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DOI
10.1007/s10955-025-03393-1

Publication date
2025

Document Version
Final published version

Published in
Journal of Statistical Physics

Citation (APA)
Giardinà, C., Redig, F., & van Tol, B. (2025). Intertwining and Propagation of Mixtures for Generalized KMP Models and Harmonic Models. *Journal of Statistical Physics*, *192*(2), Article 21. https://doi.org/10.1007/s10955-025-03393-1

Journal of Statistical Physics (2025) 192:21
https://doi.org/10.1007/s10955-025-03393-1

Intertwining and Propagation of Mixtures for Generalized KMP Models and Harmonic Models

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Received: 19 June 2024 / Accepted: 2 January 2025 / Published online: 27 January 2025

Abstract

We study a class of stochastic models of mass transport on discrete vertex set V. For these models, a one-parameter family of homogeneous product measures $\otimes_{i\in V}\nu_\theta$ is reversible. We prove that the set of mixtures of inhomogeneous product measures with equilibrium marginals, i.e., the set of measures of the form

$$\int \Big(\bigotimes_{i\in V}\nu_{\theta_i}\Big)\,\Xi\Big(\prod_{i\in V}d\theta_i\Big)$$

is left invariant by the dynamics in the course of time, and the "mixing measure" Ξ evolves according to a Markov process which we then call "the hidden parameter model". This generalizes results from De Masi et al. (Preprint arXiv:2310.01672, 2023) to a larger class of models and on more general graphs. The class of models includes discrete and continuous generalized KMP models, as well as discrete and continuous harmonic models. The results imply that in all these models, the non-equilibrium steady state of their reservoir driven version is a mixture of product measures where the mixing measure is in turn the stationary state of the corresponding "hidden parameter model". For the boundary-driven harmonic models on the chain $\{1,\dots,N\}$ with nearest neighbor edges, we recover that the stationary measure of the hidden parameter model is the joint distribution of the ordered Dirichlet distribution (cf. Carinci et al., Preprint arXiv:2307.14975, 2023), with a purely probabilistic proof based on a spatial Markov property of the hidden parameter model.

Keywords Non-equilibrium steady state · Mixtures of product states · Intertwining · Harmonic model · KMP model

Communicated by Pablo A Ferrari.

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1 Introduction

Recent developments in the study of the KMP model and related models have revealed that the non-equilibrium steady state of the boundary driven version of such models is a mixture of product measures of equilibrium marginals. In the simplest setting of the KMP model [8], this means that the non-equilibrium steady state is a mixture of products of exponential distributions, where the joint distribution of the parameters of these exponentials is in turn a stationary distribution of an auxiliary model, the so-called *hidden temperature model* [8]. For a related class of models, the generalized harmonic models [10–12], the non-equilibrium steady state of the continuous model is given in closed form in terms of products of gamma distributions, with identical shape parameters, and where the scale parameters have the ordered Dirichlet distribution [4]. In the simplest setting of the harmonic model, the non-equilibrium steady state is a product of exponential distributions, where the (scale) parameters are distributed as the order statistics of i.i.d. uniforms [5]. The structure of the stationary state as a mixture was already conjectured in [1] (for the KMP model), based on macroscopic fluctuation theory.

So far, these results are all obtained in the setting of a chain geometry, with boundary reservoirs at left and right ends. They are strongly based on dualities, which reduce the computation of moments of order n in the non-equilibrium steady state to the computation of absorption probabilities of n dual particles. For the characterization of the non-equilibrium steady state of the generalized harmonic models of a chain, an additional input came from integrability. Is it usually the latter which provides closed-form expressions for the absorption probabilities of the dual process and is only applicable in the chain geometry, whereas duality results are valid in a setting of general graphs.

In this paper, using a reformulation of duality as an *intertwining* relation, we prove that for a large class of models *on a general graph*, there exist *hidden parameter models*. As a consequence, the non-equilibrium steady state is a mixture of equilibrium product marginals where the mixing measure (i.e., the joint distribution of the parameters of these marginals) is the stationary measure of the corresponding hidden parameter model. This stationary measure is usually inaccessible in explicit form on a general graph. In the case of the harmonic model on a chain, we are able to prove that it coincides with the mixing measure found in [4], using probabilistic arguments only based on a Markovian structure in the hidden parameter model. This Markovian structure of the hidden parameter model, which we explain in Sect. 3.5 below, is the key ingredient which makes the harmonic models different from the KMP models. It allows to directly (i.e., without using explicit expressions of moments) obtain the mixture measure identified earlier in [4]. The Markovian structure of the hidden parameter model also implies that it is enough to understand the stationary state for a single site with left and right reservoirs to understand the stationary measure of a general boundary driven chain. These results show that in essence, the existence of hidden parameter models is based on duality, and therefore not restricted to integrable models. However, the identification of the mixing measure, i.e., the measure describing the joint distribution of the parameters, is only possible when there is extra structure (i.e., extra symmetries) which makes it possible e.g. to use the quantum inverse scattering method [11], or in probabilistic terms, to have a Markovian structure of the mixing measure.

The rest of our paper is organized as follows. In Sect. 1.2, we sketch the general structure of the models under consideration. In Sect. 2 we discuss the discrete and continuous generalized KMP models, recovering and generalizing the hidden temperature models in [8]. In Sect. 3 we deal with the generalized harmonic models. In particular, we identify the corresponding hidden parameter models and establish intertwining on a generic graph. For the boundary driven chain we characterize the stationary measure of the hidden parameter model by using a self-contained argument which rely on the particular structure of the model. In Sect. 4 we extend the analysis to another model, the symmetric inclusion process (SIP) and prove that it admits Poisson intertwining. As a consequence the non-equilibrium steady state is a mixture of Poisson product measures, where the mixture measure is a non-equilibrium steady state of a corresponding continuous model (the Brownian energy process). We also recover the simplest setting of boundary driven independent random walks, where the intertwined dynamics is deterministic and has a unique fixed point, which implies that the non-equilibrium steady state is a product of Poisson measures. The latter is of course well-known but we believe it is still insightful to recover it from the point of view of intertwining.

1.1 Summary of Main Results and Relation with Existing Literature

Summarizing, the main results of our paper are the following.

1. We show the existence of hidden parameter models for a general class of models on general graphs using intertwining.
2. For a one parameter family of continuous and discrete KMP models, as well as harmonic models, we show propagation of mixed product states, where the parameters evolve as hidden parameter models.
3. We reveal a dynamical Markov property for the hidden parameter model associated to the harmonic models, and derive from it in a purely probabilistic way the non-equilibrium steady state obtained in [4] for general $s > 0$, in [5] for $s = 1/2$, and predicted earlier in [1].
4. For the boundary driven symmetric inclusion process (SIP), we derive a new representation of the non-equilibrium steady state as a mixture of a product of Poisson distributions, where the mixture measure is described via a diffusion process (BEP). This result generalizes to the boundary driven case earlier Poisson intertwining between SIP and BEP obtained in [16].

The novelty of our analysis w.r.t. existing literature [1, 2, 4, 5, 8, 10–12], is thus threefold. First we derive the existence of hidden temperature models on general graphs; second we obtain in this same generality the propagation of mixed product states; third we derive the Markovian structure of the non-equilibrium steady state of harmonic models in $d = 1$ (for general parameters $s > 0$) directly via the generator, i.e., not passing via moment computations relying on integrability (such as in [4]), but rather via conditional probabilities.

1.2 General Structure of the Models

We consider a finite set of vertices V, and a symmetric irreducible collection of edge weights $p(i, j) = p(j, i) \geq 0$ where $i, j \in V$. Here, by irreducibility we mean that for every $i, j \in V$ there exists a finite discrete path $\gamma(0), \ldots, \gamma(n)$ with $\gamma(0) = i, \gamma(n) = j$ and $p(\gamma(i), \gamma(i+1)) > 0$ for all $i = 0, \ldots, n-1$. We will then consider Markov processes on either the state space $\mathbb{N}^V = \{0, 1, 2, \ldots\}^V$ (discrete models) or the state space $\mathbb{R}_+^V = [0, \infty)^V$ (continuous models). The generator of these processes will take the form

$$\sum_{i,j \in V} p(i, j) L_{ij},$$

where L_{ij} is the so-called single edge generator, which acts only on the variables η_i, η_j and models the transport of mass along the edge connecting the sites $i, j \in V$. For the boundary driven version of the models we have a generator taking the form

$$\sum_{i,j\in V} p(i,j)L_{ij} + \sum_{i\in V} c(i)L_{\theta_i^*},$$

where $c(i) \geq 0$ is a non-negative constant tuning the coupling of site $i \in V$ to a "reservoir" with parameter $\theta_i^* > 0$. The single-site generator $L_{\theta_i^*}$ is acting only on the variables η_i and models the input and output of mass at the vertex $i \in V$, by fixing the average number of particles to θ_i^*.

The system with generator $\sum_{i,j\in V} p(i,j)L_{ij}$ will have a one parameter family of product invariant measures $\bigotimes_{i\in V} \nu_\theta$, where the parameter $\theta > 0$ labels the expected number of particles (or mass) and corresponds to the conserved quantity (total number of particles or total mass). Then the system coupled to reservoirs with identical parameters ($\theta_i^* = \theta^*$ for $i \in V$) has a unique stationary measure $\bigotimes_{i\in V} \nu_{\theta^*}$. If the reservoir parameters are different, then the unique stationary measure is no longer a product measure, and is called a *non-equilibrium steady state*, where non-equilibrium refers to the absence of reversibility.

The main aim of this paper is to understand for a family of models of this type the propagation of *inhomogeneous* product measures $\bigotimes_{i\in V} \nu_{\theta_i}$ in the course of time. Given $\theta = (\theta_i)_{i\in V}$, we will then find that these measures are mapped to a stochastic mixture of the form

$$\mathbb{E}_\theta\Big(\bigotimes_{i\in V} \nu_{\theta_i(t)}\Big),$$

where $(\theta_i(t), t \geq 0, i \in V)$ will evolve as a Markov process which we then call, following [8], the "hidden parameter model". As a consequence, the unique stationary measure (non-equilibrium steady state) will be a mixture of product measures of the type

$$\int \Big(\bigotimes_{i\in V} \nu_{\theta_i}\Big)\Xi\Big(\prod_{i\in V} d\theta_i\Big).$$

The "mixing measure" Ξ is then the unique invariant measure of the hidden parameter model. Thus, in the reservoir-driven setup, the identification of the non-equilibrium steady state is reduced to the identification of the stationary measure of the hidden parameter model.

The two most important examples of models having the property that the set of mixture of equilibrium product measure is closed under the dynamics will be models of "KMP type" (Sect. 2) or models of "harmonic type" (Sect. 3). For another class of models, namely the symmetric inclusion process and the independent random walkers (Sect. 4), we will show that the same happens with a product of Poisson measures, where the evolution of the Poisson parameters is then either a Markov diffusion process or a deterministic process.

In what follows we will always use an upright L for the generator of the process under study, and the symbol $\mathcal{L}$ for the corresponding hidden parameter model. We will always use the notation $\mathbb{E}_\eta, \mathbb{E}_\xi$ for expectations for process with discrete state space such as $\mathbb{N}^V$, $\mathbb{E}_\zeta$ for the expectations of processes with continuous state space such as $[0,\infty)^V$ and $\mathbb{E}_\theta$ for expectations of processes of hidden parameter models (also with state space $[0,\infty)^V$).

2 Generalized KMP Processes

In this section we study the discrete, resp. continuous, generalized KMP models, parametrized by a non-negative number $s > 0$. These models are a one-parameter generalization of the original KMP model and were introduced in [13]. For arbitrary $s > 0$ we prove new dualities with a generalized hidden parameter model. This in turn implies that products of discrete gamma, resp. continuous gamma, distributions evolve in the course of time into mixtures of such product measures, where the mixing measure is the distribution of the corresponding hidden parameter model.

We start by first considering the bulk process and then we add reservoirs. The original discrete and continuous KMP models [14] will be recovered for $s = 1/2$.

2.1 Discrete Generalized KMP

We consider a finite set of vertices V, and irreducible edge rates $p(i, j)$, as outlined in Sect. 1.2. The discrete generalized KMP process with parameter $2s > 0$ is a Markov process on $\mathbb{N}^V$ and is defined via the generator

$$Lf(\eta) = \sum_{i,j\in V} p(i, j)L_{ij} f(\eta)\,. \tag{1}$$

Here the single edge generator L_{ij} acts on the variables η_i, η_j as

$$L_{ij} f(x, y) = \mathbb{E}\big(f(X, x + y - X) - f(x, y)\big), \tag{2}$$

where X is beta-binomial with parameters $x + y$, $2s$, $2s$, i.e.,

$$\mathbb{P}(X = k) = \int_0^1 \binom{x+y}{k} p^k (1 - p)^{x+y-k}\mathrm{Beta}(2s, 2s)[dp], \tag{3}$$

where $k \in \{0, 1, \ldots, x + y\}$ and

$$\mathrm{Beta}(2s, 2s)[dp] = \frac{1}{B(2s, 2s)} p^{2s-1}(1 - p)^{2s-1}dp \tag{4}$$

denotes the Beta distribution with parameters $(2s, 2s)$.

The discrete generalized KMP process has reversible product measures which are product of discrete Gamma distributions parametrized as follows

$$\nu_\theta(n) = \frac{1}{n!}\left(\frac{\theta}{1+\theta}\right)^n \frac{\Gamma(2s + n)}{\Gamma(2s)}\left(\frac{1}{1+\theta}\right)^{2s}. \tag{5}$$

The relation between the parameter θ and the expectation of the marginals is given by

$$\sum_{n=0}^{\infty} n\nu_\theta(n) = 2s\theta \tag{6}$$

The discrete generalized KMP process is self-dual [3] with self-duality functions given by

$$D_F(\xi, \eta) = \prod_i \frac{\eta_i!}{(\eta_i - \xi_i)!}\frac{\Gamma(2s)}{\Gamma(2s + \xi_i)}. \tag{7}$$

More precisely, we have

$$\mathbb{E}_\eta\big(D_F(\xi, \eta(t))\big) = \mathbb{E}_\xi\big(D_F(\xi(t), \eta)\big). \tag{8}$$

The subscript "F" is added to the duality function D_F to recall that, for a given $\xi \in \mathbb{N}^V$, the expectation of the duality function w.r.t. a measure on the η variables gives essentially *the multivariate factorial moments* (up to the factors $\frac{\Gamma(2s)}{\Gamma(2s+\xi_i)}$). In particular the relation between the self-duality polynomials and product measures with marginals (5) reads

$$\int D_F(\xi,\eta)\bigotimes_{i\in V}\nu_{\theta_i}[d\eta]=\prod_{i\in V}\theta_i^{\xi_i}. \tag{9}$$

This equality completely characterizes the product measure $\bigotimes_{i\in V}\nu_{\theta_i}$ via its factorial moments.

The hidden parameter model associated to the discrete generalized KMP is a process on $[0,\infty)^V$ which will determine the evolution of the parameters $\theta=(\theta_i, i\in V)$ of product measures of the type $\bigotimes_i \nu_{\theta_i}$. The process is defined in the spirit of [8] via its generator

$$\mathcal{L}f(\theta)=\sum_{i,j}p(i,j)\mathcal{L}_{ij}f(\theta), \tag{10}$$

where the single edge generator $\mathcal{L}_{ij}$ acts on the variables θ_i,θ_j as follows

$$\mathcal{L}_{ij}f(x,y)=\mathbb{E}\Big(f(xB+y(1-B),xB+y(1-B))-f(x,y)\Big), \tag{11}$$

where B has a Beta distribution with parameters $(2s,2s)$, i.e., it has density (4). More explicitly we have

$$\mathcal{L}_{ij}f(x,y)=\int_0^1\big(f(xu+y(1-u),xu+y(1-u))-f(x,y)\big)\mathrm{Beta}(2s,2s)[du]. \tag{12}$$

We then have the following duality result.

Proposition 2.1 *The discrete generalized KMP process with generator* (1) *is dual to the hidden parameter model with generator* (10) *with duality function*

$$D(\xi,\theta)=\prod_{i\in V}\theta_i^{\xi_i}. \tag{13}$$

Proof We act with the generator L_{ij} in (2) on the ξ variables and obtain, using the binomial formula

$$\begin{aligned} L_{ij}\theta_i^{\xi_i}\theta_j^{\xi_j} &= \mathbb{E}\left(\theta_i^X\theta_j^{\xi_j+\xi_i-X}\right)-\theta_i^{\xi_i}\theta_j^{\xi_j}\\ &=\sum_{k=0}^{\xi_i+\xi_j}\binom{\xi_i+\xi_j}{k}\int_0^1 p^k(1-p)^{\xi_i+\xi_j-k}\theta_i^k\theta_j^{\xi_i+\xi_j-k}\mathrm{Beta}(2s,2s)[dp]-\theta_i^{\xi_i}\theta_j^{\xi_j}\\ &=\int_0^1(p\theta_i+(1-p)\theta_j)^{\xi_i+\xi_j}\mathrm{Beta}(2s,2s)[dp]-\theta_i^{\xi_i}\theta_j^{\xi_j}. \end{aligned} \tag{14}$$

This is now clearly the same as acting with the generator $\mathcal{L}_{ij}$ in (12) on the θ variables. □

We can then state a result on the evolution of product measures of the type $\bigotimes_{i\in V}\nu_{\theta_i}$ under the discrete generalized KMP model.

Theorem 2.1 *Consider the discrete generalized KMP model with generator* (1) *and start it from a product measure* $\bigotimes_{i\in V}\nu_{\theta_i}$. *Denote by* $\big(\bigotimes_{i\in V}\nu_{\theta_i}\big)S(t)$ *the evolved measure at time* $t>0$, *where* $(S(t))_{t\geq 0}$ *is the semigroup. Then we have*

$$\Big(\bigotimes_{i\in V}\nu_{\theta_i}\Big)S(t)=\mathbb{E}_\theta\Big(\bigotimes_{i\in V}\nu_{\theta_i(t)}\Big), \tag{15}$$

where $\mathbb{E}_\theta$ denotes the expectation in the hidden parameter model with generator (10) initialized from the configuration θ. As a consequence, the set of mixtures

$$\int \Big(\bigotimes_{i\in V} \nu_{\theta_i}\Big)\Xi[d\theta]$$

is closed under the evolution of the discrete generalized KMP model.

Proof The proof uses self-duality of the discrete generalized KMP process (stated in (8)), and the duality between discrete generalized KMP and the hidden parameter model (Proposition 2.1). As a consequence of the identity (9) we obtain the following series of equality:

$$\begin{aligned}\int D_F(\xi,\eta)\Big(\bigotimes_{i\in V}\nu_{\theta_i}\Big)S(t)[d\eta] &= \int \mathbb{E}_\eta\Big(D_F(\xi,\eta(t))\Big)\Big(\bigotimes_{i\in V}\nu_{\theta_i}\Big)[d\eta]\\ &= \int \mathbb{E}_\xi\big(D_F(\xi(t),\eta)\big)\Big(\bigotimes_{i\in V}\nu_{\theta_i}\Big)[d\eta]\\ &= \mathbb{E}_\xi\Big(\prod_{i\in V}\theta_i^{\xi_i(t)}\Big)\\ &= \mathbb{E}_\theta\Big(\prod_{i\in V}\theta_i(t)^{\xi_i}\Big)\\ &= \mathbb{E}_\theta\int D_F(\xi,\eta)\Big(\bigotimes_{i\in V}\nu_{\theta_i(t)}\Big)[d\eta]. \end{aligned} \tag{16}$$

Here in the second equality we used self-duality of the discrete generalized KMP process and in the fourth equality we used Proposition 2.1. The proof is then completed by observing that the functions $\eta \to D(\xi,\eta)$ are measure determining. □

The result of Theorem 2.1 can be reformulated as an intertwining result between the hidden parameter process and the discrete generalized KMP process. We say that two Markov processes with semigroups $(S(t), t \geq 0)$ and $(\mathcal{S}(t), t \geq 0)$ are intertwined with intertwiner $\mathcal{G}$ if for all $t \geq 0$

$$\mathcal{G}S(t) = \mathcal{S}(t)\mathcal{G}. \tag{17}$$

In Theorem 2.1 we have obtained

$$\int S(t)f(\eta)\Big(\bigotimes_{i\in V}\nu_{\theta_i}\Big)[d\eta] = \mathcal{S}(t)\int f(\eta)\Big(\bigotimes_{i\in V}\nu_{\theta_i}\Big)[d\eta], \tag{18}$$

where $S(t)$ is the semigroup of the discrete generalized KMP process and where $\mathcal{S}(t)$ is the semigroup of the hidden parameter model. Therefore, if we define for a function $f : \mathbb{N}^V \to \mathbb{R}$ the "discrete-gamma" intertwiner

$$\mathcal{G}f(\theta) = \int f(\eta)\Big(\bigotimes_{i\in V}\nu_{\theta_i}\Big)[d\eta],$$

where we implicitly assumed that f is integrable w.r.t. $\bigotimes_{i\in V}\nu_{\theta_i}$, then (18) reads

$$\mathcal{G}(S(t)f) = \mathcal{S}(t)(\mathcal{G}f),$$

which is exactly the intertwining between the hidden parameter process and the discrete KMP process.

2.2 Continuous Generalized KMP

The continuous generalized KMP process with parameter $2s > 0$ is a process on $[0, \infty)^V$ and is defined via the generator

$$Lf(\zeta) = \sum_{i,j \in V} p(i, j) L_{ij} f(\zeta), \tag{19}$$

where the single edge generator L_{ij} works on the variables η_i, η_j as follows

$$L_{ij} f(x, y) = \mathbb{E}\big(f(B(x+y), (1-B)(x+y)) - f(x, y)\big). \tag{20}$$

Here B is a Beta$(2s, 2s)$ distributed random variable.

The reversible measures of the continuous generalized KMP process are products of Gamma distribution with parameters $(\theta, 2s)$, where θ is the scale parameter and where $2s$ is the shape parameter, i.e. the marginals are given by

$$\nu_\theta[dx] = \frac{x^{2s-1}}{\theta^{2s}\Gamma(2s)} e^{-x/\theta}\, dx. \tag{21}$$

The continuous and discrete generalized KMP processes are dual [3, 14] with duality function

$$D_m(\xi, \zeta) = \prod_{i \in V} \zeta_i^{\xi_i} \frac{\Gamma(2\,s)}{\Gamma(2\,s + \xi_i)}.$$

The subscript "m" is added to the duality function D_m to recall that, for a given $\xi \in \mathbb{N}^V$, the expectation of the duality function w.r.t. a measure on the ζ variables gives essentially *the multivariate moments* (up to the factors $\frac{\Gamma(2s)}{\Gamma(2s+\xi_i)}$). In particular the relation between the duality functions and product measures with marginals (21) reads

$$\int D_m(\xi, \zeta) \Big(\bigotimes_{i \in V} \nu_{\theta_i}\Big)[d\zeta] = \prod_{i \in V} \theta_i^{\xi_i}. \tag{22}$$

This equality completely characterizes the product measure $\bigotimes_{i \in V} \nu_{\theta_i}$ via its moments.

The main result on the evolution of product measures of the type $\bigotimes_{i \in V} \nu_{\theta_i}$ under the continuous generalized KMP model is stated in the following theorem.

Theorem 2.2 *Start the continuous generalized KMP model with generator* (19) *from a product measure* $\bigotimes_{i \in V} \nu_{\theta_i}[d\zeta]$. *Then at time* $t > 0$ *we have the measure*

$$\Big(\bigotimes_{i \in V} \nu_{\theta_i}\Big) S(t)[d\zeta] = \mathbb{E}_\theta\Big(\bigotimes_{i \in V} \nu_{\theta_i(t)}[d\zeta]\Big), \tag{23}$$

where $\{\theta(t), t \geq 0\}$ *is the hidden parameter model with generator* (10) *initialized from the configuration* θ.

Proof We use the duality between the continuous and discrete generalized KMP model, combined with the duality between the discrete KMP model and the hidden parameter model.

We then obtain

$$\begin{aligned}\int D_m(\xi,\zeta)\Big(\bigotimes_{i\in V}\nu_{\theta_i}\Big)S(t)[d\zeta] &= \int \mathbb{E}_\eta\big(D_m(\xi,\zeta(t))\big)\Big(\bigotimes_{i\in V}\nu_{\theta_i}\Big)[d\zeta]\\ &= \int \mathbb{E}_\xi\big(D_m(\xi(t),\zeta)\big)\Big(\bigotimes_{i\in V}\nu_{\theta_i}\Big)[d\zeta]\\ &= \mathbb{E}_\xi\Big(\prod_i \theta_i^{\xi_i(t)}\Big)\\ &= \mathbb{E}_\theta\Big(\prod_i \theta_i(t)^{\xi_i}\Big)\\ &= \mathbb{E}_\theta\int D_m(\xi,\zeta)\Big(\bigotimes_{i\in V}\nu_{\theta_i(t)}\Big)[d\zeta]. \qquad (24)\end{aligned}$$

We then conclude by observing that the functions $\zeta\to D_m(\xi,\zeta)$ are measure determining. □

We then have the analogous result of Proposition 2.1 in the setting of the continuous generalized KMP process.

Proposition 2.2 *The continuous generalized KMP process with generator* (19) *and the hidden parameter model with generator* (10) *are dual with duality function*

$$D(\theta,\zeta)=\prod_{i\in V} e^{\theta_i\zeta_i}. \qquad (25)$$

Proof It suffices to prove the duality for the single edge generators. Acting with the single edge generator of the continuous generalizd KMP model on the ζ variables gives

$$\begin{aligned}&L_{ij}e^{\theta_i\zeta_i}e^{\theta_j\zeta_j}\\ &=\int_0^1\Big(e^{\theta_i u(\zeta_i+\zeta_j)+\theta_j(1-u)(\zeta_i+\zeta_j)}-e^{\theta_i\zeta_i}e^{\theta_j\zeta_j}\Big)\,\mathrm{Beta}(2s,2s)[du]\\ &=\int_0^1\Big(e^{(u\theta_i+(1-u)\theta_j)\zeta_i+(u\theta_i+(1-u)\theta_j)\zeta_j}-e^{\theta_i\zeta_i}e^{\theta_j\zeta_j}\Big)\,\mathrm{Beta}(2s,2s)[du], \qquad (26)\end{aligned}$$

which is recognized as the action of the generator $\mathcal{L}_{ij}$ in (12) on the θ variables. □

Remark 2.1 Notice that we can find the duality function between continuous generalized KMP and the hidden parameter model also via the generating function of the duality function between discrete generalized KMP and the hidden parameter model, i.e.,

$$\sum_{n=0}^{\infty}\frac{\theta^n z^n}{n!}=e^{\theta z}.$$

Indeed, the continuous and discrete generalized KMP model are intertwined via the intertwiner

$$\Lambda f(z)=\sum_{n=0}^{\infty} f(n)\frac{z^n}{n!}.$$

More precisely denoting here by L_d the generator of the discrete generalized KMP (1) and by L_c the generator of the continuous generalized KMP (19), we have for $f:\mathbb{N}^V\to\mathbb{R}$

$$\Lambda(L_d f)=L_c(\Lambda f),$$

where with a small abuse of notation we denoted by Λ the tensorization of Λ, i.e., the Λ acting on all the variables η_i

$$\Lambda f(\zeta) = \sum_{\eta \in \mathbb{N}^V} f(\eta) \frac{\zeta^\eta}{\eta!},$$

where

$$\frac{\zeta^\eta}{\eta!} = \prod_{i \in V} \frac{\zeta_i^{\eta_i}}{\eta_i!}.$$

Also here, we can reformulate Theorem 2.2 as an intertwining result. Indeed, by considering the Gamma distribution in (21) and by defining the "Gamma" intertwiner

$$\mathcal{G} f(\theta) = \int f(\zeta) \bigotimes_{i \in V} \nu_{\theta_i}[d\zeta]$$

it follows that Theorem 2.2 can be read as an intertwining between the hidden parameter process and the continuous generalized KMP process, with intertwiner $\mathcal{G}$.

2.3 Adding Driving

We will discuss the adding of driving for the continuous generalized KMP model only. The results for the generalized discrete KMP model are completely analogous.

We start by describing the generator modelling the coupling to a reservoir. It is a generator that acts on a single variable $x \in \mathbb{R}$ as follows

$$L_{\theta^*} f(x) = \mathbb{E}(f((x+Y)B) - f(x)), \tag{27}$$

where $\mathbb{E}$ denotes expectation over the two independent random variables B, Y and where Y is distributed as ν_{θ^*} (Gamma distribution) and B is Beta$(2s, 2s)$ distributed. Thus the action of the boundary site reservoir generator is similar to the bulk edge generator, in the sense that the redistribution of energies between the site and the reservoir occurs via a Beta random variable; however now the energy of the "extra site" representing the reservoir is sampled from a Gamma distribution with mean $2s\theta^*$, which is exactly the marginal of the invariant distribution of the model without reservoirs. Reservoirs of this form were introduced originally in the setting of the KMP model (corresponding to $2s = 1$) in [1] and are different from the reservoirs in the original model [14]. Indeed, in the original model of [14] the reservoir is given by

$$\tilde{L}_{\theta^*} f(x) = \mathbb{E} f(Y) - f(x)$$

where Y is a random variable which is gamma distributed with scale parameter θ^* and shape parameter $2s$.

The corresponding boundary generator of the hidden parameter model is

$$\mathcal{L}_{\theta^*} f(\theta) = \int_0^1 \Big(f((1-u)\theta + u\theta^*) - f(\theta)\Big) \mathrm{Beta}(2s, 2s)[du], \tag{28}$$

which can be viewed as having an "extra site" from which always the value θ^* is imported.

We then have the following intertwining result.

Lemma 2.1 *For a function $f : [0, +\infty) \to \mathbb{R}$ which is integrable with respect to the Gamma distribution ν_θ define the intertwiner*

$$\mathcal{G} f(\theta) = \frac{1}{\Gamma(2s)\theta^{2s}} \int_0^\infty f(x) x^{2s-1} e^{-x/\theta} dx.$$

Then the boundary generator of the continuous generalized KMP process (27) *and the boundary generator of the hidden parameter model* (28) *are intertwined as*

$$\mathcal{G}L_{\theta^*} = \mathcal{L}_{\theta^*}\mathcal{G}. \tag{29}$$

Proof For simplicity we prove the case $2s = 1$, the general case is obtained with a similar proof. We have

$$\begin{aligned}(\mathcal{G}L_{\theta^*}f)(\theta) &= \int_0^\infty dx \frac{e^{-x/\theta}}{\theta} L_{\theta^*}f(x)\\ &= \int_0^\infty dx \frac{e^{-x/\theta}}{\theta} \int_0^\infty dy \frac{e^{-y/\theta^*}}{\theta^*} \int_0^1 du \Big(f((x+y)u) - f(x)\Big)\end{aligned} \tag{30}$$

and we also have

$$\begin{aligned}(\mathcal{L}_{\theta^*}\mathcal{G}f)(\theta) &= \int_0^1 du \Big(\mathcal{G}f((1-u)\theta + u\theta^*) - \mathcal{G}f(\theta)\Big)\\ &= \int_0^1 du \Big(\int_0^\infty dx \frac{e^{-\frac{x}{(1-u)\theta+u\theta^*}}}{(1-u)\theta+u\theta^*} f(x) - \int_0^\infty dx \frac{e^{-\frac{x}{\theta}}}{\theta} f(x)\Big).\end{aligned} \tag{31}$$

It suffices to see (29) for the functions $f_n(x) = x^n/n!$ (for all $n \in \mathbb{N}$). From the previous two equations this in turn reduces to proving the following identity

$$\int_0^\infty dx \frac{e^{-x/\theta}}{\theta} \int_0^\infty dy \frac{e^{-y/\theta^*}}{\theta^*} \int_0^1 du \frac{((x+y)u)^n}{n!} = \int_0^1 du \int_0^\infty dx \frac{e^{-\frac{x}{(1-u)\theta+u\theta^*}}}{(1-u)\theta+u\theta^*} \frac{x^n}{n!}. \tag{32}$$

The right-hand side of (32) equals

$$\int_0^1 du \int_0^\infty dx \frac{e^{-\frac{x}{(1-u)\theta+u\theta^*}}}{(1-u)\theta+u\theta^*} \frac{x^n}{n!} = \int_0^1 du (u\theta^* + (1-u)\theta)^n = \frac{1}{n+1}\sum_{k=0}^n (\theta^*)^k \theta^{n-k}, \tag{33}$$

where we used the identity

$$\int_0^1 u^k (1-u)^{n-k} du = \frac{k!(n-k)!}{(n+1)!},$$

combined with $\int_0^\infty \frac{x^n}{n!} \frac{e^{-x/\theta}}{\theta} dx = \theta^n$. The left-hand side of (32) equals

$$\begin{aligned}&\int_0^\infty dx \frac{e^{-x/\theta}}{\theta} \int_0^\infty dy \frac{e^{-y/\theta^*}}{\theta^*} \int_0^1 du \frac{((x+y)u)^n}{n!}\\ &= \int_0^\infty dx \frac{e^{-x/\theta}}{\theta} \int_0^\infty dy \frac{e^{-y/\theta^*}}{\theta^*} \frac{1}{(n+1)} \sum_{k=0}^n \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!}\\ &= \frac{1}{n+1} \sum_{k=0}^n \theta^k (\theta^*)^{n-k}.\end{aligned} \tag{34}$$

□

To define the general boundary driven model, we associate reservoirs with parameters θ_i^* at site $i \in V$ and the generator of the boundary driven continuous generalized KMP process is then given by

$$Lf(\zeta) = \sum_{i,j \in V} p(i,j) L_{ij} f(\zeta) + \sum_{i \in V} c(i) L_{\theta_i^*} f(\zeta), \tag{35}$$

where $L_{i,j}$ is read in (20) and $L_{\theta_i^*}$ is defined in (27). As a consequence of Theorem 2.1 and of the intertwining result of Lemma 2.1, we then have the following propagation of mixtures of products of Gamma distributions.

Theorem 2.3 *Consider the driven continuous generalized KMP model with generator (35). Then we have the following.*

(a) If we start the process from a product measure of the form $\bigotimes_{i \in V} \nu_{\theta_i}$, then at time $t > 0$ the distribution is given by

$$\mathbb{E}_\theta \Big(\bigotimes_{i \in V} \nu_{\theta_i(t)} \Big),$$

where the process $(\theta_i(t), i \in V, t \geq 0)$ evolves according to the generator

$$\sum_{i,j \in V} p(i,j) \mathcal{L}_{ij} + \sum_{i \in V} c(i) \mathcal{L}_{\theta_i^*}. \tag{36}$$

(b) The driven generalized KMP process converges to a unique stationary measure which reads

$$\int \bigotimes_{i \in V} \nu_{\theta_i} \, \Xi[d\theta],$$

where the mixture measure Ξ is the unique stationary measures of the associated hidden parameter model, with generator (36).

(c) In particular if all the reservoir parameters are equal to a fixed value, i.e. $\theta_i^* = \theta^*$ for all $i \in V$, then this unique stationary measure is given by $\bigotimes_{i \in V} \nu_{\theta^*}$ and is also reversible.

3 Generalized Harmonic Models

In this section we consider the generalized discrete harmonic model [11] and the associated generalized continuum harmonic model (also called integrable heat conduction model in [10]). The aim here is to prove the existence of a hidden parameter model and to derive conclusions from it about the nature of the stationary measures in the one-dimensional boundary driven set-up. Contrary to the KMP model, the invariant measure of the hidden parameter model on the chain with left and right boundary reservoirs can be obtained explicitly. The main reason is a hidden Markovian structure of the hidden parameter model, see section 3.5 and 3.8 below for details. This hidden Markovian structure can be seen as the probabilistic counterpart of the integrability of this model, which was used in previous works [4, 10, 11] to obtain the non-equilibrium steady state on the chain.

3.1 Mass Redistribution Models

In order to introduce the harmonic models, let us first consider the following general class of generators (see also [2]) acting on two variables $y_1, y_2 \geq 0$, and parametrized by a positive measure M on the interval $[0, 1]$.

$$L_{12}f(y_1,y_2)=L^M_{12}f(y_1,y_2)$$
$$=\int_0^1 M(du)[(f(y_1-uy_1,y_2+uy_1)+f(y_1+uy_2,y_2-uy_2)2f(y_1,y_2))]. \tag{37}$$

In this process, with rate $M(du)$, a fraction of mass is taken away from one of the two sites and given to the other site. Notice that in these models, different from the KMP model, only a fraction of the mass of *one site* is moved to the other site (rather than a fraction of the total mass of the two sites).

In order to introduce the associated hidden parameter model, we consider the following generator acting on two variables $\theta_1,\theta_2\geq 0$

$$\mathcal{L}_{12}f(\theta_1,\theta_2)=\int_0^1 M(du)[f(\theta_1(1-u)+u\theta_2,\theta_2)+f(\theta_1,u\theta_1+(1-u)\theta_2)-2f(\theta_1,\theta_2)]. \tag{38}$$

We see that, contrary to the hidden parameter model for the generalized KMP processes, here the parameters (or "local temperatures") θ_1,θ_2 are replaced by convex combinations only at one of the two sites, leaving the parameter at the other site untouched. Remark that the process generated by (38) preserves the order. Indeed, if $\theta_1\leq\theta_2$ then, for $0\leq u\leq 1$,

$$\theta_1(1-u)+u\theta_2\leq\theta_2,$$

and

$$\theta_1\leq u\theta_1+(1-u)\theta_2.$$

We have the following duality result.

Proposition 3.1 *The process with generator L_{12} in (37) is dual to the process with generator $\mathcal{L}_{12}$ in (38) with duality function*

$$D_c(\theta_1,\theta_2;y_1,y_2)=e^{\theta_1 y_1+\theta_2 y_2}.$$

Proof This follows from the simple observation

$$e^{\theta_1(y_1-uy_1)+\theta_2(y_2+uy_1)}=e^{(\theta_1(1-u)+\theta_2 u)y_1+\theta_2 y_2}$$

and the similar equality obtained by interchanging the sub-indices 1 and 2. □

To understand associated intertwined discrete models, let us consider the Poisson intertwiner between functions $f:\mathbb{N}^2\to\mathbb{R}$ and functions $f:[0,\infty)^2\to\mathbb{R}$

$$\Lambda_{12}f(y_1,y_2)=\sum_{k_1,k_2\in\mathbb{N}} f(k_1,k_2)\frac{y_1^{k_1}}{k_1!}\frac{y_2^{k_2}}{k_2!}. \tag{39}$$

Now we consider discrete models of mass redistribution, i.e., Markov processes on $\mathbb{N}^2$ depending on a positive measure $\mathcal{M}(k,n), k\in\mathbb{N}, n\in\mathbb{N}$ with support $\{(k,n):k\leq n\}$. The discrete models are then defined via their generator acting on functions $f:\mathbb{N}^2\to\mathbb{R}$ as follows

$$\begin{aligned} Lf(n_1, n_2) &= L_{12}^{\mathcal{M}} f(n_1, n_2) \\ &= \sum_{k=1}^{n_1} \mathcal{M}(k, n_1)(f(n_1 - k, n_2 + k) - f(n_1, n_2)) \\ &\quad + \sum_{k=1}^{n_2} \mathcal{M}(k, n_2)(f(n_1 + k, n_2 - k) - f(n_1, n_2)). \end{aligned} \tag{40}$$

We say that the discrete model (40) *is associated to the the continuum model* (37) if it is Poisson intertwined with it, i.e., if

$$\Lambda_{12}(L_{12}^{\mathcal{M}}) = L_{12}^{M}(\Lambda_{12}). \tag{41}$$

Then we have the following lemma relating dualities of the continuous models to dualities of the associated discrete models.

Lemma 3.1 *If* (41) *holds, then the process with generator* $L_{12}^{\mathcal{M}}$ *is dual to the process with generator* (38) *with duality function*

$$D_d(\theta_1, \theta_2; n_1, n_2) = \theta_1^{n_1} \theta_2^{n_2}.$$

Proof This follows by the following two facts: i) the Poissonian generating function applied to $\theta_1^{n_1}\theta_2^{n_2}$ equals $e^{\theta_1 y_1 + \theta_2 y_2}$ (cf. Remark 2.1); ii) the duality between the discrete process with generator $L_{12}^{\mathcal{M}}$ with duality functions $D_d(\theta_1, \theta_2; n_1, n_2)$ and the process with generator (38) is equivalent with duality between the continuous process with generator L_{12}^{M} and the process with generator (38) with duality function

$$D_c(\theta_1, \theta_2; n_1, n_2) = \sum_{n_1, n_2} D_d(\theta_1, \theta_2; n_1, n_2) \frac{y_1^{n_1} y_2^{n_2}}{n_1! n_2!}.$$

See e.g. [6] for a proof of this equivalence. The duality for continuum models of Proposition (3.1) therefore implies automatically the duality for discrete models which are Poisson intertwined. □

3.2 The Harmonic Models

For the simplest version of the continuous harmonic model, we have $M(du) = \frac{1}{u} du$ [10] and for the associated discrete model $\mathcal{M}(k, n) = \frac{1}{k} I(1 \le k \le n)$ [11]. We first consider the model on a general graph with vertex set V and with edge weights $p(i, j)$ and define the generator acting on functions $f : [0, \infty)^V \to \mathbb{R}$ as follows

$$\begin{aligned} Lf(\zeta) = \sum_{i,j \in V} p(i, j) \int_0^1 \frac{du}{u} &((f(\zeta - u\zeta_i\delta_i + u\zeta_i\delta_j) - f(\zeta)) \\ &+ (f(\zeta - u\zeta_j\delta_j + u\zeta_j\delta_i) - f(\zeta))). \end{aligned} \tag{42}$$

Here δ_i denotes the configuration with unit mass at site i and zero mass everywhere else. The process corresponding to the generator (42) will be called the continuous harmonic process. Its reversible product measures are products of exponentials with identical scale parameters, i.e., with marginals

$$\nu_\theta(dx) = \frac{1}{\theta} e^{-x/\theta}, \tag{43}$$

with expectation $\theta > 0$. The associated discrete model is then defined via its generator acting on functions $f : \mathbb{N}^V \to \mathbb{R}$:

$$Lf(\eta) = \sum_{i,j\in V} p(i,j) \left(\sum_{k=1}^{\eta_i} \frac{1}{k} (f(\eta - k\delta_i + k\eta_j\delta_j) - f(\eta)) + \sum_{k=1}^{\eta_j} \frac{1}{k} (f(\eta - k\delta_j + k\delta_i) - f(\zeta)) \right). \tag{44}$$

We call the corresponding process the discrete harmonic process. Its reversible product measures are products of geometric random variables with marginals

$$\nu_\theta(n) = \left(\frac{\theta}{1+\theta}\right)^n \left(\frac{1}{1+\theta}\right), \tag{45}$$

with mean θ. Finally, the corresponding hidden parameter model is defined via its generator acting on $f : [0,\infty)^V \to \mathbb{R}$:

$$Lf(\theta) = \sum_{i,j\in V} p(i,j) \int_0^1 \frac{du}{u} \left((f(\theta - u\theta_i\delta_i + u\theta_j\delta_i) - f(\theta)) + (f(\theta - u\theta_j\delta_j + u\theta_i\delta_j) - f(\theta)) \right). \tag{46}$$

This generator was also considered in the literature of integrable systems, see for instance Eq. (2.3.3) in [9] where it appears as a representation of the integrable XXX spin chain, and Sect. 2.3 in [12] where a connection between the generator (46) and the continuous harmonic generator was pointed out.

Theorem 3.1 *We have the following duality and intertwining relations:*

(a) The discrete harmonic model is self-dual with self-duality function

$$D_F(\xi,\eta) = \prod_{i\in V} \binom{\eta_i}{\xi_i}. \tag{47}$$

(b) The discrete and continuous harmonic models are dual with duality function

$$D_m(\xi,\zeta) = \prod_{i\in V} \frac{\zeta_i^{\xi_i}}{\xi_i!}. \tag{48}$$

(c) The discrete and continuous harmonic model are Poisson intertwined.

(d) The continuous harmonic process and the hidden parameter model are dual with duality function

$$D_c(\theta,\zeta) = \prod_{i\in V} e^{\theta_i\zeta_i}. \tag{49}$$

(e) The discrete harmonic process and the hidden parameter model are dual with duality function

$$D_d(\theta,\eta) = \prod_{i\in V} \theta_i^{\eta_i}. \tag{50}$$

Proof See [10, 11] for the statements (a) up to (c). From (c) it follows that (d) and (e) are equivalent via Lemma 3.1, and (d) follows from Proposition 3.1. □

Remark 3.1 Notice that the duality between the discrete and continuous harmonic model can also be used to show that the continuous model is well-defined. Indeed on a finite graph, the discrete model is clearly a well-defined system: because of the conservation law it becomes effectively a finite state space continuous Markov chain. The discrete model then defines the time evolution of polynomials in the continuous model via the duality of Theorem 3.1(b).

We can then turn the duality results into a result on propagation of mixtures of product measures, or equivalently into an intertwining result.

Theorem 3.2 *The following results hold.*

(a) Start the discrete harmonic process with generator (44) from a product measure with geometric marginals $\bigotimes_{i\in V}\nu_{\theta_i}$, where ν_θ is as in (45). Then at time $t>0$ the distribution is equal to

$$\Big(\bigotimes_{i\in V}\nu_{\theta_i}\Big)S(t)=\mathbb{E}_\theta\Big(\bigotimes_{i\in V}\nu_{\theta_i(t)}\Big), \tag{51}$$

where $(\theta(t), t\geq 0)$ is the hidden parameter process with generator (46), and $\mathbb{E}_\theta$ denotes expectation in this process starting from θ. Equivalently, considering the "geometric" intertwiner of an integrable function $f:\mathbb{N}^V\to\mathbb{R}$,

$$\mathcal{G}f(\theta)=\int f(\eta)\Big(\bigotimes_{i\in V}\nu_{\theta_i}\Big)[d\eta].$$

We have

$$\mathcal{G}(Lf)=\mathcal{L}(\mathcal{G}f)$$

which is the intertwining between the generator $\mathcal{L}$ of the hidden parameter process and the generator L of the discrete harmonic process in (44).

(b) Start the continuous harmonic process with generator (42) from a product measure with exponential marginals $\otimes\nu_{\theta_i}$, where ν_θ is as in (43). Then at time $t>0$ the distribution is equal to

$$\Big(\bigotimes_{i\in V}\nu_{\theta_i}\Big)S(t)=\mathbb{E}_\theta\Big(\bigotimes_{i\in V}\nu_{\theta_i(t)}\Big), \tag{52}$$

where $\theta(t), t\geq 0$ is the hidden parameter process with generator (46), and $\mathbb{E}_\theta$ denotes expectation in this process starting from θ. Equivalently, considering the "exponential" intertwiner of an integrable function $f:[0,\infty)^V\to\mathbb{R}$

$$\mathcal{G}f(\theta)=\int f(\zeta)\Big(\bigotimes_{i\in V}\nu_{\theta_i}\Big)[d\zeta].$$

we have

$$\mathcal{G}(Lf)=\mathcal{L}(\mathcal{G}f),$$

which is the intertwining between the generator $\mathcal{L}$ of the hidden parameter process and the generator L of the continuous harmonic process in (46).

Proof We will prove (52). The proof of (51) is analogous, replacing exponentials by geometric distributions.

The duality functions between the continuous and discrete harmonic model are given by

$$D_m(\xi,\zeta)=\prod_i\frac{\zeta_i^{\xi_i}}{\xi_i!}.$$

Then we obtain,

$$\begin{aligned}\int D_m(\xi,\zeta)\Big(\bigotimes_{i\in V}\nu_{\theta_i}[d\zeta_i]\Big)S(t) &= \int \mathbb{E}_\zeta\big(D_m(\xi,\zeta(t))\big)\Big(\bigotimes_{i\in V}\nu_{\theta_i}[d\zeta_i]\Big)\\ &= \int \mathbb{E}_\xi\big(D_m(\xi(t),\zeta)\big)\Big(\bigotimes_{i\in V}\nu_{\theta_i}[d\zeta_i]\Big)\\ &= \mathbb{E}_\xi\Big(\prod_{i\in V}\theta_i^{\xi_i(t)}\Big)\\ &= \mathbb{E}_\theta\Big(\prod_{i\in V}\theta_i(t)^{\xi_i}\Big)\\ &= \mathbb{E}_\theta\Big(\int D_m(\xi,\zeta)\Big(\bigotimes_{i\in V}\nu_{\theta_i(t)}[d\zeta_i]\Big)\Big). \end{aligned} \tag{53}$$

Here we used duality between the continuous and the discrete model in the second equality, and duality between the discrete model and the hidden parameter model in the third equality. We can then conclude (52) because the functions $\zeta \to D_m(\xi,\zeta)$ are measure determining. □

3.3 Boundary Reservoirs

We now discuss the intertwining of the boundary generator of the continuous harmonic model. This reads [10]

$$L_{\theta^*}f(x) = \int_0^1 \frac{du}{u}(f(x(1-u)) - f(x)) + \int_0^\infty \frac{du}{u}e^{-u}(f(x+u\theta^*) - f(x)). \tag{54}$$

For a discussion on the form of this generator we refer to Remark 3.2 and Lemma 3.7. This generator is reversible w.r.t. the exponential distribution with mean $\theta^* > 0$. The corresponding boundary hidden parameter generator has the same structure of the boundary hidden parameter KMP generator, the main difference being that the uniform measure of the KMP model is here replaced by the measure du/u. It reads

$$\mathcal{L}_{\theta^*}f(\theta) = \int_0^1 \Big(f((1-u)\theta + u\theta^*) - f(\theta)\Big)\frac{du}{u}. \tag{55}$$

We then have the following intertwining result.

Lemma 3.2 *For a function $f : [0,+\infty) \to \mathbb{R}$ which is integrable with respect to the Exponential distribution ν_θ define the intertwiner*

$$\mathcal{G}f(\theta) = \int_0^\infty f(x)\frac{e^{-x/\theta}}{\theta}dx.$$

The boundary generator of the continuous harmonic process (54) *and the boundary generator of the hidden parameter model* (55) *are intertwined as*

$$\mathcal{G}L_{\theta^*} = \mathcal{L}_{\theta^*}\mathcal{G}. \tag{56}$$

Proof We have

$$(\mathcal{G}L_{\theta^*}f)(\theta) = \int_0^\infty dx \frac{e^{-x/\theta}}{\theta} L_{\theta^*}f(x)$$
$$= \int_0^\infty dx \frac{e^{-x/\theta}}{\theta} \left(\int_0^1 \frac{du}{u} (f(x(1-u)) - f(x)) + \int_0^\infty \frac{du}{u} e^{-u} (f(x+u\theta^*) - f(x)) \right) \tag{57}$$

and we also have

$$(\mathcal{L}_{\theta^*}\mathcal{G}f)(\theta) = \int_0^1 \frac{du}{u} \Big(\mathcal{G}f((1-u)\theta + u\theta^*) - \mathcal{G}f(\theta) \Big)$$
$$= \int_0^1 \frac{du}{u} \Big(\int_0^\infty dx \frac{e^{-\frac{x}{(1-u)\theta+u\theta^*}}}{(1-u)\theta+u\theta^*} f(x) - \int_0^\infty dx \frac{e^{-\frac{x}{\theta}}}{\theta} f(x) \Big). \tag{58}$$

It suffices to see (56) for the functions $f(x) = x^n/n!$ (for all $n \in \mathbb{N}$). I.e., we have to prove

$$\mathcal{G}L_{\theta^*}f = \mathcal{L}_{\theta^*}\mathcal{G}f \tag{59}$$

for those f. Plugging this f into (57) we get

$$(\mathcal{G}L_{\theta^*}f)(\theta) = \int_0^\infty dx \frac{e^{-x/\theta}}{\theta} \left(\int_0^1 \frac{du}{u} \left(\frac{x^n(1-u)^n}{n!} - \frac{x^n}{n!} \right) + \int_0^\infty \frac{du}{u} e^{-u} \left(\frac{(x+u\theta^*)^n}{n!} - \frac{x^n}{n!} \right) \right)$$
$$= \theta^n \int_0^1 \frac{du}{u} ((1-u)^n - 1) + \sum_{k=1}^n \theta^{n-k} (\theta^*)^k \frac{1}{k!} \int_0^\infty \frac{du}{u} e^{-u} u^k$$
$$= \theta^n \int_0^1 \frac{du}{u} ((1-u)^n - 1) + \sum_{k=1}^n \theta^{n-k} (\theta^*)^k \frac{1}{k}. \tag{60}$$

Plugging $f(x) = x^n/n!$ into (58) we get

$$(\mathcal{L}_{\theta^*}\mathcal{G}f)(\theta) = \int_0^1 \frac{du}{u} \Big(\int_0^\infty dx \frac{e^{-\frac{x}{(1-u)\theta+u\theta^*}}}{(1-u)\theta+u\theta^*} \frac{x^n}{n!} - \int_0^\infty dx \frac{e^{-\frac{x}{\theta}}}{\theta} \frac{x^n}{n!} \Big)$$
$$= \int_0^1 \frac{du}{u} \Big(((1-u)\theta + u\theta^*)^n - \theta^n \Big)$$
$$= \theta^n \int_0^1 \frac{du}{u} ((1-u)^n - 1) + \sum_{k=1}^n \theta^{n-k} (\theta^*)^k \frac{1}{k}. \tag{61}$$

This completes the proof. □

We define the generator of the boundary driven continuous harmonic process

$$Lf(\zeta) = \sum_{i,j \in V} p(i,j) \int_0^1 \frac{1}{u} \big((f(\zeta - u\zeta_i\delta_i + u\zeta_i\delta_j) - f(\zeta)) + (f(\zeta - u\zeta_j\delta_j + u\zeta_j\delta_i) - f(\zeta)) \big)$$
$$+ \sum_{i \in V} c(i) \left(\int_0^1 \frac{du}{u} (f(\zeta - u\zeta_i\delta_i) - f(\zeta)) + \int_0^1 \frac{du}{u} e^{-u} (f(\zeta + u\theta_i^*\delta_i) - f(\zeta)) \right). \tag{62}$$

Here we associate reservoirs with parameters θ_i^* to site $i \in V$.

As a consequence of Theorem 3.2 and of the intertwining result of Lemma 3.2, we then have for the boundary driven continuous harmonic process the following propagation of mixtures of product of exponential distribution.

Theorem 3.3 *Consider the resevoir driven continuous harmonic model with generator* (62). *Start the model from a product measure of the form* $\bigotimes_{i\in V}\nu_{\theta_i}$, *where* $\nu_{\theta_i}(d\zeta_i) = \frac{e^{-x/\theta_i}}{\theta_i}d\zeta_i$. *Then at time* $t > 0$ *the distribution is given by*

$$\mathbb{E}_\theta\Big(\bigotimes_{i\in V}\nu_{\theta_i(t)}\Big),$$

where the process $(\theta_i(t), i \in V, t \geq 0)$ *evolves according to the generator*

$$Lf(\theta) = \sum_{i,j\in V} p(i,j)\int_0^1 \frac{du}{u}\Big((f(\theta - u\theta_i\delta_i + u\theta_j\delta_i) - f(\theta)) + (f(\theta - u\theta_j\delta_j + u\theta_i\delta_j) - f(\theta))\Big)$$
$$+\sum_{i\in V} c(i)\int\Big(f(\theta - u\theta_i\delta_i + u\theta_i^*\delta_i) - f(\theta)\Big)\frac{du}{u}. \tag{63}$$

When $t \to \infty$, *the reservoir driven continuous harmonic process converges to a unique stationary measure which reads*

$$\int\Big(\bigotimes_{i\in V}\nu_{\theta_i}\Big)\Xi\Big(\prod_{i\in V}d\theta_i\Big),$$

where the mixture measure Ξ *is the unique stationary measures of the associated hidden parameter model, with generator* (36).

3.4 Invariant Measure of the Single Site Hidden Parameter Model

When we consider the harmonic model with a single site in contact with two reservoirs with $\theta_L = 0$ and $\theta_R = 1$, the generator of the associated hidden parameter model reads as follows (cf. (55))

$$\mathcal{L}^{0,1}f(\theta) = \int_0^1 \frac{1}{u}(f(\theta(1-u)) - f(\theta))du$$
$$+\int_0^1 \frac{1}{u}(f(u + \theta(1-u)) - f(\theta))du. \tag{64}$$

We then prove the following.

Proposition 3.2 *The unique stationary distribution of the process with generator* (64) *is the uniform distribution on* [0, 1].

Proof To infer the stationarity of the uniform measure for the generator (64) it is convenient to consider the harmonic model on a single edge which is given by

$$L_{12}f(\zeta_1,\zeta_2) = \int_0^1 \frac{du}{u}\left(f(\zeta_1(1-u), \zeta_2 + \zeta_1 u) - f(\zeta_1,\zeta_2)\right)$$
$$+\int_0^1 \frac{du}{u}\left(f(\zeta_1 + \zeta_2 u, \zeta_2(1-u)) - f(\zeta_1,\zeta_2)\right).$$

In this model $\zeta_1 + \zeta_2$ is conserved. Therefore, if we fix $\zeta_1 + \zeta_2 = 1$ then, substituting $\zeta_2 = 1 - \zeta_1$, we see that the action of the generator L_{12} on the ζ_1 variable is exactly the same as the action of the generator (64) on the θ variable. We know that the reversible measures for the generator L_{12} are product measures with marginals exponentials with identical scale parameter, i.e., with joint density given by

$$\frac{1}{\theta^2}e^{-\zeta_1/\theta}e^{-\zeta_2/\theta}.$$

As a consequence, considering two independent exponential random variables, the distribution of the first conditional to their sum being 1 is invariant for the generator (64). This is exactly the uniform distribution. □

For general reservoir case $0 \le \theta_L \le \theta_R$ the generator of the single-site hidden parameter model reads

$$\mathcal{L}^{\theta_L,\theta_R} f(\theta) = \int_0^1 \frac{1}{u}(f(\theta_L u + \theta(1-u)) - f(\theta))du + \int_0^1 \frac{1}{u}(f(\theta_R u + \theta(1-u)) - f(\theta))du. \tag{65}$$

Using the change of variable $x \mapsto \theta_L + x(\theta_R - \theta_L)$, Proposition 3.2 implies that the uniform distribution on $[\theta_L, \theta_R]$ is invariant for the process with generator $\mathcal{L}^{\theta_L,\theta_R}$.

3.5 The Invariant Measure of the Hidden Parameter Model on the Chain

In this section we consider the geometry of the chain $\{1, \dots, N\}$ with boundary reservoirs at left and right end. The hidden parameter model is then a model on the state space $\Omega_N = [0, \infty)^{\{1,\dots,N\}}$. It is parametrized by the left and right reservoir parameters θ_L, θ_R. The generator of the hidden parameter model is given by

$$\mathcal{L} f(\theta) = \mathcal{L}_1^{\theta_L} f(\theta) + \mathcal{L}_N^{\theta_R} f(\theta) + \sum_{i=1}^{N} \mathcal{L}_{i,i+1} f(\theta), \tag{66}$$

with boundary single site generators

$$\mathcal{L}_1^{\theta_L} f(\theta) = \int_0^1 \frac{du}{u}\,(f(\theta - u\theta_1\delta_1 + u\theta_L\delta_1) - f(\theta)) \tag{67}$$

$$\mathcal{L}_N^{\theta_R} f(\theta) = \int_0^1 \frac{du}{u}\,(f(\theta - u\theta_N\delta_N + u\theta_R\delta_N) - f(\theta)) \tag{68}$$

and with single edge generators

$$(\mathcal{L}_{i,i+1} f)(\theta) = \int_0^1 \frac{du}{u}(f(\theta - u\theta_i\delta_i + u\theta_{i+1}\delta_i) + f(\theta - u\theta_{i+1}\delta_{i+1} + u\theta_i\delta_{i+1}) - 2f(\theta)). \tag{69}$$

In this subsection we identify the stationary measure of the boundary driven hidden parameter model on the chain. As a consequence of Theorem 3.3 this yields also a full characterization of the non-equilibrium steady state of the boundary driven continuous harmonic model on the chain (cf. also [4, 5, 11]).

Theorem 3.4 *The invariant measure of the hidden parameter model with generator (66) is the joint distribution of $(U_{1:1}, \dots U_{N:N})$, the order statistics of N independent uniforms* *on* $[\theta_L, \theta_R]$. *As a consequence, the invariant measure of the boundary driven continuous harmonic model on the chain is a mixture of product of exponential distributions with means* $(U_{1:1}, \dots U_{N:N})$.

Proof We will prove the case $N = 2, \theta_L = 0, \theta_R = 1$. As will turn out from the proof, there is no loss of generality in considering the case $N = 2$, due to the Markovian structure of the joint distribution of order statistics. The restriction $\theta_L = 0, \theta_R = 1$ can be generalized via elementary translation and scaling. Let us call $\Lambda^2_{0,1}$ the joint distribution of the order statistics of two independent uniforms on $[0, 1]$. Let us call $\Lambda^1_{a,b}$ the distribution of one uniform on $[a, b]$. Then we have the following conditional distributions

$$\Lambda^2_{0,1}(d\theta_2|\theta_1 = a) = \Lambda^1_{a,1}(d\theta_2),\ \Lambda^2_{0,1}(d\theta_1|\theta_2 = b) = \Lambda^1_{0,b}(d\theta_1).$$

So let us now consider the generator (66) for $N = 2, \theta_L = 0, \theta_R = 1$. Then we want to prove that

$$\int \mathcal{L}f(\theta_1, \theta_2)\Lambda^2_{0,1}(d\theta_1, d\theta_2) = 0, \tag{70}$$

where

$$\begin{aligned}\mathcal{L}f(\theta_1, \theta_2) = &\int_0^1 \frac{du}{u}\left(f((1-u)\theta_1, \theta_2) - f(\theta_1, \theta_2)\right)\\ &+ \int_0^1 \frac{du}{u}\left(f((1-u)\theta_1 + u\theta_2, \theta_2) - f(\theta_1, \theta_2)\right)\\ &+ \int_0^1 \frac{du}{u}\left(f(\theta_1, (1-u)\theta_2 + u\theta_1) - f(\theta_1, \theta_2)\right)\\ &+ \int_0^1 \frac{du}{u}\left(f(\theta_1, u + (1-u)\theta_2) - f(\theta_1, \theta_2)\right).\end{aligned} \tag{71}$$

Now we observe that for the first two terms in the right-hand side of (71) the action of the generator on the θ_1 variable is the same as the action of a reservoir generator on one site, with left parameter $\theta_L = 0$ and right parameter $\theta_R = \theta_2$ (cf. (65)). For this generator, we know that the invariant measure is uniform on $[0, \theta_2]$, which coincides with the conditional distribution $\Lambda^2_{0,1}(d\theta_1|\theta_2)$. Therefore,

$$\begin{aligned}&\int \Lambda^2_{0,1}(d\theta_1, d\theta_2)\int_0^1 \frac{du}{u}\left(f((1-u)\theta_1, \theta_2) - f(\theta_1, \theta_2)\right)\\ &+ \int \Lambda^2_{0,1}(d\theta_1, d\theta_2)\int_0^1 \frac{du}{u}\left(f((1-u)\theta_1 + u\theta_2, \theta_2) - f(\theta_1, \theta_2)\right)\\ &= \int \Lambda^{2,2}_{0,1}(d\theta_2)\Bigg(\int \Lambda^2_{0,1}(d\theta_1|\theta_2)\int_0^1 \frac{du}{u}\left(f((1-u)\theta_1, \theta_2) - f(\theta_1, \theta_2)\right)\\ &+ \int_0^1 \frac{du}{u}\left(f((1-u)\theta_1 + u\theta_2, \theta_2) - f(\theta_1, \theta_2)\right)\Bigg) = 0.\end{aligned} \tag{72}$$

Here we used the notation $\Lambda^{2,2}_{0,1}(d\theta_2)$ for the second marginal of the measure $\Lambda^2_{0,1}(d\theta_1, d\theta_2)$, and in the last step we used the invariance of the conditional distribution $\Lambda^2_{0,1}(d\theta_1|\theta_2)$ for the reservoir generator with one site and left parameter zero, right parameter θ_2. Similarly, the action of the last two terms of the generator in the right-hand side of (71) on the θ_2 variable is the same as the action of a reservoir generator on one site, with left parameter $\theta_L = \theta_1$ and right parameter $\theta_R = 1$. As a consequence

$$\begin{aligned}
&\int \Lambda^2_{0,1}(d\theta_1, d\theta_2) \int_0^1 \frac{du}{u} \left(f(\theta_1, (1-u)\theta_2 + u\theta_1) - f(\theta_1, \theta_2)\right) \\
&\quad + \int \Lambda^2_{0,1}(d\theta_1, d\theta_2) \int_0^1 \frac{du}{u} \left(f(\theta_1, u + (1-u)\theta_2) - f(\theta_1, \theta_2)\right) \\
&= \int \Lambda^{2,1}_{0,1}(d\theta_1) \int \Lambda^2_{0,1}(d\theta_2|\theta_1) \left(\int_0^1 \frac{du}{u} \left(f(\theta_1, (1-u)\theta_2 + u\theta_1) - f(\theta_1, \theta_2)\right) \right. \\
&\quad \left. + \int_0^1 \frac{du}{u} \left(f(\theta_1, u + (1-u)\theta_2) - f(\theta_1, \theta_2)\right) \right) = 0. \qquad (73)
\end{aligned}$$

Here we used $\Lambda^{2,1}_{0,1}(d\theta_1)$ for the first marginal of the measure $\Lambda^2_{0,1}(d\theta_1, d\theta_2)$, and in the last step we used the invariance of the conditional distribution $\Lambda^2_{0,1}(d\theta_2|\theta_1)$ for the reservoir generator with one site and left parameter θ_1, right parameter 1. This finishes the proof of the case $N = 2$.

The general case is now a straightforward generalization via the Markovian structure of the joint distribution of order statistics. Indeed, let us call $\Lambda^N(d\theta_1, \ldots, d\theta_N)$ the joint distribution of the order statistics of N uniforms on $[\theta_L, \theta_R]$. Then conditional on $\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_N$, the variable θ_i is uniform on $[\theta_{i-1}, \theta_{i+1}]$ and the terms in the generator acting on the variable θ_i exactly coincide with the action of the reservoir generator on a single site with left reservoir parameter θ_{i-1} and right reservoir parameter θ_{i+1}, where we made the convention $\theta_0 = \theta_L, \theta_{N+1} = \theta_R$. □

The main reason why we are able to identify the invariant measure for the hidden parameter model on the chain can be summarized as follows. Consider the chain with left boundary parameter θ_L and right boundary parameter θ_R. Then the generator of the hidden parameter model has the form

$$\mathcal{L} = \mathcal{L}_1^{\theta_0} + \sum_{i=1}^{N} \mathcal{L}_{i,i+1} + \mathcal{L}_N^{\theta_{N+1}}. \qquad (74)$$

The action of this generator on the variable θ_i coincides with the action of the generator (65) with $\theta_L = \theta_{i-1}, \theta_R = \theta_{i+1}$. In other words, the generator from (74) can be rewritten as

$$\mathcal{L} = \sum_{i=1}^{N} \mathcal{L}_i^{\theta_{i-1}, \theta_{i+1}}, \qquad (75)$$

where $\mathcal{L}_i^{\theta_{i-1}, \theta_{i+1}}$ is the generator (65) with $\theta_L = \theta_{i-1}, \theta_R = \theta_{i+1}$ acting on the θ_i variable.

Therefore, if we call $\Lambda^1_{\theta_L, \theta_R}(d\theta)$ the invariant measure of the process with generator (65) describing the action of a left and right reservoir on a single site (which is uniform for the generator (65)), then we can describe the invariant measure of the generator (74) as follows. Let $\Lambda^N_{\theta_0, \theta_{N+1}}(d\theta_1, \ldots, d\theta_N)$ be a probability measure such that its conditional distributions are given by

$$\Lambda^N_{\theta_0, \theta_{N+1}}(d\theta_i | \theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots \theta_N) = \Lambda^1_{\theta_{i-1}, \theta_{i+1}}(d\theta_i),$$

then $\Lambda^N_{\theta_0, \theta_{N+1}}(d\theta_1, \ldots, d\theta_N)$ is invariant for the generator (74).

So we conclude that for every model for which one has this structure of the generator (i.e., (75)), one can obtain its invariant measure once one has identified the invariant measure

$\Lambda^1_{\theta_L,\theta_R}(d\theta)$ of the model with a single site between a left and right reservoir. As we will see below, this is the case for the generalized harmonic models parametrized by $s > 0$.

3.6 The Generalized Harmonic Model with Parameter $s > 0$: Bulk Generator

The model from the previous section is a special case of a one-parameter family of so-called generalized “harmonic models”. For these models, the measure M in (37) reads

$$M(du) = \frac{(1-u)^{2s-1}}{u}du. \tag{76}$$

The corresponding discrete harmonic models have the measure

$$\mathcal{M}(k,n) = \frac{1}{k}\frac{\Gamma(n+1)\Gamma(n-k+2s)}{\Gamma(n+2s)\Gamma(n-k+1)}. \tag{77}$$

Considering the generalized harmonic models on a graph, we have then duality between the discrete model and the continuous model with duality functions

$$D_m(\xi,x) = \prod_{i\in V} d_m(\xi_i,x_i), \tag{78}$$

with $d_m(k,x) = \frac{x^k\Gamma(2s)}{\Gamma(2s+k)}$ and self-duality of the discrete model with self-duality functions

$$D_F(\xi,\eta) = \prod_{i\in V} d_F(\xi_i,\eta_i), \tag{79}$$

with $d_F(k,n) = I(k\leq n)\frac{n!\Gamma(2s)}{(n-k)!\Gamma(2s+k)}$. Moreover, the continuum model and the discrete model are Poisson intertwined. See [10, 11] for a proof of these dualities. As a consequence, we have the analogue of Theorem 3.1, if one replaces the duality functions D_m and D_F by the ones in (78), resp. (79). The reversible product measures are now given by products of Gamma distributions for the continuum model, and by products of discrete Gamma distributions for the discrete model. As a consequence, we also have the analogue of Theorem 3.2 if one replaces the exponential ν_θ marginals by the corresponding Gamma marginals for the continuum model, i.e.,

$$\nu_\theta(dx) = \frac{x^{2s-1}}{\Gamma(2s)\theta^{2s}}e^{-x/\theta}dx \tag{80}$$

or by the corresponding discrete Gamma marginals for the discrete model

$$\nu_\theta(n) = \frac{1}{n!}\left(\frac{\theta}{1+\theta}\right)^n\frac{\Gamma(2\,s+n)}{\Gamma(2\,s)}(1+\theta)^{-2\,s}.$$

3.7 The Generalized Harmonic Model with Parameter $s > 0$: Boundary Generator

The boundary generator with reservoir parameter θ^* is [10] (see also Remark 3.2)

$$\begin{aligned} L_{\theta^*}f(x) = &\int_0^1 \frac{du}{u}(1-u)^{2s-1}\left(f((1-u)x)-f(x)\right) \\ &+\int_0^\infty \frac{du}{u}e^{-u}\left(f(x+u\theta^*)-f(x)\right). \end{aligned} \tag{81}$$

The first term models the exit of mass to the reservoir, the second term models the input of mass from the reservoir. Denoting by ν_θ the Gamma distribution (80), the natural candidate intertwiner then reads

$$\mathcal{G} f(\theta) = \int_0^\infty f(x) \nu_\theta(dx). \tag{82}$$

The candidate generator associated to a reservoir with parameter θ^* in the corresponding hidden parameter model is given by

$$\mathcal{L}_{\theta^*} f(\theta) = \int_0^1 \frac{du}{u}(1-u)^{2s-1} \left(f((1-u)\theta + u\theta^*) - f(\theta)\right). \tag{83}$$

We can now state the analogue of the intertwining relation of Lemma 3.2.

Lemma 3.3 *The boundary generator of the generalized continuous harmonic process* (81) *and the boundary generator of the hidden parameter model* (83) *are intertwined as*

$$\mathcal{G} L_{\theta^*} f = \mathcal{L}_{\theta^*} \mathcal{G} f. \tag{84}$$

Proof It suffices to see (84) for the functions $f(x) = x^n \frac{\Gamma(2s)}{\Gamma(2s+n)}$ (for all $n \in \mathbb{N}$). We have

$$\begin{aligned}
&(\mathcal{G} L_{\theta^*} f)(\theta) \\
&= \int_0^\infty dx \frac{e^{-x/\theta}}{\theta} \left(\int_0^1 \frac{du}{u}(1-u)^{2s-1} \left(x^n (1-u)^n - x^n\right) \right. \\
&\quad \left. + \int_0^\infty \frac{du}{u} e^{-u} \left((x+u\theta^*)^n - x^n\right) \right) \frac{\Gamma(2s)}{\Gamma(2s+n)} \\
&= \theta^n \int_0^1 \frac{du}{u}(1-u)^{2s-1}((1-u)^n - 1) \\
&\quad + \sum_{k=1}^n \theta^{n-k}(\theta^*)^k \frac{n!}{k!(n-k)!} \int_0^\infty \frac{du}{u} e^{-u} u^k \frac{\Gamma(2s+n-k)}{\Gamma(2s+n)} \\
&= \theta^n \int_0^1 \frac{du}{u}(1-u)^{2s-1}((1-u)^n - 1) \\
&\quad + \sum_{k=1}^n \theta^{n-k}(\theta^*)^k \frac{1}{k} \frac{\Gamma(2s+n-k)}{\Gamma(2s+n)} \frac{\Gamma(n+1)}{\Gamma(n+2s)}
\end{aligned} \tag{85}$$

and we also have

$$\begin{aligned}
&(\mathcal{L}_{\theta^*} \mathcal{G} f)(\theta) \\
&= \int_0^1 \frac{du}{u}(1-u)^{2s-1} \Big(\int_0^\infty dx \frac{e^{-\frac{x}{(1-u)\theta+u\theta^*}}}{(1-u)\theta + u\theta^*} x^n - \int_0^\infty dx \frac{e^{-\frac{x}{\theta}}}{\theta} x^n \Big) \frac{\Gamma(2s)}{\Gamma(2s+n)} \\
&= \int_0^1 \frac{du}{u}(1-u)^{2s-1} \Big(((1-u)\theta + u\theta^*)^n - \theta^n \Big) \\
&= \theta^n \int_0^1 \frac{du}{u}(1-u)^{2s-1}((1-u)^n - 1) + \sum_{k=1}^n \theta^{n-k}(\theta^*)^k \frac{1}{k} \frac{\Gamma(2s+n-k)}{\Gamma(2s+n)} \frac{\Gamma(n+1)}{\Gamma(n+2s)}.
\end{aligned} \tag{86}$$

This completes the proof. □

The generator of the hidden parameter model associated to two reservoirs acting on a single site is then given by

$$\mathcal{L}^{\theta_L,\theta_R} f(\theta) = \mathcal{L}_{\theta_L} f(\theta) + \mathcal{L}_{\theta_R} f(\theta). \tag{87}$$

The following lemma identifies the invariant measure of $\mathcal{L}^{0,1}$.

Lemma 3.4 *The unique invariant measure of the process with generator* (87) *with* $\theta_L = 0, \theta_R = 1$ *is equal to the conditional distribution*

$$\Lambda^{0,1}(dy_1) := \nu_\theta \otimes \nu_\theta(dy_1 | y_1 + y_2 = 1).$$

In particular, this is given by the Beta distribution

$$Beta(2s, 2s)(d\theta) = \frac{\theta^{2s-1}(1-\theta)^{2s-1}}{B(2s, 2s)} d\theta,$$

where $B(a, b)$ *denotes the Beta function.*

Proof The action of $\mathcal{L}^{0,1}$ on the θ variable coincides with the action of the generator (37) with $M(du) = \frac{du}{u}(1-u)^{2s-1}$ when we start from $y_1 + y_2 = 1$, and consider the action on the y_1 variable. The product measure $\nu_\theta \otimes \nu_\theta$ is a reversible measure, and the event $y_1 + y_2 = 1$ is invariant. As a consequence, the conditioned measure $\nu_\theta \otimes \nu_\theta(dy_1 | y_1 + y_2 = 1)$ is invariant. Since ν_θ is Gamma distributed with shape parameter $2s$, then the conditional distribution $\nu_\theta \otimes \nu_\theta(dy_1 | y_1 + y_2 = 1)$ is the symmetric Beta distribution with parameter $2s$. □

3.8 Invariant Measure of the Generalized Harmonic Model on the Chain

As a consequence of the intertwining of the boundary generator described in Sect. 3.7, we obtain the analogue of Theorem 3.3 for the full set of boundary-driven generalized harmonic models with parameter s. In order to understand the structure of the invariant measure in the setting of the chain with left and right boundary reservoirs, we have to understand the invariant measure of the hidden parameter model. We have seen in Lemma 3.5 that the measure $\Lambda_1^{0,1}$ is the distribution $Beta(2s, 2s)[d\theta]$. Let us call $B_{\theta_L,\theta_R}(d\theta)$ the corresponding recentered and rescaled distribution which is such that under this distribution

$$\frac{\theta - \theta_L}{\theta_R - \theta_L}$$

is $Beta(2s, 2s)$ distributed.

Then, following the line of argument of the proof of Theorem 3.4 the invariant measure $\Lambda^N(d\theta_1, \ldots, d\theta_N)$ is such that its conditional distributions are given by

$$\Lambda^N(d\theta_i | \theta_0, \ldots, \theta_{i-1}, \ldots, \theta_{i+1}, \ldots, \theta_{N+1}) = B_{\theta_{i-1},\theta_{i+1}}(d\theta_i).$$

This yields exactly the joint distribution obtained in [4], i.e.,

$$\Lambda^N(d\theta_1, \ldots, d\theta_N) = C(N, 2s, \theta_L, \theta_R) \prod_{i=1}^{N+1} (\theta_i - \theta_{i-1})^{2s-1} \mathbb{1}(\theta_L \le \theta_1 \le \ldots \le \theta_N \le \theta_R), \tag{88}$$

where $C(N, 2s, \theta_L, \theta_R)$ is the normalization constant

$$C(N, 2s, \theta_L, \theta_R) = \frac{1}{(\theta_R - \theta_L)^{2s(N+1)-1}} \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}}.$$

We summarize the finding of this section in the following theorem.

Theorem 3.5 *The invariant measure of the hidden parameter model with generator*

$$\mathcal{L}f(\theta) = \int \frac{du}{u}(1-u)^{2s-1}\Big(f(\theta - u\theta_1\delta_1 + u\theta_L\delta_1) - f(\theta)\Big)$$
$$+ \sum_{i=1}^{N}\int_0^1 \frac{du}{u}(1-u)^{2s-1}\left(f(\theta - u\theta_i\delta_i + u\theta_{i+1}\delta_i)\right.$$
$$\left.+ f(\theta - u\theta_{i+1}\delta_{i+1} + u\theta_i\delta_{i+1}) - 2f(\theta)\right)$$
$$+ \int \frac{du}{u}(1-u)^{2s-1}\Big(f(\theta - u\theta_N\delta_N + u\theta_R\delta_N) - f(\theta)\Big) \tag{89}$$

is the measure (88)*. As a consequence, the invariant measure of the boundary driven continuous harmonic model with parameter s defined by the generator*

$$Lf(\zeta)$$
$$= \int_0^1 \frac{du}{u}(1-u)^{2s-1}\left(f(\zeta - u\delta_1) - f(\zeta)\right) + \int_0^\infty \frac{du}{u}e^{-u}\left(f(\zeta + u\theta_L\delta_1) - f(\zeta)\right)$$
$$+ \sum_{i=1}^{N}\int_0^1 \frac{du}{u}(1-u)^{2s-1}\big(f(\zeta - u\zeta_i\delta_i + u\zeta_i\delta_{i+1}) + f(\zeta - u\zeta_{i+1}\delta_{i+1} + u\zeta_{i+1}\delta_i) - 2f(\zeta)\big)$$
$$+ \int_0^1 \frac{du}{u}(1-u)^{2s-1}\left(f(\zeta - u\delta_N) - f(\zeta)\right) + \int_0^\infty \frac{du}{u}e^{-u}\left(f(\zeta + u\theta_R\delta_N) - f(\zeta)\right) \tag{90}$$

is a mixture of product of Gamma distributions with mixing measure (88)*.*

3.9 General Redistribution Rules and Reservoirs

We close this section by investigating intertwining for the general mass redistribution model with generator (37). We also discuss a general definition of reservoirs which is naturally associated to the the general mass redistribution model.

We assume that the measure M in (37) is chosen in such a way that the corresponding process has a one-parameter family of reversible product measures with marginals denoted by $\nu_\theta(dx)$. E.g. for the choice $M(du) = (1/u)du$, $\nu_\theta(dx) = \frac{1}{\theta}e^{-x/\theta}dx$; for the choice $M(du) = (1/u)(1-u)^{2s-1}du$, $\nu_\theta(dx) = \frac{1}{\theta}e^{-x/\theta}x^{2s-1}dx$. The natural boundary generator with reservoir parameter θ^* is given by

$$L_{\theta^*}f(x) = \int_0^1 M(du)\left(f((1-u)x) - f(x)\right)$$
$$+ \int_0^\infty \nu_{\theta^*}(dy)\int_0^1 M(du)\left(f(x+uy) - f(x)\right). \tag{91}$$

As in the KMP process, this choice of the reservoir in inspired by the idea that a site interacts with the reservoir as it does with the bulk sites it is connected to; however the energy of the reservoirs is random and sampled from the distribution ν_{θ^*}.

In the corresponding hidden parameter model, the candidate generator associated to a reservoir with parameter θ^* is then given by

$$\mathcal{L}_{\theta^*}f(\theta) = \int_0^1 M(du)\big(f((1-u)\theta + u\theta^*) - f(\theta)\big) \tag{92}$$

and the generator of the hidden parameter model associated to two reservoirs acting on a single site is

$$\mathcal{L}^{\theta_L,\theta_R} f(\theta) = \mathcal{L}_{\theta_L} f(\theta) + \mathcal{L}_{\theta_R} f(\theta). \tag{93}$$

The following lemma identifies the invariant measure of $\mathcal{L}^{0,1}$ in terms of the measure ν_θ.

Lemma 3.5 *The unique invariant measure of the process with generator* (93) *with* $\theta_L = 0, \theta_R = 1$ *is equal to the conditional distribution*

$$\Lambda^{0,1}(dx_1) := \nu_\theta \otimes \nu_\theta(dx_1|x_1 + x_2 = 1).$$

In particular, the latter does not depend on θ.

Proof The action of $\mathcal{L}^{0,1}$ on the θ variable coincides with the action of the generator (37) when we start from $y_1 + y_2 = 1$, and consider the action on the y_1 variable. By assumption, $\nu_\theta \otimes \nu_\theta$ is a reversible measure, and the event $y_1 + y_2 = 1$ is invariant. As a consequence, the conditioned measure $\nu_\theta \otimes \nu_\theta(dy_1|y_1 + y_2 = 1)$ is invariant. □

We introduce the natural candidate intertwiner as

$$\mathcal{G} f(\theta) = \int_0^\infty f(x)\nu_\theta(dx) \tag{94}$$

and its tensorization

$$\mathcal{G} f((\theta_i)_{i\in V}) = \int f((x_i)_{i\in V}) \bigotimes \nu_{\theta_i}\Big(\prod_i dx_i\Big). \tag{95}$$

To discuss intertwining for the boundary-driven model we need to establish conditions guaranteeing that

$$\mathcal{G} L_{\theta^*} = \mathcal{L}_{\theta^*}\mathcal{G}. \tag{96}$$

In order to obtain the intertwining (96) we make the following natural scaling assumption on the measure ν_θ:

$$\int \nu_\theta(dx)x^n = R_n\theta^n. \tag{97}$$

Here we implicitly assumed that all the moments are finite. We moreover assumed that the measures ν_θ are uniquely determined by their moments. Then we have the following.

Lemma 3.6 *The intertwining relation* (96) *is satisfied if and only if for all* n *and* $k \in \{0, \dots, n\}$ *we have*

$$R_{n-k}R_k \int_0^1 u^k\, M(du) = R_n \int_0^1 M(du)u^k(1-u)^{n-k}. \tag{98}$$

Proof We start from (96) and fill in the function $f(x) = x^n$. Then the left-hand side equals

$$\begin{aligned}(\mathcal{G}(L_{\theta^*} f))(\theta) = {} & R_n\theta^n \int_0^1 M(du)((1-u)^n - 1)\\ & + \sum_{k=1}^n \theta^{n-k}(\theta^*)^k R_{n-k}R_k \binom{n}{k} \int_0^1 M(du)u^k. \end{aligned}\tag{99}$$

The right-hand side equals

$$(\mathcal{L}_{\theta^*}(\mathcal{G}f))(\theta) = R_n\theta^n \int_0^1 ((1-u)^n - 1)M(du) + \sum_{k=1}^{n} \theta^{n-k}(\theta^*)^k R_n \binom{n}{k} \int_0^1 u^k (1-u)^{n-k} M(du). \tag{100}$$

Because both expressions have to be equal for all values of θ, θ^*, we obtain (98). □

The following corollary then proves that (98) is satisfied for the harmonic model with $s > 0$.

Corollary 3.1 *Let ν_θ be the Gamma distribution with scale parameter θ and shape parameter $2s$, i.e., the probability measure given in (80), and assume*

$$M(du) = \frac{(1-u)^{2s-1}}{u} du. \tag{101}$$

Then the moment relation (98) is satisfied.

Proof For the measure (101) we have $R_n = \frac{\Gamma(2s+n)}{\Gamma(2s)}$. Therefore, using (101), the left-hand side of (98) equals

$$R_{n-k}R_k \int_0^1 u^k \, M(du) = \frac{\Gamma(n-k+2s)}{\Gamma(2s)} \frac{\Gamma(k+2s)}{\Gamma(2s)} \frac{\Gamma(k)\Gamma(2s)}{\Gamma(k+2s)}$$

and the right-hand side of (98) equals

$$R_n \int_0^1 M(du)u^k(1-u)^{n-k} = \frac{\Gamma(n+2s)}{\Gamma(2s)} \frac{\Gamma(k)\Gamma(n-k+2s)}{\Gamma(n+2s)}.$$

Hence both expressions are indeed equal and the relation (98) is satisfied. □

Remark 3.2 1. Equation (98) can in principle allow other solutions than those of Corollary 3.1. We conjecture however that this is not the case, i.e., that (101) provides the only solution.

2. As a consequence of Corollary 3.1 we deduce that the boundary-driven generalized harmonic models with reservoirs (81) and the boundary-driven generalized harmonic models with reservoirs

$$L_{\theta^*} f(x) = \int_0^1 \frac{du}{u}(1-u)^{2s-1} \left(f((1-u)x) - f(x)\right) + \int_0^\infty \nu_{\theta^*}(dy) \int_0^1 \frac{du}{u}(1-u)^{2s-1} \left(f(x+uy) - f(x)\right) \tag{102}$$

have the same hidden parameter model. In fact, the generator (81) and the generator (102) are the same as can be seen from an explicit computation which we detail below for the readers convenience.

Lemma 3.7 *On the functions $f(x) = x^n$ the generators (81) and (102) have identical action. As a consequence they are equal.*

Proof To prove identical action on $f(x) = x^n$, we have to prove that

$$\int \nu_{\theta^*}(dy) \int_0^1 \frac{du}{u}(1-u)^{2s-1}((x+uy)^n - x^n) = \int_0^\infty \frac{du}{u} e^{-u}((x+u\theta^*)^n - x^n). \tag{103}$$

Here we recall that ν_{θ^*} is the Gamma distribution with shape parameter $2s$ and scale parameter θ^* defined in (21) above. After applying the binomial formula combined with $\int_0^\infty x^n e^{-x} dx = n!$, the right hand side of (103) equals,

$$\int_0^\infty \frac{du}{u} e^{-u}((x+u\theta^*)^n - x^n) = \sum_{k=0}^{n-1} \binom{n}{k} x^k (\theta^*)^{n-k} \Gamma(n-k).$$

After applying the binomial formula combined with exchanging the integral over y with the integral over u, the left hand side of (103) equals:

$$\begin{aligned}
&\int_0^\infty \frac{du}{u}(1-u)^{2s-1} \int_0^\infty \nu_{\theta^*}(dy) \left(\sum_{k=0}^{n-1} \binom{n}{k} x^k (uy)^{n-k} \right) \\
&= \sum_{k=0}^{n-1} \binom{n}{k} x^k \int_0^1 du(u^{n-k-1}(1-u)^{2s-1} \int_0^\infty \nu_{\theta^*}(dy) y^{n-k} \\
&= \sum_{k=0}^{n-1} \binom{n}{k} x^k \frac{\Gamma(n-k)\Gamma(2s)}{\Gamma(n-k+2s)} (\theta^*)^{n-k} \frac{\Gamma(2s+n-k)}{\Gamma(2s)} \\
&= \sum_{k=0}^{n-1} \binom{n}{k} x^k (\theta^*)^{n-k} \Gamma(n-k).
\end{aligned} \tag{104}$$

The consequence that the generators (81) and (102) are equal follows from linearity and the fact that polynomials are a core. □

4 Poisson Intertwining

The class of models for which the non-equilibrium steady state is a mixture of product measures is not limited to models of KMP or harmonic type. In this section we consider the boundary driven symmetric inclusion process (SIP) and we prove that the stationary measure is a mixture of Poisson product measures. This is a different situation compared to the previous sections, because the Poisson product measures are not the stationary measures of the SIP. The stationary measures of SIP are product of discrete Gamma distributions, which are however themselves mixtures of Poisson product measures. The Gamma intertwiners which produce hidden parameter models for KMP and harmonic models do not lead to an intertwined Markov process for SIP. One can verify by explicit computation that this Gamma intertwiner leads to a second order differential operator with non-positive definite second order part, which therefore can not be interpreted as the generator of a Markov process. To prove the intertwining result for SIP, we use the Poisson intertwiner of the classical creation and annihilation operators which transforms the boundary generators into the boundary generators of the Brownian energy process (BEP). Using this same intertwiner, we also revisit the simplest example of independent random walkers, by which we then recover the propagation of Poisson product measures, which is a version of Doob's theorem [7], or alternatively, of the random displacement theorem in point process theory [15].

4.1 Boundary Driven SIP

First, the SIP on two sites is the Markov process on $\mathbb{N}^2$ with generator

$$Lf(n_1, n_2) = n_1(2s + n_2)(f(n_1 - 1, n_2 + 1) - f(n_1, n_2)) \\ +n_2(2s + n_1)(f(n_1 + 1, n_2 - 1) - f(n_1, n_2)). \tag{105}$$

Given a vertex set V and irreducible edge weights $p(i, j)$, we define the SIP as the Markov process with generator

$$\sum_{ij} p(i, j)L_{ij}, \tag{106}$$

where, as usual, L_{ij} is the generator (105) acting on the variables η_i, η_j. The boundary generator is given by

$$L^{\alpha,\gamma} f(n) = \alpha(2s + n)(f(n + 1) - f(n)) + \gamma n(f(n - 1) - f(n)). \tag{107}$$

We assume $\alpha < \gamma$, in that case $L^{\alpha,\gamma}$ admits a unique stationary measure which is the discrete Gamma distribution (5) with $\theta = \frac{\alpha}{\gamma-\alpha}$.

The boundary driven model with boundary reservoirs is then given by

$$\sum_{ij} p(i, j)L_{ij} + \sum_{i\in V} c(i)L_i^{\alpha_i,\gamma_i}, \tag{108}$$

where L_i denotes the generator (107) acting on the variable η_i.

We first rewrite the boundary generators in terms of creation and annihilation operators. The latter are defined as acting on a function $f : \mathbb{N} \to \mathbb{R}$ via

$$af(n) = nf(n - 1) \\ a^\dagger f(n) = f(n + 1), \tag{109}$$

where in (109) it is understood $af(0) = 0$. We denote by a_i, resp. $a_i^\dagger$ these operators acting on the variable η_i. Then these operators satisfy the conjugate Heisenberg algebra commutation relations, i.e.

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \ [a_i^\dagger, a_j] = \delta_{i,j} \tag{110}$$

and we can rewrite the boundary generator (107) as

$$L^{\alpha,\gamma} = 2s\alpha(a^\dagger - I) + \alpha(aa^\dagger a^\dagger - aa^\dagger) + \gamma(a - aa^\dagger). \tag{111}$$

We first define the Poisson intertwining which turns the operators $a, a^\dagger$ into differential operators.

Lemma 4.1 *Define, for $f : \mathbb{N} \to \mathbb{R}$ and $z \geq 0$*

$$\mathcal{G} f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} e^{-z} f(n). \tag{112}$$

Then we have

$$\mathcal{G}af = A\mathcal{G}f, \ \mathcal{G}a^\dagger f = A^\dagger\mathcal{G}f, \tag{113}$$

with

$$Af(z) = zf(z),\;\; A^\dagger f(z) = f'(z) + f(z). \tag{114}$$

As a consequence

$$\mathcal{G} L^{\alpha,\gamma} = \mathcal{L}^{\alpha,\gamma}\mathcal{G}, \tag{115}$$

with

$$\mathcal{L}^{\alpha,\gamma} = \alpha z \partial_z^2 + (2s\alpha - (\gamma - \alpha)z)\partial_z. \tag{116}$$

Proof The intertwinings (113) follow from a direct computation. Then (115) follows from (111) and (113). □

Notice that $\mathcal{G} f(z) = \int f(n)\pi_z(dn)$ where π_z is the Poisson measure with parameter z. We extend as usual the intertwining Λ by tensorization, i.e., for $f : \mathbb{N}^V \to \mathbb{R}$

$$\mathcal{G} f(\zeta) = \int f(\eta)(\otimes \pi_{\zeta_i})(d\eta),$$

then we have the following intertwining result.

Theorem 4.1 *The boundary driven SIP with generator* (108) *is Poisson intertwined with the boundary driven BEP process with generator*

$$\mathcal{L} = \sum_{i,j\in V} p(i,j)\mathcal{L}_{ij} + \sum_{i\in V} c(i)\mathcal{L}_i^{\alpha_i,\gamma_i}. \tag{117}$$

Here $\mathcal{L}_{ij}$ is the single edge generator of the Brownian enery process (BEP), given by

$$\mathcal{L}_{ij} = \zeta_i\zeta_j(\partial_i - \partial_j)^2 - 2s(\zeta_i - \zeta_j)(\partial_i - \partial_j). \tag{118}$$

Here ∂_i denotes partial derivative w.r.t. ζ_i, and $\mathcal{L}_i^{\alpha_i,\gamma_i}$ is (116) *acting on the variable ζ_i.*

As a consequence we have the following result on propagation of Poisson product measures. If we start the boundary driven SIP from the product Poisson measure $\otimes_{i\in V}\pi_{\zeta_i}$ then we have

$$(\otimes_{i\in V}\pi_{\zeta_i})S(t) = \mathbb{E}_\zeta\left(\otimes_{i\in V}\pi_{\zeta_i(t)}\right), \tag{119}$$

where $\zeta(t)$ evolves according to the generator $\mathcal{L}$ in (117)*. As a further consequence the unique stationary measure of the boundary driven SIP is a mixture of Poisson measures with*

$$\int (\otimes_{i\in V}\pi_{\zeta_i})\,\Xi\Big(\prod_{i\in V} d\zeta_i\Big),$$

where the mixture measure Ξ is the unique stationary measure of the process with generator (117)*.*

Proof The intertwining follows from the combination of Lemma 4.1 with the fact that the single edge generators of SIP and BEP are Poisson intertwined see e.g. [16], or [6], i.e., for all ij

$$\mathcal{G} L_{ij} = \mathcal{L}_{ij}\mathcal{G}.$$

□

4.2 Independent Random Walkers

The independent random walk process on two sites is given by the generator

$$L_{12}f(n_1,n_2) = n_1(f(n_1-1,n_2+1) - f(n_1,n_2)) + n_2(f(n_1-1,n_2+1) - f(n_1,n_2))$$

and the boundary generator

$$Lf(n) = \alpha(f(n+1) - f(n)) + \gamma n(f(n-1) - f(n)).$$

Then the full boundary driven model for independent random walkers reads as follows,

$$L = \sum_{i,j\in V} p(i,j)L_{ij} + \sum_{i\in V} c(i)L_i^{\alpha_i,\gamma_i}. \tag{120}$$

In terms of the creation and annihilation operators the generators read

$$L_{ij} = -(a_i - a_j)(a_i^\dagger - a_j^\dagger) \tag{121}$$

for the single edge generator and

$$L_i^{\alpha_i,\gamma_i} = \alpha_i(a_i^\dagger - I) + \gamma_i(a_i - a_i a_i^\dagger) \tag{122}$$

for the boundary generator. Therefore, using Lemma 4.1, we obtain that the boundary generator is Poisson intertwined with the operator

$$\mathcal{L}_i^{\alpha_i,\gamma_i} = \alpha_i(A_i^\dagger - I) + \gamma_i(A_i - A_i A_i^\dagger) = (\alpha_i - \gamma_i z_i)\partial_i \tag{123}$$

and the single edge generator is intertwined with the operator

$$\mathcal{L}_{ij} = -(\zeta_i - \zeta_j)(\partial_i - \partial_j). \tag{124}$$

Notice that $\mathcal{L}_i^{\alpha_i,\gamma_i}$ and $\mathcal{L}_{ij}$ are first order differential operators and therefore the process build from them is a deterministic system of ODEs. We then immediately obtain the following analogue of Theorem 4.1.

Theorem 4.2 *The boundary driven independent random walkers with generator* (120) *is Poisson intertwined with the boundary driven deterministic process with generator*

$$\mathcal{L} = \sum_{i,j\in V} p(i,j)\mathcal{L}_{ij} + \sum_{i\in V} \mathcal{L}_i^{\alpha_i,\gamma_i}. \tag{125}$$

Here $\mathcal{L}_{ij}$ is the single edge generator (124) *and $\mathcal{L}_i^{\alpha_i,\gamma_i}$ is* (123).

As a consequence we have the following. When we start the boundary driven SIP from the product Poisson measure $\otimes_{i\in V}\pi_{\zeta_i}$ then we have

$$(\otimes_{i\in V}\pi_{\zeta_i})S(t) = \otimes_{i\in V}\pi_{Z_i^\zeta(t)}, \tag{126}$$

where $Z^\zeta(t)$ evolves according to the deterministic generator $\mathcal{L}$ in (125)*. As a further consequence the unique stationary measure of the boundary driven independent random walkers is a Poisson product measure*

$$\otimes_{i\in V}\pi_{\zeta_i^*}.$$

ζ^* is the unique fixed point of the deterministic system $Z^\zeta(t)$.

Acknowledgements We thank an anonymous referee for suggesting us to prove the equality of the generators (81) and (102).

Data Availibility Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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