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# Ad hoc test functions for homogenization of compressible viscous fluid with application to the obstacle problem in dimension two

Marco Bravin

*Abstract.* In this paper, we highlight a set of ad hoc test functions to study the homogenization of viscous compressible fluids in domains with very tiny holes. This set of functions allows to improve previous results in dimensions two and three. As an application, we show that the presence of a small obstacle does not influence the dynamics of a viscous compressible fluid in dimension two.

## 1. Introduction

The study of the interaction of a large number of small holes and a viscous fluid falls in the field of homogenization. Starting from the works of Tartar [21] and Allaire [1,2], it has been proved that the macroscopic behavior of an incompressible viscous fluid in a perforated domain depends on the size of the holes and their mutual distances. In particular, three regimes are possible. When the holes are tiny (subcritical case), they do not influence the dynamic of the fluid in the limit. When the holes are large (supercritical), they put large friction to the fluid and a rescaled fluid velocity satisfies a Darcy's law in the limit. The critical case leads to a Brinkman-type term in the limit equations. The works [1,2] have been extended to the non-stationary setting in [6–16].

In the case the fluid is viscous and compressible and modeled by the compressible Navier–Stokes, there is not such a complete picture as for the incompressible viscous fluid. Masmoudi tackled the supercritical case in [15]. This result was then extended in [10]. The critical case is open and the only available result is [3] where the authors consider also the low Mach number limit. While the subcritical case has been tackled in the stationary case in [7–17], the non-stationary case was studied only in dimension three in [14–19].

The goal of this work is to introduce ad hoc test functions that allow to study the homogenization in the subcritical regime of the non-stationary compressible Navier–Stokes in dimension two and also allow to extend the result [14–19] for a larger class of pressure laws.

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### 1.1. New ideas and possible applications

The homogenization of the compressible Navier–Stokes in the subcritical regime has been studied in [7–19]. In these works, the number of holes is comparable to $\varepsilon^{-d}$ where $d$ is the dimension and the small parameter $\varepsilon > 0$ is a lower bound for the distance between any couple of distinct holes. These types of results require in general a lower bound on $\gamma$ and $\alpha$ which appear respectively in the pressure law $p(\rho) = \rho^\gamma$ and on the size $\varepsilon^\alpha$ of the holes. These limitations on $\gamma$ and $\alpha$ are used in two points. The first one is to show improved pressure estimates independent of $\varepsilon$. The second one is to verify that the limit of the solutions of the compressible Navier–Stokes equations in the domain with tiny holes satisfies the same system but in the domain without holes. For example in [14], the limitations are

$$\gamma > 6, \quad \alpha > 3 \quad \text{and} \quad \frac{\gamma - 6}{2\gamma - 3}\alpha > 3,$$

in [7] they are $\gamma > 2$, $\alpha > 3$ and $\alpha(\gamma - 2) > 2\gamma - 3$. Let us notice that the more severe hypotheses are used to verify that the limit solves the compressible Navier–Stokes equations, in fact in [7–19], the authors multiply the test functions for the limiting system by cut-offs to make them compatible with domains with tiny holes. Let now for example consider the case of the non-stationary compressible Navier–Stokes system outside a tiny hole $\mathscr{S}_\varepsilon = \varepsilon\mathscr{S}$ in dimension three. Following the strategy used in [14], to show that the limit satisfies the compressible Navier–Stokes is enough to multiply any smooth test function $\varphi$ by $\eta_\varepsilon$ a scaling cut-off of the type $\eta_\varepsilon = \eta(x/\varepsilon)$ where $\eta_1$ is 1 in $\mathscr{S}$ and 0 outside $2\mathscr{S}$. In this way, $\varphi\eta_\varepsilon$ is an admissible test function for the domain with the hole $\mathscr{S}_\varepsilon$ and it remains to pass to the limit in the weak formulation. The term that gives the limitation is

$$\int \rho_\varepsilon^\gamma \operatorname{div}(\eta_\varepsilon\varphi) = \int \rho_\varepsilon^\gamma \eta_\varepsilon \operatorname{div}(\varphi) + \int \rho_\varepsilon^\gamma \varphi \cdot \nabla\eta_\varepsilon. \tag{1}$$

The difficult term to tackle is the second one on the right-hand side. Notice that $\|\nabla\eta_\varepsilon\|_{L^p}$ converges to 0 for $p < 3$. We can then conclude only if we are able to show a uniform bound for $\rho_\varepsilon$ in $L^q$ with $q > 3\gamma/2$. This condition together with the fact that the improved pressure estimate holds for $\gamma + \theta \leq 5\gamma/3 - 1$, we deduce the limitation

$$\frac{5}{3}\gamma - 1 > \frac{3}{2}\gamma \quad \text{if and only if} \quad \gamma > 6.$$

The situation is even worse in the case of dimension two because $\|\nabla\eta_\varepsilon\|_{L^p} \longrightarrow 0$ only for $p < 2$ and the improved pressure estimate holds for $\gamma + \theta < 2\gamma - 1$. In particular, $2\gamma - 1 > 2\gamma$ is false for any $\gamma$. For this reason, there are no results on the homogenization of unsteady compressible Navier–Stokes equations in this setting when the dimension is two.

To avoid this issue in dimension two, we introduce the ad hoc test function

$$\Phi_\varepsilon[\varphi] = \eta_\varepsilon\varphi + \nabla^\perp\eta_\varepsilon x^\perp \cdot \varphi.$$

This function has much better properties because if we define

$$\tilde{\Phi}^0_\varepsilon[\varphi] = (1-\eta_\varepsilon)\varphi(0) - \nabla^\perp \eta_\varepsilon x^\perp \cdot \varphi(0),$$

we have

$$\mathrm{div}\,(\tilde{\Phi}^0_\varepsilon[\varphi]) = 0.$$

This allows us to rewrite

$$\begin{aligned}\mathrm{div}\,(\Phi_\varepsilon[\varphi]) - \eta_\varepsilon \mathrm{div}\,(\varphi) &= \mathrm{div}\,(\Phi_\varepsilon[\varphi]) + \mathrm{div}\,(\tilde{\Phi}^0_\varepsilon[\varphi]) - \eta_\varepsilon \mathrm{div}\,(\varphi) \\ &= \nabla\eta_\varepsilon(\varphi - \varphi(0)) + \nabla^\perp \eta_\varepsilon \otimes (\varphi - \varphi(0)) : \nabla x^\perp \\ &\quad + \nabla^\perp \eta_\varepsilon \otimes x^\perp : \nabla\varphi.\end{aligned}$$

If, for example, $\varphi$ is Lipschitz, the term of the type

$$\nabla\eta_\varepsilon(\varphi - \varphi(0)) = |x|\nabla\eta_\varepsilon \frac{(\varphi(x) - \varphi(0))}{|x|}$$

converges to zero in any $L^p$ with $p < +\infty$. In particular, we have that

$$\begin{aligned}\int \rho_\varepsilon^\gamma \mathrm{div}\,(\Phi_\varepsilon[\varphi]) &= \int \rho_\varepsilon^\gamma \eta_\varepsilon \mathrm{div}\,(\varphi) + \int \rho_\varepsilon^\gamma (\mathrm{div}\,(\Phi_\varepsilon[\varphi]) - \eta_\varepsilon \mathrm{div}\,(\varphi)) \\ &\longrightarrow \int \overline{\rho^\gamma} \mathrm{div}\,(\varphi),\end{aligned}$$

if we have a uniform bound of $\rho_\varepsilon$ in $L^p$ for some $p > \gamma$.

In dimension three, a possible set of ad hoc test functions is

$$\Phi_\varepsilon[\varphi] = \eta_\varepsilon\varphi + \begin{pmatrix} x_2\partial_2\eta_\varepsilon\varphi_1 - x_2\partial_3\eta_\varepsilon\varphi_3 \\ x_3\partial_3\eta_\varepsilon\varphi_2 - x_2\partial_1\eta_\varepsilon\varphi_1 \\ x_1\partial_1\eta_\varepsilon\varphi_3 - x_3\partial_2\eta_\varepsilon\varphi_2 \end{pmatrix},$$

with

$$\tilde{\Phi}^0_\varepsilon[\varphi] = (1-\eta_\varepsilon)\varphi(0) - \begin{pmatrix} x_2\partial_2\eta_\varepsilon\varphi_1(0) - x_2\partial_3\eta_\varepsilon\varphi_3(0) \\ x_3\partial_3\eta_\varepsilon\varphi_2(0) - x_2\partial_1\eta_\varepsilon\varphi_1(0) \\ x_1\partial_1\eta_\varepsilon\varphi_3(0) - x_3\partial_2\eta_\varepsilon\varphi_2(0) \end{pmatrix}.$$

With the help of ad hoc test functions $\Phi_\varepsilon[\varphi]$, we can improve the hypothesis on $\gamma$ and $\alpha$ in the study of homogenization of compressible viscous fluid in the subcritical regime. Moreover, the restrictions on the parameters will not come from the pressure term, so we expect that [14] can be shown in the case $\gamma > 3$ and $\alpha > \max\{3, (2\gamma - 3)/(\gamma - 3)\}$ This result can be extended also in the case of dimension two for appropriate lower bounds of $\gamma$ and $\alpha$.

To verify that the idea introduced in this section works, we apply it to a simpler problem that is the obstacle problem in dimension two. In particular, we show that the presence of a small hole does not influence the dynamic of a viscous compressible fluid in dimension two.

Let us recall that in the case the hole is replaced by a rigid body, in [8], the authors extend the result [5] and they show that the small object does not influence the dynamics of a viscous compressible fluid in dimension three under the hypothesis $\gamma > 3/2$. Finally, let us mention that homogenization of compressible Navier–Stokes has been studied also in the setting of randomly perforated domains in [4].

## 2. The obstacle problem in dimension two

At a mathematical level, we consider $\Omega \subset \mathbb{R}^2$ an open, bounded, connected, simply connected subset of $\mathbb{R}^2$ with Lipschitz boundary such that $0 \in \Omega$. For a small parameter $\varepsilon > 0$, we consider a sequence of small holes $\mathscr{S}_\varepsilon \subset B_\varepsilon(0) \subset \Omega$ such that they are open, connected, simply connected and with Lipschitz boundary. The fluid domain is $\mathscr{F}_\varepsilon = \Omega \setminus \mathscr{S}_\varepsilon$ and to model a viscous compressible fluid in $\mathscr{F}_\varepsilon$, we consider the compressible Navier–Stokes equations that reads

$$\begin{aligned} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) &= 0 \quad \text{for} \quad x \in \mathscr{F}_\varepsilon, \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) - \operatorname{div}(\mathbb{S}(u_\varepsilon)) + \nabla p(\rho_\varepsilon) &= 0 \quad \text{for} \quad x \in \mathscr{F}_\varepsilon, \\ u_\varepsilon &= 0 \quad \text{for} \quad x \in \partial\mathscr{F}_\varepsilon, \\ \rho_\varepsilon(0,.) = \rho_\varepsilon^{in}, \quad (\rho_\varepsilon u_\varepsilon)(0) &= q^{in} \quad \text{for} \quad x \in \mathscr{F}_\varepsilon, \end{aligned} \tag{2}$$

where $u_\varepsilon : \mathbb{R}^+ \times \mathscr{F}_\varepsilon \longrightarrow \mathbb{R}^2$ describes the velocity of the fluid, $\rho_\varepsilon : \mathbb{R}^+ \times \mathscr{F}_\varepsilon \to \mathbb{R}^+$ is its density,

$$\mathbb{S}(u_\varepsilon) - p(\rho_\varepsilon)\mathbb{I} = 2\mu D(u_\varepsilon) + (\lambda - \mu)\operatorname{div}(u_\varepsilon)\mathbb{I} - \rho_\varepsilon^\gamma \mathbb{I},$$

is the stress tensor, $D(u_\varepsilon)$ is the symmetric gradient, in other words, $2D(u_\varepsilon) = \nabla u_\varepsilon + (\nabla u_\varepsilon)^T$ and $\mathbb{I}$ is the two-dimensional identity matrix. Moreover, we assume $\mu > 0$, $\lambda \geq 0$ and $\gamma > 1$. Finally, $\rho_\varepsilon^{in} \geq 0$ is the initial density and $q_\varepsilon^{in}$ is the initial momentum which satisfies the condition

$$q_\varepsilon^{in}(x) = 0 \quad \text{for any} \quad x \in \left\{ y \in \mathscr{F}_\varepsilon \text{ such that } \rho_\varepsilon^{in} = 0 \right\}. \tag{3}$$

The above system has been widely studied in the past years, and the existence of finite energy weak solutions has been proved by Lions and Feireisl see [12,18].

In this paper, we study the limit as $\varepsilon$ goes to zero for solutions of (2), in particular we show that under some mild assumption on $\rho_\varepsilon^{in}$ and $q_\varepsilon^{in}$ solutions $(\rho_\varepsilon, u_\varepsilon)$ converge in an appropriate sense to a solution $(\rho, u)$ of the system (2) with $\mathscr{F}_\varepsilon$ replaced by $\Omega$.

## 3. Definition of weak solutions and main result

In this section, we recall the definition of bounded energy weak solution for the system (2) from [18], Definition 7.3. Then, we present the main result of the paper.

In the following, we denote by $P$ the function $P(\rho) = \rho^\gamma/(\gamma - 1)$. For simplicity, we do not write the small parameter $\varepsilon$ in the next definition.

**Definition 1.** Let $T > 0$ and let $(\rho^{in}, q^{in})$ be an initial data satisfying (3) such that $P(\rho^{in}) \in L^1(\mathscr{F})$ and $|q^{in}|^2/\rho^{in} \in L^1(\mathscr{F})$. Then, a triple $(\rho, u)$ is a weak solution of (2) in $(0, T) \times \mathscr{F}$ with initial datum $(\rho^{in}, q^{in})$ if

- $\rho \in L^\infty(0, T; L^1(\mathscr{F}))$ such that $\rho \geq 0$ and $P(\rho) \in L^\infty(0, T; L^1(\mathscr{F}))$.
- $u \in L^2(0, T; W_0^{1,2}(\mathscr{F}))$.
- $(\rho, u)$ satisfies the continuity equation $\partial_t \rho + \mathrm{div}\,(\rho u) = 0$ in both a distributional sense in $[0, T) \times \mathbb{R}^2$ and in a renormalized[1] sense where we extend $\rho$ and $u$ by zero in the exterior of $[0, T) \times \mathscr{F}$.
- the momentum equation is satisfied in the weak sense

$$\int_{\mathscr{F}} q^{in}\varphi(0, .)\,dx + \int_0^T \int_{\mathscr{F}} \rho u \cdot \partial_t \varphi + [\rho u \otimes u] : D\varphi + \rho^\gamma \mathrm{div}\,\varphi\,dxdt$$
$$= \int_0^T \int_{\mathscr{F}} \mathbb{S}u : D\varphi\,dxdt,$$

  for any $\varphi \in C^\infty([0, T) \times \mathscr{F})$.
- for a.e. $\tau \in [0, T]$ the following energy equality holds

$$\int_{\mathscr{F}} \frac{1}{2}\rho|u|^2(\tau, .) + P(\rho(\tau, .))\,dx + \int_0^\tau \int_{\mathscr{F}} \mu|\nabla u|^2 + \lambda|\mathrm{div}\,u|^2\,dxdt$$
$$\leq \int_{\mathscr{F}} \frac{1}{2}\frac{|q^{in}|^2}{\rho^{in}} + P(\rho^{in})\,dx.$$

We can now recall the existence result of weak solutions.

**Theorem 1.** *Let $T > 0$ and let $(\rho^{in}, q^{in})$ be an initial data satisfying (3) such that $P(\rho^{in}) \in L^1(\mathscr{F})$ and $|q^{in}|^2/\rho^{in} \in L^1(\mathscr{F})$. Then there exists a solution $(\rho, u)$ of (2) in $(0, T) \times \mathscr{F}$ in the sense of Definition 1 with initial datum $(\rho^{in}, q^{in})$.*

The proof is classical, see, for instance, [12] or Sect. 7 of [18].

In the following, for any function or vector field $f_\varepsilon$ defined on $\mathscr{F}_\varepsilon$ we denote, with an abuse of notation, by $f_\varepsilon$ also its extension by zero in $\Omega$ or $\mathbb{R}^2$. We are now able to state our main result.

**Theorem 2.** *Let $T > 0$, let $\gamma > 2$, let $(\rho_\varepsilon^{in}, q_\varepsilon^{in})$ be a sequence of initial data satisfying (3) such that $P(\rho_\varepsilon^{in}) \in L^1(\mathscr{F}_\varepsilon)$, $|q_\varepsilon^{in}|^2/\rho_\varepsilon^{in} \in L^1(\mathscr{F}_\varepsilon)$ and let $(\rho^{in}, q^{in})$ satisfying (3) with $\mathscr{F}_\varepsilon = \Omega$ such that $P(\rho^{in}) \in L^1(\Omega)$ and $|q^{in}|^2/\rho^{in} \in L^1(\Omega)$. If*

- *$\rho_\varepsilon^{in} \longrightarrow \rho^{in}$ in $L^\gamma(\Omega)$,*
- *$|q_\varepsilon^{in}|^2/\rho_\varepsilon^{in} \longrightarrow |q^{in}|^2/\rho^{in}$ in $L^1(\Omega)$,*

[1] We refer to Sect. 6.2 of [18] for the definition and some basic properties of renormalized solutions to the continuity equation.

*then up to subsequence there exists* $(\rho, u)$ *such that*

$$\rho_\varepsilon \longrightarrow \rho \text{ in } C_w(0,T;L^\gamma(\Omega)) \quad \text{and} \quad u_\varepsilon \xrightarrow{w} u \text{ in } L^2\left(0,T;W_0^{1,2}(\Omega)\right). \tag{4}$$

*Moreover* $(\rho, u)$ *satisfies* (2) *in* $(0,T)\times\Omega$ *and with initial data* $(\rho^{in}, q^{in})$*, in the sense of Definition* 1.

Let us explain where we use the condition $\gamma > 2$.

*Remark 1.* Although the existence of weak solutions to (2) holds for $\gamma > 1$, in Theorem 2 we consider the case $\gamma > 2$. This restriction comes from the fact that in dimension two to pass to the limit in the term

$$\int_0^T \int_{\mathscr{F}_\varepsilon} \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : D\Phi_\varepsilon[\varphi]\, dxdt,$$

we use that $\rho_\varepsilon$, $u_\varepsilon$ and $D\Phi_\varepsilon[\varphi]$ are uniformly bounded, respectively, in $L^\infty(0,T;L^\gamma(\mathscr{F}_\varepsilon))$, in $L^2(0,T;L^p(\mathscr{F}_\varepsilon))$ for any $p < +\infty$ and $L^\infty(0,T;L^2(\mathscr{F}_\varepsilon))$, together with the condition

$$\frac{1}{\gamma} + \frac{1}{\infty} + \frac{1}{\infty} + \frac{1}{2} < 1 \quad \text{if and only if} \quad \gamma > 2.$$

In the remaining part of the paper, we show Theorem 2.

## 4. A priori estimates

By definition of weak solution to the system (2), any solution $(\rho_\varepsilon, u_\varepsilon)$ satisfies the inequalities

$$\|\rho_\varepsilon(t,.)\|_{L^1(\mathscr{F}_\varepsilon)} = \|\rho_\varepsilon^{in}\|_{L^1(\mathscr{F}_\varepsilon)}$$

and

$$\int_{\mathscr{F}_\varepsilon} \frac{1}{2}\rho_\varepsilon |u_\varepsilon|^2(\tau,.) + P(\rho_\varepsilon(\tau,.))\, dx + \int_0^\tau \int_{\mathscr{F}_\varepsilon} \mu|\nabla u_\varepsilon|^2 + \lambda|\operatorname{div} u_\varepsilon|^2\, dxdt \\ \leq \int_{\mathscr{F}} \frac{1}{2}\frac{|q_\varepsilon^{in}|^2}{\rho_\varepsilon^{in}} + P(\rho_\varepsilon^{in})dx.$$

By the hypothesis of Theorem 2, the right-hand side of the above inequalities are uniformly bounded in $\varepsilon$. In particular, we deduce that

$$\begin{aligned} \|\rho_\varepsilon\|_{L^\infty(0,T;L^\gamma(\Omega))} &\leq C, \\ \|\sqrt{\rho_\varepsilon} u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} &\leq C, \\ \|u_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega))} &\leq C. \end{aligned} \tag{5}$$

Moreover, we can show the following improved pressure estimates

**Lemma 1.** *Let $\gamma > 2$, under the hypothesis of Theorem 2, for any $\theta < \gamma - 1$ it holds*

$$\int_0^T \int_{\Omega \setminus B_{2\varepsilon}(0)} \rho_\varepsilon^{\gamma+\theta} \le C,$$

*where $C$ is independent of $\varepsilon$.*

We postpone the proof of this estimate in "Appendix B" because it is classical.

## 5. Some appropriate cut-off

In this section, we introduce some cut-off functions that have been considered also in [5,9]. These cut-offs have the property that they optimized the $L^2$ norm of the gradient and we denote them by $\eta_{\varepsilon,\alpha_\varepsilon}$. The parameter $\varepsilon > 0$ indicates that $\eta_{\varepsilon,\alpha_\varepsilon} = 1$ in the ball $B_\varepsilon(0)$ and $\alpha_\varepsilon$ that the support of the $\eta_{\varepsilon,\alpha_\varepsilon}$ is contained in the ball of size $\varepsilon\alpha_\varepsilon$.

**Proposition 1.** *For any $\varepsilon > 0$ and $\alpha_\varepsilon \ge 2$, there exists a cut-off function $\eta_{\varepsilon,\alpha_\varepsilon} \in C_c^\infty(B_{\varepsilon\alpha_\varepsilon}(0))$ such that $\eta_{\varepsilon,\alpha_\varepsilon}(x) = 1$ for $x \in B_\varepsilon(0)$, $\|\eta_{\varepsilon,\alpha_\varepsilon}\|_{L^\infty} \le 1$ and the following bounds hold with constant $C$ independent of $\varepsilon$ and $\alpha_\varepsilon$.*

1. *For $1 \le q < +\infty$*

$$\|\eta_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^2)} + \||x|\nabla\eta_{\varepsilon,\alpha}\|_{L^q(\mathbb{R}^2)} \le C(\varepsilon\alpha_\varepsilon)^{2/q}.$$

2. *We have*

$$\left\|\nabla\eta_{\varepsilon,\alpha_\varepsilon}\right\|^2_{L^2(\mathbb{R}^2)} + \left\||x|\nabla^2\eta_{\varepsilon,\alpha_\varepsilon}\right\|^2_{L^2(\mathbb{R}^2)} \le \frac{C}{(\log\alpha_\varepsilon)}.$$

3. *For $1 \le q < 2$,*

$$\left\|\nabla\eta_{\varepsilon,\alpha_\varepsilon}\right\|^q_{L^q(\mathbb{R}^2)} + \left\||x|\nabla^2\eta_{\varepsilon,\alpha_\varepsilon}\right\|^q_{L^q(\mathbb{R}^2)} \le \frac{C}{2-q}\frac{(\varepsilon\alpha_\varepsilon)^{2-q}}{(\log\alpha_\varepsilon)^q}.$$

4. *For $2 < q < +\infty$,, for $i = 1, 2$,*

$$\left\|\nabla\eta_{\varepsilon,\alpha_\varepsilon}\right\|^q_{L^q(\mathbb{R}^2)} + \left\|\nabla^2\eta_{\varepsilon,\alpha_\varepsilon}x_i\right\|^q_{L^q(\mathbb{R}^2)} = \frac{C}{q-2}\frac{\varepsilon^{2-q}}{(\log\alpha_\varepsilon)^q}.$$

*In particular, if $\alpha_\varepsilon \le |\log(\varepsilon)|$ and $\alpha_\varepsilon \longrightarrow +\infty$,*

$$\left\|\nabla\eta_{\varepsilon,\alpha_\varepsilon}\right\|_{L^q(\mathbb{R}^2)} + \left\||x|\nabla^2\eta_{\varepsilon,\alpha_\varepsilon}\right\|_{L^q(\mathbb{R}^2)} \longrightarrow 0 \;\; for \;\; 1 \le q \le 2$$

*and*

$$\varepsilon\alpha_\varepsilon\left\|\nabla\eta_{\varepsilon,\alpha_\varepsilon}\right\|_{L^q(\mathbb{R}^2)}, \varepsilon\alpha_\varepsilon\left\|\nabla^2\eta_{\varepsilon,\alpha_\varepsilon}x_i\right\|_{L^q(\mathbb{R}^2)} \longrightarrow 0 \;\; for \;\; 2 < q < +\infty.$$

The proof of the above proposition is a straightforward extension of Lemma 3 of [9], so let us postpone the proof in "Appendix C."

Under the assumption $\alpha_\varepsilon \leq |\log(\varepsilon)|$ and $\alpha_\varepsilon \longrightarrow +\infty$, we denote $1-\eta_{2\varepsilon,\alpha_{2\varepsilon}} = \mathfrak{n}_\varepsilon$.

Let now present another useful estimate. For a function $\varphi \in L^1(\Omega)$, we denote by

$$\Phi_\varepsilon[\varphi] = \mathfrak{n}_\varepsilon \varphi + \nabla^\perp \mathfrak{n}_\varepsilon x^\perp \cdot \varphi$$

and by

$$\Phi_\varepsilon^0[\varphi] = (1-\mathfrak{n}_\varepsilon)\langle\varphi\rangle_\varepsilon(0) - \nabla^\perp \mathfrak{n}_\varepsilon x^\perp \cdot \langle\varphi\rangle_\varepsilon(0)$$

where

$$\langle\varphi\rangle_\varepsilon(0) = \frac{1}{|B_{2\varepsilon\alpha_{2\varepsilon}}(0)|}\int_{|B_{2\varepsilon\alpha_{2\varepsilon}}(0)|} \varphi.$$

The following holds.

**Lemma 2.** *Let $p, q \in [1,+\infty]$. Then, there exist a constant $c_{p,q}(\varepsilon)$ such that $c_{p,q}(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ and for any vector field $\varphi : \Omega \longrightarrow \mathbb{R}^2$, it holds*

$$\|\Phi_\varepsilon[\varphi]-\varphi\|_{L^p(\Omega)} \leq c_{p,q}(\varepsilon)\|\varphi\|_{L^q(\Omega)} \quad \textit{and} \quad \|\Phi_\varepsilon[\varphi]-\mathfrak{n}_\varepsilon\varphi\|_{L^p(\Omega)} \leq c_{p,q}(\varepsilon)\|\varphi\|_{L^q(\Omega)}$$

*for $p < q < \infty$.*

$$\|\nabla\Phi_\varepsilon[\varphi] - \mathfrak{n}_\varepsilon\nabla\varphi\|_{L^p(\Omega)} \leq c_{p,q}(\varepsilon)\|\varphi\|_{W^{1,q}(\Omega)}$$

*for $p \leq 2$ and $q > 2$. Finally,*

$$\|\mathrm{div}\,(\Phi_\varepsilon[\varphi]) - \mathfrak{n}_\varepsilon \mathrm{div}\,(\varphi)\|_{L^p(\Omega)} \leq c_{p,q}(\varepsilon)\|\varphi\|_{L^q(\Omega)} \tag{6}$$

*for $p < q < \infty$.*

In the following, we always omit the dependence on $p$ and $q$ for $c_{p,q}(\varepsilon)$ and we write $c_{p,q}(\varepsilon) = c(\varepsilon)$.

*Proof.* The proof of these inequalities follows from the definition of $\Phi_\varepsilon[\varphi]$, Poincaré inequality and Proposition 1. Let us recall that for any functions $f \in C^\infty(B_{\varepsilon\alpha_\varepsilon}(0))$ the Poincaré inequality reads

$$\|\varphi - \langle\varphi\rangle_{\varepsilon/2}(0)\|_{L^q(B_{\varepsilon\alpha_\varepsilon(0)})} \leq C\varepsilon\alpha_\varepsilon\|\nabla f\|_{L^q(B_{\varepsilon\alpha_\varepsilon}(0))} \tag{7}$$

where $C$ is independent of $\varepsilon$ and $\alpha_\varepsilon$. The proof of (7) follows from a simple scaling argument.

The most interesting inequality is (6), so we will prove it. First of all notice that $\mathrm{div}\,(\Phi_\varepsilon^0[\varphi]) = 0$, in fact

$$\begin{aligned}\operatorname{div}(\Phi_\varepsilon^0[\varphi]) &= \operatorname{div}((1-\mathfrak{n}_\varepsilon)\langle\varphi\rangle_\varepsilon(0) - \nabla^\perp\mathfrak{n}_\varepsilon x^\perp\cdot\langle\varphi\rangle_\varepsilon(0))\\ &= \operatorname{div}(\nabla^\perp((1-\mathfrak{n}_\varepsilon)x^\perp\cdot\langle\varphi\rangle_\varepsilon(0))) = 0.\end{aligned}$$

Then,

$$\begin{aligned}\operatorname{div}(\Phi_\varepsilon[\varphi]) - \mathfrak{n}_\varepsilon\operatorname{div}(\varphi) &= \operatorname{div}(\Phi_\varepsilon[\varphi]) + \operatorname{div}(\Phi_\varepsilon^0[\varphi]) - \mathfrak{n}_\varepsilon\operatorname{div}(\varphi)\\ &= \nabla\mathfrak{n}_\varepsilon(\varphi - \langle\varphi\rangle_\varepsilon(0)) + \nabla^\perp\mathfrak{n}_\varepsilon\otimes x^\perp : \nabla\varphi\\ &\quad + \nabla^\perp\mathfrak{n}_\varepsilon\otimes(\varphi-\langle\varphi\rangle_\varepsilon(0)) : \nabla x^\perp.\end{aligned}$$

Using the above equality, we estimate for $1/s = 1/p - 1/q$

$$\begin{aligned}\|\operatorname{div}(\Phi_\varepsilon[\varphi]) - \mathfrak{n}_\varepsilon\operatorname{div}(\varphi)\|_{L^p(\Omega)} &\le \|\nabla\mathfrak{n}_\varepsilon\|_{L^s(\Omega)}\|\varphi - \langle\varphi\rangle_\varepsilon(0)\|_{L^q(B_{2\varepsilon\alpha_{2\varepsilon}}(0))}\\ &\quad + \|\nabla^\perp\mathfrak{n}_\varepsilon\otimes x^\perp\|_{L^s(\Omega)}\|\nabla\varphi\|_{L^q(\Omega)}\\ &\quad + \|\nabla^\perp\mathfrak{n}_\varepsilon\|_{L^s(\Omega)}\|\varphi-\langle\varphi\rangle_\varepsilon(0)\|_{L^q(B_{2\varepsilon\alpha_{2\varepsilon}}(0))}\\ &\le 2\varepsilon\alpha_{2\varepsilon}C\|\nabla\mathfrak{n}_\varepsilon\|_{L^s(\Omega)}\|\nabla\varphi\|_{L^q(\Omega)}\\ &\quad + C\|\nabla^\perp\mathfrak{n}_\varepsilon\otimes x^\perp\|_{L^s(\Omega)}\|\nabla\varphi\|_{L^q(\Omega)}\\ &\quad + 2\varepsilon\alpha_{2\varepsilon}C\|\nabla^\perp\mathfrak{n}_\varepsilon\|_{L^s(\Omega)}\|\nabla\varphi\|_{L^q(\Omega)}\\ &\le c(\varepsilon)\|\varphi\|_{W^{1,q}(\Omega)},\end{aligned}$$

where we use that $p < q$, the Poincaré inequality (7) and Proposition 1. □

## 6. Pass to the limit in the weak formulation

Using the estimates from (5), Lemma 1 and the fact that $(\rho_\varepsilon, u_\varepsilon)$ are solutions to the system (2) the following convergences hold.

**Lemma 3.** *Under the hypothesis of Theorem 2, we have after passing to subsequence that*

$$\begin{aligned}\rho_\varepsilon &\xrightarrow{w} \rho && \text{in } L^{2\gamma-1}([0,T]\times\Omega)\\ \rho_\varepsilon &\longrightarrow \rho && \text{in } C_w^0([0,T); L^\gamma(\Omega))\\ u_\varepsilon &\xrightarrow{w} u && \text{in } L^2(0,T; H_0^1(\Omega))\\ \mathfrak{n}_\varepsilon\rho_\varepsilon u_\varepsilon + \rho_\varepsilon u_\varepsilon\cdot\nabla^\perp\mathfrak{n}_\varepsilon x^\perp &\longrightarrow \rho u && \text{in } C_w^0([0,T); L^{2\gamma/(\gamma+1)}(\Omega))\\ \left(\mathfrak{n}_\varepsilon\rho_\varepsilon u_\varepsilon + \rho_\varepsilon u_\varepsilon\cdot\nabla^\perp\mathfrak{n}_\varepsilon x^\perp\right)\otimes u_\varepsilon &\longrightarrow \rho u\otimes u && \text{in } \mathscr{D}'((0,T)\times\Omega)\\ \mathbb{1}_{\Omega\setminus B_{2\varepsilon}(0)}\rho_\varepsilon^\gamma &\xrightarrow{w} \overline{\rho^\gamma} && \text{in } L^{(2\gamma-1)/\gamma}([0,T]\times\Omega)\end{aligned}$$

*where* $t \in (0, T)$.

*Proof.* Using (5), Lemma 1 and the fact that $(\rho_\varepsilon, u_\varepsilon)$ are solutions to the system (2), it is easy to deduce all the convergence except the fourth one. By (5), we already know that

$$\|\rho_\varepsilon u_\varepsilon\|_{L^\infty(0,T;L^{2\gamma/(\gamma+1)}(\mathscr{F}_\varepsilon))} \leq \|\sqrt{\rho_\varepsilon}\|_{L^\infty(0,T;L^{2\gamma}(\mathscr{F}_\varepsilon))}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^\infty(0,T;L^2(\mathscr{F}_\varepsilon))} \leq C.$$

We deduce that $\rho_\varepsilon u_\varepsilon$ converges weakly star in $L^\infty(0,T;L^{2\gamma/(\gamma+1)}(\Omega))$. To identify the limit, we start by noticing that the convergence $\rho_\varepsilon \to \rho$ in $C^0_w([0,T);L^\gamma(\Omega))$ implies that $\rho_\varepsilon$ converges to $\rho$ also in $C^0([0,T);H^{-1}(\Omega))$, see Lemma 6.2 of [18]. This together with the convergence $u_\varepsilon \xrightarrow{w} u$ in $L^2(0,T;H^1_0(\Omega))$ shows that $\rho_\varepsilon u_\varepsilon$ converges to $\rho u$ in a distributional sense in $(0,T)\times\Omega$. By uniqueness of the limit we have

$$\rho_\varepsilon u_\varepsilon \xrightarrow{w*} \rho u \quad \text{in} \quad L^\infty([0,T);L^{2\gamma/(\gamma+1)}(\Omega)). \tag{8}$$

Using that $|\mathfrak{n}_\varepsilon|$ and $|\nabla^\perp \mathfrak{n}_\varepsilon \otimes x^\perp|$ are bounded, we deduce that $\mathfrak{n}_\varepsilon\rho_\varepsilon u_\varepsilon + \rho_\varepsilon u_\varepsilon \cdot \nabla^\perp \mathfrak{n}_\varepsilon x^\perp$ is uniformly bounded in $L^\infty([0,T);L^{2\gamma/(\gamma+1)}(\Omega))$. Hence, we extract a weak-star convergent subsequence. To identify the limit we notice that by an Hölder inequality

$$\begin{aligned}
&\|\mathfrak{n}_\varepsilon\rho_\varepsilon u_\varepsilon + \rho_\varepsilon u_\varepsilon \cdot \nabla^\perp \mathfrak{n}_\varepsilon x^\perp - \rho_\varepsilon u_\varepsilon\|_{L^\infty(0,T;L^q(\Omega))}\\
&\quad \leq C(\|1-\mathfrak{n}_\varepsilon\|_{L^{2\gamma q/(\gamma q+q-2\gamma)}(\Omega))} + \|\nabla^\perp \mathfrak{n}_\varepsilon \otimes x^\perp\|_{L^{2\gamma q/(\gamma q+q-2\gamma)}(\Omega))})\\
&\quad \|\rho_\varepsilon u_\varepsilon\|_{L^\infty(0,T;L^{2\gamma/(\gamma+1)}(\Omega))}\\
&\quad \longrightarrow 0
\end{aligned}$$

for any $1 \leq q < 2\gamma/(\gamma+1)$ because $1-\mathfrak{n}_\varepsilon$ and $\nabla^\perp\mathfrak{n}_\varepsilon \otimes x^\perp$ are bounded uniformly in $\varepsilon$ and their support converges to zero. We deduce

$$\mathfrak{n}_\varepsilon\rho_\varepsilon u_\varepsilon + \rho_\varepsilon u_\varepsilon \cdot \nabla^\perp \mathfrak{n}_\varepsilon x^\perp \xrightarrow{w*} \rho u \quad \text{in} \quad L^\infty([0,T);L^{2\gamma/(\gamma+1)}(\Omega)).$$

To show the strong convergence in time, it is enough to prove that $\mathfrak{n}_\varepsilon\rho_\varepsilon u_\varepsilon + \rho_\varepsilon u_\varepsilon \cdot \nabla^\perp \mathfrak{n}_\varepsilon x^\perp$ is continuous and equicontinuous in some $H^{-s}(\Omega)$ for some $s$ big enough and to apply Appendix C of [11]. To do that, we will apply the following lemma. □

**Lemma 4.** *Let $H$ a Hilbert space and let $f_n : (0,T) \longrightarrow H$ a sequence of functions. If $\partial_t f_n = g^1_n + g^2_n$ where*

- $\|g^1_n\|_{L^p(0,T;H)} \leq C$ *with $C$ independent of $n$ and $p > 1$,*
- $\lim_{n\to+\infty}\|g^2_n\|_{L^1(0,T;H)} = 0.$

*Then, the functions $f_n$ are continuous and equicontinuous.*

For $\varphi \in C^\infty((0,T)\times\Omega)$, we notice that

$$\begin{aligned}
\int_0^T\int_\Omega \left(\mathfrak{n}_\varepsilon\rho_\varepsilon u_\varepsilon + \rho_\varepsilon u_\varepsilon \cdot \nabla^\perp \mathfrak{n}_\varepsilon x^\perp\right)\cdot\partial_t\varphi &= \int_0^T\int_\Omega \rho_\varepsilon u_\varepsilon \cdot \partial_t\left(\mathfrak{n}_\varepsilon\varphi + \nabla^\perp\mathfrak{n}_\varepsilon x^\perp\cdot\varphi\right)\\
&= \int_0^T\int_\Omega \rho_\varepsilon u_\varepsilon\cdot\partial_t\Phi_\varepsilon[\varphi].
\end{aligned} \tag{9}$$

We can now use the momentum equation of (2) tested with $\Phi_\varepsilon[\varphi]$ to deduce

$$\begin{aligned}\int_0^T\int_\Omega \rho_\varepsilon u_\varepsilon\cdot\partial_t\Phi_\varepsilon[\varphi] = &-\int_0^T\int_\Omega \rho_\varepsilon u_\varepsilon\otimes u_\varepsilon : D\Phi_\varepsilon[\varphi]\\ &+\int_0^T\int_\Omega \mathbb{S}u_\varepsilon : D\Phi_\varepsilon[\varphi]\\ &-\int_0^T\int_\Omega \rho_\varepsilon^\gamma \operatorname{div}(\Phi_\varepsilon[\varphi]).\end{aligned}\tag{10}$$

We now bound the terms on the right-hand side separately. Notice that

$$\begin{aligned}\int_0^T\int_\Omega \rho_\varepsilon u_\varepsilon\otimes u_\varepsilon : D\Phi_\varepsilon[\varphi] = &\int_0^T\int_\Omega \rho_\varepsilon u_\varepsilon\otimes u_\varepsilon : \mathfrak{n}_\varepsilon D\varphi\\ &+\int_0^T\int_\Omega \rho_\varepsilon u_\varepsilon\otimes u_\varepsilon : (D\Phi_\varepsilon[\varphi]-\mathfrak{n}_\varepsilon D\varphi).\end{aligned}$$

Let us recall that in dimension two $W^{1,2}\subset L^p$ for any $p<+\infty$ and that $W^{1,2}\not\subset L^\infty$, in particular, $\|f\|_{L^p}\le C\|f\|_{W^{1,2}}$ for any $p<+\infty$. In the following, we denote by $\|f\|_{L^{\infty-}}$ the norm $\|f\|_{L^p}$ for $p$ big enough. We have

$$\begin{aligned}&\left|\int_0^T\int_\Omega \rho_\varepsilon u_\varepsilon\otimes u_\varepsilon : \mathfrak{n}_\varepsilon D\varphi\right|\\ &\quad\le \|\rho_\varepsilon u_\varepsilon\|_{L^\infty(0,T:L^{2\gamma/(\gamma+1)}(\mathscr{F}_\varepsilon))}\|u_\varepsilon\|_{L^2(0,T;L^{\infty-}(\mathscr{F}_\varepsilon))}\|D\varphi\|_{L^2(0,T;L^q(\Omega))},\end{aligned}$$

for $q>2\gamma/(\gamma-1)$. Moreover,

$$\begin{aligned}&\left|\int_0^T\int_\Omega \rho_\varepsilon u_\varepsilon\otimes u_\varepsilon : (D\Phi_\varepsilon[\varphi]-\mathfrak{n}_\varepsilon D\varphi)\right|\\ &\quad\le \|\rho_\varepsilon\|_{L^\infty(0,T;L^\gamma(\Omega))}\|u_\varepsilon\|^2_{L^2(0,T;L^{\infty-}(\Omega))}\|D\Phi_\varepsilon[\varphi]-\mathfrak{n}_\varepsilon D\varphi\|_{L^1(0,T;L^{\tilde q}(\Omega))}\\ &\quad\le c(\varepsilon)\|\rho_\varepsilon\|_{L^\infty(0,T;L^\gamma(\Omega))}\|u_\varepsilon\|^2_{L^2(0,T;L^{\infty-}(\Omega))}\|\varphi\|_{L^1(0,T;W^{1,q}(\Omega))}\end{aligned}\tag{11}$$

where $2>q>\tilde q>\gamma/(\gamma-1)$ and $c(\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$. In the last inequality, we used Lemma 2. Let now move to the second term of right-hand side of (10). We have

$$\left|\int_0^T\int_\Omega \mathbb{S}u_\varepsilon : D\Phi_\varepsilon[\varphi]\right|\le c\|u_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega))}\|\varphi\|_{L^2(0,T;W^{1,2}(\Omega))}.$$

We are left with the last term of (10). Using that $\Phi_\varepsilon[\varphi]=0$ in $B_{2\varepsilon}(0)$, we rewrite

$$\begin{aligned}\int_0^T\int_\Omega \rho_\varepsilon^\gamma\operatorname{div}(\Phi_\varepsilon[\varphi]) = &\int_0^T\int_\Omega \mathbb{1}_{\Omega\setminus B_{2\varepsilon}(0)}\rho_\varepsilon^\gamma\,\mathfrak{n}_\varepsilon\operatorname{div}(\varphi)\\ &+\int_0^T\int_\Omega \mathbb{1}_{\Omega\setminus B_{2\varepsilon}(0)}\rho_\varepsilon^\gamma\left(\operatorname{div}(\Phi_\varepsilon[\varphi])-\mathfrak{n}_\varepsilon\operatorname{div}(\varphi)\right).\end{aligned}\tag{12}$$

Notice that

$$\left|\int_0^T\int_\Omega \mathbb{1}_{\Omega\setminus B_{2\varepsilon}(0)}\rho_\varepsilon^\gamma\operatorname{div}(\varphi)\right|\le\|\mathbb{1}_{\Omega\setminus B_{2\varepsilon}(0)}\rho_\varepsilon^\gamma\|_{L^p((0,T)\times\Omega)}\|\operatorname{div}(\varphi)\|_{L^q((0,T)\times\Omega)}$$

is uniformly bounded for any $1/p + 1/q = 1$ and $p < (2\gamma - 1)/\gamma$.

The same estimate holds for the second term of (12); moreover, from Lemma 2, we have

$$\begin{aligned}&\left|\int_0^T \int_\Omega \mathbb{1}_{\Omega\setminus B_{2\varepsilon}(0)}\rho_\varepsilon^\gamma \, (\mathrm{div}\,(\Phi_\varepsilon[\varphi]) - \mathfrak{n}_\varepsilon \mathrm{div}\,(\varphi))\right| \\ &\quad\le \|\mathbb{1}_{\Omega\setminus B_{2\varepsilon}(0)}\rho_\varepsilon^\gamma\|_{L^p((0,T)\times\Omega)}\|\mathrm{div}\,(\Phi_\varepsilon[\varphi]) - \mathfrak{n}_\varepsilon \mathrm{div}\,(\varphi)\|_{L^q((0,T)\times\Omega)} \\ &\quad\le c(\varepsilon)\|\mathbb{1}_{\Omega\setminus B_{2\varepsilon}(0)}\rho_\varepsilon^\gamma\|_{L^p((0,T)\times\Omega)}\|\varphi\|_{L^q(0,T;W^{1,\tilde q}(\Omega))} \end{aligned} \tag{13}$$

with $\tilde q > \max\{2, q\}$ and with $c(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

We can now apply Lemma 4 and deduce

$$\mathfrak{n}_\varepsilon \rho_\varepsilon u_\varepsilon + \rho_\varepsilon u_\varepsilon \cdot \nabla^\perp \mathfrak{n}_\varepsilon x^\perp \longrightarrow \rho u \quad \text{in } C^0_w([0,T); L^{2\gamma/(\gamma+1)}(\Omega)).$$

We will now pass to the limit in the weak formulation satisfied by $\rho_\varepsilon$ and $u_\varepsilon$. Let us recall that $(\rho_\varepsilon, u_\varepsilon)$ satisfies the continuity equation $\partial_t \rho_\varepsilon + \mathrm{div}\,(\rho_\varepsilon u_\varepsilon) = 0$ in a distributional sense in $[0,T) \times \mathbb{R}^2$ in other words for any $\psi \in C^\infty_c([0,T) \times \mathbb{R}^2)$ it holds

$$\int_{\mathbb{R}^2} \rho_\varepsilon^{in} \psi(0,.) + \int_0^T \int_{\mathbb{R}^2} \rho_\varepsilon \partial_t \psi + \rho_\varepsilon u_\varepsilon \cdot \nabla \psi = 0. \tag{14}$$

From Lemma 3, we have that $\rho_\varepsilon \longrightarrow \rho$ in $C^0_w([0,T); L^\gamma(\Omega))$, moreover $\rho_\varepsilon u_\varepsilon \xrightarrow{w*} \rho u$ in $L^\infty(0,T; L^{2\gamma/(\gamma+1)}(\Omega))$ from (8). We now pass to the limit in (14) to deduce that $(\rho, u)$ is a distributional solution of $\partial_t \rho + \mathrm{div}\,(\rho u) = 0$ in $[0,T) \times \mathbb{R}^2$. From the assumption $\gamma > 2$, we have $\rho \in L^\infty(0,T; L^2(\mathbb{R}^2))$. Lemma 6.9 of [18] implies that $(\rho, u)$ satisfies the continuity equation also in the renormalized sense.

We now explain how to pass to the limit in the momentum equation. For $\varphi \in C^\infty_c([0,T) \times \Omega)$ we test the weak formulation of the momentum equation satisfied by $\rho_\varepsilon, u_\varepsilon$ with $\Phi_\varepsilon[\varphi] = \mathfrak{n}_\varepsilon \varphi + \nabla^\perp \mathfrak{n}_\varepsilon x^\perp \cdot \varphi$. We deduce that

$$\begin{aligned}&\int_{\mathscr{F}_\varepsilon} q_\varepsilon^{in} \Phi_\varepsilon[\varphi](0,.) + \int_0^T \int_{\mathscr{F}_\varepsilon} (\rho_\varepsilon u_\varepsilon) \cdot \partial_t \Phi_\varepsilon[\varphi] \\ &\quad + \int_0^T \int_{\mathscr{F}_\varepsilon} [\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon] : D\Phi_\varepsilon[\varphi] + \rho_\varepsilon^\gamma \mathrm{div}\, \Phi_\varepsilon[\varphi] = \int_0^T \int_{\mathscr{F}_\varepsilon} \mathbb{S} u_\varepsilon : D\Phi_\varepsilon[\varphi].\end{aligned}$$

We will pass to the limit in $\varepsilon$ in any term separately. First of all, notice that

$$\Phi_\varepsilon[\varphi](0,.) = \mathfrak{n}_\varepsilon \varphi(0,.) + \nabla^\perp \mathfrak{n}_\varepsilon x^\perp \cdot \varphi(0,.) \longrightarrow \varphi(0,.) \quad \text{in} \quad L^q(\Omega)$$

for any $q < +\infty$ by dominate convergence. We deduce that

$$\int_{\mathscr{F}_\varepsilon} q_\varepsilon^{in}\Phi_\varepsilon[\varphi](0,.) = \int_{\mathscr{F}_\varepsilon} \frac{q_\varepsilon^{in}}{\sqrt{\rho_\varepsilon^{in}}}\sqrt{\rho_\varepsilon^{in}}\Phi_\varepsilon(0,.) \longrightarrow \int_\Omega \frac{q^{in}}{\sqrt{\rho^{in}}}\sqrt{\rho^{in}}\varphi(0,.) = \int_\Omega q^{in}\varphi(0,.)$$

where we used $q_\varepsilon^{in}/\sqrt{\rho_\varepsilon^{in}} \longrightarrow q^{in}/\sqrt{\rho^{in}}$ in $L^2(\Omega)$ and $\sqrt{\rho_\varepsilon^{in}} \longrightarrow \sqrt{\rho^{in}}$ in $L^{2\gamma}(\Omega)$. Using (9), we notice that

$$\int_0^T\int_{\mathscr{F}_\varepsilon}(\rho_\varepsilon u_\varepsilon)\cdot\partial_t\Phi_\varepsilon[\varphi] = \int_0^T\int_{\mathscr{F}_\varepsilon}(\mathfrak{n}_\varepsilon\rho_\varepsilon u_\varepsilon + \rho_\varepsilon u_\varepsilon\cdot\nabla^\perp\mathfrak{n}_\varepsilon x^\perp)\cdot\partial_t\varphi \longrightarrow \int_0^T\int_\Omega \rho u\cdot\partial_t\varphi$$

where we use the convergence from Lemma 3. For the next term, let us rewrite

$$\begin{aligned}\int_0^T\int_{\mathscr{F}_\varepsilon}[\rho_\varepsilon u_\varepsilon\otimes u_\varepsilon] : D\Phi_\varepsilon[\varphi] &= \int_0^T\int_{\mathscr{F}_\varepsilon}\Big[\Big(\mathfrak{n}_\varepsilon\rho_\varepsilon u_\varepsilon + \rho_\varepsilon u_\varepsilon\cdot\nabla^\perp\mathfrak{n}_\varepsilon x^\perp\Big)\otimes u_\varepsilon\Big] : D\varphi \\ &+ \int_0^T\int_{\mathscr{F}_\varepsilon}[\rho_\varepsilon u_\varepsilon\otimes u_\varepsilon] : (D\Phi_\varepsilon[\varphi] - \mathfrak{n}_\varepsilon D\varphi) \\ &- \int_0^T\int_{\mathscr{F}_\varepsilon}\Big[\rho_\varepsilon u_\varepsilon\cdot\nabla^\perp\mathfrak{n}_\varepsilon x^\perp\otimes u_\varepsilon\Big] : D\varphi.\end{aligned} \tag{15}$$

Notice that

$$\int_0^T\int_{\mathscr{F}_\varepsilon}\Big[\Big(\mathfrak{n}_\varepsilon\rho_\varepsilon u_\varepsilon + \rho_\varepsilon u_\varepsilon\cdot\nabla^\perp\mathfrak{n}_\varepsilon x^\perp\Big)\otimes u_\varepsilon\Big] : D\varphi \longrightarrow \int_0^T\int_\Omega \rho u\otimes u : D\varphi,$$

due to Lemma 3. Moreover, the second term of the right-hand side of (15) converges to zero due to (11). Finally, the last term of (15)

$$\begin{aligned}&\left|\int_0^T\int_{\mathscr{F}_\varepsilon}\Big[\rho_\varepsilon u_\varepsilon\cdot\nabla^\perp\mathfrak{n}_\varepsilon x^\perp\otimes u_\varepsilon\Big] : D\varphi\right| \\ &\le \|\rho_\varepsilon\|_{L^\infty(0,T;L^\gamma(\Omega))}\|u_\varepsilon\|^2_{L^2(0,T;L^{\infty^-}(\Omega))}\|\nabla^\perp\mathfrak{n}_\varepsilon x^\perp\|_{L^2(\Omega)} \\ &\longrightarrow 0,\end{aligned}$$

where we use Proposition 1. We deduce

$$\int_0^T\int_{\mathscr{F}_\varepsilon}[\rho_\varepsilon u_\varepsilon\otimes u_\varepsilon] : D\Phi_\varepsilon[\varphi] \longrightarrow \int_0^T\int_\Omega \rho u\otimes u : D\varphi.$$

The next term is

$$\int_0^T\int_{\mathscr{F}_\varepsilon}\rho_\varepsilon^\gamma \operatorname{div}\Phi_\varepsilon[\varphi] = \int_0^T\int_{\mathscr{F}_\varepsilon}\mathbb{1}_{\Omega\setminus B_{2\varepsilon}(0)}\rho_\varepsilon^\gamma\mathfrak{n}_\varepsilon \operatorname{div}\varphi$$

$$+\int_0^T\int_{\mathscr{F}_\varepsilon}\mathbb{1}_{\Omega\setminus B_{2\varepsilon}(0)}\rho_\varepsilon^\gamma\,(\mathrm{div}\,(\Phi_\varepsilon[\varphi])-\mathfrak{n}_\varepsilon\mathrm{div}\,(\varphi))$$
$$\longrightarrow\int_0^T\int_\Omega\overline{\rho^\gamma}\,\mathrm{div}\,\varphi,$$

where we used Lemma 3 for the convergence of the first term and (13) for the second one. Finally,

$$\int_0^T\int_{\mathscr{F}_\varepsilon}\mathbb{S}u_\varepsilon:D\Phi_\varepsilon[\varphi]=\int_0^T\int_{\mathscr{F}_\varepsilon}\mathbb{S}u_\varepsilon:\mathfrak{n}_\varepsilon D\varphi+\int_0^T\int_{\mathscr{F}_\varepsilon}\mathbb{S}u_\varepsilon:(D\Phi_\varepsilon[\varphi]-\mathfrak{n}_\varepsilon D\varphi)$$
$$\longrightarrow\int_0^T\int_\Omega\mathbb{S}u:D\varphi$$

where we used the weak convergence of $u_\varepsilon$ from Lemma 3 and the strong convergence of $\mathfrak{n}_\varepsilon D\varphi\longrightarrow D\varphi$ in $L^2$. The second term converges to zero from Lemma 2.

Putting all this convergence together, we deduce that $\rho$ and $u$ satisfy

$$\int_\Omega q^{in}\varphi(0,.)+\int_0^T\int_\Omega\rho u\cdot\partial_t\varphi+\int_0^T\int_\Omega[\rho u\otimes u]:D\varphi+\overline{\rho^\gamma}\,\mathrm{div}\,\varphi$$
$$=\int_0^T\int_\Omega\mathbb{S}u:D\varphi.$$

It now remains to show that $\overline{\rho^\gamma}=\rho^\gamma$. We will show this in the next section.

## 7. Identification of the pressure

We now show that $\overline{\rho^\gamma}=\rho^\gamma$ to do that we follow the strategy introduced by Lions in [12]. Let us recall that we are in the case $\gamma>2$ and dimension two, it is then enough to show the following lemma.

**Lemma 5.** *For any $\psi\in C_c^\infty(\Omega)$, it holds*

$$\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_\Omega\psi^2\mathfrak{n}_\varepsilon\left(\rho_\varepsilon^\gamma-(2\mu+\gamma)\mathrm{div}\,(u_\varepsilon)\right)\rho_\varepsilon=\int_0^T\int_\Omega\psi^2\left(\overline{\rho^\gamma}-(2\mu+\lambda)\mathrm{div}\,(u)\right)\rho$$

*up to subsequence.*

*Proof.* Consider $\phi_\varepsilon=\psi\Phi_\varepsilon[\nabla\Delta^{-1}[\psi\rho_\varepsilon]]=\psi\mathfrak{n}_\varepsilon\nabla\Delta^{-1}[\psi\rho_\varepsilon]+\psi\nabla^\perp\mathfrak{n}_\varepsilon x^\perp\cdot\nabla\Delta^{-1}[\psi\rho_\varepsilon]$ and $\phi=\psi\nabla\Delta^{-1}[\psi\rho]$. Here, $\nabla\Delta^{-1}$ is the integral operator defined by the singular kernel

$$K(x,y)=\frac{1}{2\pi}\frac{x-y}{|x-y|^2}\quad\text{for}\quad(x,y)\in\mathbb{R}^2\quad\text{with}\quad x\neq y.$$

From Calderón–Zygmund theory, see Sect. 5 of Chapter I of [20], we deduce that

$$\nabla\Delta^{-1}:L_c^p(\mathbb{R}^2)\longrightarrow W_{loc}^{1,p}(\mathbb{R}^2)$$

for $p \in (1,\infty)$ and $\|\nabla\Delta^{-1} f\|_{W^{1,p}(K)} \le C(R,K)\|f\|_{L^p(\mathbb{R}^2)}$ where the support of $f$ is contained in the ball $B_R(0)$ and $K \subset \mathbb{R}^2$ is compact. Moreover, for any vector field $F \in L^p_c(\mathbb{R}^2)$ it holds $\|\nabla\Delta^{-1}\mathrm{div}\,(F)\|_{L^p(K)} \le C(R,K)\|F\|_{L^p(\mathbb{R}^2)}$ where the support of $F$ is contained in the ball $B_R(0)$ and $K \subset \mathbb{R}^2$ is compact.

From the a priori estimates on the solutions $\rho_\varepsilon, u_\varepsilon$, we notice that $\nabla\Delta^{-1}[\psi\rho_\varepsilon]$ is uniformly bounded in $L^\infty(0,T;W^{1,\gamma}(\Omega))$ and $\partial_t\nabla\Delta^{-1}[\psi\rho_\varepsilon] = -\nabla\Delta^{-1}[\psi\,\mathrm{div}\,(\rho_\varepsilon u_\varepsilon)]$ is uniformly bounded is some $L^p$ spaces. We can now test the weak formulation satisfied by $\rho_\varepsilon, u_\varepsilon$ by $\phi_\varepsilon$ and the one of $\rho, u$ by $\phi$. Using the convergence of initial data, we deduce

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \rho_\varepsilon u_\varepsilon\cdot\partial_t\phi_\varepsilon + \rho_\varepsilon u_\varepsilon\otimes u_\varepsilon : \nabla\phi_\varepsilon + \rho_\varepsilon^\gamma\,\mathrm{div}\,(\phi_\varepsilon) - \mathbb{S}u_\varepsilon : \nabla\phi_\varepsilon \\ = \int_0^T\int_\Omega \rho u\cdot\phi + \rho u\otimes u : \nabla\phi + \overline{\rho^\gamma}\,\mathrm{div}\,(\phi) - \mathbb{S}u : \nabla\phi. \qquad (16)\end{aligned}$$

We will now rewrite the above equality in an appropriate way. To simplify the calculations, let us introduce the notation

$$\Phi_\varepsilon^T[\varphi] = \mathfrak{n}_\varepsilon\varphi + \varphi\cdot\nabla^\perp\mathfrak{n}_\varepsilon x^\perp, \qquad (17)$$

for any measurable velocity field $\varphi$. Notice that

$$\begin{aligned}\partial_t\phi_\varepsilon &= \psi\Phi_\varepsilon[\nabla\Delta^{-1}(\psi\partial_t\rho_\varepsilon)] \\ &= -\psi\Phi_\varepsilon[\nabla\Delta^{-1}(\mathrm{div}\,(\psi\rho_\varepsilon u_\varepsilon)] + \psi\Phi_\varepsilon[\nabla\Delta^{-1}[\nabla(\psi)\rho_\varepsilon u_\varepsilon]].\end{aligned}$$

We deduce that

$$\begin{aligned}\int_0^T\int_\Omega \rho_\varepsilon u_\varepsilon\cdot\partial_t\phi_\varepsilon &= -\int_0^T\int_\Omega \psi\rho_\varepsilon u_\varepsilon\cdot\Phi_\varepsilon[\nabla\Delta^{-1}[\mathrm{div}\,(\psi\rho_\varepsilon u_\varepsilon)]] \\ &\quad + \int_0^T\int_\Omega \rho_\varepsilon u_\varepsilon\cdot\psi\Phi_\varepsilon[\nabla\Delta^{-1}[\nabla(\psi)\rho_\varepsilon u_\varepsilon]] \\ &= -\int_0^T\int_\Omega \psi\Phi_\varepsilon^T[\rho_\varepsilon u_\varepsilon]\cdot\nabla\Delta^{-1}[\mathrm{div}\,(\psi\rho_\varepsilon u_\varepsilon)] \\ &\quad + \int_0^T\int_\Omega \psi\Phi_\varepsilon^T[\rho_\varepsilon u_\varepsilon]\cdot\nabla\Delta^{-1}[\nabla(\psi)\rho_\varepsilon u_\varepsilon].\end{aligned}$$

Lemma 3 implies that $\Phi_\varepsilon^T[\rho_\varepsilon u_\varepsilon]$ converges to $\rho u$ in $C_w(0,T;L^{2\gamma/(\gamma+1)}(\Omega))$, while $\nabla\Delta^{-1}[\nabla(\psi)\rho_\varepsilon u_\varepsilon]$ converges weakly star to $\nabla\Delta^{-1}[\nabla(\psi)\rho u]$ in $L^\infty(0,T;W^{1,2\gamma/(\gamma+1)}(\Omega))$. We deduce that

$$\begin{aligned}\int_0^T\int_\Omega \rho_\varepsilon u_\varepsilon\cdot\partial_t\phi_\varepsilon = &-\int_0^T\int_\Omega \psi\Phi_\varepsilon^T[\rho_\varepsilon u_\varepsilon]\cdot\nabla\Delta^{-1}[\mathrm{div}\,(\psi\rho_\varepsilon u_\varepsilon)] \qquad (18) \\ &+ \int_0^T\int_\Omega \psi\rho u\cdot\nabla\Delta^{-1}[\nabla(\psi)\rho u)] + c(\varepsilon),\end{aligned}$$

with $c(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Similarly, we have

$$\int_0^T \int_\Omega \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : \nabla \phi_\varepsilon = \int_0^T \int_\Omega \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : (\nabla \psi \otimes \Phi_\varepsilon[\nabla \Delta^{-1}[\psi \rho_\varepsilon]]) + \int_0^T \int_\Omega \psi \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : \nabla \Phi_\varepsilon[\nabla \Delta^{-1}[\psi \rho_\varepsilon]].$$

For the first term of the right-hand side, we rewrite

$$\begin{aligned} &\int_0^T \int_\Omega \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : (\nabla \psi \otimes \Phi_\varepsilon[\nabla \Delta^{-1}[\psi \rho_\varepsilon]]) \\ &= \int_0^T \int_\Omega \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : (\nabla \psi \otimes (\Phi_\varepsilon - \mathfrak{n}_\varepsilon \mathbb{I})[\nabla \Delta^{-1}[\psi \rho_\varepsilon]]) \\ &\quad + \int_0^T \int_\Omega \Phi_\varepsilon^T[\rho_\varepsilon u_\varepsilon] \otimes u_\varepsilon : (\nabla \psi \otimes \nabla \Delta^{-1}[\psi \rho_\varepsilon]) \\ &\quad - \int_0^T \int_\Omega \rho_\varepsilon u_\varepsilon \cdot \nabla^\perp \mathfrak{n}_\varepsilon x^\perp \otimes u_\varepsilon : (\nabla \psi \otimes \nabla \Delta^{-1}[\psi \rho_\varepsilon]) \end{aligned}$$

Using Lemma 2, using that $\Phi_\varepsilon^T[\rho_\varepsilon u_\varepsilon] \otimes u_\varepsilon$ converges to $\rho u \otimes u$ in $L^1(0, T; L^q(\Omega))$ for any $q < \gamma$, that $\nabla \Delta^{-1}[\psi \rho_\varepsilon]$ converges to $\nabla \Delta^{-1}[\psi \rho]$ in $C_w^0(0, T; W^{1,\gamma}(\Omega))$ and $\|\nabla^\perp \mathfrak{n}_\varepsilon \otimes x\|_{L^p(\mathbb{R}^2)}$ converges to zero for any $p < \infty$, we deduce

$$\begin{aligned} &\int_0^T \int_\Omega \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : (\nabla \psi \otimes \Phi_\varepsilon[\nabla \Delta^{-1}[\psi \rho_\varepsilon]]) \\ &\quad = \int_0^T \int_\Omega \rho u \otimes u : (\nabla \psi \otimes \nabla \Delta^{-1}[\psi \rho]) + c(\varepsilon), \end{aligned} \tag{19}$$

with $c(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Similarly,

$$\begin{aligned} &\int_0^T \int_\Omega \psi \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : \nabla \Phi_\varepsilon[\nabla \Delta^{-1}[\psi \rho_\varepsilon]] \\ &= \int_0^T \int_\Omega \psi \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : (\nabla \Phi_\varepsilon - \mathfrak{n}_\varepsilon \nabla)[\nabla \Delta^{-1}[\psi \rho_\varepsilon]] \\ &\quad + \int_0^T \int_\Omega \psi \Phi_\varepsilon^T[\rho_\varepsilon u_\varepsilon] \otimes u_\varepsilon : \nabla(\nabla \Delta^{-1}[\psi \rho_\varepsilon]) \\ &\quad - \int_0^T \int_\Omega \psi \rho_\varepsilon u_\varepsilon \cdot \nabla^\perp \mathfrak{n}_\varepsilon x^\perp \otimes u_\varepsilon : \nabla(\nabla \Delta^{-1}[\psi \rho_\varepsilon]) \end{aligned}$$

Lemma 2 and the convergences of $\Phi_\varepsilon^T[\rho_\varepsilon u_\varepsilon] \otimes u_\varepsilon$, $\nabla \Delta^{-1}[\psi \rho_\varepsilon]$, $\nabla^\perp \mathfrak{n}_\varepsilon \otimes x$, used to show (19), imply that

$$\begin{aligned} &\int_0^T \int_\Omega \psi \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : \nabla \Phi_\varepsilon[\nabla \Delta^{-1}[\psi \rho_\varepsilon]] \\ &= \int_0^T \int_\Omega \psi \Phi_\varepsilon^T[\rho_\varepsilon u_\varepsilon] \otimes u_\varepsilon : \nabla(\nabla \Delta^{-1}[\psi \rho_\varepsilon]) + c(\varepsilon), \end{aligned} \tag{20}$$

with $c(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. From the equalities (19) and (20), we deduce

$$\int_0^T \int_\Omega \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : \nabla \phi_\varepsilon = \int_0^T \int_\Omega \rho u \otimes u : (\nabla \psi \otimes \nabla \Delta^{-1}[\psi \rho]) + \int_0^T \int_\Omega \mathfrak{n}_\varepsilon \psi \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : \nabla(\nabla \Delta^{-1}[\psi \rho_\varepsilon]) + c(\varepsilon), \tag{21}$$

with $c(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. The next term is

$$\int_0^T \int_\Omega \rho_\varepsilon^\gamma \operatorname{div}(\phi_\varepsilon) = \int_0^T \int_\Omega \rho_\varepsilon^\gamma \nabla \psi \cdot \Phi_\varepsilon[\nabla \Delta^{-1}[\psi \rho_\varepsilon]] + \int_0^T \int_\Omega \mathfrak{n}_\varepsilon \psi^2 \rho_\varepsilon^\gamma \rho_\varepsilon + \int_0^T \int_\Omega \psi \rho_\varepsilon^\gamma \left( \operatorname{div}(\Phi_\varepsilon[\nabla \Delta^{-1}[\psi \rho_\varepsilon]]) - \mathfrak{n}_\varepsilon \operatorname{div}(\nabla \Delta^{-1}[\psi \rho_\varepsilon]) \right).$$

Using the weak convergence of $\mathbb{1}_{\mathbb{R}^2 \setminus B_{2\varepsilon}(0)} \rho_\varepsilon^\gamma$ to $\overline{\rho^\gamma}$ in $L^{(\gamma+\theta)/\gamma}$ and the convergence of $\nabla \Delta^{-1}[\psi \rho_\varepsilon]$ to $\nabla \Delta^{-1}[\psi \rho]$ in $C^0_w(0, T; W^{1,\gamma}(\Omega))$, we deduce that the first term of the right-hand side converges to

$$\int_0^T \int_\Omega \overline{\rho^\gamma} \nabla \psi \cdot \nabla \Delta^{-1}[\psi \rho],$$

and the last term converges to zero from Lemma 2. We deduce that

$$\int_0^T \int_\Omega \rho_\varepsilon^\gamma \operatorname{div}(\phi_\varepsilon) = \int_0^T \int_\Omega \overline{\rho^\gamma} \nabla \psi \cdot \nabla \Delta^{-1}[\psi \rho] + \int_0^T \int_\Omega \mathfrak{n}_\varepsilon \psi^2 \rho_\varepsilon^\gamma \rho_\varepsilon + c(\varepsilon), \tag{22}$$

with $c(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Finally

$$\int_0^T \int_\Omega \mathbb{S} u_\varepsilon : \nabla \phi_\varepsilon = \int_0^T \int_\Omega \mathbb{S} u_\varepsilon : (\nabla \psi \otimes \Phi_\varepsilon[\nabla \Delta^{-1}[\psi \rho_\varepsilon]]) + \int_0^T \int_\Omega \psi \mathfrak{n}_\varepsilon \mathbb{S} u_\varepsilon : \nabla(\nabla \Delta^{-1}[\psi \rho_\varepsilon]) + \int_0^T \int_\Omega \psi \mathfrak{n}_\varepsilon \mathbb{S} u_\varepsilon : (\nabla \Phi_\varepsilon - \mathfrak{n}_\varepsilon \nabla)[\nabla \Delta^{-1}[\psi \rho_\varepsilon]].$$

From the weak convergence of $\mathbb{S} u_\varepsilon$ to $\mathbb{S} u$ in $L^2((O, T) \times \Omega)$, the convergence $\nabla \Delta^{-1}[\psi \rho_\varepsilon]$ to $\nabla \Delta^{-1}[\psi \rho]$ in $C^0_w(0, T; W^{1,\gamma}(\Omega))$ and Lemma 2, we have

$$\int_0^T \int_\Omega \mathbb{S} u_\varepsilon : \nabla \phi_\varepsilon = \int_0^T \int_\Omega \mathbb{S} u : (\nabla \psi \otimes \nabla \Delta^{-1}[\psi \rho]) + \int_0^T \int_\Omega \psi \mathfrak{n}_\varepsilon \mathbb{S} u_\varepsilon : \nabla(\nabla \Delta^{-1}[\psi \rho_\varepsilon]) + c(\varepsilon), \tag{23}$$

with $c(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Using (18)–(21)–(22)–(23), we rewrite (16)

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega\Big(-\psi\Phi_\varepsilon^T[\rho_\varepsilon u_\varepsilon]\cdot\nabla\Delta^{-1}[\operatorname{div}(\psi\rho_\varepsilon u_\varepsilon)] \\ +\psi\Phi_\varepsilon^T[\rho_\varepsilon u_\varepsilon]\otimes u_\varepsilon:\nabla(\nabla\Delta^{-1}[\psi\rho_\varepsilon])+\mathfrak{n}_\varepsilon\psi^2\rho_\varepsilon^\gamma\rho_\varepsilon-\mathfrak{n}_\varepsilon\psi\mathbb{S}u_\varepsilon:\nabla(\nabla\Delta^{-1}[\psi\rho_\varepsilon])\Big) \\ =\int_0^T\int_\Omega\Big(-\psi\rho u\cdot\nabla\Delta^{-1}[\operatorname{div}(\psi\rho u)]+\psi\rho u\otimes u:\nabla(\nabla\Delta^{-1}[\psi\rho]) \\ +\psi^2\overline{\rho^\gamma}\rho-\psi\mathbb{S}u:\nabla(\nabla\Delta^{-1}[\psi\rho])\Big). \tag{24}$$

We will now show that

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega-\psi\Phi_\varepsilon^T[\rho_\varepsilon u_\varepsilon]\cdot\nabla\Delta^{-1}[\operatorname{div}(\psi\rho_\varepsilon u_\varepsilon)]+\psi\Phi_\varepsilon^T[\rho_\varepsilon u_\varepsilon]\otimes u_\varepsilon:\nabla(\nabla\Delta^{-1}[\psi\rho_\varepsilon]) \\ =\int_0^T\int_\Omega-\psi\rho u\cdot\nabla\Delta^{-1}[\operatorname{div}(\psi\rho u)]+\psi\rho u\otimes u:\nabla(\nabla\Delta^{-1}[\psi\rho]). \tag{25}$$

In the case $\gamma > 2$, this equality can be verified by using the commutator estimates from Step 3 of proof of Theorem 5.1 of [12].

Finally, notice that

$$\int_0^T\int_\Omega\mathfrak{n}_\varepsilon\psi\mathbb{S}u_\varepsilon:\nabla(\nabla\Delta^{-1}[\psi\rho_\varepsilon])=\int_0^T\int_\Omega\mathfrak{n}_\varepsilon\psi\mu Du_\varepsilon:\nabla(\nabla\Delta^{-1}[\psi\rho_\varepsilon]) \\ +\int_0^T\int_\Omega\mathfrak{n}_\varepsilon\psi^2(\mu+\lambda)\operatorname{div}(u_\varepsilon)\rho_\varepsilon.$$

From some integrations by parts and using the density of smooth functions in Sobolev spaces, we have

$$\int_0^T\int_\Omega\mathfrak{n}_\varepsilon\psi\mu Du_\varepsilon:\nabla(\nabla\Delta^{-1}[\psi\rho_\varepsilon])-\int_0^T\int_\Omega\mathfrak{n}_\varepsilon\psi^2\mu\operatorname{div}(u_\varepsilon)\rho_\varepsilon \\ =\int_0^T\int_\Omega\psi\mu Du:\nabla(\nabla\Delta^{-1}[\psi\rho])-\int_0^T\int_\Omega\psi^2\mu\operatorname{div}(u)\rho+c(\varepsilon)$$

with $c(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. We deduce that

$$\int_0^T\int_\Omega\mathfrak{n}_\varepsilon\psi\mathbb{S}u_\varepsilon:\nabla(\nabla\Delta^{-1}[\psi\rho_\varepsilon])=\int_0^T\int_\Omega\mathfrak{n}_\varepsilon\psi^2(2\mu+\lambda)\operatorname{div}(u_\varepsilon)\rho_\varepsilon \\ -\int_0^T\int_\Omega\psi\mu Du:\nabla(\nabla\Delta^{-1}[\psi\rho])+\int_0^T\int_\Omega\psi^2\mu\operatorname{div}(u)\rho+c(\varepsilon). \tag{26}$$

The statement of the lemma follows from (24)–(25) and (26). □

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## Declarations

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## Appendix A: The Bogovskiĭ operator in domains with holes

In this appendix, we recall a definition of Bogovskiĭ operator for domains with a hole. Moreover, we show estimates independent of the size of the hole when it is assumed to be small enough.

Let us recall that a Bogovskiĭ operator is a right inverse of the divergence on $\tilde{L}^p$ which is the space of $L^p$ functions with integral zero. Due to the non-uniqueness of this operator, we choose $\mathscr{B}_\Omega$ to satisfy the following extra properties.

**Theorem 3.** *There exists a Bogovskiĭ operator $\mathscr{B}_\Omega$ such that*

$$\mathscr{B}_\Omega : \tilde{L}^p(\Omega) \longrightarrow W_0^{1,p}(\Omega)$$

*and it is linear and continuous for any $1 < p < +\infty$,*

$$\operatorname{div}(\mathscr{B}_\Omega[f]) = f \text{ for any } f \in \tilde{L}^p(\Omega) \quad \text{and} \quad \|\mathscr{B}_\Omega[f]\|_{L^\infty(\Omega)} \leq \|f\|_{L^2(\Omega)}.$$

*Moreover, for any vector field $F \in L^p(\Omega)$ such that $F \cdot n = 0$ on $\partial\Omega$, it holds*

$$\|\mathscr{B}_\Omega[\operatorname{div}(F)]\|_{L^p(\Omega)} \leq \|F\|_{L^p(\Omega)}.$$

We refer to Sect. 3.3.1.2 of [18] for a proof of the above theorem and more details.

To define the Bogovskiĭ operator on the domain with hole $\Omega\backslash B_\varepsilon(0)$, we use an idea from [14], more precisely we define $\mathscr{B}_{\Omega\backslash B_\varepsilon(0)}$ as the composition of three operators. The extension by zero operator $\mathscr{E}_\varepsilon : \tilde{L}^p(\Omega\backslash B_\varepsilon(0)) \longrightarrow \tilde{L}^p(\Omega)$, the Bogovskiĭ operator on $\Omega$ and the restriction operator $\mathscr{R}_\varepsilon : W_0^{1,p}(\Omega) \longrightarrow W_0^{1,p}(\Omega\backslash B_\varepsilon(0))$ which is defined as follows.

Let $\eta : [0,+\infty) \longrightarrow [0,1]$ an increasing smooth function such that $\eta(x) = 0$ for $x \in [0,1]$ and $\eta(x) = 1$ for $x \in [2,+\infty)$ and let $B_1 = \mathscr{B}_{B_2(0)\backslash B_1(0)}$ a Bogovskiĭ operator. For $\varepsilon > 0$ let introduce the functions $\eta_\varepsilon(x) = \eta(x/\varepsilon)$ and similarly $B_\varepsilon[f](x) = \varepsilon B_1[f(\varepsilon y)](x/\varepsilon)$. We define the restriction operator

$$\mathscr{R}_\varepsilon[F] = \eta_\varepsilon F + B_\varepsilon[\operatorname{div}(1-\eta_\varepsilon)F) - \ll \operatorname{div}((1-\eta_\varepsilon)F) \gg],$$

where

$$\ll f \gg = \frac{1}{|\mathscr{B}_{2\varepsilon}(0)\setminus B_\varepsilon(0)|}\int_{\mathscr{B}_{2\varepsilon}(0)\backslash B_\varepsilon(0)} f.$$

We can define the Bogovskiĭ operator on the domain with hole $\Omega\backslash B_\varepsilon(0)$.

$$\mathscr{B}_{\Omega\backslash B_\varepsilon(0)}[f] = \mathscr{B}_\varepsilon[f] = \mathscr{R}_\varepsilon \circ \mathscr{B}_\Omega \circ \mathscr{E}_\varepsilon[f]. \tag{27}$$

Moreover, they satisfy the following estimates uniformly in $\varepsilon$.

**Proposition 2.** *The operators $\mathscr{B}_\varepsilon$ defined in (27) are Bogovskiĭ operators; moreover, for $1 < p \leq 2$ they satisfy the uniform bounds*

$$\|\mathscr{B}_\varepsilon[f]\|_{W_0^{1,p}(\Omega\backslash B_\varepsilon(0))} \leq C\|f\|_{L^p(\Omega\backslash B_\varepsilon(0))} \quad \text{and} \quad \|\mathscr{B}_\varepsilon[f]\|_{L^\infty(\Omega\backslash B_\varepsilon(0))} \leq C\|f\|_{L^2(\Omega\backslash B_\varepsilon(0))},$$

*with $C$ independent of $\varepsilon$. For any vector field $F \in L^q(\Omega\backslash B_\varepsilon(0))$ such that $F\cdot n = 0$ on $\partial\Omega \cup \partial B_\varepsilon(0)$, it holds*

$$\|\mathscr{B}_\varepsilon[\operatorname{div}(F)]\|_{L^q(\Omega\backslash B_\varepsilon(0))} \leq \|F\|_{L^q(\Omega\backslash B_\varepsilon(0))}. \tag{28}$$

*for any $1 < q < +\infty$.*

*Proof.* The proof follows from the definition of the operator $\mathscr{B}_\varepsilon$. Compared with the correspondent result in [14] (Proposition 2.2), we notice that (28) holds also for $1 < q \leq 2$. So let us show this result. By definition, we have

$$\begin{aligned}\mathscr{B}_\varepsilon[\operatorname{div}(F)] &= \mathscr{R}_\varepsilon \circ \mathscr{B}_\Omega[\operatorname{div}(F)] = \eta_\varepsilon\mathscr{B}_\Omega[\operatorname{div}(F)] + B_\varepsilon[\operatorname{div}((1-\eta_\varepsilon)\mathscr{B}_\Omega[\operatorname{div}(F)])] \\ &= \eta_\varepsilon\mathscr{B}_\Omega[\operatorname{div}(F)] - B_\varepsilon[\nabla\eta_\varepsilon \cdot \mathscr{B}_\Omega[\operatorname{div}(F)]] + B_\varepsilon[\operatorname{div}((1-\eta_\varepsilon)F)] \\ &\quad + B_\varepsilon[\nabla\eta_\varepsilon \cdot F]\end{aligned}$$

We estimate the right-hand side separately. It is straightforward to see that

$$\|\eta_\varepsilon\mathscr{B}_\Omega[\operatorname{div}(F)]\|_{L^q(\mathscr{F}_\varepsilon)} \leq C\|F\|_{L^q(\mathscr{F}_\varepsilon)}$$

For the second term, we denote by $q^* = 2q/(2-q)$ and we notice that the support of $B_\varepsilon$ is contained in $B_{2\varepsilon}(0)\backslash B_\varepsilon(0) = A_\varepsilon$. Then,

$$\begin{aligned}\|B_\varepsilon[\nabla\eta_\varepsilon \cdot \mathscr{B}_\Omega[\mathrm{div}\,(F)]]\|_{L^q(A_\varepsilon)} &\le C\varepsilon\|B_\varepsilon[\nabla\eta_\varepsilon \cdot \mathscr{B}_\Omega[\mathrm{div}\,(F)]]\|_{L^{q^*}(A_\varepsilon)}\\ &\le C\varepsilon\|\nabla B_\varepsilon[\nabla\eta_\varepsilon \cdot \mathscr{B}_\Omega[\mathrm{div}\,(F)]]\|_{L^q(A_\varepsilon)} \le C\varepsilon\|[\nabla\eta_\varepsilon \cdot \mathscr{B}_\Omega[\mathrm{div}\,(F)]]\|_{L^q(A_\varepsilon)}\\ &\le C\varepsilon\|\nabla\eta_\varepsilon\|_{L^\infty(A_\varepsilon)}\|\mathscr{B}_\Omega[\mathrm{div}\,(F)]\|_{L^q(A_\varepsilon)} \le \|F\|_{L^q(\mathscr{F}_\varepsilon)}.\end{aligned}$$

The same strategy gives

$$\|B_\varepsilon[\nabla\eta_\varepsilon \cdot F]\|_{L^q(A_\varepsilon)} \le C\|F\|_{L^q(\mathscr{F}_\varepsilon)}.$$

We are left to show the estimates for $B_\varepsilon[\mathrm{div}\,((1-\eta_\varepsilon)F)]$. To simplify the notation let $G = (1-\eta_\varepsilon)F$. By definition of $B_\varepsilon$, we have

$$B_\varepsilon[\mathrm{div}_x(G)](x) = \varepsilon B_1[\mathrm{div}_x(G(\varepsilon y))](x/\varepsilon) = B_1[\mathrm{div}_y(G(\varepsilon y))](x/\varepsilon)$$

We deduce that

$$\begin{aligned}\|B_\varepsilon[\mathrm{div}_x(G)](x)\|_{L^q(A_\varepsilon)} &= \varepsilon^{2/q}\|B_1[\mathrm{div}_y(G(\varepsilon y))]\|_{L^q(A_1)} \le C\varepsilon^{2/q}\|G\|_{L^q(A_1)}\\ &= \|G\|_{L^q(A_\varepsilon)}.\end{aligned}$$

After recalling that $G = (1-\eta_\varepsilon)F$, we obtain the desired result.

□

## Appendix B: Improved pressure estimates

This section is devoted to the proof of the improved pressure estimates from Proposition 1.

*Proof.* Let us recall that classical regularity ensures that

$$\rho_\varepsilon \in L^q((0,T)\times\mathscr{F}_\varepsilon) \quad \text{for any} \quad q < 2\gamma - 1. \tag{29}$$

Here, we are interested in showing a bound independent of $\varepsilon$. The idea is then to test the momentum equation of (2) with

$$\varphi_\varepsilon = \phi\mathscr{B}_\varepsilon[\psi_\varepsilon\rho_\varepsilon^\theta - \langle\psi_\varepsilon\rho_\varepsilon^\theta\rangle], \tag{30}$$

where $\phi \in C_c^\infty([0,T))$, $\psi_\varepsilon = (\tilde\psi_\varepsilon)^2$ with $\tilde\psi_\varepsilon(x) = 1 - \eta_\varepsilon(|x|)$ and

$$\langle\psi_\varepsilon\rho_\varepsilon^\theta\rangle = \frac{1}{|\Omega \setminus B_\varepsilon(0)|}\int_{\Omega\setminus B_\varepsilon(0)} \psi_\varepsilon\rho_\varepsilon^\theta.$$

The functions $\varphi_\varepsilon$ are not smooth enough in the time variable to be test functions in the weak formulations, so to be rigorous we should smooth them out by using a convolution kernel as in Sect. 7.9.5 of [18]. We will not consider this regularization here because it will not influence the estimates we are going to do.

If we use (30) in the momentum equation of (2), we deduce

$$\int_0^T \int_{\mathscr{F}_\varepsilon} \phi\psi_\varepsilon\rho_\varepsilon^{\gamma+\theta} = \int_0^T \int_{\mathscr{F}_\varepsilon} \phi\rho^\gamma \langle\psi_\varepsilon\rho_\varepsilon^\theta\rangle - \int_{\mathscr{F}_\varepsilon} q_\varepsilon^{in}\cdot\varphi_\varepsilon(0,.) + 2\mu\int_0^T\int_{\mathscr{F}_\varepsilon} Du_\varepsilon : D\varphi_\varepsilon$$
$$+(\lambda+\mu)\int_0^T\int_{\mathscr{F}_\varepsilon} \mathrm{div}\,(u_\varepsilon)\mathrm{div}\,(\varphi_\varepsilon) - \int_0^T\int_{\mathscr{F}_\varepsilon} \rho_\varepsilon u_\varepsilon\otimes u_\varepsilon : D\varphi_\varepsilon$$
$$-\int_0^T\int_{\mathscr{F}_\varepsilon} \rho_\varepsilon u_\varepsilon \partial_t\varphi_\varepsilon = \sum_{i=1}^6 I_i.$$

We will now show that the right-hand side of the above expression is bounded by a constant independent of $\varepsilon$ multiplied by the norm of the initial data. To do that, we estimate the $I_i$ separately.

$$|I_1| = \left|\int_0^T \int_{\mathscr{F}_\varepsilon} \phi\rho_\varepsilon^\gamma\langle\psi_\varepsilon\rho_\varepsilon^\theta\rangle\right| \le C\|\rho_\varepsilon\|^{\gamma+\theta}_{L^\infty(0,T;L^\gamma(\mathscr{F}_\varepsilon))} \le C$$

where we use $\theta\le\gamma$. Using the definition of $\varphi_\varepsilon$, we have

$$|I_2| \le \left|\int_{\mathscr{F}_\varepsilon} q_\varepsilon^{in}\cdot\phi(0)\mathscr{B}_\varepsilon\left[\psi_\varepsilon(\rho_\varepsilon^{in})^\theta - \langle\psi_\varepsilon(\rho_\varepsilon^{in})^\theta\rangle\right]\right|$$
$$\le \left\|\frac{q_\varepsilon^{in}}{\sqrt{\rho_\varepsilon^{in}}}\right\|_{L^2(\mathscr{F}_\varepsilon)} \left\|\sqrt{\rho_\varepsilon^{in}}\right\|_{L^{2\gamma}(\mathscr{F}_\varepsilon)} \|\mathscr{B}_\varepsilon\left[(\psi_\varepsilon\rho_\varepsilon^{in})^\theta - \langle\psi_\varepsilon(\rho_\varepsilon^{in})^\theta\rangle\right]\|_{L^{2\gamma/(\gamma-1)}(\mathscr{F}_\varepsilon)}$$
$$\le C\left\|\frac{q_\varepsilon^{in}}{\sqrt{\rho_\varepsilon^{in}}}\right\|_{L^2(\mathscr{F}_\varepsilon)} \left\|\sqrt{\rho_\varepsilon^{in}}\right\|_{L^{2\gamma}(\mathscr{F}_\varepsilon)} \|(\psi_\varepsilon\rho_\varepsilon^{in})^\theta - \langle\psi_\varepsilon(\rho_\varepsilon^{in})^\theta\|_{L^{2\gamma/(2\gamma-1)}(\mathscr{F}_\varepsilon)}$$
$$\le C,$$

for $2\theta\le 2\gamma-1$. In the third inequality, we use that

$$\|\mathscr{B}_\varepsilon[f]\|_{L^{p^*}} \le \|\mathscr{B}_\varepsilon[f]\|_{W^{1,p}} \le \|f\|_{L^p} \quad \text{for} \quad 2 < p^* = 2p/(2-p).$$

$$|I_3| \le \left|\mu\int_0^T\int_{\mathscr{F}_\varepsilon} Du_\varepsilon : D\mathscr{B}_\varepsilon\left[\phi\psi_\varepsilon\rho_\varepsilon^\theta - \langle\phi\psi_\varepsilon\rho_\varepsilon^\theta\rangle\right]\right|$$
$$\le C\|Du_\varepsilon\|_{L^2(0,T;L^2(\mathscr{F}_\varepsilon))}\|(\phi\psi_\varepsilon)^{1/\theta}\rho\|^{1/2}_{L^{2\theta}(0,T;L^{2\theta}(\mathscr{F}_\varepsilon))}.$$

Notice that the last term of the right-hand side can be absorbed in the left-hand side for any $\theta\le\gamma$.

$$|I_4| \le \left|(\mu+\lambda)\int_0^T\int_{\mathscr{F}_\varepsilon} \mathrm{div}\,(u_\varepsilon)\left[\phi\psi_\varepsilon\rho_\varepsilon^\theta - \langle\phi\psi_\varepsilon\rho_\varepsilon^\theta\rangle\right]\right|$$
$$\le C\|\mathrm{div}\,(u_\varepsilon)\|_{L^2(0,T;L^2(\mathscr{F}_\varepsilon))}\|(\phi\psi_\varepsilon)^{1/\theta}\rho_\varepsilon\|^{1/2}_{L^{2\theta}(0,T;L^{2\theta}(\mathscr{F}_\varepsilon))}.$$

As before, we can absorb the last term of the right-hand side in the left-hand side if $\theta \leq \gamma$. The next term is

$$
\begin{aligned}
|I_5| &\leq \left| \int_0^T \int_{\mathscr{F}_\varepsilon} \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : D\mathscr{B}_\varepsilon \left[ \phi \psi_\varepsilon \rho_\varepsilon^\theta - \langle \phi \psi_\varepsilon \rho_\varepsilon^\theta \rangle \right] \right| \\
&\leq \|\rho_\varepsilon\|_{L^\infty(0,T;L^\gamma(\mathscr{F}_\varepsilon))} \|u_\varepsilon\|^2_{L^2(0,T;L^{\infty^-}(\mathscr{F}_\varepsilon))} \|\rho_\varepsilon^\theta\|_{L^\infty(0,T;L^{\gamma/\theta}(\mathscr{F}_\varepsilon))},
\end{aligned}
$$

in the above inequality we need $\gamma/\theta \leq 2$ for the estimates on $D\mathscr{B}_\varepsilon$; moreover, from

$$
\frac{1}{\gamma} + \frac{1}{\infty^-} + \frac{1}{\infty^-} + \frac{\theta}{\gamma} \leq 1
$$

we have the classical bound $\theta < \gamma - 1$ and from $\gamma/\theta \leq 2$ we have also $\gamma > 2$.

We are now left with the estimates of $I_6$. Recall from (29) that for any $\varepsilon$ we already know that $\rho_\varepsilon \in L^q((0,T) \times \mathscr{F}_\varepsilon)$ for any $q < 2\gamma - 1$ but we do not have a control of this norm uniform in $\varepsilon$. Lemma 6.9 of [18] ensures that $\rho_\varepsilon$ and $u_\varepsilon$ satisfy the equation

$$
\partial_t \rho_\varepsilon^\theta + \operatorname{div}(u_\varepsilon \rho_\varepsilon^\theta) + \operatorname{div}(u_\varepsilon)(\theta - 1)\rho_\varepsilon^\theta = 0,
$$

in a distributional sense for any $\theta < \gamma - 1/2$. Using the equation, we have that

$$
\begin{aligned}
\partial_t(\varphi_\varepsilon) &= \partial_t \phi \mathscr{B}_\varepsilon \left[ \psi_\varepsilon \rho_\varepsilon^\theta - \langle \psi_\varepsilon \rho_\varepsilon^\theta \rangle \right] + \phi \mathscr{B}_\varepsilon \left[ \psi_\varepsilon \partial_t \rho_\varepsilon^\theta - \langle \psi_\varepsilon \partial_t \rho_\varepsilon^\theta \rangle \right] \\
&= \partial_t \phi \mathscr{B}_\varepsilon \left[ \psi_\varepsilon \rho_\varepsilon^\theta - \langle \psi_\varepsilon \rho_\varepsilon^\theta \rangle \right] - \phi \mathscr{B}_\varepsilon \left[ \operatorname{div}(\psi_\varepsilon \cdot u_\varepsilon \rho_\varepsilon^\theta) - \langle \operatorname{div}(\psi_\varepsilon \cdot u_\varepsilon \rho_\varepsilon^\theta) \rangle \right] \\
&\quad + \phi \mathscr{B}_\varepsilon \left[ \nabla \psi_\varepsilon \cdot u_\varepsilon \rho_\varepsilon^\theta - \langle \nabla \psi_\varepsilon \cdot u_\varepsilon \rho_\varepsilon^\theta \rangle \right] \\
&\quad - \phi \mathscr{B}_\varepsilon \left[ \psi_\varepsilon \operatorname{div}(u_\varepsilon)(\theta - 1)\rho_\varepsilon^\theta - \langle \psi_\varepsilon \operatorname{div}(u_\varepsilon)(\theta - 1)\rho_\varepsilon^\theta \rangle \right] \\
&= \sum_{j=1}^4 J_j.
\end{aligned}
$$

We can now estimate

$$
\begin{aligned}
|I_6| \leq \sum_{j=1}^4 \left| \int_0^T \int_{\mathscr{F}_\varepsilon} \rho_\varepsilon u_\varepsilon \cdot J_j \right| &\leq \|\rho_\varepsilon\|_{L^\infty(0,T;L^\gamma(\mathscr{F}_\varepsilon))} \|u_\varepsilon\|_{L^2(0,T;L^{\infty^-}(\mathscr{F}_\varepsilon))} \\
&\quad \|J_j\|_{L^2(0,T;L^{(\gamma/\theta)-}(\mathscr{F}_\varepsilon))}.
\end{aligned}
$$

We are left with the estimates of $J_j$. Notice that

$$
\|J_1\|_{L^2(0,T;L^{(\gamma/\theta)-}(\mathscr{F}_\varepsilon))} \leq C \|\partial_t \phi\|_{L^2(0,T)} \|\rho_\varepsilon^\theta\|_{L^\infty(0,T;L^{\gamma/\theta}(\mathscr{F}_\varepsilon))}.
$$

Then we have

$$
\begin{aligned}
\|J_2\|_{L^2(0,T;L^{(\gamma/\theta)-}(\mathscr{F}_\varepsilon))} &\leq C \|u_\varepsilon \rho_\varepsilon^\theta\|_{L^\infty(0,T;L^{(\gamma/\theta)-}(\mathscr{F}_\varepsilon))} \\
&\leq \|u_\varepsilon^\theta\|_{L^2(0,T;L^{\infty^-}(\mathscr{F}_\varepsilon))} \|\rho_\varepsilon^\theta\|_{L^\infty(0,T;L^{\gamma/\theta}(\mathscr{F}_\varepsilon))}.
\end{aligned}
$$

Using that

$$\left(\left(\frac{2\gamma}{\gamma+2\theta}\right)^{-}\right)^{*} = \left(\frac{\gamma}{\theta}\right)^{-} \quad \text{for} \quad \gamma > 2\theta,$$

we have

$$\begin{aligned}\|J_3\|_{L^2(0,T;L^{(\gamma/\theta)-}(\mathscr{F}_\varepsilon))} &\le C\|\mathrm{div}\,(u_\varepsilon)\rho_\varepsilon^{\theta}\|_{L^2(0,T;L^{2\gamma/(\gamma+2\theta)}(\mathscr{F}_\varepsilon)} \\ &\le C\|u_\varepsilon\|_{L^2(0,T;L^2(\mathscr{F}_\varepsilon))}\|\rho_\varepsilon^{\theta}\|_{L^\infty(0,T;L^{\gamma/\theta}(\mathscr{F}_\varepsilon))}\end{aligned}$$

and similarly

$$\begin{aligned}\|J_4\|_{L^2(0,T;L^{(\gamma/\theta)-}(\mathscr{F}_\varepsilon))} &\le C\|\nabla\psi_\varepsilon u_\varepsilon\rho_\varepsilon^{\theta}\|_{L^2(0,T;L^{(2\gamma/(\gamma+2\theta)-}(\mathscr{F}_\varepsilon)} \\ &\le C\|\nabla\psi_\varepsilon\|_{L^2(\mathscr{F}_\varepsilon)}\|u_\varepsilon\|_{L^2(0,T;L^{\infty-}(\mathscr{F}_\varepsilon))}\|\rho_\varepsilon^{\theta}\|_{L^\infty(0,T;L^{\gamma/\theta}(\mathscr{F}_\varepsilon))}.\end{aligned}$$

For $\theta = \gamma/2$, we use a different estimate. First of all notice that from interpolation we have

$$\|\rho_\varepsilon u_\varepsilon\|_{L^6(0,T;L^{(3/2)+}(\mathscr{F}_\varepsilon))} \le \|\rho_\varepsilon u_\varepsilon\|^{1/3}_{L^2(0,T;L^{\gamma-}(\mathscr{F}_\varepsilon))}\|\rho_\varepsilon u_\varepsilon\|^{2/3}_{L^\infty(0,T;L^{2\gamma/(\gamma+1)})}$$

under the hypothesis $\gamma > 2$. For $j = 3, 4$, we have

$$\left|\int_0^T\int_{\mathscr{F}_\varepsilon}\rho_\varepsilon u_\varepsilon\cdot J_j\right| \le \|\rho_\varepsilon u_\varepsilon\|_{L^6(0,T;L^{(3/2)+}(\mathscr{F}_\varepsilon))}\|J_j\|_{L^{6/5}(0,T;L^{3-}(\mathscr{F}_\varepsilon)}$$

We then estimate

$$\|J_3\|_{L^{6/5}(0,T;L^{3-}(\mathscr{F}_\varepsilon)} \le \|\mathrm{div}\,(u_\varepsilon)\|_{L^2(0,T;L^2(\mathscr{F}_\varepsilon)}\|\phi\psi_\varepsilon\rho_\varepsilon^{\gamma/2}\|_{L^3(0,T;L^3(\mathscr{F}_\varepsilon))}$$

in particular we can absorb the last term on the right hand side.

Recall that we assume $\psi_\varepsilon = \tilde{\psi}_\varepsilon^2$. Similarly,

$$\begin{aligned}\|J_4\|_{L^{6/5}(0,T;L^{3-}(\mathscr{F}_\varepsilon)} &\le C\varepsilon\|\nabla\psi_\varepsilon u_\varepsilon\psi_\varepsilon^{\gamma/2}\|_{L^2(0,T;L^{2-}(\mathscr{F}_\varepsilon)} \\ &\le C\|\nabla\tilde{\psi}_\varepsilon\|_{L^2(\mathscr{F}_\varepsilon)}\|u_\varepsilon\|_{L^2(0,T;L^{\infty-}(\mathscr{F}_\varepsilon))}\|\phi\tilde{\psi}_\varepsilon\rho_\varepsilon^{\gamma/2}\|_{L^3(0,T;L^3(\mathscr{F}_\varepsilon))}.\end{aligned}$$

□

## Appendix C: Proof of Proposition 1

In this section, we prove Proposition 1 which is a straightforward extension of Lemma 3 of [9]. First of all for $A, B \in \mathbb{R}$ with $0 < A < B$, we denote by $\alpha = B/A > 1$ and we define the functions

$$f_{A,B}(z) = \begin{cases} 1 & \text{for} \quad 0 \le z < A, \\ \frac{\log z - \log B}{\log A - \log B} & \text{for} \quad A \le z \le B, \\ 0 & \text{for} \quad z > B. \end{cases}$$

It holds that $f_{A,B} \in W^{1,\infty}(\mathbb{R}^+)$. We define the cut-off

$$\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}(x) = f_{\varepsilon,\alpha_\varepsilon\varepsilon}(|x|),$$

for $x \in \mathbb{R}^2$ and $\alpha_\varepsilon > 1$.

**Proposition 3.** *Under the hypothesis that $\alpha_\varepsilon \le |\log(\varepsilon)|$ and $\alpha_\varepsilon \longrightarrow +\infty$, it holds*

1. *The functions $1 - \tilde{\eta}_{\varepsilon,\alpha_\varepsilon} \longrightarrow 0$ in $L^q(\mathbb{R}^2)$ for $1 \le q < +\infty$.*
2. *For $1 \le q < 2$,*

$$\left\|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\right\|^q_{L^q(\mathbb{R}^2)} = \frac{2\pi}{2-q}\frac{\alpha_\varepsilon^{2-q}-1}{(\log\alpha_\varepsilon)^q}\varepsilon^{2-q}.$$

3. *We have*

$$\left\|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\right\|^2_{L^2(\mathbb{R}^2)} = \frac{2\pi}{\log\alpha_\varepsilon}.$$

4. *For $2 < q < +\infty$, for $i = 1, 2$,*

$$\left\|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\right\|^q_{L^q(\mathbb{R}^2)} + \left\|\nabla^2\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}x_i\right\|^q_{L^q(B_{\varepsilon\alpha_\varepsilon}(0))} \le \frac{C}{q-2}\frac{1}{(\log\alpha_\varepsilon)^q}\varepsilon^{2-q}\left(1-\frac{1}{(\alpha_\varepsilon)^{q-2}}\right).$$

*In particular*

$$\left\|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\right\|_{L^q(\mathbb{R}^2)} \longrightarrow 0 \quad \textit{for} \quad 1 \le q \le 2$$

*and*

$$\varepsilon\alpha_\varepsilon\left\|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\right\|_{L^q(\mathbb{R}^2)}, \varepsilon\alpha_\varepsilon\left\|\nabla^2\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}x_i\right\|_{L^q(B_{\varepsilon\alpha_\varepsilon}(0))} \longrightarrow 0 \quad \textit{for} \quad 2 < q < +\infty.$$

*Proof.* After passing to radial coordinates, the proof is straightforward. For example to show part 3., we compute

$$\begin{aligned}
\left\|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\right\|^2_{L^2(\mathbb{R}^2)} &= \int_0^{2\pi}\int_\varepsilon^{\varepsilon\alpha_\varepsilon}\left|\frac{1}{r}\frac{1}{\log(\varepsilon)-\log(\varepsilon\alpha_\varepsilon)}\right|^2 r\,drd\theta \\
&= \frac{2\pi}{(\log(\alpha_\varepsilon))^2}\left[\log(r)\right]_\varepsilon^{\varepsilon\alpha_\varepsilon} \\
&= \frac{4\pi}{\log(\alpha_\varepsilon)}.
\end{aligned}$$

□

The cut-offs $\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}$ satisfy all the bounds of Proposition 1, but they are not smooth; in particular, they are not $C^2$ on $\partial B_\varepsilon(0) \cup \partial B_{\varepsilon\alpha_\varepsilon}(0)$. To solve this issue, we modify these functions as in [9]. Let introduce a function $g \in C_c^\infty([0, 12/10))$ such that $0 \le g \le 1$ and $g(y) = 1$ for $y \in [0, 11/10]$. Then, we define

$$\eta_{\varepsilon,\alpha_\varepsilon}(x) = 1 + \left(1 - g\left(\frac{|x|}{\varepsilon}\right)\right)\left(\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}(x)g\left(\frac{13}{10}\frac{|x|}{\alpha_\varepsilon\varepsilon}\right) - 1\right), \tag{31}$$

which rewrites

$$\eta_{\varepsilon,\alpha_\varepsilon}(x)=\begin{cases}1 & \text{for} \quad |x|<\frac{11}{10}\varepsilon,\\ 1+\left(1-g\left(\frac{|x|}{\varepsilon}\right)\right)\left(\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}(x)-1\right) & \text{for} \quad \frac{11}{10}\varepsilon\leq|x|<\frac{12}{10}\varepsilon,\\ \tilde{\eta}_{\varepsilon,\alpha_\varepsilon}(x) & \text{for} \quad \frac{12}{10}\varepsilon\leq|x|<\frac{11}{13}\varepsilon\alpha_\varepsilon,\\ \tilde{\eta}_{\varepsilon,\alpha_\varepsilon}(x)g\left(\frac{13}{10}\frac{|x|}{\alpha_\varepsilon\varepsilon}\right) & \text{for} \quad \frac{11}{13}\varepsilon\alpha_\varepsilon\leq|x|<\frac{12}{13}\varepsilon\alpha_\varepsilon,\\ 0 & \text{for} \quad |x|\geq\frac{12}{13}\varepsilon\alpha_\varepsilon.\end{cases}$$

The functions $\eta_{\varepsilon,\alpha_\varepsilon}$ are smooth. It remains to show that they satisfy all the properties stated in Proposition 1.

*Proof of Proposition 1.* We verify that the family $\eta_{\varepsilon,\alpha_\varepsilon}$ defined in (31) satisfies all the properties stated in Proposition 1. First of all by definition $\eta_{\varepsilon,\alpha_\varepsilon}\in C_c^\infty(B_{\varepsilon\alpha_\varepsilon}(0))$, $\eta_{\varepsilon,\alpha_\varepsilon}(x)=1$ for $x\in B_\varepsilon(0)$ and $\|\eta_{\varepsilon,\alpha_\varepsilon}\|_{L^\infty}\leq 1$. Let us now bound the $L^q$ norm of $\nabla\eta_{\varepsilon,\alpha_\varepsilon}$. As in [9], we denote by

$$g_\varepsilon^1(x)=\left(1-g\left(\frac{|x|}{\varepsilon}\right)\right),\quad g_\varepsilon^2(x)=g\left(\frac{13}{10}\frac{|x|}{\alpha_\varepsilon\varepsilon}\right)$$

and by $A_{r,R}=B_R(0)\setminus B_r(0)$ the annulus for $0<r<R$. Finally, we notice that

$$\|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}-1\|_{L^\infty\left(A_{\frac{11}{10}\varepsilon,\frac{12}{10}\varepsilon}\right)}=\left\|\frac{\log(|x|/\varepsilon)}{\log(\alpha_\varepsilon)}\right\|_{L^\infty\left(A_{\frac{11}{10}\varepsilon,\frac{12}{10}\varepsilon}\right)}\leq\frac{C}{\log(\alpha_\varepsilon)}$$

and similarly

$$\|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^\infty\left(A_{\frac{11}{13}\varepsilon\alpha_\varepsilon,\frac{12}{13}\varepsilon\alpha_\varepsilon}\right)}=\left\|\frac{\log(|x|/(\varepsilon\alpha_\varepsilon))}{\log(\alpha_\varepsilon)}\right\|_{L^\infty\left(A_{\frac{11}{13}\varepsilon\alpha_\varepsilon,\frac{12}{10}\varepsilon\alpha_\varepsilon}\right)}\leq\frac{C}{\log(\alpha_\varepsilon)}.$$

For $1\leq q<+\infty$, we estimate

$$\begin{aligned}\|\nabla\eta_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^2)}&\leq\|\nabla(g_\varepsilon^1(\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}-1))\|_{L^q\left(A_{\frac{11}{10}\varepsilon,\frac{12}{10}\varepsilon}\right)}+\|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q\left(A_{\frac{12}{10}\varepsilon,\frac{11}{13}\varepsilon\alpha_\varepsilon}\right)}\\&\quad+\|\nabla(g_\varepsilon^2\tilde{\eta}_{\varepsilon,\alpha_\varepsilon})\|_{L^q\left(A_{\frac{11}{13}\varepsilon\alpha_\varepsilon,\frac{12}{10}\varepsilon\alpha_\varepsilon}\right)}\\&\leq\|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^2)}\left(\|g_\varepsilon^1\|_{L^\infty(\mathbb{R}^2)}+1+\|g_\varepsilon^2\|_{L^\infty(\mathbb{R}^2)}\right)\\&\quad\|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}-1\|_{L^\infty\left(A_{\frac{11}{10}\varepsilon,\frac{12}{10}\varepsilon}\right)}\|\nabla g_\varepsilon^1\|_{L^q(\mathbb{R}^2)}\\&\quad+\|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^\infty\left(A_{\frac{11}{13}\varepsilon\alpha_\varepsilon,\frac{12}{13}\varepsilon\alpha_\varepsilon}\right)}\|\nabla g_\varepsilon^2\|_{L^q(\mathbb{R}^2)}\\&\leq C\|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^2)}+\frac{C}{\log(\alpha_\varepsilon)}((\alpha_\varepsilon\varepsilon)^{(2-q)/q}+\varepsilon^{(2-q)/q}),\end{aligned}$$

where we use that $1-g_\varepsilon^1$ and $g_\varepsilon^2$ are appropriate rescaling of $g$ to estimate the $L^q$ norm of $\nabla g_\varepsilon^1$ and $\nabla g_\varepsilon^2$. The bounds of the $L^q$ norm of $\nabla\eta_{\varepsilon,\alpha_\varepsilon}$ follows from the above estimate and Proposition 3, after noticing that in the case $q \leq 2$ it holds $(\alpha_\varepsilon \varepsilon)^{(2-q)/q} \geq \varepsilon^{(2-q)/q}$, while for $q \geq 2$ it holds $(\alpha_\varepsilon \varepsilon)^{(2-q)/q} \leq \varepsilon^{(2-q)/q}$. This explains the slightly different bounds in points 3. and 4.

Similarly, we estimate for $q \neq 2$

$$\begin{aligned}
\||x|\nabla \eta_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^2)} \leq & \||x|\nabla (g^1_\varepsilon (\tilde{\eta}_{\varepsilon,\alpha_\varepsilon} - 1))\|_{L^q\left(A_{\frac{11}{10}\varepsilon, \frac{12}{10}\varepsilon}\right)} + \||x|\nabla \tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q\left(A_{\frac{12}{10}\varepsilon, \frac{11}{13}\varepsilon\alpha_\varepsilon}\right)} \\
& + \||x|\nabla (g^2_\varepsilon \tilde{\eta}_{\varepsilon,\alpha_\varepsilon})\|_{L^q\left(A_{\frac{11}{13}\varepsilon\alpha_\varepsilon, \frac{12}{10}\varepsilon\alpha_\varepsilon}\right)} \\
\leq & \||x|\nabla \tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^2)} \left(\|g^1_\varepsilon\|_{L^\infty(\mathbb{R}^2)} + 1 + \|g^2_\varepsilon\|_{L^\infty(\mathbb{R}^2)}\right) \\
& + \|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon} - 1\|_{L^\infty\left(A_{\frac{11}{10}\varepsilon, \frac{12}{10}\varepsilon}\right)} \||x|\nabla g^1_\varepsilon\|_{L^q(\mathbb{R}^2)} \\
& + \|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^\infty\left(A_{\frac{11}{13}\varepsilon\alpha_\varepsilon, \frac{12}{13}\varepsilon\alpha_\varepsilon}\right)} \||x|\nabla g^2_\varepsilon\|_{L^q(\mathbb{R}^2)} \\
\leq & C\left(\||x|\nabla \tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^2)}\right) + \frac{C}{\log(\alpha_\varepsilon)}((\alpha_\varepsilon \varepsilon)^{2/q} + \varepsilon^{2/q}).
\end{aligned}$$

where we use that $1 - g^1_\varepsilon$ and $g^2_\varepsilon$ are appropriate rescaling of $g$ to estimate the $L^\infty$ norm of $|x|\nabla g^1_\varepsilon$ and $|x|\nabla g^2_\varepsilon$. Finally, we estimate for $q \neq 2$

$$\begin{aligned}
\||x|\nabla^2 \eta_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^2)} \leq & \||x|\nabla^2 (g^1_\varepsilon (\tilde{\eta}_{\varepsilon,\alpha_\varepsilon} - 1))\|_{L^q\left(A_{\frac{11}{10}\varepsilon, \frac{12}{10}\varepsilon}\right)} + \||x|\nabla^2 \tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q\left(A_{\frac{12}{10}\varepsilon, \frac{11}{13}\varepsilon\alpha_\varepsilon}\right)} \\
& + \||x|\nabla^2 (g^2_\varepsilon \tilde{\eta}_{\varepsilon,\alpha_\varepsilon})\|_{L^q\left(A_{\frac{11}{13}\varepsilon\alpha_\varepsilon, \frac{12}{10}\varepsilon\alpha_\varepsilon}\right)} \\
\leq & \||x|\nabla^2 \tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(B_{\varepsilon\alpha_\varepsilon}(0))} \left(\|g^1_\varepsilon\|_{L^\infty(\mathbb{R}^2)} + 1 + \|g^2_\varepsilon\|_{L^\infty(\mathbb{R}^2)}\right) \\
& + \|\nabla \tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^2)} \left(\||x|\nabla g^1_\varepsilon\|_{L^\infty(\mathbb{R}^2)} + \||x|\nabla g^2_\varepsilon\|_{L^\infty(\mathbb{R}^2)}\right) \\
& + \|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon} - 1\|_{L^\infty\left(A_{\frac{11}{10}\varepsilon, \frac{12}{10}\varepsilon}\right)} \||x|\nabla^2 g^1_\varepsilon\|_{L^q(\mathbb{R}^2)} \\
& + \|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^\infty\left(A_{\frac{11}{13}\varepsilon\alpha_\varepsilon, \frac{12}{13}\varepsilon\alpha_\varepsilon}\right)} \||x|\nabla^2 g^2_\varepsilon\|_{L^q(\mathbb{R}^2)} \\
\leq & C\left(\||x|\nabla^2 \tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(B_{\varepsilon\alpha_\varepsilon}(0))} + \|\nabla \tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^2)}\right) \\
& + \frac{C}{\log(\alpha_\varepsilon)}((\alpha_\varepsilon \varepsilon)^{(2-q)/q} + \varepsilon^{(2-q)/q}).
\end{aligned}$$

where as before we use that $1 - g^1_\varepsilon$ and $g^2_\varepsilon$ are appropriate rescaling of $g$ to estimate the $L^q$ norm of $|x|\nabla^2 g^1_\varepsilon$ and $|x|\nabla^2 g^2_\varepsilon$. □

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