Shallow Water Waves

Deriving model equations for shallow water waves in a continuously stratified fluid over variable bottom topography

F.A. Brouwer

TU Delft

Shallow Water Waves

Deriving model equations for shallow water waves in a continuously stratified fluid over variable bottom topography

by

F.A. Brouwer

to obtain the degree of Bachelor of Science

at the Delft University of Technology,

to be defended publicly on Wednesday July 3, 2024 at 11:00 AM.

Student number:	5170621	
Project duration:	April 22, 2024 – July 3, 2024	
Thesis committee:	Dr. A. Geyer,	TU Delft, supervisor
	Dr. J. G. Spandaw,	TU Delft

Cover image: By Jeremy Bishop on Unsplash

TU Delft

Abstract

Layman's Abstract

This thesis focuses on understanding how waves behave in shallow water, particularly near the equator, where the water's density changes with depth, and the bottom of the ocean isn't flat. The thesis is divided into two parts. In the first part, we develop a mathematical model to describe water waves in shallow water that move in two directions and where the water's density changes with depth. In the second part, we develop a mathematical model to describe water waves in shallow water that move only in one direction, where the water's density changes with depth and where we incorporate that the bottom of the ocean may vary. These two models help us predict the speed and shape of the water waves at different depths. For the second model, we find exact solutions so we can explore the effect of change in the density of the water and an uneven bottom on the waves.

Abstract

The aim of this thesis is to derive two distinct model equations for shallow water waves in a continuously stratified fluid and with variable bottom topography. This thesis is divided into two parts. First, we derive a KdV-type shallow water model equation for bi-directional shallow water waves along the equator with a continuous depth-dependent density. Secondly, we derive a KdV-type shallow water model equation for uni-directional shallow water waves along the equator with a continuous depth-dependent density and a bottom that may vary in the direction of wave propagation. We derive both model equations from the governing equations using asymptotic expansion. We obtain model equations that describe the horizontal velocity component for each fixed depth. We derive exact solutions for the second model equation under the assumption that the bottom is slowly varying and perform an analysis of the effect of the varying bottom and the change of density on the waves.

Contents

1 Introduction

In 1844, John Scott Russel reported on the discovery he called the "Wave of Translation" [10]. While conducting experiments to determine the most efficient designs for canal boats, he made a remarkable discovery. Russel observed a boat suddenly stopping in a narrow canal, causing a mass of water to surge forward, forming a smooth, solitary wave. He followed the wave on his horse, noting it maintained shape and speed over a significant distance.

This wave of translation has been of great scientific interest for the past two centuries. Such a wave is now known as a *solitary wave*, which is a solution of a nonlinear wave equation that maintains its shape and speed over long distances. The theory of solitary waves also caught the interest of two Dutch scientists, Korteweg and De Vries, who published a paper on soliton theory in 1895 [9]. Korteweg en De Vries described the solitary wave in shallow water as a solution of the nonlinear Partial Differential Equation (PDE) of the third-order, as follows:

$$2u_\tau + 3uu_\xi + \frac{1}{3}u_{\xi\xi\xi} = 0,$$

where τ and ξ are time-like and space-like variables, respectively. The equation can be derived from the governing Euler's equation and the equation of mass conservation, together with appropriate boundary conditions. For this endeavour to succeed, one has to make several simplifying assumptions regarding the fluid and the involved forces; for instance, that the fluid is incompressible and inviscid.

Many model equations for water waves are derived under the assumption that the water has a constant density or that the bottom of the fluid domain is completely flat. However, the density of the water can change with temperature and salinity, and the bottom does not have to be completely flat. Studies by Geyer and Quirchmayr, and Johnson have taken these factors into account, when deriving a KdV-like model equation. Geyer and Quirchmayr recently derived a KdV-type equation modelling the propagation of shallow water waves in a fluid with continuous density stratification [6]. Johnson derived both a two-dimensional KdV equation and a KdV type equation for a variable bottom topography [7]. The aim of this thesis is to derive two different model equations for shallow water waves. The first model equation is a KdV-type equation that models bi-directional shallow water waves in a fluid with continuous density stratification. The second model equation is a KdV-type equation that models uni-directional shallow water waves in a fluid with continuous density stratification and variable bottom topography. In both model equations we consider equatorial flows, taking into account the relatively small Coriolis force due to the Earth's rotation.

This thesis adheres to the following structure. Chapter 2 provides essential theoretical background. We will discuss the governing equations necessary for deriving the model equations. Then, we consider the standard and 2D-KdV equation. Lastly, two key concepts, stratification and variable bottom topography, and their importance to this thesis are discussed. In Chapter 3, we derive the first model equation for two-dimensional shallow water waves with continuous density stratification. This model equation will be derived by means of asymptotic expansion of certain variables. Chapter 4 will derive the second model equation for shallow water waves with continuous density stratification and variable bottom topography. This derivation will again be done by means of asymptotic expansion of certain variables. We will also consider an exact travelling wave solution of this model equation. Lastly, the main takeaways and discussion points can be found in Chapter 5.

2

Theoretical Background

Before starting with the derivation and analysis for both model equations, we will need some understanding of the theory. First, we will introduce the governing equations and boundary conditions. Next, we will describe the standard KdV and 2D-KdV equation. After that, we will explore the concept of stratification and see why it is important to take stratification into consideration. Finally, we will discuss the effects of variable bottom topography on waves.

2.1. Governing Equations

2.1.1. Euler's Equation

We start by considering Euler's equation, which is fundamental to fluid dynamics. Euler's equation describes the motion of an inviscid fluid, and many models and model equations are derived from this. Euler's equation will also be the starting point of our derivations of the shallow water model equations.
Euler's equation can be derived from Newton's Second Law of motion applied to an *inviscid* fluid (that is, the fluid has zero viscosity). Newton's Second Law balances the rate of change of momentum of the fluid against the forces that are acting on the fluid. We refer to Johnson [7] for a detailed derivation and explanation of Euler's equation. Euler's equation then becomes:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla P + \mathbf{F}, \tag{2.1}$$

where $\frac{D\mathbf{u}}{Dt}$ is the material derivative. The material derivative is denoted as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}. \tag{2.2}$$

Euler's equation describes the flow of a fluid. We have a spatial vector $\mathbf{x} \equiv (x, y, z)$. The vector $\mathbf{u} \equiv (u, v, w)$ denotes the velocity in the x, y, and z direction, respectively. Time is described by t. The pressure of the fluid is denoted by P. The variable ρ is the fluid density. The force vector is denoted by $\mathbf{F}$ and is written in three components as $\mathbf{F} \equiv (0, 0, -g)$, where g is the gravitational acceleration. We expand Euler's equation into three components, resulting in:

$$\begin{aligned}\frac{Du}{Dt} &= -\frac{1}{\rho}\frac{\partial P}{\partial x}\\ \frac{Dv}{Dt} &= -\frac{1}{\rho}\frac{\partial P}{\partial y}\\ \frac{Dw}{Dt} &= -\frac{1}{\rho}\frac{\partial P}{\partial z} - g.\end{aligned} \tag{2.3}$$

Euler's equation forms a basis for the derivations of the two model equations for shallow water waves. However, we need more equations to derive the model equations, namely the equation of mass conservation and the boundary conditions. We start with the equation of mass conservation.

2.1.2. Equation of Mass Conservation

Besides Euler's equation, we need the equation of mass conservation. The derivation of this equation is again described in Johnson [7]. This equation states that the system's mass must remain constant for a closed system. That implies neither mass can be destroyed nor created. The equation of mass conservation becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \tag{2.4}$$

Euler's equation, together with the equation of mass conservation, form the basis for our derivations to arrive at a model equation for shallow water waves.

2.1.3. Boundary Conditions

To finalize the governing equations, we need boundary conditions for our model. We keep in mind that we are modelling waves in shallow water, which is a fluid. Boundary conditions play a crucial role in describing the behaviour of the fluid near the boundaries of the domain. In both of our models, we will have a *dynamic* and a *kinematic* boundary condition at the surface of the water. We will have one more boundary condition at the bottom of the ocean. We will explain the boundary conditions in more detail when we derive our two model equations.

2.2. The Standard KdV Equation

One way to model shallow water waves is by the Korteweg-de Vries (KdV) equation. The history of the KdV equation is already discussed in the Introduction of this thesis. The solutions of the KdV equation describe the phenomenon of solitary waves. The KdV equation can be derived from Euler's equation together with the equation of mass conservation while imposing important assumptions. The standard KdV assumes that the waves propagate only in the x-direction and that the density, ρ, is constant throughout the fluid domain. Then the equation of mass conservation (2.4) simplifies to

$$u_x + w_z = 0, \tag{2.5}$$

and Euler's equation is applied to the velocity vector $\mathbf{u} \equiv (u, w)$ in spatial direction $\mathbf{x} \equiv (x, z)$. The standard form of the KdV equation is as follows:

$$2\eta_\tau + 3\eta\eta_\xi + \frac{1}{3}\eta_{\xi\xi\xi} = 0, \tag{2.6}$$

where η describes the deviation of the free surface from the average water level at leading order. The variables τ and ξ are time-like and space-like variables, respectively. We will define the variables in Section 3.3.

2.2.1. The 2D-KdV Equation

The standard KdV equation (2.6) describes nonlinear waves that propagate in the x-direction. In the physical situation, however, the waves propagate on a two-dimensional surface in the x- and y-direction. One important assumption is made to construct a KdV equation for waves that propagate in two directions, namely that the waves propagate predominantly in the x-direction. The 2D KdV equation is again derived by Johnson [7] and stated as:

$$\left(2\eta_\tau + 3\eta\eta_\xi + \frac{1}{3}\eta_{\xi\xi\xi}\right)_\xi + \eta_{yy} = 0. \tag{2.7}$$

2.3. Stratification

In many models, the fluid density is assumed to be constant throughout the entire fluid body. However, factors such as temperature and salinity affect the density and give rise to *stratified flows* [3]. Stratified flows are characterized by different horizontal layers of the fluid with varying densities. These different layers are caused by variations in temperature and salinity. The temperature of the ocean decreases rapidly between 200m and 1.000m. This oceanic water layer is known as the thermocline. Furthermore, the halocline is the vertical zone in the ocean where the salinity changes rapidly with depth. Both the thermocline and the halocline give rise to the pycnocline - a layer that separates surface water of lower density from deep ocean water of higher density. The pycnocline is often considered an infinitely thin interface between two layers of different but constant densities. In contrast, this research assumes that density is a continuous function of depth, providing a more physically accurate model in the case where density difference happens gradually.

Such stratification phenomena occur mainly in oceanic regions near the equator [4]. For this reason, we take the *Coriolis force* into account. The Coriolis force is caused by the rotation of the Earth around its axis. This rotation deflects the wind to the right in the Northern Hemisphere and the left in the Southern Hemisphere. The wind's movement over the ocean's surface drives surface ocean currents. So, when modelling water waves in the ocean near the equator, we need to consider the Coriolis force. We may use the f-plane approximation when considering the Coriolis force in our model, which is valid near the equator [1].

The research of Geyer and Quirchmayr [6] examined a KdV-type equation with continuous stratification. They found a different model equation for the horizontal component of the velocity field for each fixed water depth. The model equation for the free surface η at leading-order is stated as follows:

$$2\eta_\tau - 2\Omega_0\eta_\xi + \nu(\rho)\eta\eta_\xi + \frac{1}{3}\eta_{\xi\xi\xi} = 0, \tag{2.8}$$

where

$$\nu(\rho) = 2 - \rho_z(1) + \int_0^1 \left(\rho_z + \frac{(z\rho)_z}{\rho^2}\right) dz \in \mathbb{R}, \tag{2.9}$$

and the Coriolis force is captured in Ω_0.

Then, the equation for the horizontal velocity u at leading-order at any depth has a depth-dependent coefficient of the nonlinear term:

$$2u_\tau - 2\Omega_0 u_\xi + \rho(z)\nu(\rho)uu_\xi + \frac{1}{3}u_{\xi\xi\xi} = 0. \tag{2.10}$$

Geyer and Quirchmayr discovered that changes in density only affect the amplitude of solitary waves.

2.4. Variable Bottom Topography

Besides the common assumption that the density is constant throughout the fluid body, the bottom of the fluid volume is often assumed to be flat. However, the bottom topography does have a significant impact on the propagation of the waves. For example, when a solitary wave is approaching shallower water, the amplitude of the wave increases [11]. We can model a wave in shallow water over a variable bottom topography with the KdV equation for variable depth. This equation is derived by Johnson [7] and is stated as follows:

$$2(D^{\frac{1}{4}}\eta_0)_X + \frac{3}{D^{\frac{5}{4}}}\eta_0\eta_{0\xi} + \frac{1}{3}D^{\frac{3}{4}}\eta_{0\xi\xi\xi} = 0, \tag{2.11}$$

where $D = D(X)$ is the depth. In this equation, X can be considered as τ, and thus as a time-like variable. The variable D is the depth of the water and it may vary with X.

Johnson examined the solutions for both slowly and rapidly varying bottom topographies. He found that over a slowly varying bottom, a solitary wave experiences an increase in amplitude and becomes narrower. Conversely, over a rapidly varying bottom, the solitary wave splits into an N-soliton wave.

3

A Two-Dimensional Shallow Water Model Equation with Continuous Stratification

In this chapter, we will derive a two-dimensional shallow water model equation with continuous stratification. We will first explain why we can use Euler's equation and the equation of mass conservation with a continuous density function as the basis of our derivation for a shallow water model equation near the equator. We will examine the dimensional form of these equations, after which we introduce the dimensionless variables. After we have nondimensionalised the model, we will scale the model. Following the method of Johnson [7], we will see that the leading-order approximation of the free surface satisfies the linear wave equation. We will then introduce *far-field variables* and perform an asymptotic expansion on all involved variables. From the resulting system, we derive an equation for the free-surface elevation and the horizontal velocity of bi-directional shallow water waves in fluids with continuous stratification.

3.1. Constructing the Governing Equations and Boundary Conditions

3.1.1. Assumptions

Before constructing a model, it is essential to establish a set of assumptions while keeping in mind that we want to model shallow water waves near the equator. Firstly, we will assume that the water is inviscid, meaning that the water has zero viscosity. Secondly, we consider a continuous change in density. In other research, the density function is often assumed to be constant throughout the entire fluid body. As explained in section 2.3, this research assumes the density to be a continuous function of the depth. Finally, we will assume that the waves both propagate in the direction of increasing azimuth as well as in the direction of increasing latitude. We look at waves near the equator, where often it is assumed that waves only propagate in the direction of increasing azimuth. In this paper, we assume that the waves propagate in both directions, but we impose the restriction that the waves propagate predominantly in the direction of increasing azimuth.

Combining all those assumptions makes it possible to use Euler's equation with a continuous density function. We follow the approach of Johnson [7], but include a continuous change of density. Before we look at these equations, we will need to consider the coordinate system and the domain of the governing equations.

3.1.2. Coordinate System and Domain

First, we need to define our coordinate system in which we define Euler's equation. We will use the coordinate system as used in [1] and [7]. The variables $\bar{x}$, $\bar{y}$, and $\bar{z}$ denote the directions of increasing azimuth, increasing latitude, and vertical elevation, respectively. This is illustrated by figure 3.1.

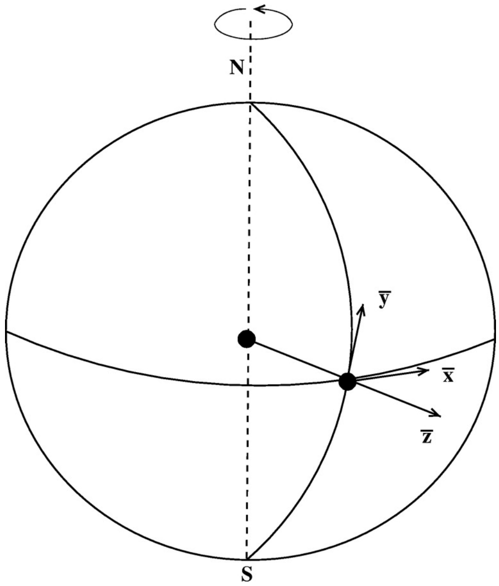


Figure 3.1: The rotating frame of reference, with the x-axis chosen horizontally due East, the y-axis horizontally due North and the z-axis upward.

Note. From "The dynamics of waves interacting with the Equatorial Undercurrent," by A. Constantin & R.S. Johnson, 2015, Geophysical and Astrophysical Fluid Dynamics, 109(4), 311-358.

The coordinate $\bar{t}$ for changes in time completes our coordinate system. Furthermore, we need to examine the fluid domain on which our equations will be defined. Since we look at water flows in the ocean, we work in an infinite fluid domain in the x and y direction. Thus, we only consider boundary conditions in the $\bar{z}$ direction. From below, the fluid is bounded by a flat bed located at $\bar{z} = 0$ and above by the free water surface at $\bar{h}_0 + \bar{\eta}(\cdot, \bar{t})$. Here $\bar{\eta}(\bar{x}, \bar{y}, \bar{t})$ measures the deviation of the free surface from the average water depth $\bar{h}_0$ at $(\bar{x}, \bar{y}, \bar{t})$. The fluid domain is illustrated in Figure 3.2a. As presented in Section 3.1.1, we describe the density as a continuously differentiable function of the depth. An example of a density function is illustrated in figure 3.2b.

3.1.3. Equations of Motion

In the previous two subsections, we made some assumptions for our model and we defined the coordinate system and domain of the model. We concluded that we could use Euler's equation with a continuous density function together with the equation for mass conservation. We combine Euler's equation (2.3) and the equation of mass conservation (2.4) together with the Coriolis effect. This yields the following equations of motion with continuous stratification:

$$\begin{aligned} \bar{u}_{\bar{t}} + \bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}} + \bar{w}\bar{u}_{\bar{z}} + 2\bar{\Omega}\bar{w} &= -\bar{\rho}^{-1}\bar{P}_{\bar{x}} \\ \bar{v}_{\bar{t}} + \bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}} + \bar{w}\bar{v}_{\bar{z}} &= -\bar{\rho}^{-1}\bar{P}_{\bar{y}} \\ \bar{w}_{\bar{t}} + \bar{u}\bar{w}_{\bar{x}} + \bar{v}\bar{w}_{\bar{y}} + \bar{w}\bar{w}_{\bar{z}} - 2\bar{\Omega}\bar{u} &= -\bar{\rho}^{-1}\bar{P}_{\bar{z}} - \bar{g} \\ \bar{\rho}_{\bar{t}} + (\bar{\rho}\bar{u})_{\bar{x}} + (\bar{\rho}\bar{v})_{\bar{y}} + (\bar{\rho}\bar{w})_{\bar{z}} &= 0, \end{aligned} \tag{3.1}$$

where the bars distinguish the physical variables from the dimensionless variables later on.

In Equation (3.1) $\bar{x}$, $\bar{y}$ and $\bar{z}$ denote the directions of increasing azimuth, increasing latitude and vertical elevation respectively. $\bar{u}$, $\bar{v}$, and $\bar{w}$ denote the fluid velocity in the directions of increasing azimuth, increasing latitude, and vertical elevation, respectively. The pressure is denoted by $\bar{P} = \bar{P}(\bar{x}, \bar{y}, \bar{z}, \bar{t})$. We consider a prescribed but arbitrary continuously differentiable density function $\bar{\rho}$ of the water which may vary with depth

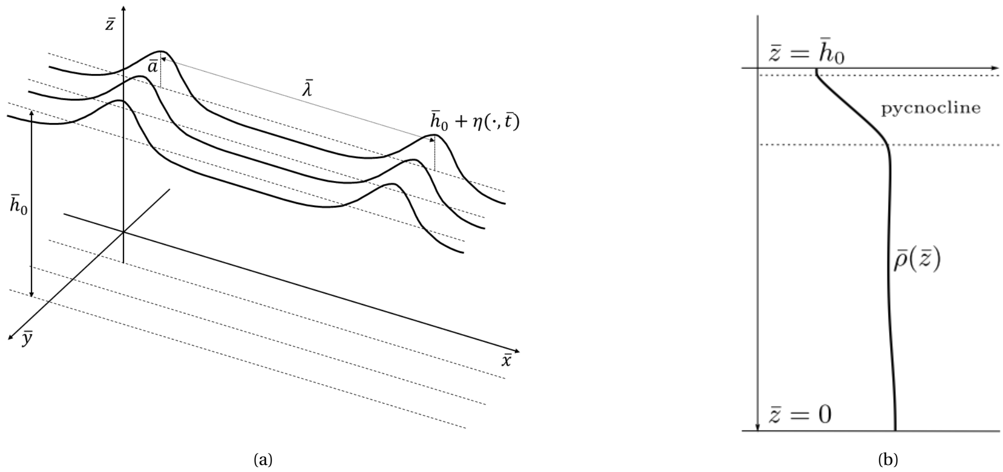


Figure 3.2: Figure 3.2a illustrates the fluid domain. The typical depth of the water is denoted by $\bar{h}_0$. The typical wavelength of the surface wave is denoted by $\bar{\lambda}$, and $\bar{a}$ is the typical amplitude of a surface wave. Figure 3.2b shows a density function varying with depth, having a significant increase in the pycnocline.

Note. Fig 3.2b from "Shallow water models for stratified flows," by A. Geyer & R. Quirchmayr, 2019, Discrete and continuous dynamical systems, 39(8), 4533-4545.

only, as illustrated in Figure 3.2b,

$$\bar{\rho} = \bar{\rho}(\bar{z}) > 0.$$

The gravitational acceleration is denoted by $\bar{g} \approx 9.81 ms^{-2}$. The Coriolis effect is taken into account with the two $\bar{\Omega}$ terms. These two terms follow from the f-plane approximation of the Coriolis force. The Coriolis force is captured by $\bar{\Omega}$, which is the rotational speed of the Earth around the polar axis towards the East and has the approximate magnitude of 7.29×10^{-5}rads^{-1}.

Since ρ only depends on z, the equation of mass conservation simplifies to

$$\bar{u}_{\bar{x}} + \bar{v}_{\bar{y}} + \bar{w}_{\bar{z}} + \frac{\bar{\rho}_{\bar{z}}\bar{w}}{\bar{\rho}} = 0.$$

3.1.4. Boundary Conditions

To complete our model, we need boundary conditions. As mentioned in Section 3.1.2, we only have boundary conditions in the $\bar{z}$ direction. The fluid is bounded below by a flat bed, located at $\bar{z} = 0$ and bounded from above by the free water surface at $\bar{h}_0 + \bar{\eta}(\cdot, \bar{t})$. At the surface of inviscid water, the atmosphere exerts pressure on the surface. Hence the *dynamic boundary condition* requires that $\bar{P} = \bar{P}_{atm}$ at the free surface $\bar{z} = \bar{h}_0 + \bar{\eta}(\cdot, \bar{t})$. Furthermore, we require that fluid particles that are initially on the surface stay on the surface; that is, the fluid particles cannot leave the fluid domain. We call this the *kinematic boundary condition.* This kinematic boundary conditions states that $\bar{w} = \bar{\eta}_{\bar{t}} + \bar{u}\bar{\eta}_{\bar{x}} + \bar{v}\bar{\eta}_{\bar{y}}$ on $\bar{z} = \bar{h}_0 + \bar{\eta}(\cdot, \bar{t})$. Finally, we have the boundary condition for the flat bed. We assume that the bottom surface is fixed and rigid; that is, the fluid velocity will be zero here. Together, the boundary conditions are described by the following set of equations:

$$\bar{P} = \bar{P}_{atm} \text{ on } \bar{z} = \bar{h}_0 + \bar{\eta}(\cdot, \bar{t}) \tag{3.2}$$

$$\bar{w} = \bar{\eta}_{\bar{t}} + \bar{u}\bar{\eta}_{\bar{x}} + \bar{v}\bar{\eta}_{\bar{y}} \text{ on } \bar{z} = \bar{h}_0 + \bar{\eta}(\cdot, \bar{t}) \tag{3.3}$$

$$\bar{w} = 0 \text{ on } \bar{z} = 0. \tag{3.4}$$

3.2. Nondimensionalisation and Scaling

3.2.1. Nondimensionalisation

To analyse the governing equations and boundary conditions described in Section 3.1, we need to nondimensionalise the equations. Nondimensionalisation ensures that the equations have no physical dimensions. This helps us to get insight into the relations between different variables and quantities in our system. We will make use of length scales $\bar{h}_0$ and $\bar{\lambda}$. We take $\bar{h}_0$ to be the typical depth of the water and $\bar{\lambda}$ to be the typical wavelength of the surface wave. We scale $\bar{\eta}$ with $\bar{a}$, which is the typical amplitude of a surface wave. We introduce the following nondimensional variables, following the same method as in [6], but including the nondimensional variables for $\bar{y}$ and $\bar{v}$:

$$\begin{aligned}
\bar{x} = \bar{\lambda}x, \quad \bar{y} = \bar{\lambda}y, \quad \bar{z} = \bar{h}_0 z, \quad \bar{t} = \frac{\bar{\lambda}}{\sqrt{\bar{g}\bar{h}_0}}t,\\
\bar{u} = \sqrt{\bar{g}\bar{h}_0}u, \quad \bar{v} = \sqrt{\bar{g}\bar{h}_0}v, \quad \bar{w} = \sqrt{\bar{g}\bar{h}_0}\frac{\bar{h}_0}{\bar{\lambda}}w, \quad \bar{\eta} = \bar{a}\eta,\\
\bar{P} = \bar{P}_{atm} - \bar{g}\int_{\bar{h}_0}^{\bar{z}} \bar{\rho}(s)ds = \bar{g}\bar{h}_0\bar{\rho}P, \quad \bar{\Omega} = \frac{\sqrt{\bar{g}\bar{h}_0}}{\bar{h}_0}\Omega.
\end{aligned} \tag{3.5}$$

Furthermore, the density is scaled according to

$$\bar{\rho}(\bar{z}) = \bar{\rho}_0\rho(z), \quad \text{where} \quad \bar{\rho}_0 = \bar{\rho}(\bar{h}_0). \tag{3.6}$$

Lastly, we introduce two dimensionless parameters

$$\epsilon := \frac{\bar{a}}{\bar{h}_0}, \qquad \delta := \frac{\bar{h}_0}{\bar{\lambda}}, \tag{3.7}$$

expressing respectively the *amplitude* and *shallowness* parameter.

When we plug in the dimensionless variables and the amplitude and shallowness parameter in (3.1) and (3.2), we get the following dimensionless form:

$$\begin{aligned}
u_t + uu_x + vu_y + wu_z + 2\Omega w &= -P_x\\
v_t + uv_x + vv_y + wv_z &= -P_y\\
\delta^2(w_t + uw_x + vw_y + ww_z) - 2\Omega u &= -\frac{(\rho P)_z}{\rho}\\
u_x + v_y + w_z + \frac{\rho_z w}{\rho} &= 0\\
P = \frac{1}{\rho}\int_1^z \rho(s)ds \quad &\text{on} \quad z = 1 + \epsilon\eta(x,y,t)\\
w = \epsilon(\eta_t + u\eta_x + v\eta_y) \quad &\text{on} \quad z = 1 + \epsilon\eta(x,y,t)\\
w = 0 \quad &\text{on} \quad z = 0.
\end{aligned} \tag{3.8}$$

A problem that occurs now is that we have two boundary conditions at the free surface $z = 1 + \epsilon\eta(x,y,t)$, which is unknown. We will use Taylor expansions about $z = 1$ to rewrite the boundary conditions at the free surface to a fixed boundary. We obtain the following set of boundary conditions

$$\begin{aligned} P + \epsilon\eta P_z + \frac{\epsilon^2\eta^2}{2} P_{zz} &= \epsilon\eta - \epsilon^2\eta^2 \frac{\rho_z}{2} + \mathcal{O}(\epsilon^3) \quad \text{on} \quad z = 1 \\ w &= \epsilon(\eta_t + u\eta_x + v\eta_y - \eta w_z) + \mathcal{O}(\epsilon^2) \quad \text{on} \quad z = 1, \end{aligned}$$

where we have used that $\rho(1) = 1$.

3.2.2. Scaling

When we examine the boundary conditions, we see that P and w are proportional to ϵ. This makes sense since when there are no disturbances on the free surface, it becomes horizontal on which $w = 0 = P$. So, following [2], [6] and [7], we will scale P and w with ϵ. Since the equation of mass conservation must hold, we will also scale u and v with ϵ. We write

$$P \mapsto \epsilon P, \quad w \mapsto \epsilon w, \quad (u, v) \mapsto \epsilon(u, v), \quad \Omega \mapsto \epsilon\Omega_0, \tag{3.9}$$

where Ω_0 is an appropriate constant. This scaling is valid since Ω and ϵ are of the same order of magnitude, as discussed in [5].

We apply this scaling and the Taylor expansion of Section 3.2.1 in (4.6), resulting in the following system:

$$\begin{aligned} u_t + \epsilon(uu_x + vu_y + wu_z) + 2\epsilon\Omega_0 w &= -P_x \\ v_t + \epsilon(uv_x + vv_y + wv_z) &= -P_y \\ \delta^2(w_t + \epsilon(uw_x + vw_y + ww_z)) - 2\epsilon\Omega_0 u &= -\frac{(\rho P)_z}{\rho} \\ u_x + v_y + w_z + \frac{\rho_z w}{\rho} &= 0 \\ P = \eta - \epsilon\eta(\frac{\eta}{2}\rho_z + P_z) \quad &\text{on} \quad z = 1 \\ w = \eta_t + \epsilon(u\eta_x + v\eta_y - \eta w_z) \quad &\text{on} \quad z = 1 \\ w = 0 \quad &\text{on} \quad z = 0. \end{aligned} \tag{3.10}$$

Next, we follow the approach of Johnson [7]. They scale out δ in favour of ϵ, making the following transformation.

$$\begin{aligned} x \mapsto \frac{\delta}{\sqrt{\epsilon}} x, \quad y \mapsto \frac{\delta}{\sqrt{\epsilon}} y, \quad z \mapsto z \quad , \quad t \mapsto \frac{\delta}{\sqrt{\epsilon}} t, \\ P \mapsto P, \quad \eta \mapsto \eta, \quad u \mapsto u, \quad v \mapsto v, \quad w \mapsto \frac{\sqrt{\epsilon}}{\delta} w. \end{aligned} \tag{3.11}$$

When we perform this transformation, we note that it is the same as replacing δ^2 by ϵ. As in [7], we consider the relation between ϵ and δ that describe *shallow water waves of small amplitude*:

$$\delta \ll 1, \qquad \epsilon = \mathcal{O}(\delta^2). \tag{3.12}$$

When we apply the scaling (3.11) to the system (3.10) we get

$$
\begin{aligned}
u_t + \epsilon(uu_x + vu_y + wu_z) + 2\epsilon\Omega_0 w &= -P_x \\
v_t + \epsilon(uv_x + vv_y + wv_z) &= -P_y \\
\epsilon(w_t + \epsilon(uw_x + vw_y + ww_z)) - 2\epsilon\Omega_0 u &= -\frac{(\rho P)_z}{\rho} \\
u_x + v_y + w_z + \frac{\rho_z w}{\rho} &= 0
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
P = \eta - \epsilon\eta(\frac{\eta}{2}\rho_z + P_z) \quad &\text{on} \quad z = 1 \\
w = \eta_t + \epsilon(u\eta_x + v\eta_y - \eta w_z) \quad &\text{on} \quad z = 1 \\
w = 0 \quad &\text{on} \quad z = 0.
\end{aligned}
$$

The system (3.13) encompasses the nondimensionalised and scaled governing equations, based on Euer's equation and the equation of mass conservation, and boundary conditions valid in the regime where $\epsilon = \mathcal{O}(\delta^2)$; that is, in shallow water. This system is still too difficult to analyse. Therefore, in the next chapter, we will derive equations for the leading-order approximation of η and u.

3.3. Derivation of a Shallow Water model Equation

3.3.1. Leading-Order Problem

The basis of the following discussion is the system (3.13), obtained in the previous section. We will examine the Leading-order problem of this system. If we let $\epsilon \to 0$, we arrive at the following Leading-order Problem:

$$
\begin{aligned}
u_t &= -P_x \\
v_t &= -P_y \\
0 &= (\rho P)_z \\
u_x + v_y + \frac{\rho_z}{\rho} w &= 0
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
P = \eta \quad &\text{on} \quad z = 1 \\
w = \eta_t \quad &\text{on} \quad z = 1 \\
w = 0 \quad &\text{on} \quad z = 0.
\end{aligned}
$$

We will see that this Leading-order Problem satisfies the linear wave equation:

$$
\eta_{tt} - (\eta_{xx} + \eta_{yy}) = 0. \tag{3.15}
$$

To get this linear wave equation, we note that the third equation in (3.14) together with the dynamic boundary condition on $z = 1$ lead to $\eta = \rho P$ for all $z \in [0, 1]$. Plugging this into the first and second equation of (3.14) leads to the equations $\rho u_t = -\eta_x$ and $\rho v_t = -\eta_y$. Note that since η is independent of z, we know that ρu_t and ρv_t must be independent of z. Furthermore, the equation of mass conservation yield that $(\rho w)_z = -\rho u_x - \rho v_y$, and hence

$$
w(x, y, z, t) = -\rho(z)^{-1} \int_0^z (\rho(s) u_x(x, y, s, t) + \rho(s) v_y(x, y, s, t)) ds.
$$

When we integrate this with respect to t, we get:

$$\frac{\partial}{\partial t} w(x,y,z,t) = \frac{\partial}{\partial t}\Big(-\rho(z)^{-1}\int_0^z (\rho(s)u_x(x,y,s,t)+\rho(s)v_y(x,y,s,t))ds\Big)$$
$$\Leftrightarrow$$
$$w_t(x,y,z,t) = -\rho(z)^{-1}\int_0^z \frac{\partial}{\partial t}(\rho(s)u_x(x,y,s,t)+\rho(s)v_y(x,y,s,t))ds$$
$$\Leftrightarrow$$
$$w_t(x,y,z,t) = -\rho(z)^{-1}\int_0^z (\rho(s)u_{xt}(x,y,s,t)+\rho(s)v_{yt}(x,y,s,t))ds.$$

When considering the kinematic boundary condition at $z=1$, we note that we have $w_t = \eta_{tt}$ on $z=1$. Thus on $z=1$, we have that

$$\eta_{tt} = w_t = -\int_0^1 (\rho(s)u_{xt}(x,y,s,t)+\rho(s)v_{yt}(x,y,s,t))ds.$$

We then get

$$\begin{aligned}\eta_{tt}-(\eta_{xx}+\eta_{yy}) &= -\int_0^1 \big(\rho(s)u_{xt}(x,y,s,t)+\rho(s)v_{yt}(x,y,s,t)\big)\,ds - \big[(-\rho u_t)_x+(-\rho v_t)_y\big]\\ &= -\int_0^1 \big(\rho(s)u_{xt}(x,y,s,t)+\rho(s)v_{yt}(x,y,s,t)\big)\,ds + \rho u_{xt}+\rho v_{yt}\\ &= -\int_0^1 \big(\rho(s)u_{xt}(x,y,s,t)+\rho(s)v_{yt}(x,y,s,t)\big)\,ds + \int_0^1 \big(\rho(s)u_{xt}(x,y,s,t)+\rho(s)v_{yt}(x,y,s,t)\big)\,ds\\ &= 0,\end{aligned} \tag{3.16}$$

where the second-to-last step is justified by the fact that ρu_{xt} and ρv_{yt} are independent of z. So, for $z=1$, the Leading-order Problem satisfies the linear wave equation. However, this equation lacks essential properties for modelling more complex wave behaviours. The Linear wave equation cannot model dispersion, where wave components with different wave numbers travel at different speeds. Additionally, it cannot model solitary waves, periodic waves with prolonged roughs and sharper crests, or the nonlinear interaction between solitary waves. Therefore, in the next section, we will derive a nonlinear model equation for shallow water waves by considering an asymptotic expansion of the unknown variables.

3.3.2. Asymptotic Expansion

In this section, we will asymptotically expand variables η, u, v, w and P and then consider the leading-order and first-order systems. Before we do this, we consider the assumption that the waves propagate predominantly in the x-direction. We follow the method of Johnson [7] and consider the solution of the linear wave equation (3.15). This equation has a solution

$$\eta \propto e^{i(kx+ly-\omega t)}.$$

Upon substitution of this solution into the linear wave equation (3.15), we get that (3.15) is a solution of the linear wave equation if $\omega^2 = k^2 + l^2$. This relation is called the *dispersion relation*. In this identity, k and l are the *wave numbers* in the x and y direction, respectively. The wave number is defined as the number of wavelengths per unit distance. The variable ω is the *angular frequency* or *angular velocity* of the wave.

We assumed that waves predominantly propagate in the x direction, so we require l to be small. The dispersion relation then gives

$$\omega = \sqrt{k^2 + l^2} = k\sqrt{1 + \frac{l^2}{k^2}} \sim k\left(1 + \frac{1}{2}\frac{l^2}{k^2}\right) \quad \text{as} \quad l \to 0,$$

and thus

$$\frac{\omega}{k} \sim 1 + \frac{1}{2}\frac{l^2}{k^2}.$$

This last relation represents the propagation at the speed of unity in the x-direction, together with a correction provided by the wave-number component in the y-direction. We can deduce that waves with different wave numbers $\mathbf{k} \equiv (k, l)$ travel at different speeds. We want the correction in the y-direction to be the same size as the nonlinearity and dispersion. In order to get this, we require $l^2 = \mathcal{O}(\epsilon)$. Thus, we must transform y and v accordingly. We follow the approach of Johnson [7], and transform $y \to \epsilon^{\frac{1}{2}} y$, and $v \to \epsilon^{-\frac{1}{2}} v$.

Furthermore, we introduce the *far-field variables*. The far-field variables are used to study the behaviour of the nonlinear terms. At Leading-order, the model behaves like the linear wave equation. To gain insight into the nonlinear effects, we use the far-field variables. In this problem, we use a slow time variable $\tau = \epsilon t$. This slow time variable allows us to observe the nonlinear effects over a longer period rather than the linear effects that are dominant in the near-field region. We also choose to follow the right-going wave so we can study how the wave evolves over a long time when the nonlinear effects become apparent.

Thus, we introduce the variables

$$\xi = x - t, \qquad \tau = \epsilon t, \qquad Y = \sqrt{\epsilon} y, \qquad v = \sqrt{\epsilon} V, \tag{3.17}$$

which we substitute into equation (3.13). This system then becomes

$$\begin{aligned}
-u_\xi + \epsilon(u_\tau + u u_\xi + \epsilon V u_Y + w u_z) + 2\epsilon\Omega_0 w &= -P_\xi \\
-V_\xi + \epsilon(V_\tau + u V_\xi + \epsilon V V_Y + w V_z) &= -P_Y \\
\epsilon(-w_\xi + \epsilon(w_\tau + u w_\xi + \epsilon V w_Y + w w_z)) - 2\epsilon\Omega_0 u &= -\frac{(\rho P)_z}{\rho} \\
u_\xi + \epsilon V_Y + w_z + \frac{\rho_z w}{\rho} &= 0
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
P = \eta - \epsilon\eta(\frac{\eta}{2}\rho_z + P_z) \quad &\text{on} \quad z = 1 \\
w = -\eta_\xi + \epsilon(\eta_\tau + u\eta_\xi + \epsilon V \eta_Y - \eta w_z) \quad &\text{on} \quad z = 1 \\
w = 0 \quad &\text{on} \quad z = 0.
\end{aligned}$$

Now, we can follow the approach of Johnson [7] and asymptotically expand the variables η, u, v, w and P. Expanding the variables asymptotically means we expand the variables in the form $q \sim \sum_{n=0}^{\infty} q_n \epsilon^n$. At leading order, meaning we let $\epsilon \to 0$, we obtain the following linear system:

$$\begin{aligned}
u_{0\xi} &= P_{0\xi} \\
V_{0\xi} &= P_{0Y} \\
0 &= (\rho P_0)_z \\
(\rho w_0)_z &= -\rho u_{0\xi}
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
P_0 = \eta_0 \quad &\text{on} \quad z = 1 \\
w_0 = -\eta_{0\xi} \quad &\text{on} \quad z = 1 \\
w_0 = 0 \quad &\text{on} \quad z = 0.
\end{aligned}$$

From integration of the third equation and the dynamic boundary condition on $z = 1$, we obtain that $\eta_0 = \rho P_0$ for all $z \in [0,1]$. The first and second equation give then $u_{0\xi} = \rho^{-1}\eta_{0\xi}$ and $V_{0\xi} = \rho^{-1}\eta_{0Y}$. Lastly, we observe that

$$u_0 = \rho^{-1}\eta_0, \tag{3.20}$$

since the perturbation of u is only caused by the passage of the wave. This is, we must have that $u_0 = 0$ whenever $\eta + 0 = 0$. From this relation, the fourth equation and the bottom condition, we deduce that $w_0 = -z\rho^{-1}\eta_{0\xi}$.

We see that P_0, u_0, v_0 and w_0 all depend on η_0, which is still unknown. To determine η_0, we will have a look at the first order system. We consider the first order system, with the leading order solutions of u_0, v_0, and w_0 substituted in the equation:

$$\begin{aligned} -u_{1\xi} + \rho^{-1}\eta_{0\tau} + \rho^{-2}(1 + z\rho^{-1}\rho_z)\eta_0\eta_{0\xi} - 2\Omega_0 z\rho^{-1}\eta_{0\xi} &= -P_{1\xi} \\ -V_{1\xi} + V_{0\tau} + \rho^{-2}\eta_0\eta_{0Y} - z\rho^{-1}\eta_{0\xi}V_{0z} &= -P_{1Y} \\ z\eta_{0\xi\xi} - 2\Omega_0\eta_0 &= -(\rho P_1)_z \\ (\rho w_1)_z &= -\rho u_{1\xi} - V_{0Y} \\ P_1 = \eta_1 + \frac{\rho_z}{2}\eta_0^2 \quad &\text{on} \quad z = 1 \\ w_1 = -\eta_{1\xi} + \eta_{0\tau} + (2 - \rho_z)\eta_0\eta_{0\xi} \quad &\text{on} \quad z = 1 \\ w_1 = 0 \quad &\text{on} \quad z = 0. \end{aligned} \tag{3.21}$$

When we consider the third equation as a linear ODE in P_1 and use the dynamic boundary condition in $z = 1$, we get

$$\begin{aligned} P_1(z) &= \rho^{-1}\Big[\int_0^z (2\Omega_0\eta_0 - s\eta_{0\xi\xi})ds\Big] \\ &= \rho^{-1}\Big[P_1(1) + \int_1^z (2\Omega_0\eta_0 - s\eta_{0\xi\xi})ds\Big] \\ &= \rho^{-1}\Big[\eta_1 + \frac{\rho_z}{2}\eta_0^2 + \int_1^z (2\Omega_0\eta_0 - s\eta_{0\xi\xi})ds\Big] \\ &= \rho^{-1}\Big[\eta_1 + \frac{\rho_z}{2}\eta_0^2 + 2\Omega_0\eta_0(z-1) - \frac{z^2-1}{2}\eta_{0\xi\xi}\Big]. \end{aligned} \tag{3.22}$$

Next, we consider the conservation of mass as a linear ODE in w_1, and we get

$$\begin{aligned} w_1(z) &= -\rho^{-1}\int_0^z (\rho u_{1\xi} - \rho V_{0Y})ds \\ &= -\rho^{-1}\int_0^z \rho\Big[\rho^{-1}\eta_{0\tau} + \rho^{-2}(1 + s\rho^{-1}\rho_z)\eta_0\eta_{0\xi} - 2\Omega_0 s\rho^{-1}\eta_{0\xi} + P_{1\xi} - V_{0Y}\Big]ds \\ &= -\rho^{-1}\int_0^z \Big[\eta_{1\xi} + \eta_{0\tau} - 2\Omega_0\eta_{0\xi} - \frac{s^2-1}{2}\eta_{0\xi\xi\xi} + (\rho_z + \rho^{-1} + \frac{s\rho_z}{\rho^2})\eta_0\eta_{0\xi} - V_{0Y}\Big]ds \\ &= -\rho^{-1}(\eta_{1\xi} + \eta_{1\tau} - 2\Omega_0\eta_{0\xi} - V_{0Y})z + \rho^{-1}(\frac{1}{6}z^3 - \frac{1}{2}z)\eta_{0\xi\xi\xi} - \rho^{-1}\int_0^z \eta_0\eta_{0\xi}(\rho_z + \rho^{-1} + \frac{s\rho_z}{\rho^2})ds, \end{aligned} \tag{3.23}$$

where we used the first equation of (3.21), the bottom boundary condition and equation (3.22). When we evaluate this integral at $z = 1$ and compare the outcome with the kinematic boundary condition at the free surface, we arrive at the following equation:

$$2\eta_{0\tau} - 2\Omega_0\eta_{0\xi} - V_{0Y} + \frac{1}{3}\eta_{0\xi\xi\xi} + \eta_0\eta_{0\xi}(2 - \rho_z + \int_0^1 \Big(\rho_z + \frac{(z\rho)_z}{\rho^2}\Big)dz = 0. \tag{3.24}$$

Lastly, we want to eliminate V_{0Y} in this equation. We use the fact that $V_{0\xi} = \eta_{0Y}$. This yields the weakly nonlinear fourth-order partial differential equation

$$(2\eta_\tau - 2\Omega_0\eta_\xi + \frac{1}{3}\eta_{\xi\xi\xi} + \nu(\rho)\eta\eta_\xi)_\xi - \rho^{-1}(z)\eta_{YY} = 0, \tag{3.25}$$

where

$$\nu(\rho) = 2 - \rho_z(1) + \int_0^1 \Big(\rho_z + \frac{(z\rho)_z}{\rho^2}\Big)dz \quad \in \mathbb{R}, \tag{3.26}$$

which governs the free surface η at leading order, where $\eta \sim \eta_0 + \mathcal{O}(\epsilon)$. By the relation between u_0 and η_0, see Equation (3.20), we get an equation for the leading-order approximation of the horizontal velocity u:

$$(2u_\tau - 2\Omega_0 u_\xi + \frac{1}{3}u_{\xi\xi\xi} + \rho(z)\nu(\rho)uu_\xi)_\xi - \rho^{-1}(z)u_{YY} = 0, \tag{3.27}$$

where we observe that this equation has a depth-dependent coefficient of the nonlinear term and a depth-dependent coefficient of the term dependent on Y. These terms dependent on depth represent a difference compared to the standard KdV equation. The standard KdV equation models both the free surface η and the horizontal velocity u at any depth. The obtained equations (3.25) and (3.27) in this research model the free surface η and the horizontal velocity u at a given depth, and the wave profiles change with depth.

In the case where the density is constant throughout the fluid domain; that is $\rho \equiv 1$, and vanishing Coriolis force, we get that $\nu(\rho) = 3$ and equation (3.27) reduces to the 2D KdV equation

$$(2u_\tau + \frac{1}{3}u_{\xi\xi\xi} + 3uu_\xi)_\xi + u_{YY} = 0. \tag{3.28}$$

This equation is also known as the Kadomtsev-Petviashvili (KP) equation. Then, if we reduce the equation further and assume no dependence on y, we retrieve the classical KdV equation

$$2u_\tau + 3uu_\xi + \frac{1}{3}u_{\xi\xi\xi} = 0. \tag{3.29}$$

4

A Shallow Water Model Equation with Continuous Stratification and Variable Bottom Topography

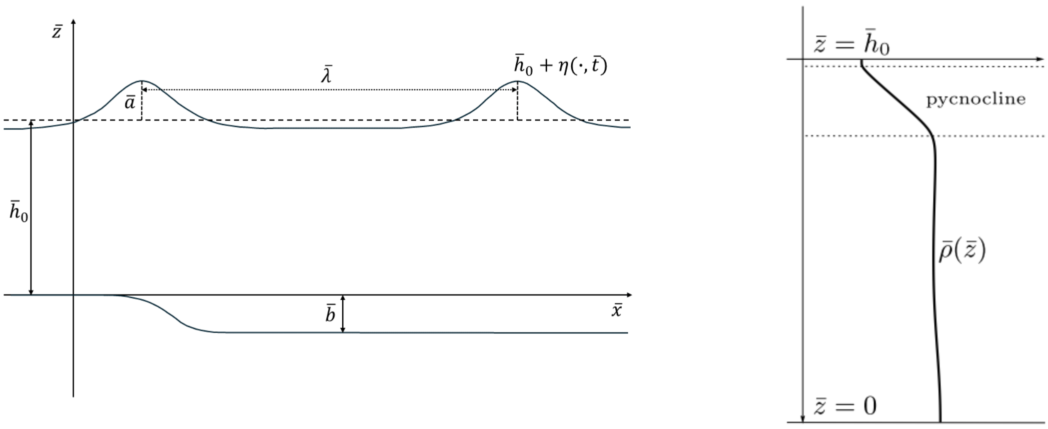
In this chapter, we will explain why we can use Euler's equation together with the equation of mass conservation with a continuous density function and boundary conditions with variable bottom topography to start our derivation for the shallow water model equation near the equator. We will derive the model equation in the same manner as we did in Chapter 3. We will again look at the dimensional form, after which we introduce the dimensionless variables. After we have nondimensionalised, we will scale the model. We will see that the leading-order approximation of the free surface satisfies the linear wave equation. We then introduce *far-field variables* and perform an asymptotic expansion on all involved variables. From the resulting system, we derive an equation for the free-surface elevation and an equation for the horizontal velocity of shallow water waves in fluids with continuous stratification and variable bottom topography. Finally, we will consider an exact travelling wave solution of these equations.

4.1. Constructing the Governing Equations and Boundary Conditions

4.1.1. Assumptions

Again, we need to establish a set of assumptions while keeping in mind that we want to model shallow water waves. As in Chapter 3, we assume that the water is inviscid, and we assume the density to be a continuous function of depth. As opposed to Chapter 3, we consider uni-directional waves instead of bi-directional waves. Additionally, we do not assume that the bottom is flat, but we assume that it may vary in the direction of x. As discussed in Chapter 2, a variation in depth affects the propagation of waves. Therefore, in some cases, it is more realistic to include a variable bottom topography in the model. Finally, we go back to the one-dimensional wave propagation and assume that the waves only propagate in the direction of increasing azimuth, so that there is no variation in the direction of increasing latitude.

Combining all those assumptions makes using Euler's equation with a continuous density function possible. We follow the approach of Johnson [7] but include a continuous change of density. The variable bottom topography will be incorporated into the boundary conditions. Before we look at these equations, we must consider the coordinate system and the domain of the governing equations.



(a) (b)

Figure 4.1: Figure 4.1a illustrates the fluid domain. The typical depth of the water is denoted by $\bar{h}_0$. The typical wavelength of the surface wave is denoted by $\bar{\lambda}$, $\bar{a}$ is the typical amplitude of a surface wave, and $\bar{b}$ is the bottom surface. Figure 3.2b shows a density function varying with depth, having a significant increase in the pycnocline.

Note. Fig 4.1b from "Shallow water models for stratified flows," by A. Geyer & R. Quirchmayr, 2019, Discrete and continuous dynamical systems, 39(8), 4533-4545.

4.1.2. Coordinate System and Domain

First, we need to define the coordinate system in which we define Euler's equation. We will use the coordinate system as used in the research of Geyer and Quirchmayr [6]. The variables $\bar{x}$ and $\bar{z}$ denote the directions of increasing azimuth and vertical elevation, respectively. Recall Figure 3.1 for an illustration of these directions.

The coordinate $\bar{t}$ for changes in time completes our coordinate system. Furthermore, we need to examine the fluid domain on which our equations will be defined. Since we look at water flows in the ocean, we consider an infinite fluid domain in the x-direction. Thus, we only consider boundary conditions in the $\bar{z}$ direction. From below, the fluid is bounded by a variable bottom that may vary with x, located at $\bar{z} = \bar{b}(\bar{x})$ and above by the free water surface at $\bar{h}_0 + \bar{\eta}(\bar{x}, \bar{z})$. Here $\bar{\eta}(\bar{x}, \bar{z})$ measures the deviation of the free surface from the average water depth $\bar{h}_0$ at $(\bar{x}, \bar{z})$. The fluid domain is illustrated in Figure 4.1a. As presented in Section 4.1.1, we describe the density as a continuous differentiable function of the depth. An example of a density function is illustrated in Figure 4.1b.

4.1.3. Equations of Motion

In the previous two subsections, we made some assumptions for our model, and we defined the coordinate system and domain of the model. We concluded that we could use Euler's equation with a continuous density function together with the equation for mass conservation. As explained in Section 4.1.1, the waves propagate only in the $\bar{x}$ direction. That implies that we can assume that the velocity of the wave in the $\bar{y}$ direction is zero, hence the $\bar{y}$-component in Euler's equation and the equation of mass conservation vanishes. We combine Euler's equation and the equation of mass conservation with vanishing $\bar{y}$-component together with the Coriolis effect. This yields to the following equations of motion with continuous stratification:

$$\begin{aligned} \bar{u}_{\bar{t}} + \bar{u}\bar{u}_{\bar{x}} + \bar{w}\bar{u}_{\bar{z}} + 2\bar{\Omega}\bar{w} &= -\bar{\rho}^{-1}\bar{P}_{\bar{x}} \\ \bar{w}_{\bar{t}} + \bar{u}\bar{w}_{\bar{x}} + \bar{w}\bar{w}_{\bar{z}} - 2\bar{\Omega}\bar{u} &= -\bar{\rho}^{-1}\bar{P}_{\bar{z}} - \bar{g} \\ \bar{\rho}_{\bar{t}} + (\bar{\rho}\bar{u})_{\bar{x}} + (\bar{\rho}\bar{w})_{\bar{z}} &= 0, \end{aligned} \tag{4.1}$$

where the bars distinguish the physical variables from the dimensionless variables later on.

In the system (4.1), $\bar{x}$ and $\bar{z}$ denote the directions of increasing azimuth and vertical elevation, respectively. The variables $\bar{u}$ and $\bar{w}$ denote the fluid velocity in the directions of increasing azimuth and vertical elevation, respectively. The pressure is denoted by $\bar{P} = \bar{P}(\bar{x}, \bar{z})$. We consider a prescribed but arbitrary continuously differentiable density function $\bar{\rho}$ of the water, which may vary with depth only, as illustrated in Figure 4.1b,

$$\bar{\rho} = \bar{\rho}(\bar{z}).$$

The gravitational acceleration is denoted by $\bar{g} \approx 9.81 ms^{-2}$. The Coriolis effect is taken into account with the two $\bar{\Omega}$ terms. These two terms follow from the f-plane approximation of the Coriolis force. The Coriolis force is captured by $\bar{\Omega}$, which is the rotational speed of the Earth around the polar axis towards the East and has the approximate magnitude of $7.29 \times 10^{-5} \text{rad} s^{-1}$.

Since $\bar{\rho}$ only depends on $\bar{z}$, the equation of mass conservation simplifies to

$$\bar{u}_{\bar{x}} + \bar{w}_{\bar{z}} + \frac{\bar{\rho}_{\bar{z}} \bar{w}}{\bar{\rho}} = 0.$$

4.1.4. Boundary Conditions

To complete the model, we need boundary conditions. As mentioned in Section 4.1.2, we only have boundary conditions in the direction of $\bar{z}$. The fluid is bounded below by a variable bottom, located at $\bar{z} = \bar{b}(\bar{x})$ and bounded from above by the free water surface at $\bar{h}_0 + \bar{\eta}(\cdot, \bar{t})$. Similar to what is described in Section 3.1.4, we again have a *dynamic boundary condition* and a *kinematic boundary condition*. The only difference with the previous model is that the $\bar{y}$ component vanishes in the kinematic boundary condition.

There is a difference with the previous model in the *bottom boundary condition* since we have a variable bottom. The bottom boundary is defined as a surface moving with the fluid. For one-dimensional propagation, this boundary condition becomes $\bar{w} = \bar{u}\bar{b}_{\bar{x}}$ on $\bar{z} = \bar{b}(\bar{x})$. Together, the boundary conditions are described by the following set of equations:

$$\begin{aligned} \bar{P} &= \bar{P}_{atm} \quad \text{on} \quad \bar{z} = \bar{h}_0 + \bar{\eta}(\bar{x}, \bar{t}) \\ \bar{w} &= \bar{\eta}_{\bar{t}} + \bar{u}\bar{\eta}_{\bar{x}} \quad \text{on} \quad \bar{z} = \bar{h}_0 + \bar{\eta}(\bar{x}, \bar{t}) \\ \bar{w} &= \bar{u}\bar{b}_{\bar{x}} \quad \text{on} \quad \bar{z} = \bar{b}(\bar{x}). \end{aligned} \tag{4.2}$$

4.2. Nondimensionalisation and Scaling

4.2.1. Nondimensionalisation

To analyse the governing equations and boundary conditions described in Section 4.1, we need to nondimensionalise the equation. Nondimensionalisation ensures that the equations have no physical dimensions. This helps us get insight into the relations between different variables and quantities in our system. We will make use of length scales $\bar{h}_0$ and $\bar{\lambda}$, just like in Chapter 3. We take $\bar{h}_0$ to be the typical depth of the water and $\bar{\lambda}$ to be the typical wavelength of the surface wave. We scale $\bar{\eta}$ with $\bar{a}$, which is the typical amplitude of a surface wave. We introduce the following nondimensional variables, following the same method as in [6] and [7]:

$$\begin{aligned}\bar{x} = \bar{\lambda}x, \quad \bar{z} = \bar{h}_0 z, \quad \bar{t} = \frac{\bar{\lambda}}{\sqrt{\bar{g}\bar{h}_0}}t,\\ \bar{u} = \sqrt{\bar{g}\bar{h}_0}u, \quad \bar{w} = \sqrt{\bar{g}\bar{h}_0}\frac{\bar{h}_0}{\bar{\lambda}}w, \quad \bar{\eta} = \bar{a}\eta, \quad \bar{b} = \bar{h}_0 b\\ \bar{P} = \bar{P}_{atm} - \bar{g}\int_{\bar{h}_0}^{\bar{z}} \bar{\rho}(s)ds + \bar{g}\bar{h}_0\bar{\rho}P, \quad \bar{\Omega} = \frac{\sqrt{\bar{g}\bar{h}_0}}{\bar{h}_0}\Omega.\end{aligned} \tag{4.3}$$

Furthermore, the density is scaled according to

$$\bar{\rho}(\bar{z}) = \bar{\rho}_0\rho(z), \quad \text{where} \quad \bar{\rho}_0 = \bar{\rho}(\bar{h}_0). \tag{4.4}$$

Lastly, we introduce two dimensionless parameters

$$\epsilon := \frac{\bar{a}}{\bar{h}_0}, \qquad \delta := \frac{\bar{h}_0}{\bar{\lambda}}, \tag{4.5}$$

expressing respectively the *amplitude* and *shallowness* parameter.

When we plug in the dimensionless variables and the amplitude and shallowness parameter in (3.1) and (3.2), we get the following dimensionless form:

$$\begin{aligned} u_t + uu_x + wu_z + 2\Omega w &= -P_x\\ \delta^2(w_t + uw_x + ww_z) - 2\Omega u &= -\frac{(\rho P)_z}{\rho}\\ u_x + w_z + \frac{\rho_z w}{\rho} &= 0\\ \\ P = \frac{1}{\rho}\int_1^z \rho(s)ds \quad &\text{on} \quad z = 1 + \epsilon\eta(x,t)\\ w = \epsilon(\eta_t + u\eta_x) \quad &\text{on} \quad z = 1 + \epsilon\eta(x,t)\\ w = ub'(x) \quad &\text{on} \quad z = b(x).\end{aligned} \tag{4.6}$$

A problem now is that we have two boundary conditions at the free surface $z = 1 + \epsilon\eta(x, y, t)$, which is unknown. We will use Taylor expansions about $z = 1$ to rewrite the boundary conditions at the free surface to a fixed boundary. We obtain the following set of boundary conditions

$$\begin{aligned} P + \epsilon\eta P_z + \frac{\epsilon^2\eta^2}{2}P_{zz} = \epsilon\eta - \epsilon^2\eta^2\frac{\rho_z}{2} + \mathcal{O}(\epsilon^3) \quad &\text{on} \quad z = 1\\ w = \epsilon(\eta_t + u\eta_x - \eta w_z) + \mathcal{O}(\epsilon^2) \quad &\text{on} \quad z = 1,\end{aligned}$$

where we have used that $\rho(1) = 1$.

4.2.2. Scaling

As before, we scale all variables with ϵ. We write

$$P \mapsto \epsilon P, \quad w \mapsto \epsilon w, \quad u \mapsto \epsilon u, \quad \Omega = \epsilon\Omega_0, \tag{4.7}$$

Furthermore, we assumed that the bottom varies in the direction of x. In this research, we will consider the case where the scale of the bottom is the same scale on which the wave will naturally propagate over a constant bottom. Following the approach of Johnson [7], we set the bottom to be

$$b(x) = B(\epsilon x), \tag{4.8}$$

and we define $B(\epsilon x) = 0$ in $x \leq 0$ so that the wave propagates from a region with a constant bottom.

Apply the scaling (4.7) and transformation (4.8) and the Taylor expansion of Section 4.2.1 to the system (4.6), resulting in the following system:

$$\begin{aligned}
u_t + \epsilon(uu_x + wu_z) + 2\epsilon\Omega_0 w &= -P_x \\
\delta^2(w_t + \epsilon(uw_x + ww_z)) - 2\epsilon\Omega_0 u &= -\frac{(\rho P)_z}{\rho} \\
u_x + w_z + \frac{\rho_z w}{\rho} &= 0 \\
\\
P = \eta - \epsilon\eta(\frac{\eta}{2}\rho_z + P_z) \quad &\text{on} \quad z = 1 \\
w = \eta_t + \epsilon(u\eta_x - \eta w_z) \quad &\text{on} \quad z = 1 \\
w = uB'(\epsilon x) \quad &\text{on} \quad z = B(\epsilon x).
\end{aligned} \tag{4.9}$$

Next, we follow the approach of Johnson in [7]. He scales out δ in favor of ϵ, making the following transformation:

$$\begin{aligned}
&x \mapsto \frac{\delta}{\sqrt{\epsilon}}x, \quad z \mapsto z \quad , \quad t \mapsto \frac{\delta}{\sqrt{\epsilon}}t, \quad P \mapsto P, \\
&\eta \mapsto \eta, \quad B \mapsto B, \quad u \mapsto u, \quad w \mapsto \frac{\sqrt{\epsilon}}{\delta}w.
\end{aligned} \tag{4.10}$$

When we perform this transformation, we note that it is the same as replacing δ^2 by ϵ. As in [7], we consider the relation between ϵ and δ that describe *shallow water waves of small amplitude*:

$$\delta \ll 1, \qquad \epsilon = \mathcal{O}(\delta^2). \tag{4.11}$$

When we apply the scaling (4.10) to the system (4.9) we get

$$\begin{aligned}
u_t + \epsilon(uu_x + wu_z) + 2\epsilon\Omega_0 w &= -P_x \\
\epsilon(w_t + \epsilon(uw_x + ww_z)) - 2\epsilon\Omega_0 u &= -\frac{(\rho P)_z}{\rho} \\
u_x + w_z + \frac{\rho_z w}{\rho} &= 0 \\
\\
P = \eta - \epsilon\eta(\frac{\eta}{2}\rho_z + P_z) \quad &\text{on} \quad z = 1 \\
w = \eta_t + \epsilon(u\eta_x - \eta w_z) \quad &\text{on} \quad z = 1 \\
w = uB'(\epsilon x) \quad &\text{on} \quad z = B(\epsilon x).
\end{aligned} \tag{4.12}$$

The system (4.12) encompasses the nondimensionalised and scaled governing equations, based on Euer's equation and the equation of mass conservation, and boundary conditions valid in the regime where $\epsilon = \mathcal{O}(\delta^2)$; that is, in shallow water. This system is still too difficult to analyse. Therefore, in the next chapter, we will derive equations for the leading-order approximation of η and u.

4.3. Derivation of a Shallow Water Model Equation

4.3.1. Leading-Order Problem

The basis of the following discussion is the system (4.12), obtained in the previous section. We will examine the Leading-order problem of this system. If we let $\epsilon \to 0$, we arrive at the following Leading-order Problem:

$$\begin{aligned} u_t &= -P_x \\ 0 &= (\rho P)_z \\ u_x + w_z + \frac{\rho_z w}{\rho} &= 0 \\ \\ P = \eta &\quad \text{on} \quad z = 1 \\ w = \eta_t &\quad \text{on} \quad z = 1 \\ w = 0 &\quad \text{on} \quad z = 0, \end{aligned} \tag{4.13}$$

where we used that $B(0) = 0$.

We will show that this Leading-order Problem satisfies the linear wave equation:

$$\eta_{tt} - \eta_{xx} = 0.$$

To get this linear wave equation, we note that the third equation in (4.13) together with the dynamic boundary condition on $z = 1$ lead to $\eta = \rho P$ for all $z \in [0,1]$. Plugging this into the first and second equation of (4.13) leads to the equation $\rho u_t = -\eta_x$. Note that since η is independent of z, we know that ρu_t must be independent of z. Furthermore, the equation of mass conservation yields $(\rho w)_z = -\rho u_x$, and hence

$$w(x,z,t) = -\rho(z)^{-1} \int_0^z \big(\rho(s) u_x(x,s,t)\big)\, ds.$$

When we integrate this with respect to t, we get:

$$\begin{aligned} \frac{\partial}{\partial t} w(x,z,t) &= \frac{\partial}{\partial t}\left(-\rho(z)^{-1} \int_0^z \big(\rho(s) u_x(x,s,t)\big)\, ds\right) \\ &\Leftrightarrow \\ w_t(x,z,t) &= -\rho(z)^{-1} \int_0^z \frac{\partial}{\partial t}\big(\rho(s) u_x(x,s,t)\big)\, ds \\ &\Leftrightarrow \\ w_t(x,z,t) &= -\rho(z)^{-1} \int_0^z \big(\rho(s) u_{xt}(x,s,t)\big)\, ds. \end{aligned}$$

When considering the kinematic boundary condition at $z = 1$, we note that we have $w_t = \eta_{tt}$ on $z = 1$. Thus on $z = 1$ we have that

$$\eta_{tt} = w_t = -\int_0^1 \big(\rho(s) u_{xt}(x,s,t)\big)\, ds.$$

We then get

$$
\begin{aligned}
\eta_{tt} - \eta_{xx} &= -\int_0^1 \big(\rho(s) u_{xt}(x,y,s,t)\big)\, ds - (-\rho u_t)_x \\
&= -\int_0^1 \big(\rho(s) u_{xt}(x,y,s,t)\big)\, ds + \rho u_{xt} \\
&= -\int_0^1 \big(\rho(s) u_{xt}(x,y,s,t)\big)\, ds + \int_0^1 \big(\rho(s) u_{xt}(x,y,s,t\big)\, ds \\
&= 0,
\end{aligned} \tag{4.14}
$$

where the second-to-last step is justified by the fact that ρu_{xt} is independent of z. So, for $z = 1$, the Leading-order Problem satisfies the linear wave equation. However, this equation lacks essential properties for modelling more complex wave behaviours. The linear wave equation cannot model dispersion, where wave components with different wave numbers travel at different speeds. Additionally, it cannot model solitary waves, periodic waves with prolonged roughs and sharper crests, or the nonlinear interaction between solitary waves. Therefore, in the next section, we will derive a nonlinear model equation for shallow water waves by considering an asymptotic expansion of the unknown variables.

4.3.2. Asymptotic Expansion

This section will asymptotically expand the variables η, u, w, and P and then consider the leading-order and first-order systems. Before we do this, we introduce the *far-field variables*. For this model, we need far-field variables different from those in the previous chapter since we consider a variable bottom topography. We still use the far-field variables to gain insight into the nonlinear effects, but instead of seeking a slow-time solution, we seek a slow-distance solution. This slow distance solution will then be used to study the behaviour of the nonlinear terms. Let's consider our original objective, modelling waves that propagate in shallow water over a variable bottom. The bottom is given as a function of the distance x. Therefore, instead of using a slow time variable, we will be using a slow distance variable. Since we want the waves to vary on the same scale as the bottom, we introduce the variables

$$
\xi = \frac{1}{\epsilon}\chi(X) - t, \quad X = \epsilon x, \tag{4.15}
$$

where $\chi(X)$ is to be determined from the analysis later on.

With these variables, we have that

$$
\frac{\partial}{\partial x} \equiv \chi' \frac{\partial}{\partial \xi} + \epsilon \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial t} \equiv -\frac{\partial}{\partial \xi}. \tag{4.16}
$$

We substitute these into system (4.12), which then becomes:

$$
\begin{aligned}
-u_\xi + \epsilon\left[u(\chi' u_\xi + \epsilon u_X) + w u_z + 2\Omega_0 w\right] &= -(\chi' P_\xi + \epsilon P_X) \\
\epsilon\left[-w_\xi + \epsilon(u(\chi' w_\xi + \epsilon w_X) + w w_z)\right] - 2\epsilon\Omega_0 u &= -\frac{(\rho P)_z}{\rho} \\
\chi' u_\xi + \epsilon u_X + w_z + \frac{\rho_z}{\rho} w &= 0 \\
P = \eta - \epsilon\eta\left(\frac{\eta}{2}\rho_z + P_z\right) &\quad \text{on} \quad z = 1 \\
w = \eta_\xi + \epsilon\left(u(\chi'\eta_\xi + \epsilon\eta_X) - \eta w_z\right) &\quad \text{on} \quad z = 1 \\
w = u\epsilon B'(X) &\quad \text{on} \quad z = B(X)
\end{aligned}
\tag{4.17}
$$

Now, we can follow the approach of Johnson [7] and asymptotically expand the variables η, u, w, and P. Expanding the variables asymptotically means we expand the variables in the form $q \sim \sum_{n=0}^{\infty} q_n \epsilon^n$. At Leading-order, meaning we let $\epsilon \mapsto 0$, we obtain the following linear system:

$$
\begin{aligned}
u_{0\xi} &= \chi' P_{0\xi} \\
0 &= (\rho P_0)_z \\
(\rho w_0)_z &= -\rho\chi' u_{0\xi} \\
P_0 = \eta_0 &\quad \text{on} \quad z = 1 \\
w_0 = -\eta_{0\xi} &\quad \text{on} \quad z = 1 \\
w_0 = 0 &\quad \text{on} \quad z = B(X).
\end{aligned}
\tag{4.18}
$$

From integration of the second equation and the dynamic boundary condition on $z = 1$ in the system (4.18), we obtain that $\eta_0 = \rho P_0$ for all $z \in [0, 1]$. The first equation gives then $u_{0\xi} = \chi' \rho^{-1} \eta_{0\xi}$, which implies that

$$
u_0 = \chi' \rho^{-1} \eta_0, \tag{4.19}
$$

since the perturbation of u is only caused by the passage of the wave. That is, we must have that $u_0 = 0$ whenever $\eta_0 = 0$. From this relation, the third equation, and the bottom condition, we deduce that

$$
w_0 = \rho^{-1} \chi'^2 \eta_{0\xi} \left(B(X) - z\right).
$$

Now we can define χ'. From the surface boundary condition, we get that

$$
\chi'^2 = \frac{1}{D(X)}, \quad \text{with} \quad D(X) = 1 - B(X), \tag{4.20}
$$

where $D(X)$ is the local depth. And thus Equation (4.20) implies

$$
\chi(X) = \int_0^X \frac{1}{\sqrt{D(\tilde{X})}} d\tilde{X}. \tag{4.21}
$$

We see in Equation (4.18) that P_0, u_0 and w_0 all depend on η_0, which is still unknown. To determine η_0, we will look at the first-order system. We consider the first-order system, with the leading-order solutions of u_0 and w_0 substituted in the system:

$$\begin{aligned}
-u_{1\xi} + \chi'^3\rho^{-2}(1+\rho^{-1}(B(X)-z))\eta_0\eta_{0\xi} + 2\Omega_0\chi'\rho^{-1}(B(X)-z)\eta_{0\xi} &= -\chi' P_{1\xi} - \rho^{-1}\eta_{0X} \\
-\chi'^2\rho^{-1}(B(X)-z)\eta_{0\xi\xi} - 2\Omega_0\chi'\rho^{-1}\eta_0 &= -\frac{(\rho P_1)_z}{\rho} \\
\chi'\rho u_{1\xi} - \frac{1}{2}D'(X)\chi'^3\eta_0 + \chi'\eta_{0X} + (\rho w_1)_z &= 0
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
P_1 &= \eta_1 + \frac{\eta_0^2}{2}\rho_z, \quad \text{on} \quad z=1 \\
w_1 &= -\eta_{1\xi} + 2\chi'\eta_0\eta_{0\xi}, \quad \text{on} \quad z=1 \\
w_1 &= \chi'\rho^{-1}B'(X)\eta_0, \quad \text{on} \quad z=B(X).
\end{aligned}$$

From the second equation, together with the dynamic boundary condition, we get that

$$P_1(z) = \rho^{-1}\left\{\frac{1}{D}\eta_{0\xi\xi}\left[\frac{1}{2}\left(1-z^2\right) + B\left(z-1\right)\right] + 2\Omega_0\chi'(z-1)\eta_0 + \eta_1 + \eta_0^2\frac{\rho_z}{2}\right\} \tag{4.23}$$

Now the third equation of (4.22) together with the first equation gives us the following:

$$\begin{aligned}
\chi'^4\rho^{-1}\left(1+\rho^{-1}\left(B(X)-z\right)\right)\eta_0\eta_{0\xi} + 2\Omega_0\chi'^2\left(B(X)-z\right)\eta_{0\xi} - \\
\frac{1}{2}D'(X)\chi'^3\eta_0 + \chi'\eta_{0X} + (\rho w_1)_z \quad &= -\chi'^2\rho P_{1\xi} - \chi'\eta_{0X}
\end{aligned} \tag{4.24}$$

Writing this equation in terms of η_0, collecting the terms and moving the terms with η_0 to the right side:

$$(\rho w_1)_z = -\frac{1}{D^2}\left(\frac{1}{2}(1-z^2) + B(z-1)\right)\eta_{0\xi\xi\xi} + 2\Omega_0\frac{1}{\sqrt{D}}\eta_{0\xi} - \frac{1}{D}\eta_{1\xi} \tag{4.25}$$

$$-\frac{1}{D}\left(\rho_z + \frac{1}{D}\left(\rho^{-1} - \rho^{-2}\rho_z(B-z)\right)\right)\eta_0\eta_{0\xi} - 2\frac{1}{\sqrt{D}}\eta_{0X} \tag{4.26}$$

$$+\frac{1}{2}\frac{D'}{D^{\frac{3}{2}}}\eta_0 \tag{4.27}$$

We then integrate this equation, resulting in

$$\begin{aligned}
w_1(z) = \rho^{-1}\int_B^z \Bigg[&-\frac{1}{D^2}\left(\frac{1}{2}(1-s^2) + B(s-1)\right)\eta_{0\xi\xi\xi} + 2\Omega_0\frac{1}{\sqrt{D}}\eta_{0\xi} - \frac{1}{D}\eta_{1\xi} \\
&-\frac{1}{D}\left(\rho_z + \frac{1}{D}\left(\rho^{-1} - \rho^{-2}\rho_z(B-s)\right)\right)\eta_0\eta_{0\xi} - 2\frac{1}{\sqrt{D}}\eta_{0X} \\
&+\frac{1}{2}\frac{D'}{D^{3/2}}\eta_0\Bigg] ds \\
= \rho^{-1}(z-B)&\left\{2\Omega_0\frac{1}{\sqrt{D}}\eta_{0\xi} - \frac{1}{D}\eta_{1\xi} - 2\frac{1}{\sqrt{D}}\eta_{0X} + \frac{1}{2}\frac{D'}{D^{3/2}}\eta_0\right\} \\
-\rho^{-1}\frac{1}{D^2}&\left(\frac{1}{2}(z-\frac{1}{3}z^3) + B(\frac{1}{2}z^2 - z) - \frac{1}{3}B^3 + B^2 - \frac{1}{2}B\right)\eta_{0\xi\xi\xi} \\
-\rho^{-1}\frac{1}{D}&\left\{\int_B^z\left(\rho_z + \frac{1}{D}\left(\rho^{-1} - \rho^{-2}\rho_z(B-s)\right)\right)ds\right\}\eta_0\eta_{0\xi} + \rho^{-1}c(\xi,X),
\end{aligned} \tag{4.28}$$

where $c(\xi, X)$ is an integration constant depending on ξ and X. Using the bottom boundary condition gives that $c(\xi, X) = \frac{1}{\sqrt{D}}B'\eta_0$. Using this, we evaluate $w_1(z)$ at the surface and compare the outcome with the corresponding boundary condition. Furthermore, we write every B in terms of D via relation (4.20).

We then arrive at the weakly nonlinear third-order partial differential equation

$$2\sqrt{D}\eta_X - 2\Omega_0\sqrt{D}\eta_\xi + \frac{1}{2}\frac{D'}{\sqrt{D}}\eta + \frac{1}{3}D\eta_{\xi\xi\xi} + \frac{1}{D}\mu(D,\rho)\eta\eta_\xi = 0, \tag{4.29}$$

where

$$\mu(D,\rho) = 2 + \int_{1-D}^{1}\left[\rho_z + \frac{1}{D}\left(\rho^{-1} - \rho^{-2}\rho_z(1-D-s)\right)\right]ds \quad \in \mathbb{R}, \tag{4.30}$$

which governs the free surface η at leading order, where $\eta \sim \eta_0 + \mathcal{O}(\epsilon)$. By the relation between u_0 and η_0,

$$u_0 = \chi'\rho^{-1}\eta_0 = \frac{1}{\sqrt{D}}\rho^{-1}\eta_0,$$

we observe that for every depth, we get a different equation for u_0. The equation for the leading order approximation of the horizontal velocity u is given by:

$$2u_X - 2\Omega_0 u_\xi + \frac{1}{2}D'u + \frac{1}{3}\sqrt{D}u_{\xi\xi\xi} + \frac{1}{D^2}\rho(z)\mu(D,\rho)uu_\xi = 0, \tag{4.31}$$

where we observe that the coefficient of the nonlinear term depends on the density. Thus, the strength of the nonlinearity depends on the density, and since the density depends on the depth, the strength of the nonlinearity depends on the depth.

In the case where the density is constant throughout the fluid domain; that is $\rho \equiv 1$, and vanishing Coriolis force, we get that $\mu(D,\rho) = 3$ and equation (4.29) reduces to the KdV equation for variable depth as in [7]

$$2\sqrt{D}\eta_X + \frac{1}{2}\frac{D'}{\sqrt{D}}\eta + \frac{1}{3}D\eta_{\xi\xi\xi} + \frac{3}{D}\eta\eta_\xi = 0. \tag{4.32}$$

When we introduce a constant depth, $D \equiv 1$, this equation reduces to the classical KdV equation

$$2\eta_X + 3\eta\eta_\xi + \frac{1}{3}\eta_{\xi\xi\xi} = 0, \tag{4.33}$$

where X van be interpreted as τ. This can be explained by considering that $X = \epsilon x$, and $\tau = \epsilon t$. We know that x and t are of order unity, so X and τ are both of order ϵ.

4.4. Explicit Traveling Wave Solutions

4.4.1. Computing Solutions

We will try to compute explicit travelling wave solutions of equations (4.29) and (4.31) over a slowly varying depth. Before considering travelling wave solutions, we modify equation (4.29) a little to make computations easier. We multiply the equation by $D^{-\frac{1}{4}}$, and then introduce

$$H(\xi, X) = D^{\frac{1}{4}}\eta.$$

We then search for a travelling wave solution to H. When we obtain that solution, we can multiply the solution with $D^{-\frac{1}{4}}$ to retrieve the travelling wave solution for η.

Multiplying equation (4.29) with $D^{-\frac{1}{4}}$ and writing $H(\xi, X) = D^{\frac{1}{4}}$ gives the following equation:

$$2H_X - 2\Omega_0 H_\xi + \frac{1}{3}D^{\frac{1}{2}} H_{\xi\xi\xi} + \frac{1}{D^{\frac{7}{4}}}\mu(D,\rho)HH_\xi = 0. \tag{4.34}$$

Now, let's consider what it means to consider a slowly varying depth. It means that the depth variation occurs on a scale that is slower than the evolution scale of the wave, which is X. So then, the wave will essentially evolve at the (local) constant depth. For further computations, we consider $D(X) = \hat{D}(\hat{X})$, where $\hat{X} = \sigma X$ and $\sigma \mapsto 0$. We now consider the depth to be independent of X. Equation (4.34) then becomes

$$2H_X - 2\Omega_0 H_\xi + \frac{1}{3}\hat{D}^{\frac{1}{2}} H_{\xi\xi\xi} + \frac{1}{\hat{D}^{\frac{7}{4}}}\mu(D,\rho)HH_\xi = 0, \tag{4.35}$$

where $\hat{D} = \hat{D}(\hat{X})$. The next step will be to find a travelling wave solution to this equation. We assume that the travelling wave solution of (4.35) is of the form $H(\xi, X) = \varphi(\xi - cX)$, with wave speed $c > 0$. Substituting this travelling wave into equation (4.35) yields the ordinary differential equation

$$-2(c+\Omega_0)\varphi' + \frac{1}{3}\sqrt{\hat{D}}\varphi''' + \hat{D}^{-\frac{7}{4}}\mu(\hat{D},\rho)\varphi\varphi' = 0, \tag{4.36}$$

with $\mu(\hat{D},\rho)$ defined in (4.30).

First, we write $\varphi\varphi'$ as $\left[\frac{\varphi^2}{2}\right]'$, resulting in

$$-2(c+\Omega_0)\varphi' + \frac{1}{3}\sqrt{\hat{D}}\varphi''' + \hat{D}^{-\frac{7}{4}}\mu(\hat{D},\rho)\left[\frac{\varphi^2}{2}\right]' = 0,$$

then integration gives

$$-2(c+\Omega_0)\varphi + \frac{1}{3}\sqrt{\hat{D}}\varphi'' + \frac{1}{2\hat{D}^{\frac{7}{4}}}\mu(\hat{D},\rho)\varphi^2 = c_1,$$

where c_1 is a constant of integration. In order to be able to integrate again, we multiply this equation by integration factor φ'. Then we get

$$-2(c+\Omega_0)\varphi\varphi' + \frac{1}{3}\sqrt{\hat{D}}\varphi''\varphi' + \frac{1}{2\hat{D}^{\frac{7}{4}}}\mu(\hat{D},\rho)\varphi^2\varphi' = c_1\varphi'$$
$$-2(c+\Omega_0)\left[\frac{\varphi^2}{2}\right]' + \frac{1}{3}\sqrt{\hat{D}}\left[\frac{(\varphi')^2}{2}\right]' + \frac{1}{2\hat{D}^{\frac{7}{4}}}\mu(\hat{D},\rho)\left[\frac{\varphi^3}{3}\right]' = c_1\varphi'.$$

Integration on both sides leads to

$$(c+\Omega_0)\varphi^2 + \frac{1}{6}\sqrt{\hat{D}}(\varphi')^2 + \frac{\mu(\hat{D},\rho)}{6\hat{D}^{\frac{7}{4}}}\varphi^3 = c_1\varphi + c_2, \tag{4.37}$$

where c_2 is again a constant of integration. Now, we assume that the wave decays at infinity, from where it must follow that $c_1 = c_2 = 0$. Now equation (4.37) is a solitary travelling wave equation and can be written as

$$(\varphi')^2 = \left(\frac{6(c+\Omega_0)}{\sqrt{\hat{D}}} - \frac{\mu(\hat{D},\rho)}{\hat{D}^{\frac{9}{4}}}\varphi\right)\varphi^2. \tag{4.38}$$

The next step is to find a solution to this first-order differential equation. We follow the steps performed by Geyer and Quirchmayr [6]. We integrate the equation by writing

$$\int \frac{d\varphi}{\varphi\sqrt{\frac{6(c+\Omega_0}{\sqrt{\hat{D}}} - \frac{\mu(\hat{D},\rho)}{D^{\frac{9}{4}}}\varphi}} = \pm\int ds,$$

where $s = \xi - cX$. We use the substitution $\varphi = \frac{6(c+\Omega_0)}{\mu(\hat{D},\rho)}\hat{D}^{\frac{7}{4}}\operatorname{sech}^2(\theta)$ in the former equation, which results in

$$\begin{aligned}
\pm\int ds &= \int \frac{d\varphi}{\varphi\sqrt{\frac{6(c+\Omega_0}{\sqrt{\hat{D}}} - \frac{\mu(\hat{D},\rho)}{\hat{D}^{\frac{9}{4}}}\varphi}} \\
&= \int \frac{-2}{\sqrt{\frac{6(c+\Omega_0)}{\sqrt{\hat{D}}}(1-\operatorname{sech}^2(\theta)}} \cdot \frac{\sinh(\theta)}{\cosh^3(\theta)}d\theta \\
&= -\frac{2\hat{D}^{\frac{1}{4}}}{\sqrt{6(c+\Omega_0)}}\int d\theta \\
&= -\frac{2\hat{D}^{\frac{1}{4}}}{\sqrt{6(c+\Omega_0)}}\theta
\end{aligned}$$

Now we use the identity for φ, to get the final result:

$$\varphi(\xi - cX) = \frac{6(c+\Omega_0)}{\mu(\hat{D},\rho)}\hat{D}^{\frac{7}{4}}\operatorname{sech}^2\left(\frac{\sqrt{6(c+\Omega_0)}}{2\hat{D}^{\frac{1}{4}}}(\xi - cX)\right). \tag{4.39}$$

We now found a traveling wave solution for H, in equation (4.35). We originally wanted to find a solution for the free-surface η and the horizontal velocity u. We can do this by considering the relation between H and η, $H(\xi,X) = D^{\frac{1}{4}}\eta$ and the relation betweeon η and u: (4.19). Therefore, we obtain traveling wave solutions of (4.29) and (4.31) given by

$$\eta(\xi,X) = \hat{D}^{-\frac{1}{4}}\varphi(\xi - cX) = \frac{6(c+\Omega_0)}{\mu(\hat{D},\rho)}\hat{D}^{\frac{3}{2}}\operatorname{sech}^2\left(\frac{\sqrt{6(c+\Omega_0)}}{2\hat{D}^{\frac{1}{4}}}(\xi - cX)\right), \tag{4.40}$$

and

$$u(\xi,X,z) = \hat{D}^{-\frac{1}{2}}\rho^{-1}(z)\eta(\xi,X) = \frac{6(c+\Omega_0)}{\rho(z)\mu(\hat{D},\rho)}\hat{D}\operatorname{sech}^2\left(\frac{\sqrt{6(c+\Omega_0)}}{2\hat{D}^{\frac{1}{4}}}(\xi - cX)\right), \tag{4.41}$$

respectively. From equation (4.41), we observe that ρ only appears as a multiplicative factor in front of the solution. This is different from the Coriolis parameter, wave speed and depth that also appear inside the sech-function. Hence, we can deduce that ρ only affects the amplitude of the wave, and Ω_0, c, and $\hat{D}$ affect

both the amplitude and the narrowness of the wave.
When substituting the solutions of η and u into the obtained equations (4.29) and (4.31), we can check that these solutions are indeed solutions to our equations. We need to note that these are solutions in the case where D is independent of X. We will comment more on this in Section 4.4.3, where we will look at some plots of the solutions that were obtained.

4.4.2. Plotting a Solitary Wave Moving over a Slowly Varying Depth

We will plot the free surface of the wave η and the horizontal velocity u to see how the solutions behave in different depths and with a continuously changing density. First, we choose Ω_0 to be equal to $2.5 \cdot 10^{-4}$, based on observations of Constantin and Johnson in [1]. Furthermore, we choose a linear density function, according to [6]. We have that

$$\rho(z) = 2 - z, \tag{4.42}$$

such that $\rho = 1$ at the surface where $z = 1$, and the density increases with depth. We will consider a slowly varying bottom, where we examine the wave propagation over a bottom that varies from depth 1 to depth 0.5.

The free-surface over a slowly varying depth
First, we will consider the free surface of a solitary wave propagating over a slowly varying depth. At the free surface, we have that the density is $\rho = 1$.

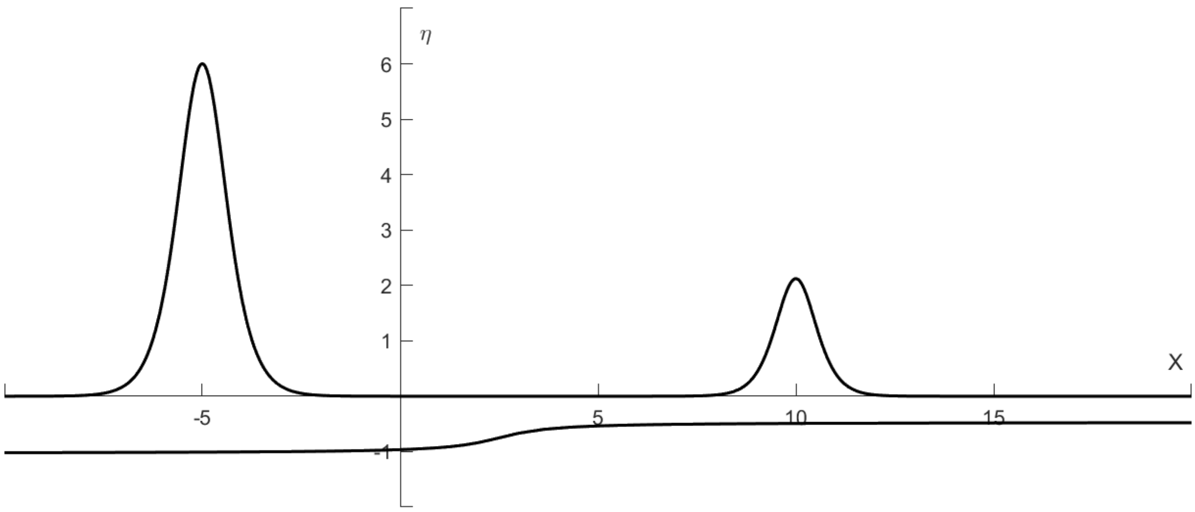
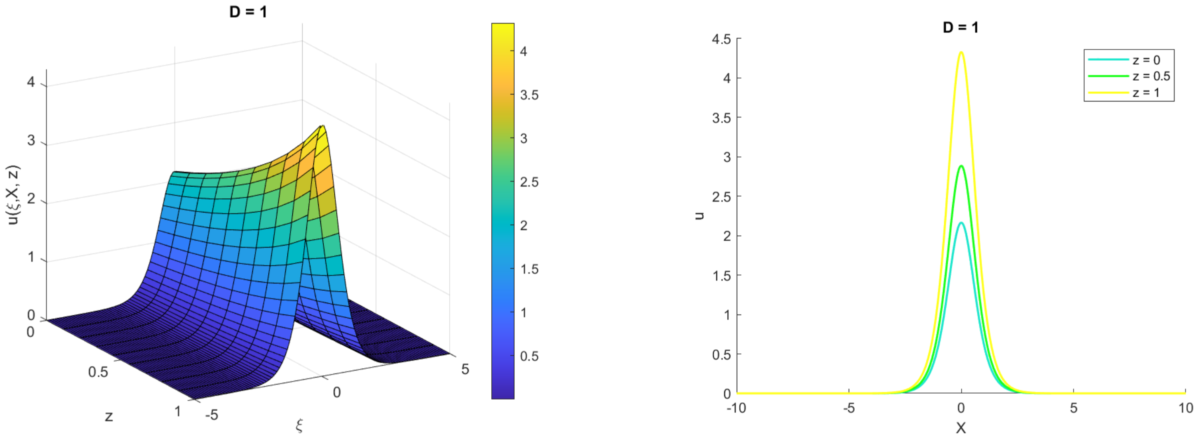


Figure 4.3: A solitary wave moving over a slowly varying depth, from depth 1 to depth 0.5.

In Figure 4.3, we can observe a right-moving solitary wave moving over a slowly varying depth. Equation (4.40) was evaluated at $X = -5$, where the depth is 1, and at $X = 10$, where the depth is equal to 0.5 (see Appendix A). An important observation is that the wave's amplitude decreases with a smaller depth. This can be confirmed mathematically by inspecting the solution (4.40).

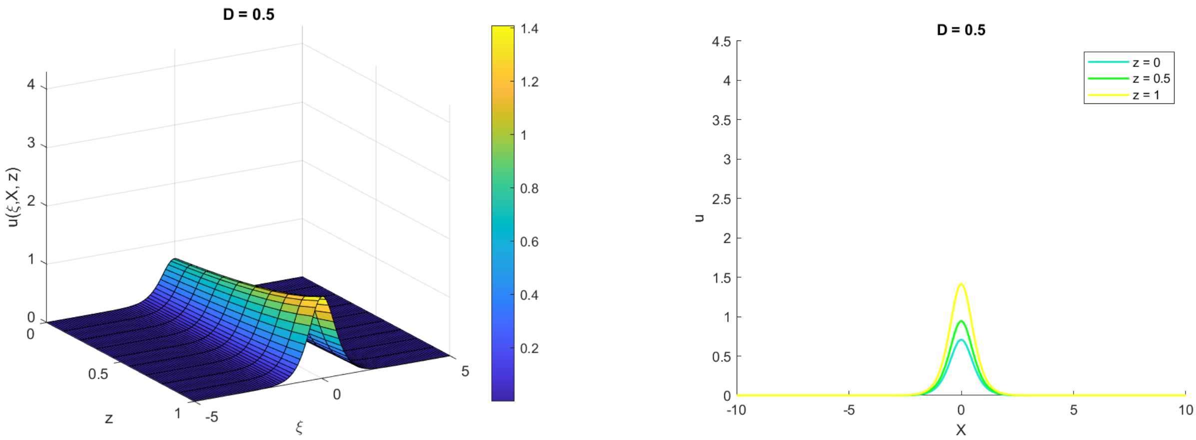
The horizontal velocity over a slowly varying depth with stratification
We can obtain a travelling wave solution of the horizontal velocity for each depth $z \in [0, 1]$, given by equation (4.41). We recall that the amplitude of the horizontal velocity component decreases with depth. To see this, we plot the wave's horizontal velocity component for two different depths, $\hat{D} = 1$ and $\hat{D} = 0.5$. We make a surface plot for the horizontal velocity for depth $z \in [0, 1]$ (see Appendix B.1). Besides that, we plot the horizontal velocity for three different depths, $z = 1$, $z = 0$, and $z = 0$ (see Appendix B.2. We make these plots for both depth 1 and depth 0.5.

From Figure 4.4 and 4.5, we clearly see that the amplitude of solutions decreases with depth. Additionally, from Figures 4.4b and 4.5b, we observe that ρ does not affect the width of the solutions, as mentioned at the end of Section 4.4.1.



(a) (b)

Figure 4.4: Solitary traveling wave solutions (4.41) of the horizontal velocity equation (4.31) with linear density function $\rho(z) = 2 - z$ and a slowly varying depth. In Figure 4.4a, we see a travelling wave over water with depth 1, with a decreasing amplitude with depth $z \in [0, 1]$. In Figure 4.4b, we see the same travelling wave over water with depth 1, but evaluated only at three different depths: $z = 1, z = 0.5$, and $z = 0$.



(a) (b)

Figure 4.5: Solitary traveling wave solutions (4.41) of the horizontal velocity equation (4.31) with linear density function $\rho(z) = 2 - z$ and a slowly varying depth. In Figure 4.5a, we see a travelling wave over water with depth 0.5, with a decreasing amplitude with depth $z \in [0, 1]$. In Figure 4.5b, we see the same travelling wave over water with depth 0.5, but evaluated only at three different depths: $z = 1, z = 0.5$, and $z = 0$.

4.4.3. Comparing Solutions to Other Research

Looking at Figure 4.3, we see that the wave amplitude decreases when the wave moves from a bigger depth to a smaller depth. This is not what we would expect from research, including the research from Johnson [7]. Johnson showed that over a slowly varying depth, a solitary traveling wave attains a higher amplitude and becomes narrower when the depth decreases. We suspect our solution to be wrong due to the assumption that the depth D does not depend on X. Because of that assumption, we can find an exact travelling wave solution by direct integration. The found solution (4.40) is a correct solution of equation (4.29), *under the assumption* that D is not dependent on X, which is not an assumption that holds in physical reality. We expect to find a correct solution by assuming that D depends on X and by using a more elaborate solving technique, for example, following the method from Johnson in [8].

From Figures 4.4a and 4.5a, we can deduce that the amplitude of solutions decreases with depth $z \in [0, 1]$ like z^{-1}. This is exactly as we would expect from the research of Geyer and Quirchmayr [6].

5

Conclusion and discussion

In this research, we derived two model equations for shallow water waves. The first equation models bi-directional shallow water waves over a continuously stratified fluid. The second equation models uni-directional shallow water waves over a continuously stratified fluid with variable bottom topography. These models were obtained by first considering Euler's equation, the equation of mass conservation, and the boundary conditions of the different models. We followed the approaches of Geyer and Quirchmayr [6] and Johnson [7] to obtain both model equations.

The first model equation models bi-directional shallow water waves over a continuously stratified fluid. We derived the weakly nonlinear KdV-type shallow water equation

$$(2\eta_\tau - 2\Omega_0\eta_\xi + \frac{1}{3}\eta_{\xi\xi\xi} + \nu(\rho)\eta\eta_\xi)_\xi - \rho^{-1}\eta_{YY} = 0, \tag{5.1}$$

for an arbitrary prescribed depth-dependent smooth density distribution ρ. The function $\nu(\rho)$ is described in (3.26). For a constant density, $\rho \equiv 1$ and vanishing Coriolis force, the model equation (3.25) reduces to the 2D-KdV equation as stated in [7]. This finding is reassuring since it suggests that the model equation is accurate. One (trivial) solution of this equation will be the solution of the unidirectional KdV equation with continuous stratification with no y-dependence. However, we have not examined more solutions. Additional research is needed to compute more, non-trivial, solutions to this model equation to understand better the bi-directional wave propagation in a continuously stratified fluid.

The second model equation models uni-directional shallow water waves over a continuously stratified fluid with variable bottom topography. We derived the weakly nonlinear KdV-type shallow water equation

$$2\sqrt{D}\eta_X - 2\Omega_0\sqrt{D}\eta_\xi + \frac{1}{2}\frac{D'}{\sqrt{D}}\eta + \frac{1}{3}D\eta_{\xi\xi\xi} + \frac{1}{D}\mu(D,\rho)\eta\eta_\xi = 0, \tag{5.2}$$

for an arbitrary prescribed depth-dependent smooth density distribution ρ and X-dependent variable depth D. In this equation, the variables X and ξ can be interpreted as time-like and space-like variables, respectively. The function $\mu(D,\rho)$ is described in (4.30). This second model equation also proved consistent with research from Johnson [8], amongst others, for a constant density, $\rho \equiv 1$, and vanishing Coriolis force. This implies that the model equation is accurate.

Under the assumption that the depth is not dependent on X, we found a solitary travelling wave solution using the direct integration method. The solution

$$\eta(\xi,X)=\hat{D}^{-\frac{1}{4}}\varphi(\xi-cX)=\frac{6(c+\Omega_0)}{\mu(\hat{D},\rho)}\hat{D}^{\frac{3}{2}}\,\mathrm{sech}^2\left(\frac{\sqrt{6(c+\Omega_0)}}{2\hat{D}^{\frac{1}{4}}}(\xi-cX)\right), \tag{5.3}$$

describes a solitary travelling wave in a continuously stratified fluid over a slowly varying depth, independent of X. We conclude that this wave attains the property that the density only influences the wave's amplitude in a way that we expect from the research from Geyer and Quirchmayr [6]. However, depth affects both the wave's amplitude and its narrowness in a way that contradicts our expectations. Specifically, we would expect the wave's amplitude to increase and to become narrower as the depth decreases, considering the research of Johnson [7]. In our model equation, the wave becomes narrower, but the amplitude decreases for a wave propagating over a decreasing depth. We suspect this inconsistency arises from the incorrect assumption that D is independent of X. This assumption validates the use of the direct integration method. Therefore, a further study focusing on another solving technique, such as the technique used in [8], is suggested.

The two model equations as described in this thesis can be used to model waves in shallow water. The first model equation applies to shallow water waves with bi-directional wave propagation and continuous stratification. The second model equation applies to shallow water waves with uni-directional wave propagation, continuous stratification, and variable bottom topography. These models are an improvement over those in previous research in the sense that they provide a better approximation of two-directional wave propagation in continuously stratified fluids or with variable bottom topography. However, further research could focus on finding even more improvements in the models. For instance, one could combine the two model equations to obtain a model equation for bi-directional shallow water waves in a continuously stratified fluid with variable bottom topography. Another suggestion for further research is to incorporate the effects of wind flows at the water's surface or larger oceanic currents, such as gyres, into the models.

References

[1] A. Constantin and R. S. Johnson. "The dynamics of waves interacting with the Equatorial Undercurrent". In: *Geophysical and Astrophysical Fluid Dynamics* 109 (4 July 2015), pp. 311–358. ISSN: 10290419. DOI: 10.1080/03091929.2015.1066785.

[2] A. Constantin and R.S. Johnson. *On the Non-Dimensionalisation, Scaling and Resulting Interpretation of the Classical Governing Equations for Water Waves.* 2008.

[3] The Editors of Encyclopaedia Britannica. *Pycnocline | Thermocline, Stratification salinity.* June 2009. URL: https://www.britannica.com/science/pycnocline.

[4] A.V. Fedorov and J.N. Brown. "Equatorial Waves". In: Elsevier, 2009, pp. 271–287. DOI: 10.1016/B978-012374473-9.00610-X.

[5] A. Geyer and R. Quirchmayr. "Shallow water equations for equatorial tsunami waves". In: *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* 376 (2111 Jan. 2018). ISSN: 1364503X. DOI: 10.1098/rsta.2017.0100.

[6] A. Geyer and R. Quirchmayr. "Shallow water models for stratified equatorial flows". In: *Discrete Continuous Dynamical Systems - Series A* 39 (8 2019), pp. 4533–4545. ISSN: 1553-5231. DOI: 10.3934/dcds.2019186.

[7] R. S. Johnson. *A Modern Introduction to the Mathematical Theory of Water Waves.* Cambridge University Press, Oct. 1997. ISBN: 9780521591720. DOI: 10.1017/CBO9780511624056.

[8] R. S. Johnson. "On the development of a solitary wave moving over an uneven bottom". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 73 (1 1973), pp. 183–203. ISSN: 14698064. DOI: 10.1017/S0305004100047605.

[9] D. J. Korteweg and G. de Vries. "XLI. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves". In: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 39 (240 May 1895), pp. 422–443. ISSN: 1941-5982. DOI: 10.1080/14786449508620739.

[10] J.S. Russel. *Report on Waves. Report of the 14th Meeting of the British Association for the Advancement of Science.* 1844.

[11] P. Webb. *10.3 Waves on the Shore.* URL: https://rwu.pressbooks.pub/webboceanography/chapter/10-3-waves-on-the-shore/.

A

Matlab code Solitary Wave over a Slowly Varying Depth

```
% Define parameters
c = 1;
Omega0 = (2.5) .*10.^ -4;
mu_rho = @(rho) 2 - rho;
rho = 1;

% Define arrays of X and D values
Xs = [-5, 10];
Ds = [1,0.5];

% Compute mu
mu_value = mu_rho(rho);

% Define the range for xi
xi_ranges = {linspace(-10, 0, 1000), linspace(0, 20, 1000)}; % Different ranges for different D values

% plot
figure;
hold on;

% Loop over the pairs of X and D values
for i = 1:length(Xs)
    X = Xs(i);
    D = Ds(i);
    xi_range = xi_ranges{i};

    % Define the function for phi(xi)
    u = @(xi) (6 * (c + Omega0) * D.^(3/2)/ (rho*mu_value)) * sech((sqrt(6 * (c + Omega0))) / (2*D.^(1/4)) * (xi - c * X)).^2;

    % Evaluate the function
    u_values = u(xi_range);

    % Plot the function
    plot(xi_range, u_values, 'k', 'LineWidth', 1.5);
end


% Define the piecewise function for a slowly varying depth from 1 to 0.5
y = @(x) (-0.75) + 0.18 .* atan(x-2.5);

% Plot the function using fplot
fplot(y, [-10, 20], 'k', 'LineWidth', 1.5);
xlabel('X');
```

```
ylabel('\eta');
x0=170;
y0=170;
width=1000;
height=400;
set(gcf,'position',[x0,y0,width,height])
ax = gca;
ax.XAxisLocation = 'origin';
ax.YAxisLocation = 'origin';

hold off
```

B

Matlab code Solitary Waves with continuous density function

B.1. A 3D plot of a solitary wave with continuous stratification

```
% Define parameters
syms z

c = 1;
Omega0 = (2.5) * 10^(-4);

% Define rho(z) and rho_z(z).
rho = @(z) 2 - z;
rho_z = @(z) -1;

% Define the integrand
integrand = @(D, z) rho_z(z) + 1./D .* (1./rho(z) - 1./rho(z).^2 .* rho_z(z) .* (1 - D
    - z));

% Define Mu(D, rho(z))
mu = @(D, z) 2 + arrayfun(@(D_val) integral(@(z_val) integrand(D_val, z_val), 1 - D_val
    , 1), D);

% Define the range for xi and z
xi_range = linspace(-5, 5, 100);
z_range = linspace(1,0, 10);

% Define D values
D_values = [1, 0.5];

% Loop over D values to create surface plots
for k = 1:length(D_values)
    D = D_values(k);
    X = 0;

    % Create meshgrid for xi and z
    [XI, Z] = meshgrid(xi_range, z_range);

    % Evaluate mu for each z value
    MU = mu(D, Z);

    % Evaluate u(xi, z) for each (xi, z) pair
    U = (6 * (c + Omega0) * D ./ (rho(Z) .* MU)) .* sech((sqrt(6 * (c + Omega0))) / (2
    * D^(1/4)) .* (XI - c * X)).^2;

    % Create figure
    figure;
```

```
    surf(XI, Z, U);
    az_angle =-30;
    el_angle =20;
    xlabel('\xi');
    ylabel('z');
    zlabel('u(\xi,X, z)');
    title(['D = ' num2str(D)]);
    zlim([0,4.3]);
    xlim([-5,5]);
    set(gca, 'YDir', 'reverse');
    view(az_angle ,el_angle);
    colorbar;
end
```

B.2. A solitary wave with continuous stratification evaluated at three depths

```
% Define parameters
syms z

c = 1;
Omega0 = (2.5) .*10.^ -4;

%define rho(z) and rho_z(z).
rho = @(z) 2-z;
rho_z = @(z) -1;

%define the integrand
integrand = @(D, z) rho_z(z) + 1./D .* (1./rho(z) - 1./rho(z).^2 .* rho_z(z) .* (1 - D
    - z));

%define mu(D,rho(z))
mu = @(D, z) 2 + arrayfun(@(D) integral(@(z) integrand(D, z), 1 - D, 1), D);


% Define arrays of X, D and z values
Xs = [-5, 10];
Ds = [1,0.5];
Zs = [0, 0.5, 1];

%Define range for xi
xi_range = linspace(-10, 10, 1000);

% Plot for X = 0, D = 1
X1 = 0;
D1 = 1;

% Create figure for X = 0, D = 1
colors = [0.1 0.9 0.8
    0.1 1 0.1
    1 1 0.1];
figure;
hold on;

for j = 1:length(Zs)
    z = Zs(j);
    mu_value1 = mu(D1, z); % Evaluate mu at specific D and z

    % Define the function for u(xi)
    u1 = @(xi) (6 * (c + Omega0) * D1 / (rho(z) * mu_value1)) * sech((sqrt(6 * (c +
    Omega0))) / (2 * D1.^(1/4)) * (xi - c * X1)).^2;

    % Evaluate the function
    u_values1 = u1(xi_range);

    % Plot the function
    plot(xi_range, u_values1, 'LineWidth', 1.5, 'Color', colors(j,:));
end

```

```
xlabel('X');
ylabel('u');
title('D = 1');
legend('z = 0', 'z = 0.5', 'z = 1');
set(gca, 'XAxisLocation', 'origin');
ylim([0,4.5]);
hold off;

% Plot for X = 0, D = 0.5
X2 = 0;
D2 = 0.5;

% Create figure for X = 0, D = 0.5
figure;
hold on;
for j = 1:length(Zs)
    z = Zs(j);
    mu_value2 = mu(D2, z); % Evaluate mu at specific D and z

    % Define the function for u(xi)
    u2 = @(xi) (6 * (c + Omega0) * D2 / (rho(z) * mu_value2)) * sech((sqrt(6 * (c +
    Omega0))) / (2 * D2.^(1/4)) * (xi - c * X2)).^2;

    % Evaluate the function
    u_values2 = u2(xi_range);

    % Plot the function
    plot(xi_range, u_values2, 'LineWidth', 1.5, 'Color', colors(j,:));
end

xlabel('X');
ylabel('u');
title('D = 0.5');
legend('z = 0', 'z = 0.5', 'z = 1');
set(gca, 'XAxisLocation', 'origin');
ylim([0,4.5]);
hold off;
```