Self-Similar Solutions to the Thin-Film Equation

Y. H. Boesveld

TU Delft

Self-Similar Solutions to the Thin-Film Equation

by

Y. H. Boesveld

to obtain the degree of Bachelor of Science
at the Delft University of Technology,
to be publicly defended on Wednesday, 25 June 2025 at 16:00h.

Student number:	5614953	
Project duration:	April 22, 2025 – June 25, 2025	
Thesis committee:	Prof. M. V. Gnann,	TU Delft, supervisor
	Prof. Y. Murakami,	TU Delft

An electronic version of this thesis is available at `http://repository.tudelft.nl/`.

TU Delft

Abstract

Technical Summary

This thesis considers the thin-film equation $\partial_t u = -\partial_x(|u|^n \partial_x^3 u)$ with respect to time $t \geq 0$ and one dimensional space $x \in \mathbb{R}$ where $n > 0$. A special case of the thin-film equation is when the initial condition is $u_0(x,0) = c\delta(x)$. A solution with this initial condition is called a source type solution. A source type solution describes how a viscous droplet spreads over a solid flat surface with volume $c > 0$. Source type solutions are expected to have a self-similar form with $u(x,t) = t^{-\alpha} f(\mu)$, $\mu = xt^{-\alpha}$ and $\alpha = \frac{1}{n+4}$ which reduces the equation into an ordinary boundary-value problem $(|f(\mu)|^n f'''(\mu))' = \alpha(\mu f(\mu))'$ with $\mu f(\mu) \to 0$ as $\mu \to \pm\infty$ and $\int_{-\infty}^{\infty} f(\mu)\, d\mu = c$. A solution of this boundary-value problem is called a self-similar solution. This thesis presents a detailed discussion of the paper by Bernis, Peletier & Williams [4] on the existence and uniqueness of self-similar solutions to the thin-film equation together with its qualitative properties. Here, existence will be proven by using a shooting method. Additionally, the thin-film equation will be derived from the Navier-Stokes equations using a lubrication approximation. Furthermore, a numerical construction of the self-similar solution is presented to visualize its behavior. The results demonstrate that the solution exhibits key qualitative features such as compact support and conservation of mass.

Non-Technical Summary

This thesis explores how a liquid droplet spreads over a flat surface, for instance, a droplet of oil spreading out on a flat table. The thesis focuses on a mathematical model called the thin-film equation, which helps to describe the spreading process of a droplet. The solution to this problem is expected to spread out in a predictable pattern over time, which is called a self-similar solution. This means that even though the droplet gets wider and thinner over time, the universal shape of the droplet stays the same. This kind of behavior makes it much easier to study and understand complex spreading processes, because only one shape needs to be understood, not how it changes at every moment. To understand this better, the thesis reviews a scientific paper by Bernis, Peletier, & Williams [4] which proves that a mathematical self-similar solution exists and is unique. Proving uniqueness is also important as this gives scientists and engineers confidence that they are using the right model. The self-similar solution will also be simulated on a computer. The simulations demonstrate that the droplet behaves as expected, that is, it is spreading symmetrically while keeping the same total amount of liquid.

Y. H. Boesveld
Delft, June 2025

Contents

1
Introduction

In this thesis, we study the existence and uniqueness of nonnegative source type solutions of the one dimensional thin-film equation

$$\begin{cases} \partial_t u = -\partial_x(|u|^n \partial_x^3 u) & \text{in } S = \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where n is a positive constant. A solution of equation (1.1) with initial condition

$$u_0(x) = c\delta(x) \qquad \text{for } x \in \mathbb{R}, \tag{1.2}$$

where δ denotes the Dirac mass centered at the origin and c a nonzero constant, is called a source type solution. For this thesis, $c > 0$ shall be assumed.

The thin-film equation describes how a fluid that is very thin in height compared to the length travels over time on a one-dimensional flat solid surface. It provides the height $u(x,t)$ depending on the location x and time t. The source type solution is a special type solution of the thin-film equation which describes how a droplet spreads over time on the surface. Here c denotes the volume of the droplet.

The exponent n is called the mobility exponent and is related to the boundary condition in the original Navier-Stokes equation, which is also called the slip condition. The focus of this thesis lies on the values between $0 < n < 3$. For $n \leq 0$ the fluid travels with infinite speed, while for values $0 < n < 1$, the film can become negative, meaning it goes through the surface. Since the absolute value term $|u|^n$ is used, mathematical analysis can still be done for the values $0 < n < 1$, even though it is not physically justified. It will even be shown in section 1.3 that nonnegativity holds for source type solutions. For the value $n = 1$, there is free slip, meaning that the surface provides no friction in the tangential direction. For $n = 3$ (no slip) and $n > 3$, the free boundary of the film cannot move [16]. In section 1.1, we will be show that the derivation is only valid for $0 < n \leq 3$. For values $1 < n < 3$, the fluid has partial slip, meaning that it moves slower at the surface. See Figure 1.1 for a visualization. In chapter 3, it will also be mathematically proven that the self-similar solution does not have a non-trivial solution for $n \geq 3$.

1.1. Derivation of the thin-film equation

In this section the thin-film equation will be derived, following [6, 22, 23]. A derivation from a more physical viewpoint can be found in [3]. The thin-film equation can be derived from the incompressible Navier-Stokes equations. The incompressible Navier-Stokes equations describe how a fluid moves in three dimensions in time assuming that the fluid has constant viscosity (Newtonian), is mass conserved and the density remains constant. For the thin-film equation it is assumed that the fluid is uniform in the direction perpendicular to the plane concerning the height and x direction meaning that the Navier-Stokes equations can be reduced to two dimensions. The thin-film equation will be derived using a lubrication approximation. This means that

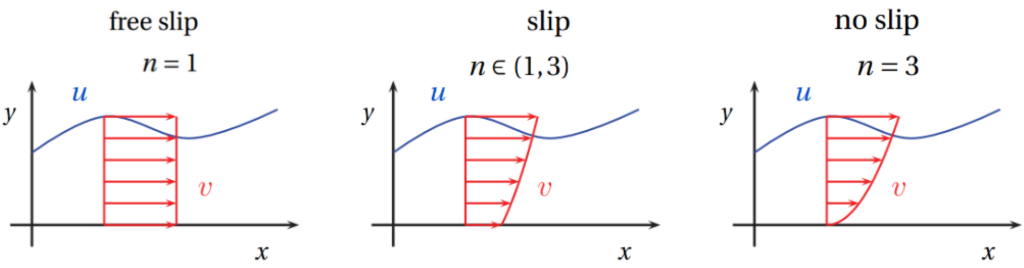

Figure 1.1: Schematics of the flow fields v close to the liquid–solid interface for different values of the mobility exponent n. Adapted from [13].

one dimension is much smaller compared to the other dimension which can be used to simplify the Navier-Stokes equations. For the thin-film equation the height is much smaller than the length of the fluid.

Consider a fluid with a free surface $y = h(x,t)$ that lies on a flat area. The governing equations for the in-compressible Navier-Stokes equations are the velocities u and v concerning the x and y direction and the pressure p which is normalized by the density of the fluid. The incompressible Navier-Stokes equations are given by

$$\begin{aligned} \partial_t u + u\partial_x u + v\partial_y u - \mu(\partial_x^2 u + \partial_y^2 u) + \partial_x p = 0, \\ \partial_t v + u\partial_x v + v\partial_y v - \mu(\partial_x^2 v + \partial_y^2 v) + \partial_y p = 0, \\ \partial_x u + \partial_y v = 0. \end{aligned} \tag{1.3}$$

Here, μ is the kinematic viscosity, which describes how easily a fluid flows per unit mass or per unit density. For a derivation of the Navier-Stokes equations, see for instance [9].

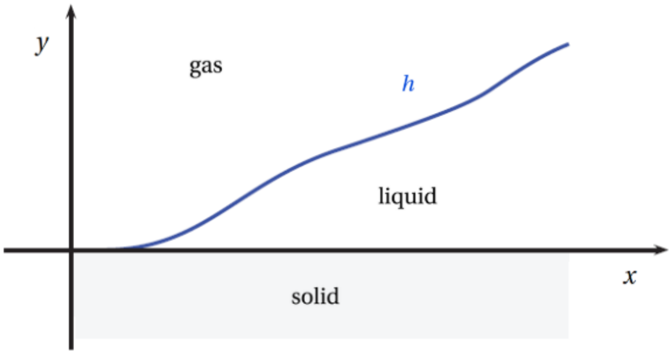

Figure 1.2: Schematic of the liquid thin-film, where liquid, gas and solid are shown. Adapted from [13].

To work with the equation, boundary conditions must be specified for the solid-liquid and liquid-gas interfaces, as shown in Figure 1.2. On the solid-liquid interface, it is required that the fluid cannot go through the solid. This means that the velocity v, which goes in the y direction, should be zero. So

$$v = 0 \qquad \text{when } y = 0. \tag{1.4}$$

Imposing a no-slip boundary condition at the solid-liquid interface ($u = 0$ when $y = 0$) leads to infinite energy dissipation near the contact line, where the contact line represents the place the liquid, gas and solid meet. Therefore, to let the liquid slip on the surface, a non-zero velocity in the x direction is allowed which is proportional to derivative $\partial_y u$

$$u - k(h)\partial_y u = 0 \qquad \text{when } y = 0. \tag{1.5}$$

Here $k(h) = \lambda^{3-n}h^{n-2}$, where λ represents the slip length which indicates how far you have to go below the surface for the fluid's velocity to be zero and $n \in (0,3)$ is the mobility exponent which was discussed before.

On the liquid-gas interface, we require that the fluid moves smoothly. Therefore, the tangential component of the shear stress should be continuous, meaning that the tangential forces between the liquid and gas are balanced. This can be done using

$$\partial_y u = 0 \qquad \text{when } y = h. \tag{1.6}$$

At the liquid-gas interface, there is a pressure jump proportional to the mean curvature of the interface which is called Laplace's law. This is denoted as

$$p_0 - p = \gamma \partial_x^2 h \qquad \text{when } y = h, \tag{1.7}$$

where p_0 is the constant air pressure and γ the liquid-gas surface tension constant.

Furthermore, a kinematic boundary condition is employed which ensures that the fluid stays on the free surface

$$\partial_t h + u\partial_x h = v \qquad \text{when } y = h. \tag{1.8}$$

Now that all necessary boundary conditions are considered, the thin-film equation will be derived. It is assumed that the thickness of the fluid H in the y direction is very small compared to the length of the fluid L in the x direction. Defining U as the velocity scale of u, V as the velocity scale of v, P as the pressure scale of p, and T as the time scale of t, the following transformations will be applied:

$$x \to Lx, \quad y \to Hy, \quad u \to Uu, \quad v \to Vv, \quad p \to Pp, \quad t \to Tt.$$

Note that the time scale T can be seen as how long it takes for the fluid to travel length L with velocity U. So T can be described as

$$T = \frac{L}{U}.$$

The Reynolds number, defined as

$$Re = \frac{UH}{\nu},$$

is also small.

Inserting these transformations in (1.3) and using that $H \ll L$ and $Re \ll 1$, results in

$$\begin{aligned} \mu\partial_y^2 u - \partial_x p &= 0 \qquad && \text{for } 0 < y < h, \\ -\partial_y p &= 0 \qquad && \text{for } 0 < y < h, \\ \partial_x u + \partial_y v &= 0 \qquad && \text{for } 0 < y < h, \end{aligned} \tag{1.9}$$

with

$$P = \frac{LU}{H^2}.$$

The second equation of (1.9) tells us that the pressure p is constant with respect to y. Using condition (1.7) gives

$$p = p_0 - \gamma \partial_x^2 h \qquad \text{for } 0 < y < h.$$

Differentiating p with respect to x gives

$$\partial_x p = -\gamma \partial_x^3 h \qquad \text{for } 0 < y < h.$$

Substituting $\partial_x p$ using the first equation of (1.9) results in

$$\mu \partial_y^2 u = -\gamma \partial_x^3 h \qquad \text{for } 0 < y < h.$$

Integrating this equation over (y, h) with respect to y and dividing both sides by μ gives, using the Fundamental Theorem of Calculus A.1,

$$\begin{aligned}
&\int_y^h \partial_y^2 u \, dz = \int_y^h -\frac{\gamma}{\mu} \partial_x^3 h \, dz \\
\Longrightarrow \; &\partial_y u\big|_{y=h} - \partial_y u = -(h-y)\frac{\gamma}{\mu}\partial_x^3 h \\
\Longrightarrow \; &\partial_y u = (h-y)\frac{\gamma}{\mu}\partial_x^3 h.
\end{aligned} \tag{1.10}$$

Here, condition (1.6) is used. Integrating this over $(0, h)$ with respect to y and using condition (1.5) gives

$$\begin{aligned}
&\int_0^h \partial_y u \, dz = \int_0^y (h-y)\frac{\gamma}{\mu}\partial_x^3 h \, dz \\
\Longrightarrow \; &u - u|_{y=0} = (hy - \frac{1}{2}y^2)\frac{\gamma}{\mu}\partial_x^3 h \\
\Longrightarrow \; &u = \left(hy - \frac{1}{2}y^2\right)\frac{\gamma}{\mu}\partial_x^3 h + k(h)\partial_y u\big|_{y=0}.
\end{aligned}$$

Using (1.10) for $k(h)\partial_y u\big|_{y=0}$ results in

$$u = \frac{\gamma}{\mu}\left(hy - \frac{1}{2}y^2 + hk(h)\right)\partial_x^3 h.$$

The averaged horizontal velocity can now be calculated using

$$\bar{u} = \frac{1}{h}\int_0^h u \, dy.$$

Therefore, we arrive at

$$\bar{u} = \frac{\gamma}{\mu}\left(\frac{1}{3}h^2 + hk(h)\right)\partial_x^3 h. \tag{1.11}$$

Integrating the third equation of (1.9) over $(0, h)$ with respect to y and using the conditions (1.4) and (1.8) gives

$$\begin{aligned}
&\int_0^h \partial_x u + \partial_y v \, dy = 0 \\
\Longrightarrow \; &\partial_x\left(\int_0^h u \, dy\right) + v|_{y=h} - v|_{y=0} = 0 \\
\Longrightarrow \; &\partial_x\left(\int_0^h u \, dy\right) + \partial_t h + u\partial_x h|_{y=h} = 0 \\
\Longrightarrow \; &\partial_x(h\bar{u}) + \partial_t h + u\partial_x h|_{y=h} = 0.
\end{aligned}$$

Note that, since $H \ll L$, $\partial_x h$ varies slowly in the x direction and so we can neglect this value. Therefore, using (1.11) and $k(h) = \lambda^{3-n} h^{n-2}$, we arrive at

$$\frac{\gamma}{\mu}\partial_x\left(\left(\frac{1}{3}h^3 + \lambda^{3-n}h^n\right)\partial_x^3 h\right) + \partial_t h = 0.$$

When rescaling the variables, the constants can be eliminated. Furthermore, for $n \in (0,3)$, the term h^n dominates over the h^3 term if $h \ll \lambda$. Thus, we arrive at the thin-film equation

$$\partial_x(h^n \partial_x^3 h) + \partial_t h = 0.$$

Note that u is used in this thesis for the height instead of h.

1.2. Applications of the thin-film equation

Applications of the thin-film equation are usually found in the field of fluid dynamics. An important application is spin coating, a mechanism used to apply thin-films to substrates. During spin coating, a liquid is deposited onto a flat substrate, which is then rapidly spun. This results in a thin-film covering the substrate uniformly. Spin coating is widely used in many industries, especially in photolithography for making semiconductors and nanotechnology devices [15].

Applications also arise in biology. An example is that it helps to understand cell spreading. This is because cell spreading has, in some sense, similar properties to the spreading of a droplet [17]. Another example lies in the pulmonary surfactant dynamics in the lungs. The alveoli, which are small air sacs in the lungs, are lined by a thin-film coated with surfactant, a substance that reduces surface tension to keep the alveoli open. A small percentage of firstborn babies have not produced enough surfactant yet, which causes the alveoli to collapse, making it much harder to breathe. To solve this, synthetic lung surfactant can be delivered in the form of a droplet to the lungs. Modeling this process helps predict the behavior of the fluid [10].

1.3. Mass conservation and nonnegativity

In [5], equation (1.1) was also studied for a more general case. They studied the equation in a cylinder $Q_{T_0} = \Omega \times (0, T_0)$, where Ω is a bounded interval. They also assumed that $u_0 \in H^1(\Omega)$, meaning that $\int_\Omega u_0^2\,dx < \infty$ and $\int_\Omega (\partial_x u_0)^2\,dx < \infty$, and that $u_x = u_{xxx} = 0$ on the lateral boundary. For this, they proved a weak positivity property:

$$u_0(x) \geq 0 \quad \Longrightarrow \quad u(x,t) \geq 0 \qquad \text{in } Q_T$$

and a conservation law:

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0\,dx \qquad \text{for } 0 < t < T_0.$$

In view of these results, it is natural that this should also hold for the extension $\Omega = \mathbb{R}$ assuming that $u_{xxx} \to 0$ as $x \to \pm\infty$ for all $t > 0$. Now $\delta(x) \notin H^1(\mathbb{R})$, but the sequence of Gaussian's which approximates $\delta(x)$ in a distributional sense as $t \to 0$,

$$f_t(x) = \frac{1}{\sqrt{2\pi t^2}} e^{-\frac{x^2}{2t^2}},$$

is smooth and satisfies $f_t(x) \geq 0$ and $f_t(x) \in H^1(\mathbb{R})$ for all $t > 0$. In particular, $c\delta(x)$ is a nonnegative distribution, since $c\delta(\varphi(x)) \geq 0$ for all $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi \geq 0$. Here, $C_c^\infty(\mathbb{R})$ represents the functions that are infinitely differentiable and have compact support, see Definition 2.4. The volume of the approximated initial data f_t is

$$\int_{-\infty}^{\infty} c f_t\,dx = c$$

for all $t > 0$. Hence by continuity, the limiting function u should also satisfy the conservation law for all $t > 0$

$$\int_{-\infty}^{\infty} u(x,t)\,dx = c. \tag{1.12}$$

1.4. Self-similar solutions

Due to the scaling properties of equation (1.1) with initial condition (1.2) and the conservation law (1.12), solutions of equation (1.1) are expected to have a self-similar form

$$u(x,t) = t^{-\alpha} f(\mu), \qquad \mu = xt^{-\beta}. \tag{1.13}$$

Now, to be consistent, it should also follow the conservation law (1.12)

$$\begin{aligned} &\int_{-\infty}^{\infty} u(x,t)\,dx = c \\ \implies &\int_{-\infty}^{\infty} t^{-\alpha} f(xt^{-\beta})\,dx = c \\ \implies &t^{-\alpha+\beta}\int_{-\infty}^{\infty} f(\mu)\,d\mu = c. \end{aligned}$$

Here, a change of variables $\mu = xt^{-\beta}$ is used. To hold for all $t > 0$, we have

$$\alpha = \beta.$$

It is now substituted in equation (1.1). For $\partial_t u$, using the product rule,

$$\begin{aligned} \partial_t u &= -\alpha t^{-\alpha-1} f(\mu) + t^{-\alpha} f'(\mu)(-\alpha x t^{-\alpha-1}) \\ &= -\alpha t^{-\alpha-1} f(\mu) - \alpha\mu t^{-\alpha-1} f'(\mu) \\ &= -\alpha t^{-\alpha-1}(\mu f(\mu))'. \end{aligned}$$

For $\partial_x^3 u$,

$$\partial_x^3 u = t^{-4\alpha} f'''(\mu).$$

This yields

$$\begin{aligned} \partial_x(|u|^n \partial_x^3 u) &= (t^{-n\alpha}|f|^n t^{-4\alpha} f'''(\mu))' t^{-\alpha} \\ &= t^{-\alpha(n+5)}(|f|^n f'''(\mu))'. \end{aligned}$$

To have $\partial_t u = -\partial_x(|u|^n \partial_x^3 u)$, it should hold that

$$t^{-\alpha-1} = t^{-\alpha(n+5)}.$$

Therefore, their powers should be the same. This can be used to find α

$$\begin{aligned} &-\alpha - 1 = -\alpha(n+5) \\ \implies &\alpha = \frac{1}{n+4}. \end{aligned} \tag{1.14}$$

To satisfy the initial condition (1.2), we note that $\sup_{x\neq 0}\delta(x) = 0$. Therefore, for $x \neq 0$,

$$\lim_{t\to 0^+} u(x,t) = 0.$$

If $x \neq 0$, the variable μ behaves as

$$\lim_{t\to 0^+} \mu = xt^{-\alpha} = \pm\infty,$$

where the sign of ∞ depends on the sign of x. This means that

$$0 = \lim_{t\to 0^+} u(x,t) = \lim_{\mu\to\pm\infty} t^{-\alpha} f(\mu).$$

Since $t^{-\alpha} \to \infty$ as $t \to 0^+$, the only way this limit can be zero is if f decays faster than $t^{-\alpha}$. t can be written as $t = \left(\frac{x}{\mu}\right)^{\frac{1}{\alpha}}$ using (1.13). So

$$t^{-\alpha} f(\mu) = \left(\frac{x}{\mu}\right)^{-\frac{\alpha}{\alpha}} f(\mu) = \frac{\mu}{x} f(\mu).$$

Therefore,

$$\lim_{\mu\to\pm\infty} \mu f(\mu) = 0. \tag{1.15}$$

Using (1.13) with (1.14) and (1.15), equation (1.1) can be reduced to

$$\begin{cases} (|f(\mu)|^n f'''(\mu))' = \alpha(\mu f(\mu))' & \text{for } -\infty < \mu < \infty, \\ \mu f(\mu) \to 0 & \text{as } \mu \to \pm\infty, \end{cases} \tag{1.16}$$

which satisfies the integral condition

$$\int_{-\infty}^{\infty} f(\mu)\, d\mu = c. \tag{1.17}$$

When f is a solution of the given equation, it means that f is continuously differentiable and $\int_{-\infty}^{\infty} |f|\, dx < \infty$ ($f \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$) such that f''' exists and is continuous when $f \neq 0$ and $|f|^n f''' \in C^1(\mathbb{R})$.

Note that for a broad class of initial conditions, including the source type solutions, solutions to the thin-film equation are expected to evolve toward a self-similar form due to scaling properties, although it has not yet been proven analytically for general n. Understanding and analyzing self-similar solutions is still relevant since this reduces the problem to studying one universal shape independent of time. In [8], convergence to the unique source type self-similar solution was established for strong solutions with more general initial data $u_0 \geq 0$ under certain conditions when $n = 1$. The source type self-similar solution is conjectured to act as a universal attractor for a certain class of initial data satisfying suitable conditions, such as nonnegativity, finite mass, and sufficient decay. Outside this class, different asymptotic behaviors or self-similar profiles may emerge. The focus of this thesis therefore lies in studying the existence and uniqueness of source type self-similar solutions of equation (1.16).

1.5. Overview

This thesis is mostly based on the paper *"Source type solutions of a fourth order nonlinear degenerate parabolic equation"* by Bernis, Peletier & Williams [4]. In this paper, the following main theorems will be proven for nonnegative even self-similar solutions:

Theorem 1.1. *If* $n \geq 3$*, then there exists no non-trivial solution for equation (1.16).*

Theorem 1.2. *If* $0 < n < 3$*, then there exists precisely one nonnegative even solution of equation (1.16). This solution has compact support. If* $0 < n \leq 2$*, uniqueness holds without assuming that u is even.*

The main goal of this thesis is to expand the content of [4] by rewriting it and providing more details to the proofs. Note that these theorems prove that there exists a nonnegative source type solution for equation (1.1) which is unique if and only if $0 < n < 3$. The solution of equation (1.16) also has compact support, see Definition 2.4. The paper of Bernis, Peletier & Williams [4] state that that there is no explicit closed-form formula for $f(\mu)$, except when $n = 1$, which is found by [21]

$$f(\mu) = \begin{cases} \frac{1}{120}\left(a^2 - \mu^2\right)^2 & \text{on } (-a, a), \\ 0 & \text{elsewhere,} \end{cases} \tag{1.18}$$

where $a^5 = \frac{225c}{2}$. They did, however, proof estimates of how $f(\mu)$ behaves near the boundary of its compact support for $0 < n < 3$, which will be shown in chapter 7.

The structure of this thesis is as follows. In chapter 2, the prerequisite knowledge in this thesis is presented. In chapter 3, Theorem 1.1 will be proven. In chapter 4, relevant preliminary results will be proven about equation (1.16), which helps to set up the existence proof. In chapter 5, a solution is constructed using the shooting method when $0 < n < 3$. In chapter 6, we will prove uniqueness for even self-similar solutions when $0 < n < 3$, and we will also show that every solution is even for $0 < n \leq 2$, thereby completing the proof of Theorem 1.2. In chapter 7, the self-similar solution will be numerically constructed using the shooting method derived in chapter 5. In chapter 8, a conclusion will be given.

2

Prerequisites

The following chapter gives an overview of the necessary knowledge needed for the analysis of self-similar solutions of the thin-film equation (1.16). This is done by stating the essential definitions and theorems, without giving the proof. These results can also be found in the literature. For each definition and theorem, an example of a book or article where it can be found is mentioned. Additional definitions and theorems that are less central to the main text are collected in Appendix A.2.

2.1. Definitions

Definition 2.1. *(Uniformly bounded) [7] A sequence of functions $\{f_n\}$, where each $f_n : D \to \mathbb{R}$, is said to be uniformly bounded if, there exists a $K > 0$ such that $|f_n(x)| \leq K$ for all $x \in D$ and $n \in \mathbb{N}$.*

Definition 2.2. *(Equicontinuity) [7] A sequence of functions $\{f_n\}$, where each $f_n : D \to \mathbb{R}$, is said to be equicontinuous if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$, whenever $x, y \in D$ satisfy $|x - y| < \delta$ for all $n \in \mathbb{N}$.*

Definition 2.3. *(Lipschitz continuity) [7] Let $f : D \to \mathbb{R}$. f is said to be Lipschitz continuous in D if, there exists a $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in D$. Or rather, $\left|\frac{df}{dx}\right| \leq K$ for all $x \in D$.*

Definition 2.4. *(Compact support) [12] A function f is said to be compactly supported if there exists a compact set M for which f is zero outside of the set M.*

2.2. Theorems

Theorem 2.1. ***Picard-Lindelöf Theorem*** *[26] Consider the following initial value problem*

$$\boldsymbol{y}'(t) = \boldsymbol{f}(t, \boldsymbol{y}(t)), \qquad \boldsymbol{y}(t_0) = \boldsymbol{y}_0.$$

Suppose $\boldsymbol{f} \in C(U, \mathbb{R}^n)$, where $U \subset \mathbb{R}^{n+1}$ is open and $(t_0, \mathbf{y}_0) \in U$. If $\mathbf{f}$ is locally Lipschitz continuous in the second argument on a neighborhood $D = [t_0 - a, t_0 + a] \times \overline{B}(\mathbf{y}_0, b) \subset U$, then there exists a unique local solution $\bar{\mathbf{y}}(t) \in C^1(I)$ to the initial value problem, where I is an interval containing t_0.

Theorem 2.2. *(Continuous dependence) [20] Consider the following initial value problem*

$$\boldsymbol{y}'(t) = \boldsymbol{f}(t, \boldsymbol{y}(t)), \qquad \boldsymbol{y}(t_0) = \boldsymbol{y}_0.$$

Suppose $\boldsymbol{f}(t, \boldsymbol{y}(t))$ is continuous on an open set $E \subset \mathbb{R}^{n+1}$, and that for every $(t_0, \boldsymbol{y}_0) \in E$, the initial value problem has a unique solution $\boldsymbol{y}(t) \equiv \boldsymbol{\mu}(t, t_0, \boldsymbol{y}_0)$. Then the solution map $\boldsymbol{\mu}(t, t_0, \boldsymbol{y}_0)$ is continuous in $(t_0, \boldsymbol{y}_0) \in E$ and $t \in I$, where I is an open interval which is the maximal interval of existence of $\boldsymbol{\mu}$.

Theorem 2.3. ***Arzelà-Ascoli Theorem*** *[11] Let X be a compact set and let $\{f_n\}$ be a uniformly bounded, equicontinuous sequence of real-valued functions in X. Then $\{f_n\}$ has a subsequence that converges uniformly in X to a continuous function f in X.*

3

Nonexistence

3.1. Simplifying equation (1.16)

In this chapter, the following equation is considered

$$\begin{cases} \left(|u|^n u'''\right)' = (xu)' & \text{for } -\infty < x < \infty, \\ xu(x) \to 0 & \text{as } x \to \pm\infty. \end{cases} \tag{3.1}$$

This equation uses $\alpha = 1$ to make computations easier. Equation (1.16) can be rewritten to equation (3.1) using the following rescaling:

$$f(\mu) = ku(l\mu), \qquad k^n l^4 = \alpha. \tag{3.2}$$

Here, k scales the height of the solution and l scales the width of the solution. To determine the criterion for k and l such that it corresponds to α, $f(\mu) = ku(l\mu)$ can be substituted into equation (1.16) to get the equality. This means that all results proven for equation (3.1) also apply to equation (1.16). Note that there is only one k and l possible that also satisfies the mass conservation law (1.17). To determine the unique values of k and l, we observe that the solution to equation (3.1) should be in $L^1(\mathbb{R})$, otherwise the mass conservation integral would not be valid. This condition can then be used to deduce the value of l

$$\begin{aligned} &\int_{-\infty}^{\infty} f(\mu)\,d\mu = c \\ \Longrightarrow &\int_{-\infty}^{\infty} ku(l\mu)\,d\mu = c \\ \Longrightarrow &\frac{k}{l}\int_{-\infty}^{\infty} u(y)\,dy = c \\ \Longrightarrow &\frac{k|u|_1}{l} = c \\ \Longrightarrow &\frac{k|u|_1}{l} = c \\ \Longrightarrow &\, l = \frac{k|u|_1}{c}. \end{aligned}$$

Here, change of variables is performed with $y = l\mu$ and $|\cdot|_1$ denotes the norm in $L^1(\mathbb{R})$. To derive k, the condition from (3.2) is used

$$
\begin{aligned}
& k^n l^4 = \alpha \\
\implies & k^n \left(\frac{k|u|_1}{c}\right)^4 = \alpha \\
\implies & k^{n+4} \left(\frac{|u|_1^4}{c^4}\right) = \alpha \\
\implies & k^{n+4} = \alpha \left(\frac{c^4}{|u|_1^4}\right) \\
\implies & k = \left(\frac{\alpha c^4}{|u|_1^4}\right)^{1/n+4}.
\end{aligned}
$$

This ensures that there is a unique k and l for which the mass conservation law holds.

Suppose u is a solution of equation (3.1). When u is a solution to the given equation, it means that u is a continuously differentiable function ($u \in C^1(\mathbb{R})$) such that u''' exists and is continuous when $u \neq 0$ and $|u|^n u''' \in C^1(\mathbb{R})$. If u is now substituted into equation (3.1) and integrated, we obtain, using the Fundamental Theorem of Calculus A.1,

$$
\begin{aligned}
& \int (|u|^n u''')' \, dx = \int (xu)' \, dx \\
\implies & |u|^n u''' = xu + C,
\end{aligned}
$$

where $C \in \mathbb{R}$ is an integration constant. To determine the constant, the condition $xu(x) \to 0$ as $x \to \infty$ is used. If $C \neq 0$, the term $|u|^n u'''$ dominates as $x \to \infty$, since $|u|^n u''' \approx C$ for large x. This means that for large x we have $|u|^n \neq 0$ if $C \neq 0$ and therefore $u \neq 0$. So $\lim_{x\to\infty} u(x) \neq 0$. This contradicts the condition that $xu(x) \to 0$ as $x \to \infty$ and hence $C = 0$. If $u \neq 0$, the equation can be written as

$$u''' = \frac{xu}{|u|^n}. \tag{3.3}$$

If $u > 0$, we obtain

$$u''' = xu^{1-n}. \tag{3.4}$$

3.2. Proving there exists no non-trivial solution when $n \geq 3$

Using equation (3.4), the nonexistence theorem will be proven.

Theorem 3.1. *If $n \geq 3$, then equation (3.1) has no non-trivial solution.*

To prove this, a helping result is needed to help prove this statement.

Lemma 3.2. *Suppose $n \geq 3$. Let $x_0 \geq 0$ and let u be a solution of (3.4) in a neighborhood of x_0 such that $u(x_0) > 0$. Then $u(x) > 0$ for all $x \geq x_0$ and*
(a) u can be continued to the entire half-line $[x_0, \infty)$.
(b) $\lim_{x\to\infty} u(x) = \infty$.

Proof. First, (a) will be proven using contradiction. Suppose that there exists a maximal right-neighborhood of existence $[x_0, a)$ with $a < \infty$, so u cannot be extended further. Since u is nonnegative and $n \geq 3$, if $u \to 0$ at a certain point in the interval $[x_0, a)$, the right-hand side of equation (3.4) diverges, causing u''' to diverge. This means $u > 0$ on $[x_0, a)$. Therefore, equation (3.4) can be used to get the following: $u''' = xu^{1-n} > 0$. So u''' has a constant (positive) sign on the interval $[x_0, a)$. This means that $u(a^-) = \lim_{x\to a^-} u(x)$ and $u'(a^-) = \lim_{x\to a^-} u'(x)$ exist. To see why, note that $u''' > 0$ on $[x_0, a)$, meaning that u'' is strictly increasing. This means that $\lim_{x\to a^-} u''(x)$ exists and is finite if u'' is bounded above or is infinite if u'' is not bounded above on the interval $[x_0, a)$. Now, using the Fundamental Theorem of Calculus A.1, note that u' is defined as

$$u'(x) = u'(x_0) + \int_{x_0}^{x} u''(t) \, dt. \tag{3.5}$$

If u'' is bounded above, the limit $\lim_{x\to a^-} u''(x) = L$ is finite. This means that u'' can be extended to $[x_0, a]$ using $u''(a) = L$. Since u'' is increasing, u'' is bounded above by L in $[x_0, a]$. Therefore,

$$\begin{aligned}\lim_{x\to a^-} u'(x) &= u'(x_0) + \lim_{x\to a^-}\int_{x_0}^{x} u''(t)\,dt\\ &= u'(x_0) + \int_{x_0}^{a} u''(t)\,dt\\ &\leq u'(x_0) + \int_{x_0}^{a} L\,dt\\ &= u'(x_0) + L(a-x_0) < \infty.\end{aligned}$$

If u'' is not bounded, then $\lim_{x\to a^-} u''(x) = \infty$. Depending on the growth rate of u'', the integral in (3.5) diverges to ∞ or stays bounded. Therefore, the limit for u' as $x \to a^-$ also exists and is bounded or unbounded depending on the growth rate of u''. A similar argument, applied to u and u', shows that the limit for u as $x \to a^-$ also exists.

Since u cannot be extended further than $x = a$, there are two options: (i) the solution blows up, i.e., $|u| \to \infty$ as $x \to a^-$, (ii) the solution tends to zero, i.e., $u \to 0$ as $u \to a^-$, which causes a singularity due to the term u^{1-n} in equation (3.4). Both cases will be discussed.

If $u(a^-) = \infty$ then, since $n \geq 3$, $u^{1-n} \to 0$ as $x \to a^-$. This means that $u''' \to 0$ when $x \to a^-$ and so u'' will be bounded above in the interval $[x_0, a)$. Hence, using the previous argument, $\lim_{x\to a^-} u(x)$ will be finite. This contradicts that $u(a^-) = \infty$.

Suppose $u(a^-) = 0$. Since $u > 0$ and $u \to 0$ as $x \to a^-$, u must be decreasing near a, so $u'(a^-) \leq 0$. Furthermore, $u'(x)$ can be written as

$$u'(x) = u'(x_0) + \int_{x_0}^{x} u''(x)\,dx.$$

Since $u''' > 0$, u'' is increasing. Therefore, we obtain

$$\begin{aligned}u'(x) &\geq u'(x_0) + \int_{x_0}^{x} u''(x_0)\,dx\\ &= u'(x_0) + (x-x_0)u''(x_0).\end{aligned}$$

This means that $u'(x)$ is bounded below for any $x \in [x_0, a)$ and hence $u'(a^-) \geq -C$ for some positive constant C. To check how the solution behaves near $x = a$, $u'(a^-) \geq -C$ will be integrated on both sides

$$\begin{aligned}&\lim_{x^*\to a^-}\int_{x}^{x^*} u'(t)\,dt \geq \lim_{x^*\to a^-}\int_{x}^{x^*} -C\,dt\\ &\implies u(a^-) - u(x) \geq -C(a-x)\\ &\implies u(x) \leq C(a-x).\end{aligned}$$

This result will now be used for equation (3.4). Using that $u^{1-n} \geq C^{1-n}(a-x)^{1-n}$ is valid near $x = a$, and considering $x \geq x_0$, which is greater than 0, we obtain

$$u''' \geq \frac{\tilde{C}}{(a-x)^{n-1}},$$

where $\tilde{C} = x_0 C^{1-n}$. Integrating this near $x = a$ with a small $\delta > 0$ gives

$$\int_{a-\delta}^{x} u'''(t)\,dt \geq \int_{a-\delta}^{x} \frac{\tilde{C}}{(a-t)^{n-1}}\,dt$$
$$\Longrightarrow u''(x) \geq \frac{\tilde{C}}{n-2}\left(\frac{1}{(a-x)^{n-2}} - \frac{1}{\delta^{n-2}}\right) + u''(a-\delta)$$
$$\Longrightarrow u''(x) \geq \frac{A}{(a-x)^{n-2}} + B,$$

where $A = \frac{\tilde{C}}{n-2}$ and $B = -\frac{\tilde{C}}{(n-2)\delta^{n-2}} + u''(a-\delta)$. To integrate this again, both $n = 3$ and $n > 3$ must be taken into consideration. For $n = 3$, the integration gives

$$\int_{a-\delta}^{x} u''(t)\,dt \geq \int_{a-\delta}^{x} \frac{A}{(a-x)^{n-2}} + B\,dt$$
$$\Longrightarrow u'(x) \geq A\log\left(\frac{\delta}{a-x}\right) + Bx - B(a-\delta) + u'(a-\delta).$$

As $x \to a^-$, the logarithm will diverge to ∞ which means that $u'(x) \to \infty$ as $x \to a^-$. This contradicts that $u'(a^-) \leq 0$. For $n > 3$, the integral will be taken again

$$\int_{a-\delta}^{x} u''(t)\,dt \geq \int_{a-\delta}^{x} \frac{A}{(a-x)^{n-2}} + B\,dt$$
$$\Longrightarrow u'(x) \geq \frac{A}{n-3}\left(\frac{1}{((a-x)^{n-3}} - \frac{1}{\delta^{n-3}}\right) + Bx - B(a-\delta) + u'(a-\delta).$$

Here, u' will again diverge to ∞ as x approaches a^- which leads to a contradiction.

Since both cases (i) and (ii) lead to a contradiction, we can conclude that u can be continued to the half line $[x_0, \infty)$.

To prove (b), we note that $\lim_{x\to\infty} u(x)$ exists using the same argument given in (a) for the existence of $\lim_{x\to a^-} u(x)$. Suppose that $\lim_{x\to\infty} u(x) = L < \infty$. Using equation (3.4), for large x, $u''' \approx xL^{1-n}$. This means that $\lim_{x\to\infty} u'''(x) = \infty$. So there exists some α and large C such that, for all $x \geq \alpha$, we obtain $u'''(x) \geq C$. Integrating this on the interval $[\alpha, x]$ gives $u''(x) \geq u''(\alpha) + C(x-\alpha)$. This also means that $\lim_{x\to\infty} u''(x) = \infty$. Repeating this argument for u' and u gives $\lim_{x\to\infty} u(x) = \infty$, which is a contradiction. Hence $\lim_{x\to\infty} u(x) = \infty$. □

Note that we can prove the same results in Lemma 3.2 analogously for the left half line $(-\infty, x_0]$, where $x_0 \leq 0$.

Now, Theorem 3.1 can be proven.

Proof. Suppose a non-trivial solution exists for equation (3.1) with $n \geq 3$. By Lemma 3.2, $\lim_{x\to\infty} u(x) = \infty$. The condition $xu(x) \to 0$ as $x \to \infty$ states that u should go to 0 as x approaches ∞ and therefore we arrive at a contradiction. Hence there exists no non-trivial solution for equation (3.1) when $n \geq 3$. □

Remark. *Note that, by applying the scaling argument of* (3.2), *it follows that equation* (1.16) *also admits no non-trivial solution. This completes the proof concerning the nonexistence of a non-trivial solution for $n \geq 3$.*

4

Qualitative properties of solutions

Before we will prove existence and uniqueness for $0 < n < 3$, some relevant preliminary observations will be proven in this chapter about nonnegative solutions of equation (3.1). These insights will contribute to the proof of existence and uniqueness.

4.1. Compactly supported solutions

First, we will proof that a nonnegative solution u is compactly supported, see Definition 2.4. To prove this, the following lemma will be used.

Lemma 4.1. *Let u be a nonnegative solution of equation (3.1) and let $u(x_0) = 0$ for some $x_0 \geq 0$. Then $u(x) = 0$ for all $x \geq x_0$.*

Proof. Let $x_0 \geq 0$ and suppose on the contrary that there exists a point $x_1 > x_0$ such that $u(x_1) > 0$. Define the most left point where $u > 0$ as

$$\zeta_0 = \inf\{x < x_1 : u > 0 \text{ on } (x, x_1)\}.$$

Since $u(x_0) = 0$, it implies that $\zeta_0 \geq x_0$. Since u is continuous and nonnegative, it means that $u(\zeta_0) = 0$. To show that $u'(\zeta_0) = 0$, we will discuss two cases. If $\zeta_0 > x_0$, we have that $u = 0$ on the interval $[x_0, \zeta_0]$ which implies that $u'(\zeta_0^-) = 0$. Also, $u'(\zeta_0^+) = 0$ by continuity of u'. Therefore, $u'(\zeta_0) = 0$. If $\zeta_0 = x_0$, then since $u(x_0) = 0$, continuity implies that $u'(x_0) > 0$ would lead to $u(x) < 0$ for some $x < x_0$, violating nonnegativity. Similarly, $u'(x_0) < 0$ would imply that $u(x) < 0$ for some $x > x_0$, again contradicting nonnegativity. Therefore, it must be that $u'(\zeta_0) = 0$.

Now, the Mean Value Theorem A.2 states that, since $u(\zeta_0) = 0$ and $\zeta_0 < x_1$, there exists a point $\zeta_1 \in (\zeta_0, x_1)$ such that

$$u'(\zeta_1) = \frac{u(x_1) - u(\zeta_0)}{x_1 - \zeta_0} > 0.$$

Using equation (3.4), we have that $u''' > 0$ on (ζ_0, x_1) because $u > 0$ on that interval. Therefore, u'' is increasing on the interval (ζ_0, x_1). Since $u'(\zeta_0) = 0$ and $u'(\zeta_1) > 0$, it follows that $u''(\zeta_1) > 0$. Given that $u''' > 0$, u'' remains positive and continues to increase on (ζ_1, x_1). Consequently, $u' > 0$ on (ζ_1, x_1), implying that u is strictly increasing on (ζ_1, x_1). Moreover, since $u > 0$ and is strictly increasing, the positivity of u''' will persist beyond x_1, ensuring that both u'' and u' continue to increase and so remain positive for all $x > \zeta_1$. As a result, $u(x) \geq u(\zeta_1) > 0$ for all $x > \zeta_1$. This contradicts that $xu \to 0$ as $x \to \infty$. Hence $u(x) = 0$ for all $x \geq x_0$. □

Note that, using the same arguments, we can also show that $u(x) = 0$ for all $x \leq x_0$ with $x_0 \leq 0$ if $u(x_0) = 0$.

With this, we can prove that a nonnegative solution u is compactly supported.

Lemma 4.2. *Let u be a nonnegative non-trivial solution of equation (3.1). Then*

$$u(x) = \begin{cases} > 0 & \text{on } x \in (a^-, a^+), \\ = 0 & \text{elsewhere,} \end{cases} \tag{4.1}$$

where $-\infty < a^- < 0 < a^+ < \infty$.

Proof. To prove this, Lemma 4.1 states that if there is a point $a^- \le 0$ and $a^+ \ge 0$ such that $u(a^-) = u(a^+) = 0$, $u(x) = 0$ for all $x \le a^-$ and $x \ge a^+$. If $u(0) = 0$, this implies that $u(x) = 0$ for all x. This cannot happen by the assumption that u is a non-trivial solution. So $a^- < 0$ and $a^+ > 0$. It remains to prove that a^- and a^+ are finite. We will prove it for a^+, since the proof is analogous for a^-.

Suppose that $a^+ = \infty$. This implies that $u > 0$ for all $x > 0$. As a consequence, using equation (3.4), $u''' > 0$ for all $x > 0$. This means that u'' is strictly increasing for all $x > 0$ and so $\lim_{x\to\infty} u''(x) = L$ exists. If $L > 0$, there exists an α such that for all $x \ge \alpha$, $u''(x) \ge \frac{L}{2}$. Using the same argument as in Lemma 3.2(b), this will result in $\lim_{x\to\infty} u(x) = \infty$, violating $xu \to 0$ as $x \to \infty$. Suppose $L < 0$. This means that $u(x)$ should be lower than $\frac{L}{2}$ at a certain point, since $\frac{L}{2} > L$ for $L < 0$. So there exists an α such that for all $x \ge \alpha$, $u''(x) \le \frac{L}{2}$. Again, using the same argument as in Lemma 3.2, but this time with a negative coefficient, this results in $\lim_{x\to\infty} u(x) = -\infty$, violating nonnegativity. Therefore, $L = 0$. As u'' is strictly increasing for all $x > 0$, this implies that $u'' < 0$ for all $x > 0$. So u' is strictly decreasing for all $x > 0$.

Now, either $u'(0) \le 0$ or $u'(0) > 0$. If $u(0)' \le 0$, since u' is strictly decreasing, there exists an $x_0 \in [0,\infty)$ such that $u'(x_0) \le -C < 0$ for all $x > 0$, where C is a positive constant. Integrating this gives

$$\int_{x_0}^{x} u'(t)\,dt \le \int_{x_0}^{x} -C\,dt$$
$$\Longrightarrow u(x) \le u(x_0) - C(x - x_0).$$

An x can now be found such that the right-hand side becomes zero:

$$u(x_0) - C(x - x_0) = 0$$
$$\Longrightarrow x = x_0 + \frac{u(x_0)}{C}.$$

As $\frac{u(x_0)}{C} > 0$, a finite point further than x_0 can be found such that $u\left(x_0 + \frac{u(x_0)}{C}\right) \le 0$. Therefore, $a^+ < \infty$. This contradicts that $a^+ = \infty$. If $u'(0) > 0$, it is not clear yet if there exists an $x_0 \in [0,\infty)$ such that $u'(x_0) \le -C < 0$ for all $x > 0$. Since $xu \to 0$ as $x \to \infty$, there must exist a large constant M such that $u(0) > u(x)$ for all $x \ge M$. Since u is continuous on the set $[0, M]$, the Extreme Value Theorem A.3 states that there should be a maximum value in $[0, M]$. Therefore, there exists an $x_{max} \in [0, M]$ such that $u'(x_{max}) = 0$. Since u' is strictly decreasing, u' would become negative for all $x > x_{max}$. The argument used for $u'(0) \le 0$ can now be applied to conclude again that $a^+ < \infty$, which is a contradiction. Therefore, $a^+ < \infty$ and Lemma 4.1 can be applied. □

Remark. *Note that Lemma 4.2 implies that $u'(a^+) = 0$ since $u'(x) = 0$ for all $x > a^+$, so by continuity of u', $u'(a^+) = 0$. It also implies that $\lim_{x\uparrow a^+} u''(x) > 0$, since $u \to 0$ as $x \to a^+$ implies that $u''' \to \infty$. But $\lim_{x\downarrow a^+} u''(x) = 0$, as $u''(x) = 0$ for all $x > a^+$. Therefore, u is not twice differentiable at $x = a^+$. The same result holds for a^-.*

4.2. Symmetric solutions

If the nonnegative solution u is symmetric, another property can be proven for u.

Lemma 4.3. *Let u be a nonnegative non-trivial solution of equation (3.1) and let $u(x) = u(-x)$ for all $x \in \mathbb{R}$. Then*

$$u'(x) < 0 \qquad \text{for } 0 < x < a^+.$$

Proof. Since $u(x) = u(-x)$, we have $u'(x) = -u'(-x)$. For $x = 0$, we obtain $u'(0) = -u'(0)$. Therefore, $u'(0) = 0$. Lemma 4.2 also implies that $u'(a^+) = 0$ as seen in the remark. Since $u > 0$ on $(0, a^+)$, $u''' > 0$ using equation (3.4). This means that u' is strictly convex. By Definition A.3

$$u'(\alpha x_1 + (1-\alpha)x_2) \le \alpha u'(x_1) + (1-\alpha)u'(x_2) \qquad \text{for all } x_1, x_2 \in [0, a^+], \quad \alpha \in [0,1]. \tag{4.2}$$

Suppose $u'(x)$ has roots on the interval $(0, a^+)$. Let c be the first root on that interval, so $u'(c) = 0$. If $x \in (0, c)$, x can be written as

$$x = \alpha \cdot 0 + (1-\alpha)c,$$

for some $\alpha \in [0,1]$. Using (4.2), we obtain

$$u'(x) \le \alpha u'(0) + (1-\alpha)u'(c) = 0.$$

If $x \in (c, a^+)$, x can be written as

$$x = \alpha c + (1-\alpha)a^+,$$

for some $\alpha \in [0,1]$. Therefore,

$$u'(x) \le \alpha u'(c) + (1-\alpha)u'(a^+) = 0.$$

This means that $u'(x) \le 0$ for all $x \in [0, a^+]$. Now, choose $x_0 \in (0, c)$ and $x_1 \in (c, a^+)$. This means that $u'(x_0) < 0$ and $u'(x_1) \le 0$ as c was the first root on the interval $(0, a^+)$. c can now be written as $c = \alpha x_0 + (1-\alpha)x_1$ for some $\alpha \in [0,1]$. By convexity

$$u'(c) \le \alpha u'(x_0) + (1-\alpha)u'(x_1) < 0.$$

This violates that $u'(c) = 0$. Therefore, u' must have a constant sign on $(0, a^+)$. Since $u(0) > 0$ and $u(a^+) = 0$, we obtain that $u'(x) < 0$ on $0 < x < a^+$. □

4.3. New formulation of the problem

For the existence of nonnegative solutions of equation (3.1) symmetry shall be assumed and later proved in chapter 6 for $0 < n \le 2$. Therefore, using all the results proven in this chapter, equation (3.1) can be reduced to the following: find a number $a > 0$ and a function $u \in C^1([0, a]) \cap C^3((0, a))$ such that

$$\begin{cases} u^{n-1}u''' = x & u > 0, \qquad \text{for } 0 < x < a, \\ u(0) = 1, & u'(0) = 0, \\ u(a) = 0, & u'(a) = 0. \end{cases} \tag{4.3}$$

To show that solving equation (4.3) leads to solving (3.1) and vice versa, both implications will be proven.

Suppose u is a solution of equation (4.3). If u is extended by $u(x) = 0$ for $x > a$ and reflected at $x = 0$, u will also be a solution of equation (3.1). If u is a nonnegative symmetric solution of equation (3.1), a scaling function is considered

$$u_h(x) = h^\alpha u\left(\frac{x}{h}\right). \tag{4.4}$$

To solve h, the initial condition is considered. Since $u_h(0) = 1$ is required to solve equation (4.3), the following result holds:

$$\begin{aligned} h^\alpha u(0) &= 1 = u_h(0) \\ \Longrightarrow h &= u(0)^{-\frac{1}{\alpha}}. \end{aligned}$$

To solve for α, (4.4) will be substituted into equation (4.3)

$$
\begin{aligned}
&u_h^{n-1} u_h''' = x \\
\Longrightarrow\ &h^{\alpha(n-1)+\alpha-3} u^{n-1}\left(\frac{x}{h}\right) u'''\left(\frac{x}{h}\right) = x \\
\Longrightarrow\ &h^{\alpha(n-1)+\alpha-3} u^{n-1}(y) u'''(y) = hy.
\end{aligned}
$$

Here, a change of variables is performed with $x = hy$. To match equation (4.3), it must hold that

$$h^{\alpha(n-1)+\alpha-3} = h.$$

Therefore, their power should be the same. This can be used to find α

$$
\begin{aligned}
&\alpha(n-1) + \alpha - 3 = 1 \\
\Longrightarrow\ &\alpha = \frac{4}{n}.
\end{aligned}
$$

Hence finding a nonnegative symmetric solution u of equation (3.1) gives

$$u_h(x) = h^{\frac{4}{n}} u\left(\frac{x}{h}\right), \qquad h = u(0)^{-\frac{n}{4}},$$

as scaling function. Using Lemma 4.2, the restriction of $u_h(x)$ to $[0, ha^+]$ is a solution of equation (4.3) with $a = ha^+$.

Therefore, since solving equation (4.3) also yields a solution for (3.1), and solving (3.1) in turn leads to a solution of equation (1.16), we will use equation (4.3) to establish the existence of a nonnegative symmetric solution u.

5

Existence

5.1. The shooting method

To establish the existence of a solution of equation (4.3), we employ a shooting method. Equation (4.3) is therefore converted into an initial value problem

$$\begin{cases} u''' = xu^{1-n} & u > 0, \quad x > 0, \\ u(0) = 1, & u'(0) = 0, \quad u''(0) = -\gamma, \end{cases} \tag{5.1}$$

where $\gamma > 0$ is a parameter that can be chosen appropriately. Now, for every $\gamma > 0$ there exists a local solution $u(x,\gamma)$ for equation (5.1) near $x = 0$ using the Picard-Lindelöf Theorem 2.1. This is because equation (5.1) can be written as a system of first order Ordinary Differential Equations (ODEs) and, since $u \approx 1$ near $x = 0$, the system is locally Lipschitz continuous, see Definition 2.3. Therefore, the theorem implies that there exists a unique local solution near $x = 0$. Now, the solution can be extended uniquely as long as the solution stays positive and does not blow up, since the system remains Lipschitz continuous. Set

$$\begin{aligned} a(\gamma) &= \sup\{x > 0 : u(\cdot,\gamma) > 0 \text{ on } [0,x)\}, \\ b(\gamma) &= \sup\{0 < x < a(\gamma) : u'(\cdot,\gamma) < 0 \text{ on } [0,x)\}. \end{aligned} \tag{5.2}$$

The question of existence is now reduced to finding a number γ_0 such that

$$u'(a(\gamma_0)) = 0, \tag{5.3}$$

where $u'(a(\gamma_0))$ denotes the limit of $u'(x)$ as $x \uparrow a(\gamma_0)$. The reason why this γ_0 still needs to be found is that, even though the unique solutions can be extended as long as $u > 0$ for all $\gamma > 0$, it is not known if these solutions satisfy Lemma 4.1 and Lemma 4.3. Therefore, the following sets are defined

$$\begin{aligned} S^+ &= \{\gamma > 0 : u'(\cdot,\gamma) = 0 \text{ before } u(\cdot,\gamma) = 0\}, \\ S^- &= \{\gamma > 0 : u(\cdot,\gamma) = 0 \text{ before } u'(\cdot,\gamma) = 0\}, \end{aligned} \tag{5.4}$$

or, in terms of $a(\gamma)$ and $b(\gamma)$

$$\begin{aligned} S^+ &= \{\gamma > 0 : b(\gamma) < a(\gamma)\}, \\ S^- &= \{\gamma > 0 : b(\gamma) = a(\gamma) \text{ and } u'(a(\gamma)) < 0\}. \end{aligned} \tag{5.5}$$

S^+ is a set that violates Lemma 4.3 and S^- is a set that violates Lemma 4.1, since $u(\cdot,\gamma) = 0$ before $u'(\cdot,\gamma) = 0$ implies that $u < 0$ for slightly higher values.Therefore, $\mathbb{R}^+ \setminus (S^+ \cap S^-) \neq \emptyset$ implies that there exists a number γ_0 such that (5.3) holds. Since the condition $xu \to 0$ as $x \to \infty$ implied that Lemma 4.1 and Lemma 4.3 hold for equation (3.1), an γ_0 can be found which mimics $xu \to 0$ as $x \to a(\gamma_0)^-$ which is not in S^+ or S^- and so it can be concluded from Lemma 4.2 that $a(\gamma_0) < \infty$. Therefore, the existence of a solution is found for equation (4.3). The following theorem will be proven:

Theorem 5.1. *Let S^+ and S^- be defined as in (5.4) and let* $0 < n < 3$. *Then* $\mathbb{R}^+ \setminus (S^+ \cap S^-)$ *is nonempty, since the following assertions hold:*

(a) $S^+ \cap S^-$ is empty.
(b) S^+ and S^- are nonempty.
(c) S^+ and S^- are open.

(a) is true by definition. To show that (b) and (c) hold, more mathematical arguments are required.

5.2. Proving S^+ is nonempty and open for $0 < n < 3$ and S^- is nonempty and open for $0 < n \leq 1$

Proving that S^+ is nonempty and open for $0 < n < 3$ and S^- is nonempty and open for $0 < n \leq 1$ will be done first. This is because solutions in S^+ are well defined on the interval of interest $[0, b(\gamma)]$ for $0 < n < 3$ as $u > 0$ on that interval. The same applies for S^- when $0 < n \leq 1$ as u''' does not blow up as $u \to 0$.

Lemma 5.2. *S^+ defined as in (5.4) is nonempty and open for* $0 < n < 3$.

Proof. To prove that S^+ is nonempty, we note that $u > 0$ for the whole interval up to and including $u'(\cdot, \gamma) = 0$. Taylor expansion around $x = 0$ implies

$$u(x) = u(0) + xu'(0) + \frac{x^2}{2}u''(0) + \frac{x^3}{6}u'''(\zeta),$$

where $u'''(\zeta)$ is the remainder term with $\zeta \in (0, x)$. Since $u''' > 0$ for $0 < x < a(\gamma)$, we obtain the following by filling in the initial conditions

$$u(x) \geq 1 - \frac{\gamma}{2}x^2. \tag{5.6}$$

Note that $u(x) \leq 1$ as long as $u'(x) \leq 0$. Also note that $u(x) > 0$ when $x < \sqrt{\frac{2}{\gamma}}$ by (5.6). For $0 < n \leq 1$, we have $u^{1-n} \geq u$ when $u \leq 1$. So

$$u''' = xu^{1-n} \geq xu \geq x(1 - \frac{\gamma}{2}x^2).$$

To simplify the equation, since $x \geq 0$, the following inequality is derived

$$\begin{aligned} & 1 - \frac{\gamma}{2}x^2 \geq \frac{1}{2} \\ \iff & -\frac{\gamma}{2}x^2 \geq -\frac{1}{2} \\ \iff & \frac{\gamma}{2}x^2 \leq \frac{1}{2} \\ \iff & x^2 \leq \frac{1}{\gamma} \\ \iff & x \leq \sqrt{\frac{1}{\gamma}}. \end{aligned}$$

This means that if $x \leq \sqrt{\frac{1}{\gamma}}$, then

$$u''' \geq x(1 - \frac{\gamma}{2}x^2) \geq \frac{1}{2}x.$$

Integrating this two times and using the initial values gives

$$u' \ge -\gamma x + \frac{1}{12}x^3. \tag{5.7}$$

The first positive root of (5.7) is

$$x'_{root} = \sqrt{12\gamma}.$$

To satisfy $x'_{root} \le \sqrt{\frac{1}{\gamma}}$ and so also $x'_{root} < \sqrt{\frac{2}{\gamma}}$, γ will be derived

$$\begin{aligned} \sqrt{12\gamma} &\le \sqrt{\frac{1}{\gamma}} \\ \implies 12\gamma &\le \frac{1}{\gamma} \\ \implies \gamma^2 &\le \frac{1}{12} \\ \implies \gamma &\le \sqrt{\frac{1}{12}}. \end{aligned}$$

And so there exists a $\gamma > 0$ such that $\gamma \in S^+$ for $0 < n \le 1$.

For $1 < n < 3$, we have $u^{1-n} \ge 1$ when $u \le 1$. So

$$u''' \ge x.$$

Integrating this two times and using the initial values gives

$$u' \ge -\gamma x + \frac{1}{6}x^3. \tag{5.8}$$

The first positive root of (5.8) is

$$x'_{root} = \sqrt{6\gamma}.$$

In order to satisfy $x'_{root} < \sqrt{\frac{2}{\gamma}}$, γ will be derived

$$\begin{aligned} \sqrt{6\gamma} &< \sqrt{\frac{2}{\gamma}} \\ \implies 6\gamma &< \frac{2}{\gamma} \\ \implies \gamma^2 &< \frac{1}{3} \\ \implies \gamma &< \sqrt{\frac{1}{3}}. \end{aligned}$$

Therefore, there exists a $\gamma > 0$ such that $\gamma \in S^+$ for $1 < n < 3$.

To show that S^+ is open, it suffices to show that for every $\gamma \in S^+$ there exists an $r > 0$ such that all $\hat{\gamma}$ with $|\gamma - \hat{\gamma}| < r$ implies $\hat{\gamma} \in S^+$, see Definition A.2. Let $\gamma \in S^+$. Since $u > 0$ for the whole interval up to and including $u'(\cdot, \gamma) = 0$, the unique local solution near $x = 0$ can be extended uniquely to this interval. Therefore, Theorem 2.2 states that the solution depends continuously on the initial values, including γ. By continuous dependence, slightly higher and lower values $\hat{\gamma}$ of γ have the same property that u' should be 0 before u, so $\hat{\gamma} \in S^+$. Hence S^+ is open for $0 < n < 3$. □

To prove that S^- is nonempty and open, we will split the proof into two parts. In the first part where $0 < n \leq 1$, equation (5.1) remains bounded when u approaches zero. In the second part where $1 < n < 3$, equation (5.1) becomes unbounded when u approaches zero. We will begin with the first part.

Lemma 5.3. *S^- defined as in (5.4) is nonempty and open for* $0 < n \leq 1$.

Proof. First, we will prove that S^- is open. In the set S^-, u will be zero before u'. The unique local solution can be extended while $u > 0$, but it will run into problems when $u = 0$, as xu^{1-n} will not be Lipschitz continuous. But since $0 < n \leq 1$, $u''' \to 0$ when $u \to 0$ using equation (5.1). This means that u''' stays bounded and u can be continuously extended to $u = 0$ at $x = a(\gamma)$. Therefore, continuous dependence also holds in this case. So if $\gamma \in S^-$, by continuous dependence, slightly higher and lower values $\hat{\gamma}$ of γ have the same property that u should be 0 before u', so $\hat{\gamma} \in S^-$. Hence S^- is open for $0 < n \leq 1$.

To show that S^- is nonempty, we note that $u(x) \leq 1$ as long as $u'(x) \leq 0$ and since $0 < n \leq 1$,

$$u''' = xu^{1-n} \leq x.$$

Integrating this two times and using the initial values gives

$$u'(x,\gamma) \leq -\gamma x + \frac{1}{6}x^3. \tag{5.9}$$

Integrating this again gives

$$u(x,\gamma) \leq 1 - \frac{\gamma}{2}x^2 + \frac{1}{24}x^4. \tag{5.10}$$

The first positive root of the right-hand side of (5.10) is

$$x_{root} = \sqrt{6\gamma - \sqrt{36\gamma^2 - 24}}.$$

If $\gamma = \sqrt{\frac{2}{3}}$, the term $36\gamma^2 - 24$ becomes greater or equal to 0. The only root when $x > 0$ of the right-hand side of (5.9) is

$$x'_{root} = \sqrt{6\gamma}.$$

For u' to remain negative in the interval $[0, x_{root}]$, it must hold that $x_{root} < x'_{root}$. This happens when $\gamma > \sqrt{\frac{2}{3}}$. Therefore, $\left(\sqrt{\frac{2}{3}}, \infty\right) \subset S^-$. Hence S^- is nonempty for $0 < n \leq 1$. □

5.3. Analyzing $a(\gamma)$ and $b(\gamma)$

It remains to show that S^- is nonempty and open when $1 < n < 3$. Since u''' blows up when $u \to 0$, another method is needed to prove this. For this, equation (5.1) will be rewritten to a second order ordinary differential equation. Before this is done, $a(\gamma)$ and $b(\gamma)$ will be analyzed more closely which will help later on when the second order ordinary differential equation is introduced.

Lemma 5.4. *Let $a(\gamma)$ and $b(\gamma)$ be defined as in (5.2) and* $0 < n < 3$.
(a) If $b(\gamma) < a(\gamma)$, then $a(\gamma) = \infty$.
(b) If $a(\gamma) = \infty$ for all $\gamma > 0$, then $b(\gamma) \to 0$ as $\gamma \to 0$ and $\sqrt{\gamma}b(\gamma) \to \sqrt{2}$ as $\gamma \to \infty$.

Proof. For (a), suppose that $b(\gamma) < a(\gamma)$. Then $u(b(\gamma)) > 0$ and $u'(b(\gamma)) = 0$. Since u' is negative for $0 < x < b(\gamma)$, we obtain that $u''(b(\gamma)) \geq 0$. Now, $u''' > 0$ using equation (5.1) and therefore $u'' > 0$ for all $x > b(\gamma)$ which results in $u' > 0$ for all $x > b(\gamma)$. So u will be strictly increasing and will never approach 0. Thus, $a(\gamma) = \infty$.

For (b), suppose that $a(\gamma) = \infty$ for all $\gamma > 0$. This means that $u > 0$ for all $x > 0$. So the unique local solution near $x = 0$ can be extended on the whole interval $x > 0$ and therefore continuous dependence on compact subsets holds on the whole interval by Theorem 2.2. For $\gamma = 0$, since $u''' > 0$ for all $x > 0$, we obtain $u'' > 0$ for all $x > 0$. Therefore, u' will also be strictly positive for all $x > 0$ and u' will only hit zero at $x = 0$. In this case $b(\gamma) = 0$. Hence by continuous dependence, $b(\gamma) \to 0$ as $\gamma \to 0$.

To prove $\sqrt{\gamma} b(\gamma) \to \sqrt{2}$ as $\gamma \to \infty$, a scaling argument is used. Set $\xi = \sqrt{\gamma} x$, then

$$u_\gamma(\xi) = u(x) = u\left(\frac{\xi}{\sqrt{\gamma}}\right).$$

If u is a solution to equation (5.1), then u_γ can be substituted into equation (5.1), where x becomes $\frac{\xi}{\sqrt{\gamma}}$. So u_γ is a solution of

$$\begin{cases} u_\gamma''' = \gamma^{-2} \xi u_\gamma^{1-n} & u_\gamma > 0, \quad \xi > 0, \\ u_\gamma(0) = 1, & u_\gamma'(0) = 0, \quad u_\gamma''(0) = -1. \end{cases}$$

Note that for large γ we have $u_\gamma''' \approx 0$. So the limiting function at "$\gamma = \infty$" is, by integrating $u_\gamma''' = 0$ three times and using the initial conditions

$$\bar{u}(\xi) = 1 - \frac{\xi^2}{2}. \tag{5.11}$$

This equation has a positive root $\sqrt{2}$, but since the assumption is that $a(\gamma) = \infty$, the function is not well defined anymore on $[\sqrt{2}, \infty)$. So analysis is done on $[0, \sqrt{2})$. We will now show that $u_\gamma \to \bar{u}$ as $\gamma \to \infty$ uniformly in $C^3(K)$ for any compact set $K = [0, b] \subset [0, \sqrt{2})$, meaning that up to and including the third derivative converges uniformly (this is called a compactness argument). It suffices to show that for every sequence $\gamma_m \to \infty$, we have that $u_{\gamma_m} \to \bar{u}$ in $C^3(K)$ uniformly. Let $\{\gamma_m\}$ be a sequence that goes to ∞. The Arzelà-Ascoli Theorem 2.3 will be used to show that there exists a subsequence γ_{m_k} such that $u_{\gamma_{m_k}}$ converges uniformly to $\bar{u}$ in $C^3(K)$. To use this theorem, it must be shown that u_{γ_m}, u'_{γ_m}, u''_{γ_m} and u'''_{γ_m} are uniformly bounded and equicontinuous on the compact interval K, see Definition 2.1 and Definition 2.2.

To show this, we note that $u_{\gamma_m}^{1-n}$ is bounded for all $\xi > 0$ and $m \in \mathbb{N}$ as $u_{\gamma_m} > 0$. So on the interval K, there exists an $M > 0$ such that $|u_{\gamma_m}^{1-n}| \le M$ for all $\xi \in K$ and $m \in \mathbb{N}$. Now, $\gamma_m > 0$ for all $m \in \mathbb{N}$, so there exists a $D > 0$ such that $\gamma_m^{-2} \le D$. Therefore, $|u'''_{\gamma_m}| \le DbM < \infty$ for all $\xi \in K = [0, b]$ and $m \in \mathbb{N}$. Hence u'''_{γ_m} is uniformly bounded. To show that u''_{γ_m} is uniformly bounded on K, u'''_{γ_m} is integrated and its bound is used

$$\int_0^\xi u'''_{\gamma_m}\, ds = \int_0^\xi \gamma_m^{-2} s u_{\gamma_m}^{1-n}\, ds \le \int_0^\xi DbM\, ds$$
$$\implies u''_{\gamma_m}(\xi) \le u''_{\gamma_m}(0) + DbM\xi.$$

This implies that $|u''_{\gamma_m}| \le |-1 + Db^2M|$ for all $\xi \in K$ and $m \in \mathbb{N}$. So u''_{γ_m} is also uniformly bounded. Using the same argument, it can also be shown that u'_{γ_m} and u_{γ_m} are uniformly bounded.

Next, we will show that u'''_{γ_m} is equicontinuous. For this, the derivative is taken of u'''_{γ_m}

$$u''''_{\gamma_m} = \frac{d}{d\xi}(\gamma_m^{-2} \xi u_{\gamma_m}^{1-n}) = \gamma_m^{-2} u_{\gamma_m}^{1-n} + \gamma_m^{-2} \xi u_{\gamma_m}^{-n} u'_{\gamma_m}.$$

Since $u_{\gamma_m} > 0$ and $u'_{\gamma_m} > 0$ are uniformly bounded, this means that u''''_{γ_m} is also uniformly bounded by a number $A > 0$. Using the Mean Value Theorem A.2 and uniform boundedness of u''''_{γ_m}, for each $\xi_1, \xi_2 \in K$ with $\xi_1 \neq \xi_2$ there exists a $c \in (\xi_1, \xi_2)$ such that

$$|u'''_{\gamma_m}(\xi_1) - u'''_{\gamma_m}(\xi_2)| = |u''''(c)||\xi_1 - \xi_2| \le A|\xi_1 - \xi_2|.$$

This implies that u'''_{γ_m} is Lipschitz continuous and therefore also equicontinuous. Using the same argument with the Mean Value Theorem and uniform boundedness for u''_{γ_m}, u'_{γ_m} and u_{γ_m}, we can conclude that they are also equicontinuous.

Now, since u_{γ_m}, u'_{γ_m}, u''_{γ_m} and u'''_{γ_m} are uniformly bounded and equicontinuous on the compact interval K, the Arzelà-Ascoli Theorem can be applied and so there exists a subsequence $\{\gamma_{m_k}\}$ such that $u_{\gamma_{m_k}} \to \bar{u}$ in $C^3(K)$ uniformly. Note that, since $\gamma_m \to \infty$, it means that for every subsequence $\{\gamma_{m_l}\}$, we have $\gamma_{m_l} \to \infty$. Using the same arguments, it implies that for all subsequences $\{\gamma_{m_l}\}$, there exists a further subsequence $\{\gamma_{m_{l_k}}\}$ such that $u_{\gamma_{m_{l_k}}} \to \bar{u}$ in $C^3(K)$ uniformly. So by Theorem A.6, $u_{\gamma_m} \to \bar{u}$ in $C^3(K)$ uniformly. Since the sequence $\{\gamma_m\}$ was taken arbitrarily, it holds for all sequences and therefore $u_\gamma \to \bar{u}$ in $C^3(K)$ uniformly as $\gamma \to \infty$.

To prove $\lim_{\gamma\to\infty} \sqrt{\gamma}b(\gamma) = \sqrt{2}$, it will be shown that $\liminf_{\gamma\to\infty} \sqrt{\gamma}b(\gamma) \geq \sqrt{2}$ and $\limsup_{\gamma\to\infty} \sqrt{\gamma}b(\gamma) \leq \sqrt{2}$. Define ξ_0 as $u'_\gamma(\xi_0) = 0$. For $\liminf_{\gamma\to\infty} \sqrt{\gamma}b(\gamma) \geq \sqrt{2}$, we note that $\bar{u}' = -\xi$ which is always negative for $\xi > 0$. Since $u_\gamma \to \bar{u}$ uniformly in $C^3(K)$ for any compact set $K = [0, b]$ with $b < \sqrt{2}$, it follows that for all $\epsilon > 0$ and all $\xi \in [0, \sqrt{2})$, there exists a γ^* such that for all $\gamma > \gamma^*$, $|u_\gamma(\xi) - \bar{u}(\xi)| < \epsilon$ and $|u'_\gamma(\xi) - \bar{u}'(\xi)| < \epsilon$. Therefore, for sufficiently large γ and $\xi < \sqrt{2}$, we have $u_\gamma \approx \bar{u} = 1 - \frac{\xi^2}{2} > 0$ and $u'_\gamma \approx \bar{u}' = -\xi$. The derivative for large γ is negative when $\xi \in (0, \sqrt{2})$. As $\bar{u}$ is well defined on $\xi \in [0, \sqrt{2})$, it means that for large γ, the turning point for u'_γ becoming nonnegative is at least on $\xi_0 \geq \sqrt{2}$. So $\liminf_{\gamma\to\infty} \xi_0 = \sqrt{\gamma}b(\gamma) \geq \sqrt{2}$.

To show $\limsup_{\gamma\to\infty} \sqrt{\gamma}b(\gamma) \leq \sqrt{2}$, we claim that for a fixed $\delta \in (0, \sqrt{2})$

$$\liminf_{\gamma\to\infty} u'_\gamma(\sqrt{2}+\delta) \geq 0. \tag{5.12}$$

Suppose this is not the case, so $\liminf_{\gamma\to\infty} u'_\gamma(\sqrt{2}+\delta) < 0$. Let $\{\gamma_n\}$ be the sequence where this holds. The value $\xi = \sqrt{2}-\delta$ is on the interval $[0, \sqrt{2})$ and so $u'_{\gamma_n}(\sqrt{2}-\delta) \to \bar{u}'(\sqrt{2}-\delta) = -\sqrt{2}+\delta < 0$ as $\gamma_n \to \infty$. Also for large n, $u'_{\gamma_n}(\sqrt{2}+\delta) < 0$ by the contradiction assumption. So for large n, as u'' is monotone increasing

$$u'_{\gamma_n} \leq -C < 0 \qquad \text{on } (\sqrt{2}-\delta, \sqrt{2}+\delta).$$

Now, even though $\bar{u}$ is only well defined as a limiting function on the interval $[0, \sqrt{2})$, u_{γ_n} approaches 0 if $\xi \to \sqrt{2}$ for large n. And since u_{γ_n} is well defined for all $\xi > 0$, it means that, since u_{γ_n} is strictly decreasing on the interval $(\sqrt{2}-\delta, \sqrt{2}+\delta)$, u_{γ_n} will become negative. This contradicts that $a(\gamma) = \infty$ and so (5.12) must hold. This does not prove yet that $\limsup_{\gamma\to\infty} \sqrt{\gamma}b(\gamma) \leq \sqrt{2}$ as $\liminf_{\gamma\to\infty} u'_\gamma(\sqrt{2}+\delta)$ can be 0 and so there exists a subsequence $\{\gamma_n\}$ where $u'_{\gamma_n}(\sqrt{2}+\delta)$ approaches zero, but may never actually reach zero. So $\liminf_{\gamma\to\infty} u'_\gamma(\sqrt{2}+\delta) > 0$ is needed.

Note that u''_γ is increasing and so by the Mean Value Theorem there exists a $c \in (\sqrt{2}-\delta, \sqrt{2}+\delta)$ such that

$$\frac{u'_\gamma(\sqrt{2}+\delta) - u'_\gamma(\sqrt{2}-\delta)}{(\sqrt{2}+\delta-(\sqrt{2}-\delta))} = u''_\gamma(c) \leq u''_\gamma(\sqrt{2}+\delta).$$

Then for sufficiently large γ and using (5.12), $u'_\gamma(\sqrt{2}-\delta) \approx \bar{u}'(\sqrt{2}-\delta) = -\sqrt{2}+\delta$ and $u'_\gamma(\sqrt{2}+\delta) \geq 0$. This implies

$$\liminf_{\gamma\to\infty} u''_\gamma(\sqrt{2}+\delta) \geq \frac{\sqrt{2}-\delta}{2\delta}.$$

As u''_γ is increasing, for large γ and $\xi \geq \sqrt{2}+\delta$ it holds that

$$u''_\gamma(\xi) \geq u''_\gamma(\sqrt{2}+\delta) \geq \frac{\sqrt{2}-\delta}{2\delta}.$$

Integrating u''_γ on the interval $(\sqrt{2}+\delta, \sqrt{2}+2\delta)$ for large γ gives

$$\int_{\sqrt{2}+\delta}^{\sqrt{2}+2\delta} u''_\gamma \, ds \geq \int_{\sqrt{2}+\delta}^{\sqrt{2}+2\delta} \frac{\sqrt{2}-\delta}{2\delta} \, ds$$
$$\implies u'_\gamma(\sqrt{2}+2\delta) \geq u'_\gamma(\sqrt{2}+\delta) + \frac{\sqrt{2}-\delta}{2} \geq \frac{\sqrt{2}-\delta}{2}.$$

Therefore,

$$\liminf_{\gamma\to\infty} u'_\gamma(\sqrt{2}+2\delta) \geq \frac{\sqrt{2}-\delta}{2} > 0.$$

This means that u' must cross $u'=0$ somewhere on the interval $(\sqrt{2}, \sqrt{2}+2\delta)$ for a large enough γ as u'' is monotone increasing. So $\xi_0 < \sqrt{2}+2\delta$ for large γ. Since δ may be chosen arbitrarily small, we conclude that $\limsup_{\gamma\to\infty} \xi_0 = \sqrt{\gamma} b(\gamma) \leq \sqrt{2}$.

Since $\liminf_{\gamma\to\infty} \sqrt{\gamma} b(\gamma) \geq \sqrt{2}$ and $\limsup_{\gamma\to\infty} = \sqrt{\gamma} b(\gamma) \leq \sqrt{2}$, we have proven that $\lim_{\gamma\to\infty} \sqrt{\gamma} b(\gamma) = \sqrt{2}$. □

With Lemma 5.4, a bound can be formulated for $b(\gamma)$.

Lemma 5.5. *Let $a(\gamma)$ and $b(\gamma)$ be defined as in (5.2) and $0 < n < 3$. Define $I = \{\gamma > 0 : a(\gamma) = \infty\}$. Then*

$$M_1 = \sup_{\gamma\in I} b(\gamma) < \infty.$$

Proof. Suppose to the contrary that there exists a sequence $\{\gamma_k\} \subset I$ such that $b(\gamma_k) \to \infty$ as $k \to \infty$. Now if $\gamma_k \to \infty$, by Lemma 5.4(b), $\sqrt{\gamma_k} b(\gamma_k) \to \sqrt{2}$. But if $b(\gamma_k) \to \infty$, it must mean that $\gamma_k \to 0$ contradicting $\gamma_k \to \infty$. Furthermore, if $\gamma_k \to 0$, by Lemma 5.4(b), it must mean that $b(\gamma) \to 0$, contradicting $b(\gamma_k) \to \infty$. So $\{\gamma_k\}$ must be bounded and stays away from zero. The Bolzano-Weierstrass Theorem A.5 then states that $\{\gamma_k\}$ has a convergent subsequence $\{\gamma_{k_l}\}$ which converges to $\gamma^* > 0$ and also $b(\gamma_{k_l}) \to \infty$ as $l \to \infty$. By the same logic as the compactness argument in Lemma 5.4(b) where $a(\gamma) = \infty$, we have

$$u(\cdot, \gamma_{k_l}) \to u(\cdot, \gamma^*) \qquad \text{as } k \to \infty \qquad \text{in } C^3(K) \text{ uniformly},$$

where K is any compact subset of $[0, a(\gamma^*))$, since for $x \geq a(\gamma^*)$ the solution crosses $u = 0$ and leads to irregularities for u'''. This means that on K, $b(\gamma_{k_l}) \to b(\gamma^*)$ as $l \to \infty$ and so $b(\gamma^*) = \infty$. Suppose that $a(\gamma^*) = \infty$, then $\gamma^* \in I$. An argument similar as in the proof of Lemma 4.2 will be used. As $u'''(\cdot, \gamma^*) > 0$ for all $x > 0$, it means that $u''(\cdot, \gamma^*)$ is strictly increasing for all $x > 0$, so $\lim_{x\to\infty} u''(x, \gamma^*) = L$ exists. If $L > 0$, it would mean that $\lim_{x\to\infty} u(x, \gamma^*) = \infty$ and so $u'(\cdot, \gamma^*)$ must have switched from negative to positive in finite time and so $b(\gamma^*) < \infty$, which is a contradiction. If $L < 0$, $\lim_{x\to\infty} u(x, \gamma^*) = -\infty$. So $u(\cdot, \gamma^*)$ crosses 0 in finite time, contradicts $a(\gamma^*) = \infty$. So $L = 0$. As u'' is strictly increasing for all $x > 0$, this would mean that $u'' < 0$ for all $x > 0$. So u' is strictly decreasing for all $x > 0$. The proof of Lemma 4.2 then shows that $a(\gamma^*)$ should be finite and so contradicting $a(\gamma^*) = \infty$.

Since $a(\gamma^*) < \infty$, using the same argument steps as in Lemma 5.4(b), it can be shown that $b(\gamma_{k_l}) \to a(\gamma^*)$ as $l \to \infty$. But this would mean that since $b(\gamma_{k_l}) \to b(\gamma^*)$ as $l \to \infty$, $b(\gamma^*) < \infty$, which contradicts that $b(\gamma^*) = \infty$. Hence $\sup_{\gamma\in I} b(\gamma) < \infty$. □

A bound for $a(\gamma)$ is now also formulated.

Lemma 5.6. *Let $a(\gamma)$ and $b(\gamma)$ be defined as in (5.2) and $0 < n < 3$. Define $J = \{\gamma > 0 : a(\gamma) < \infty\}$. Then*

$$M_2 = \sup_{\gamma\in J} a(\gamma) < \infty.$$

Proof. Suppose to the contrary that there exists a sequence $\{\gamma_k\} \subset J$ such that $a(\gamma) \to \infty$ as $k \to \infty$. It will be shown that γ_k is bounded and stays away from zero. If $\gamma_k \to 0$ we note that, since it was proved that S^+ is open and nonempty in Lemma 5.2 for small γ, it would mean that for large k, $b(\gamma_k) < a(\gamma_k)$. Lemma 5.4(a) implies that $a(\gamma_k) = \infty$. By definition, $a(\gamma_k) < \infty$ for all k, so this is not possible. If $\gamma_k \to \infty$, the same scaling argument as in Lemma 5.4(b) can be used. We now note that u_{γ_k} is defined on $[0, \sqrt{\gamma_k} a(\gamma_k))$ instead of $[0, \infty)$. Since $\gamma_k \to \infty$, using the same compactness argument, the solution u_{γ_k} converges uniformly to the same limiting function (5.11) in $C^3(K)$, where $K = [0, b] \subset [0, \sqrt{2})$. Since $\bar{u}$ has a zero on $\sqrt{2}$, it means that $\sqrt{\gamma_k} a(\gamma_k) \to \sqrt{2}$ as $\gamma_k \to \infty$. But that would imply that $a(\gamma_k) \to 0$, which contradicts that $a(\gamma_k) \to \infty$. Therefore, $\{\gamma_k\}$ must be bounded and stays away from zero.

The Bolzano-Weierstrass Theorem A.5 states that $\{\gamma_k\}$ has a convergent subsequence $\{\gamma_{k_l}\}$ which converges to $\gamma^* > 0$ and also $a(\gamma_{k_l}) \to \infty$ as $l \to \infty$. So again by the compactness argument in Lemma 5.4(b)

$$u(\cdot,\gamma_{k_l}) \to u(\cdot,\gamma^*) \qquad \text{as } k \to \infty \qquad \text{in } C^3(K) \text{ uniformly,}$$

where K is any compact subset of $[0, a(\gamma^*))$. As $a(\gamma_{k_l}) \to \infty$, it means that $u(\cdot,\gamma^*) > 0$ for all $x > 0$. So $a(\gamma^*) = \infty$. As in the proof of Lemma 5.5, $a(\gamma^*) = \infty$ leads to a contradiction because $\lim_{x\to\infty} u''(x,\gamma^*) = L$ exists and should be $L = 0$. This means that $u'' < 0$ for all $x > 0$, since u'' is strictly increasing. The proof of Lemma 4.2 then shows that $a(\gamma^*)$ should be finite and so contradicting $a(\gamma^*) = \infty$. Therefore, there cannot exist such a sequence and hence $\sup_{\gamma\in J} a(\gamma) < \infty$. □

5.4. Reformulating equation (5.1)

Now that these bounds are proven, equation (5.1) will be rewritten into a second order ordinary differential equation. u will be an independent variable, while $(u')^2$ becomes a dependent variable. This is possible as $u''(0) = -\gamma < 0$, so $u' < 0$ initially. This means that u is strictly decreasing on the interval $[0,\xi]$, where $\xi = \min\{a(\gamma), b(\gamma)\}$. Set $\theta = u(\xi,\gamma)$. Then $\theta = 0$ if $a(\gamma) = b(\gamma)$ and $0 < \theta < 1$ if $a(\gamma) > b(\gamma)$. As u is strictly monotone and continuous on the interval interval $[0,\xi]$ ($[0,\xi)$ if $\theta = 0$), we obtain that u is bijective on that interval. So an inverse can be defined $\sigma : [\theta,1] \to [0,\xi]$ where

$$\sigma(u(x,\gamma)) = x \qquad \text{for } 0 \le x \le \xi. \tag{5.13}$$

Note that the σ is defined on $(\theta,1] \to [0,\xi)$ if $\theta = 0$. σ has the following properties

$$\sigma(u) > 0, \qquad \sigma'(u) < 0 \qquad \text{for } \theta < u < 1 \qquad \text{and } \sigma(1) = 0. \tag{5.14}$$

Next, define $y : [\theta,1] \to [0,\infty)$ by

$$y(u,\gamma) = (u'(\sigma(u),\gamma))^2. \tag{5.15}$$

As $\sigma(u(x,\gamma)) = x$ and $u' < 0$ on $[0,\xi]$, taking the square root on both sides gives

$$u'(x,\gamma) = -\sqrt{y(u,\gamma)}. \tag{5.16}$$

Differentiating this twice using the chain rule and (5.15) and (5.16) gives

$$u''(x,\gamma) = \frac{1}{2}y'(u,\gamma), \qquad u'''(x,\gamma) = -\frac{1}{2}\sqrt{y(u,\gamma)}y''(u,\gamma). \tag{5.17}$$

If u is a solution of (5.1), substituting these new variables gives

$$\begin{cases} y'' + \phi(u)y^{-\frac{1}{2}} = 0 & \theta < u < 1, \\ y(1,\gamma) = 0, & y'(1,\gamma) = -2\gamma, \end{cases} \tag{5.18}$$

where

$$\phi(u) = 2u^{1-n}\sigma(u).$$

For the boundary conditions we have $u = 1$ at $x = 0$. So using (5.17)

$$\begin{aligned} u(0,\gamma) = 1 &\implies \sigma(1) = 0, \\ u'(0,\gamma) = 0 &\implies y(1,\gamma) = (u'(\sigma(1),\gamma))^2 = (u'(0,\gamma))^2 = 0, \\ u''(0,\gamma) = -\gamma \text{ and } u''(0) = \frac{1}{2}y'(1,\gamma) &\implies y'(1,\gamma) = -2\gamma. \end{aligned}$$

The sets S^+ and S^- can now also be reformulated. Since θ was defined as $\theta = u(\xi,\gamma)$, if $\theta > 0$, it means that $u(\xi,\gamma) > 0$ while $u'(\xi,\gamma) = 0$. So all γ with this property are in S^+. If $\theta = 0$ and $\lim_{u\to 0^+} y(u,\gamma) > 0$, it means by (5.16) that $\lim_{x\to\xi^-} u'(x,\gamma) < 0$ while $\lim_{x\to\xi^-} u(x,\gamma) = 0$. So if γ has this property, it means that it is in S^-. The desired solution is therefore characterized as $\theta = 0$ and $\lim_{u\to 0^+} y(u,\gamma) = 0$. Even though $\theta = u(\xi,\gamma)$ is not a fixed boundary, we have more structural information available at θ. Equation (5.18) is therefore easier to work with as we now also have a second order ordinary and we can obtain bounds easier for the function y However, it should be noted that the function $\sigma(u)$, which appears in the coefficient $\phi(u)$, is not given explicitly. It depends implicitly on the original solution $u(x,\gamma)$ through inversion. Therefore, the equation retains some complexity due to the implicit nature of σ.

Before equation (5.18) is used to prove S^- is nonempty and open for $1 < n < 3$, we proceed to derive bounds for y.

Lemma 5.7. *Let y be a solution of equation* (5.18) *and let* $0 < n < 3$. *Then*

$$y(u,\gamma) < 2\gamma(1-u) \qquad \textit{for } \theta < u < 1.$$

Proof. Since $\sigma(u) > 0$ on $(\theta,1)$ by (5.14), it follows that $\phi(u) > 0$ on $(\theta,1)$. Also, $u' < 0$ on $(\theta,1)$ and so $\phi(u)y^{-\frac{1}{2}} > 0$. Using equation (5.18), we have $y'' < 0$ on $(\theta,1)$. So y' is strictly decreasing. Therefore,

$$y'(u,\gamma) > y'(1,\gamma) = -2\gamma \qquad \text{on } \theta < u < 1.$$

Integrating this over $(u,1)$ with $u > \theta$ gives

$$\begin{aligned}
&\int_u^1 y'(s,\gamma)\,ds > \int_u^1 -2\gamma\,ds \\
\implies &y(1,\gamma) - y(u,\gamma) > -2\gamma(1-u) \\
\implies &y(u,\gamma) < 2\gamma(1-u).
\end{aligned}$$

Where $y(1,\gamma) = 0$ using the boundary condition of equation (5.18). Therefore, the strict inequality holds on $(\theta,1)$. □

The following two functionals will now be defined

$$\begin{aligned}
E(u) &= \frac{1}{2}(y')^2 + 2\phi(u)\sqrt{y}, \\
G(u) &= \frac{(y')^2}{2\phi(u)} + 2\sqrt{y}.
\end{aligned} \tag{5.19}$$

If y is a solution of equation (5.18), then

$$\phi'(u) = -2(n-1)u^{-n}\sigma(u) + 2u^{1-n}\sigma'(u).$$

Note that $\phi'(u) < 0$ on $(\theta,1)$, since $\sigma'(u) < 0$ on $(\theta,1)$. Computing the derivatives of E and G gives, using equation (5.18) for y'' and the chain rule,

$$
\begin{aligned}
E'(u) &= (y')y'' + 2\phi'(u)\sqrt{y} + y'\phi(u)y^{-\frac{1}{2}} \\
&= y'(-\phi(u)y^{-\frac{1}{2}}) + 2\phi'(u)\sqrt{y} + y'\phi(u)y^{-\frac{1}{2}} \\
&= 2\phi'(u)\sqrt{y}, \\
G'(u) &= \frac{4(y')y''\phi(u) - 2(y')^2\phi'(u)}{4(\phi(u))^2} + y'y^{-\frac{1}{2}} \\
&= \frac{2(y')(-\phi(u)y^{-\frac{1}{2}})\phi(u) - (y')^2\phi'(u)}{2(\phi(u))^2} + y'y^{-\frac{1}{2}} \\
&= -y'y^{-\frac{1}{2}} - \frac{(y')^2\phi'(u)}{2(\phi(u))^2} + y'y^{-\frac{1}{2}} \\
&= -\frac{(y')^2\phi'(u)}{2(\phi(u))^2}.
\end{aligned}
\tag{5.20}
$$

As $\phi'(u) < 0$ on $(\theta, 1)$, we obtain

$$
E'(u) < 0 \qquad \text{and} \qquad G'(u) > 0 \qquad \text{for } \theta < u < 1. \tag{5.21}
$$

Using this, the following bound can be proven:

Lemma 5.8. *Let y be a solution of equation* (5.18) *and let* $0 < n < 3$. *Then*

$$
(y')^2 + 4\phi(u)\sqrt{y} > 4\gamma^2 \qquad \textit{for } \theta < u < 1.
$$

Proof. Since $E'(u) < 0$ on $(\theta, 1)$, it means that $E(u)$ is strictly decreasing on $(\theta, 1)$. Now using the boundary conditions of equation (5.18)

$$
E(1) = \frac{1}{2}(y'(1,\gamma))^2 + 2\phi(1)\sqrt{y(1,\gamma)} = \frac{(-2\gamma)^2}{2} + 2 \cdot 0 \cdot \sqrt{0} = 2\gamma^2.
$$

So on $(\theta, 1)$ we have

$$
E(u) = \frac{1}{2}(y')^2 + 2\phi(u)\sqrt{y} > E(1) = 2\gamma^2.
$$

Therefore, on $(\theta, 1)$ we obtain

$$
(y')^2 + 4\phi(u)\sqrt{y} > 4\gamma^2.
$$

□

5.5. Proving S^- is nonempty and open for $1 < n < 3$

With equation (5.18) and the bounds that were proven for y, $b(\gamma)$, and $a(\gamma)$, we can finally be prove that S^- is nonempty and open for $1 < n < 3$.

Lemma 5.9. *S^- defined as in* (5.4) *is nonempty for* $1 < n < 3$.

Proof. Using Lemma 5.8 on $\theta < u < 1$

$$
\begin{aligned}
&(y')^2 + 4\phi(u)\sqrt{y} > 4\gamma^2 \\
\Longrightarrow\ &(y')^2 > 4\gamma^2 - 4\phi(u)\sqrt{y} \\
\Longrightarrow\ &(y')^2 > 4\gamma^2 - 8u^{1-n}\sigma(u)\sqrt{y}.
\end{aligned}
\tag{5.22}
$$

Note that $\sigma(u)$ is a strictly decreasing function as u is also strictly decreasing. This means that the highest value is $\sigma(\theta) = \xi$, where $\xi = \min\{a(\gamma), b(\gamma)\}$. If $a(\gamma) = \infty$, using Lemma 5.5, it means that $\xi \leq M_1$. If $a(\gamma) < \infty$, using Lemma 5.6, it means that $\xi \leq M_2$. Therefore, $\sigma(u) \leq M < \infty$ on $\theta < u < 1$, where $M = \max\{M_1, M_2\}$. This means that $-\sigma(u) \geq -M$.

Using Lemma 5.7 on $\theta < u < 1$ we get

$$\begin{aligned} & y(u,\gamma) < 2\gamma(1-u) \\ \Longrightarrow & \sqrt{y(u,\gamma)} < \sqrt{2\gamma(1-u)} \\ \Longrightarrow & -\sqrt{y(u,\gamma)} > -\sqrt{2\gamma(1-u)} \\ \Longrightarrow & -\sqrt{y(u,\gamma)} > -\sqrt{2\gamma}. \end{aligned} \tag{5.23}$$

Filling this in (5.22) gives

$$\begin{aligned} (y')^2 &> 4\gamma^2 - 8u^{1-n}\sigma(u)\sqrt{y} \\ &> 4\gamma^2 - 8Mu^{1-n}\sqrt{2\gamma} \\ &= 4\gamma^2\left(1 - \left(\frac{2}{\gamma}\right)^{\frac{3}{2}} Mu^{1-n}\right). \end{aligned} \tag{5.24}$$

Now, to see for which values the term $4\gamma^2$ dominates, the inequality

$$\left(\frac{2}{\gamma}\right)^{\frac{3}{2}} Mu^{1-n} < \frac{1}{2}$$

is solved. Therefore, since $1 < n < 3$

$$\begin{aligned} & \left(\frac{2}{\gamma}\right)^{\frac{3}{2}} Mu^{1-n} < \frac{1}{2} \\ \Longleftrightarrow & u^{1-n} < \frac{1}{2M}\left(\frac{\gamma}{2}\right)^{\frac{3}{2}} \\ \Longleftrightarrow & u > \left(\frac{\gamma^{\frac{3}{2}}}{2^{\frac{5}{2}}M}\right)^{\frac{1}{1-n}} \\ \Longleftrightarrow & u > (2^{-\frac{5}{2}}M^{-1})^{\frac{1}{1-n}}\gamma^{\frac{3}{2(1-n)}} \\ \Longleftrightarrow & u > (2^{\frac{5}{2}}M)^{\frac{1}{n-1}}\gamma^{-\frac{3}{2(n-1)}}. \end{aligned}$$

Define

$$A = (2^{\frac{5}{2}}M)^{\frac{1}{n-1}}, \qquad v = \frac{3}{2(n-1)}, \qquad u_0(\gamma) = A\gamma^{-v}. \tag{5.25}$$

Note that $u_0(\gamma) \to 0$ as $\gamma \to \infty$. So for large γ, filling this in (5.24) on $u_0(\gamma) < u < 1$ gives

$$(y')^2 > 4\gamma^2\left(1 - \left(\frac{2}{\gamma}\right)^{\frac{3}{2}} Mu^{1-n}\right) > 2\gamma^2. \tag{5.26}$$

On $u_0(\gamma) < u < 1$ we have

$$\begin{aligned} & (y')^2 > 2\gamma^2 \\ \Longrightarrow & |y'| > \sqrt{2}\gamma \\ \Longrightarrow & y' < -\sqrt{2}\gamma \quad \text{or} \quad y' > \sqrt{2}\gamma. \end{aligned} \tag{5.27}$$

If $y' > \sqrt{2}\gamma$, we would have that $y' > 0$ on $u_0(\gamma) < u < 1$. But then $\lim_{u \to 1^-} y'(u,\gamma) \neq -2\gamma$. So by continuity of y', this cannot hold and so $y' < -\sqrt{2}\gamma$ on $u_0(\gamma) < u < 1$. By continuity of y', we obtain $y'(u_0(\gamma),\gamma) \leq -\sqrt{2}\gamma$. Integrating this over $(u_0(\gamma), 1)$ and using $y(1,\gamma) = 0$ gives

$$\begin{aligned}&\int_{u_0(\gamma)}^{1} y'(u,\gamma)\,du < \int_{u_0(\gamma)}^{1} -\sqrt{2}\gamma\,du\\ \implies & y(1,\gamma) - y(u_0(\gamma),\gamma) < -\sqrt{2}\gamma(1-u_0(\gamma))\\ \implies & y(u_0(\gamma),\gamma) > \sqrt{2}\gamma(1-u_0(\gamma)) = \sqrt{2}\gamma - \sqrt{2}A\gamma^{1-\nu}.\end{aligned}$$

To bridge the interval $(\theta, u_0(\gamma))$, define

$$u_1 = \inf\{\theta < u < 1 : y' < 0 \text{ on } (u,1]\}. \tag{5.28}$$

Since $y'(u_0(\gamma),\gamma) < 0$ by (5.27), we obtain that $u_1 < u_0(\gamma)$ by continuity. Since y is strictly decreasing on $(u_1, u_0(\gamma))$

$$y(u_1,\gamma) > y(u_0(\gamma),\gamma) > \sqrt{2}\gamma - \sqrt{2}A\gamma^{1-\nu}. \tag{5.29}$$

Suppose that $u_1 = \theta$. If $1 < n \le \frac{5}{2}$ we have $\nu \ge 1$, which implies that for large γ, $\lim_{u\to 0^+} y(u,\gamma) > 0$. If $\frac{5}{2} < n < 3$ we have $0 < \nu < 1$, meaning that $1-\nu > 0$. However, since the first term has a γ of order 1, the first term will outgrow the second term which will also result in $\lim_{u\to\theta^+} y(u,\gamma) > 0$ for sufficiently large γ. Since $\theta > 0$ implies that $y(\theta,\gamma) = 0$, we obtain that $\theta = 0$. So this means that S^- is nonempty by the reformulation. Therefore, we assume that $u_1 > \theta$.

Suppose that S^- is empty and so $\theta > 0$ or $\theta = 0$ and $y(\theta,\gamma) = 0$ for all $\gamma > 0$. Now for $u < u_1$ and using (5.19), we obtain

$$\begin{aligned}G(u) &= \frac{(y'(u,\gamma))^2}{2\phi(u)} + 2\sqrt{y(u,\gamma)}\\ \implies (y'(u,\gamma))^2 &= 2\phi(u)\left(G(u) - 2\sqrt{y(u,\gamma)}\right).\end{aligned}$$

By (5.21), $G'(u) > 0$ on $(\theta,1)$. So

$$\begin{aligned}&(y'(u,\gamma))^2 = 2\phi(u)\left(G(u) - 2\sqrt{y(u,\gamma)}\right)\\ \implies &(y'(u,\gamma))^2 < 2\phi(u)\left(G(u_1) - 2\sqrt{y(u,\gamma)}\right)\\ \implies &(y'(u,\gamma))^2 < 2\phi(u)G(u_1).\end{aligned}$$

Computing $G(u_1)$ using (5.19) and using that $y'(u_1,\gamma) = 0$ by definition, the following holds

$$\begin{aligned}(y'(u,\gamma))^2 &< 2\phi(u)G(u_1)\\ &< 2\phi(u)\left(\frac{(y'(u_1,\gamma))^2}{2\phi(u_1)} + 2\sqrt{y(u_1,\gamma)}\right)\\ &< 4\phi(u)\sqrt{y(u_1,\gamma)}.\end{aligned}$$

Using (5.23) and $\sigma(u) \le M$ gives

$$\begin{aligned}(y'(u,\gamma))^2 &< 4\phi(u)\sqrt{y(u_1,\gamma)}\\ &< 8u^{1-n}\sigma(u)\sqrt{y(u_1,\gamma)}\\ &< 8\sqrt{2\gamma}Mu^{1-n}.\end{aligned}$$

Taking the square root gives

$$\begin{aligned}&(y'(u,\gamma))^2 < 8\sqrt{2\gamma}Mu^{1-n}\\ \implies &|y'(u,\gamma)| < 2^{\frac{7}{4}}\gamma^{\frac{1}{4}}\sqrt{M}u^{\frac{1-n}{2}}\\ \implies &y'(u,\gamma) > -2^{\frac{7}{4}}\gamma^{\frac{1}{4}}\sqrt{M}u^{\frac{1-n}{2}} \quad\text{and}\quad y'(u,\gamma) < 2^{\frac{7}{4}}\gamma^{\frac{1}{4}}\sqrt{M}u^{\frac{1-n}{2}}.\end{aligned}$$

So on (θ, u_1)

$$y'(u,\gamma) < 2^{\frac{7}{4}}\gamma^{\frac{1}{4}}\sqrt{M}u^{\frac{1-n}{2}}.$$

Integrating over (θ, u_1) yields

$$\begin{aligned}
&\int_\theta^{u_1} y'(u,\gamma)\,du < \int_\theta^{u_1} 2^{\frac{7}{4}}\gamma^{\frac{1}{4}}\sqrt{M}u^{\frac{1-n}{2}}\,du \\
&\Longrightarrow y(u_1,\gamma) - y(\theta,\gamma) < \frac{2}{3-n}2^{\frac{7}{4}}\gamma^{\frac{1}{4}}\sqrt{M}u_1^{\frac{3-n}{2}} - \frac{2}{3-n}2^{\frac{7}{4}}\gamma^{\frac{1}{4}}\sqrt{M}\theta^{\frac{3-n}{2}} \\
&\Longrightarrow y(u_1,\gamma) < C\gamma^{\frac{1}{4}}u_1^{\frac{3-n}{2}} - C\gamma^{\frac{1}{4}}\theta^{\frac{3-n}{2}} + y(\theta,\gamma),
\end{aligned}$$

where $C = \frac{2}{3-n}2^{\frac{7}{4}}\sqrt{M} > 0$. Since $u_1 < u_0(\gamma)$ and $\frac{3-n}{2} > 0$ for $1 < n < 3$

$$\begin{aligned}
y(u_1,\gamma) &< C\gamma^{\frac{1}{4}}u_1^{\frac{3-n}{2}} - C\gamma^{\frac{1}{4}}\theta^{\frac{3-n}{2}} + y(\theta,\gamma) \\
&< C\gamma^{\frac{1}{4}}u_0(\gamma)^{\frac{3-n}{2}} - C\gamma^{\frac{1}{4}}\theta^{\frac{3-n}{2}} + y(\theta,\gamma) \\
&= CA\gamma^{\frac{1}{4}-\frac{v(3-n)}{2}} - C\gamma^{\frac{1}{4}}\theta^{\frac{3-n}{2}} + y(\theta,\gamma),
\end{aligned}$$

where the definition of $u_0(\gamma)$ is used (5.25). The power of γ in the first term can now be rewritten as

$$\begin{aligned}
\frac{1}{4} - \frac{v(3-n)}{2} &= \frac{1}{4} - \frac{\frac{3}{2(n-1)}(3-n)}{2} \\
&= \frac{1}{4} - \frac{3(3-n)}{4(n-1)} = \alpha,
\end{aligned} \tag{5.30}$$

where the power of γ in the first term is denoted as α. Note that $\alpha < 0$ when $1 < n < \frac{5}{2}$ and $0 \leq \alpha < 1$, when $\frac{5}{2} \leq n < 3$. Also, since $\theta \geq 0$, we obtain that $C\gamma^{\frac{1}{4}}\theta^{\frac{3-n}{2}} > 0$. Therefore,

$$y(u_1,\gamma) < CA\gamma^{\alpha} + y(\theta,\gamma).$$

Furthermore, by assumption S^- is empty, so $y(\theta,\gamma) = 0$. So

$$y(u_1,\gamma) < CA\gamma^{\alpha}. \tag{5.31}$$

Now, (5.29) and (5.31) contradict each other for large γ. To see this, note that for $1 < n \leq \frac{5}{2}$ we have $v \geq 1$, so the second term in (5.29) goes to zero or 1 for large γ, while the first term has a γ of order 1. Since (5.31) has a power of $\alpha < 1$, it would mean that, for γ large enough, (5.29) will overtake (5.31) which cannot happen by the strict inequality. The same also applies when $\frac{5}{2} < n < 3$. Even though, $0 < v < 1$, (5.29) will still overtake (5.31) for sufficiently large γ. Thus, there must exist a γ such that $\theta = 0$ and $\lim_{u\to 0^+} y(u,\gamma) > 0$. Thus, S^- is nonempty. □

Lemma 5.10. *S^- defined as in* (5.4) *is open for* $1 < n < 3$.

Proof. Let $\gamma_0 \in S^-$. Then, by the reformulation of S^-, $\theta(\gamma_0) = 0$ and $y(0,\gamma_0) > 0$. It must be shown that for γ in the neighborhood of γ_0, it should also satisfy $\theta(\gamma) = 0$ and $y(0,\gamma) > 0$ and so $\gamma \in S^-$. As $u'_{\gamma_0} < 0$ for $[0,1)$, it means that $y(u,\gamma_0) = (u'_{\gamma_0})^2 > 0$ on the same interval. So there exists a positive number c such that $y(u,\gamma_0) > 4c$ for $0 \leq u \leq \frac{1}{2}$.

Since $u(\cdot,\gamma)$, depends continuously on γ as long as $u > 0$ by Theorem 2.2, $\sigma(u)$ also depends continuously on γ as long as $u > 0$. So a solution $y(u,\gamma)$ of equation (5.18) also depends continuously on γ for $0 < u \leq 1$.

Therefore, as $\gamma \to \gamma_0$, $y(u,\gamma) \to y(u,\gamma_0)$ on $\theta(\gamma) < u \le 1$. Since $\theta(\gamma_0) = 0$ by definition, we obtain that $\theta(\gamma) \to 0$ as $\gamma \to \gamma_0$.

Let $\delta > 0$ be a small positive number. Let γ be a number that is close enough to γ_0 such that $\theta(\gamma) < \delta$ and $y(\delta,\gamma) > c$. This is possible as $\theta(\gamma) \to 0$ when $\gamma \to \gamma_0$ by the previous argument and since $\delta > 0$, it implies that there is continuous dependence on $y(\delta,\gamma)$. So we can get close enough such that $|y(\delta,\gamma) - y(\delta,\gamma_0)| < 3c$. And since $y(\delta,\gamma_0) > 4c$, this implies that $y(\delta,\gamma) > c$. Integrating equation (5.18) over (u,δ) gives

$$\begin{aligned}
\int_u^\delta y''(s,\gamma)\,ds &= -\int_u^\delta \phi(s) y^{-\frac{1}{2}}(s,\gamma)\,ds \\
\implies y'(\delta,\gamma) - y'(u,\gamma) &= -\int_u^\delta \phi(s) y^{-\frac{1}{2}}(s,\gamma)\,ds \\
\implies y'(u,\gamma) &= y'(\delta,\gamma) + \int_u^\delta \phi(s) y^{-\frac{1}{2}}(s,\gamma)\,ds.
\end{aligned}$$

Integrating this again over (u,δ) gives

$$\begin{aligned}
\int_u^\delta y'(s,\gamma)\,ds &= \int_u^\delta y'(\delta,\gamma)\,ds + \int_u^\delta \left(\int_s^\delta \phi(r) y^{-\frac{1}{2}}(r,\gamma)\,dr\right) ds \\
\implies y(\delta,\gamma) - y(u,\gamma) &= y'(\delta,\gamma)(\delta - u) + \int_u^\delta \left(\int_s^\delta \phi(r) y^{-\frac{1}{2}}(r,\gamma)\,dr\right) ds \\
\implies y(u,\gamma) &= y(\delta,\gamma) - y'(\delta,\gamma)(\delta - u) - \int_u^\delta \left(\int_s^\delta \phi(r) y^{-\frac{1}{2}}(r,\gamma)\,dr\right) ds.
\end{aligned}$$

To tackle the double integral, Fubini's Theorem A.4 is used

$$\begin{aligned}
y(u,\gamma) &= y(\delta,\gamma) - y'(\delta,\gamma)(\delta - u) - \int_u^\delta \left(\int_s^\delta \phi(r) y^{-\frac{1}{2}}(r,\gamma)\,dr\right) ds \\
&= y(\delta,\gamma) - y'(\delta,\gamma)(\delta - u) - \int_u^\delta \left(\int_u^r ds\right) \phi(r) y^{-\frac{1}{2}}(r,\gamma)\,dr \\
&= y(\delta,\gamma) - y'(\delta,\gamma)(\delta - u) - \int_u^\delta (r-u)\phi(r) y^{-\frac{1}{2}}(r,\gamma)\,dr \\
&= y(\delta,\gamma) - y'(\delta,\gamma)(\delta - u) - \int_u^\delta (r-u)2\sigma(r) r^{1-n} y^{-\frac{1}{2}}(r,\gamma)\,dr.
\end{aligned} \tag{5.32}$$

Observe that as long as $y(u,\gamma) > c$ and noting that $\sigma(u) \le M$, we obtain

$$\int_u^\delta (r-u)2\sigma(r) r^{1-n} y^{-\frac{1}{2}}(r)\,dr < \int_u^\delta (r-u)2M r^{1-n} \frac{1}{\sqrt{c}}\,dr.$$

Integrating the right-hand side for $n \neq 2$ gives

$$\begin{aligned}
\int_u^\delta (r-u)2M r^{1-n} \frac{1}{\sqrt{c}}\,dr &= \frac{2M}{\sqrt{c}} \int_u^\delta (r-u) r^{1-n}\,dr \\
&= \frac{2M}{\sqrt{c}} \left(\frac{\delta^{3-n}}{3-n} - \frac{u^{3-n}}{3-n} - \frac{u\delta^{2-n}}{2-n} + \frac{u^{3-n}}{2-n}\right).
\end{aligned}$$

For $2 < n < 3$, the last three terms are less than 0 and so

$$\int_u^\delta (r-u)2M r^{1-n} \frac{1}{\sqrt{c}}\,dr < \frac{2M}{(3-n)\sqrt{c}} \delta^{3-n} \qquad \text{for } 0 < u < \delta.$$

For $1 < n < 2$, the fourth term is greater than 0, but since $u < \delta$ and $2 - n > 0$

$$\frac{u\delta^{2-n}}{2-n} > \frac{u^{3-n}}{2-n}.$$

This means that for $1 < n < 2$

$$\int_u^\delta (r-u)2Mr^{1-n}\frac{1}{\sqrt{c}}\,dr < \frac{2M}{(3-n)\sqrt{c}}\delta^{3-n} \qquad \text{for } 0 < u < \delta.$$

For $n = 2$, the integral is taken again

$$\begin{aligned}\int_u^\delta (r-u)2Mr^{1-n}\frac{1}{\sqrt{c}}\,dr &= \frac{2M}{\sqrt{c}}\int_u^\delta (r-u)r^{1-n}\,dr \\ &= \frac{2M}{\sqrt{c}}\left(\frac{\delta^{3-n}}{3-n} - \frac{u^{3-n}}{3-n} - u\log(\delta) + u\log(u)\right).\end{aligned}$$

Since $\delta > u$ and $\log(\cdot)$ is a strictly increasing function

$$u\log(\delta) > u\log(u).$$

Therefore, for $n = 2$,

$$\int_u^\delta (r-u)2Mr^{1-n}\frac{1}{\sqrt{c}}\,dr < \frac{2M}{(3-n)\sqrt{c}}\delta^{3-n} \qquad \text{for } 0 < u < \delta. \tag{5.33}$$

Also, note that $\lim_{u\to 0^+} u\log(u) = 0$. So for $0 < n < 3$ the strict inequality holds. Since this term is greater than 0, a δ can now be chosen small enough such that

$$\frac{2M}{(3-n)\sqrt{c}}\delta^{3-n} = \frac{1}{2}c. \tag{5.34}$$

Furthermore, since $\delta > 0$, by continuous dependence there exists a γ close enough to γ_0 such that

$$|y(\delta,\gamma) - y(\delta,\gamma_0)| + \delta|y'(\delta,\gamma) - y'(\delta,\gamma_0)| < c. \tag{5.35}$$

Now, $y_0(u,\gamma)$ is subtracted from $y(u,\gamma)$ and the absolute values are taken. Using (5.32) we get

$$|y(u,\gamma)-y(u,\gamma_0)| = \left|y(\delta,\gamma) - y(\delta,\gamma) - y'(\delta,\gamma_0)(\delta-u) + y'(\delta,\gamma)(\delta-u) - \int_u^\delta (r-u)\phi(r)(y^{-\frac{1}{2}}(r,\gamma) - y^{-\frac{1}{2}}(r,\gamma_0))\,dr\right|.$$

By the triangle inequality for real numbers and integrals we obtain

$$|y(u,\gamma) - y(u,\gamma_0)| \le |y(\delta,\gamma) - y(\delta,\gamma)| + (\delta-u)|y'(\delta,\gamma_0) - y'(\delta,\gamma)| + \int_u^\delta |(r-u)\phi(r)(y^{-\frac{1}{2}}(r,\gamma) - y^{-\frac{1}{2}}(r,\gamma_0))|\,dr.$$

Since $(r-u) \ge 0$ and $\phi(r) > 0$ on $r \in [u,\delta]$, using the triangle equality again inside the integral gives

$$|y(u,\gamma)-y(u,\gamma_0)| \le |y(\delta,\gamma)-y(\delta,\gamma)|+(\delta-u)|y'(\delta,\gamma_0)-y'(\delta,\gamma)|+\int_u^\delta (r-u)\phi(r)|y^{-\frac{1}{2}}(r,\gamma)|\,dr+\int_u^\delta (r-u)\phi(r)|y^{-\frac{1}{2}}(r,\gamma_0)|\,dr.$$

Now, as long as $y(u,\gamma) > c$, using the strict inequality for the integral (5.33) and using that $\delta - u < \delta$ gives

$$|y(u,\gamma) - y(u,\gamma_0)| < |y(\delta,\gamma) - y(\delta,\gamma)| + \delta|y'(\delta,\gamma_0) - y'(\delta,\gamma)| + \frac{2M}{(3-n)\sqrt{c}}\delta^{3-n} + \frac{2M}{(3-n)\sqrt{c}}\delta^{3-n}.$$

Using (5.34) and (5.35) yields

$$|y(u,\gamma) - y(u,\gamma_0)| < c + 2 \cdot \frac{1}{2} c = 2c. \tag{5.36}$$

Using that $y(u,\gamma_0) > 4c$ on $u \in [0, \frac{1}{2}]$ and using (5.36) results in

$$y(u,\gamma) = y(u,\gamma_0) - (y(u,\gamma_0) - y(u,\gamma)) \geq y(u,\gamma_0) - |y(u,\gamma_0) - y(u,\gamma)| > 4c - 2c = 2c.$$

Since assuming $y(u,\gamma) > c$ implied a stronger bound $y(u,\gamma) > 2c$, by a bootstrap argument, $y(u,\gamma)$ can be continued down to $u = 0$ and $y(0,\gamma) \geq 2c$. Therefore, $y(0,\gamma) > 0$ so there exists a neighborhood of γ_0 which lies in S^-. Hence S^- is open for $1 < n < 3$. □

Having proved all these results, the main theorem can finally be proven.

Theorem 5.11. *Suppose* $0 < n < 3$. *Then there exists a solution of equation* (4.3).

Proof. In the first section of this chapter it was argued that if there exists a γ_0 with solution u_{γ_0} of equation (5.1) such that $a(\gamma_0) < \infty$ and $u'(a(\gamma_0)) = 0$, there exists a solution to equation (4.3). It suffices to show that $\mathbb{R}^+ \setminus (S^+ \cap S^-) \neq \emptyset$. Theorem 5.1 states that this is true with the help of Lemma 5.2, Lemma 5.3, Lemma 5.9 and Lemma 5.10. Therefore, a solution exists of the equation (4.3). □

Remark. *Note that if the nonnegative solution u of equation* (4.3) *is extended by $u(x) = 0$ for $x > a$ and reflected at $x = 0$, by Lemma 4.2, this will yield a non-trivial nonnegative solution of equation* (3.1). *Using the scaling argument of* (3.2), *it will result in a nonnegative solution of equation* (1.16). *This completes the proof concerning the existence of a non-trivial nonnegative even solution when* $0 < n < 3$.

6
Uniqueness

To prove statements about the uniqueness of solutions, the following problem is considered: find two positive numbers a and b and a function $u \in C^1([-b,a]) \cap C^3((-b,a))$ such that

$$\begin{cases} u''' = xu^{1-n} \quad u > 0, \quad \text{for } -b < x < a, \\ u(-b) = 0, \quad u'(-b) = 0, \quad u(a) = 0, \quad u'(a) = 0, \\ u(0) = 1. \end{cases} \tag{6.1}$$

The following two theorems will be proven:

Theorem 6.1. *If* $0 < n \leq 2$*, there exists a unique nonnegative solution of equation* (6.1)*. This solution is even.*

Theorem 6.2. *If* $2 < n < 3$*, there exists a unique nonnegative even solution of equation* (6.1)*.*

These two theorems will be proven using a series of mathematical results.

6.1. Proving the solution is unique and even for $0 < n \leq 1$

The statements will first be proven for $0 < n \leq 1$.

Lemma 6.3. *If* $n = 1$*, there exists a unique nonnegative solution of equation* (6.1)*. This solution is even.*

Proof. For $n = 1$, equation (6.1) becomes $u''' = x$. Using that the boundary values give $u = 0$ and $u' = 0$ on $x = a$ and $x = -b$, $u(x)$ can be written of the form

$$u(x) = \frac{1}{24}(a-x)^2(b+x)^2.$$

Differentiating this three times gives

$$u'''(x) = x + \frac{b}{2} - \frac{a}{2}.$$

Note that $u'''(0) = 0$ when $u''' = x$. Therefore,

$$\begin{aligned} 0 = u'''(0) &= 0 + \frac{b}{2} - \frac{a}{2} \\ \implies \frac{b}{2} &= \frac{a}{2} \\ \implies a &= b. \end{aligned}$$

Using that $u(0) = 1$, gives

$$
\begin{aligned}
1 = u(0) &= \frac{1}{24}(a-0)^2(a+0)^2 \\
\Longrightarrow 1 &= \frac{a^4}{24} \\
\Longrightarrow 24 &= a^4 \\
\Longrightarrow a &= 24^{\frac{1}{4}}.
\end{aligned}
$$

Therefore, u becomes the following unique solution

$$u(x) = \frac{1}{24}(24^{\frac{1}{4}} - x)^2(24^{\frac{1}{4}} + x)^2.$$

This solution is also even since

$$u(-x) = \frac{1}{24}(24^{\frac{1}{4}} - (-x))^2(24^{\frac{1}{4}} + (-x))^2 = \frac{1}{24}(24^{\frac{1}{4}} + x)^2(24^{\frac{1}{4}} - x)^2 = u(x).$$

Note that this result is the same as equation (1.18) when extended by $u(x) = 0$ for $x > a$ and $x < -a$ and rescaled using (3.2). □

Lemma 6.4. *If* $0 < n < 1$*, any solution of equation* (6.1) *is even.*

Proof. For a function to be even, it must hold that every odd derivative is zero in $x = 0$. Let (u, a, b) be a solution of (6.1). Suppose that u is not even. So there exists an odd derivative of u which is not zero in $x = 0$. Define the function

$$v(x) = u(x) - u(-x). \tag{6.2}$$

Then

$$v(0) = 0, \qquad v'(0) = 2u'(0), \qquad v''(0) = 0 \tag{6.3}$$

and using equation (6.1)

$$v'''(x) = x(u^{1-n}(x) - u^{1-n}(-x)). \tag{6.4}$$

Note that all even derivatives are zero in $x = 0$ for v. For small $x \neq 0$, the lowest nonzero odd derivative is dominating. Suppose that, without loss of generality, the lowest nonzero odd derivative of u is greater than 0 (if the lowest odd derivative is lower than 0, a similar contradiction can be derived using $u(-x) > u(x)$). This implies that $v(x) > 0$ for small $x > 0$. Therefore, using (6.2), $u(x) > u(-x)$ for small $x > 0$. Since $0 < n < 1$, the function u^{1-n} is increasing in u and so $v''' > 0$ for small $x > 0$. Now, note that all derivatives below the lowest nonzero odd derivative of v are zero at $x = 0$. Since $v''' > 0$ for small $x > 0$, it follows that $v'' > 0$ for small $x > 0$ and thus $v' > 0$ for small $x > 0$. Consequently, v is increasing and remains positive on the entire interval $x > 0$, as long as both $u(x)$ and $u(-x)$ are defined. Therefore, we conclude that $u(x) > u(-x)$ holds throughout this interval.

If $a \leq b$, then at $x = a$, since $u > 0$ on $(-b, a)$, we have

$$u(a) = 0 \qquad \text{and} \qquad u(-a) \geq 0.$$

But, since $u(x) > u(-x)$ by (6.2), this implies that

$$0 = u(a) > u(-a) \geq 0.$$

This is a contradiction and so $a > b$ must hold. This means that $v''' > 0$, $v'' > 0$, $v' > 0$ and $v > 0$ on $(0, b)$. Since $v(0) = 0$, we obtain $v(b) > 0$. Using the boundary conditions of (6.1), we get

$$\begin{aligned} v(a) &= u(a) - u(-a) = -u(-a), \\ v(b) &= u(b) - u(-b) = u(b). \end{aligned}$$

This implies that $u(b) > 0$ as $v(b) > 0$. Since $a > b$ and v is increasing, we obtain

$$\begin{aligned} & v(a) > v(b) \\ \implies & -u(-a) > u(b) > 0 \\ \implies & u(-a) < -u(b) < 0. \end{aligned}$$

So $u(-a) < 0$. But, since $u > 0$ on $(-b, a)$, we arrive at a contradiction. Therefore, all even derivatives should be zero at $x = 0$ for u. Thus, u must be even. □

Lemma 6.5. *If* $0 < n < 1$*, the nonnegative even solution of equation* (6.1) *is unique.*

Proof. Let (u_1, a_1, a_1) and (u_2, a_2, a_2) be two even solutions. Define

$$v = u_1 - u_2. \tag{6.5}$$

Then

$$\begin{aligned} v(0) &= u_1(0) - u_2(0) = 1 - 1 = 0, \\ v'(0) &= u_1'(0) - u_2'(0) = 0 - 0 = 0. \end{aligned}$$

Assume for contradiction that $u_1 \neq u_2$. If $v''(0) = 0$, then $v(x) = 0$. To show this, we note that v''' is

$$v'''(x) = x(u_1^{1-n}(x) - u_2^{1-n}(x)).$$

This is a third order ordinary differential equation. Near $x = 0$, the right-hand side is locally Lipschitz continuous, as $u_1 \approx 1$ and $u_2 \approx 1$ near $x = 0$, and therefore it has a unique local solution by the Picard-Lindelöf Theorem 2.1. Since $v(0) = v'(0) = v''(0) = 0$, it means that $v = 0$ is the unique solution near $x = 0$. So $v = 0$ on $(-\epsilon, \epsilon)$ for small $\epsilon > 0$. This implies that all derivatives are 0 on $x = 0$. Therefore, the Taylor expansion around $x = 0$ gives

$$\begin{aligned} v(x) &= v(0) + xv'(0) + \frac{x^2}{2}v''(0) + \frac{x^3}{6}v'''(0) + \frac{x^4}{4!}v''''(0) + \dots \\ &= 0 + x0 + \frac{x^2}{2}0 + \frac{x^3}{6}0 + \frac{x^4}{4!}0 + \dots = 0. \end{aligned}$$

This results in $v(x) = 0$. This gives $u_1 = u_2$, which is a contradiction. Therefore, $v''(0) \neq 0$ must hold. We assume that, without loss of generality, $v''(0) > 0$ (if $v''(0) < 0$, a similar contradiction can be derived using $u_2 < u_1$). This implies that $v' > 0$ for small $x \neq 0$ and so $v > 0$ for small $x \neq 0$. Consequently, $u_1 > u_2$ for small $x \neq 0$ using (6.5). Using the definition of v''', like the proof in Lemma 6.4 with $a_1 = a$ and $a_2 = b$, it can be deduced that $v''' > 0$, $v'' > 0$, $v' > 0$ and $v > 0$ as long as $u_1(x)$ and $u_2(x)$ are both defined on the interval. Moreover, the condition $a_2 < a_1$ can also be derived using the same argument as in the proof of Lemma 6.4. Now,

$$v'(a_2) = u_1'(a_2) - u_2'(a_2) = u_1'(a_2) - 0 = u_1'(a_2).$$

Since v' is strictly increasing on $(0, a_2)$, we have

$$u_1'(a_2) = v'(a_2) > v'(0) = 0.$$

However, Lemma 4.3 states that $u_1'(x) < 0$ for $0 < x < a_1$. This is a contradiction and therefore the solution must be unique. □

6.2. Proving the even solution is unique for $1 < n < 3$ and all solutions are even for $1 < n \leq 2$

To prove the even solution is unique for $1 < n < 3$ and all solutions are even for $1 < n \leq 2$, a helping result is needed to help prove this.

Lemma 6.6. *Let $1 < n < 3$. Suppose that (u_1, a_1, b_1) and (u_2, a_2, b_2) are nonnegative solutions that satisfy equation (6.1) without the assumption $u(0) = 1$. Suppose that $a_1 = a_2 = a$ and $u_1(0) \geq u_2(0)$. Then the difference $v = u_1 - u_2$ has the following property:*

$$\begin{cases} \text{either } v \equiv 0 & \text{on } [0, a], \\ \text{or } v(0) > 0, \quad v'(0) < 0 & \text{and } v''(0) > 0. \end{cases}$$

Proof. Differentiating v three times and using equation (6.1) gives

$$v''' = x(u_1^{1-n} - u_2^{1-n}). \tag{6.6}$$

Now, define the following function

$$\Phi = v v'' - \frac{1}{2}(v')^2. \tag{6.7}$$

The derivative of this function is

$$\Phi' = v'v'' + v v''' - v'v'' = v v'''.$$

Because $n > 1$, the function u^{1-n} is decreasing in u. This means that if $u_1 \geq u_2$, we have $u_1^{1-n} \leq u_2^{1-n}$. So v and v''' are always opposite signs except when $v = 0$ or $x = 0$. This means that

$$\Phi' = v v''' \leq 0 \qquad \text{on } [0, a)$$

and therefore Φ is decreasing on $[0, a)$. As v'' evaluated in $x = a$ may not even be defined, $\lim_{x \to a^-} \Phi(x)$ will be studied. It is known by the boundary conditions and by continuity that $\lim_{x \to a^-} v(x) = 0$ and $\lim_{x \to a^-} v'(x) = 0$. To show that $\lim_{x \to a^-} v(x) v''(x) = 0$, we must show that v'' is bounded near $x = a$. Since u_1 and u_2 are solutions to equation (6.1) without the assumption that $u(0) = 1$, their third derivatives are both bounded near $x = a$. So the difference v''' must also be bounded near $x = a$. This implies that $|v'''(x)| \leq C$ for $x \in [a - \delta, a)$ for some $C > 0$ and small $\delta > 0$. Integrating this over the interval $[a - \delta, x)$ gives

$$v''(x) = v''(a - \delta) + \int_{a-\delta}^{x} v'''(t)\, dt.$$

Taking the absolute value and using the triangle inequality gives

$$
\begin{aligned}
|v''(x)| &= \left| v''(a-\delta) + \int_{a-\delta}^{x} v'''(t)\,dt \right| \\
&\leq |v''(a-\delta)| + \left| \int_{a-\delta}^{x} v'''(t)\,dt \right| \\
&\leq |v''(a-\delta)| + \int_{a-\delta}^{x} |v'''(t)|\,dt \\
&\leq |v''(a-\delta)| + \int_{a-\delta}^{x} C\,dt \\
&= |v''(a-\delta)| + C(x-a+\delta).
\end{aligned}
$$

Since $|v''(a-\delta)|$ is bounded, it follows that v'' is bounded near $x = a$. Therefore, $\Phi(x) \to 0$ as $x \to a^-$. Hence $\Phi \geq 0$ on $[0, a)$.

Suppose $\Phi(c) = 0$ for some $c \in [0, a)$. Since Φ is decreasing and $\Phi \geq 0$ on $[0, a)$, it follows that $\Phi(x) = 0$ on $x \in [c, a)$. This implies that $\Phi' = vv''' = 0$ on $[c, a)$. As a result, $v = 0$ on (c, a) since $v''' = 0$ only applies when $v = 0$. Since $v = 0$ on an interval, it implies that all derivatives are 0 for $\xi \in (c, a)$. The Taylor expansion around the point $x = \xi$, suggests that $v \equiv 0$ on $[0, a]$.

Suppose that $\Phi > 0$ on $[0, a)$. Then on $[0, a)$, by the definition of Φ,

$$vv'' > \frac{1}{2}(v')^2 \geq 0.$$

This means that v and v'' have the same sign on $[0, a)$ and are both nonzero. Since $v(0) \geq 0$, it follows that $v > 0$ and $v'' > 0$ on the interval $[0, a)$. To show that $v'(0) < 0$, note that $v'' > 0$ on $[0, a)$, so v' is strictly increasing on this interval. Since $v'(a) = 0$, it follows that $v' < 0$ on $[0, a)$. Hence $v(0) > 0$, $v'(0) < 0$ and $v''(0) > 0$. This completes the proof where both cases are discussed. □

With this lemma, the following two results will be proven.

Lemma 6.7. *If* $1 < n < 3$*, the nonnegative even solution of equation* (6.1) *is unique.*

Proof. Let (u_1, a_1, a_1) and (u_2, a_2, a_2) be two even solutions. u_1 will be scaled such that Lemma 6.6 can be applied. To get the same boundary point a_2, define

$$\mu = \frac{a_1}{a_2}, \qquad \hat{u}_1(x) = \lambda u_1(\mu x). \tag{6.8}$$

So

$$u_1(\mu x) = \frac{\hat{u}_1(x)}{\lambda}. \tag{6.9}$$

Now,

$$
\begin{aligned}
\hat{u}_1(a_2) &= \lambda u_1(\mu a_2) = \lambda u_1\left(\frac{a_1 \cdot a_2}{a_2}\right) = \lambda u_1(a_1) = 0, \\
\hat{u}_1(-a_2) &= \lambda u_1(-\mu a_2) = \lambda u_1\left(\frac{a_1 \cdot -a_2}{a_2}\right) = \lambda u_1(-a_1) = 0.
\end{aligned}
$$

This implies that $\hat{u}_1$ is defined on $[-a_2, a_2]$. Since u_1 satisfies equation (6.1), we obtain the equation

$$u_1'''(\mu x) = \mu x u_1(\mu x)^{1-n}. \tag{6.10}$$

Now, using (6.8),

$$\hat{u}_1'''(x) = \lambda \mu^3 u_1'''(\mu x).$$

Using (6.9), we get

$$u_1(\mu x)^{1-n} = \left(\frac{\hat{u}_1(x)}{\lambda}\right)^{1-n} = \lambda^{n-1}\hat{u}_1(x)^{1-n}.$$

Therefore, using (6.10),

$$\hat{u}_1'''(\mu x) \equiv \lambda\mu^3 u_1'''(\mu x) = \lambda\mu^4 x u_1(\mu x)^{1-n} \equiv \lambda^n\mu^4 x\hat{u}_1(x)^{1-n}.$$

To satisfy equation (6.1), we require

$$\lambda^n\mu^4 = 1.$$

Therefore,

$$\lambda = \mu^{-\frac{4}{n}} = \left(\frac{a_2}{a_1}\right)^{\frac{4}{n}}.$$

Set $v = \hat{u}_1 - u_2$. Since both $\hat{u}_1$ and u_2 satisfy equation (6.1), without the assumption $u(0) = 1$, and both have the boundary condition on a_2, it only needs to be shown that $\hat{u}_1(0) \geq u_2(0)$. At $x = 0$, since $u_1(0) = 1$

$$\hat{u}_1(0) = \left(\frac{a_2}{a_1}\right)^{\frac{4}{n}} u_1(0) = \left(\frac{a_2}{a_1}\right)^{\frac{4}{n}}.$$

Note that $u_2(0) = 1$. If $a_1 > a_2$, $-v$ satisfies the conditions of Lemma 6.6. If $a_1 \leq a_2$, v satisfies the conditions of Lemma 6.6. Now, by evenness, $v'(0) = 0$. It follows that $v \equiv 0$ on $[0, a_2]$ by Lemma 6.6. This means that $\hat{u}_1 \equiv u_2$. Since $\hat{u}_1(0)$ must be 1, it holds that

$$\left(\frac{a_2}{a_1}\right)^{\frac{4}{n}} = 1.$$

Therefore, $a_1 = a_2 = a$. This means that $\hat{u}_1 \equiv u_1 \equiv u_2$ on $[0, a]$ and so on $[-a, a]$ by evenness. Hence the even solution is unique. □

Lemma 6.8. *If* $1 < n \leq 2$, *any solution of equation* (6.1) *is even.*

Proof. Let (u, a, b) be a solution of equation (6.1). To apply Lemma 6.6, a similar function like in the proof of Lemma 6.7 is used

$$u_1 = \left(\frac{a}{b}\right)^{\frac{4}{n}} u\left(-\frac{bx}{a}\right). \tag{6.11}$$

Now,

$$u_1(a) = \left(\frac{a}{b}\right)^{\frac{4}{n}} u\left(-\frac{ba}{a}\right) = \left(\frac{a}{b}\right)^{\frac{4}{n}} u(-b) = 0,$$
$$u_1\left(-\frac{a^2}{b}\right) = \left(\frac{a}{b}\right)^{\frac{4}{n}} u\left(\frac{ba^2}{ab}\right) = \left(\frac{a}{b}\right)^{\frac{4}{n}} u(a) = 0.$$

So the function is defined on $[-\frac{a^2}{b}, a]$ and satisfies equation (6.1) without the assumption that $u_1(0) = 1$. Set $v = u_1 - u$. This satisfies the conditions for Lemma 6.6 for either v or $-v$ depending on whether $a > b$ or $a \leq b$. Differentiating v on $x = 0$ gives

$$
\begin{aligned}
v(0) &= \left(\left(\frac{a}{b}\right)^{\frac{4}{n}} - 1\right) u(0), \\
v'(0) &= -\left(\left(\frac{a}{b}\right)^{\frac{4-n}{n}} + 1\right) u'(0), \\
v''(0) &= \left(\left(\frac{a}{b}\right)^{\frac{4-2n}{n}} - 1\right) u''(0).
\end{aligned} \tag{6.12}
$$

If $n = 2$, (6.12) yields that $v''(0) = 0$. So $v \equiv 0$ by lemma 6.6. This means that $u_1 \equiv u$ on $[0, a]$ and so $u_1(0) = 1$. Using (6.11), we obtain $a = b$. Also, by the construction of u_1, $u_1 \equiv u$ implies that $u(-x) = u(x)$ on $[0, a]$ and therefore it holds on $[-a, a]$. Hence u is even. This proves the lemma for $n = 2$.

Next, suppose that $1 < n < 2$. It will first be proven that $a = b$. Suppose by contradiction that $a > b$ without loss of generality (if $a < b$ a similar contradiction can be derived using $-v$). It follows from (6.12) that $v(0) > 0$. By Lemma 6.6, we obtain that $v'(0) < 0$ and $v''(0) > 0$. Since $n < 2$, it must hold that $u(0) > 0$, $u'(0) > 0$ and $u''(0) > 0$ by (6.12). Since $u''' > 0$ on $(0, a)$ by equation (6.1), it implies that u is strictly increasing on $(0, a)$. Since $u(0) > 0$, it means that $u(a) > 0$, which is a contradiction with the boundary condition $u(a) = 0$. Therefore, $a = b$. Therefore, $v(0) = 0$ by (6.12). So $v \equiv 0$ on $[0, a]$ by Lemma 6.6. Hence $u(-x) = u(x)$ on $[-a, a]$ using the same argument with $n = 2$. Thus, u must be even. □

All these results combined prove Theorem 6.1 and Theorem 6.2.

Remark. *Note that if the unique nonnegative even solution u of equation* (6.1) *is extended by* $u(x) = 0$ *for* $x > a$ *and* $x < -a$*, by Lemma 4.2, this will yield in a unique nonnegative even solution of equation* (3.1)*. Using the scaling argument of* (3.2)*, it will result in a unique nonnegative even solution of equation* (1.16)*. Using the same transformations, we have also proven that every solution is even for equation* (1.16) *when* $0 < n \leq 2$*. This completes the proof concerning the uniqueness of a nonnegative even solution when* $0 < n < 3$ *and the proof that all solutions are even when* $0 < n \leq 2$*.*

7

Constructing self-similar solutions numerically

In this chapter, the self-similar solution of equation (1.16) will be constructed numerically using the shooting method. This is done by searching for a γ_0 where the solution of equation (5.1) satisfies the properties of equation (4.3). In chapter 5 and 6, it was shown that this γ_0 is existent and unique. When γ_0 is found, the solution of equation (4.3) with that corresponding γ_0 will be extended by $u(x) = 0$ for $x > a_{\gamma_0}$ and will be reflected at $x = 0$. Using the scaling argument of (3.2), this will result in a unique nonnegative even solution for (1.16). Methods and results in this chapter can be found in more detail in [14].

To solve equation (5.1), we will rewrite the equation as a system of equations. This is done by defining the functions u_1, u_2 and u_3 as

$$u_1 = u, \qquad u_2 = u', \qquad u_3 = u''.$$

Then

$$u_1' = u_2, \qquad u_2' = u_3, \qquad u_3' = x u_1^{1-n}.$$

Therefore, the equation can be written as

$$\mathbf{u}(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \end{pmatrix}, \quad \mathbf{u}'(x) = \mathbf{F}(x, \mathbf{u}) = \begin{pmatrix} u_2(x) \\ u_3(x) \\ x\, u_1(x)^{1-n} \end{pmatrix}, \tag{7.1}$$

with initial condition

$$\mathbf{u}(0) = \begin{pmatrix} 1 \\ 0 \\ -\gamma \end{pmatrix}. \tag{7.2}$$

7.1. Stability analysis

First, we check the analytical stability. A system is called stable if a small perturbation of the parameters (including the initial condition) results in a small difference in the solution. This results in numerical errors not getting amplified as the solution evolves, ensuring that the numerical solution stays close to the true solution. Analytical stability is obtained when all the eigenvalues have nonpositive real part. Since equation (7.1) is nonlinear, the stability can be checked locally in the point $(\hat{x}, \hat{\mathbf{u}})$ by computing the Jacobian matrix in that point and computing the eigenvalues. The Jacobian matrix is defined as

$$\mathbf{J}(\hat{x},\hat{\mathbf{u}}) = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \frac{\partial F_1}{\partial u_3} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \frac{\partial F_2}{\partial u_3} \\ \frac{\partial F_3}{\partial u_1} & \frac{\partial F_3}{\partial u_2} & \frac{\partial F_3}{\partial u_3} \end{bmatrix}.$$

Evaluating this in the point $(\hat{x},\hat{\mathbf{u}})$ gives

$$\mathbf{J}(\hat{x},\hat{\mathbf{u}}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ (1-n)\hat{x}\hat{u}_1^{-n} & 0 & 0 \end{bmatrix}.$$

Computing the eigenvalues yields

$$\begin{aligned} \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ (1-n)\hat{x}\hat{u}_1^{-n} & 0 & -\lambda \end{bmatrix} &= 0 \\ \Longrightarrow -\lambda^3 + (1-n)\hat{x}\hat{u}_1{}^{-n} &= 0 \\ \Longrightarrow \lambda^3 = (1-n)\hat{x}\hat{u}_1{}^{-n}. \end{aligned}$$

For $0 < n \leq 1$, this results in

$$\lambda = r, \qquad \lambda = re^{\frac{2}{3}i\pi}, \qquad \lambda = re^{\frac{4}{3}i\pi},$$

where

$$r = \left|(1-n)\hat{x}\hat{u}_1{}^{-n}\right|^{\frac{1}{3}}. \tag{7.3}$$

For $1 < n < 3$, this leads to

$$\lambda = re^{i\pi}, \qquad \lambda = re^{\frac{1}{3}i\pi}, \qquad \lambda = re^{\frac{5}{3}i\pi},$$

where

$$r = \left|(1-n)\hat{x}\hat{u}_1{}^{-n}\right|^{\frac{1}{3}}. \tag{7.4}$$

By shifting the root by $\frac{2\pi}{3}$ in the complex plane, we observe that there are always eigenvalues with positive real part, except when $n = 1$, where all the eigenvalues are zero. Consequently, the solution is not analytically stable for $0 < n < 3$, except at $n = 1$. Although the solution is analytically unstable for $0 < n < 3$ (except at $n = 1$), numerical integration can still provide valuable insight into the system's behavior. By carefully selecting the numerical methods and controlling the step sizes, we aim to approximate the solution trajectories and investigate whether the computed solutions retain the qualitative properties established analytically in chapter 4.

7.2. The numerical integration method

Next, a numerical integration method will be chosen. A built-in Scipy function will be used to integrate the system in steps of size Δx_n, where Δx_n differs for each step. Since it is an approximation, $\mathbf{u}$ will be denoted as $\mathbf{w}$. This built-in function uses the Runge-Kutta (RK4) method which approximates the solution $\mathbf{w}_{n+1}$ at step x_{n+1} by using the approximated solution $\mathbf{w}_n$ evaluated in the previous step x_n. This method is defined as

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \frac{1}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4),$$

where $\mathbf{k}_1$ to $\mathbf{k}_4$ are given by

$$\mathbf{k}_1 = \Delta x_n \mathbf{F}(x_n, \mathbf{w}_n),$$
$$\mathbf{k}_2 = \Delta x_n \mathbf{F}(x_n + \frac{1}{2}\Delta x_n, \mathbf{w}_n + \frac{1}{2}\mathbf{k}_1),$$
$$\mathbf{k}_3 = \Delta x_n \mathbf{F}(x_n + \frac{1}{2}\Delta x_n, \mathbf{w}_n + \frac{1}{2}\mathbf{k}_2),$$
$$\mathbf{k}_4 = \Delta x_n \mathbf{F}(x_n + \Delta x_n, \mathbf{w}_n + \mathbf{k}_3).$$

For $\mathbf{w}_0$, the initial condition is used. Note that this is the explicit RK4 method. The built-in function also has an implicit RK4 method, but this method is much more computationally expensive, while giving identical results as the explicit RK 4 method, see Figures A.1–A.8. Therefore, the explicit RK4 method is used.

The RK4 method has a local truncation error of $\mathcal{O}(\Delta x_n^4)$, see Definition A.4. The local truncation error indicates the new error per integration step. Since Δx_n is typically smaller than 1, higher-order methods like RK4 generally yield a lower error per timestep compared to lower-order methods. The built-in function also uses local extrapolation such that the local truncation error becomes $\mathcal{O}(\Delta x_n^5)$. Using the amplification matrix $Q(A\Delta x_n)$ of the RK4 method, numerical stability can be acquired when $|Q(A\Delta x_n)| \leq 1$, which controls or dampens errors that are caused by small perturbations. However, since our eigenvalues λ have positive real parts, analyzing stability is not meaningful in our case. Instead, we will use a small step size and observe the behavior of the numerical solution.

7.3. The bisection method

To find the γ_0 numerically, the bisection method is used. In the proof of Lemma 5.2, it was shown that γ should be small enough in order to be in the set S^+ and in the proof of Lemma 5.3 and Lemma 5.9 it was shown that γ should be large enough in order to be in the set S^-. This means that there is a transition point γ_0 such that all values below γ_0 are in S^+, while all values above γ_0 are in S^-. Therefore, two initial solutions will be used. One solution where γ will be chosen small enough such that the solution is in S^+ and one solution where γ will be chosen large enough such that the solution is in S^-. Call γ_+ the γ where the solution is in S^+ and γ_- the γ where the solution is in S^-. This means that γ_0 is in the interval (γ_+, γ_-). The bisection method keeps dividing the interval into two equal parts and continues with the interval that contains γ_0. Set $\gamma_+^0 = \gamma_+$ and $\gamma_-^0 = \gamma_-$. A new approximation for γ_0 is iteratively computed by

$$\gamma^m = \frac{\gamma_+^m + \gamma_-^m}{2}.$$

For each γ^m, the solution will be constructed numerically. If u and u' are simultaneously vanishing, γ_0 is found. If not, a new interval $[\gamma_+^{m+1}, \gamma_-^{m+1}]$ is created using $\gamma_+^{m+1} = \gamma^m$ and $\gamma_-^{m+1} = \gamma_-^m$ if the solution γ^m is in S^+, and $\gamma_+^{m+1} = \gamma_+^m$ and $\gamma_-^{m+1} = \gamma^m$ if the solution γ^m is in S^-. The bisection method makes sure γ^m converges to γ_0. Since u and u' cannot be exactly zero when computed numerically, a very small bound is used in order to check whether u and u' become zero. This bound will be the stopping criterion.

7.4. The results

The self-similar solution will now be constructed using the methods described above. When γ_0 is found, the solution of equation (4.3) with that γ_0 will be extended by $u(x) = 0$ for $x > a_{\gamma_0}$ and will be reflected at $x = 0$. The solution is then scaled using the scaling argument of (3.2) to get the self-similar solution for equation (1.16). The integral is calculated using a composite trapezoidal rule. The corresponding code is provided in Appendix A.3. The results for $c = 1$, where c represents the volume, are given in Figures 7.1–7.8 and Table 7.1 for different n values.

In Table 7.1 the γ_0 is given for multiple n values together with the value a where the solution becomes zero along with the values for f and f' evaluated in the point $\mu = a$. The fact that f and f' are numerically close to zero at $\mu = a$, and that mass is conserved to high precision, provides strong numerical evidence that this is the self-similar solution of equation (1.16), up to a small numerical error. Note that the same values also apply for $\mu = -a$ by symmetry.

Figure 7.1: The numerical solution of the self-similar thin-film equation for $n = 0.1$. Here, $c = 1$.

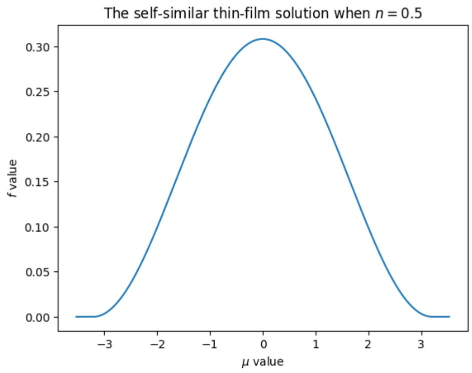

Figure 7.2: The numerical solution of the self-similar thin-film equation for $n = 0.5$. Here, $c = 1$.

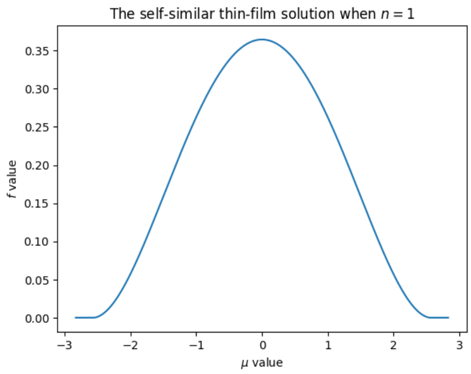

Figure 7.3: The numerical solution of the self-similar thin-film equation for $n = 1$. Here, $c = 1$.

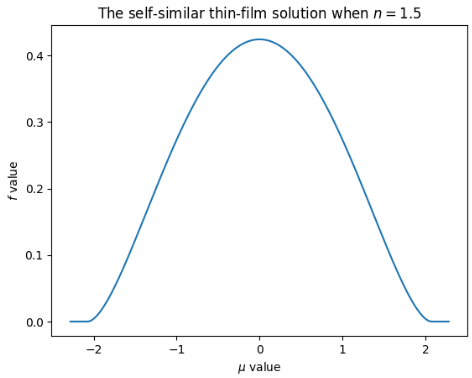

Figure 7.4: The numerical solution of the self-similar thin-film equation for $n = 1.5$. Here, $c = 1$.

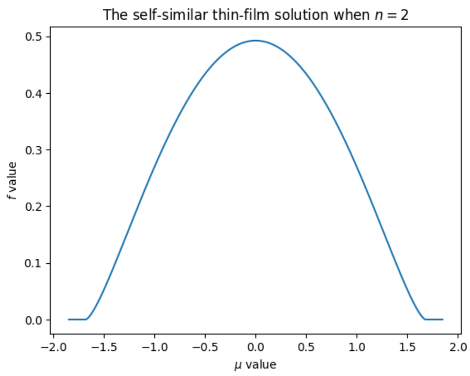

Figure 7.5: The numerical solution of the self-similar thin-film equation for $n = 2$. Here, $c = 1$.

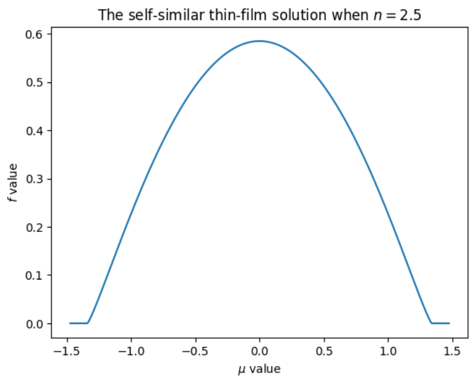

Figure 7.6: The numerical solution of the self-similar thin-film equation for $n = 2.5$. Here, $c = 1$.

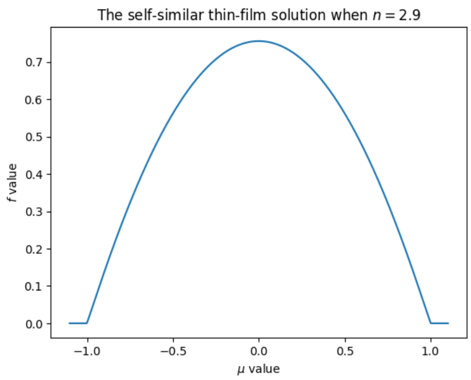

Figure 7.7: The numerical solution of the self-similar thin-film equation for $n = 2.9$. Here, $c = 1$.

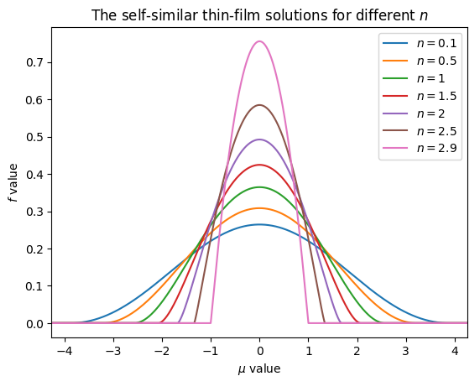

Figure 7.8: The numerical solution of the self-similar thin-film equation for different n. Here, $c = 1$.

n	γ_0	a	$f(a)$	$f'(a)$	$\int_{-\infty}^{\infty} f(\mu)\,d\mu$
0.1	0.67506	3.87868	$-2.1 \cdot 10^{-9}$	$-8.6 \cdot 10^{-17}$	1.000001443
0.5	0.72822	3.20979	$2.1 \cdot 10^{-10}$	$9.5 \cdot 10^{-17}$	1.000002351
1.0	0.81650	2.57176	$-1.4 \cdot 10^{-8}$	$5.6 \cdot 10^{-17}$	0.999988783
1.5	0.94708	2.07448	$3.4 \cdot 10^{-11}$	$1.7 \cdot 10^{-16}$	1.000004076
2.0	1.16895	1.67889	$2.7 \cdot 10^{-10}$	$-1.4 \cdot 10^{-12}$	0.999999996
2.5	1.67716	1.33930	$5.5 \cdot 10^{-10}$	$3.7 \cdot 10^{-10}$	0.999994000
2.9	3.59439	1.00174	$5.3 \cdot 10^{-10}$	$2.4 \cdot 10^{-9}$	0.999967202

Table 7.1: The γ_0 for different n values together with the value a where the solution becomes zero and the values for f and f' in the point $\mu = a$. The integral is also given which shows that mass conservation holds. Here, $c = 1$.

n	γ_0	a	$f(a)$	$f'(a)$	$\int_{-\infty}^{\infty} f(\mu)\,d\mu$
0.1	0.67506	4.59044	$-1.8 \cdot 10^{-7}$	$-6.1 \cdot 10^{-14}$	1000.001
0.5	0.72822	6.91529	$1.0 \cdot 10^{-8}$	$2.0 \cdot 10^{-14}$	1000.002
1.0	0.81650	10.23836	$-3.5 \cdot 10^{-7}$	$3.6 \cdot 10^{-15}$	999.988
1.5	0.94708	13.64871	$5.1 \cdot 10^{-10}$	$3.9 \cdot 10^{-15}$	1000.003
2.0	1.16895	16.78887	$2.7 \cdot 10^{-9}$	$-1.4 \cdot 10^{-11}$	1000.003
2.5	1.67716	19.08639	$3.8 \cdot 10^{-9}$	$1.8 \cdot 10^{-9}$	1000.002
2.9	3.59439	18.26528	$2.9 \cdot 10^{-9}$	$7.1 \cdot 10^{-9}$	1000.006

Table 7.2: The γ_0 for different n values together with the value a where the solution becomes zero and the values for f and f' in the point $\mu = a$. The integral is also given which shows that mass conservation holds. Here, $c = 1000$.

The self-similar solution is also computed for $c = 1000$. The results are given in Figures 7.9–7.16 and Table 7.2.

Although Table 7.2 shows slightly reduced accuracy in the computed integral and in the values f and f' at $\mu = a$, the results remain sufficiently accurate to support the conclusion that the numerical solution exhibits the properties of the self-similar solution. Note that a larger mass c does not impact the γ_0 value. This is because the function is first solved for equation (5.1), and after that, it is rescaled to match the volume c.

In chapter 1, we mentioned that there is no closed form solution for equation (1.16) except when $n = 1$, which is described in (1.18). Comparing the exact solution of equation (1.18) with the numerical approximation is found in Figure 7.17 and Figure 7.18. The figures indicate that the exact closed form solution is the same as our numerical approximation.

Observe that, even though the differential equation was not stable, we still obtain numerical solutions that satisfy the qualitative properties such as compact support and mass conservation. This consistency enforces the reliability of the computed results.

7.5. Behavior near $\mu = a$

Even though both Table 7.1 and Table 7.2 show that the function value at $\mu = a$ drops to practically zero for both f and f', Figure 7.7 and Figure 7.15 indicate that there is a sharp edge near $\mu = a$ for large n, while Figure 7.1 and Figure 7.9 indicate that the solution smoothly translates to zero near $\mu = a$ for small n, which is what would be expected. This can be explained both mathematically and physically.

In the paper of Bernis, Peletier & Williams [4], the behavior near $x = a_l$ is analyzed for a solution of equation (4.3), where $a_l = al$ denotes the boundary of the support of the solution, and l denotes the scaling coefficient from (3.2). This is done by using results from [25]. As a result, the following theorem was proven.

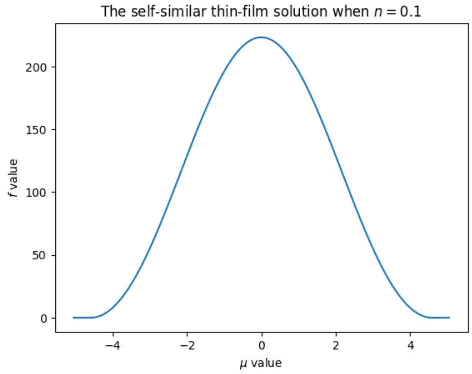

Figure 7.9: The numerical solution of the self-similar thin-film equation for $n = 0.1$. Here, $c = 1000$.

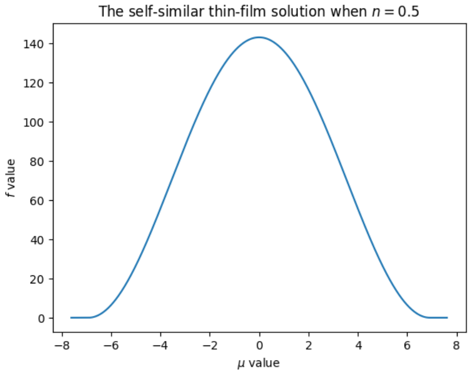

Figure 7.10: The numerical solution of the self-similar thin-film equation for $n = 0.5$. Here, $c = 1000$.

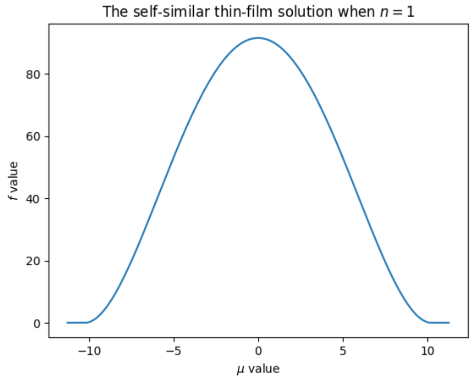

Figure 7.11: The numerical solution of the self-similar thin-film equation for $n = 1$. Here, $c = 1000$.

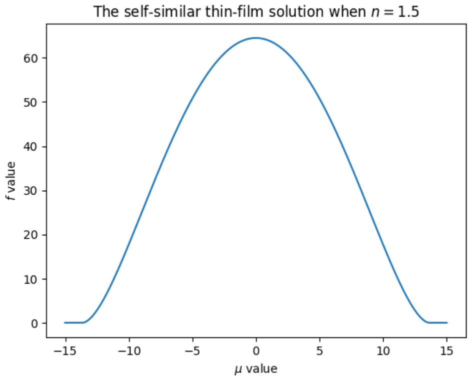

Figure 7.12: The numerical solution of the self-similar thin-film equation for $n = 1.5$. Here, $c = 1000$.

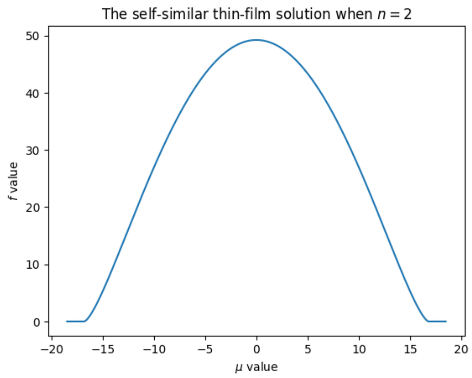

Figure 7.13: The numerical solution of the self-similar thin-film equation for $n = 2$. Here, $c = 1000$.

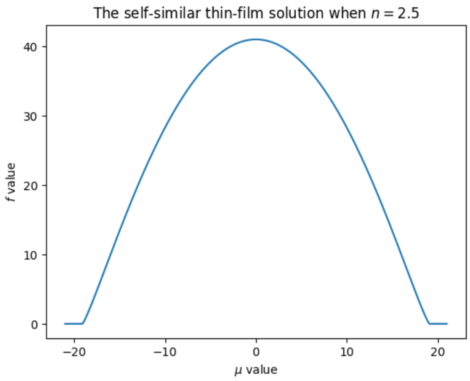

Figure 7.14: The numerical solution of the self-similar thin-film equation for $n = 2.5$. Here, $c = 1000$.

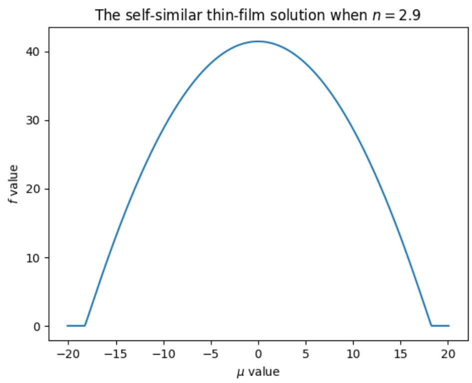

Figure 7.15: The numerical solution of the self-similar thin-film equation for $n = 2.9$. Here, $c = 1000$.

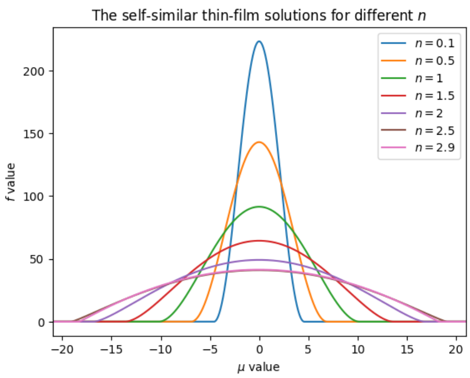

Figure 7.16: The numerical solution of the self-similar thin-film equation for different n. Here, $c = 1000$.

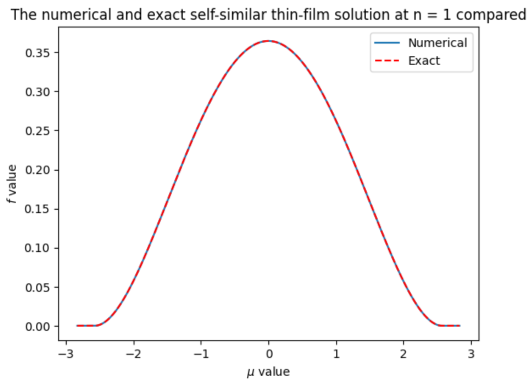

Figure 7.17: The numerical self-similar thin-film solution compared with the exact self-similar thin-film solution. Here, $c = 1$.

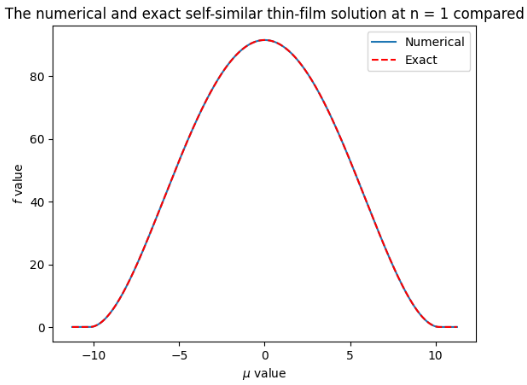

Figure 7.18: The numerical self-similar thin-film solution compared with the exact self-similar thin-film solution. Here, $c = 1000$.

Theorem 7.1. *Let u be a nonnegative solution of equation* (4.3).
(a) If $0 < n < \frac{3}{2}$*, then there exists a constant* $\xi > 0$ *such that*

$$u(x) \sim \xi (a_l - x)^2 \qquad \text{as } x \to a_l.$$

(b) If $n = \frac{3}{2}$*, then*

$$u(x) \sim \left(\frac{3a_l}{4}\right)^{\frac{2}{3}} (a_l - x)^2 \left(\log\left(\frac{1}{a_l - x}\right)\right)^{\frac{2}{3}} \qquad \text{as } x \to a_l.$$

(c) If $\frac{3}{2} < n < 3$*, then*

$$u(x) \sim B(a_l - x)^{\frac{3}{n}} \qquad \text{as } x \to a_l,$$

where

$$B = \left(\frac{n^3 a_l}{3(3-n)(2n-3)}\right)^{\frac{1}{n}}.$$

Note that the same behavior applies to solutions of equation (1.16), as rescaling does not effect the qualitative behavior of the solution near the edge of the support, it only modifies the associated constants. This theorem shows that for small n, the derivative will smoothly decay to zero as $x \to a_l$. For large n, the exponent $\frac{3}{n} - 1$ becomes close to zero, and since $(a_l - x) < 1$ for values close to a_l, the solution exhibits a sharp cutoff despite vanishing at $x = a_l$.

A more physical way to understand this is that when there is little or no slip at the surface, the fluid encounters high resistance near the free boundary. This leads to a sharp cutoff at the edge of the film, as the fluid cannot easily move where it's very thin. In contrast, with free slip, the resistance is lower, allowing the solution to decay more smoothly towards zero near the free boundary.

Another interesting aspect to note is that for small values of c, the value a tends to be larger, while the maximum value f is smaller for small values of n compared to large n. For large c, the opposite is observed. This can be understood by considering the effect of the nonlinear mobility $|f|^n$. For small c, the function f is small at the origin by mass conservation. For small n, the mobility $|f|^n$ remains relatively large even when f is small, allowing the profile to diffuse more efficiently and spread further. For larger n, the mobility is much more compressed for small f, resulting in a more localized and less spread-out profile. For large c, the function f is large at the origin by mass conservation. Therefore, the mobility $|f|^n$ increases rapidly as n increases. Thus, for higher n values, the profile spreads more, which is in contrast to small c values.

7.6. Behavior as $n \to 0$

To study the behavior as $n \to 0$ for self-similar solutions, we will compare it with the biharmonic heat equation with a source type initial condition

$$\begin{cases} \partial_t u = -\partial_x^4 u & \text{in } S = \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = c\delta(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{7.5}$$

where $c > 0$. Observe that this equation corresponds to $n = 0$ for equation (1.1) which serves as a linear reference for understanding the behavior of solutions as $n \to 0$. The biharmonic heat equation also describes a diffusive process, but one that spreads more slowly and smoothly than the regular heat equation. The biharmonic heat equation also preserves mass conservation. We can solve the equation analytically using Fourier transforms and known Fourier integral identities, drawing on definitions and results presented in [19]. The Fourier transform in space $\mathscr{F}[u(x,t)](\omega,t) = \hat{u}(\omega,t)$ is defined as

$$\mathscr{F}[u(x,t)](\omega,t) = \int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx.$$

Taking the Fourier transform on both sides of equation (7.5) gives

$$\mathscr{F}[\partial_t u(x,t)](\omega,t) = \partial_t \hat{u}(\omega,t),$$
$$\mathscr{F}[-\partial_x^4 u(x,t)](\omega,t) = -\omega^4 \hat{u}(\omega,t).$$

The initial condition becomes

$$\mathscr{F}[c\delta(x)](\omega,0) = c.$$

This reduces the problem to

$$\partial_t \hat{u}(\omega,t) = -\omega^4 \hat{u}(\omega,t).$$

Solving this results in

$$\hat{u}(\omega,t) = ce^{-\omega^4 t}.$$

Applying the inverse Fourier transform yields

$$u(x,t) = \frac{c}{2\pi}\int_{-\infty}^{\infty} e^{-\omega^4 t}e^{i\omega x}\,d\omega. \tag{7.6}$$

Applying a change of variables $\omega = st^{-\frac{1}{4}}$ gives

$$u(x,t) = t^{-\frac{1}{4}}\frac{c}{2\pi}\int_{-\infty}^{\infty} e^{-s^4}e^{isxt^{-\frac{1}{4}}}\,ds.$$

Therefore, equation (7.5) has a self-similar form

$$u(x,t) = t^{-\frac{1}{4}}H(\eta), \qquad \eta = xt^{-\frac{1}{4}}, \tag{7.7}$$

where $H(\eta)$ is defined as

$$H(\eta) = \frac{c}{2\pi}\int_{-\infty}^{\infty} e^{-s^4}e^{is\eta}\,ds.$$

Therefore, to compare the self-similar solution of the thin-film equation with the solution of the biharmonic heat equation, we can set $t = 1$. At this time scale, the biharmonic heat kernel attains its canonical self-similar profile, allowing us to directly compare the two solutions. We set $n = 10^{-16}$ and we will solve the solutions for both $c = 1$ and $c = 1000$. The results are shown in Figure 7.19 and Figure 7.20.

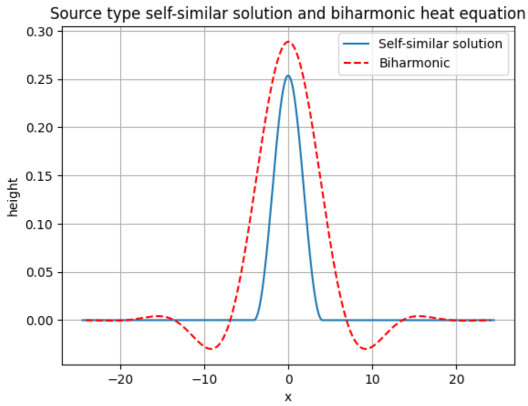

Figure 7.19: The self-similar thin-film solution compared to the biharmonic heat equation solution. Here, $c = 1$.

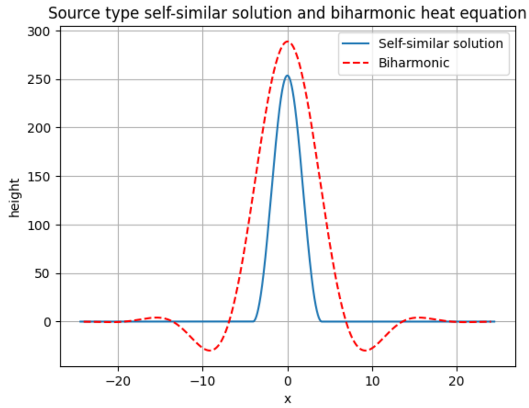

Figure 7.20: The self-similar thin-film solution compared to the biharmonic heat equation solution. Here, $c = 1000$.

We observe that the self-similar solution remains positive, while the biharmonic heat equation solution develops oscillations and takes on negative values. As a result, its peak at the origin is higher to ensure mass conservation. Furthermore, the volume c does not change the shape of the solution. This qualitative difference in profile shape confirms that the limit $n \to 0$ does not exist in a meaningful analytical sense. Therefore, we can conclude that the nonlinear thin-film equation with $n > 0$ and the linear biharmonic heat equation with $n = 0$ give rise to solution profiles that belong to fundamentally different classes, and that the biharmonic heat equation cannot be viewed as a true limit of the thin-film equation as $n \to 0$.

8 Conclusion

The main goal of this thesis was to expand the content of the paper from Bernis, Peletier & Williams [4] by rewriting it and providing more detail to the proofs. The paper focuses on the existence and uniqueness of nonnegative, even, source type self-similar solutions

$$\begin{cases} (|f(\mu)|^n f'''(\mu))' = \alpha(\mu f(\mu))' & \text{for } -\infty < \mu < \infty, \\ \mu f(\mu) \to 0 & \text{as } \mu \to \pm\infty, \end{cases}$$

which satisfy the integral condition

$$\int_{-\infty}^{\infty} f(\mu)\, d\mu = c.$$

This is done by first proving that no non-trivial solution exists when $n \geq 3$ and that for $0 < n < 3$, the function has qualitative properties such as compact support. After that, a shooting method is established to find a solution that satisfies those qualitative properties. Furthermore, uniqueness is proven together with the fact that all nonnegative self-similar solutions are symmetric for $0 < n \leq 2$.

This thesis also constructed the nonnegative symmetric self-similar solution numerically, which visualizes how the self-similar solution looks and how it behaves. It also shows that the numerically constructed solution indeed satisfies the qualitative properties which were proven analytically. Moreover, we analyzed the behavior near $\mu = a$ to understand the sharp cutoff at the edge of its support as n gets larger and we also studied the behavior as $n \to 0$ where we concluded that the biharmonic heat equation cannot be viewed as a true limit of the thin-film equation.

Furthermore, the one-dimensional thin-film equation was derived from the Navier-Stokes equations using a lubrication approximation to give a more physical insight into how the solution behaves.

A

Appendix

A.1. Self-similar solutions constructed using an implicit RK4 method

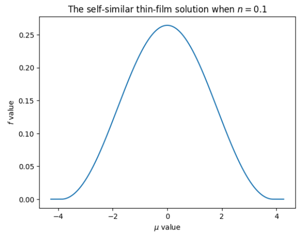

Figure A.1: The numerical solution of the self-similar thin-film equation for $n = 0.1$ using an implicit RK4 method. Here, $c = 1$.

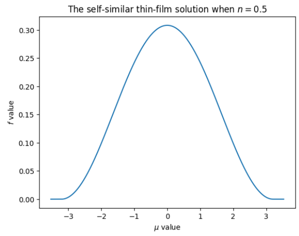

Figure A.2: The numerical solution of the self-similar thin-film equation for $n = 0.5$ using an implicit RK4 method. Here, $c = 1$.

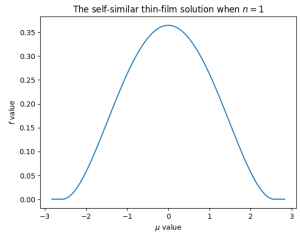

Figure A.3: The numerical solution of the self-similar thin-film equation for $n = 1$ using an implicit RK4 method. Here, $c = 1$.

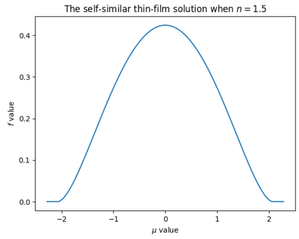

Figure A.4: The numerical solution of the self-similar thin-film equation for $n = 1.5$ using an implicit RK4 method. Here, $c = 1$.

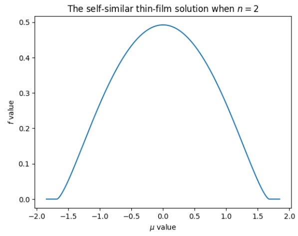

Figure A.5: The numerical solution of the self-similar thin-film equation for $n = 2$ using an implicit RK4 method. Here, $c = 1$.

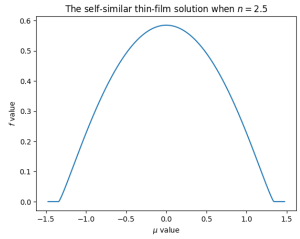

Figure A.6: The numerical solution of the self-similar thin-film equation for $n = 2.5$ using an implicit RK4 method. Here, $c = 1$.

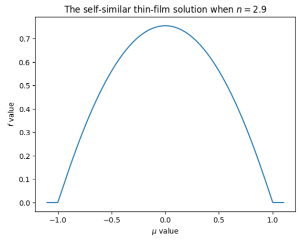

Figure A.7: The numerical solution of the self-similar thin-film equation for $n = 2.9$ using an implicit RK4 method. Here, $c = 1$.

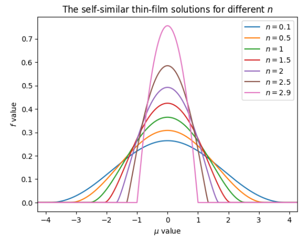

Figure A.8: The numerical solution of the self-similar thin-film equation for different n using an implicit RK4 method. Here, $c = 1$.

A.2. Additional definitions and theorems

Definition A.1. *(Open and closed balls) [7] Let M be a set and $r > 0$. The set $B(x,r) = \{y \in M : |x-y| < r\}$ is called the open ball around x with radius r. The set $\overline{B}(x,r) = \{y \in M : |x-y| \leq r\}$ is called the closed ball around x with radius r.*

Definition A.2. *(Open set) [7] Let M be a set. The set M is called open if, for all $x \in M$, there exists an $\epsilon > 0$ such that $B(x,\epsilon) \subset M$.*

Definition A.3. *(Convexity) [1] Let $f : D \to \mathbb{R}$. f is said to be a convex function if and only if, for all $x_1, x_2 \in D$ and $\alpha \in [0,1]$, we have*

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2).$$

Therefore, f is also convex if $f'' \geq 0$. If there is a strict inequality for $\alpha \in (0,1)$, then f is called strictly convex.

Definition A.4. *($\mathcal{O}$-symbol) [14] Let f and g be given functions. Then $f = \mathcal{O}(g(x))$ for $x \to 0$, if there exist a $r > 0$ and $M > 0$ such that*

$$|f(x)| \leq M|g(x)| \qquad \text{for all } x \in [-r,r].$$

Theorem A.1. ***Fundamental Theorem of Calculus*** *[24] Let f be integrable on $[a,b]$. For each $x \in [a,b]$, let*

$$F(x) = \int_a^x f(t)\,dt.$$

If f is continuous at $c \in [a,b]$, then F is differentiable at c and

$$F'(c) = f(c).$$

Also, if F is differentiable on $[a,b]$ and f is integrable on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(a) - F(b).$$

Theorem A.2. ***Mean Value Theorem*** *[24] Let f be a continuous function on $[a,b]$ that is differentiable on (a,b). Then there exists at least one point $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{a-b}.$$

Theorem A.3. ***Extreme Value Theorem*** *[2] Let f be a continuous function on $[a,b]$. Then f is bounded and it attains a $c,d \in [a,b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a,b]$.*

Theorem A.4. ***Fubini's Theorem*** *[18] Let $B = [a_1,b_1] \times [a_2,b_2]$ be a closed triangle on $\mathbb{R}^2$ and let $f : B \to \mathbb{R}$ be an integrable function. Then*

$$\begin{aligned}\iint_B f(x,y)\,d(x,y) &= \int_{a_1}^{b_1} \left\{ \int_{a_2}^{b_2} f(x,y)\,dy \right\} dx \\ &= \int_{a_2}^{b_2} \left\{ \int_{a_1}^{b_1} f(x,y)\,dx \right\} dy.\end{aligned}$$

Theorem A.5. ***Bolzano-Weierstrass Theorem*** *[24] Every bounded sequence* $\{s_n\} \subset \mathbb{R}$ *has a convergent subsequence.*

Theorem A.6. *[7] If every subsequence of* $\{s_n\}$ *has a further subsequence that converges to s, then* $\{s_n\}$ *converges to s.*

A.3. Numerical solution code

```
# Self-Similar Solutions to the Thin-Film Equation

# Implementing the Shooting Method

# The self-similar solution for the thin-film equation is constructed using numerical methods.
# In order to use it, one should change the c value to its desired value
# and also add the desired n value with 0 < n < 3 in the list nvalues.
# The last codeblock returns a gif which plots the n values ranging
# from 0.1 to 2.9 with step size 0.1 with the chosen c value.
# The code was made using a Jupyter Notebook file.
# Each code cell is enclosed between lines starting with ######.

##############################################################

# Loading the needed packages

import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp
from scipy.interpolate import interp1d
from matplotlib.animation import FuncAnimation
from scipy.fft import fft, ifft, fftfreq
import warnings

warnings.filterwarnings("ignore")

###############################################################

# Defining the parameters

#Defining the different 0 < n < 3
nvalues = [0.1, 0.5, 1, 1.5, 2, 2.5, 2.9]

#Setting the mass of the droplet
c = 1




#Defining the starting parameters gamma, where gamma_plus is in S+ and gamma_minus is in S-
gamma_minus = 10
gamma_plus = 0.001

#Setting the maximum step size and xmax
Dx = 0.01
xmax = 1e6
```

```
#Defining a bound for u and u_prime
bound = 1e-10

#Max iterations for the bisection method
max_iter = 1000

#################################################################

# The Non-linear System

def f(x: float, n: float, u: np.ndarray) -> np.ndarray:
    """
    The non-linear system.

    Parameters:
        x (float): x value.
        n (float): n value.
        u (np.ndarray): array with value u, u' and u'' at point x.

    Returns:
        f (np.ndarray): array with the values evaluated in the non-linear system at x.
    """
    u1, u2, u3 = u
    return np.array([u2,u3,x*float(u1)**(1-n)])

#################################################################

# Time Integration

#These two functions make sure that the solver terminates when u or u' hit zero
def u_is_zero(x: float, y: float) -> float:
    return y[0]

u_is_zero.terminal = True
u_is_zero.direction = -1

def u_prime_is_zero(x: float, y: float) -> float:
    return y[1]

u_prime_is_zero.terminal = True
u_prime_is_zero.direction = 1


def TimeIntegration(Dx: float, xmax: float, n: float, u_0: np.ndarray) -> tuple:
    """
    The time integration method. Uses the built-in scipy function "solve_ivp"
    with standard method explicit RK4.

    Parameters:
        Dx (float): update step size.
        xmax (float): maximum x value before the function terminates.
        n (float): n value.
        u_0 (np.ndarray): array with the initial values.

    Returns:
        u_values, x_values, u, u_prime (tuple): u_values and x_values after time integration
```

```
        and u and u_prime at the last integration timestep.
    """
    sol = solve_ivp(
        lambda x, y: f(x, n, y),
        [0,xmax],
        u_0,
        events =[u_is_zero,u_prime_is_zero],
        rtol = 1e-16,
        atol = 1e-16,
        max_step = Dx
    )

    return sol.y[0], sol.t, sol.y[0][-1], sol.y[1][-1]

#################################################################

# The Bisection Method

def Bisection(Dx: float, xmax: float, n: float, gamma_minus: float, gamma_plus: float,
              bound: float, max_iter: float) -> tuple:
    """
    The bisection method. Used to find the correct gamma such that u and u' become
    simultaniously zero.

    Parameters:
        Dx (float): update step size.
        xmax (float): maximum x value before the function in TimeIntegration terminates.
        n (float): n value.
        gamma_minus (float): gamma value which is in the set S-.
        gamma_plus (float): gamma value which is in the set S+.
        bound (float): bound for u and u' which they need to satisfy in order to find
        the correct gamma.
        max_iter (float): maximum number of iterations of the bisection method before
        it terminates.

    Returns:
        u_values, x_values, opt_gamma, u, u_prime (tuple): u_values of the optimal gamma
        x_values of the optimal gamma and u and u_prime are the exact values at
        the last timestep of the optimal gamma.
    """

    #Defining the initial values for S- and S+ and performing time integration
    u_0_minus = np.array([1,0,-gamma_minus])
    u_0_plus = np.array([1,0,-gamma_plus])

    _, _, _, u_prime = TimeIntegration(Dx = Dx, xmax = xmax, n = n, u_0 = u_0_plus)
    _, _, u, _ = TimeIntegration(Dx = Dx, xmax = xmax, n = n, u_0 = u_0_minus)

    #Defining the iteration parameter
    iter = 0

    #A while loop to keep performing the bisection method until it hits the correct gamma
    #or max iterations
    while True:
        #Defining a new gamma by taking the average of gamma+ and gamma- and
        #performing time integration
```

```
        new_gamma = (gamma_minus + gamma_plus)/2
        u_0 = np.array([1,0,-new_gamma])
        u_values, x_values, u, u_prime, = TimeIntegration(Dx = Dx, xmax = xmax, n = n,
                                                          u_0 = u_0)

        #If the correct gamma is found which satisfy the bounds, the values are returned
        if u<bound and -u_prime<bound:
            return u_values, x_values, new_gamma, u, u_prime

        #Updating the iteration step and if it hits the max iterations, the function terminates
        iter += 1
        if iter == max_iter:
            print("Max␣Iterations␣hit")
            return u_values, x_values, new_gamma, u, u_prime

        #Checking in which set our new gamma is and updating our old gamma in order to apply
        #the bisection method again
        if u < bound:
            gamma_minus = new_gamma
        else:
            gamma_plus = new_gamma

################################################################

# Mass Conversation

def TrapezoidalRule(x_values: np.ndarray, u_values: np.ndarray) -> float:
    """
    The trapezoidal rule for integrals. Uses the trapezoidal rule with composition.

    Parameters:
        x_values (np.ndarray): x values of the numerically approached ODE.
        u_values (np.ndarray): u values of the numerically approached ODE.

    Returns:
        integralvalue (float): value of the integral using trapezoidal rule.
    """
    res = 0
    for i in range(len(x_values)-1):
        res += (x_values[i+1]-x_values[i])/2 * (np.abs(u_values[i])+np.abs(u_values[i+1]))
    return res



def Original_ODE(c: float, alpha: float, x_values: np.ndarray, u_values: np.ndarray,
                 HeatEquation: bool = False) -> tuple:
    """
    Returns the original ODE.

    Parameters:
        c (float): mass of the droplet.
        alpha (float): self-similar value.
        x_values (np.ndarray): x values of the numerically approached ODE.
        u_values (np.ndarray): u values of the numerically approached ODE.
        HeatEquation (bool): this parameter is used to change the grid size when comparing
        to the biharmonic heat equation. Standard value is False.
```

```
    Returns:
        new_x_values, new_u_values, support, l, k (tuple): new_x_values is an array of the new
        x values, new_u_values is an array of the original u values, support is the x value
        where u and u' hit zero for the original ODE, l and k are the values that are needed
        for the transformation to the original ODE.
    """

    #Calculating the k and l value
    norm1 = 2*TrapezoidalRule(x_values = x_values, u_values = u_values)
    k = ((alpha*c**4)/norm1**4)**(alpha)
    l = (k*norm1)/c

    #Determining the x value where u and u' hit zero for the original ODE
    support = x_values[-1]/l

    #Creating an interpolation function in order to get get the transformation function
    #f(x) = k*u(l*x)
    outside_support_x = np.linspace(x_values[-3],support+10, num = 100000)
    outside_support_u = np.zeros(100000)
    u_interp = interp1d(np.concatenate((x_values[:-10],outside_support_x)),
                        np.concatenate((u_values[:-10],outside_support_u)), kind='cubic',
                        fill_value='extrapolate', bounds_error = False)

    #Getting the new x and u values
    if HeatEquation:
        new_x_values = np.linspace(0,support+5*support, num = 3000)
    else:
        new_x_values = np.linspace(0,support+support/10, num = 300)
    new_u_values = k*u_interp(l*new_x_values)

    #Mirroring the values by symmetry
    new_u_values = np.concatenate((new_u_values[::-1], new_u_values[1:]))
    new_x_values = np.concatenate((-new_x_values[::-1], new_x_values[1:]))

    return new_x_values, new_u_values, support, l, k

###############################################################

# The Results

#Defining lists where the optimal values will be stored
u_zero = []
u_prime_zero = []
gamma_opts = []
x_value_zero = []
masses = []

#Storing the support and x and u values for one plot with all n values
x_values_stored = []
u_values_stored = []
support_stored = []

#A loop is done for all the n values
```

```
for n in sorted(nvalues):

    #Here the optimal gamma with the x and u values are calculated
    u_values, x_values, gamma_opt, u, u_prime = Bisection(Dx = Dx, xmax = xmax, n = n,
                                                          gamma_minus = gamma_minus,
                                                          gamma_plus = gamma_plus,
                                                          bound = bound, max_iter = max_iter)

    gamma_opts.append(gamma_opt)

    #Here the orginal x and u values of the ODE are returned
    alpha = 1/(n+4)
    original_x_values, original_u_values, support, l, k = Original_ODE(c = c, alpha = alpha,
                                                                       x_values = x_values,
                                                                       u_values = u_values)

    u_zero.append(u*k)
    u_prime_zero.append(u_prime*k*l)
    x_value_zero.append(support)

    #Here the mass is calculated
    masses.append(TrapezoidalRule(x_values = original_x_values, u_values = original_u_values))

    support_stored.append(support)
    u_values_stored.append(original_u_values.tolist())
    x_values_stored.append(original_x_values.tolist())

    #Plotting the different n values
    plt.plot(original_x_values, original_u_values)
    plt.title(f"The self-similar thin-film solution when $n = {n}$")
    plt.xlabel("$\mu$ value")
    plt.ylabel("$f$ value")
    plt.show()

#Plotting the different n values in one plot
for i, n in enumerate(nvalues):

    x_values_stored = np.array(x_values_stored)
    u_values_stored = np.array(u_values_stored)

    outside_support_x1 = np.linspace(-max(support_stored)-2,x_values_stored[i,0], num = 10)
    outside_support_x2 = np.linspace(x_values_stored[i,-1], max(support_stored)+2, num = 10)
    outside_support_u = np.zeros(10)

    plt.plot(np.concatenate((outside_support_x1, x_values_stored[i], outside_support_x2)),
             np.concatenate((outside_support_u, u_values_stored[i], outside_support_u)),
             label = f"$n = {n}$")



plt.xlim(-max(support_stored)-max(support_stored)/10,
         max(support_stored)+max(support_stored)/10)
plt.ylim(-np.max(u_values_stored)/20, np.max(u_values_stored)+np.max(u_values_stored)/20)
plt.title(f"The self-similar thin-film solutions for different $n$")
plt.xlabel("$\mu$ value")
plt.ylabel("$f$ value")
```

```
plt.legend()
plt.show()


#Printing the values
print("{:>10s}␣|␣{:>23s}␣|␣{:>23s}␣|␣{:>23s}␣|␣{:>23s}␣|␣{:>23s}".format('n␣value',
    'gamma_0', 'a', 'f(a)', "f'(a)", 'integral␣of␣f'))
print("-"*140)
for n, gamma_opt, x_zero, u, u_prime, mass in zip(nvalues, gamma_opts, x_value_zero, u_zero,
                                                    u_prime_zero, masses):
    print('{:>10f}␣|␣{:>23.18f}␣|␣{:>23.18f}␣|␣{:>23.18f}␣|␣{:>23.18f}␣|{:>23.18f}'.format(n,
        gamma_opt, x_zero, u, u_prime, mass))

##############################################################

#  Checking if n = 1 corresponds to the analytical solution

def f_n1(mu: np.ndarray, c: float) -> np.ndarray:
    """
    The exact closed form self-similar solution when n = 1

    Parameters:
        mu (np.ndarray): mu values.
        c (float): the volume c.

    Returns:
        f (np.ndarray): array with the values evaluated in the exact closed form at mu when n =1.
    """
    a = (225*c/2)**(1/5)
    res = []
    for elt in mu:
        if abs(elt) > a:
            res.append(0)
        else:
            res.append(1/120*(a**2-elt**2)**2)
    return np.array(res)

n = 1

u_values, x_values, _, _, _ = Bisection(Dx = Dx, xmax = xmax, n = n,
                                        gamma_minus = gamma_minus, gamma_plus = gamma_plus,
                                        bound = bound, max_iter = max_iter)

alpha = 1/(n+4)
original_x_values, original_u_values, support, _, _ = Original_ODE(c = c, alpha = alpha,
                                                                    x_values = x_values,
                                                                    u_values = u_values)

f1 = f_n1(mu = original_x_values, c = c)
plt.plot(original_x_values, original_u_values, label="Numerical")
plt.plot(original_x_values, f1, label="Exact", linestyle='--', color='red')
plt.title(f"The␣numerical␣and␣exact␣self-similar␣thin-film␣solution␣at␣n␣=␣1␣compared")
plt.xlabel("$\mu$␣value")
plt.ylabel("$f$␣value")
plt.legend()
plt.show()
```

```
##################################################################

# Behavior as n tends to zero

#First we compute the self-similar solution for a very small n size

#Choosing a very small n to observe its behavior
n = 1e-16

u_values, x_values, _, _, _ = Bisection(Dx = Dx, xmax = xmax, n = n,
                                        gamma_minus = gamma_minus,
                                        gamma_plus = gamma_plus,
                                        bound = bound, max_iter = max_iter)

alpha = 1/(n+4)
original_x_values, original_u_values, support, _, _ = Original_ODE(c = c, alpha = alpha,
                                                                   x_values = x_values,
                                                                   u_values = u_values,
                                                                   HeatEquation = True)


#Defining new parameters

#The number of gridpoints must be even for a Fourier transform
N = 2**10

#Length of the x values
L = support + 20

#Time
t = 1

#Grid space
dx = L / N

x = np.linspace(-L, L, N)

#Defining the initial condition
#At t = 0, the mass is concentrated at 1 point due to the dirac delta function
u0 = np.zeros(N)
u0[N//2] = c / dx

#Computing the correct wavenumbers for the fast Fourier transform (FFT)
k = 2 * np.pi * fftfreq(N, dx)

#Computing the solution in Fourier space
u0_hat = fft(u0)
u_hat = u0_hat * np.exp(-k**4 * t)

#Applying the inverse FFT and ensuring we only take real values
u = ifft(u_hat).real

#Computing the integral to see if mass conservation holds
integral = np.sum(u) * dx
print(f"Integral of u(x,t) for the biharmonic heat equation at t = {t} is: {integral}")
```

```
#Plotting the self-similar solution and biharmonic heat equation next to each other
plt.plot(original_x_values, original_u_values, label="Self-similar solution")
plt.plot(x, u, label="Biharmonic", linestyle='--', color='red')
plt.xlabel("x")
plt.ylabel("height")
plt.title("Source type self-similar solution and Biharmonic heat equation")
plt.legend()
plt.grid()
plt.show()

##############################################################

# Animation

#n values that will be plotted
n_values = np.linspace(0.1, 2.9, 29)
values = np.linspace(1, 29, 29)


#Storing the support and x and u values
x_values_stored = []
u_values_stored = []
support_stored = []

#Going trough all the n_values
for n in n_values:
    u_values, x_values, _, _, _ = Bisection(Dx = Dx, xmax = xmax, n = n,
                                            gamma_minus = gamma_minus,
                                            gamma_plus = gamma_plus,
                                            bound = bound,
                                            max_iter = max_iter)

    alpha = 1/(n+4)

    original_x_values, original_u_values, support, _, _ = Original_ODE(c = c, alpha = alpha,
                                                                        x_values = x_values,
                                                                        u_values = u_values)

    x_values_stored.append(original_x_values.tolist())
    u_values_stored.append(original_u_values.tolist())
    support_stored.append(support)

x_values_stored = np.array(x_values_stored)
u_values_stored = np.array(u_values_stored)

#Initializing the figure
x = np.linspace(-np.max(x_values_stored)-1, np.max(x_values_stored)+1, 500)
fig, ax = plt.subplots()
line, = ax.plot(x, np.zeros_like(x))
title = ax.set_title("")

ax.set_xlim(-max(support_stored)-1, max(support_stored)+1)
ax.set_ylim(-np.max(u_values_stored)/20, np.max(u_values_stored) + np.max(u_values_stored)/20)

ax.set_xlabel("$\mu$ value")
```

```
ax.set_ylabel("$f$␣value")

#Defining the updating step
def update(i):

    outside_support_x1 = np.linspace(-max(support_stored)-2,x_values_stored[int(i-1),0],
                                     num = 10)
    outside_support_x2 = np.linspace(x_values_stored[int(i-1),-1], max(support_stored)+2,
                                     num = 10)
    outside_support_u = np.zeros(10)

    line.set_data(np.concatenate((outside_support_x1, x_values_stored[int(i-1)],
                  outside_support_x2)), np.concatenate((outside_support_u,
                  u_values_stored[int(i-1)], outside_support_u)))
    title.set_text(f"The␣self-similar␣thin-film␣solution␣with␣$c␣=␣{c}$␣and␣$n␣=␣{0.1*i:.2f}$")
    return line, title

#Defining the animation
ani = FuncAnimation(fig, update, frames=values, interval=200, blit=True)

#Saving the animation to the device
ani.save("Self-Similar_Solution.gif", writer="pillow")

###############################################################
```

Bibliography

[1] Aardal, K., van Iersel, L. & Janssen, R. (2023). *Optimization* [Lecture Notes]. TU Delft.

[2] Andreescu, T., Mortici, C. & Tetiva, M. (2017). *Mathematical Bridges.* Birkhäuser.

[3] Bankoff, S. G., Davis, S. H. & Oron A. (1997, July 1). Long-scale evolution of thin liquid films. *Review of Modern Physics, 69*(3), 931-980. https://doi.org/10.1103/RevModPhys.69.931.

[4] Bernis, F., Peletier, L. A. & Williams, S. M. (1992, February). Source type solutions of a fourth order nonlinear degenerate parabolic equation. *Nonlinear Analysis: Theory, Methods & Applications, 18*(3), 217-234. https://doi.org/10.1016/0362-546X(92)90060-R.

[5] Bernis F. & Friedman A. (1990, January). Higher order nonlinear degenerate parabolic equations. *Journal of Differential Equations 83*(1), 179-206. http://doi.org/10.1016/0022-0396(90)90074-Y.

[6] Bertozzi, A., Buckingham, R. & Shearer, M. (2003). Thin film traveling waves and the Navier slip condition. *SIAM journal on Applied Mathematics, 63*(2), 722-744. https://doi.org/10.1137/S0036139902401409.

[7] Carothers, N. L. (2000). *Real Analysis.* Cambridge University Press.

[8] Carrillo, J. A. & Toscani, G. (2002, February). Long-time asymptotics for strong solutions of the thin film equation. *Communications in Mathematical Physics, 225*(3), 551–571. https://doi.org/10.1007/s002200100591.

[9] Chorin, A. J. & Marsden J. E. (1993). *A Mathematical Introduction to Fluid Mechanics: Third Edition.* Springer.

[10] Chugunova, M., Kang, D., Nadim, A., Walther, F. J. & Waring, A. J. (2018, September 20). Modeling Coating Flow and Surfactant Dynamics inside the Alveolar Compartment. *Journal of Engineering Mathematics, 113*(3), 23-43. https://doi.org/10.1007/s10665-018-9972-8.

[11] Fitzpatrick P. M. & Royden, H. L. (2010). *Real Analysis: Fourth Edition.* China Machine Press.

[12] Folland, G. B. (1999). *Real Analysis Modern Techniques and Their Applications.* John Wiley & Sons.

[13] Giacomelli, L., Gnann, M. V. & Otto F. (2016, July 13). Rigorous asymptotics of traveling-wave solutions to the thin-film equation and Tanner's law. *Nonlinearity, 29*(9), 2497-2536. https://doi.org/10.1088/0951-7715/29/9/2497.

[14] van Gijzen, M. B., Vermolen, F. J., Vuik, C. & Vuik, M. J. (2023). *Numerical Methods for Ordinary Differential Equations.* TU Delft OPEN Publishing. https://doi.org/10.5074/t.2023.001.

[15] Gnann, M. V., Kim, H. J. & Knüpfer, H. (2018, June). Travelling wave solutions for a thin-film equation related to the spin-coating process. *European Journal of Applied Mathematics, 29*(3), 369-392. https://doi.org/10.1017/S0956792517000195.

[16] Gnann, M. V. & Wisse A. C. (2025, March 19). Classical solutions to the thin-film equation with general mobility in the perfect-wetting regime. *Journal of Functional Analysis, 289*(8), 110941. https://doi.org/10.1016/j.jfa.2025.110941.

[17] Greenspan, H. P. (1978, January 16). On the motion of a small viscous droplet that wets a surface. *Journal of Fluid Mechanics, 84*(1), 125-143. https://doi.org/10.1017/S0022112078000075.

[18] Groenevelt, W., Janssens, B., & de Pagter, B. (2020). *Analysis 2* [Lecture Notes]. TU Delft.

[19] Haberman, R. (2012). *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems: Fifth Edition.* Pearson.

[20] Hartman, P. (2002). *Ordinary Differential Equations: Second Edition.* Society for Industrial and Applied Mathematics.

[21] Hill, J. M. & Smyth, N. F. (1988, May 1). High-Order Nonlinear Diffusion. *IMA Journal of Applied Mathematics, 40*(2), 73-86. https://doi-org.tudelft.idm.oclc.org/10.1093/imamat/40.2.73.

[22] John, D. (2013, December 5). *Uniqueness and Stability near Stationary Solutions for the Thin-Film Equation in Multiple Space Dimensions with Small Initial Lipschitz Perturbations* [PhD thesis]. Rheinischen Friedrich–Wilhelms–Universität Bonn. https://hdl.handle.net/20.500.11811/5764.

[23] Knüpfer, H. (2008). *Classical solutions for a thin-film equation* [PhD thesis]. Rheinischen Friedrich–Wilhelms–Universität Bonn. https://hdl.handle.net/20.500.11811/3558.

[24] Lay, S.R. (2014). *Analysis with an introduction to proof.* Pearson.

[25] Taliaferro, S. D. (1979, November). A nonlinear singular boundary value problem. *Nonlinear Analysis 3*(6), 897-904. https://doi.org/10.1016/0362-546x(79)90057-9.

[26] Teschl, G. (2021). *Ordinary Differential Equations and Dynamical Systems.* American Mathematical Society.