}^{(3)}(\Gamma) = \left\{ [\omega(e_a^x \otimes e_b^y \otimes e_c^z)]_{a,b,c;x,y,z} : \omega \in S\left(\bigotimes_{\max}^{3} *_{x=1}^{k} \ell_\infty^m \right) \right\}. \tag{9}$$

Proof. Let $P(a,b,c|x,y,z) \in C_{qs}^{(3)}(\Gamma)$ and Φ_A, Φ_B, Φ_C the ucp maps describing the measurements that exist by Proposition 6.1.2. Let $\omega \in S(\mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C))$ be the shared state. To show that the functional

$$\tilde{\omega}(\lambda_1 \otimes \lambda_2 \otimes \lambda_3) := \omega(\Phi_A(\lambda_1) \otimes \Phi_B(\lambda_2) \otimes \Phi_C(\lambda_3))$$

is a state, we need to show that the map $\Phi_A \otimes_{\min} \Phi_B \otimes_{\min} \Phi_C : \lambda_1 \otimes \lambda_2 \otimes \lambda_3 \mapsto \Phi_A(\lambda_1) \otimes \Phi_B(\lambda_2) \otimes \Phi_C(\lambda_3)$ is positive. This follows from two applications of Proposition 6.2.1. The proof for $C_{qa}^{(3)}(\Gamma)$, i.e. for limits of these quantum correlations, is exactly the same as in Proposition 6.2.3.

For the converse, a representation $\pi : *_{x=1}^{k} \ell_\infty^m \to \mathbb{B}(\mathcal{H})$ induces a representation $\bigotimes_{\max}^3 *_{x=1}^{k} \ell_\infty^m \to \mathbb{B}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$ by definition of the $\otimes_{\min}$ norm. The measurement operators are defined as $A_x^a := \pi(e_a^x)$, $B_y^b := \pi(e_b^y), C_z^c := \pi(e_c^z)$. Now if $P(a,b,c|x,y,z)$ is given by a state ω on $\bigotimes_{\max}^3 *_{x=1}^{k} \ell_\infty^m \to \mathbb{B}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$, we can again use Theorem A.1.1 to approach ω by vector states on $\mathcal{B}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$.

Next we consider (9). The inclusion $\subseteq$ follows similarly as the one for (8), with maximal tensor products and two applications of Proposition 6.2.2. For the converse, let ω be a state on $\mathcal{G}_{\max}(\Gamma)$. Just as in Proposition 6.2.3, we can use the GNS-representation of ω to define a vector state $\tilde{\omega}$ on $\mathcal{H}_\omega$ satisfying $\omega = \tilde{\omega} \circ \pi_\omega$. We define the operators $A_a^x := \pi_\omega(e_a^x \otimes \mathbb{1} \otimes \mathbb{1})$, $B_b^y := \pi_\omega(\mathbb{1} \otimes e_b^y \otimes \mathbb{1})$ and $C_c^z := \pi_\omega(\mathbb{1} \otimes \mathbb{1} \otimes e_c^z)$. Then the operators commute pairwise and $\tilde{\omega}(A_a^x B_b^y C_c^z) = \omega(e_a^x \otimes e_b^y \otimes e_c^z) = P(a,b,c|x,y,z)$, as required. □

The above proposition shows that conjecture (T3) is implied by the tripartite variant of Kirchberg's Conjecture (K3), i.e.

$$\bigotimes_{\min}^{3} *_{x=1}^{k} \ell_\infty^m = \bigotimes_{\max}^{3} *_{x=1}^{k} \ell_\infty^m.$$

8.4 Characterisation in terms of LP

Next we use the results of chapter 7 to find a characterisation for (T3) (or for (K3) to be more precise), and see what it means for a potential proof of (T2) $\Rightarrow$ (T3).

The link between QWEP and (K2) came mostly from the tensorial characterisations of the WEP given in Corollary 7.1.5 and 7.2.2. We will use this approach again here. Assuming QWEP (and thus (K2)), we define the unique tensor product space

$$\mathcal{B} = *_{x=1}^{k} \ell_\infty^m \otimes_{\min} *_{x=1}^{k} \ell_\infty^m = *_{x=1}^{k} \ell_\infty^m \otimes_{\max} *_{x=1}^{k} \ell_\infty^m.$$

Then $\bigotimes_{\min}^3 *_{x=1}^{k} \ell_\infty^m$ and $\bigotimes_{\max}^3 *_{x=1}^{k} \ell_\infty^m$ are equal to $*_{x=1}^{k} \ell_\infty^m \otimes_{\min} \mathcal{B}$ and $*_{x=1}^{k} \ell_\infty^m \otimes_{\max} \mathcal{B}$ respectively. By Corollary 7.2.2, it suffices to show that $\mathcal{B}$ has the WEP - or, by Proposition 7.1.10, the LP.

However, this does not seem to follow directly from the QWEP-conjecture. As Ozawa pointed out, it is not known whether the maximal or minimal tensor product preserves the LP [18, p. 15]. Therefore, it seems that to prove (K3) and thus (T3) we need the QWEP conjecture and additionally the LP on the unique space $\mathcal{B}$ which is implied by QWEP.

Let us demonstrate a proof idea and highlight what the problem is. The tensor product of separable spaces is still separable; therefore, by Lemma 7.2.3, it suffices to show that the identity map $\pi : \mathcal{B} \to C^*(\mathbb{F})/J$ is liftable. By Theorem 7.2.4, $*_{x=1}^{k} \ell_\infty^m$ has the LP, so the restrictions π_1, π_2 given by $\pi_1(a) = \pi(a \otimes 1), \pi_2(b) = \pi(1 \otimes b)$ are liftable. Let $\psi_1, \psi_2 : *_{x=1}^{k} \ell_\infty^m \to C^*(\mathbb{F})$ be the ucp liftings. Then to finish the proof, we need to combine these in a ucp lifting $\psi : \mathcal{B} \to C^*(\mathbb{F})$. However, in order to apply Proposition A.2.8, we need ψ_1 and ψ_2 to have commuting ranges. This seems unlikely to be possible, especially as it is generally hard for things in $C^*(\mathbb{F})$ to commute.

8.5 Converse implications

The true difficulty of the (T3) conjecture comes to light when considering converse implications. Essentially, the problem comes from the fact that the implication (T3) $\Rightarrow$ (K3) is open, as far as we know:

Conjecture 8.5.1. *The following statements are equivalent:*

1. *For all* $\Gamma = (k, m)$, *it holds that* $C_{qa}^{(3)}(\Gamma) = C_{qc}^{(3)}(\Gamma)$
2. *For all* $k, m \in \mathbb{N}$, *we have*

$$\bigotimes_{\min}^{3} *_{x=1}^{k} \ell_\infty^m = \bigotimes_{\max}^{3} *_{x=1}^{k} \ell_\infty^m$$

In the two-player case, Ozawa in [19] originally proved that (T2) implies the Connes Embedding Conjecture, which in turn implies (K2). As the number of players increases, we can just increase the number of tensor products in the FJO algebras, but there is no clear way to expand the Connes Embedding Conjecture or the QWEP conjecture. Therefore it is not clear how to extend the proof of (T2) $\Rightarrow$ (K2). Perhaps there exists some stronger version of Connes Embedding that can be used as a link.

The lack of proof for this conjecture leads to several issues with statements that we would expect to be true but cannot prove (or at least in the same way as in the two-player case). For example, as in the 2-player case (see Corollary 7.2.6), (K3) can be equivalently formulated as

$$\bigotimes_{\min}^{1\leq i\leq 3} *_{x=1}^{k_i} \ell_\infty^{m_{i,x}} = \bigotimes_{\max}^{1\leq i\leq 3} *_{x=1}^{k_i} \ell_\infty^{m_{i,x}}$$

But since we do not have an implication (T3) $\Rightarrow$ (K3), it is not clear whether Tsirelson 3 depends on whether every player has the same number of inputs and outputs. Therefore it is also not entirely clear if (T3) even implies (T2)! Given a 2-player scenario, one would expect the proof to add a 3rd 'trivial' player with 1 in/output, but it is not clear whether this is allowed.

Fortunately, the matrix-valued Tsirelson problem does survive the extension to three players. Define the correlation sets

$$C_{qa}^{n,(3)}(\Gamma) := \{[\omega(E_{i,j} \otimes e_a^x \otimes e_b^y \otimes e_c^z)]_{a,b,c;x,y,z;i,j} : \omega \in S(M_n(\bigotimes_{\min}^{3} *_{x=1}^{k} \ell_\infty^{m}))\} \tag{10}$$

and

$$C_{qc}^{n,(3)}(\Gamma) := \{[\omega(E_{i,j} \otimes e_a^x \otimes e_b^y \otimes e_c^z)]_{a,b,c;x,y,z;i,j} : \omega \in S(M_n(\bigotimes_{\max}^{3} *_{x=1}^{k} \ell_\infty^{m}))\}. \tag{11}$$

Then the following analogue of theorem 6.3.1 holds:

Theorem 8.5.2. *Let $\Gamma = (k, m)$ be fixed. The following statements are equivalent:*

(i) We have

$$\bigotimes_{\min}^{3} *_{x=1}^{k} \ell_\infty^{m} = \bigotimes_{\max}^{3} *_{x=1}^{k} \ell_\infty^{m}.$$

(ii) The matrix-valued three-player Tsirelson problem (T3M): for all n, we have $C_{qa}^{n,(3)}(\Gamma) = C_{qc}^{n,(3)}(\Gamma)$.

To give the proof, we need a tripartite version of the Pisier Linearisation Trick (Theorem A.3.3). We include this in the following Lemma

Lemma 8.5.3. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be C^*-algebras that are generated by the respective sets $(u_x), (v_y), (w_z)$. Define $S = \mathrm{Span}(\mathrm{u_x} \otimes \mathrm{v_y} \otimes \mathrm{w_z})$. The space $M_n(S)$ inherits $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ norms from $M_n(\mathcal{A} \otimes_{\min} \mathcal{B} \otimes_{\min} \mathcal{C})$ and $M_n(\mathcal{A} \otimes_{\max} \mathcal{B} \otimes_{\max} \mathcal{C})$ respectively. Now the following statements are equivalent:*

i) $\mathcal{A} \otimes_{\min} \mathcal{B} \otimes_{\min} \mathcal{C} = \mathcal{A} \otimes_{\max} \mathcal{B} \otimes_{\max} \mathcal{C}$

ii) $\|x\|_{\min} = \|x\|_{\max}$ for all self-adjoint $x \in M_n(S)$, $n \in \mathbb{N}$.

Proof. One can copy the entire proof of Theorem A.3.3 and just replace every instance of $\mathcal{A} \otimes_\alpha \mathcal{B}$ by $\mathcal{A} \otimes_\alpha \mathcal{B} \otimes_\alpha \mathcal{C}$ (where α is min or max) to get a correct proof. $\square$

Proof of Theorem 8.5.2. By the definitions of (10) and (11) we need only prove $(ii) \Rightarrow (i)$. The proof runs mostly along the same lines as in Theorem 6.3.1. We now define the set S by $\mathrm{Span}\{e_a^x \otimes e_b^y \otimes e_c^z\} \subseteq \bigotimes^3 *_{x=1}^{k} \ell_\infty^{m}$. Then for $x \in M_n(S)^{\mathrm{sa}}$, we can write

$$x = \sum_{i,j,a,b,c,x,y,z} \lambda_{i,j} E_{i,j} \otimes e_a^x \otimes e_b^y \otimes e_c^z$$

Define (as in Chapter 6) $\mathcal{G}^3_{\min}(\Gamma) = \bigotimes^3_{\min} *_{x=1}^k \ell^m_\infty$ and $\mathcal{G}^3_{\max}(\Gamma) = \bigotimes^3_{\max} *_{x=1}^k \ell^m_\infty$. By the assumption, it follows that for each $\omega \in S(M_n(\mathcal{G}^3_{\min}(\Gamma)))$ there exists a $\tilde{\omega} \in S(M_n(\mathcal{G}^3_{\max}(\Gamma)))$ such that $\omega(x) = \tilde{\omega}(x)$, and vice versa. Thus:

$$\|x\|_{\min} = \sup_{\omega \in S(M_n(\mathcal{G}^3_{\min}(\Gamma)))} |\omega(x)| = \sup_{\omega \in S(M_n(\mathcal{G}^3_{\max}(\Gamma)))} |\omega(x)| = \|x\|_{\max}.$$

The conclusion now follows from Lemma 8.5.3 □

So fortunately, the matrix-valued three player Tsirelson problem (T3M) does not have the problems that (T3) does; in particular, it does imply (T2). To summarise, here is a diagram of conjectures and the known implication arrows between them:

$$\begin{tikzcd}
(T3M) \arrow[dr, Rightarrow] \arrow[drr, Rightarrow] \arrow[ddd, Rightarrow] & & & \\
& (T3) & (K3) \arrow[l, Rightarrow] \arrow[d, Rightarrow] & \\
& & (K2) \arrow[dll, Rightarrow, "6.2.3"'] & QWEP \arrow[l, Rightarrow, "7.1.10"'] \\
(T2) \arrow[rr, Rightarrow] & & CE \arrow[ur, Rightarrow] &
\end{tikzcd}$$

8.6 Intermediate cases

In terms of choice of models, we have so far looked at the ‘extreme’ (and most natural) cases in the sense that all measurements lie either in separate tensor legs or are all in the same Hilbert space. In other words, we assume either the tensor product model or the commuting operator model, which in the Kirchberg picture corresponds to either maximal or minimal tensor products. For the sake of mathematical interest, we can also look at intermediate cases of FJO-algebras combining minimal and maximal tensor products and ask ourselves if there is a sensible way to represent this in the Tsirelson picture as a combination of the tensor product model and the commuting operator model. This intermediate case was considered for example by Fritz in [7].

For three players, there are (modulo symmetry) two cases we can consider, corresponding to which tensor norm we take first. Let us first consider the case

$$(*_{x=1}^k \ell^m_\infty \otimes_{\max} *_{x=1}^k \ell^m_\infty) \otimes_{\min} *_{x=1}^k \ell^m_\infty.$$

If we look at the correspondence between the Kirchberg and Tsirelson picture, we would expect this scenario to correspond to joint measurement operators of the form $(A^x_a B^y_b) \otimes C^z_c$. In terms of representations of these operators, A^x_a and B^y_b should be commuting sets of operators on a joint Hilbert space $\mathcal{H}_{AB}$, whereas C^z_c should be defined on a separate tensor leg $\mathcal{H}_C$. This leads to the following correlation set:

$$C_{qca}(\Gamma) = \text{closure}\Big(\Big\{ \big\langle \psi \big| (A^x_a B^y_b) \otimes C^z_c \big| \psi \big\rangle : A^x_a, B^y_b \in \mathbb{B}(\mathcal{H}_{AB})^+, C^z_c \in \mathbb{B}(\mathcal{H}_C)^+, [A^x_a, B^y_B] = 0, ... \Big\}\Big)$$

where due to a lack of space we ommitted the following requirements: $\psi \in \text{Ball}(\mathcal{H}_{\text{AB}} \otimes \mathcal{H}_{\text{C}})$, $\sum_a A^x_a = \sum_b B^y_b = \sum_c C^z_c = \mathbb{1}$, and $x, y, z \in \{1, \ldots, k\}$, $a, b, c \in \{1, \ldots, m\}$.
Indeed, a relation similar to Proposition 6.2.3 turns out to hold in this case

Proposition 8.6.1. *We have*

$$C_{qca}(\Gamma) = \{\omega(e_a^x \otimes e_b^y \otimes e_c^z) : \omega \in S((*_{x=1}^k \ell_\infty^m \otimes_{\max} *_{x=1}^k \ell_\infty^m) \otimes_{\min} *_{x=1}^k \ell_\infty^m).$$

Proof. The inclusion $\subseteq$ is almost the same as in Proposition 6.2.3. Let $P(a,b,c|x,y,z) \in C_{qca}(\Gamma)$ be a quantum correlation with associated Hilbert spaces $\mathcal{H}_{AB}$ and $\mathcal{H}_C$ and vector state ω. We define ucp maps Φ_A, Φ_B and Φ_C describing the corresponding sets of measurements. We have to show that the linear functional

$$\tilde{\omega}(\lambda_1 \otimes \lambda_2 \otimes \lambda_3) := \omega\big([\Phi_A(\lambda_1) \cdot \Phi_B(\lambda_2)] \otimes \Phi_C(\lambda_3)\big), \qquad \lambda_1, \lambda_2, \lambda_3 \in *_{x=1}^k \ell_\infty^m$$

is positive. By an application of Proposition 6.2.1 followed by an application of Proposition 6.2.1, we conclude that the map $(\Phi_A \otimes_{\max} \Phi_B) \otimes_{\min} \Phi_C : \lambda_1 \otimes \lambda_2 \otimes \lambda_3 \mapsto (\Phi_A(\lambda_1) \cdot \Phi_B(\lambda_2)) \otimes \Phi_C(\lambda_3)$ is ucp, and thus $\tilde{\omega}$ is positive. The proof for a limit of such quantum correlations is completely analogous to the one in Proposition 6.2.3.

For the converse inclusion, we need to find a faithful representation of the FJO-algebra of the form $\mathcal{H}_{AB} \otimes \mathcal{H}_C$; then by Theorem A.1.1 we can approach ω by vector states. Take any faithful representation $*_{x=1}^k \ell_\infty^m \otimes_{\max} *_{x=1}^k \ell_\infty^m \subseteq \mathbb{B}(\mathcal{H}_{AB})$ (this does not need to be the GNS representation). Then we find measurement operators $A_a^x := \pi(e_a^x \otimes \mathbb{1})$ and $B_b^y := \pi(\mathbb{1} \otimes e_b^y)$. Let $*_{x=1}^k \ell_\infty^m \subseteq \mathbb{B}(\mathcal{H}_C)$ be another faithful representation; then by definition of $\otimes_{\min}$ we have a faithful representation $(*_{x=1}^k \ell_\infty^m \otimes_{\max} *_{x=1}^k \ell_\infty^m) \otimes_{\min} *_{x=1}^k \ell_\infty^m \to \mathbb{B}(\mathcal{H}_{AB} \otimes \mathcal{H}_C)$. With an application of Theorem A.1.1 we approach ω by vector states and we are done. $\square$

It turns out that we have a satisfying relation in this case. So surely, one would expect to have one also in the other case, concerning the FJO-algebra

$$(*_{x=1}^k \ell_\infty^m \otimes_{\min} *_{x=1}^k \ell_\infty^m) \otimes_{\max} *_{x=1}^k \ell_\infty^m.$$

This corresponds to joint measurement operators of the form $(A_a^x \otimes B_b^y) C_c^z$, where the operators A_a^x and B_b^y should be represented on separate tensor legs $\mathcal{H}_A, \mathcal{H}_B$. The operators C_c^z need to be defined on the same Hilbert space as the operators $A_a^x \otimes B_b^y$ to be able to compose them; therefore we should have $C_c^z \in \mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. This leads to the following correlation set:

$$C_{qac}(\Gamma) = \text{closure}\Big(\Big\{ \langle \psi | (A_a^x \otimes B_b^y) C_c^z | \psi \rangle : A_a^x \in \mathbb{B}(\mathcal{H}_A)^+, B_B^y \in \mathbb{B}(\mathcal{H}_B)^+, C_c^z \in \mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B)^+, \\ [C_c^z, A_a^x \otimes B_B^y] = 0, ... \Big\}\Big)$$

However, we have not been able to prove the following statement:

Conjecture 8.6.2. We have

$$C_{qac}(\Gamma) = \{\omega(e_a^x \otimes e_b^y \otimes e_c^z) : \omega \in S((*_{x=1}^k \ell_\infty^m \otimes_{\min} *_{x=1}^k \ell_\infty^m) \otimes_{\max} *_{x=1}^k \ell_\infty^m).$$

Let us attempt to prove this conjecture in similar fashion as the previous one. The inclusion $\subseteq$ poses no problems; it is essentially the same as above. For the backward inclusion, we need to find a faithful representation of the FJO-algebra of the form $\mathcal{H}_A \otimes \mathcal{H}_B$, on which the operators are defined as in the definition of C_{qac}. Starting from a faithful representation $\pi : *_{x=1}^k \ell_\infty^m \subseteq \mathbb{B}(\mathcal{H})$, with the minimal tensor product we get a faithful representation $*_{x=1}^k \ell_\infty^m \otimes_{\min} *_{x=1}^k \ell_\infty^m \subseteq \mathbb{B}(\mathcal{H} \otimes \mathcal{H})$. Now how to represent the final maximal tensor product?

One plausible trick would be to 'enlarge' the second Hilbert space to make space for the final copy of $*_{x=1}^k \ell_\infty^m$; i.e. we define $\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}$ and redefine $B_b^y := \pi(e_b^y) \otimes \mathbb{1}_{\mathcal{H}}$ to be the identity on the second tensor leg. This gives a faithful representation $*_{x=1}^{k}\ell_\infty^m \otimes_{\min} *_{x=1}^{k}\ell_\infty^m \subseteq \mathbb{B}(\mathcal{H}\otimes\tilde{\mathcal{H}})$. We can now define a representation $*_{x=1}^{k}\ell_\infty^m \subseteq \mathbb{B}(\mathcal{H}\otimes\tilde{\mathcal{H}})$ by setting $e_c^z \mapsto \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{H}} \otimes \pi(e_c^z) =: C_c^z$. We now have two representations with commuting ranges - by the universal property, this gives a representation $(*_{x=1}^{k}\ell_\infty^m \otimes_{\min} *_{x=1}^{k}\ell_\infty^m) \otimes_{\max} *_{x=1}^{k}\ell_\infty^m \subseteq \mathbb{B}(\mathcal{H}\otimes\tilde{\mathcal{H}})$. However, this representation need not be faithful!

In general, when $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, the usual trick to get a faithful representation on the maximal tensor product is to take $\mathcal{A}\otimes_{\max}\mathcal{B} \subseteq \mathbb{B}(\hat{\mathcal{H}})$, where $\hat{\mathcal{H}} = \bigoplus_\pi H_\pi$. Here the direct sum is taken over all representations $\pi = (\pi_A, \pi_B)$ with $\pi_A : \mathcal{A} \to \mathbb{B}(\mathcal{H}), \pi_B : \mathcal{B} \to \mathbb{B}(\mathcal{H})$ and $[\pi_A, \pi_B] = 0$. In our case, we have $\mathcal{A} = *_{x=1}^{k}\ell_\infty^m \otimes_{\min} *_{x=1}^{k}\ell_\infty^m$, where we need the representation to be of the form $\pi_A : \mathcal{A} \to \mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. In general, as far as we know, a representation of a minimal tensor product cannot always be written of this form. For that reason, it is not clear how to construct a faithful representation here!

It could well be that there is no suitable Tsirelson picture corresponding to this FJO-algebra. After all, it is somewhat artificial to combine the two quantum models in this way; usually one either assumes one model or the other. In particular, assuming a commuting operator model *after* a tensor product model leads to seemingly unnatural situations. To further illustrate this point, let us consider one more FJO-algebra with 4 players:

$$(*_{x=1}^{k}\ell_\infty^m \otimes_{\min} *_{x=1}^{k}\ell_\infty^m) \otimes_{\max} (*_{x=1}^{k}\ell_\infty^m \otimes_{\min} *_{x=1}^{k}\ell_\infty^m).$$

If D_d^w are the measurement operators of the fourth player Dirk, then the joint measurement operator would have to look like $(A_a^x \otimes B_b^y)\cdot(C_c^z \otimes D_d^w)$. This means that the operators $A_a^x \otimes B_b^y$ and $C_c^z \otimes D_d^w$ need to be defined in the same 'universal' Hilbert space while commuting with each other. At the same time, each operator needs to be defined on its own Hilbert space; to summarise, we need $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathcal{H}_C \otimes \mathcal{H}_D$. While this sort of makes sense from a mathematical view, how would we physically imagine this experiment? At the one hand each player needs to have its own Hilbert space (the tensor product model). On the other hand, when Alice and Bob respectively Charlie and Dirk combine their operators they need to end up in a sort of universal Hilbert space.

Appendices

A Proofs of related theorems

In the following, $\mathcal{A}$ and $\mathcal{B}$ are always C^*-algebras.

A.1 General useful theorems

We start by showing density of vector states in the state space with the weak-$*$ topology. Note that in the theorem below this concerns mixed states, not pure states. In proofs of e.g. Proposition 6.2.3 we assume pure states, so there seems to be a bit of a gap here. This is solved, however, by remark 3.5.3, which shows that in the definition of correlation sets it would be equivalent to consider mixed states instead.

Theorem A.1.1. *Let $\mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$ be a concretely represented C^*-algebra with state space $\mathcal{S}(\mathcal{A})$. Let*

$$\mathcal{VS}(\mathcal{A}) := \{\rho : a \mapsto \sum_{i=1}^{n} \lambda_i \langle a\xi_i, \xi_i\rangle \mid \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, \xi_i \in \mathcal{H}\}$$

be the set of (mixed) vector states of $\mathcal{A}$. Then for every state $\rho \in \mathcal{S}(\mathcal{A})$, every finite set of elements $x_1, \ldots, x_n \in \mathcal{A}$ and every $\varepsilon > 0$, there is a vector state $\rho' \in \mathcal{VS}(\mathcal{A})$ such that

$$|\rho(x_i) - \rho'(x_i)| < \varepsilon, \qquad i = 1, \ldots, n.$$

This implies that $\mathcal{VS}(\mathcal{A})$ is weak-$*$ dense in $\mathcal{S}(\mathcal{A})$.

Proof. Consider the finite-dimensional subspace $S = \mathrm{Span}\{\mathbb{1}, x_1, \ldots, x_n\}$ and let S_+ be the cone of positive elements within S. Note that every state $\rho \in \mathcal{S}(\mathcal{A})$ can be reduced to an element of S_+^* by restricting to S. Similarly, elements in $\mathcal{VS}(\mathcal{A})$ can be restricted to S_+ and thus considered elements of S_+^*.

Let $V = \{\lambda\rho : \rho \in \mathcal{VS}(\mathcal{A}), \lambda \geq 0\}$ be the set of unnormalised vector states. We claim that V, as a subset of S_+^*, is dense in S_+^* (in the norm topology, which is anyway the same as the weak-$*$ topology on finite dimensional spaces). Indeed, if we assume otherwise, then there is an element $\rho \in S_+^* \setminus \bar{V}$, which we can assume to be a state. Now by (a corollary of) the Hahn-Banach theorem, there exists a 'separating element' $x \in S^{**}$. Because S is finite dimensional, we have $S^{**} \cong S$, so we can assume $x \in S$. This means that $\rho'(x) = 0$ for all $\rho' \in V$ and $\rho(x) = 1$.

But then in particular $\langle(\mathbb{1} - 2x)\xi, \xi\rangle = 1 > 0$ for all $\xi \in \mathcal{H}$ with $\|\xi\| = 1$, which implies that $\mathbb{1} - 2x$ is positive. On the other hand, $\rho(\mathbb{1} - 2x) = -1$, which is a contradiction since ρ is also a positive linear functional.

The finite-dimensional subspaces form a net in $\mathcal{A}$. If $\rho \in \mathcal{S}(\mathcal{A})$ is a state, then for any finite-dimensional subspace $E \subseteq \mathcal{A}$ we can find a state ρ_E such that $|\rho(x) - \rho_E(x)| < 1/|E|$ for all $x \in E$. Then the ρ_E form a weak-$*$ convergent net to ρ. (Note that this is no longer a norm convergence!) $\square$

The Stinespring dilation theorem is a fundamental result in quantum information theory.

Theorem A.1.2 (Stinespring dilation theorem)**.** *Let $\Phi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ be a ucp map. Then $\Phi(a) = P_{\mathcal{H}}\pi(a)P_{\mathcal{H}}$, where $\pi : \mathcal{A} \to \mathbb{B}(\hat{\mathcal{H}})$ is some $*$-homomorphism to a Hilbert space $\hat{\mathcal{H}}$ that isometrically contains $\mathcal{H}$.*

The proof is well-known, and can be seen as a generalisation of the GNS construction, where the ucp map Φ takes the role of the state. Note here that the second $P_{\mathcal{H}}$ is really the inclusion of $\mathcal{H}$ in $\hat{\mathcal{H}}$ (or the $*$ of the projection $\hat{\mathcal{H}} \to \mathcal{H}$).

Proof. In the GNS representation the Hilbert space was constructed from $\mathcal{A}$. Here, we construct the Hilbert space from the tensor product of complex vector spaces $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{H}$ instead. This way we can construct a 'potential' inner product from the map Φ by defining:

$$\langle a \otimes \xi, b \otimes \xi' \rangle := \langle \Phi(b^* a)\xi, \xi' \rangle_{\mathcal{H}} \tag{12}$$

and extending by sesquilinearity. Conjugate symmetry follows because positive maps preserve involution, i.e. $\Phi(x)^* = \Phi(x^*)$ (one can see this by writing x as a linear combination of 4 positive elements). To check positive semidefiniteness, we really need complete positivity:

$$\left\langle \sum_{i=1}^{n} a_i \otimes \xi_i, \sum_{i=1}^{n} a_i \otimes \xi_i \right\rangle = \sum_{i,j=1}^{n} \langle \Phi(a_i^* a_j)\xi_i, \xi_j \rangle.$$

If we define $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{H}^n$ and $a = \begin{pmatrix} a_1 & \ldots & a_n \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{pmatrix} \in M_n(\mathcal{A})$, then one can check that $a^* a = (a_i^* a_j)_{i,j}$ and the RHS is equal to $\langle \Phi^{(n)}(a^* a)\xi, \xi \rangle_{\mathcal{H}^n} \geq 0$. Here $\Phi^{(n)} = \mathbb{1}_{M_n} \otimes \Phi$.

Of course, (12) is not yet positive definite since there are non-zero elements with zero inner product. Therefore, we quotient out the space of 'zero elements'. Define

$$\mathcal{N} := \{x \in \mathcal{A} \otimes_{\mathbb{C}} \mathcal{H} : \langle x, x \rangle = 0\}.$$

Then (12) defines an inner product on the space $(\mathcal{A} \otimes_{\mathbb{C}} \mathcal{H})/\mathcal{N}$. After completion with respect to this inner product, we have finally got our desired Hilbert space: $\hat{\mathcal{H}} := \overline{(\mathcal{A} \otimes_{\mathbb{C}} \mathcal{H})/\mathcal{N}}$. This Hilbert space isometrically contains $\mathcal{H}$ via the embedding $\xi \mapsto \mathbb{1} \otimes \xi + \mathcal{N}$.

We define $\pi(a)(b \otimes \xi + \mathcal{N}) = ab \otimes \xi + \mathcal{N}$ (the 'left regular representation'). We need to check that this is well-defined and bounded (so that we can extend to the completion). If $\sum_{i=1}^{n} a_i \otimes \xi_i \in \mathcal{A} \otimes_{\mathbb{C}} \mathcal{H}$ and $b \in \mathcal{A}$, then

$$\left\langle \sum_{i=1}^{n} b a_i \otimes \xi_i, \sum_{i=1}^{n} b a_i \otimes \xi_i \right\rangle = \sum_{i,j=1}^{n} \langle \Phi(a_i^* b^* b a_i)\xi_i, \xi_j \rangle \leq \sum_{i,j=1}^{n} \|b^* b\| \langle \Phi(a_i^* a_i)\xi_i, \xi_j \rangle,$$

where the last estimate follows from Theorem 3.3.7 from Murphy by defining positive linear functionals $x \mapsto \langle \Phi(x)\xi_i, \xi_j \rangle$ (that are only positive because Φ is positive). If $\sum_{i=1}^{n} a_i \otimes \xi_i \in \mathcal{N}$, then the RHS is 0, which shows well-definedness. At the same time, the estimate shows that π is bounded, so it defines a $*$-homomorphism $\mathcal{A} \to \mathbb{B}(\hat{\mathcal{H}})$.

Finally, we check that $\Phi(a) = P_{\mathcal{H}} \pi(a) P_{\mathcal{H}}$ holds. To be precise, we need to have $\Phi(a)\xi = P_{\mathcal{H}} \pi(a) P_{\mathcal{H}}(\mathbb{1} \otimes \xi + \mathcal{N})$ for any $\xi \in \mathcal{H}$. We check this via the inner product:

$$\langle P_{\mathcal{H}} \pi(a) P_{\mathcal{H}} \mathbb{1} \otimes \xi, \mathbb{1} \otimes \xi \rangle_{\hat{\mathcal{H}}} = \langle \pi(a) \mathbb{1} \otimes \xi, P_{\mathcal{H}} \mathbb{1} \otimes \xi \rangle_{\hat{\mathcal{H}}} = \langle a \otimes \xi, \mathbb{1} \otimes \xi \rangle_{\hat{\mathcal{H}}} = \langle \Phi(a)\xi, \xi \rangle.$$

This finishes the proof. ☐

Remark A.1.3. For contractive completely positive (so non-unital) maps, we also have a Stinespring Dilation Theorem. However, there is no longer an isometric inclusion $\mathcal{H} \subseteq \hat{\mathcal{H}}$, so instead of the projection $P_{\mathcal{H}}$ we have a map $V : \mathcal{H} \to \hat{\mathcal{H}}$ such that $\Phi(a) = V^*\pi(a)V$.

Theorem A.1.4 (Arveson extension theorem)**.** *Let $S \subseteq A$ be a self-adjoint subspace containing $\mathbb{1}_A$. Then any ucp map $\Phi : S \to \mathbb{B}(\mathcal{H})$ can be extended to a ucp map $\tilde{\Phi} : A \to \mathbb{B}(\mathcal{H})$.*

We refer to [6, Thm. 4.1.5] for the proof.

A.2 Tensor products on operator systems

In this section, we give a brief introduction of tensor products of abstract operator systems, culminating in Theorems A.2.7 and A.2.8. More details can be found in [13].

Definition A.2.1. *Let V be a $*$-algebra. $\{C_n\}_{n=1}^{\infty}$ is called a* matrix ordering *on V if*

(1) C_n is a cone in $M_n(V)_h$, the hermitian elements of $M_n(V)$,

(2) $C_n \cap -C_n = \{0\}$ for $n \in \mathbb{N}$,

(3) $X^*C_nX \subseteq C_m$ for each $X \in M_{n,m}$ and $m, n \in \mathbb{N}$.

In this case we call $(V, \{C_n\}_{n=1}^{\infty})$ an (abstract) operator system*.*

In the following, let $(\mathcal{S}, \{P_n\}_{n=1}^{\infty})$, $(\mathcal{T}, \{Q_n\}_{n=1}^{\infty})$ be operator systems.

Definition A.2.2. *An operator system structure on $\mathcal{S} \otimes \mathcal{T}$ consists of a family of cones $\tau = \{C_n\}_{n=1}^{\infty}$, with $C_n \subseteq M_n(\mathcal{S} \otimes \mathcal{T})$, satisfying:*

(i) $(\mathcal{S} \otimes \mathcal{T}, \{C_n\}_{n=1}^{\infty})$ is an operator system denoted by $\mathcal{S} \otimes_\tau \mathcal{T}$,

(ii) $P_n \otimes Q_m \subseteq C_{nm}$, for all $n, m \in \mathbb{N}$,

(iii) If $\Phi : \mathcal{S} \to \mathbb{B}(\mathcal{H}_A)$ and $\Psi : \mathcal{T} \to \mathbb{B}(\mathcal{H}_B)$ are ucp maps, then $\Phi \otimes \Psi : \mathcal{S} \otimes_\tau \mathcal{T} \to \mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is also an ucp map.

Analogously to the C^*-algebra case, we will define a minimal and a maximal tensor product on operator systems.

Definition A.2.3. *For each $n \in \mathbb{N}$, we let*

$$C_n^{\min} = \{t \in M_n(\mathcal{S} \otimes \mathcal{T}) : (\Phi \otimes \Psi)^{(n)}(t) \geq 0 \text{ for all ucp maps } \Phi : \mathcal{S} \to \mathbb{B}(\mathcal{H}_A),\ \Psi : \mathcal{T} \to \mathbb{B}(\mathcal{H}_B)\}.$$

We define the minimal tensor product operator system $\mathcal{S} \otimes_{\mathrm{OSmin}} \mathcal{T}$ to be the algebraic tensor product $\mathcal{S} \otimes \mathcal{T}$ with operator system structure $\{C_n^{\min}\}_{n=1}^{\infty}$.

Intuitively, this is the operator system structure generating the minimal amount of ucp maps. Indeed, we define every element $t \in M_n(\mathcal{S} \otimes \mathcal{T})$ to be positive as long as it can be made positive without violating requirement (*iii*), so that we have the largest possible number of positive elements. The more positive elements, the harder it is for a map to be completely positive.

Definition A.2.4. *For each $n \in \mathbb{N}$, we let*

$$\begin{aligned} C_n^{\mathrm{cmax}} = \{& t \in M_n(\mathcal{S} \otimes \mathcal{T}) : (\Phi \cdot \Psi)^{(n)}(t) \geq 0 \text{ for all ucp maps } \Phi : \mathcal{S} \to \mathbb{B}(\mathcal{H}),\ \Psi : \mathcal{T} \to \mathbb{B}(\mathcal{H}) \\ & \text{with commuting ranges}\}. \end{aligned}$$

We define the commuting maximal tensor product operator system $\mathcal{S} \otimes_{\mathrm{cmax}} \mathcal{T}$ to be the algebraic tensor product $\mathcal{S} \otimes \mathcal{T}$ with operator system structure $\{C_n^{\mathrm{cmax}}\}_{n=1}^{\infty}$.

Intuitively, this is the operator system structure generating the least amount of ucp maps such that in addition to satisfying requirement (iii), every map of the form $\Phi \cdot \Psi$ is ucp, where (Φ, Ψ) are pairs of ucp maps on $\mathcal{S}$ respectively $\mathcal{T}$. Note that we have not called this the maximal operator system tensor product. The reason is that it is not actually 'maximal', in the sense that it does not contain the least number of positive elements (i.e. the largest number of ucp maps). However, when $\mathcal{S}$ and $\mathcal{T}$ are C^*-algebras, this is actually true. For more details and a more thorough discussion about tensor products on operator systems, we refer to [13]. In particular, we will use the following two theorems:

Theorem A.2.5. [13, 4.10] Let $\mathcal{A}$, $\mathcal{B}$ be C^*-algebras. Then the minimal operator system tensor product $\mathcal{A} \otimes_{\text{OSmin}} \mathcal{B}$ is completely order isomorphic to the image of $\mathcal{A} \otimes \mathcal{B}$ inside the minimal C^*-algebraic tensor product $\mathcal{A} \otimes_{\min} \mathcal{B}$.

Theorem A.2.6. [13, 5.12 and 6.6] Let $\mathcal{A}$, $\mathcal{B}$ be C^*-algebras. Then the commuting maximal operator system tensor product $\mathcal{A} \otimes_{\text{cmax}} \mathcal{B}$ is completely order isomorphic to the image of $\mathcal{A} \otimes \mathcal{B}$ inside the maximal C^*-algebraic tensor product $\mathcal{A} \otimes_{\max} \mathcal{B}$.

For this last theorem, note that Kavruk et al. denote the commuting maximal tensor product by $\otimes_c$, and the actual maximal tensor product by $\otimes_{\max}$.

Theorem A.2.7. Let $\varphi : \mathcal{A} \to \mathcal{C}$ and $\psi : \mathcal{B} \to \mathcal{D}$ be ucp maps of C^*-algebras. Then the map $\varphi \otimes \psi : a \otimes b \to c \otimes d$ extends to a ucp map on both the minimal and maximal tensor products.

Proof. Let $S = \mathcal{A} \otimes \mathcal{B}$ be the algebraic tensor product, and let the operator systems $S_{\min}$ and $S_{\max}$ be the subspaces of $\mathcal{A} \otimes_{\min} \mathcal{B}$ respectively $\mathcal{A} \otimes_{\max} \mathcal{B}$. Then by Theorems A.2.5 and A.2.6, $S_{\min}$ and $S_{\max}$ are completely order isomorphic to $\mathcal{A} \otimes_{\text{OSmin}} \mathcal{B}$ and $\mathcal{A} \otimes_{\text{cmax}} \mathcal{B}$.

We define maps $\Phi_A \otimes_{\min} \Phi_B$ and $\Phi_A \otimes_{\max} \Phi_B$ on $S_{\min}$ respectively $S_{\max}$ to $\mathcal{C} \otimes \mathcal{D}$ by mapping $a \otimes b$ to $\Phi_A(a) \otimes \Phi_B(b)$. Because $S_{\min}$ and $S_{\max}$ are also abstract operator systems, they satisfy requirement (iii) of Definition A.2.4. Therefore, $\Phi_A \otimes_{\min} \Phi_B$ and $\Phi_A \otimes_{\max} \Phi_B$ are ucp maps (to be precise, we should choose representations $\mathcal{C} \subseteq \mathcal{H}_C$ and $\mathcal{D} \subseteq \mathcal{H}_D$ to use requirement (iii), and then note that we can restrict the range of $\Phi_A \otimes \Phi_B$). By Proposition 2.1.2, these maps are contractive and can thus be extended to unital maps

$$\Phi_A \otimes_{\min} \Phi_B : \mathcal{A} \otimes_{\min} \mathcal{B} \to \mathcal{C} \otimes_{\min} \mathcal{D}$$

and

$$\Phi_A \otimes_{\max} \Phi_B : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{C} \otimes_{\max} \mathcal{D}.$$

Because positive elements in a closure are precisely the limits of positive elements in the original space, these maps are again completely positive (and thus ucp). □

See [1, 3.5.3] for a different proof using the Stinespring dilation theorem.

Theorem A.2.8. If $\varphi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ and $\psi : \mathcal{B} \to \mathbb{B}(\mathcal{H})$ are ucp maps with commuting ranges, then the map $\varphi \cdot \psi : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathbb{B}(\mathcal{H})$ is well-defined and ucp.

Proof. This proof is similar to the previous one. Here we use that in Definition A.2.4, the positive elements are chosen in such a way that for ucp maps $\Phi : \mathcal{S} \to \mathbb{B}(\mathcal{H})$ and $\Psi : \mathcal{T} \to \mathbb{B}(\mathcal{H})$, the map $\Phi \cdot \Psi$ is again ucp. This allows us to show via Theorem A.2.6 that the map $\varphi \cdot \psi$ is ucp and thus contractive on the algebraic tensor product with respect to the maximal norm, so that we can extend it to a ucp map on the maximal one. □

A different proof using a variant of the Stinespring dilation theorem can be found in [7, B.12]

A.3 The Pisier Linearisation trick

Here we prove the Pisier Linearisation trick (Theorem A.3.3) as used in Theorem 6.3.1. We start by introducing the concept of multiplicative domains.

Definition A.3.1. *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a ucp map between C^*-algebras. Then the set of all elements $a \in \mathcal{A}$ satisfying*

$$\Phi(a^*a) = \Phi(a^*)\Phi(a), \quad \text{and} \quad \Phi(aa^*) = \Phi(a)\Phi(a)^*$$

is called the multiplicative domain *of Φ.*

Theorem A.3.2 (Multiplicative Domains)**.** ([7, A.6]) *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a ucp map between C^*-algebras. Then for each $x \in \mathcal{A}$ and a in the multiplicative domain, we have $\Phi(ax) = \Phi(a)\Phi(x)$ and $\Phi(xa) = \Phi(x)\Phi(a)$. Moreover, the multiplicative domain of Φ is a C^*-subalgebra of $\mathcal{A}$, and the restriction of Φ to the multiplicative domain is a $*$-homomorphism.*

Proof. Let $\mathcal{B} \subseteq \mathbb{B}(\mathcal{H})$ be a faithful representation. Then by the Stinespring Dilation Theorem A.1.2, there is a Hilbert space $\hat{\mathcal{H}} \supseteq \mathcal{H}$ and a $*$-homomorphism $\pi : \mathcal{A} \to \mathbb{B}(\hat{\mathcal{H}})$ such that $\Phi(a) = P_{\mathcal{H}}\pi(a)P_{\mathcal{H}}$. The difference between $\Phi(a^*a)$ and $\Phi(a)^*\Phi(a)$ in terms of this Stinespring formulation is given by

$$\Phi(a^*a) - \Phi(a)^*\Phi(a) = P_{\mathcal{H}}\pi(a)^*(\mathbb{1}_{\hat{\mathcal{H}}} - P_{\mathcal{H}})\pi(a)P_{\mathcal{H}} = \left[(\mathbb{1}_{\hat{\mathcal{H}}} - P_{\mathcal{H}})\pi(a)P_{\mathcal{H}}\right]^* (\mathbb{1}_{\hat{\mathcal{H}}} - P_{\mathcal{H}})\pi(a)P_{\mathcal{H}}$$

Therefore, the above is 0 iff $(\mathbb{1}_{\hat{\mathcal{H}}} - P_{\mathcal{H}})\pi(a)P_{\mathcal{H}} = 0$. But this implies that for a in the multiplicative domain and $x \in \mathcal{A}$

$$\Phi(xa) - \Phi(x)\Phi(a) = P_{\mathcal{H}}\pi(x)(\mathbb{1}_{\hat{\mathcal{H}}} - P_{\mathcal{H}})\pi(a)P_{\mathcal{H}} = 0.$$

Switching the roles of a^* and a we also deduce that $\Phi(ax) = \Phi(a)\Phi(x)$. By linearity of Φ, this property is preserved under linear combinations, and it is clearly preserved under products. It was already clear that the multiplicative domain is $*$-closed, so it follows that the multiplicative domain is a C^*-subalgebra of $\mathcal{A}$ and that the restriction of Φ to the multiplicative domain is a $*$-homomorphism. □

Theorem A.3.3 (Pisier's trick)**.** [24, Thm. 1] & [7, B.14] *Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras that are generated by the respective sets $(u_x), (v_y)$. Define $S = \mathrm{Span}(\mathrm{u_x} \otimes \mathrm{v_y})$. The space $M_n(S)$ inherits $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ norms from $M_n(\mathcal{A} \otimes_{\min} \mathcal{B})$ and $M_n(\mathcal{A} \otimes_{\max} \mathcal{B})$ respectively. Now the following statements are equivalent:*

i) $\mathcal{A} \otimes_{\min} \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$

ii) $\|x\|_{\min} = \|x\|_{\max}$ *for all self-adjoint $x \in M_n(S)$, $n \in \mathbb{N}$.*

Proof. The direction i) $\Rightarrow$ ii) is obvious. For the other direction, note that the identity $\mathcal{A} \otimes_{\text{alg}} \mathcal{B} \to \mathcal{A} \otimes_{\min} \mathcal{B}$ is contractive. Therefore, it extends to a $*$-homomorphism $\Phi : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{A} \otimes_{\min} \mathcal{B}$. We will prove i) by constructing an inverse to this map.

We denote $S_{\min}$ and $S_{\max}$ as the set S equipped with the min- and max norm respectively. We consider the identity mapping $S_{\min} \to \mathcal{A} \otimes_{\max} \mathcal{B}$, which is clearly unital. We first show that this map is also completely positive, so that we can call on Arveson's extension theorem A.1.4.

Let $n \in \mathbb{N}$ be fixed and let $a \in M_n(S_{\min})^+ \subseteq (\mathcal{A} \otimes_{\min} \mathcal{B})^+$. Then, by Lemma 2.2.2 from Murphy, we have $\|a - \|a\| \mathbb{1}_{\mathcal{A}}\|_{\min} \leq \|a\|_{\min}$. Since $a - \|a\| \mathbb{1}_{\mathcal{A}} \in M_n(S)$, we have

$$\|a - \|a\| \mathbb{1}_{\mathcal{A}}\|_{\max} \overset{ii)}{=} \|a - \|a\| \mathbb{1}_{\mathcal{A}}\|_{\min} \leq \|a\|_{\min} = \|a\|_{\max}.$$

Again by Lemma 2.2.2 from Murphy, we find that $a \in M_n(S_{\max})^+ \subseteq (\mathcal{A} \otimes_{\max} \mathcal{B})^+$, which implies that the identity mapping is completely positive.

Embed $\mathcal{B} \subseteq \mathbb{B}(\mathcal{H})$ and note that S contains only self-adjoint elements. Then by Arveson's extension theorem, we can extend the identity mapping to a ucp map $\Psi : \mathcal{A} \otimes_{\min} \mathcal{B} \to \mathbb{B}(\mathcal{H})$. Now we claim that the range of Ψ is contained in $\mathcal{A} \otimes_{\max} \mathcal{B}$. Indeed, since Ψ is the identity on S, it is clear that S is in the multiplicative domain of Ψ. Now since $S_{\min}$ generates $\mathcal{A} \otimes_{\min} \mathcal{B}$, it is also true that $\Psi(S)$ generates the range of Ψ, which is also a C^*-algebra. Since S is contained in $\mathcal{A} \otimes_{\max} \mathcal{B}$, it follows that Ψ maps into $\mathcal{A} \otimes_{\max} \mathcal{B}$.

By Theorem A.3.2, the multiplicative domain has to be a C^*-subalgebra of $\mathcal{A} \otimes_{\min} \mathcal{B}$. But since $S_{\min}$ generates $\mathcal{A} \otimes_{\min} \mathcal{B}$ as a C^*-algebra, it follows that the multiplicative domain is the whole of $\mathcal{A} \otimes_{\min} \mathcal{B}$, and thus (again by Theorem A.3.2) Ψ is a $*$-homomorphism. Since $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity on S, it follows that they must be the identity on $\mathcal{A} \otimes_{\min} \mathcal{B}$ respectively $\mathcal{A} \otimes_{\max} \mathcal{B}$ (again using the fact that $S_{\min}$ respectively $S_{\max}$ are generating sets). Thus these are both $*$-isomorphisms, which are automatically isometric (see Theorem 2.1.7 from Murphy). $\square$

A.4 Factors over finite dimensional Hilbert spaces

Here we prove that every factor over a finite dimensional Hilbert space is spacially isomorphic to $\mathbb{1}_{M_d(\mathbb{C})} \otimes M_q(\mathbb{C})$ for suitable $d, q \in \mathbb{N}$ (Lemma A.4.2). This result was used in Theorem 5.2.1.

We first prove the following Lemma, which is a simple case of [12, 6.6.1], stating that type $\mathrm{I_n}$ factors are $*$-isomorphic to $\mathcal{B}(\mathcal{K})$ where $\mathcal{K}$ has dimension n. We prove our simple case without using any theory not in Murphy's book:

Lemma A.4.1. Let $\mathcal{M} \subseteq \mathbb{B}(\mathcal{H})$ be a factor, where $\mathcal{H}$ is a finite dimensional Hilbert space. Then $\mathcal{M}$ is $*$-isomorphic to $\mathcal{B}(\mathcal{K})$ for some finite dimensional Hilbert space $\mathcal{K}$.

Proof. One property of a von Neumann algebra is that is has many projections; for example, by Theorem 4.1.9 from Murphy, it contains all the range projections of its elements. Since $\mathcal{H}$ is finite dimensional, there must exist minimal projections (i.e. non-zero projections so that 0 is its only proper subprojection). Let $\{p_i\}$ be a maximal set of minimal (orthogonal) projections. Then $\sum_{i=1}^q p_i = \mathbb{1}_{\mathcal{H}}$; indeed, if $\mathbb{1}_{\mathcal{H}} - \sum_{i=1}^q p_i \neq 0$, then it is either a minimal projection or has a minimal subprojection, which contradicts maximality. We will use this fact at the very end of the current proof.

Let p' be any minimal projection in $\mathcal{M}'$. We show that the map $\mathcal{M} \to \mathcal{M}p'$ given by $A \mapsto Ap'$ is a $*$-isomorphism. Using commutativity of p' and $\mathcal{M}$, it can be easily checked that this map is a $*$-homomorphism, and surjectivity is clear. It remains to prove injectivity. To show this, let $T \in \mathcal{M}$ such that $Tp' = 0$.

Define Γ as the set of all projections E' in $\mathcal{M}'$ such that $TE' = 0$. Let Q be the union over all projections in Γ. Note that Q is the range projection of $\sum_{E' \in \Gamma} E' \in \mathcal{M}'$, therefore $Q \in \mathcal{M}'$. Also, it is clear that $TQ = 0$.

We show that $Q \in \mathcal{M} \cap \mathcal{M}'$, for which it suffices to show that $Q \in \mathcal{M}'' = \mathcal{M}$. To this end, let $F' \in \mathcal{M}'$ be a fixed projection. Then $TF'Q = F'TQ = 0$. Since $F'Q$ and thus its range projection is in $\mathcal{M}'$, it follows by the construction of Q that $\mathrm{ran}(F'Q) \subseteq \mathrm{ran}(Q)$. We show that $F'Q = QF'$ with the following surprising trick:

$$F'Q = QF'Q = (QF'Q)^* = (F'Q)^* = QF'.$$

It follows that $Q \in \mathcal{M} \cap \mathcal{M}'$. But since $\mathcal{M}$ is a factor, it follows that $Q = \lambda \mathbb{1}_{\mathcal{H}}$. Thus $\lambda T \mathbb{1}_{\mathcal{H}} = 0$, which means $T = 0$. This shows injectivity.

The above shows that the space $\mathcal{M}p'$, acting on the Hilbert space $\mathcal{K} := p'(\mathcal{H})$, is a von Neumann algebra. We next show that $\mathcal{M}p' = \mathcal{B}(\mathcal{K})$. The first step is to show that the commutant of $\mathcal{M}p'$ is $p'\mathcal{M}'p'$. Let $T' \in \mathcal{B}(\mathcal{K})$ such that T' commutes with $\mathcal{M}p'$, and denote by T' the extension to $\mathcal{H}$ by setting it 0 on $(\mathbb{1}_{\mathcal{H}} - p')(\mathcal{H})$. If $T \in \mathcal{M}$, then

$$T'T = T'p'T = T'(Tp') = (Tp')T' = TT'.$$

Therefore $T' \in \mathcal{M}'$. Since, clearly, $T' = p'T'p'$, we have $T' \in p'\mathcal{M}'p'$. Conversely, if $p'T'p' \in p'\mathcal{M}'p'$ and $Tp' \in \mathcal{M}p'$, then

$$(p'T'p')(Tp') = p'T'Tp' = p'TT'p' = (Tp')(p'T'p').$$

Therefore $p'\mathcal{M}'p'$ is the commutant of $\mathcal{M}p'$.

For the second step, note that the only projections in $p'\mathcal{M}'p'$ are 0 and p', since p' is minimal. Since a von Neumann algebra is the closed linear span of its projections, it follows that $p'\mathcal{M}'p' = \mathbb{C}p' = \mathbb{C}\mathbb{1}_{\mathcal{K}}$. As $\mathcal{M}p'$ is a von Neumann algebra with trivial commutant, we find $\mathcal{M}p' = \mathcal{B}(\mathcal{K})$.

Combining our findings, we find a $*$-isomorphism $\Phi : \mathcal{M} \to \mathcal{B}(\mathcal{K})$. Note that all orthogonal projections $\Phi(p_i)$ are minimal, and thus rank 1 projections in $\mathcal{B}(\mathcal{K})$. Since the q orthogonal rank-one projections $\Phi(p_i)$ add up to the identity, we find that $\mathcal{K} \cong \mathbb{C}^q$, and thus $\mathcal{M} \cong M_q(\mathbb{C})$. □

Next, we show how this implies a spatial isomorphism onto $\mathbb{1}_{M_d(\mathbb{C})} \otimes M_q(\mathbb{C})$, where $d = \dim(\mathcal{H})/q$.

Lemma A.4.2. *Let $\mathcal{M} \subseteq \mathbb{B}(\mathcal{H})$ be a factor, where $\mathcal{H}$ is a finite dimensional Hilbert space. Then there exists a unitary $u \in \mathbb{B}(\mathcal{H})$ such that $u\mathcal{M}u^* = \mathbb{1}_{M_d(\mathbb{C})} \otimes M_q(\mathbb{C})$ for some $d, q \in \mathbb{N}$ (and thus $\mathcal{H} \cong \mathbb{C}^{dq} \cong \mathbb{C}^d \otimes C^q$).*

Proof. Let p be any of the projections $p_1, \ldots, p_n$ from the previous proof. First, we show that for $x \in p(\mathcal{H})$, the projection on $\mathcal{H}_x := [\mathcal{M}x]$ is a minimal projection in $\mathcal{M}'$ (which we can then use for the previous proof). This is essentially Proposition 6.4.4 in [12]

Let $x \in p(\mathcal{H})$ be non-zero and let $G \in \mathcal{B}(p(\mathcal{H}))$ be the (minimal) projection on the one-dimensional span of x. Switching the roles of $\mathcal{M}$ and $\mathcal{M}'$, we can use the same proof as before to show that $\mathcal{M}'p = \mathcal{B}(p(\mathcal{H}))$, therefore $G \in \mathcal{M}'p$. Also as before, we can show that the map $\mathcal{M}' \to \mathcal{M}'p$ given by $T' \mapsto T'p$ is a $*$-isomorphism; this means that there is some minimal projection $G' \in \mathcal{M}'$ such that $G'p = G$.

We have $G'x = G'px = Gx = x$. So if $T \in \mathcal{M}$, then $Tx = TG'x = G'Tx \in G'(\mathcal{H})$. This means that $\mathcal{H}_x \subseteq G'(\mathcal{H})$. If we define p' as the projection on $\mathcal{H}_x$, then it follows that $p' \leq G'$. Since

$\mathcal{H}_x$ is invariant under $\mathcal{M}$, it follows that $p' \in \mathcal{M}'$. Since G' is a minimal projection in $\mathcal{M}'$ and $p' \neq 0$, it follows by the claim that $p' = G'$ and thus p' is a minimal projection in $\mathcal{M}'$. By the previous proof, we conclude that

$$\text{if } x \in p(\mathcal{H}), \text{ then } \mathcal{H}_x \cong \mathbb{C}^q \text{ and } \mathcal{M} \cong \mathbb{B}(\mathcal{H}_x). \tag{13}$$

We now use a trick similar to the one in Theorem 5.1.3 in Murphy. Let $\Lambda = \{x_1, \ldots, x_d\}$ be a maximal set of non-zero elements of $\bigcup_{i=1}^q p_i(\mathcal{H})$ such that the spaces $\mathcal{H}_x$ are orthogonal for all $x \in \Lambda$ (in infinite dimensions this would require Zorn's Lemma, but in the finite dimensional case we do not need it).

Let $y \in \left(\bigcup_{x\in\Lambda} \mathcal{H}_x\right)^{\perp}$. Then for $x \in \Lambda$ and $a, b \in \mathcal{A}$ arbitrarily, we have

$$\langle ay, bx\rangle = \langle y, a^*bx\rangle = 0.$$

Therefore $\mathcal{H}_y$ is orthogonal to $\mathcal{H}_x$. Let $i \in \{1, \ldots, q\}$. Since $\mathcal{H}_{p_i(y)} \subseteq \mathcal{H}_y$ for all i, this means that $\mathcal{H}_{p_i(y)}$ is orthogonal to $\mathcal{H}_x$ for all $x \in \Gamma$.
But due to our maximality assumption, this means that $p_i(y) = 0$ for all $i = 1, \ldots, q$. Since $\sum_{i=1}^q p_i = \mathbb{1}_{\mathcal{H}}$, this means that $y = 0$. Therefore we can write $\mathcal{H} \cong \bigoplus_{i=1}^d \mathcal{H}_{x_i}$ as the direct sum of $\mathcal{M}$-invariant orthogonal subspaces. Through a suitable basis transformation, we can thus write $T = \bigoplus_{i=1}^d T_i$ for $T \in \mathcal{M}$, where $T_i \in \mathbb{B}(\mathcal{H}_{x_i})$.

By (13) we have that $\mathcal{M} \cong \mathbb{B}(\mathcal{H}_x) \cong M_q(\mathbb{C})$ for each $x \in \Gamma$. This gives automorphisms $\varphi_{i,j} : M_q(\mathbb{C}) \to M_q(\mathbb{C})$ such that an element $T = \bigoplus_{i=1}^d T_i$ has to satisfy $T_j = \varphi_{i,j}T_i$. All automorphisms of $M_q(\mathbb{C})$ are inner, so there are unitaries $u_{i,j}$ such that $T_j = u_{i,j}^* T_i u_{i,j}$. Now the unitary $\bigoplus_{i=1}^d u_{i,1} \in \mathbb{B}(\mathcal{H})$ (where $u_{1,1}$ is the identity) gives rise to a spatial $*$-isomorphism, which combined with the earlier basis transformation gives the spatial $*$-isomorphism

$$\mathcal{M} \cong \{\bigoplus_{i=1}^d T : T \in M_q(\mathbb{C})\} = \mathbb{1}_{M_p(\mathbb{C})} \otimes M_q(\mathbb{C}).$$

□

A.5 Kirchberg equivalent statements

The Kirchberg conjecture on tensor products of maximal C^*-group algebras can be formulated in several equivalent ways. Here are a few:

Theorem A.5.1. *The following statements are equivalent.*

1. $C^*(\mathbb{F}_2) \otimes_{\min} C^*(\mathbb{F}_2) = C^*(\mathbb{F}_2) \otimes_{\max} C^*(\mathbb{F}_2)$
2. $C^*(\mathbb{F}_k) \otimes_{\min} C^*(\mathbb{F}_k) = C^*(\mathbb{F}_k) \otimes_{\max} C^*(\mathbb{F}_k)$ *for any* $k \geq 2$ *or* $k = \infty$.
3. *For some* $k, m \geq 2$, $(k, m) \neq (2, 2)$ *we have*
$$*_{x=1}^k \ell_\infty^m \otimes_{\min} *_{x=1}^k \ell_\infty^m = *_{x=1}^k \ell_\infty^m \otimes_{\max} *_{x=1}^k \ell_\infty^m.$$

We start with several Lemmas. We state the first two without proof. The first one leads to the method of showing that groups are subgroups of each other to conclude that the tensor norms are equal.

Lemma A.5.2. [7, C.4] *Let $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ be subgroups. Then*

$$C^*(G_1) \otimes_{\min} C^*(G_2) = C^*(G_1) \otimes_{\max} C^*(G_2)$$

implies that

$$C^*(H_1) \otimes_{\min} C^*(H_2) = C^*(H_1) \otimes_{\max} C^*(H_2).$$

The second lemma gives a way to check if the subgroup generated by specific elements is free.

Lemma A.5.3 (Ping-pong lemma)**.** *Let G be a group acting on a set X and $g_1, \dots, g_k$ be elements of infinite order. Suppose there are non-empty pairwise disjoint subsets*

$$X_1^+, \dots, X_k^+, \text{ and } X_1^-, \dots, X_k^-$$

such that

$$g_i(X \setminus X_i^-) \subseteq X_i^+$$
$$g_i^{-1}(X \setminus X_i^+) \subseteq X_i^-$$

for $i = 1, \dots, k$, then the subgroup H generated by the g_i is free (with generators $g_1, \dots, g_k$).

Proofs can be found in many places, such as [3, II.B] (or even Wikipedia).

Lemma A.5.4. *For any $k \in \mathbb{N}$ or $k = \infty$, the group $\mathbb{F}_2$ has a subgroup isomorphic to $\mathbb{F}_k$.*

Proof. Let a and b be the generators of $\mathbb{F}_2$. Then the elements

$$g_i = a^i b^{-i}, \quad i = 1, \dots, k$$

(where $i \in \mathbb{N}$ if $k = \infty$) generate a subgroup of $\mathbb{F}_2$. We can consider this subgroup to be acting on $\mathbb{F}_2$ by left multiplication. Define X_i^- and X_i^+ as the classes of words of reduced form $a^i x$ respectively $b^i x$. These subsets satisfy the conditions of the Ping-pong lemma, so therefore the group generated by the g_i is a free group with k generators, and therefore isomorphic to $\mathbb{F}_k$. □

Lemma A.5.5. *The Fourier transform gives a $*$-isomorphism $*_{x=1}^k \ell_\infty^m \cong C^*(\mathbb{Z}_m * \dots * \mathbb{Z}_m)$.*

Proof. In Lemma 7.2.5, we showed that $\ell_\infty^m \cong C^*(\mathbb{Z}_m)$. Taking the free product, we get $*_{x=1}^k \ell_\infty^m \cong *_{x=1}^k C^*(\mathbb{Z}_m)$. Upon writing out what elements from the latter space look like, we see that this space is the same as $C^*(\mathbb{Z}_m * \dots * \mathbb{Z}_m)$. □

Lemma A.5.6. *Any group $G = *_{x=1}^k \mathbb{Z}_m$ for $k, m \geq 2$, $(k, m) \neq (2, 2)$ has a subgroup isomorphic to $\mathbb{F}_2$.*

Proof. Note that for any $k \geq 2$, $\mathbb{Z}_m * \mathbb{Z}_m$ is a subgroup of $*_{x=1}^k \mathbb{Z}_m$. Thus, it suffices to show that $\mathbb{Z}_m * \mathbb{Z}_m$ has a subgroup isomorphic to $\mathbb{F}_2$. However, this will only work for $m \geq 3$. In that case, let a and b be the generators of the first respectively second instance of $\mathbb{Z}_m$. We define elements $g_1 = ab$, $g_2 = ba$ and sets $X_1^-, X_1^+, X_2^-, X_2^+$ as classes of words of reduced forms respectively $b^{-1}x, ax, a^{-1}x, bx$. These satisfy the conditions of the ping-pong lemma, thus the subgroup generated by $\{g_1, g_2\}$ is isomorphic to $\mathbb{F}_2$.

For $m = 2$, the ping-pong lemma cannot be applied in the same way since $a = a^{-1}$ (hence the exclusion of the case $(k, m) \neq (2, 2)$). Therefore we can only prove this case for $k \geq 3$; here it suffices to consider $k = 3$, i.e. $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$. Let a, b, c be the respective generators. Now, we define $g_1 = abc$ and $g_2 = acb$ and sets $X_1^-, X_1^+, X_2^-, X_2^+$ as classes of words of reduced forms respectively cbx, abx, bcx, acx. Again, these satisfy the conditions of the ping-pong lemma, thus the elements g_1, g_2 generate the required free subgroup isomorphic to $\mathbb{F}_2$. □

In addition to these lemmas, we will need some results from section 7.

Proof of Theorem A.5.1. 1. $\Rightarrow$ 2. This is a combination of lemmas A.5.2 and A.5.4.

2. $\Rightarrow$ 3. Let $k, m \in \mathbb{N}$. Note that $*_{x=1}^k \ell_\infty^m$ is separable. Indeed, the set of finite words in $\mathbb{Q}^m$ is dense, and since it corresponds to finite $\mathbb{Q}^m$-valued sequences, it is also countable. Therefore, by Corollary 7.2.1, we have $*_{x=1}^k \ell_\infty^m \cong C^*(\mathbb{F}_k)/J$ for some countable free group $\mathbb{F}_k$ and some ideal J. By Theorem 7.2.4, there exists a ucp lifting $\psi : *_{x=1}^k \ell_\infty^m \to C^*(\mathbb{F}_k)$.

Now note that by Theorem A.2.7, the map $\psi \otimes \psi : *_{x=1}^k \ell_\infty^m \otimes_{\min} *_{x=1}^k \ell_\infty^m \to C^*(\mathbb{F}_k) \otimes_{\min} C^*(\mathbb{F}_k)$ is ucp (specifically contractive). If $\pi : C^*(\mathbb{F}_k) \to *_{x=1}^k \ell_\infty^m$ is the quotient map, then the map

$$*_{x=1}^k \ell_\infty^m \otimes_{\min} *_{x=1}^k \ell_\infty^m \xrightarrow{\psi\otimes\psi} C^*(\mathbb{F}_k) \otimes_{\min} C^*(\mathbb{F}_k) = C^*(\mathbb{F}_k) \otimes_{\max} C^*(\mathbb{F}_k) \xrightarrow{\pi\otimes\pi} *_{x=1}^k \ell_\infty^m \otimes_{\max} *_{x=1}^k \ell_\infty^m$$

is a contraction. Since the minimum norm is obviously smaller than the maximum norm, this means that both are equal, i.e. $*_{x=1}^k \ell_\infty^m \otimes_{\min} *_{x=1}^k \ell_\infty^m = *_{x=1}^k \ell_\infty^m \otimes_{\max} *_{x=1}^k \ell_\infty^m$. (Compare to the proof of Corollary 7.2.2)

3. $\Rightarrow$ 1. This is a combination of lemmas A.5.5, A.5.6 and A.5.2. $\square$

A.6 Theory behind QWEP

Theorem A.6.1. *For any countable free group $\mathbb{F}_n$ (i.e. $n \in \mathbb{N}$ or $n = \infty$), the group C^*-algebra $C^*(\mathbb{F}_n)$ has the LP.*

Proof. We prove only that every $*$-homomorphism $\theta : C^*(\mathbb{F}_n) \to \mathcal{B}/J$ is liftable. The generalisation to ucp maps requires some heavy machinery such as the Kasparov dilation theorem and the noncommutative Tietze extension theorem; we refer to [18, Thm. 3.8].

Let us assume $n = \infty$ (the case $n \in \mathbb{N}$ is the same). We begin by choosing representants $x_1, x_2, \dots \in \mathcal{B}$ from the equivalence classes $\theta(U_1), \theta(U_2), \dots$. Since the $\theta(U_i)$ have norm ≤ 1, we can also take the x_i to have norm ≤ 1. Our aim is to extend this choice to a u.c.p. lifting of θ. The easiest way would be to extend to a $*$-homomorphism, but the problem is that the x_i need not be unitary. Therefore, we use what are called unitary dilations:

$$\hat{x}_i = \begin{pmatrix} x_i & (1 - x_i x_i^*)^{1/2} \\ (1 - x_i^* x_i)^{1/2} & -x_i^* \end{pmatrix} \in M_2(\mathcal{B}).$$

One can easily check that these are indeed unitary. Therefore, we can extend the map $U_i \mapsto \hat{x}_i$ to a unitary representation $\mathbb{F}_n \to M_2(\mathcal{B})$. By the universal property of the maximal group C^*-algebra, there is a $*$-homomorphism $\rho : C^*(\mathbb{F}_n) \to M_2(\mathcal{B})$ extending this map.

We now claim that the (1,1)-corner of ρ is a ucp lifting of θ. The completely positive part follows from the fact that the upper left square matrix of a positive matrix is positive. To show that it is a lifting, we need to prove that $q \circ \rho_{(1,1)} = \theta$, where $q : \mathcal{B} \to \mathcal{B}/J$ is the quotient map. This is clear for linear combinations of the generators U_i. We show that it is also true for a product $U_i U_j$. Indeed,

$$\begin{aligned} q(\rho(U_i U_j)_{(1,1)}) &= q(x_i x_j + (1 - x_i x_i^*)^{1/2}(1 - x_j^* x_j)^{1/2}) \\ &= \theta(U_i U_j) + q(1 - x_i x_i^*)^{1/2} q(1 - x_j^* x_j)^{1/2} = \theta(U_i U_j) \end{aligned}$$

(since $q(x_i^* x_i) = U_i U_i^* = 1$). $\square$

Before stating the next theorem we introduce the notion of *completely bounded maps*, which is similar in spirit to completely positive maps. Similar to completely positive maps, completley bounded maps can be defined on a more general sort of space (operator spaces) but we will not need that here.

Definition A.6.2. A map $\Phi : \mathcal{A} \to \mathcal{B}$ is called *completely bounded* if

$$\|\Phi\|_{\mathrm{cb}} := \sup_{n\in\mathbb{N}} \|\mathbb{1}_{M_n(\mathbb{C})} \otimes \Phi\| < \infty.$$

If $\|\Phi\|_{\mathrm{cb}} \leq 1$ we call Φ *completely contractive.*

Proposition A.6.3. *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be completely bounded and let $\mathcal{H}$ be any Hilbert space. Then*

$$\|\mathbb{1}_{\mathbb{B}(\mathcal{H})} \otimes \Phi\| \leq \|\Phi\|_{\mathrm{cb}}.$$

Proof. Let Λ be the net of finite dimensional projections on $\mathcal{H}$ and let $\mathcal{B} \subseteq \mathbb{B}(\mathcal{H}_B)$ be a faithful representation. Let $z \in \mathbb{B}(\mathcal{H}) \otimes \mathcal{A}$. We claim that

$$\|(\mathbb{1}_{\mathbb{B}(\mathcal{H})} \otimes \Phi)(z)\| = \lim_{p\in\Lambda} \|(p \otimes \mathbb{1}_{\mathcal{H}_B})(\mathbb{1}_{\mathbb{B}(\mathcal{H})} \otimes \Phi)(z)(p \otimes \mathbb{1}_{\mathcal{H}_B})\|.$$

Let $\varepsilon > 0$. Let $\xi = \sum_i \xi_i \eta_i \in \mathcal{H} \otimes \mathcal{H}_B$ be such that

$$|(\mathbb{1}_{\mathbb{B}(\mathcal{H})} \otimes \Phi)(z)\xi\| \geq |(\mathbb{1}_{\mathbb{B}(\mathcal{H})} \otimes \Phi)(z)\| - \varepsilon,$$

and write $(\mathbb{1}_{\mathbb{B}(\mathcal{H})} \otimes \Phi)(z)\xi = \hat{\xi} = \sum_i \hat{\xi}_i \hat{\eta}_i$. Now let p be a finite rank projection such that $\xi_i, \hat{\xi}_i \in p(\mathcal{H})$ for all i. Then

$$\begin{aligned}
\|(p \otimes \mathbb{1}_{\mathcal{H}_B})(\mathbb{1}_{\mathbb{B}(\mathcal{H})} \otimes \Phi)(z)(p \otimes \mathbb{1}_{\mathcal{H}_B})\| &\geq \|(p \otimes \mathbb{1}_{\mathcal{H}_B})(\mathbb{1}_{\mathbb{B}(\mathcal{H})} \otimes \Phi)(z)(p \otimes \mathbb{1}_{\mathcal{H}_B})\xi\| \\
&= \|(p \otimes \mathbb{1}_{\mathcal{H}_B})(\mathbb{1}_{\mathbb{B}(\mathcal{H})} \otimes \Phi)(z)\xi\| \\
&= \|(\mathbb{1}_{\mathbb{B}(\mathcal{H})} \otimes \Phi)(z)\xi\| \\
&\geq |(\mathbb{1}_{\mathbb{B}(\mathcal{H})} \otimes \Phi)(z)\| - \varepsilon
\end{aligned}$$

Hence the claim holds. Thus we have

$$\begin{aligned}
\|(\mathbb{1}_{\mathbb{B}(\mathcal{H})} \otimes \Phi)(z)\| &= \lim_{p\in\Lambda} \|(p \otimes \mathbb{1}_{\mathcal{H}_B})(\mathbb{1}_{\mathbb{B}(\mathcal{H})} \otimes \Phi)(z)(p \otimes \mathbb{1}_{\mathcal{H}_B})\| \\
&= \lim_{p\in\Lambda} \|(\mathbb{1}_{\mathbb{B}(\mathcal{H})} \otimes \Phi)((p \otimes \mathbb{1}_{\mathcal{H}_A})z(p \otimes \mathbb{1}_{\mathcal{H}_A}))\| \\
&= \lim_{p\in\Lambda} \|(\mathbb{1}_{\mathbb{B}(p(\mathcal{H}))} \otimes \Phi)((p \otimes \mathbb{1}_{\mathcal{H}_A})z(p \otimes \mathbb{1}_{\mathcal{H}_A}))\| \\
&\leq \|\Phi\|_{\mathrm{cb}} \lim_{p\in\Lambda} \|(p \otimes \mathbb{1}_{\mathcal{H}_A})z(p \otimes \mathbb{1}_{\mathcal{H}_A})\| \leq \|\Phi\|_{\mathrm{cb}}\|z\|
\end{aligned}$$

where the second step holds by writing z explicitly as sum of elementary tensors and calculating out. This concludes the proof. ☐

Theorem A.6.4. *For any free group $\mathbb{F}$ and any Hilbert space $\mathcal{H}$, we have*

$$C^*(\mathbb{F}) \otimes_{\max} \mathbb{B}(\mathcal{H}) = C^*(\mathbb{F}) \otimes_{\min} \mathbb{B}(\mathcal{H}).$$

We give the proof from [24]. We start with some preparations before stating one lemma. Let $(U_i)_{i\in I}$ be the unitary generators of $C^*(\mathbb{F})$. We define $S = \mathrm{Span}\{U_i \otimes x; x \in \mathcal{B}(\mathcal{H})\}$. Note that S is of the required form of Theorem A.3.3. Let $z = \sum_{i=1}^n U_i \otimes x_i \in S$. We define an associated operator $T_z : \ell_\infty^n \to \mathcal{B}(\mathcal{H})$ by $(\alpha_i)_{i=1}^n \mapsto \sum_{i=1}^n \alpha_i x_i$. The first lemma states that there is a duality relation between the elements z and the operators T_z.

Lemma A.6.5. *For any $z = \sum_{i=1}^n U_i \otimes x_i$, the norm equality $\|z\|_{\min} = \|T_z\|_{\mathrm{cb}}$ holds.*

Proof. Let $E = \mathrm{Span}\{U_i : i = 1, \dots, n\}$. We note that $z = (\mathbb{1}_E \otimes T_z)((U_i)_{i=1}^n)$. Since $\|(U_i)_{i=1}^n\|_{E \otimes \ell_\infty^n} = 1$, we have by Proposition A.6.3

$$\|z\|_{\min} = \|(\mathbb{1}_E \otimes T_z)((U_i)_{i=1}^n)\| \leq \|T_z\|_{\mathrm{cb}}.$$

For the converse inequality, let $k \in \mathbb{N}$ and $(a_i)_{i=1}^n \in M_k(\mathbb{C}) \otimes \ell_\infty^n$. Let $\theta : E \to M_k(\mathbb{C})$ be defined as $\theta(U_i) = a_i$. Then we have

$$(\mathbb{1}_{M_k(\mathbb{C})} \otimes T_z)((a_i)_{i=1}^n) = \sum_{i=1}^n a_i \otimes x_i = \sum_{i=1}^n \theta(U_i) \otimes x_i = (\theta \otimes \mathbb{1}_{\mathbb{B}(\mathcal{H})})(z).$$

Therefore we are done if we can prove that θ is completely contractive. We do this via a trick by Ozawa: let $\hat{a}_i$ be the unitary dilations of a_i. Similarly to Theorem A.6.1, the map φ given by $U_i \mapsto \hat{a}_0^{-1}\hat{a}_i$ extends to a $*$-homomorphism on $C^*(\mathbb{F})$. Note that $\theta(U_i) = (\hat{a}_i)_{(1,1)} = (\hat{a}_0\varphi(U_i))_{(1,1)}$. Since $*$-homomorphisms are completely contractive and $\hat{a}_0$ has norm 1, it follows that θ is completely contractive. □

Proof of Theorem A.6.4. Recall that $S = \mathrm{Span}\{U_i \otimes x; x \in \mathbb{B}(\mathcal{H})\}$; by Theorem A.3.3 it suffices to prove that $\|z\|_{\min} = \|z\|_{\max}$ for all $z \in M_n(S)$. Note that S can in fact be written as $\mathrm{Span}\{U_i\} \otimes \mathbb{B}(\mathcal{H})$, so

$$M_n(S) \cong M_n(\mathbb{C}) \otimes \mathrm{Span}\{U_i\} \otimes \mathbb{B}(\mathcal{H}) \cong \mathrm{Span}\{U_i\} \otimes M_n(\mathbb{B}(\mathcal{H})) \cong \mathrm{Span}\{U_i\} \otimes \mathbb{B}(\mathcal{H}^n).$$

But since $\mathcal{H}$ and $\mathcal{H}^n$ have the same cardinality (they are both separable), they are actually isomorphic Hilbert spaces, so we can assume that $z \in S$. Let $\|z\|_{\min} = 1$. Then by the preceding lemma, $\|T_z\|_{cb} = 1$; in other words, T_z is completely contractive. Now, we refer to a generalisation of the Stinespring theorem known as the factorization theorem for completely bounded maps (see [1, Thm. A.6]). This theorem says that we can write $T_z(\alpha) = V^*\pi(\alpha)W$, where $\pi : \ell_\infty^n \to \mathbb{B}(\hat{\mathcal{H}})$ is a $*$-homomorphism and $V, W \in \mathbb{B}(\mathcal{H}, \hat{\mathcal{H}})$ are isometries with $\|V\|\|W\| = \|T_z\|_{\mathrm{cb}}$. Again, $\hat{\mathcal{H}}$ can be taken separable and therefore isomorphic to $\mathcal{H}$, so we can assume that $\mathcal{H} = \hat{\mathcal{H}}$.

Next, we define $a_i = V^*\pi(e_i)$ and $b_i = \pi(e_i)W$, so that $x_i = a_i b_i$. Note that $\sum_{i=1}^n a_i a_i^* = V^*\pi(\sum_{i=1}^n e_i)V = V^*\mathbb{1}_{\mathcal{H}}V = \mathbb{1}_{\mathcal{H}}$. Similarly, $\sum_{i=1}^n b_i^* b_i = \mathbb{1}_{\mathcal{H}}$. Then we can estimate as follows:

$$\begin{aligned}
\|z\|_{\max} = \left\|\sum_{i=1}^n U_i \otimes x_i\right\|_{\max} &= \left\|\sum_{i=1}^n (\mathbb{1} \otimes a_i)(U_i \otimes b_i)\right\|_{\max} \\
&\overset{C-S}{\leq} \left\|\sum_{i=1}^n (1 \otimes a_i)(1 \otimes a_i)^*\right\|_{\max}^{1/2} \left\|\sum_{i=1}^n (U_i \otimes b_i)^*(U_i \otimes b_i)\right\|_{\max}^{1/2} \\
&= \left\|1 \otimes \sum_{i=1}^n a_i a_i^*\right\|_{\max}^{1/2} \left\|1 \otimes \sum_{i=1}^n b_i^* b_i\right\|_{\max}^{1/2} = 1.
\end{aligned}$$

And thus $\|z\|_{\max} = \|z\|_{\min} = 1$. This finishes the proof. □

A.7 Triple tensor products

Here, we review in detail some properties about norms on triple tensor products.

Hilbert spaces

If $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C$ are Hilbert spaces, then we can naturally define an inner product on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ by $\langle a_1 \otimes b_2 \otimes c_2 | a_2 \otimes b_2 \otimes c_2 \rangle = \langle a_1 | a_2 \rangle \langle b_1 | b_2 \rangle \langle c_1 | c_2 \rangle$. We denote the Hilbert space completion as $\mathcal{H}_A \hat{\otimes} \mathcal{H}_B \hat{\otimes} \mathcal{H}_C$.

We can also define a Hilbert space by using two consecutive completions on two tensor legs: $\mathcal{H}_A \hat{\otimes} (\mathcal{H}_B \hat{\otimes} \mathcal{H}_C)$. In other words, we first form a Hilbert space $\mathcal{H}_B \hat{\otimes} \mathcal{H}_C$ taking the completion, and then tensor that with the Hilbert space $\mathcal{H}_A$ using another completion. One can easily check that the inner product defined this way is the same as before. To conclude that both Hilbert spaces are the same, we need to show that $\mathcal{H}_A \otimes (\mathcal{H}_B \hat{\otimes} \mathcal{H}_C) \subseteq \mathcal{H}_A \hat{\otimes} \mathcal{H}_B \hat{\otimes} \mathcal{H}_C$.

If $z = \sum_{i \in I} a_i \otimes y_i$, for $a_i \in \mathcal{H}_A$ and $y_i \in \mathcal{H}_B \hat{\otimes} \mathcal{H}_C$, then we can find sequences $(y_i^n)_{n \in \mathbb{N}} \in \mathcal{H}_B \otimes \mathcal{H}_C$ such that $y_i^n \to y_i$ in $\mathcal{H}_B \hat{\otimes} \mathcal{H}_C$. Thus

$$\begin{aligned}
\|z - \sum_{i \in I} a_i \otimes y_i^n\|_{\mathcal{H}_A \hat{\otimes} \mathcal{H}_B \hat{\otimes} \mathcal{H}_C} &\leq \sum \|a_i \otimes (y_i - y_i^n)\|_{\mathcal{H}_A \hat{\otimes} \mathcal{H}_B \hat{\otimes} \mathcal{H}_C} \\
&= \sum \|a_i \otimes (y_i - y_i^n)\|_{\mathcal{H}_A \hat{\otimes} (\mathcal{H}_B \hat{\otimes} \mathcal{H}_C)} \\
&= \sum \|a_i\| \|y_i - y_i^n\| \to 0
\end{aligned}$$

Therefore $z \in \mathcal{H}_A \hat{\otimes} \mathcal{H}_B \hat{\otimes} \mathcal{H}_C$ and thus $\mathcal{H}_A \otimes (\mathcal{H}_B \hat{\otimes} \mathcal{H}_C) \subseteq \mathcal{H}_A \hat{\otimes} \mathcal{H}_B \hat{\otimes} \mathcal{H}_C$ holds . Finally, we take the Hilbert space completion of the left tensor leg; since the associated norm is the same as the norm on $\mathcal{H}_A \hat{\otimes} \mathcal{H}_B \hat{\otimes} \mathcal{H}_C$, it follows that

$$\mathcal{H}_A \hat{\otimes} \mathcal{H}_B \hat{\otimes} \mathcal{H}_C = \mathcal{H}_A \hat{\otimes} (\mathcal{H}_B \hat{\otimes} \mathcal{H}_C) = (\mathcal{H}_A \hat{\otimes} \mathcal{H}_B) \hat{\otimes} \mathcal{H}_C.$$

(The last equality follows by symmetry).

Minimal tensor norm on C^*-algebras

We assume for simplicity that we have 3 C^*-algebras $\mathcal{A} \subseteq \mathbb{B}(\mathcal{H}_A)$, $\mathcal{B} \subseteq \mathbb{B}(\mathcal{H}_B)$ and $\mathcal{C} \subseteq \mathbb{B}(\mathcal{H}_C)$ (otherwise such an identification can be found through Gelfand-Naimark).

The straightforward way to define the minimal tensor norm on the triple tensor product $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$ is via the embedding $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \subseteq \mathbb{B}(\mathcal{H}_A \hat{\otimes} \mathcal{H}_B \hat{\otimes} \mathcal{H}_C)$; we define the completion as $\mathcal{A} \otimes_{\min} \mathcal{B} \otimes_{\min} \mathcal{C}$. On the other hand, we can proceed again by doing two C^*-completion steps of 2-leg tensor norms. First, $\mathcal{A} \otimes (\mathcal{B} \otimes_{\min} \mathcal{C})$ is the tensor product of C^*-algebras on $\mathbb{B}(\mathcal{H}_A)$ and $\mathbb{B}(\mathcal{H}_B \hat{\otimes} \mathcal{H}_C)$, respectively. After the final C^*-completion, we obtain the space

$$\mathcal{A} \otimes_{\min} (\mathcal{B} \otimes_{\min} \mathcal{C}) \subseteq \mathbb{B}(\mathcal{H}_A \hat{\otimes} (\mathcal{H}_B \hat{\otimes} \mathcal{H}_C)) = \mathbb{B}(\mathcal{H}_A \hat{\otimes} \mathcal{H}_B \hat{\otimes} \mathcal{H}_C)$$

where in the last step we used the Hilbert space result. We see that the norms on both spaces are the same. Again, we need to check whether $\mathcal{A} \otimes (\mathcal{B} \otimes_{\min} \mathcal{C}) \subseteq \mathcal{A} \otimes_{\min} \mathcal{B} \otimes_{\min} \mathcal{C}$; the proof is precisely the same as in the Hilbert space setting. Therefore, we can conclude (by symmetry) that

$$\mathcal{A} \otimes_{\min} \mathcal{B} \otimes_{\min} \mathcal{C} = \mathcal{A} \otimes_{\min} (\mathcal{B} \otimes_{\min} \mathcal{C}) = (\mathcal{A} \otimes_{\min} \mathcal{B}) \otimes_{\min} \mathcal{C}.$$

Maximal tensor norm on C^*-algebras

Let $\mathcal{A} \subseteq \mathbb{B}(\mathcal{H}_A)$, $\mathcal{B} \subseteq \mathbb{B}(\mathcal{H}_B)$ and $\mathcal{C} \subseteq \mathbb{B}(\mathcal{H}_C)$ again be (unital) C^*-algebras. The maximal norm on $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$ is defined as $\| \cdot \|_{\max} = \sup_\pi \pi(\cdot)$, where the supremum is taken over all representations $\pi : \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \to \mathbb{B}(\mathcal{H})$ of the algebraic tensor product. The completion is denoted as $\mathcal{A} \otimes_{\max} \mathcal{B} \otimes_{\max} \mathcal{C}$.

We prove that $\mathcal{A} \otimes_{\max} \mathcal{B} \otimes_{\max} \mathcal{C} = \mathcal{A} \otimes_{\max} (\mathcal{B} \otimes_{\max} \mathcal{C})$ by constructing $*$-homomorphisms from left to right and right to left that are each other's inverses.

First, define $\pi : \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \to \mathcal{A} \otimes_{\max} (\mathcal{B} \otimes_{\max} \mathcal{C})$ by the natural action $a \otimes b \otimes c \mapsto a \otimes (b \otimes c)$. Upon a choice of faithful representation $\mathcal{A} \otimes_{\max} (\mathcal{B} \otimes_{\max} \mathcal{C}) \subseteq \mathbb{B}(\mathcal{H})$, π defines a representation of $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$, so $\|\pi(z)\| \leq \|z\|_{\max}$ for all $z \in \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$. In other words, π is a contraction; this means that π can be extended to a (contractive) $*$-homomorphism on $\mathcal{A} \otimes_{\max} \mathcal{B} \otimes_{\max} \mathcal{C}$.

For the inverse mapping we need to take a two-step approach and use the universal property of the maximal tensor norm. We start with a map $\varphi' : \mathcal{B} \otimes \mathcal{C} \to \mathcal{A} \otimes_{\max} \mathcal{B} \otimes_{\max} \mathcal{C}$ given by $\varphi'(x) = \mathbb{1}_A \otimes x$. Then upon a choice of faithful representation $\mathcal{A} \otimes_{\max} \mathcal{B} \otimes_{\max} \mathcal{C} \subseteq \mathbb{B}(\mathcal{H})$, φ defines a representation of $\mathcal{B} \otimes \mathcal{C}$, so $\|\varphi'(x)\| \leq \|x\|_{\max}$. Therefore, again, φ' is a contraction and can be extended to a $*$-homomorphism on $\mathcal{B} \otimes_{\max} \mathcal{C}$.

Now define $\bar{\varphi} : \mathcal{A} \to \mathcal{A} \otimes_{\max} \mathcal{B} \otimes_{\max} \mathcal{C}$ by $\bar{\varphi}(a) = a \otimes \mathbb{1}_B \otimes \mathbb{1}_C$. Clearly, every element of $\bar{\varphi}(\mathcal{A})$ commutes with every element of $\varphi'(\mathcal{B} \otimes_{\max} \mathcal{C})$. So by the universal property of the maximal tensor norm, there exists a (unique) $*$-homomorphism $\varphi : \mathcal{A} \otimes_{\max} (\mathcal{B} \otimes_{\max} \mathcal{C}) \to \mathcal{A} \otimes_{\max} \mathcal{B} \otimes_{\max} \mathcal{C}$ such that

$$\varphi(a \otimes x) = \bar{\varphi}(a)\varphi'(x).$$

When restricted to $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$, it is clear that $\varphi \circ \pi = \pi \circ \varphi = \mathbb{1}_{\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}}$. By density of $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \subseteq \mathcal{A} \otimes_{\max} \mathcal{B} \otimes_{\max} \mathcal{C}$, we find that $\varphi \circ \pi = \mathbb{1}_{\mathcal{A} \otimes_{\max} \mathcal{B} \otimes_{\max} \mathcal{C}}$. For the converse, we show that $\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$ is dense in $\mathcal{A} \otimes_{\max} (\mathcal{B} \otimes_{\max} \mathcal{C})$. By definition, $\mathcal{A} \otimes (\mathcal{B} \otimes_{\max} \mathcal{C})$ is dense in $\mathcal{A} \otimes_{\max} (\mathcal{B} \otimes_{\max} \mathcal{C})$, hence it suffices to show that $\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$ is dense in $\mathcal{A} \otimes (\mathcal{B} \otimes_{\max} \mathcal{C})$, with respect to the max norm on the outer tensor product.

Indeed, let $\sum_i a_i \otimes x_i \in \mathcal{A} \otimes (\mathcal{B} \otimes_{\max} \mathcal{C})$. Then for each x_i there is a sequence $(y_i^n) \in \mathcal{B} \otimes \mathcal{C}$ such that $y_i^n \to x_i$ with respect to the max norm. By Corollary 2.3.6, we have

$$\| \sum_i a_i \otimes x_i - \sum_i a_i \otimes y_i^n \|_{\max} \leq \sum_i \|a_i\|_A \|x_i - y_i^n\|_{\max} \to 0.$$

Note here that the max norm in the left hand side is the one from the outer tensor product, while the one in the right hand side is from the inner tensor product $\mathcal{B} \otimes_{\max} \mathcal{C}$. This shows that $\pi \circ \varphi = \mathbb{1}_{\mathcal{A} \otimes_{\max} (\mathcal{B} \otimes_{\max} \mathcal{C})}$. Hence we have

$$\mathcal{A} \otimes_{\max} \mathcal{B} \otimes_{\max} \mathcal{C} \cong \mathcal{A} \otimes_{\max} (\mathcal{B} \otimes_{\max} \mathcal{C}) \cong (\mathcal{A} \otimes_{\max} \mathcal{B}) \otimes_{\max} \mathcal{C}.$$

B An introduction to Quantum Mechanics

States

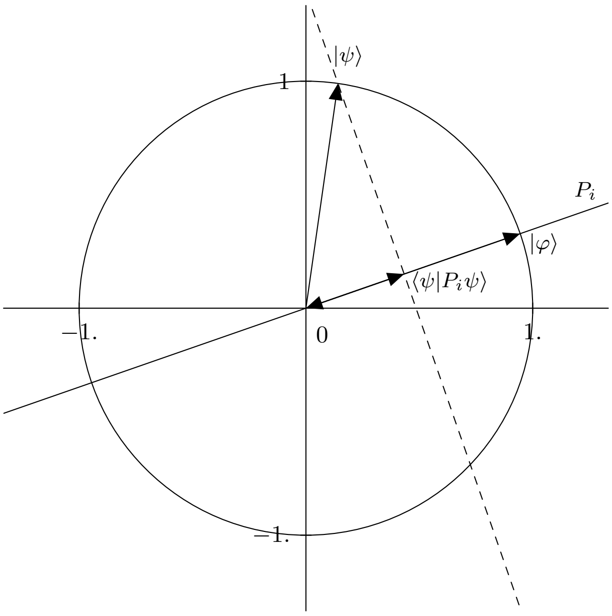
At the beginning, we have *pure states*, represented as unit vectors om some Hilbert space $\mathcal{H}$. In quantum mechanics, they are usually denoted as $|\psi\rangle$, the so-called *ket* notation. A *qubit* is the equivalent of a pure state for $\mathcal{H} = \mathbb{C}^2$.

There is also a corresponding 'bra' notation $\langle\psi|$, which stands for the linear functional that sends a state $|\varphi\rangle$ to the inner product $\langle\psi|\varphi\rangle$. Note how the notations 'morph' to become the inner product. This is also the idea behind the 'bra-ket' word choice; once put together, they form the word 'bracket'. Typically, operators are not put inside the bra/ket but instead written as $A\,|\psi\rangle$ and correspondingly $\langle\psi|\,A^*$.

One can do two things with states: one can either do a measurement, or one can do a unitary operation. A measurement allows us to actually get some concrete information out of a quantum state, at the cost of destroying part of the information. A unitary operation changes a quantum state without destroying any information, but we cannot get any concrete information from it. We will not go into operations here.

Measurements

A **measurement** corresponds to a set of orthogonal projections $\{P_1, \ldots, P_n\}$ such that $\sum_{i=1}^n P_i = \mathbb{1}_{\mathcal{H}}$, with an associated set of '**outcomes**' $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Measuring a state $|\psi\rangle$ with this measurement yields one of the results λ_i. Intuitively, the probability of getting a result λ_i depends on how 'close' the state ψ is to the projection range P_i. To be precise, the probability to get a specific λ_i is given by $\|P_i\,|\psi\rangle\,\|^2 = \langle\psi|P_i^* P_i|\psi\rangle = \langle\psi|P_i|\psi\rangle$ (using that a projection satisfies $P_i^* = P_i$ and $P_i^2 = P_i$). This is illustrated in the picture below.



The state after the measurement will collapse to the projection range that resulted from the measurement. In the picture, it would be the unit vector indicated by $|\varphi\rangle$.

We can associate to this measurement a matrix $M = \sum_{i=1}^{n} \lambda_i P_i \in M_n(\mathbb{C})$, called an **observable**. Note that this matrix is self-adjoint, and that its set of eigenvalues is $\{\lambda_1, \dots, \lambda_n\}$ and the corresponding eigenspaces are $\{\mathrm{ran}(P_1), \dots, \mathrm{ran}(P_n)\}$.

In fact, measurements and observables are in one-to-one correspondence; indeed, a self-adjoint matrix M has a spectral decomposition $M = \sum_{i=1}^{m} \lambda_i P_i$, where P_i is the projection on the eigenspace corresponding to the eigenvalue λ_i. So the result of a measurement corresponding to an observable is some eigenvalue of the observable; the state collapses to the corresponding eigenspace. The terms measurement and observables are sometimes used interchangeably.

Entangled states

A state is called bipartite if it is defined on a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ that is a tensor product of Hilbert spaces. These Hilbert spaces are usually attributed to fictional people 'Alice' and 'Bob' who can do measurements or unitary operations on their respective Hilbert spaces. This means that Alice has operators $A_i \in \mathbb{B}(\mathcal{H}_A)$ that, when applied to a state on $\mathcal{H}$, are the identity on the other tensor leg, i.e. $A_i \otimes \mathbb{1}_{\mathcal{H}_B}$. Similarly, Bob has operators $\mathbb{1}_{\mathcal{H}_A} \otimes B_i$.

Sometimes, a state can be seen as a combination of a state on Alice's and a state on Bob's Hilbert space, i.e. $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$. When this is not the case, so when a state cannot be written in tensor product form (which easily happens when taking linear combinations of states of this form), we say that it is *entangled*.

Entangled states can lead to strange, non-local behaviour: say Alice and Bob are so far apart that they cannot communicate in any way, and Alice performs a measurement on an entangled state. Then this will have influence on measurements that Bob does immediately after that, even though no information has passed between them! This is what makes quantum mechanics so 'scary' but at the same time so interesting - it leads to a a whole new realm of possibilities, like the quantum correlations (see Chapter 3).

Mixed states

Next, we turn our attention to *mixed states* and *density matrices*. A mixed state is, so to speak, a probability distribution over a set of pure states. Think of this as the situation where we don't know in which pure state the system is, but we only know the probabilities that the system is in a certain state.

In the case of finite dimensional Hilbert spaces, a common way to describe a mixed state is through a *density matrix*. To show how this works, we first describe how a pure state can be written down as a density matrix. If $|\psi\rangle$ is a pure state, then its corresponding density matrix is $|\psi\rangle\langle\psi|$, i.e. the projection on the one-dimensional subspace spanned by $|\psi\rangle$. (Indeed: if φ is another pure state, then $|\psi\rangle\langle\psi| \, |\varphi\rangle = |\psi\rangle \, \langle\psi|\varphi\rangle = \langle\psi|\varphi\rangle \, |\psi\rangle$, which corresponds to the projection of $|\varphi\rangle$ on the one-dimensional subspace spanned by $|\psi\rangle$).

Since it is the projection on a one-dimensional subspace, the trace of such a matrix $|\psi\rangle\langle\psi|$ is 1. Now, if a mixed state is in state $|\psi_j\rangle$ with probability p_j, then its density matrix is given by $\sum_{j=1}^{m} p_j \, |\psi_j\rangle\langle\psi_j|$. One can check that the trace of this matrix is still 1, since the trace is linear and the p_j's add up to 1. Also, since a projection is positive, this matrix has to be positive definite. This leads to the general definition of a density matrix:

Definition B.0.1. A *density matrix* $\Phi \in \mathcal{L}(\mathbb{C}^n) = M_n(\mathbb{C})$ is a matrix such that:

1. $\mathrm{Tr}(\Phi) = 1$,
2. $\Phi \geq 0$ (positive definite).

Note that for a pure state, we have that (using the cyclic property of the trace):

$$\mathrm{Tr}(|\psi\rangle\langle\psi| \cdot P_i) = \mathrm{Tr}(\langle\psi|P_i|\psi\rangle) = \langle\psi|P_i|\psi\rangle \, .$$

So we can define the probability of a measurement of $|\psi\rangle$ to yield λ_i by the trace. We can extend this to density matrices as follows: doing a measurement of a mixed state with density matrix $\Phi = \sum_{j=1}^{n} p_i \, |\psi_j\rangle\langle\psi_j|$ should yield λ_i with probability

$$\sum_j p_j \, \langle\psi_j|P_i\psi_j\rangle = \sum_j p_j \mathrm{Tr}(|\psi\rangle\langle\psi| \cdot P_i) = \mathrm{Tr}\left(\sum_j p_j \, |\psi\rangle\langle\psi| \cdot P_i\right) = \mathrm{Tr}(\Phi \cdot P_i).$$

POVMs

The definition of measurement given above is also called a *projective measurement*. There exists a more general form of measurement called a *Positive Operator-Valued Measurement*, or POVM in short. A POVM is defined as a set of positive operators $\{A_1, \ldots, A_n\}$ such that $\sum_{i=1}^{n} A_i = \mathbb{1}_{\mathcal{H}}$. Every operator A_i still corresponds to an outcome λ_i, which is attained upon measuring state $|\psi\rangle$ with probability $\langle\psi|A_i|\psi\rangle$.

The physical intuition behind POVMs might not be clear immediately. One way that they naturally appear has to do with restrictions from measurements on a bipartite state to one of the tensor legs. If ρ is a bipartite mixed state on a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, then we can 'trace out' Bob's part of that state by applying $\mathbb{1}_{\mathcal{H}_A} \otimes \mathrm{Tr}$. The resulting state, which describes Alice's part of the state, is denoted by ρ_A. Now imagine we have a projective measurement on $\mathcal{H}$ and we want to describe what it does on Alice's part of the state. We can define this again by tracing out, but the outcome is no longer a projective measurement; however, it is still a POVM.

In fact, a converse statement to this construction holds true: every POVM can be extended to a projective measurement by 'enlarging' the state, i.e. adding an ancilla. This is known as Naimark's dilation theorem, and can be seen as a consequence of Stinespring's dilation theorem (cf. Theorem A.1.2 and Propositions 6.1.2 and 6.1.3).

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