Delft University of Technology

The Cognitive Infrastructures of Markets

Empirical Studies on the Role of Categories in Valuation and Competition, and a Formal Theory of Classification Systems Based on Lattices and Order

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Proefschrift

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus prof.dr.ir. T.H.J.J. van der Hagen,
voorzitter van het College voor Promoties,
in het openbaar te verdedigen op donderdag 22 februari 2018 om 10:00 uur

door

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What one wishes to gain from one's categories
is a great deal of information about the environment
while conserving finite resources as much as possible.

Eleanor Rosch on the principle of cognitive economy

CONTENTS

SUMMARY

T HIS DISSERTATION addresses the question of how the information en-
coded by category labels is interpreted by agents in a market for
the purpose of decision-making. To this end, we first examine the
influence of categorization on economic and strategic outcomes with two
empirical studies, and then use the insights provided by these studies to
develop a formal theory of classification systems. Consistently with Formal
Concept Analysis (FCA), this theory builds on the fundamental mathemati-
cal notions of lattices and order, and it is thus uniquely suited to yield an
ontological perspective on category representations. As a result, we are
much better equipped to understand how categories serve as the "cognitive
infrastructures" of markets and affect economic activity. Chapter 1 offers a
concise overview of the extant research on categorization in cognitive psy-
chology, economic sociology, and organization theory. We build extensively
on this diverse literature during the course of our exposition.

The first part of this thesis includes our empirical studies. In Chapter 2,
we synthesize insights from industrial economics, strategic management,
and organizational ecology to examine the effects of product proliferation
strategies. Conceptualizing the market as a multidimensional (Lancastrian)
space of product features, we argue that product categories guide firms'
strategic decisions by partitioning the space into subsets or regions. Prod-
uct proliferation occurs when a firm bids to occupy a product category
at the expense of competitors by saturating the corresponding region of
space. Consistently with game-theoretic models of product competition in
differentiated markets, we predict proliferation to have a negative effect
on the likelihood of rival product introductions in the targeted category;
however, we also predict that this effect is weaker if the region of space to
which the category maps is more complex (i.e., heterogeneous in terms of
product features). Our analysis of firms' patterns of new product introduc-
tions in the US recording industry supports these hypotheses; in addition,
it suggests that product proliferation effectively deters competitors who
can alter their positioning in feature space, but those who are constrained
to particular positions remain virtually unaffected.

In Chapter 3, we turn to consumers' perspective and examine how the cat-
egorization of products according to different classification systems affects

the attribution of value. Focusing on the distinction between categories based on prototypes and categories based on goals, we argue that these category labels of these two kinds map to structurally different regions of the feature space. Valuation requires consumers to infer the location of products from their labels, but because type- and goal-based categories have different internal structures, they enable different sorts of inferences. Building on this argument, we theorize that under particular conditions spanning type-based categories has a U-shaped effect on consumers' evaluations, whereas spanning goal-based categories has a negative effect. At the same time, we predict that spanning goal-based categories can moderate the U-shaped effect of spanning type-based categories by enabling consumers to make more precise inferences from fewer type-based labels. Our analysis of product ratings on a popular music website offers empirical support for these hypotheses.

In the second part of this thesis, we develop a formal theory of categorization that accounts for the key aspects highlighted by our empirical studies. In Chapter 4, we introduce an order-theoretic account of classification systems as RS-frames. These are algebraic structures based on RS-polarities, which we enrich with additional relations to interpret modalities. Consistently with FCA, we propose to interpret an RS-polarity as a database consisting of a set of objects (such as products or organizations in a market), a set of features, and an incidence relation linking objects with their features. All the possible categories whereby the objects and the features may be grouped arise as the Galois-stable sets of this polarity, just like formal concepts in FCA. An agent's perception of the objects and their features, which can be unique, incomplete, or even mistaken, is modeled by a relation giving rise to a normal modal operator that expresses an agent's beliefs about a category's intensional and extensional meaning. The fixed points of the iterations of belief modalities are used to model categories whose meaning is shared as they arise from social interaction.

In Chapter 5, we clarify how the order-theoretic perspective on concepts enabled by FCA complements the geometric perspective allowed by the theory of conceptual spaces. In addition to introducing a sound and complete epistemic-logical language, we refine the framework presented in the previous chapter both technically and conceptually: Technically, because we free its semantics from the restrictions imposed by the RS-conditions and generalize to more natural Kripke-style frames. This makes our formalism better suited to represent formal contexts (i.e., databases) as they occur in real-world domains. Conceptually, because we enhance our theory of classification systems as concept lattices and propose formalizations for

some of the most important theoretical constructs in the categorization literature, including typicality, similarity, contrast, and leniency. In particular, we elaborate our interpretation of the fixed-point construction introduced before by tying it directly to the notion of typicality. Possible extensions are discussed, especially with regard to dynamic updates.

Chapter 6 summarizes the main findings of this dissertation, elucidates their implications for organizational research, identifies key areas for improvement, and presents promising directions for future study. Special consideration is given to the possibility of unifying FCA and conceptual spaces using the framework of correspondence theory. We conclude with a general reflection on the role of logic in the social sciences.

SAMENVATTING

DIT PROEFSCHRIFT gaat in op de vraag hoe de informatie van categorielabels wordt geïnterpreteerd door marktactoren ten behoeve van hun besluitvorming. Hiertoe onderzoeken we eerst de invloed van categorisatie op economische en strategische uitkomsten met twee empirische studies en gebruiken vervolgens de inzichten van deze studies om een formele theorie van classificatiesystemen te ontwikkelen. Overeenkomstig de aanpak van Formeleconceptanalyse (FCA) bouwt deze theorie voort op de fundamentele wiskundige concepten van tralies en orde en is daarom uiterst geschikt voor een ontologisch perspectief op categorierepresentaties. Als gevolg hiervan zijn we veel beter in staat om te begrijpen hoe categorieën dienen als de "cognitieve infrastructuren" van markten en de economische activiteit beïnvloeden. Hoofdstuk 1 biedt een beknopt overzicht van het bestaande onderzoek naar categorisatie in cognitieve psychologie, economische sociologie, en organisatietheorie. We bouwen uitgebreid voort op deze diverse literatuur in deze studie.

Het eerste deel van dit proefschrift bevat onze empirische studies. In Hoofdstuk 2 synthetiseren we inzichten uit de industriële economie, strategisch management, en de organisatie-ecologie om de effecten van productproliferatie-strategieën te onderzoeken. Als we ons de markt kunnen voorstellen als een multidimensionale (Lancastriaanse) ruimte van producteigenschappen, stellen we dat productcategorieën de strategische beslissingen van bedrijven sturen door de ruimte te verdelen in deelverzamelingen of regio's. Productproliferatie treedt op wanneer een bedrijf poogt een productcategorie te bezetten ten koste van concurrenten door het bijbehorende deel van de ruimte te verzadigen. Consistent met speltheoretische modellen van productconcurrentie op gedifferentieerde markten voorspellen we dat proliferatie een negatief effect zal hebben op de waarschijnlijkheid van concurrerende productintroducties in de betroffen categorie; we voorspellen echter ook dat dit effect zwakker is in de delen van de ruimte waar de categorizering meer complex is (d.w.z., meer heterogeen in termen van productkenmerken). Onze analyse van de patronen van nieuwe productintroducties van bedrijven in de Amerikaanse muziekindustrie ondersteunt deze hypothesen; bovendien suggereert het dat productproliferatie concurrenten, die hun positionering in kenmerk-

ruimte kunnen wijzigen, effectief afschrikt, maar weinig effect heeft op degenen die hun posities niet of moeilijk kunnen veranderen.

In Hoofdstuk 3 komt het consumentenperspectief centraal te staan en onderzoeken we hoe de indeling van producten volgens verschillende classificatiesystemen van invloed is op de bepaling van waarde. Door te focussen op het onderscheid tussen categorieën op basis van prototypen en categorieën op basis van doelen, stellen we dat deze categorielabels van deze twee soorten verwijzen naar structureel verschillende regio's van de kenmerk-ruimte. Om waarde te bepalen moet de consument de locatie van producten van hun labels afleiden, maar omdat categorieën op basis van typen en doelgroepen verschillende interne structuren hebben, maken ze verschillende soorten gevolgtrekkingen mogelijk. Voortbouwend op dit argument theoretiseren we dat onder bepaalde omstandigheden type gebaseerde categorieën het hebben van meerdere categorielabels tegelijk een U-vormig effect op consumentenevaluaties heeft, terwijl het hebben van meerdere op doel gebaseerde categorieën een negatief effect heeft. Tegelijkertijd voorspellen we dat het hebben van meerdere labels van doel gebaseerde categorieën het U-vormige effect van het hebben van meerdere labels van type-categorieën kan modereren door consumenten in staat te stellen meer precieze conclusies te trekken op basis van de gecombineerde type gebaseerde labels. Onze analyse van productbeoordelingen op een populaire muziekwebsite ondersteunt deze hypothesen.

In het tweede deel van dit proefschrift ontwikkelen we een formele theorie van categorisering die rekening houdt met de belangrijkste aspecten die worden benadrukt door onze empirische studies. In Hoofdstuk 4 introduceren we een op ordetheorie gebaseerde beschrijving van classificatiesystemen als RS-frames. Dit zijn algebraïsche structuren op basis van RS-polariteiten, die we verrijken met extra relaties om modaliteiten te interpreteren. In overeenstemming met FCA stellen we voor om een RS-polariteit te interpreteren als een database die bestaat uit een reeks objecten (zoals producten of organisaties op een markt), een reeks kenmerken en een incidentie-relatie die objecten koppelt aan hun kenmerken. Alle mogelijke categorieën waarin de objecten en de kenmerken gegroepeerd kunnen worden, ontstaan als de Galois-stabiele verzamelingen van deze polariteit, net zoals formele concepten in FCA. De perceptie van een agent van de objecten en hun kenmerken, die uniek, onvolledig of zelfs fout kan zijn, wordt gemodelleerd door een relatie die aanleiding geeft tot een normale modale operator die de opvattingen van een agent over de intensionale en extensionale betekenis van een categorie uitdrukt. De dekpunten van de iteraties van de modaliteiten van geloof worden gebruikt om categorieën

te modelleren waarvan de betekenis wordt gedeeld als ze voortkomen uit sociale interactie.

In Hoofdstuk 5 verduidelijken we hoe het ordetheoretische perspectief op begrippen die door FCA mogelijk worden gemaakt, een aanvulling vormt op het geometrische perspectief dat door de theorie van conceptuele ruimten is toegestaan. Naast het introduceren van een correct en volledig epistemisch-logische taal, verfijnen we het raamwerk dat in het vorige hoofdstuk werd gepresenteerd, zowel technisch als conceptueel: technisch gezien, omdat we de semantiek bevrijden van de beperkingen opgelegd door de RS-condities en generaliseren naar meer natuurlijke Kripke-achtige frames. Dit maakt ons formalisme beter geschikt om formele contexten (d.w.z., databases) weer te geven zoals deze in de werkelijkheid voorkomen. Conceptueel, omdat we onze theorie van classificatiesystemen als tralies van concepten verbeteren en formalisaties voorstellen voor enkele van de belangrijkste theoretische constructies in de categoriseringsliteratuur, waaronder typicaliteit, gelijkenis, contrast, en toegevendheid. In het bijzonder werken we onze interpretatie van de eerder geïntroduceerde dekpunt-constructie uit door deze direct aan de notie van typicaliteit te koppelen. Mogelijke uitbreidingen worden besproken, vooral met betrekking tot dynamische updates.

Hoofdstuk 6 vat de belangrijkste bevindingen van dit proefschrift samen, licht hun implicaties voor organisatorisch onderzoek toe, identificeert belangrijke verbeterpunten, en suggereert veelbelovende richtingen voor toekomstig onderzoek. Speciale aandacht wordt geschonken aan de mogelijkheid om FCA en conceptuele ruimtes te verenigen met behulp van het algemene kader van correspondentietheorie. We sluiten af met een algemene reflectie op de rol van logica in de sociale wetenschappen.

PREFACE

SUMMARIZING my research into a short pitch to be deployed in elevator conversations that never seem to happen is not a task that puts me at ease. I believe many academics dread this exercise, but it feels all the more frustrating in my case because this dissertation is atypical by any standard. I find myself at a loss whenever I am requested to assign it to a particular category or domain of scholarship. The irony is not lost on me that its topic is precisely categorization, and I do not know what the reader will make of my abilities as a researcher upon learning that I am unable to classify my own doctoral thesis, but such is the case. To give a sense of why this task is difficult for me, I would like to borrow an insightful metaphor Peter Gärdenfors used in his preface to *Conceptual Spaces: The Geometry of Thought* (MIT Press, Cambridge, MA, 2000, p. ix):

> While writing the text [of my book], I felt like a centaur, standing on four legs and waving two hands. The four legs are supported by four disciplines: philosophy, computer science, psychology, and linguistics (and there is a tail of neuroscience). Since these disciplines pull in different directions—in particular when it comes to methodological questions—there is a considerable risk that my centaur has ended up in a four-legged split.

> A consequence of this split is that I will satisfy no one. Philosophers will complain that my arguments are weak; psychologists will point to a wealth of evidence about concept formation that I have not accounted for; linguistics will indict me for glossing over the intricacies of language in my analysis of semantics; and computer scientists will ridicule me for not developing algorithms of the various processes that I describe.

> I plead guilty to all four charges. My aim is to unify ideas from different disciplines into a general theory of representation. This is a work within cognitive science and not one in philosophy, psychology, linguistics, or computer science. My ambition here is to present a coherent research program that others will find attractive and use as a basis for more detailed investigations.

I hope to be forgiven for my hubris in borrowing this incipit, but it is indeed useful to think of this thesis as a creature that stands on four legs and uses two hands to engage with the object of its inquiry. Unlike Gärdenfors' centaur, however, this creature can be pictured as having not one but two heads. I find this picture more faithful and agreeable than any category label, if a little grotesque.

In my case, the four legs represent the fields of sociology, economics, psychology, and mathematics. The combination of these research domains is far from haphazard: some of them co-occur quite regularly in the study of social phenomena, as in the case of economic sociology, social psychology, behavioral economics, and game theory. The four of them together, however, make a relatively uncommon sight. I hope to present a good case that their match is worthwhile. The two hands represent two scientific methodologies commonly used in the disciplines above to answer their respective research questions, i.e., statistical modeling and logical formalization. Because of the relative independence with which these are deployed toward a common objective, this dissertation consists of two parts. Finally, the two heads represent two separate but (I argue) mutually compatible ways of reasoning about categories. One of them views categories geometrically as the regions of a conceptual space; the other views them order-theoretically as the elements of a concept lattice. Each perspective is powerful enough on its own to yield an insightful account of categorization, but together they offer an unparalled glimpse into its mathematical nature.

In light of all the elements incorporated in this thesis, my reluctance to apply a category label is perhaps slightly more understandable. Above all, I believe it would be restrictive to consider its scope limited to the field of business: markets are but one of many settings where categories exert their influence, and many studies cited in the following chapters concern people's behavior in different contexts. The aim of the Applied Logic Group at the Delft University of Technology, in whose womb my dissertation took shape, is to weave together insights from the social and the exact sciences in order to shed light on the epistemic foundations of social behavior. My thesis is a step toward this general purpose: although it addresses questions related to economic decision-making, its implications are much broader in principle. My aim is not to "explain" categorization in markets in a way that appears satisfactory to sociologists, economists, psychologists, and mathematicians, but to show a novel, integrative, and rigorous way to study categorization and other social phenomena.

My main ambition is to introduce scholars in sociology and management to a formalism developed by Bernhard Ganter and Rudolf Wille during

the Nineties and widely known as Formal Concept Analysis (FCA). This is an algebraic method for representing categories or concepts that proved suitable for a wide range of interdisciplinary applications, including economics (*Formal Concept Analysis: Foundations and Applications*, Springer, Berlin/Heidelberg, Germany, 2005). Given the current enthusiasm among social scientists for machine-learning techniques, it is surprising that FCA remains obscure. I believe this is partly because social scientists overwhelmingly favor verbal theorizing: in fact, it has been pointed out repeatedly to me that publishing on journals in sociology and management requires the scientist to be a little bit of a novelist. Storytelling is crucial to making one's findings interesting to a journal's readership. I think this embellishment would dignify scientific research were not natural language so unbearably ambiguous. Given the current state of our knowledge about social behavior, it would be preferable to set aside the "storylines" to better spend one's cognitive resources on the formalization of causal relationships.

There is undoubtedly a learning curve to using formal language, but the potential benefits for social science are immense. Much like a line of code allows a machine to perform in seconds operations that would take months to accomplish manually, logical formulas can express relationships, conditions, and deductive steps that would be cumbersome if not impossible to render verbally with the same level of accuracy. Moreover, like good code, formal language has the advantage of being succinct. By using it in theory construction we stand to gain not only expressive power but also mathematical beauty. There are also clear benefits in terms of generality: as a prime example, FCA is agnostic to the nature of the objects to be sorted into categories. This dissertation usually assumes they are products, but the same framework can be used to reason about categories of services, patents, organizations, or even firm strategies, architectures, and routines. Given the amount of (informal) theories put forward by sociologists and management scientists on a yearly basis—a large number of which turn out to be too vague, context-dependent, inconsistent, circular, redundant, or not theories at all—the adoption of symbolic language and formal rules of inference appears long overdue.

Fortunately, I am neither the sole nor by any means the most qualified researcher to draw attention to this problem. The idea that social science is at present too ambiguous for its own good was the premise of a grant awarded in 2013 by the Netherlands Organization for Scientific Research to Alessandra Palmigiano (a logician) and Nachoem Wijnberg (a management scientist), under whose joint supervision this dissertation was produced. The same concern had been expressed before by three eminent sociologists,

Michael Hannan, László Pólos, and Glenn Carroll, in *Logics of Organization Theory: Audiences, Codes, and Ecologies* (Princeton University Press, Princeton, NJ, 2007). Even earlier, bridging logic and the social sciences was among the objectives of the Center for Computer Science in Organization and Management, an interdisciplinary research venture started at the University of Amsterdam in 1990. This dissertation owes much to these early attempts to introduce logic to an audience overly accustomed to verbal theory and does not necessarily realize its limitations.

As if to close a circle, toward the end of my Ph.D. I had the privilege of reading the draft of a new book by Hannan, Pólos, and their colleagues, provisionally titled *Concepts and Categories: Foundations for Sociological Analysis*. Like this thesis builds on FCA, Hannan *et al.* build on Gärdenfors' spatial approach to study concept formation and inference, offering a mathematical framework to scholars interested in social categories. I find most of their arguments to be compatible with those presented here, especially with regard to the lattice-based interpretation of classification systems. Hannan *et al.* model these systems as (upper) semilattices whereas we opt for complete lattices, which essentially amounts to adding a lower bound. They define a category's prototype as the region of conceptual space where the likelihood that agents recognize an object as a member of the category (or an instance of the concept) is maximal; following the same intuition, we define it as the set of features that every agent attributes to the concept and knows to be attributed to the concept by every other agent. Both frameworks allow us to connect the cognitive-psychological notion of typicality to the sociological construct of taken-for-grantedness. Hannan *et al.* define this as the extent to which the agents assume to share the same meaning for a given concept and thus need not observe each other's categorization decisions to know if they agree with them. Though we do not formalize taken-for-grantedness here, the epistemic language we build on top of FCA is capable of describing it in similar terms.

In addition, there are important complementarities between the two approaches. Hannan *et al.* focus on the cognitive mechanisms whereby an agent comes to believe that an object has particular features and hence constitutes an instance of a certain concept. To make this problem tractable, they restrict the agent's consideration to a given "root" concept, which sets the boundaries of the cognitive domain (e.g., ROCK MUSIC), and to a cohort of concepts that are immediately subordinate to the root and represent the possible alternatives (HARD ROCK, POST-ROCK, etc.). In this thesis, we do not ask why agents believe some objects to be instances of particular concepts, but we examine the consequences of these beliefs for their entire

classification system, including the concepts subordinate to the focal cohort and those superordinate to the root. Therefore, while Hannan *et al.* examine concept formation and inference, we focus on knowledge organization. I can think of no better way to understand the formal nature of categorization than to try and unify these two perspectives.

At present, the connection between conceptual spaces and FCA is purely intuitive. Its formalization is not easy to achieve because the two frameworks draw on separate branches of mathematics—geometry and algebra, respectively—and thus differ in some fundamental respects. One of them is that, at least in Ganter and Wille's formulation, FCA does not allow us to encode the gradedness of category memberships. In this view, an object is considered either within or outside the category and partiality is impossible. By contrast, conceptual spaces are well-suited to model partial memberships by exploiting the metric nature of space and the real-valued distance of the category members from a category's prototype. Being tools of discrete mathematics, lattice-based methods such as FCA are less naturally equipped to account for this information.

Nonetheless, I expect this divide will be bridged in time as there are ongoing efforts to merge FCA and metric spaces into a generalized theory of representation (e.g., Dusko Pavlovic, *Quantitative Concept Analysis*). This goes to show not only that research on categories and concepts pushes the limits of scientific knowledge, but also that major breakthroughs in this direction require a concerted effort by scholars engaged on different fronts. In light of these difficulties, this dissertation may only be considered a building block. The objective of developing a formally and empirically faithful account of categorization in social domains is far from being attained. But from our vantage point, the horizon looks promising.

Michele Piazzai
Delft, December 2017

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INTRODUCTION

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1.1. Categories: Why Bother?

Describing categories as cognitive devices that simplify decision-making is fundamentally correct, but it is a massive understatement. Categorization is the basic epistemic recourse that allows human beings to entertain whatever hope of survival, both individually and as a species. It is the reason why someone who experiences pain as a result of mishandling a KNIFE can expect to suffer the same pain from mishandling any other KNIFE, or indeed, any other SHARP EDGE. It is also the reason why one is usually able to open a DOOR without necessarily having seen this particular door before, or to drive a new CAR after having learned how to drive another. While the capacity to update one's behavior on the basis of past experience is what makes an individual capable of driving the same car, opening the same door, or handling the same knife differently in the future, it is the nontrivial ability to recognize some relevant patterns in the perceived structure of the world [1] that enables one to redeploy the accumulated knowledge to novel situations and tasks.

The human brain is hardwired to detect correlations in the characteristics of the objects about which it is called to make decisions. It is readily apparent to us that the objects' attributes do not coincide with equal probabilities: "some pairs, triples, etc., are quite probable, appearing in combination sometimes with one, sometimes another attribute; others are rare; others logically cannot or empirically do not occur" [2, p. 253]. *Categorization* is the process whereby these regularities are acknowledged and the objects that display them sorted into sets. The consequence of this process is that some of the perceivable distinctions between objects, which are deemed irrelevant to the decision at hand, are effectively suppressed. One advantage of imposing such filters on the information coming in from the senses and to the brain is that sets of objects can be assigned a default behavioral response, such as preference or aversion, which saves the agent cognitive effort in future decisions involving similar objects. Another advantage is that semantic identifiers, or *category labels*, can be attached to the sets so as to allow multiple agents to communicate efficiently. It is thanks to category labels, for instance, that airplane passengers can easily determine whether a meal is consistent with their dietary restrictions upon reading that it is LACTOSE-FREE, VEGETARIAN, or HALAL, and that they generally know what do at border control after reading that one line is reserved to LOCAL NATIONALS and one is for other passport holders.

Given that the need for coordination underlies most human activities, it is hardly surprising that categories dominate many aspects of social life [3]. We congregate in PUBLIC PLACES, seek FULL-TIME or PART-TIME EMPLOYMENT,

visit ART MUSEUMS, and in some cases, read PEER-REVIEWED RESEARCH and perform REPLICABLE STUDIES to earn DOCTORAL DEGREES from TECHNICAL UNIVERSITIES. The influence of categories is inescapable: anyone who happens to forget what constitutes FORMAL ATTIRE and tries to defend a doctoral thesis wearing a pirate costume [4] will be promptly reminded of this fact. Categorization is, by extension, at the heart of the institutions that form the very fabric of society. Deciding what makes a GOVERNMENT, a MARRIAGE, a SUSTAINABLE TECHNOLOGY, or—moving to the realm of economics—a MARKET as opposed to an ORGANIZATION [5], is essentially a classification problem. The very execution of economic transactions rests upon the expectations induced by category labels, not only because these allow BUYERS and SELLERS to carry out a codified sequence of actions that results in a legitimate transfer of property, but also because words like BANKING, FREELANCING, and RETAIL underpin the flow of capital [6] and labor [7, 8].

Yet categories regulate economic activity well beyond the effecting of transactions. It is because of category-induced expectations, for instance, that customers do not consider a CHINESE RESTAURANT inauthentic because its staff refuses to perform ACUPUNCTURE, but they very well might if the restaurant serves ENCHILADAS [cf. 9]. Likewise, it is because of consumer segmentation (a synonym of categorization) that companies like Maserati, Bugatti, and Lamborghini do not advertise their products to TEENAGERS. Such default expectations are not necessarily correct: some teenagers can afford to buy LUXURY CARS, but the generalization is nonetheless useful to optimize a firm's use of resources. The existence of *strategic groups*, i.e., categories of firms that maintain a similar competitive positioning [10], is the reason why Maserati executives are likely to be well aware of which cars are sold by Lamborghini and Bugatti, where, when, and at what price, but may be relatively indifferent to what companies like Dacia and Suzuki are up to [11, 12]. On the demand side, categories are the reason why fans of HEAVY METAL may assume that other fans of HEAVY METAL are worthy of social interaction whereas fans of TECHNO are not [cf. 13]. It is in the attempt to exploit categorization processes that, as Bourdieu [14] reminds us, some people buy OPERA tickets even though they have no taste for the subject, but simply because they wish to be seen sitting beside people who do.

Explaining how categories affect the functioning of markets is currently among the foremost objectives for students of organization theory [15]. Two streams of literature stand out for their significant contributions to this research agenda. One originates from social psychology [16], and primarily concerns itself with the mechanisms whereby firms and their constituents use category labels to define their own identity [17, 18]. The other is rooted

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in economic sociology and revolves around the notion of "categorical imperative." In Zuckerman's formulation [19], this refers to the pressure exerted on firms by their external stakeholders, like analysts, critics, investors, and customers, who expect conformity to their category definitions. This latter perspective is widely represented in the work of organizational ecologists [20], who seek to explain firms' success in the market through the opposing forces of competition, which promotes differentiation, and legitimation, which rewards consistency with established schemata [21]. As ecologists tend to place the locus of identity outside the organization itself [22], they prioritize the viewpoint of external audiences who control the material and symbolic resources firms must acquire in order to survive [e.g., 23].

The two perspectives can be regarded as complementary: indeed, many spectacular failures of products and organizations can be explained by the mismatch between what people on the inside believe they are doing and what those on the outside perceive [24]. Given specific research questions and empirical contexts, however, it can be reasonable to privilege one perspective over the other. For example, in the markets for creative goods it is often the case that producers' claims to particular category memberships are irrelevant to consumers' decisions, while the judgment of critics [25] and other consumers [13] is supremely important. To illustrate this point, most of us have favorite musicians, painters, or film directors, and we would be able to allocate them to particular genres if requested to do so, but we are oblivious to how most of these artists would define their own work. This is not necessarily the case in other contexts: for example, in high-technology industries, the information producers convey about themselves and their offerings is extremely relevant to investors [26]. Even in these settings the expectations of external audiences tend to be important—e.g., because investors are likely to consider what kind of patents the firms apply for [27]—but organizations are granted more leeway [28].

The potential to explain how products are affected by consumers' default expectations [29], how these expectations can be exploited to firms' benefit [30], and how firm managers can sway them strategically [31], is the reason why the study of categories has practical implications for business. Understanding the consequences of categorization helps researchers address the question of why some innovations succeed [32, 33] whereas others are misunderstood [34, 35]. At a more theoretical level, studying categories and their effects can shed fundamental insights on the mechanisms that drive the evolution of industries [6], such as audience members' interaction [36, 37]. Further, examining competitive processes through the lens of categorization can illuminate why new markets and submarkets emerge [38],

and why existing ones split, merge, dissolve, or become obsolete [39]. It can also be useful to clarify why some clusters of products or services [e.g. 40] fail to meet the requirements that mark their transition into full-fledged competitive arenas: in this sense, research on categories can help scholars explain the puzzle of "markets that weren't" [41].

Such practical and theoretical considerations motivate the research presented in this thesis. We aim to attain a more complete understanding of the role played by categories in the ordering of markets by clarifying how the information encoded by category labels is decoded by agents for the purpose of decision-making. To this end, we proffer two empirical studies and a formal theory. More specifically, in Chapter 2 we empirically analyze how the properties of categories affect the outcomes of firms' competitive strategies. In Chapter 3, we consider how consumers derive information from category labels and how this information is combined in the case of multiple category memberships. Switching to formal methods, in Chapter 4 we argue that classification systems can be mathematically represented as RS-frames, and that agents' consensus about the meaning of category labels is simultaneously determined by factual information, subjective perception, and social interaction. In Chapter 5, we refine this order-theoretic representation by generalizing to Kripke-style frames, and present a sound and complete epistemic-logical language that can be used to accurately describe category labels and their meaning. Finally, in Chapter 6, we summarize our findings, reflect on their implications for organizational research, and suggest directions for further study.

Attesting to the breadth of extant literature on categorization, our arguments build on previous research in a variety of domains. Figure 1.1 offers an indication of the interdisciplinary scope of this thesis: to produce this chart, the references of each chapter were assigned to particular subjects based on Clarivate's 2016 *Journal Citation Reports* (JCR) classification.[1] As the figure shows, there is always a significant proportion of references from the core disciplines of sociology, management, and business, but Chapters 2 and 3 draw more extensively from economic theory, Chapters 3, and 5 build more on cognitive psychology, and Chapters 4 and 5 are more strongly oriented toward mathematics and logic. Many additional disciplines are lumped into the other category, including statistics, linguistics, biology, philosophy, musicology, acoustics, engineering, computer science, and

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[1]References belonging to multiple JCR subjects are listed once per subject. References for which no JCR subjects were available, such as unclassified journals, books, book chapters, conference proceedings, online sources, and unpublished manuscripts, were manually assigned to particular JCR subjects based on their keywords.
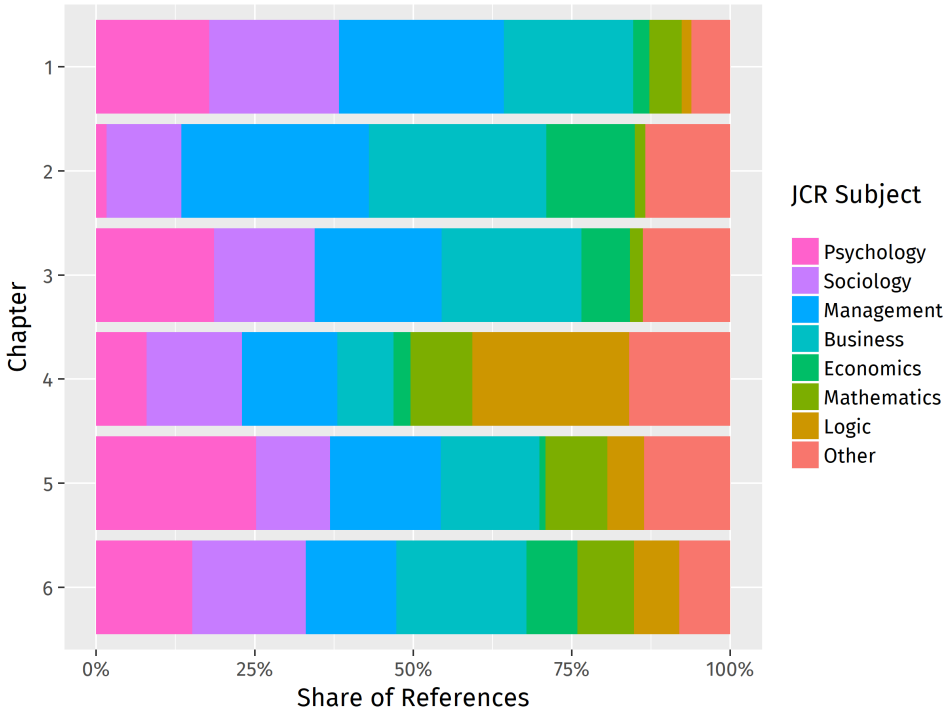
Figure 1.1: Distribution of references by subject category

information science, among others. Notwithstanding this bibliographic variation, four overarching themes can be taken to represent the common threads that bind this dissertation together. These broadly relate to:

1. the power of categories to encode the features of objects in a particular domain, which enables cognitively limited agents to overcome information asymmetries and make rational decisions [42, 43];

2. the arrangement of categories into partially ordered structures, or *classification systems*, which hinge on specific rules for categorization [44] and normally consist of multiple levels of abstraction [45];

3. the inherently dynamic nature of categories [46], classification systems [6, 47], and category properties [48]; and

4. the amenability of categories to various kinds of formal representation, e.g., logical [49, 50], geometric [51], and set-theoretic [52].

None of these themes applies exclusively to categorization in markets. In fact, the very same notions are relevant to categorization in other domains, such as natural language semantics [53–55]. Precisely because of their generality, they are useful to highlight that categorization is not a context-specific phenomenon to be explained by certain theories in psychology, others in linguistics, and others yet in sociology or management, but a deep-seated cognitive mechanism that permeates every aspect of social behavior, including economic processes, and is governed by general rules whose mark is discernible regardless of context [2].

Before providing an overview of the four themes and outlining their relations to the individual chapters, it is worth clarifying my personal contribution to the research presented in Chapters 2–5, which is co-authored with other members or collaborators of the Applied Logic Group at the Delft University of Technology. Chapters 2 and 3 are the extended versions of research papers where I am formally the lead author. Chapters 4 and 5 are the outcome of joint work by a group of co-authors in which there is no formal lead. My role in this team has been to coordinate the inputs from mathematics and social science, bridging the two worlds and articulating their connections. In particular, I directed the logicians' formalizations so as to keep our theory of classification systems true to extant literature on categorization in sociology, management, and cognitive psychology.

1.2. Main Themes

1.2.1. Cognitive Economy

In describing the contents of a fictitious Chinese encyclopedia fantastically titled *Celestial Emporium of Benevolent Knowledge*, Jorge Luis Borges [56, p. 103] reports a very peculiar and, by now, very well-known taxonomy:

> Animals are divided into (a) those that belong to the Emperor, (b) embalmed ones, (c) those that are trained, (d) suckling pigs, (e) mermaids, (f) fabulous ones, (g) stray dogs, (h) those that are included in this classification, (i) those that tremble as if they were mad, (j) innumerable ones, (k) those drawn with a very fine camel's hair brush, (j) others, (m) those that have just broken a flower vase, (n) those that resemble flies from a distance.

In the author's intention, the list serves to show that any attempt at categorizing the objects that populate the world is bound to be arbitrary and conjectural, because the world, for all its complexity and the cognitive limitations to which humans are subject, is unfathomable to mankind. The

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passage was later popularized by Foucault [57], who pinpointed exactly why it sounds so outlandish: not because of the categories themselves, which are clearly understandable even though some are imaginary, but because of the accompanying alphabetical sequence, which gives the illusion of coherence. The French author observed that the wondrous quality of the taxonomy resides chiefly in the narrowness of the interstitial space that its inventor must have perceived between STRAY DOGS and FABULOUS ANIMALS, SUCKLING PIGS and MERMAIDS, to think that this list could help someone make sense of the animal kingdom.

The classification system presented in the *Celestial Emporium* has been widely discussed by sociologists and literary critics, and almost as if it were real, it has been invoked with some regularity by the critics of structuralism to support their claim that Western and non-Western cultures tend to organize the world using radically different principles [58]. This proposition was found questionable by cognitive psychologists, particularly Eleanor Rosch and her colleagues [45, 59–61], who devoted substantial effort to uncovering scientific evidence of the universal, cross-cultural rules that guide the formation of categories. In her seminal writings, Rosch thusly commented the *Celestial Emporium*: "Conceptually, the most interesting aspect of this classification system is that it does not exist. Certain types of categorizations may appear in the imagination of poets, but they are never found in the practical or linguistic classes of organisms or of man-made objects used by any of the cultures of the world" [2, p. 27].

In Rosch's view, two principles lie at the core of any categorization. The first principle is that, as mentioned before, the objects that constitute a cognitive domain display a perceivable correlational structure. For example, given three attributes of products such as sweetness, sourness, and wireless connectivity, it is empirically the case that the former two attributes co-occur with each other more often than either of them co-occurs with the third. This implies that the categories used by different agents to sort products and organizations are not accidental: they are (idiosyncratic) constructions based on objective patterns. The second principle, termed *cognitive economy*, holds that the purpose of any categorization is to yield an accurate but parsimonious representation of the cognitive domain. Therefore, the categories considered meaningful by decision-makers are merely a subset of all the possible categories that could be used to sort objects: namely, they are the ones that most closely capture the objects' perceived correlational structure [52]. Clearly, the animal categories of the *Celestial Emporium* do not reflect this optimization. They strike us as absurd precisely because they are not economical.

In the past few years, researchers in strategy and organization theory increasingly emphasized the subjectiveness of market categories [12, 62, 63]. There is merit to these observations, as the process of fitting complex entities into neat cognitive schemata can give rise to difficulties that must be arbitrarily resolved [64]. Organizations are certainly complex enough to make these inconsistencies conspicuous in some cases [e.g., 19, Figure 4]. However, the subjective component of categorization is sometimes exaggerated to the point that one is left to wonder how can markets function at all: for example, strategy scholars recently proposed the notion of "infinite dimensionality" [12, p. 66], arguing that, because infinite distinctions can exist between objects [65], there is an infinite number of ways in which the objects can be sorted by different agents. Leaving aside the question of whether the perceivable distinctions between objects are ever truly infinite,[2] it is reasonable to presume that agents' need for coordination would encourage them to seek common ground through social interaction [13, 37]. In the absence of some minimal level of consensus about category definitions, it is hard to envision how category labels can be fruitfully used in communication. Like the *Celestial Emporium* taxonomy, the notion that infinite dimensions can be used to distinguish two objects sounds improbable because it is uneconomical.

Relation to the chapters.   Cognitive economy appears in Chapter 2 as we discuss how firms view the market as a partitioned landscape in lieu of a continuous surface. The regions that comprise this landscape can be considered subsets of a feature space, inasmuch as they map to particular product characteristics [67], as well as subsets of a resource space, as they map to particular consumer preferences [68, 69]. In both cases, their boundaries denote variations relevant to firms' strategic decisions. In Chapter 3, cognitive economy is addressed with regard to consumers' evaluations: here, we study how category labels underpin the attribution of worth and examine how audience members economically combine information from multiple categories [70–73]. In Chapter 4, the theme rises to prominence as we consider how meaningful categories stand out from the much broader set of possible categories, depending on the agents' perceptions of objects and features as well as social interaction. In Chapter 5, the theme is further exemplified by our efforts to generalize our formal theory by relaxing some of its technical restrictions and to build an epistemic-logical language that accommodates different perspectives on categorization.

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[2]Psychological research suggests they are not [66]. In fact, they tend to be quite limited because humans have finite cognitive resources at their disposal.

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1.2.2. STRUCTURE OF CLASSIFICATION SYSTEMS

Beyond the reason pointed out by Foucault [57], there are two factors that further contribute to the awkwardness of the *Celestial Emporium*. One of these is that the categories it includes are likely to overlap though the classification system does not seem to be hierarchically ordered [74]. In other words, it appears devoid of vertical structure. On the contrary, the alphabetical sequence reinforces the impression that relatively generic and relatively specific categories belong to exactly the same level of abstraction. For example, ANIMALS THAT BELONG TO THE EMPEROR is quite specific, but ANIMALS THAT ARE INCLUDED IN THIS CLASSIFICATION is exceedingly broad, especially because the category OTHERS extends the system so as to encompass all animals. Real taxonomies [cf. 45] are not normally structured in this confusing fashion: instead, they consist of multiple levels of abstraction and hence distinguish subordinate categories from superordinate ones [75]. The relations between categories located at different levels of the hierarchy are governed by set-theoretic inclusion [52, 76]: because coherently applying this rule to organize knowledge is one of the signs of mature reasoning [77], listing subordinate and superordinate categories in a way that ignores their obvious vertical relations is bizarre.

The hierarchical arrangement of classification systems is extremely relevant to the study of categories in organizational contexts. Many of the systems commonly examined by organization theorists, such as industry codes [19], art genres [78], and patent classes [35], have an explicit vertical ordering. Much like cognitive psychologists, organization scholars face the question of how to select the most important level of abstraction for the decisions that agents (managers, analysts, investors, consumers) are required to make in a given situation. In most cases, the categories that are subordinate and superordinate to whatever level of abstraction is chosen by researchers are empirically disregarded. For example, many studies of category spanning in the creative industries use genres as the level of analysis [21, 23, 79, 80] but do not take into account that these categories can be genealogically linked [81].[3] As a result, researchers tend to overlook the fact that agents in a market can derive a great deal of information from vertical structure [27]. Arguably, strategy scholars have been much more

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[3]In recent years, it has become standard practice in empirical studies to account for the similarity of spanned categories by way of co-occurrences [e.g., 82]. This partly addresses the problem because hierarchically linked categories tend to co-occur more often; however, some categories rarely co-occur even though they are clearly related, as in the case of WEST COAST RAP and EAST COAST RAP music. The inclusion of two or more categories' into the same superordinate category is thus imperfectly captured by similarity.

mindful of the vertical dimension of classification systems: it has been acknowledged since the work of Richard Rumelt [83, 84], for instance, that firms diversifying across related vs. unrelated market segments effectively engage in two separate strategies.

The final reason why the *Celestial Emporium* "taxonomy" appears incoherent is that its categories have radically different internal (as opposed to external or vertical) structures. Some of them, like SUCKLING PIGS and MERMAIDS, hinge on the same sorting rules that drive the distinction of real biological taxa, that is, family resemblance [61]; others, like EMBALMED ANIMALS, emphasize features that have little to do with biology but are consistent with a causal model of categorization [85, 86]; others yet, like ANIMALS THAT HAVE JUST BROKEN A FLOWER VASE, seem to be constructed entirely *ad hoc* [87]. While it is relatively common for people to use categories with different internal structures to organize cognitive domains [88], these categories are rarely perceived to be part of a single classification system, and indeed, the act of mentally switching from one system to another tends to be evident in subjects engaged in experimental assignments [89]. Objects that belong to multiple categories within the same system can be confusing to the audience because multiple labels give rise to uncertainty when they encode conflicting information [43, 90], but category labels that belong to different systems do not necessarily engender confusion [72, 88, 91]. For example, it makes perfect sense for a car to be considered a SEDAN according to a system based on family resemblance, a CAR WITH A STICK SHIFT according to a causal model, and a CAR RELIABLE FOR LONG-DISTANCE TRAVEL according to an *ad hoc* or goal-based perspective [cf. 92].

Agents in a market can take advantage of different categorizing rules to more fully encode the perceived correlational structure of products and firms [62]. In some cases, the objects' category memberships in one classification system allow agents to disambiguate the inconsistencies arising from multiple memberships in another [63]. Although cognitive psychologists have long acknowledged that considering family resemblance to be the sole relevant criterion for categorization is grossly inadequate [93], organization scholars have largely privileged this perspective over the past two decades [15]. This is partly because classification systems that are highly institutionalized and thus "sociologically real" [22, p. 478] tend to be grounded on this rationale. Empirical research has only recently begun to address the limitations of this perspective [63, 94, 95]. This dissertation aims to contribute to such an "ontological turn" [96] by investigating how agents in a market leverage the different external or internal structures of categories in order to make better decisions.

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Relation to the chapters.   In Chapter 2, we focus on a classification system based on family resemblance and show that its verticality has direct implications for the effects of firms' competitive strategies. In particular, we use the hierarchical relations between superordinate and subordinate categories as an indicator of the *complexity* [cf. 97] of the regions of feature space to which the superordinate categories correspond. In Chapter 3, we address the consequences of cross-classification [98] by examining the effects of category spanning when the categories have similar or different internal structures. Focusing on the distinction between categories based on family resemblance [61] and categories based on consumers' goals [91, 92], we argue that the two classification systems radically differ in terms of the information they allow evaluators to derive [44]. In Chapter 4, we present the order-theoretic foundations of a formal theory of classification systems that is capable of accommodating both kinds of internal structure, as well as categories generated by a causal model [85, 86]. In Chapter 5, we refine this mathematical framework and demonstrate its versatility by formalizing the psychological notion of *typicality*.

1.2.3. CATEGORY DYNAMICS

Another recent trend in organizational research concerns the study of how classification systems mutate over time [15]. While marketing scholars have long acknowledged that sociocognitive dynamics of categorization fuel the evolution of industries [36], organization theorists only lately begun to ask programmatic questions about how new market categories emerge [38], acquire legitimacy [99–101], become associated with particular evaluation criteria [102, 103], and consequently endure as valuable tools for sensemaking [46]. Of even more recent origin is the systematic study of processes whereby existing categories lose their explanatory power [104]—and sometimes their legitimacy [e.g., 105]—which can forewarn of their dissolution [39, 96]. These efforts have been instrumental in pushing organizational research beyond the naive conception of categories as static schemata, which is especially simplistic in the case of innovation-driven industries. As a result of this shift, organization theorists begun to examine market categories as having a lifecycle of their own [106].

Adopting categories as the unit of analysis, as opposed to the objects they encompass, is arguably necessary to explain how the information encoded by category labels is interpreted by agents for the purpose of decision-making [107]. It can also be useful to identify which changes in a particular environment can foretell changes in the informational content of the labels [108]. Organizational ecologists have been especially prolific in

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their pursuit of this research agenda, as many of their studies were geared toward the reduction of categories to fundamental properties that correlate with meanings but can be measured empirically and generalized across contexts [109]. Some of these properties have become well-established theoretical constructs in the organizational literature [15], as in the case of *category leniency* [26] and *contrast* [110, 111], which relate to the flexibility of category boundaries, or *category similarity* [7], which rather refers to the distance between category labels in a metric space where the axes or dimensions represent encoded information [82].

If changes in the meaning of categories are precursory to changes in classification systems, and thus deserving of researchers' attention, longitudinal variation in the properties of categories is all the more important to consider because it can anticipate change in their meanings. Empirical studies that point to the relevance of such cascading effects already exists in the organizational literature [48, 111–113], but cognate disciplines appear to have lagged behind. In strategy, for instance, empirical studies concerned with category dynamics [e.g., 114] remain few and far between, and they are scarcer yet in innovation management, even though the sociocognitive aspects of categorization are known to be crucial determinants of products' convergence around a dominant design [115, 116]. Even more surprisingly, category dynamics are conspicuously absent in industrial economics, although previous research in this field resorted to categories to better analyze competitive interactions [117]. This dissertation advocates for more thorough consideration of category dynamics in the answering of questions outside the traditional scope of organization theory. In our studies, we devote special attention to their time-variant properties and to the sociocognitive mechanisms that trigger their emergence.

Relation to the chapters.   Category dynamics figure prominently in Chapter 2 as we analyze how a distinctly changeable aspect of product categories, namely their complexity [cf. 97], affects the consequences of product proliferation. Building on extant research in strategy, industrial economics, and organization theory, we relate the complexity of categories to the complexity of the underlying region of the feature space and suggest possible connections with the study of complexity in organizational evolution [118]. The theme moves to the background in Chapter 3, where category properties like similarity and contrast appear as controls in our analysis of category spanning; however, it returns to the forefront in Chapter 4 as we develop a formal account of category emergence through social interaction. We argue that categories arise en masse from a set of objects with certain

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features and a set of agents who interact with each other but may have an idiosyncratic perception of objects and features. The theme remains dominant in Chapter 5, where we use our mathematical machinery to formalize well-known category properties, including contrast, leniency, and similarity. The time-variant nature of these properties is accommodated by tying their definitions to objects and features as well as the agents' subjective perceptions thereof, all of which are subject to change.

1.2.4. LOGICS FOR CATEGORIZATION

The idea that is perhaps most vehemently emphasized throughout this dissertation is that category representations are formally tractable. As a matter of fact, half of the research presented in the following chapters is aimed at developing a suitable symbolic language. Although the use of formal theory is not new to sociological research [119], nor to the study of categories in particular [20], there is much to be gained from a more comprehensive deployment of formal methods as the field continues to grow and the scope of existing theories expands. Among the methods that have been successfully deployed in organization theory [120], formal logic occupies a prominent spot [20, 121–127]. One of the reasons for this primacy is that logical formalization allows organization scholars to determine whether views that are commonly understood as competing can actually be reconciled [128]. In addition to the obvious potential for theory advancement, this has the desirable effect of shattering the barriers between different "camps" and preventing the field's compartmentalization. Another reason is that formal logic strips organizational theories of the ambiguity inherent to natural language and makes them accessible to scrutiny, evaluation, and repair [129, 130]. The quality and explanatory thrust of the theories can be much improved by this exercise.

Scholars who promote the use of logical methods in organizational research tend to emphasize the advantages they offer in terms of replicability, comparability, and generalizability of insights. Indeed, as noted by Hannan [131, p. 147] it can be "a humbling experience" to realize how much of one's arguments depend on tacit assumptions once they are even partly formalized. Organizational research can immensely benefit from the meticulousness imposed by logic, as researchers' quest for precision in theory construction is fundamentally hampered by the equivocacies of informal language. Embracing methods of inquiry that forcefully rid causal statements of their vagueness can help propelling scholarship forward. "It can be hard to abide by whatever these formal, logical, or methodological standards demand. Yet in practice, they are what keep the theory under

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control. Perhaps counterintuitively, by establishing limits they are also what allow for the creative development of new ideas" [132, p. 119]. It is important to remark, however, that the perks of cross-fertilization between logic and organization theory are not unidirectional. Logicians can also benefit from using the messy world of organizations as a testing ground for their formalisms: in fact, the peculiar challenges posed by the inexactness of social behavior [126, 133, 134] can lead to findings of mathematical import and pave the way to new, unconventional applications.

Two formal approaches are considered especially germane to the study of categories and are hence discussed in this thesis. The first one builds on Gärdenfors' notion of *conceptual spaces* [51, 55], a method for representing cognitive domains based on geometric distance. In recent years, sociologists have begun to explore the potential of conceptual spaces to yield an intuitive depiction of markets [67, 82, 108, 135–137], and there is little doubt that this line of research will come to full bloom in the future. The second framework builds on Birkhoff's representation theorem [138] and is widely known in mathematics [e.g., 74] as *Formal Concept Analysis*. First developed by Ganter and Wille [139, 140], this method draws from order theory and characterizes categories as nodes of a concept lattice. Although this approach is uniquely suited to describe the ontological nature of classification systems [96], it is still relatively foreign to sociologists. There is ground to believe that this application is fruitful, however, because such an order-theoretic interpretation of categories can explain aspects of agents' reasoning that are less naturally captured by geometric arguments. One of the objectives of this dissertation is to stimulate appreciation for this valuable method in organizational research, not as an alternative but rather a complement to conceptual spaces.

Relation to the chapters. In the empirical part of this thesis, the theme of formal representation is expressed by numerous (explicit and implicit) references to conceptual spaces. In particular, Chapter 2 explores the link between this framework and other geometric models familiar to sociologists [68] and industrial economists [141]. Chapter 3 relies on the same geometric framework to clarify how agents derive and combine information from categories that belong to different classification systems. Though the arguments presented in these two chapters are not (yet) formalized, they lend themselves well to logical reconstruction. The non-empirical part of this thesis is devoted to the application of Formal Concept Analysis to the research on categories in the social sciences. More specifically, in Chapter 4 we define the lattice-based structures underpinning our formalism and

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introduce their associated RS-semantics [142], In Chapter 5, we generalize to more natural Kripke-style semantics and presents a sound and complete epistemic-logical language to be used in formalizations.

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I

EMPIRICAL STUDIES

2

CATEGORIZATION AND STRATEGIC DETERRENCE

2

2.1. Motivation

By *product proliferation*, industrial economists refer to the competitive strategy whereby a firm extends its product offer in a specific market (or submarket) so as to fill the product space and minimize unmet demand [1]. This is an especially common strategy in industries characterized by non-price competition, such as food [2, 3], chemical [4], or cigarette manufacturing [5], where it is often used by the dominant incumbents to achieve and perpetuate a differentiated oligopoly [6–8]. Schmalensee [9] offered an example from the US market for ready-to-eat cereals, where four conglomerates, namely Kellogg, General Foods, General Mills, and Quaker Oats, introduced such a large number of marginally different products during the years 1950–1972 that they were brought to trial by the Federal Trade Commission.[1] Shaw [4] reported a similar pattern in the UK chemical industry, where three large incumbents—Imperial Chemical, Fisons, and Shell—came to dominate the fertilizer market by proliferating extensively in 1958–1978. A few years later, Brander and Eaton [10] presented a compelling game-theoretic model whereby sequential strategic decisions with regard to new product introductions naturally lead to equilibria where a single firm monopolizes close substitutes. The rationale is that, if the firm will not offer certain products, then its competitors will [cf. 11]. Proliferation serves to signal one's commitment to a particular submarket or product category. To the extent that firms can be assumed to behave rationally, this signal represents a credible threat in the eyes of rivals [12].

These studies were the origin of an enormous stream of literature on product proliferation, which now ranges from game theory [13] to operations research [14], marketing [15, 16], strategic management [17], and organizational ecology [18]. The empirical evidence amassed over nearly four decades suggests that proliferation strategies can have positive effects on a variety of firm outcomes, such as profitability [19], market share [20], market power [3], and survival in the industry [21], provided that firms manage to contain the relative increase in coordination costs [18, 22] and do not irritate consumers by burdening them with overchoice [2]. The mechanisms through which proliferation can beget competitive advantage include the exploitation of scope economies [8] and learning effects [23], the capacity to cater to more customized requests [21], and the "contrived deterrence" [6] of competition, which affects *de novo* and *de alio* entrants as well as

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[1]Interestingly, the author found no evidence of collusion as the firms did not cooperate toward the purpose of becoming oligopolists. The anticompetitive effect that they enjoyed was "an unforseen, but presumably not unwelcome, consequence of a mode of behavior that arose more or less naturally from the industry's structure" [9, p. 316].

more established rivals. This latter effect, which arguably has the most direct and macroscopic implications for the structure of an industry [7], represents the focus of our arguments in this chapter.

We believe this effect deserves additional consideration because, while game theorists agree that product proliferation makes a credible deterrent strategy [13, 24–26], empirical research found little or no evidence of its power to discourage rival product introductions [15]. It is worthwhile to revisit this relationship because the determinants of new product entry have become increasingly important for organization theory as scholars started to address questions related to product demography [e.g., 27]. One of the most compelling insights offered by this line of research is that the survival rate of products in the market depends on distinctly ecological factors, such as the density of firms' offerings in the various categories or niches whereby the market is partitioned [28]. It thus seems reasonable to conjecture that, by causing the saturation of particular product categories, proliferation strategies negatively affect the likelihood of rival product introductions. In light of this, the lack of evidence in favor of the "deterrence hypothesis" [15, p. 149] seems all the more surprising. We suspect that the absence of empirical support for this hypothesized causal link is due to the moderating influence of category-level properties: these have not been accounted for by previous research on proliferation; however, as shown by many studies in organizational ecology [29–32], they can substantively alter the outcome of firms' product strategies.

Our objective is to theorize and test this moderating effect with regard to product proliferation. More specifically, we analyze how the deterrent power of proliferation strategies varies with the level of complexity of the region of the product space where the strategy is enacted. In examining the link between product proliferation and performance, previous research has suggested that product space complexity, i.e., the degree of product heterogeneity within the targeted (sub)market, tends to multiply the strategy's effects [17]. On the one hand, this is because greater returns accrue to organizational learning if the product spaces is complex, as specialized knowledge is more difficult to acquire [33] and consumers are more willing to pay for quality [16]. On the other hand, complex product spaces encourage consumers to adopt simpler routines in their purchasing decisions, such as submarket loyalism [34], which enables firms with focused product lines to exploit psychological associations between their brand and the focal (sub)market [35]. Although this research has illuminated the moderating role of product space complexity, it has considered only some of the beneficial mechanisms underlying product proliferation, namely learning

effects and demand synergies, and did not analyze the possible effects on the strategy's capacity to deter competition. This gap is especially glaring if we consider that preventing rival product introductions is usually regarded as the main purpose of product proliferation [1].

Beyond filling this gap, we contribute to the broader research on proliferation in two important respects: First, we analyze the strategy's effect on the number of future product introductions by new entrants as well as established rivals. This is relevant because, even decades after Schmalensee's [9] foundational work, studies on proliferation largely concerned themselves with entry deterrence and did not consider the barriers to firms' movement across submarkets. In doing so, they missed what Richard Caves and Michael Porter [6, p. 249] termed "a great opportunity for generality." Second, we examine product space complexity not as a characteristic of markets or industries [cf. 17], but rather as a property of individual product categories. To this purpose, we bridge the industrial-economic literature on market structure and the burgeoning research on categories in organization theory [36]. We also clarify how complexity differs from other properties of categories previously examined by organization scholars.

Consistently with economic research [2–5, 9], we focus on product proliferation strategies enacted by the dominant incumbents in a differentiated oligopoly. This allows for a more accurate analysis of strategic deterrence because these firms are normally active in multiple submarkets [37]: hence, their choice to saturate a particular submarket or category with their products can be interpreted as a deliberate attempt to "seize" a region of the market and keep out rival companies.[2] Moreover, it is usually the dominant incumbents within an industry who have the means to retaliate against competitors who encroach upon their territory: therefore, it is especially in the hands of these firms that product proliferation serves as a credible threat [8, 24, 26]. Our empirical setting is the US recording industry, 2004–2014. This is an appropriate context for a study on product proliferation because the market for recorded music is notoriously oligopolistic [39, 40] and the various categories where record companies compete can vary in their degree of internal heterogeneity [41]. This setting is also convenient because the dominant incumbents (termed *majors*) form a recognizable strategic group and tend to monitor each other's product line decisions [42, 43]. The companies outside this group (termed *indies*) form a motley collection that is generally observed only in the aggregate [cf. 44].

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[2]A similar interpretation may be unwarranted in the case of small or focused organizations, who may choose to extend their product offer in certain categories because this is the only option available to try and survive [38].

    This chapter is structed as follows: In Section 2.2, we discuss the extant literature on product proliferation and market structure in economics and organizational ecology. We establish a theoretical connection between the concepts of niche, category, and submarket, and build upon this link to develop our hypotheses about proliferation strategies, category complexity, and strategic deterrence. In Section 2.3, we describe our empirical setting, data, sample, and statistical methods. In Section 2.4, we report the results of our analysis, and in Section 2.5 we discuss their implications for strategic management and organization theory.

2.2. Theory and Hypotheses

2.2.1. The Geometric Structure of Markets

As shown by previous studies in industrial economics [4, 9, 13, 24], product proliferation can be appropriately conceptualized from a spatial or locational-analog perspective. This involves picturing the market as a multidimensional metric space where each axis or dimension represents a relevant product characteristic [8], or feature [45], along which the objects can perceivably differ. In this abstract variant of Hotelling's model [46], known in the economic literature as a locational-analog or Lancastrian[3] model, consumers are defined as singletons, or points in space, which intuitively correspond to the consumers' ideal product specifications. Firms, instead, are defined as sets of points, which correspond to the products they currently offer on the market. The location of products is usually considered fixed because significant costs tend to be associated with repositioning: compatibly with this, firms normally prefer to abandon unprofitable products rather than trying to move them [9]. However, both consumers and firms can change their positions over time: in the one case, this is because consumers' preferences are inherently dynamic [28, 50]; in the other, it is because firms can introduce new products in order to keep up with shifts in demand [38] and react to other firms' behavior [51].

    Just like Hotelling's example of a "market on Main Street" [46, p. 45], the proportion of demand captured by a firm at any given time is a function of the distance between the location of its products and the distribution of consumers. Because consumers are more likely to select products closer to their preferences, every product is associated with a *catchment area* that extends around its position in the feature space[4] and ends as soon as

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[3]After Kelvin Lancaster, who critically contributed to its development [8, 47–49].
[4]Throughout the course of this chapter, the terms "product space" and "feature space" are used interchangeably. This equivalence is consistent with extant literature [17, 28].
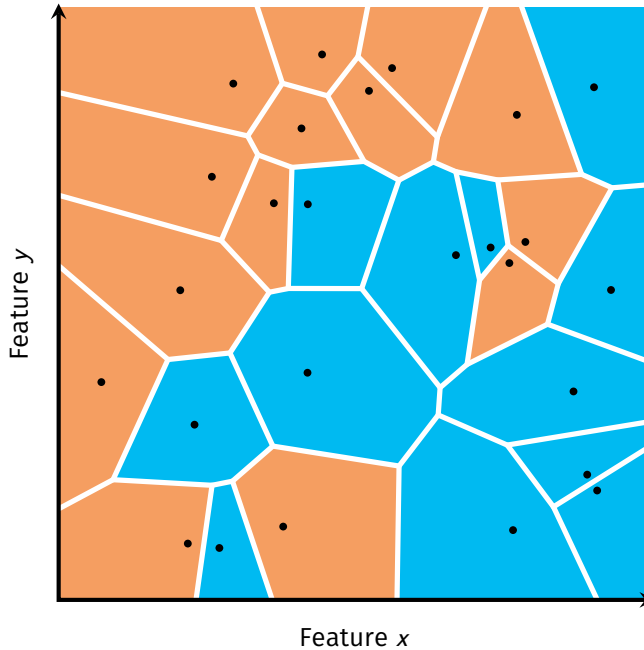
Figure 2.1: A two-dimensional product space populated by two firms

any other product becomes closer. For the sake of illustration, consider a simplified market where the products can differ along two dimensions *x* and *y*. As displayed in Figure 2.1, the entire space can be represented as a Euclidean plane and the catchment area of each product is modeled as a Voronoi cell [52]. Competition ensues between the firms that populate this space (represented by different colors in the figure above) because demand is finite and a product's likelihood to catch it depends not only on its distance from consumers' preferences but also on the proximity of competing products [53]. If demand were uniformly distributed across the space, the profitability of each product would be directly proportional to the size of its catchment area and thus the firm controlling more territory would outcompete the other. This assumption is clearly unrealistic, however: in most markets, demand is unevenly and polymodally distributed [54], and its concentration in particular areas of space can lead to the formation of a *market center* [55], that is, a highly competitive location in the space where large incumbents tend to flock because no other region can provide enough resources to guarantee their sustenance [56–58].

It is generally advantageous for firms to introduce new products at particular coordinates if, by doing so, they intercept demand that would

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otherwise be caught by competitors' products [24]. Of course, the potential benefits of this course of action must always be weighted against the costs that developing, marketing, and distributing a greater variety of products entails. The costs of coordination [22] can be especially high because firm managers may have to divide their attention across a greater number of production units. In extreme cases, these costs can grow so large as to cripple the firm's operation and increase its rate of failure in the short term [18]. For this reason, firms cannot compete with one another by offering an ever-increasing number of products, thereby partitioning the space into ever-smaller catchment areas. An equilibrium is reached when the division of space is such that further product introductions are unprofitable.

Strategic decision-making thus involves searching for an optimal distribution of products across the feature space [25, 38]. This tends to be a very uncertain endeavor because firm managers do not know *ex ante* how demand is distributed, nor the timing of its shifts [59]. From this perspective, product competition is akin to a repeated game where firms periodically place their bids by positioning their offerings at chosen locations. At the end of each period, firm performance is non-negative if the total demand met by the firm's products is sufficient to recoup the total costs, including production, distribution, marketing, and coordination. Anything in excess of this threshold is profit and can be either reinvested in the next round or stored as a buffer, but if performance is negative and the buffer is depleted, the firm incurs failure [60]. Organizational learning occurs because the firms that survive consider past outcomes in future decisions [33]. Nevertheless, learning may only lead to a fleeting advantage that does not guarantee survival in the face of rising competition [61]: in order to sustain their edge, firms must not only scout for profitable positions in the feature space but also properly defend them against rivals' intrusion.

As argued both by the strategic literature on reference groups [62, 63] and by the ecological literature on resource partitioning [55, 58, 64], it is overly simplistic to presume that each player in a market competes in equal measure against every other player in the same market. Indeed, firms that occupy distant positions in the feature space do not target the same demand and may not even perceive one another as competitors [65]. What other firms are doing in distant regions is thus relatively inconsequential compared to what one's neighbors are up to [66]. Similar considerations apply to the demand side: consumers do not necessarily perceive their peers as worthy of social interaction if they have different preferences, and rather seek to establish ties with peers located in their proximity [67, 68]. Such mechanisms engender the compartmentalization of the feature space into

bounded regions, which are relatively homogeneous in terms of both product features and consumer tastes. Industrial economists commonly refer to these regions as submarkets [69, 70], whereas organizational ecologists term them niches [71–73] or (product) categories [45].

Inasmuch as consumers represent a resource that firms ought to acquire in order to sustain their operation, the Lancastrian model described above bears a direct correspondence to the ecological model that conceptualizes the market as a geometric resource space [57]. From an economic perspective, each category can be viewed as a separate competitive arena within the broader landscape of an industry, much as if it were a regional market in a true locational (i.e., geographical) model [69]. Just like multinational corporations have a presence in multiple regional markets, firms within a given industry can maintain a foothold in multiple categories. If they encounter other firms in the same category, then they compete for the same share of resources [37]. This connection between economic and sociological views on market structure is useful to clarify the link between the concepts of niche, category, and submarket: these have been used somewhat interchangeably in previous research; in fact, each of them refers to a bounded region of the feature/resource space, and they are interchangeable insofar as consumers in that region form a niche, products in that region form a category, and firms with products in that region vie to meet the same demand. This conceptual integration allows one to interpret the properties of categories studied by organization theorists, like leniency [74] and contrast [75], as attributes pertaining to submarkets within an industry, and conversely, to redefine the structural attributes of markets, like product space complexity [17], at the level of submarkets so that they can be reinterpreted as category properties.

Further, this theoretical connection allows one to distinguish product strategies by the changes they exert on a firm's distribution of products across the various categories that tessellate the feature space. For example, diversification occurs either when the firm laterally expands by launching products in new categories [76], or when it pursues a more even distribution of products across the various categories where it currently competes [77]. Product proliferation, instead, occurs when the firm increases the number of products offered in a single category, so as to reinforce its presence in the corresponding region of the feature space [4, 9, 13].

A proliferation strategy can be advantageous for many reasons: First, launching similar products in a relatively short time frame provides the firm with ample opportunities to sharpen its design skills and streamline its routines, which allows it to improve the quality of its products [23] and

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thereby outcompete its rivals in the focal niche [71]. Second, offering more variations of a certain kind of product allows firms to meet consumers' more unusual or sophisticated preferences, stimulating their loyalty [17, 21]. Third, and most important to the purpose of this chapter, filling a particular region of the feature space with one's own products helps the firm increase its market share by tightening the gaps in its current product offer. Smaller gaps result in smaller catchment areas, which means that smaller pockets of demand are available for rival products: therefore, competitors are less likely to improve their performance by positioning themselves in the firm's neighborhood and are eventually pushed out of the category (Figure 2.2). This process, distinctly ecological in nature as it pivots on the density of firms' product offer [28], is the reason why game-theoretic models find product proliferation to be a viable deterrent strategy [e.g., 13]. Although it can erodes the sales of individual products [53], it can keep competitors away from the firm's business: in this sense, firms can deploy this strategy to trade current profits for future submarket leadership [10, 11].

Quite surprisingly, this game-theoretic prediction is not corroborated by empirical findings. For example, Bayus and Putsis [15] reported no evidence of deterrence in their analysis of the personal computer industry, 1981–1992: the authors justified their null result by arguing that deterrence may only emerge empirically as an artefact of model misspecification. In this chapter, we propose an alternative explanation grounded in ecological theory, namely that the level of complexity of the feature space undermines proliferation strategies' capacity to deter rival product introductions. In the next section, we elaborate on this argument and theorize a moderating effect on the relationship between product proliferation and new product entry. For brevity, we sometimes use the word "complexity" in reference to the complexity of the feature space. It should be remembered that this complexity, which relates to the degree of heterogeneity in product attributes [17] as opposed to the sophistication of firms' strategies [78] or of their organizational architectures [79], is the only kind we consider.

2.2.2. COMPLEXITY AS A CATEGORY PROPERTY

It is a widely accepted notion in economics and decision theory that rational agents in a market tend to structure their decision-making process so as to minimize their cognitive effort [80, 81]. Partitioning the competitive landscape into categories is crucial to this purpose because it enables cognitively limited agents to reduce the virtually infinite distinctions between products on offer [cf. 82]. Owing to their influence on such fundamental cognitive processes, categories play a major role in the evolution and
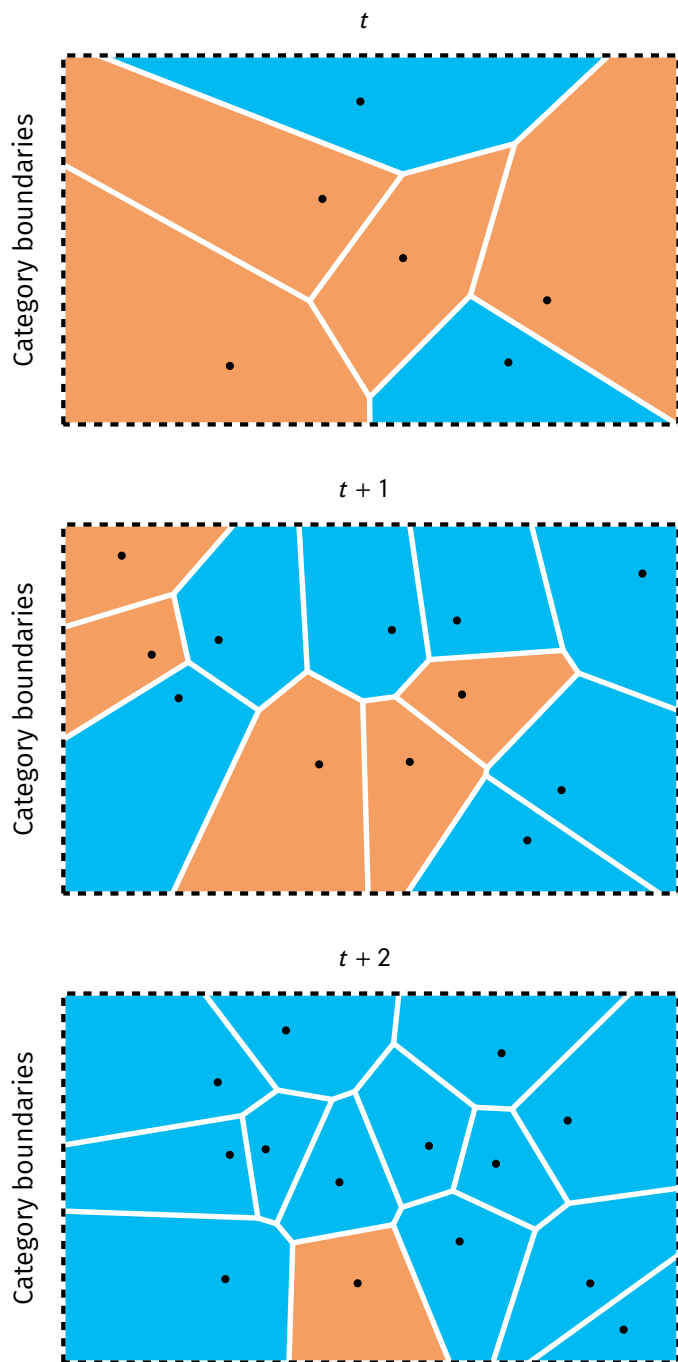
Figure 2.2: Spatial consequences of product proliferation

consolidation of industries. This is testified by the importance placed on classification systems by analysts [83], critics [84, 85], and trade associations [86]: maintaining clear distinctions is so vital to these gatekeepers that they actively discourage challenges to their schemata by penalizing firms that straddle category boundaries. For example, winemakers that mix different styles of vineyard management tend to incur lower evaluations by critics [87] and patent applications that span technological domains are more readily denied by examiners [88].

Though firms sometimes begrudgingly comply with the prescriptions of these "boundary patrollers" [e.g., 89], they also find categorization necessary to their efficiency. By viewing the competitive landscape as a collection of relatively homogeneous regions instead of a continuous surface, firm managers can more easily select appropriate strategies [90], identify their rivals [65], and interpret other firms' behavior [91]. Indeed, categories affect firm managers' very perception of what constitutes an industry [92], as well as the various niches or submarkets it comprises [93]. Compatibly with this, category boundaries act as guidelines in their strategic choices; for example, by demarcating portions of the feature space where diversification or proliferation strategies may be coherently pursued [17].

Having limited cognitive resources at their disposal, consumers also find themselves in need of categorization to achieve cognitive economy and make boundedly rational decisions. In point of fact, the way they search, compare, and evaluate products in a market tends to be governed by simple heuristics [94]. In particular, they heavily rely on categories to determine products' likelihood to fit their preferences [45]: by sifting, encoding, and conveying relevant product characteristics, category labels allow consumers to curtail the dimensionality of the feature space and discard whatever distinction between alternatives is irrelevant to their utility function [95]. Thanks to this filter, categories decrease the information load imposed by purchasing decisions to behaviorally and cognitively manageable proportions [cf. 82]. Given the centrality of this role, it is hardly surprising that categories come to determine the way consumers aggregate [67, 68] and communicate about products [96], steer the drift of their tastes [cf. 50], and guide the evolution of niches [28].

Marketing research suggests that consumers' purchasing decisions tend to be more onerous in complex environments because these defy simplification [97–99]. This resonates with cognitive-psychological literature: if the objects populating a domain have more heterogeneous features, fewer dimensions of the feature space can be ignored or compressed for the sake of cognitive economy. If a (sub)market is more complex, consumers

must cope with a greater information load and they are likely to adopt even simpler heuristics, like restricting their consideration to products that possess certain attributes [34]. As noted by previous research [17], this tendency can ultimately benefit proliferating firms because it allows them to exploit stronger psychological associations between their brand and the focal category [35]. As a result, firms that proliferate in spaces of greater complexity are more likely to be selected by demand. In addition, these firms experience greater payoffs from organizational learning because specialized knowledge tends to be more valuable in complex environments [33] and consumers who care about subtler distinctions are more willing to pay for products that meet their sophisticated expectations [16].

In previous studies, complexity was assumed to be a property of markets or industries [17]. A market is more complex than another (or than itself at a previous time period) if there is greater variance in the attributes of products: in our spatial model, this occurs either if the range of possible values for existing features becomes wider, so that the products can differ more substantially along the space's current dimensions, or if entirely new features become relevant, so that the total number of dimensions increases. Very similar arguments underpin the evolutionary literature on fitness landscapes [61]. The main difference between our model and the NK-models commonly adopted in this literature is that, in NK-models, firms are characterized as singletons in a space of organizational features. The complexity of a fitness landscape is considered a function of the number N and interrelatedness K of the firms' attributes, and it is often studied in relation to the firms' ability to move around in search for a better architecture [79]. There are interesting connections between these two models, as firms' position in a fitness landscape partly depends on the coordinates of their products in a Lancastrian space. Moreover, the complexity of a product space can be closely related to the sophistication of the organizational architecture a firm requires to compete successfully.

Complexity is thus a variable related to the structure of the feature space, and it is a variable that can change over time. Its value can increase as a result of shifts in laws and regulations [17], as well as radical innovation and technological discontinuities, which can lead to greater product heterogeneity. For example, the feature space of personal computers grew considerably more complex during the years 1981–1992, as the number of machines with distinct technical specifications increased from approximately two hundreds to two thousands [15]. It is also possible for complexity to decrease over time, especially as a result of imitation or competitive convergence [51]. For example, the market for popular music

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became gradually simpler in 1971–1988, as the artists' output became more and more similar in terms of aesthetic features [100]. One of the key argument we make about complexity, therefore, is that it should be considered a dynamic or time-variant property.

It is possible to measure product space complexity at the level of an industry, and it is arguably correct to do so when studying competition among firms that engage in unrelated diversification [90]. In examining firms' strategies within a single industry, however, it is crucial to note that the tessellation the feature space can lead to variance in complexity even within the boundaries of a particular industry. This can ensue because the products included in each category can differ more or less substantially along particular dimensions. For example, the disk storage capacity of LAPTOPS varies in a relatively small interval compared to DESKTOP COMPUTERS as the number of hard drives a LAPTOP can accommodate is constrained by the size of its chassis. It can also ensue because the dimensions relevant to product comparisons differ across categories; for example, weight may not be a relevant feature to consider in the case of DESKTOP COMPUTERS but it is decidedly important for LAPTOPS. As a result of this variation, the strategic outcomes experienced by a firm within a certain category can differ from those experienced in another category within the same industry—or even those experienced in the same category at a previous time period.

Given these considerations, we propound that complexity affects the degree to which product proliferation strategies are conducive to deterrence. This argument is justified by the spatial model above: more complex regions of the product space are naturally harder to occupy because consumers tend to care about finer-grained distinctions and competitors are more likely to find gaps to exploit. As mentioned above, greater complexity arises either if a category encompasses a broader portion of the feature space or if more features are relevant to membership in the category.[5] Either condition makes the space included in the category more difficult to hold against competitors. An analogy with military strategy may be particularly useful to illustrate this point: given a fixed amount of outposts (products) to be positioned across the territory, it is harder for an army (firm) to control a state (category) like Kansas than one like North Dakota; although both are relatively flat, Kansas includes nearly 20-percent more land. It may be harder yet for the army to control a state like Utah, which is similar to Kansas in land size but encompasses a much more rugged terrain. Securing a vast and mountainous territory like Alaska, instead, may prove well-nigh

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[5]This is equivalent to saying that the category has greater dimensionality [101].

impossible, because the topology of the landscape limits the contribution of each outpost to the army's capacity to keep watch.[6]

   Consistently with this example, we expect firms that engage in product proliferation within more complex regions of the feature space to experience a decrease in their capacity to keep competitors at bay. In other words, the deterrent power of product proliferation should be reduced by the complexity of the targeted category. Consistently with the game-theoretic literature [13], we anticipate a baseline negative relationship between the firm's number of new product introductions in a given category and the likelihood of new product introductions in the same category by the firm's rivals: however, we expect this relationship to be positively moderated by the level of complexity of the focal category. In other words, we predict that regions of the feature space that are more heterogeneous in terms of product features witness a weaker deterrent effect as a result of product proliferation strategies. If the category's complexity is sufficiently high, it may even be possible for this effect to disappear entirely, leading to an apparently null relationship [cf. 15]. Hence our two hypotheses:

Hypothesis 2.1. *Engaging in product proliferation within a certain category decreases the likelihood of rival product introductions in that category.*

Hypothesis 2.2. *The more a category is complex, the less product proliferation decreases the likelihood of rival product introductions in that category.*

2.3. Methodology

2.3.1. Empirical Setting

We test our predictions by analyzing firms' patterns of new product introductions in the US recording industry, 2004–2014. This is an appropriate setting for our study because the industry is dominated by non-price competition. Indeed, copyright protection ensures that record companies cannot sell exactly the same product as their rivals. Just like food [2, 9] or chemical manufacturing [4], the companies primarily compete through differentiation, and while their pricing conduct is approximately cooperative—because products that adhere to the same technological format tend to be equally priced regardless of their features—their choices with respect to advertising [102] and positioning [43] are most definitely not. Another reason for choosing the recording industry is that firms can surmise consumers'

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[6]For simplicity, our example assumes that rivals can enter the territory at any location and not just at its boundaries, much like firms can introduce new products at any coordinates within a certain region of the feature space [13].

preferences from the information made available by Billboard in the form of ranking charts [103]. These charts are regularly used to motivate decision with regard to the allocation of organizational resources: to quote Roger Karshner [104, p. 115], former vice-president of Capitol Records, "Everybody in the record business is constantly lipping chart potentials, trade picks, and chart life. In fact, the entire industry rises and falls upon the waves of this silly number game." This means that record companies need not rely on monitoring [38] to keep track of consumers' shifting tastes, and those occupying the more profitable positions in the feature space, such as the market center, must face the threat of rival product introductions.

The US market for recorded music is also suitable for our analysis because it is decidedly oligopolistic. The business has been historically concentrated in the hands of a few major record companies [39, 40]: large, diversified, and vertically integrated organizations [105], which outclass the independent companies in terms of production, marketing, and distribution capacity [43, 106]. Each major directly or indirectly controls a host of subsidiaries and imprints, and in the way they allocate resources across these different production units they are fully comparable to multidivisional corporations [39]. At the outset of our study period, the group of majors included Sony BMG Music Entertainment (renamed Sony Music Entertainment in 2008), Warner Music Group, Universal Music Group, and EMI Group. At the end of 2011, EMI disbanded and its assets were acquired by Sony and Universal. Currently, the three surviving majors collectively control between 64 and 86-percent of the market,[7] account for two-thirds of all the sales, and tally 3.2 billion USD in yearly revenue [107].

Because the tastes of consumers in this market are notoriously mercurial [108], music products tend to have a short lifecycle. While some continue to sell for decades after their original release, the vast majority remains unnoticed and the few that reach some degree of popularity usually exhaust it within the course of a year [106, 109]. Owing to this rapid turnover, the yearly production of new records represents "the bedrock assumption on which the entire commercial music industry is constructed" [103, p. 280]. To be efficient in this respect, record companies maintain Artists & Repertoire (A&R) departments, the industry's equivalent of R&D, whose mission is to scout for talent, offer contracts to artists, and work as the companies' liaison during the recording process. The marketing and distribution of the

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[7]They directly own about 64-percent of all the active record companies; however, they control 86-percent via ownership of the firms' distribution channels [44]. More than 50-percent of the indies rely on majors to distribute their records because they lack the infrastructure to be entirely independent.

artists' output is usually the responsibility of record companies: despite the opening of digital channels, self-released or self-distributed products are rare and they are only tolerated by the record companies because they serve to stimulate retail purchases [110].

Music products are commonly classified according to a well-established genre system [111]. Consumers care about products' memberships in genres because these categories are key to their social identity [67, 68]; in addition, genres are of utmost relevance to gatekeepers such as music critics, who tend to maintain their institutionalized boundaries [84]. As in other creative industries [51], the major companies are active in all genres that have sufficient appeal for mainstream consumers, and the close distance they maintain between each other in terms of market positioning [42] suggests a strong proclivity to imitation. Although intellectual property law prevents them from offering exactly the same products as their rivals, nothing stops them from producing relatively similar records. Deterrence is thus extremely important: the majors' A&R departments have been known to "hoard" artists proficient in a certain genre purely for the purpose of preventing other firms from releasing similar music. As reported by Casey Rae, former deputy director of the Future of Music Coalition [112]:

> Maybe it makes sense to sign you, get you under contract, and keep you off the streets, so nobody else has you. But they do not actually care if they do anything with you. If a garage sound was popular like *The White Stripes* and now *The Black Keys*, then maybe they just sign up all the *White Stripes*- and *Black Keys*- sounding bands. Lord knows that happened during the grunge era, where A&R guys were literally jumping out of airplanes with briefcases over the city of Seattle. "Sign anything with a goatee!"

In summary, four characteristics make the recording industry comparable to other contexts where proliferation strategies were previously examined by researchers, namely product differentiation, oligopolistic structure, multiple point competition, and an emphasis on strategic deterrence. In our study, we analyze the extent to which product proliferation by the major record companies turns out to be effective at preventing rival product introductions, including both those by other majors and those by independents. Our choice to focus on the majors is partly dictated by the specifics of our empirical context: because proliferation requires firms to sign a greater number of artists, only companies with a sufficient amount of resources can afford to pursue this strategy. The indies tend to be excluded from this group because they are usually smaller, more specialized, and they

produce a relatively small number of records per year [43]. Though this limits the generalizability of our findings to proliferation strategies enacted by large and diversified organizations, it makes our analysis consistent with previous research in industrial economics, which similarly focused on the dominant incumbents [e.g., 3–5, 9].

2.3.2. Sample and Variables

Our data is drawn from two online databases. The first is *Billboard.com*: as mentioned above, this is an authoritative source of information on the commercial success of music products. Owing to their central role in the music business, Billboard charts are frequently used in empirical research to compute proxies for product sales [106, 113] or popularity [114]. In our analysis, we use data from the *Billboard 200*, one of the website's signature charts, which lists the top 200 albums every week by number of units sold in the US. This includes both physical and digital sales, as measured by Nielsen SoundScan through a sample of physical retailers and online music stores. Our study period begins in 2004, when Sony and Bertelsmann launched their joint venture and became a unified force in the US market, and terminates in 2014, when the *Billboard 200* underwent important changes in the way Nielsen data is aggregated. This amounts to 574 weeks of chart information.

Our second source of data is *Discogs.com*, an extensive, user-contributed archive that provides release dates, genre memberships, and links to record companies for more than 1.2 million original products. This data has been used in previous research to analyze firms' product strategies in the music industry because it adequately reflects the boundaries of genres according to both consumers and record companies, and it is reasonably free from retrospective bias [115, p. 959]. To ensure the accuracy of firm-product relationships and release dates, we cross-referenced our data with another online archive, *MusicBrainz.com*. Our final dataset includes 73,722 original products released in the US during our study period. No more than 4,330 of these ever appeared on the *Billboard 200*. The products are distributed across 14 categories: BLUES; BRASS AND MILITARY; CHILDREN'S; CLASSICAL; ELECTRONIC; FOLK, WORLD, & COUNTRY; FUNK/SOUL; HIP HOP; JAZZ; LATIN; POP; ROCK; REGGAE; STAGE AND SCREEN.[8] Figure 2.3 reports the number (both logged and untransformed) of products in our dataset assigned to each category. It is possible for a product to be assigned to multiple genres: in this case, it is counted once per category. Figure 2.4 shows the genres' positions relative to

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[8]Discogs also includes a fifteenth category, NON-MUSIC, which is used for commentaries and interviews. As this does not represent a music genre, it is excluded from our analysis.
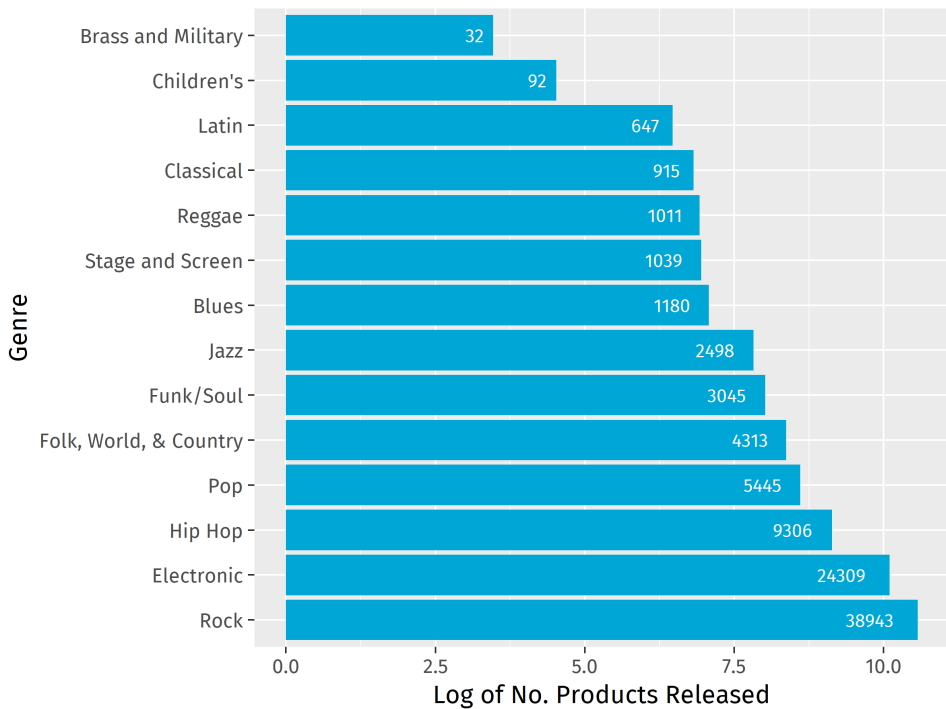
Figure 2.3: New product introductions in each genre, 2004–2014

one another, computed via Kruskal's non-metric multi-dimensional scaling (NMDS) algorithm [116]. The distance between two categories on this map decreases with the number of products assigned to both.

Approximately 11.2-percent of records in our dataset were produced by one of the four major record companies, or any of their respective subsidiaries and imprints. All the others were produced by 15,491 independents. Consistently with the industry's reputation as a highly concentrated business, most of the independents' products turn out to be commercial flops: out of all those that appeared on the *Billboard 200*, 45.3-percent were produced by one of the four majors or any of their subsidiaries, and the rest were produced by 840 indies. Figure 2.5 presents the yearly number of new products introduced by each major across all genres during our study period. As the red line shows, only a small number of products were released in the US by EMI (which was UK-based) in the years prior to its break-up. In our analysis, we retain the company within the group of observed majors, but its exclusion does not affect our empirical results.
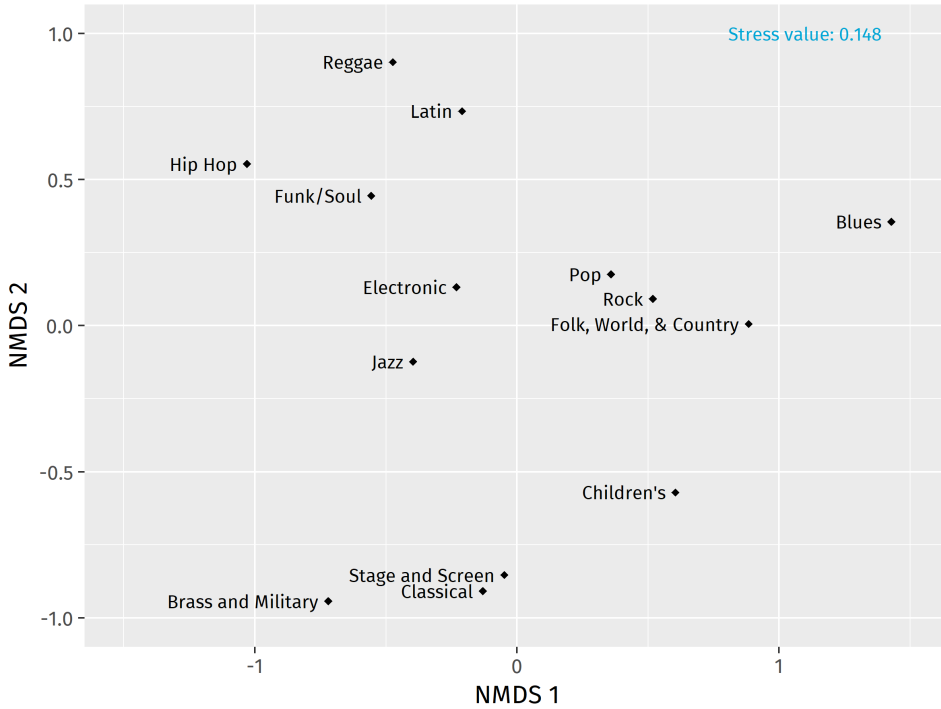
Figure 2.4: Map of Discogs genres computed via Kruskal's NMDS

Dependent variable. Each observation in our analysis is a major-genre-year triple. As three of the four majors are followed for 11 years, one is followed for eight, and the data covers 14 genres in total, our sample includes 574 observations. The dependent variable corresponds to the total number of products released by rival record companies within the focal genre during the given year. This includes both products released by other majors and those released by independents. Formally, for each major $m_1 \dots m_4$, each category $c_1 \dots c_{14}$, and each year $t_1 \dots t_{11}$ (or $t_1 \dots t_8$, in the case of EMI), we compute the following sum:

$$\text{RivalProducts}_{mct} = \sum_{i=1}^{J} p_{ict} + \sum_{i=1}^{D} p_{ict}, \quad (2.1)$$

where J is the set of majors other than m, D is the set of indies, and p_{ict} is the number of products released by firm i in category c during year t.
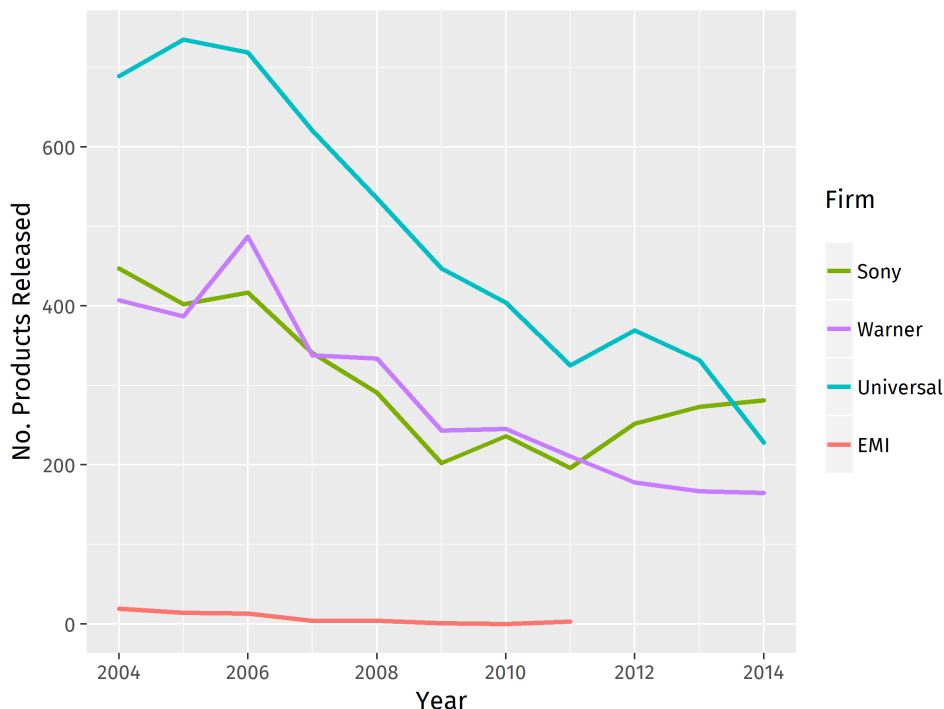
Figure 2.5: Yearly number of releases by the major record companies

Independent variables.   To test our hypotheses we compute two predictors of theoretical interest, namely the number of records released by the focal major in the focal genre during the year of observation, and the complexity of the focal genre during the same year. The first variable (OwnProducts) is a simple count of the products introduced by major m in category c during year t. The second variable (Complexity) is more sophisticated and requires detailed explanation. Ideally, a category-based measure of feature space complexity should increase with the degree of heterogeneity in the attributes of category members [cf. 17], and it should be allowed to vary from year to year depending on the characteristics of new products released within the category.

   Although we have no access to detailed information about product attributes, the hierarchical nature of the genre system allows us to obtain a proxy for the categories' internal diversity. In fact, each genre on Discogs is associated with a number of subcategories, or styles, whose boundaries correlate with variations in the musical attributes of products in the genre, such as lyrical themes, instrumentation, symbolic elements, and harmonic,

melodic, or rhythmic conventions [111]. The degree of heterogeneity within a genre is thus inversely related to the concentration of category members across the styles subordinate to the genre. More specifically, the diversity of product attributes is minimal if all the products belong to the same style, and maximal if they are uniformly distributed across every possible style. Consistently with this reasoning, we measure complexity by way of a diversity index. To this purpose we use a measure known in ecology and information theory as Shannon's H [117]: this index, which allows the measurement of entropy in data, was used in previous research to quantify precisely the diversity of recorded music [e.g., 100]. Formally, for each year t and category c, the variable is computed as follows:

$$\text{Complexity}_{ct} = -\sum_{i=1}^{S} \frac{p_{it}}{p_{ct}} \ln\left(\frac{p_{it}}{p_{ct}}\right), \tag{2.2}$$

where S is the set of styles $s_1 \dots s_n$ subordinate to genre c. Because the argument of the sum is a non-positive number, the resulting variable is always non-negative. Figure 2.6 presents the yearly level of complexity for the five genres for which the mean level of the variable is highest.

Control variables. In addition to complexity, genre categories can vary in a number of other properties. Especially relevant to our arguments are contrast [30, 32] and leniency [74], which concern the fuzziness of category boundaries. In particular, contrast measures the extent to which products' membership in a category tends to be exclusive, in the sense that category members do not simultaneously belong to other categories in the same domain. Leniency, instead, measures the number of other categories with which the focal category overlaps (if any). This is different from contrast because while a high-contrast category tends to overlap with a small number of others, a low-contrast category can overlap with few or many. Both properties are important to control for because firms can be more inclined to introduce new products in high-contrast or lenient categories because of the advantages they offer [32, 118]. Consistently with extant research [75], we calculate Contrast for each category c during year t as the mean grade of membership of products released in c at t, where a product's grade of membership is the reciprocal of the number of genres to which it belongs. The resulting variable ranges between zero and one, with a higher value representing a category with sharper boundaries. Leniency, instead, is computed as the natural logarithm of the total number of genres to which
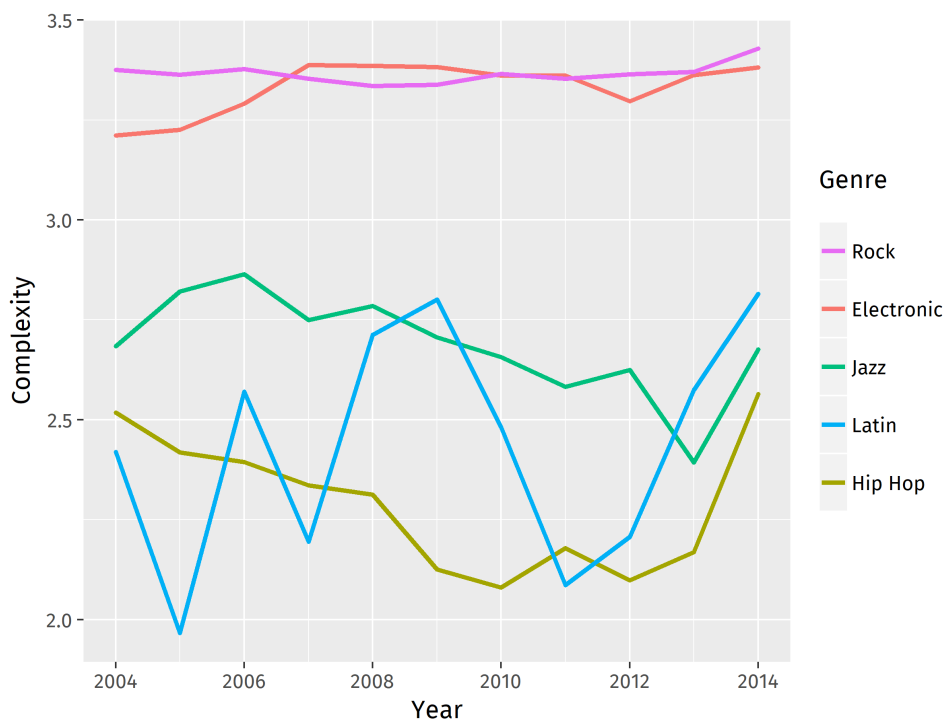
Figure 2.6: Yearly level of complexity for the five most complex genres

the products released in c at t belong.[9] The measure is non-negative, with a higher value indicating greater leniency.

Another determinant of new product introductions that varies at the genre level is the concentration of demand. This is important to control for because categories for which there is greater demand can support a greater number of products before they are saturated [28]. For each category c and year t, we measure DEMAND by the proportion of products released in c during t that enter the *Billboard 200*, relative to the total number of products released during t that enter the *Billboard 200*. This closely mirrors the measure used by Kennedy [51] in his analysis of the broadcasting industry. Figure 2.7 displays the yearly level of demand for the top five categories in our sample. As the plot lines show, ROCK was by far the most popular genre in 2004–2014, accounting for over 50-percent of products on the *Billboard 200* during any given year. The category POP also

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[9]In previous research [74], leniency was computed by multiplying this logarithm with the reverse of contrast. We avoid this approach because contrast is a separate variable in our analysis and the multiplication would induce collinearity.
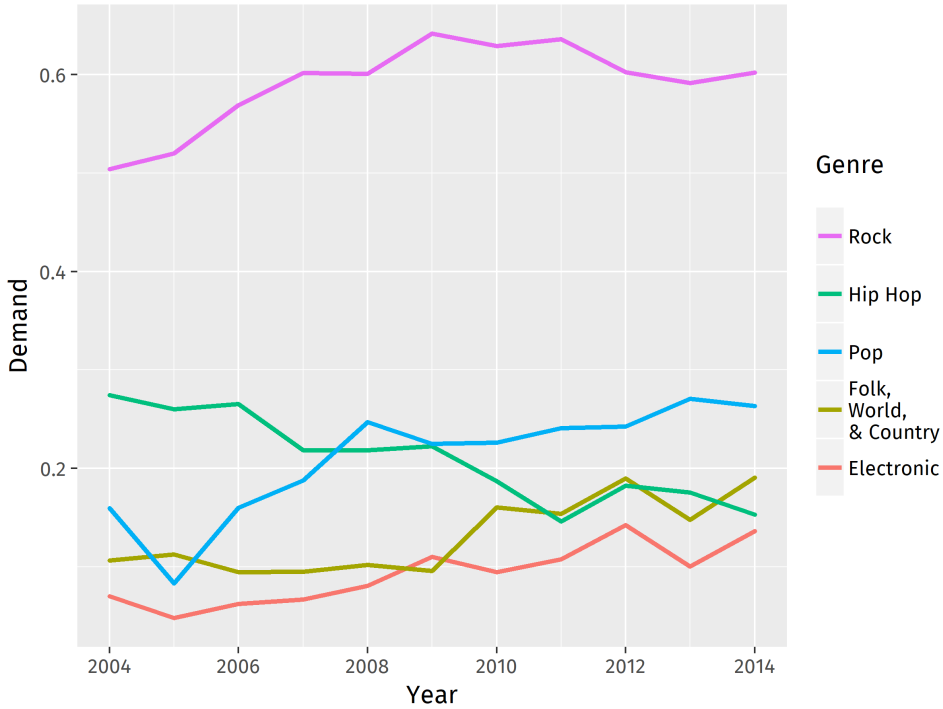
Figure 2.7: Yearly level of demand for the five most popular genres

gained some traction over the course of our study period, whereas demand for HIP HOP consistently decreased. All the other genres are clustered at values between zero and 20-percent.

At the firm level, we compute three control variables that capture yearly variation in the firms' positioning and competencies. The first such variable (INCUMBENT) measures a firm's presence in the focal genre: for each major m, category c, and year t, this equals the number of products released by m in c during t over the total number of products released by m during t [cf. 51]. The second variable (STRENGTH) measures the firm's proficiency in the focal genre, and for each m, c, t it equals the number of products released by m in c during t that enter the *Billboard 200*, over the total number of products released by m in c during t [cf. 51]. The third and final variable (SPAN) measures the extent to which the focal major straddles the boundaries of a genre. This is relevant to control for because products in multiple genres are located in-between different regions of the feature space [119] and thus they tend to be constrained in their feature space coordinates. A firm may not be able to adequately fill the region of space

that corresponds to a genre if it only releases products with multiple genre memberships. For each m, c, t, this variable equals the mean number of genres of products released by m in c during t.

2.3.3. ESTIMATION PROCEDURE

Because our dependent variable (Equation 2.1) is a count, an appropriate statistical model is necessary to accommodate its bounded and discrete nature. A common approach is to apply a logarithmic or square-root transformation and use the OLS estimator, but this is often suboptimal because the transformed variable remains non-negative and this can violate the assumption of normality in the distribution of OLS residuals [120]. An alternative solution is to use a Poisson model [121], which is estimated through the log-likelihood parameterization of the Poisson probability distribution. A distinguishing trait of this model is that the variance of the distribution is assumed to be equal to the mean: though this assumption is reasonable in some cases, it can lead to biased estimates when the true distribution whence the dependent variable is drawn has a variance that exceeds the mean. In these circumstances, referred to as overdispersion, other models should be used that allow the variance and the mean to differ [122].

Two models commonly used to account for overdispersion are the quasi-Poisson and the negative binomial. Both belong to the family of generalized linear models, and they express the expected value $E(Y) = \mu$ of the dependent variable as a function of the linear combination of regressors, namely $\mu = g^{-1}(\beta X)$, where g^{-1} is the exponential function, β is a vector of parameters, and X is the model matrix. Alternatively, $g(\mu) = \beta X$, where g is the logarithmic function. Quasi-Poisson regression assumes the observed value of the dependent $Y \sim \text{Poi}(\mu, \theta)$, and $\text{var}(Y) = \theta\mu$; negative binomial regression, instead, assumes $Y \sim \text{NB}(\mu, \kappa)$, and $\text{var}(Y) = \mu + \kappa\mu^2$. The difference, therefore, is that the quasi-Poisson model assumes the variance to be a linear function of the mean, whereas the negative binomial model assumes it to be a quadratic function. As a result, the negative binomial gives observations with a smaller value of the dependent greater weight in the regression relative to the quasi-Poisson, and conversely, the quasi-Poisson attributes greater weight to the observations with higher values. The choice between these two models thus depends on which of the two weighings is preferable for the research question at hand [123].

As shown in Figure 2.8, our dependent variable (RivalProducts) is overdispersed, which makes the standard Poisson model inappropriate. Among the two alternatives discussed above, the quasi-Poisson appears preferable because it gives smaller weights to peripherical genres (cf. Figure 2.3), where
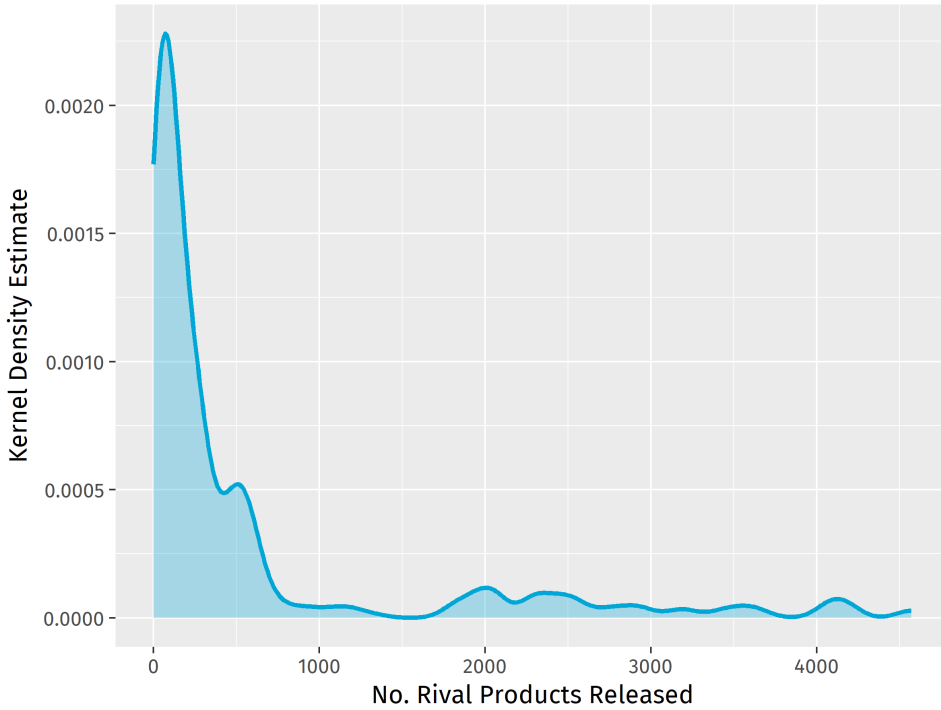
Figure 2.8: Probability function of the dependent variable

the record companies' activity is low. We use a fixed-effects specification to account for systematic differences across majors, genres, and years. All the regressors are lagged by one year, which causes the loss of 56 observations relative to 2004. The full model takes the form:

$$
\begin{aligned}
\ln \mathrm{RivalProducts}_{mc(t+1)} = {} & \alpha + \beta_1 \mathrm{Major}_2 \cdots \beta_3 \mathrm{Major}_4 + \beta_4 \mathrm{Genre}_2 \cdots \beta_{16} \mathrm{Genre}_{15} \\
& + \beta_{17} \mathrm{Year}_2 \cdots \beta_{25} \mathrm{Year}_{10} + \beta_{26} \mathrm{Contrast}_{ct} + \beta_{27} \mathrm{Leniency}_{ct} \\
& + \beta_{28} \mathrm{Demand}_{ct} + \beta_{29} \mathrm{Incumbent}_{mct} + \beta_{30} \mathrm{Strength}_{mct} \\
& + \beta_{31} \mathrm{Span}_{mct} + \beta_{32} \mathrm{OwnProducts}_{mct} + \beta_{33} \mathrm{Complexity}_{ct} \\
& + \beta_{34} \mathrm{OwnProducts}_{mct} \times \mathrm{Complexity}_{ct} + \varepsilon.
\end{aligned}
$$

$$(2.3)$$

where ε is drawn from a distribution $\mathrm{N}(0, \sigma_\varepsilon)$. Our results suggest that this model is appropriate: the dispersion parameter $\hat{\theta}$ is much larger than one.

Table 2.1: Descriptive statistics

	Mean	Std. dev.	Min.	Max.
RivalProducts$_{mc(t+1)}$	581.41	992.02	0	4570
Major	2.38	1.08	1	4
Genre	7.50	4.04	1	14
Year	2009.22	2.79	2005	2014
Contrast$_{ct}$	0.62	0.16	0.26	1.00
Leniency$_{ct}$	2.44	0.46	0	2.77
Demand$_{ct}$	0.10	0.15	0	0.64
Incumbent$_{mct}$	0.07	0.09	0	0.50
Strength$_{mct}$	0.15	0.17	0	0.67
Span$_{mct}$	1.66	1.25	0	12.00
Complexity$_{ct}$	1.85	0.94	0	3.39
OwnProducts$_{mct}$	21.70	36.37	0	258

2.4. RESULTS

Table 2.1 reports the descriptive statistics of the variables involved in our analysis. The pairwise correlations between these variables are presented in Table 2.2. We find highly significant correlations between the outcome variable and most of the predictors; in addition, some of the predictors are strongly correlated with one another, especially in the case of Incumbent and Demand ($r = 0.764$, $p = 0.000$), and Incumbent and OwnProducts ($r = 0.775$, $p = 0.000$). To assess the risk of multicollinearity, we perform conditioning diagnostics on the model matrix. The condition number is 43.27, which is above the threshold of 30 recommended by Belsley, Kuh, and Welsh [124]. Standardizing the matrix helps returning this number to a more acceptable 5.34. In the tables below, we strictly report estimates from standardized variables. For robustness, we also replicate our analysis after excluding the problematic variable Incumbent: in this case, the condition number is further reduced to 4.89, but the results are nearly identical.

Our main analysis involves the estimation of six nested quasi-Poisson models. The results from our control-only specifications are reported in Table 2.3. We begin by estimating a baseline model that includes only control variables and fixed effects for observation years (Model 1). The model's residual deviance of 54,043 suggests a relatively poor fit for the data, but this value decreases substantially with the addition of the other fixed effects: in Model 2, where the genre dummies are included, it is equal

Table 2.2: Pairwise correlations matrix

		1	2	3	4
1	RivalProducts$_{mc(t+1)}$				
2	Major	−0.01			
3	Genre	0.35•••	0.00		
4	Year	0.06	−0.15•••	0.00	
5	Contrast$_{ct}$	0.60•••	0.01	0.23•••	−0.07
6	Leniency$_{ct}$	0.24•••	0.00	0.40•••	0.02
7	Demand$_{ct}$	0.84•••	0.00	0.50•••	0.03
8	Incumbent$_{mct}$	0.65•••	−0.06	0.41•••	−0.01
9	Strength$_{mct}$	0.24•••	−0.27•••	0.20•••	0.16•••
10	Span$_{mct}$	0.00	−0.35•••	0.14••	0.06
11	Complexity$_{ct}$	0.67•••	0.00	0.42•••	−0.01
12	OwnProducts$_{mct}$	0.52•••	−0.10•	0.33•••	−0.11•

		5	6	7	8
6	Leniency$_{ct}$	−0.13••			
7	Demand$_{ct}$	0.49•••	0.27•••		
8	Incumbent$_{mct}$	0.46•••	0.26•••	0.76•••	
9	Strength$_{mct}$	0.12••	0.30•••	0.37•••	0.35•••
10	Span$_{mct}$	−0.07	0.32•••	0.00	0.10•
11	Complexity$_{ct}$	0.55•••	0.57•••	0.52•••	0.48•••
12	OwnProducts$_{mct}$	0.43•••	0.22•••	0.65•••	0.77•••

		9	10	11	12
10	Span$_{mct}$	0.37•••			
11	Complexity$_{ct}$	0.32•••	0.22•••		
12	OwnProducts$_{mct}$	0.30•••	0.09•	0.41•••	

Note: • $p < 0.05$; •• $p < 0.01$; ••• $p < 0.001$

to 3,572.3, and in Model 3, where the major dummies are also included, it is further reduced by 58. The estimates from this latter model suggest that Contrast has a positive effect on RivalProducts ($\hat{\beta} = 0.462$, $p = 0.000$): a one-std. dev. increase in the value of Contrast multiplies the expected number of rival product introductions in the following year by $e^{0.462} = 1.587$, which means that the dependent variable increases by 58.7-percent. We also find a positive effect for Leniency ($\hat{\beta} = 0.238$, $p = 0.000$), where a one-std. dev. increase leads to a 26.9-percent increase in the dependent, and Demand

Table 2.3: Quasi-Poisson model results: Controls

	Model 1		Model 2		Model 3	
	Dependent variable:					
	RivalProducts$_{mc(t+1)}$					
Intercept	4.66 •••	(0.10)	4.99 •••	(0.05)	4.99 •••	(0.05)
Contrast$_{ct}$	1.13 •••	(0.04)	0.45 •••	(0.05)	0.46 •••	(0.05)
Leniency$_{ct}$	2.40 •••	(0.17)	0.24 •••	(0.05)	0.24 •••	(0.05)
Demand$_{ct}$	0.14 •••	(0.03)	0.43 •••	(0.03)	0.43 •••	(0.03)
Incumbent$_{mct}$	−0.01	(0.02)	−0.00	(0.00)	−0.00	(0.01)
Strength$_{mct}$	−0.03	(0.03)	−0.02 ••	(0.01)	−0.02 •	(0.01)
Span$_{mct}$	0.08	(0.04)	0.00	(0.01)	0.02	(0.01)
Dispersion θ	158.91		7.40		7.33	
Major dummies	Not included		Not included		Included	
Genre dummies	Not included		Included		Included	
Year dummies	Included		Included		Included	
No. observations	518		518		518	
Deviance	54043		3572.3		3514.3	

Note: • $p < 0.05$; •• $p < 0.01$; ••• $p < 0.001$; std. errors in parentheses

($\hat{\beta} = 0.427$, $p = 0.000$), where a one-std. dev. increase leads to a 53.3-percent increase in the dependent. Finally, we find a negative effect for Strength ($\hat{\beta} = -0.018$, $p = 0.036$), where a one-std. dev. increase is associated with a 1.8-percent decrease in the dependent. This suggests that rival firms tend to avoid categories where the focal firm is more successful.

In the following models, we add our predictors of theoretical interest to the list of regressors. The results of these estimations are reported in Table 2.4. In Model 4, we add the first predictor, Complexity, and find that this significantly contributes to model fit as the deviance of residuals decreases by 308.61. The variable has a positive effect on RivalProducts ($\hat{\beta} = 0.426$, $p = 0.000$): namely, a one-std. dev. increase in Complexity causes the expected count of rival products released during the following year to increase by 53.1-percent. In Model 5, we include our second predictor of interest, OwnProducts, and detect a negative but non-significant effect ($\hat{\beta} = -0.001$, $p = 0.879$). The addition of this variable to the model leads only to a marginal decrease in deviance, which seems to suggest that the number of products released by the firm in the focal category has no effect

Table 2.4: Quasi-Poisson model results: Main effects and interaction

	Dependent variable:		
	RivalProducts$_{mc(t+1)}$		
	Model 4	Model 5	Model 6
Intercept	4.99 ••• (0.05)	4.99 ••• (0.05)	4.99 ••• (0.05)
Contrast$_{ct}$	0.43 ••• (0.05)	0.43 ••• (0.05)	0.44 ••• (0.05)
Leniency$_{ct}$	0.18 ••• (0.05)	0.18 ••• (0.05)	0.19 ••• (0.05)
Demand$_{ct}$	0.36 ••• (0.03)	0.36 ••• (0.03)	0.38 ••• (0.03)
Incumbent$_{mct}$	−0.00 (0.00)	0.00 (0.01)	−0.00 (0.01)
Strength$_{mct}$	−0.01 (0.01)	−0.01 (0.01)	−0.02 • (0.01)
Span$_{mct}$	0.02 (0.01)	0.02 (0.01)	0.02 (0.01)
Complexity$_{ct}$	0.42 ••• (0.06)	0.43 ••• (0.06)	0.44 ••• (0.06)
OwnProducts$_{mct}$		−0.00 (0.01)	−0.03 • (0.01)
Complexity$_{ct}$ × OwnProducts$_{mct}$			0.02 •• (0.01)
Dispersion θ	6.77	6.78	6.70
Major dummies	Included	Included	Included
Genre dummies	Included	Included	Included
Year dummies	Included	Included	Included
No. observations	518	518	518
Deviance	3205.7	3205.6	3156.5

Note: • $p < 0.05$; •• $p < 0.01$; ••• $p < 0.001$; std. errors in parentheses

on the number of products released by the firm's rivals in the following year. This null result is consistent with the analysis of the personal computer industry performed by Bayus and Putsis [15], but the question arises of whether the lack of empirical evidence is due to the actual absence of deterrence or rather to the fact that this model ignores the interaction with Complexity, as we suggest in this chapter.

We investigate this possibility by adding the interaction term to our list of regressors (Model 6). The model estimates support our proposition: the deviance of residuals decreases by a more substantial 49.1, the negative effect of OwnProducts becomes significant ($\hat{\beta} = -0.028$, $p = 0.041$), and the variable's interaction with Complexity has a positive and significant effect ($\hat{\beta} = 0.023$, $p = 0.007$). These coefficients imply that if the value of OwnProducts increases by one std. dev. while Complexity is held at its

mean, the expected count of rival products in the following year decreases by 2.8-percent. If Complexity simultaneously increases by one std. dev., however, then the negative effect of OwnProducts is offset by a 2.3-percent increase in the dependent: therefore, the expected value of RivalProducts only decreases by 0.5-percent. If, instead, the value of Complexity is one-std. dev. lower, then the negative effect of OwnProducts further increases by 2.3-percent, which means that RivalProducts decreases by 5.1-percent. Thus, the extent to which product proliferation reduces the likelihood of rival product introductions is greater in less complex regions of space.

Additional analysis. Though these results support our argument that the deterrent power of proliferation strategies is contingent on the level of complexity of the feature space, the parameter estimates strike us as small and it is reasonable to ask whether this is because proliferation strategies are relatively ineffective or because our outcome variable does not differentiate between products released by other majors and those released by independents (Equation 2.1). It could be the case that the effect is primarily driven by the other majors: these firms are diversified enough to actually have a choice as to where to position their products in the feature space, whereas indies may be constrained in their capacity to reposition themselves because they tend to be more specialized and thus constrained to particular categories. As a result, moving to a different region of the feature space can pose a greater threat to their survival than holding their ground and fighting an uphill battle against the proliferating major. If this were the case, the inclusion of the indie component in the dependent variable would blunt the effects of OwnProducts and its interaction with Complexity, pulling their estimated coefficients downward.

    We formally test this possibility by splitting our dependent variable into MajorProducts$_{mc(t+1)}$ and IndieProducts$_{mc(t+1)}$. Consistently with Equation 2.1, the first variable is computed as $\sum_{i=1}^{J} p_{ic(t+1)}$, where J is the set of other majors, and the second is computed as $\sum_{i=1}^{D} p_{ic(t+1)}$, where D is the set of indies. We enter these variables as regressands in two quasi-Poisson models with nearly identical specifications, which differ only in that the first also includes the total number of products released by indies in the same category during the previous year (IndieProducts$_{mct}$), whereas the second includes the total number of products released by other majors during the previous year (MajorProducts$_{mct}$). These extra variables are necessary to take into account that, in addition to the products introduced by the focal major, future product introductions by other majors (resp. by independents) are partly driven by the number of products introduced in the same category

Table 2.5: Quasi-Poisson model results: Additional analysis

	Dependent variable:			
	$\text{MajorProducts}_{mc(t+1)}$		$\text{IndieProducts}_{mc(t+1)}$	
	Model 6		Model 7	
Intercept	3.55 •••	(0.09)	4.68 •••	(0.05)
$\text{IndieProducts}_{mct}$	−0.14 •••	(0.04)		
$\text{MajorProducts}_{mct}$			0.02	(0.01)
Contrast_{ct}	0.48 •••	(0.08)	0.45 •••	(0.05)
Leniency_{ct}	0.02	(0.09)	0.22 •••	(0.05)
Demand_{ct}	0.61 •••	(0.07)	0.24 •••	(0.03)
Incumbent_{mct}	−0.03 •	(0.01)	0.00	(0.01)
Strength_{mct}	−0.01	(0.02)	−0.01	(0.01)
Span_{mct}	−0.03	(0.02)	0.01	(0.01)
Complexity_{ct}	0.79 •••	(0.02)	0.22 ••	(0.06)
OwnProducts_{mct}	−0.08 •••	(0.02)	0.02	(0.01)
Complexity_{ct} $\times \text{OwnProducts}_{mct}$	0.04 ••	(0.01)	−0.01	(0.01)
Dispersion θ	2.82		5.48	
Major dummies	Included		Included	
Genre dummies	Included		Included	
Year dummies	Included		Included	
No. observations	518		518	
Deviance	1328.1		2590.2	

Note: • $p < 0.05$; •• $p < 0.01$; ••• $p < 0.001$; std. errors in parentheses

by independents (resp. by other majors). Their inclusion does not create a collinearity problem as the condition number of the model matrix is 6.23 for the MajorProducts model and 6.93 for the IndieProducts model after standardization. We expect the effects of OwnProducts and its interaction with Complexity to be stronger in the MajorProducts model.

The results of these additional regressions are presented in Table 2.5 (Models 6–7). The estimates support our conjecture that deterrence mostly affects the other majors: in point of fact, it seems to affect only these companies because the coefficient of OwnProducts is negative and significant in Model 6 ($\hat{\beta} = -0.079$, $p = 0.022$) but not significantly different from zero in Model 7 ($\hat{\beta} = 0.017$, $p = 0.199$). The parameters imply that, all else being

equal, a one-std. dev. increase in the number of products released by the focal major in a particular genre is associated with a 7.6-percent decrease in the number of products released by the other majors in the same genre during the following year but with a negligible change in the number of products released by independents. The interaction effect is also stronger in the MajorProducts model ($\hat{\beta} = 0.045$, $p = 0.001$): if Complexity simultaneously increases by one std. dev., the net effect of a one-std. dev. increase in OwnProducts only amounts to a 3-percent decrease in the dependent variable. If, instead, Complexity decreases by one std. dev., then a one-std. dev. increase in OwnProducts lowers the expected value of MajorProducts by 12.2-percent. These results indicate a deterrent effect that is more than twice as strong than our main analysis suggested, albeit limited to a specific group of rivals. Post-estimation diagnostics (available upon request) suggest that our estimates are consistent and safely interpretable. No discernible pattern appears in the plot of residuals and no observation has an excessive leverage in the regression.

2.5. Discussion

This chapter examined the capacity of product proliferation to reduce competitive pressure by discouraging rival product introductions in a particular submarket or category, an effect Caves and Porter [6] referred to as the "contrived deterrence" of competition. Although previous studies in game theory have suggested that occupying (regions of) the feature space with one's products has a substantive deterrent effect [9, 13, 24–26], empirical evidence of this relationship is scarce. Consequently, scholars came to question the notion that product proliferation is actually useful to ward off competitors [15]. Bridging research in strategic management and organizational ecology, we proposed an original explanation for this inconsistency between game-theoretic predictions and empirical results. We argued that a product market cannot be regarded as a continuous surface but it rather consists of multiple subsets or submarkets, i.e., regions of the feature space, which represent product categories. These regions can be variably complex; some are internally homogeneous and therefore easy to saturate, and in these categories proliferation can be helpful to secure a dominant position. Other categories, however, are internally diverse, and because consumers are able to make finer-grained distinctions between products on offer the firms find it harder to saturate the space. In this case, product proliferation tends to be a less effective deterrent, in the sense that the viability of rival product introductions is less strongly affected.

Based on this argument, we hypothesized that product proliferation indeed has a deterrent effect (Hypothesis 2.1), but that this is negatively moderated by the complexity of the category where the strategy is pursued (Hypothesis 2.2). Our empirical analysis of new product introductions in the US recording industry supports these two hypotheses: we found that the proliferation strategies enacted by major record companies discourage product introductions in the same category by rival companies; however, this effect tends to disappear if the category presents greater variance in product attributes. In other words, the number of new products introduced in the same category by competitors does not change if the category is complex enough. Our additional analysis showed that deterrence primarily affects the other majors: hence, proliferation may only be a viable deterrent strategy against competitors who are sufficiently diversified to reposition themselves in feature space. It is not necessarily effective against rivals who are constrained to their current positions by specialization: for these firms, there may not be a real choice between fight and flight.

These findings, which allow us to accept our hypotheses, point to two possible explanations as to why the deterrent power of product proliferation seems to be virtually inexistent in some markets. The first is that the market in question may be too heterogeneous in terms of product features for proliferation to serve as a credible deterrent strategy. This is especially likely in technological markets during their emergent phase, such as the market for personal computer during the time period analyzed by Bayus and Putsis [15]. Indeed, the computer industry suffered a veritable "complexity catastrophe" [79] during the Eighties and Nineties as the firms became more vertically integrated and increasingly favored the use of proprietary components. As a result, the variance in technical specifications among the personal computer models available to consumers sharply increased. A similar rise in complexity can occur in non-emergent markets when they undergo important changes in regulatory frameworks. For example, proliferation may have temporarily ceased to be a viable deterrent strategy in the Spanish automobile market in after Spain's entry into the European Economic Community, as a large number of foreign car models with different technologies and designs became suddenly available to Spanish buyers [17]. An increase in complexity may have also rendered proliferation less effective in our very own empirical context, music recording, during the turbulent years that followed digitization [106]. In this case, it was not a shift in regulations that triggered the increase but rather a radical technological innovation, that is, digital distribution, which suddenly made available an unprecedented variety of products.

The second explanation as to why proliferation strategies can have little deterrent power hinges on the ecological distinction between generalists and specialists [71]. Ecological theory suggests that only diversified or generalist firms can easily change their level of engagement across different regions or categories, though this capacity tends to decrease with firm age [50, 125]. In the case of specialists, movements across market segments can be exceedingly hazardous because these firms are ill-equipped to handle change in the competitive environment [64]. As a consequence of this differentail tolerance, specialists may be undeterred by product proliferation, and this is not because they have the resources to effectively counterattack but rather because they do not have any other choice. In markets where specialists account for the vast majority of new products, such as the craft beer industry [58, 126], one may find no empirical evidence of strategic deterrence because only a relatively small number of rival product introductions is prevented.

Beyond explaining variation in the deterrent power of proliferation strategies, our findings contribute to the literature on product proliferation by exploring the implications of product space complexity. Previous research [17] analyzed complexity for the moderating role it exerts on the relationship between proliferation and performance: we advance this line of research by theorizing and testing the influence of complexity on the relationship between proliferation and rival product introductions. This effect is relevant to the strategic literature because creating barriers for competitors is among the primary reasons to pursue proliferation in the first place [1]. We also add to extant literature by arguing that complexity should not be examined at the level of markets or industries as a whole, but rather as a property of individual submarkets or categories. This perspective can be useful to generate new theory about competitive interactions at a subordinate level of analysis. In this regard, it is likely that complexity also affects the outcomes of other product strategies, such as diversification. Future research is needed to explore this possibility.

Defining the complexity of space as a category property contributes to the growing literature on categories in organization theory. This stream of research, which by now has reached the proportions of a self-standing research domain [36], concerns itself with the identification of category-level sources of heterogeneity that affect firm- [e.g., 127] and product-level outcomes [e.g., 128]. A number of category properties have already been defined and extensively analyzed by organization theorists [cf. 75]: we believe space complexity deserves to be added to this list because, while its variance across categories is not properly captured by the properties

examined before, it can affect many outcomes of interest to organization scholars, including the longevity of demand for particular kinds of offerings [27, 28], the competitive pressure experienced by organizations [115], and even the likelihood that new product categories emerge from existing ones by means of "subdivision" or "subtraction" [129, p. 389]. Our time-variant treatment of complexity speaks to the importance of considering categories and their properties from a dynamic perspective.

Our arguments are generalizable to other product markets that are sufficiently differentiated for the boundaries of categories to be discernible to firms and consumers. This is normally the case in mature industries [83], and it certainly is the case in markets for cultural products [45]; however, it is not necessarily the case in nascent industries where the products are too innovative for meaningful distinctions between their characteristics to be fully established. Our study is limited in that we analyze competitive dynamics in an oligopolistic market and restrict our consideration to the strategies enacted by the dominant incumbents: our conclusions are thus maximally applicable to similar firms. In our empirical setting, these firms are the only ones who can effectively pursue a proliferation strategy due to the costs associated with releasing a greater number of products in a fixed time frame; in other settings, however, the costs of introducing new products can be considerably lower, so that product proliferation may be likewise available to smaller firms. The question remains open of whether the outcomes experienced by these firms are comparable to those suggested by our empirical analysis.

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2

3

PROTOTYPES, GOALS, AND CROSS-CLASSIFICATION

3.1. MOTIVATION

Classification systems are ordered structures that organize cognitive domains by sorting objects into relatively homogeneous groups, or categories [1]. Rational agents in a market defer to these systems when making decisions about products and organizations: in some contexts, the categories they use are institutionalized and formally enforced [2], whereas in others they arise informally from communication and discourse [3, 4]. Either way, the category labels assigned to an object affect consumers' perception of value because they encode information that is relevant to economic decision-making and costly to obtain otherwise. This information can be especially critical in markets characterized by uncertainty, such as the creative [5] or technological industries [6], where consumers may be forced to choose among products with partly obscure characteristics [cf. 7]. In these circumstances, categories ease economic decisions by inducing expectations and default beliefs [8]: in doing so, they reduce information asymmetries and enable appropriate comparisons.

Sociologists showed that consumers heavily rely on classification systems when predicting if a product fits their taste [9], and because consumers are generally risk-averse, they tend to devalue objects with ambiguous category labels [4, 10]. This normally occurs when a product is assigned to multiple categories, as different labels convey partial or conflicting information [11]. Precisely for this reason, products that straddle the boundaries of categories have been found to receive lower evaluations by audiences regardless of their actual quality [12, 13]. This "multiple-category discount" is consistent with cognitive-psychological insights on categorization, according to which categories coalesce around specific reference points in a cognitive space [14]. In the eyes of consumers, the objects that resemble these points are familiar and easy to process [cf. 15]; deviant products, instead, tend to be cognitively taxing. In accord with economic theory [16], the uncertainty that consumers incur when attempting to derive information about multi-categorical products engenders a value discount.

This negative perspective on category spanning has long dominated organizational research. From the markets for creative goods [17] to those for technology [18], labor [19, 20] and capital [21], firms have been advised to focus themselves and their offerings in order to avoid being perceived as misfits. Yet many studies offered examples of products, organizations, or job candidates that span category boundaries—and succeed [22–25]. How can this evidence be reconciled with a constraining view of category labels? To answer this question, some scholars invoked the moderating role of producer-level constructs, such as status [26], identity [27], and

tenure in the industry [20]. Others appealed to category properties like contrast [28], leniency [4], and similarity [29]. In this chapter, we examine an alternative explanation, namely that different classification systems can be simultaneously relevant to evaluators, so that products belonging to multiple categories according to one system can appear less ambiguous as a result of the information one derives from another.

Such an instance of cross-classification [30] in multiple category systems can easily occur in markets where products are categorized according to the goals they fulfill as well as to the prototypes they resemble [31]. For example, foods are normally classified in a type-based system that includes GRAINS, MEATS, FRUIT, DAIRY, and VEGETABLES, but they can also be categorized in a system that comprises such labels as FUNCTIONAL FOODS [32], FOODS TO EAT WHILE DRIVING [33], and FOODS TO SERVE AT A PARTY [34]. Similar considerations apply to the categorization of firms: high-technology ventures, for example, are commonly sorted by investors based on the type of inventions they patent, but also (and perhaps more importantly) based on whether they pursue commercial product applications vs. fundamental science [35]. Consistency within a goal-based classification system can affect the evaluations received by category spanners: investors who are interested in high-tech start-ups, for instance, may not care that the technology is atypical as long as it can lead to marketable products.

Unlike categories based on prototypes, categories based on goals are seldom institutionalized and can be difficult to observe empirically. Unsurprisingly, most studies on category spanning so far only considered categorization from a prototype-centered view, which is not always adequate [36]. Organization scholars occasionally considered goal-based categories, but the few extant studies that reflect this broader perspective are limited in that they only examined evaluations by specific audience members who could be assumed to have particular goals [23, 24]. In this chapter, we adopt a more general approach and analyze product evaluations in a setting where both type- and goal-based categories are explicit and observable. This allows us to examine how products' positioning according to distinct classification systems affects their evaluations by a heterogeneous audience. Our analysis contributes to the organizational literature by showing that these different systems jointly contribute to sensemaking and to the formulation of a value order. In particular, we show that multi-categorical products can be easier to make sense of and receive higher evaluations either when goal-based category labels indicate goal-consistency, or when the products span such a large number of type-based categories that only a few combinations of features are possible.

To test our predictions, we analyze product ratings on *AllMusic.com*, a popular online platform where consumers can browse, sample, and evaluate thousands of music records. In this context, the type-based classification system corresponds to a well-established genre taxonomy where each membership implies conformity to a particular aesthetic canon [37]. The goal-based system, instead, revolves around the emotions or *moods* evoked by the music [38]. Mood categories have become increasingly important in the music business after digitization [39–41], as consumers increasingly use them to search for new products [42] and streaming services use them to provide goal-directed playlists [43]. On AllMusic, each record can span genre and mood categories independently: some products, like Morrissey's *You Are the Quarry*, have only two genre labels (POP/ROCK and R&B) but a large number of moods (including HUMOROUS, GLOOMY, INTIMATE, and BITTERSWEET); others, like Deep Forest's *Boheme*, have only one mood (HYPNOTIC) but span several genres (ELECTRONIC, POP/ROCK, and NEW AGE). We exploit this variance to isolate the effects of spanning in one or both systems.

This chapter is structed as follows: In Section 3.2, we review the cognitive-psychological literature that underpins our theory and present a geometric model of consumers' cognition. We explain how type-based categories and goal-based categories map to structurally different regions of this cognitive space, and why such different internal structures lead to different effects on evaluations for products with multiple labels. In addition, we theorize the effects of spanning type- and goal-based categories simultaneously. Section 3.3 presents the setting of our observational study, as well as our data and statistical methods. Section 3.4 reports our empirical findings, and Section 3.5 discusses their implications for organizational research.

3.2. Theory and Hypotheses

3.2.1. The Feature Space

Although from a psychological viewpoint categories affect decision-making by filtering information and helping agents prevent an overwhelming barrage of stimuli [1], in economic contexts they often serve a different purpose, namely compensating for the paucity of information by encoding facts about products that would be costly to obtain through different means [e.g., 6]. By virtue of this role, organization theorists termed categories the "cognitive infrastructures" of markets [44, p. 255]: this definition is appropriate not only because it captures the profound influence of categorization on human perception [45], but also because it (correctly) suggests that category labels hardly exist in a vacuum; instead, they arise in broader,

ordered structures, which we refer to as classification systems. Examples of such systems that are familiar to organization scholars include the *R. G. Dun* rating schema [2], the USPTO patent classes [18, 35, 46], the varietal categories of wines [47, 48] and cuisines [49], feature film genres [17, 20, 22, 27], and standard industrial codes [50]. Objects that belong to more than one category within the same classification system, as in the case of patents filed under different USPTO classes [18] or movies assigned by critics to different genres [22], are termed *category spanners*.

Understanding the effects of category spanning is key to fully appraising the role that categories play in the ordering of markets. In a concerted effort to clarify these effects, previous studies analyzed the relationship between category spanning and consumers' evaluations in a variety of empirical setting, reporting consistent evidence of a negative relationship [11]. Early studies explained this phenomenon by invoking the ecological principle of allocation: because producers have finite resources, trying to appeal to different audience groups causes them to have a lower fit with the tastes of each group on average [17]. Subsequent research refined this explanation by acknowledging that perceptual effects are simultaneously at play [10, 12, 13]: from this angle, multiple category labels negatively affect evaluations because they induce ambiguity and increase audiences' risk of forming erroneous expectations about products. If the audience members are risk-averse, as is normally the case of consumers [4], they are likely to react to this uncertainty by ascribing lower value to the products [16].

This perspective is consistent with seminal research on categories in cognitive psychology, according to which category labels function as cues that people can interpret to retrieve and possibly combine [51–53] information previously stored in their minds. Building directly on prototype theory [54] and the related literature on conceptual spaces [55, 56], sociologists recently proposed a geometric model of the market as a multidimensional space where each category maps to a particular region and where prototypes act as shared reference points [9]. The location of products in this cognitive space depends on their features, of which category labels can offer an inkling [57]. If there is only one (type-based) category label, meaning that the product is highly typical of a certain category, consumers can easily surmise its location thanks to their familiarity with the category prototype. The presence of multiple labels, however, tends to undermine this process by signaling that the product's position is intermediate [29] and thus harder to predict [58]. This can vex evaluators, who need clear information about products' features to estimate their worth. Clear-cut memberships help making these features known to consumers *ex ante*.

3

This model of the market as a space of product features has illustrious precedents in economics [59–63] and can be considered akin to a Lancastrian or locational-analog model of demand (cf. Chapter 2). In this chapter, we aim to address one of the current limitations of this geometric approach: inasmuch as this model is only used to conceptualize categories that revolve around prototypes, it can misrepresent real-world domains where other classification systems exist and are simultaneously relevant to consumers [30]. In fact, the limitations of prototype theory and the relevance of categories that do not possess a prototypical structure have long been acknowledged in psychology [36, 64] and consumer research [65]: here, we consider the implications of categories based on consumers' goals, which are common in many product markets [31, 32] but do not necessarily hinge on prototypical representations [66]. While type-based categories encode the products' conformity to some cognitive reference point [14]—in other words, they signal proximity to a prototype in the feature space—goal-based categories express suitability for particular purposes and cannot be used to infer a distance. In point of fact, such categories differ from their type-based counterparts precisely because *central tendency* is not a good predictor of category membership [67]: the same label can be equally applicable to very dissimilar (i.e., distant) objects [68].

This major distinction notwithstanding, goal-based categories encode information that helps consumers determine a product's utility [33], and because consumer behavior is often goal-directed [69], their impact on product evaluations is likely to be vast. At present, this impact is largely unaccounted for in organizational theories. We fill this gap in extant research by extending the spatial model above so as to account for goal-based categories in a geometric fashion. We use this amended model to derive testable predictions about the effects of spanning categories in a type-based classification system, a goal-based classification system, or both. Moreover, we highlight this model gives rise to an original parsimonious explanation as to why products that span type-based categories can receive better evaluations by audiences, all else being equal.

Our arguments pivot on the elementary notion, widely acknowledged in economics since the work of Herbert Simon [70], that "anticipating future consequences of present decisions is often subject to substantial error" [71, p. 589]. In our case, the future consequences relate to the utility derived from a product that consumers know little about before consumption [72]. One of the main reasons why category labels influence economic behavior is that they allow agents in a market to make better predictions vis-à-vis a product's value: in the spatial model we co-opt from sociological research,

this prediction involves pinpointing the product's location in feature space with reasonable accuracy. Compatibly with Akerlof's classic argument about uncertainty in markets [16], some of the variance in consumers' disposition toward products depends on how easily they manage to solve this informational problem [6]: If the product has a single type-based category label, meaning that it closely resembles a specific prototype, the solution to this problem is relatively straightforward because prototypes are already familiar to consumers. Yet, under particular circumstances it is possible to derive the product's location with reasonable accuracy even if the product has multiple type-based category labels, which may give rise to a nonmonotonic effect on evaluations. This effect is impossible in the case of goal-based categories because of fundamental differences in these categories' internal structures [cf. 67].

The objective of our study is to provide empirical evidence of this structural difference. In the next section, we detail the cognitive mechanism through which consumers can infer a product's features by combining information from multiple prototypes. While this mechanism is justified by the geometric nature of the feature space, it requires the following assumptions about evaluators' knowledge and behavior:

(a) Evaluators must know at least some of the prototypes associated with the products they evaluate. Notice that they are not required to know all the prototypes in the domain at hand, nor are they required to know all the prototypes associated with the product they consider. It suffices to know only some.

(b) Evaluators must able to tell how far a product can be from a prototype and still be considered a member of the corresponding type-based category. In other words, they must know the category's contrast [11].

(c) Evaluators must be able and willing to combine information from multiple categories in order to make better inferences about products' features. To this purpose, they must not resort to simpler strategies, like defaulting to the most salient category label.

Assumptions (a) and (b) imply a minimal level of familiarity with the cognitive domain: in our case, the market where the products belong. In this sense, the mechanism we propose is only available to an audience that knows at least some of the categories. Assumption (c) implies that evaluators engage in multiple-category induction [73]. Extant research in cognitive psychology suggests that this assumption is reasonable, especially when the category labels are listed explicitly [74, 75].

It is worth remarking explicitly that the primary objective of this chapter is not to identify an empirical relationship that occurs in settings where these assumptions hold true. The assumptions above are restrictive and do not hold in most markets. Our aim is rather to offer empirical proof of a key distinction between type- and goal-based categories, which is expected to manifest itself through a differential effect on consumers' evaluations under the aforementioned conditions. The violation of these assumptions does not imply that the distinction wanes or that its implications for evaluators' uncertainty become negligible; only that there may not be a similar effect on the value consumers attribute to products.

3.2.2. ATYPICALITY AND ITS CONSEQUENCES

We first consider the consequences of spanning categories that encode family resemblance. Consistently with previous research [e.g., 9], we characterize the market as a D-dimensional space where D is the total number of features $f_1 \ldots f_D$ that distinguish the products in the eyes of consumers. The value of D represents the space's dimensionality. Each product occupies a point or position in this feature space, which is uniquely identified by a vector $\vec{f} = \langle f_1 \ldots f_D \rangle$, termed *feature profile* [57]. Consumers do not necessarily know the feature profiles of the products they evaluate, but they are required to infer them with reasonable accuracy in order to compute the products' distance from their tastes [63]. For the sake of illustration, consider a simple market where products are distinguishable by as little as four features, as in the case of balloons that differ by the color of four quadrants on their surface ($D = 4$). Without loss of generality, suppose that the color of each quadrant can be either blue ($f = 0$) or orange ($f = 1$), thus giving rise to a space of 16 positions, as presented in Figure 3.1. Each node in this graph corresponds to a unique feature profile, and the edges connect profiles that differ by only one value, meaning that they occupy adjacent positions in the feature space.

Suppose that ORANGE BALLOONS were a relevant type-based category in this context, with prototype $\vec{f}_1 = \langle 1\,1\,1\,1 \rangle$. To emphasize the centrality of this prototype in category generation, we refer to ORANGE BALLOONS in notation as TYPE($\vec{f}_1$). If products were required to have exactly four orange quadrants in order to be considered members of TYPE($\vec{f}_1$), then the category would have very high contrast, meaning that it would be easy to tell whether a product belonged to the category or not [11]. If the members of TYPE($\vec{f}_1$) were allowed to have one blue quadrant, instead, contrast would be lower and the category would also encompass $\vec{f}_{2-5}$. As a result, its boundaries would be fuzzier [cf. 17] and membership would no longer be clear-cut.
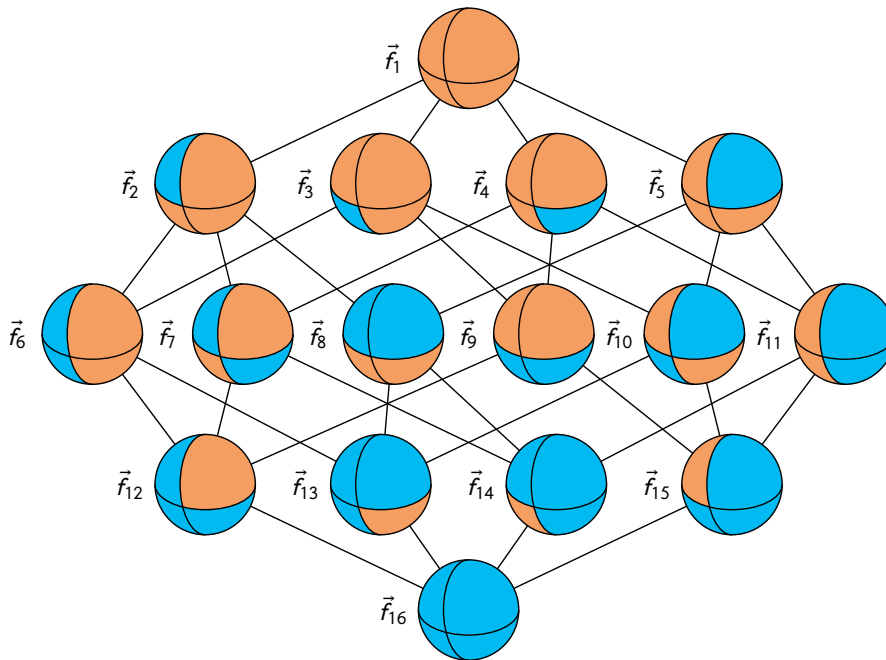
Figure 3.1: A four-dimensional space of binary features

Irrespective of contrast, the positions encompassed by a type-based category are always clustered in a *convex* region of the feature space [76]. Convexity is a geometric property implying that, for any two non-adjacent points that belong to the category, there are positions in-between these two points that also belong to the category. In a Euclidean space, for example, shapes like circles, squares, and triangles are convex, but those that have indents or holes such as stars, lunes, and annuli are not. In practice, convexity implies that consumers can trace the location of any category member to a subset of adjacent and therefore similar feature profiles. The geometric mean of these positions (i.e., the center of the region) corresponds to the category prototype. Familiarity with this geometric center is the reason why consumers can derive information about products with a single category membership with relative ease. This process is all the more accurate if the category has higher contrast, because high-contrast categories tolerate smaller deviations from their centroids. As a result, products (and organizations) in high-contrast categories tend to receive better evaluations by their audiences [77].

The risk of misjudging a product's position can be higher if the product has multiple type-based category labels. This is because consumers infer

that the product imperfectly resembles several prototype; however, they do not know the extent to which it resembles each—and even if they did, there could be many positions in the feature space that are equally consistent with this information. For example, suppose that a second type-based category existed in our market for balloons, with prototype $\vec{f}_{15} = \langle 0\ 1\ 0\ 0 \rangle$. If consumers knew that a product belonged to TYPE($\vec{f}_1$) and TYPE($\vec{f}_{15}$), they would not be able to locate its position with the same accuracy as the previous scenario because multiple profiles are compatible with these partial cues. Indeed, $\vec{f}_{3-5}$ and $\vec{f}_{9-11}$ are all located between the two prototypes, hence they are equally plausible in the eyes of a rational agent [75]. Intuitively, this means that the features associated with TYPE($\vec{f}_1$) and TYPE($\vec{f}_{15}$) can be combined in multiple ways that are equally compatible with both category labels, but in lack of any additional information, the evaluator is unable to tell which combination is correct. In this sense, our model agrees with extant research in organization theory [e.g., 10]: spanning type-based categories renders products' features ambiguous.

Our model diverges from conventional wisdom, however, in that this uncertainty-increasing effect is expected to reverse if the number of type-based category labels assigned to the product is sufficiently large. This is because more and more labels make fewer and fewer options appear plausible in the eyes of consumers. For example, suppose that a third type-based category existed in our fictitious market, with prototype $\vec{f}_6 = \langle 1\ 0\ 0\ 1 \rangle$. If consumers knew that the product in question belonged to TYPE($\vec{f}_1$), TYPE($\vec{f}_{15}$), and TYPE($\vec{f}_6$), the range of possible solutions would be restricted to only one feature profile, namely $\vec{f}_3$, because this is not only located between $\vec{f}_1$ and $\vec{f}_{15}$, but also closer to $\vec{f}_6$. The process whereby consumers selectively combine the information from different category labels follows the set-theoretic rules that Hampton [51] referred to as the "necessity" and "impossibility" of feature inheritance. These imply that a product belonging to multiple categories must have some of the features of each category prototype, but this combination cannot be arbitrary because some of the features are incompatible. Given a sufficient amount of prototypes, consumers may eventually obtain a unique solution to this informational problem. As the number of prototypes (and hence of type-based category labels) approaches this critical threshold, evaluators' uncertainty about the features a product actually possesses should decrease.

From a geometric standpoint, this cognitive mechanism is comparable to the geometric process of trilateration of a variable point on a Euclidean plane: given the distance from two other fixed points (the prototypes), multiple solutions are possible, but knowing the distance from another

fixed point is sufficient to single one out. In our model, type-based category labels convey information about distances: more specifically, they encode that the distance between a product and the category prototype is smaller than some minimal threshold required for category membership. Generalizing the trilateration example to a space of $D > 2$ dimensions, more than three distances can be required to derive the coordinates of a variable point; however—and this is the gist of our argument—the minimum number of labels required for a single solution to be determinable is constrained by the space's dimensionality. Formally, this is because the coordinates of a variable point in an D-dimensional Euclidean space can be determined by solving a system of $D + 1$ equations (assuming that the point exists):

$$\begin{cases} d(x, y_1)^2 = (f_{1x} - f_{1y_1})^2 + \ldots + (f_{Dx} - f_{Dy_1})^2 \\ \vdots \\ d(x, y_i)^2 = (f_{1x} - f_{1y_i})^2 + \ldots + (f_{Dx} - f_{Dy_i})^2 \end{cases}, \qquad (3.1)$$

where d is a Euclidean distance, $f_1 \ldots f_D$ are the coordinates (features) of a point in the space, x is the variable point (the product), y_i is the i-th fixed point (the i-th prototype), and $i = 1 \ldots D+1$. A cognate way to conceptualize this problem is by picturing the boundaries of each type-based category as a D-sphere centered on the prototype. The intersection of $D - 1$ D-spheres (e.g., two regular spheres in a three-dimensional space) is a circle. Hence, any product to which the $D - 1$ type-based category labels are applied ought to be in this circle. Given an additional D-sphere, the intersection reduces to two points, and given yet another, it reduces to only one point. Informally, this means that as consumers are provided with more and more type-based category labels they see fewer and fewer ways in which the features associated with membership in these categories can be combined in a way that remains consistent all the category labels.

Our conjecture is that, if evaluators are given a number of labels that exceeds the space's dimensionality, the feature profile of a product can be determined with the same level of accuracy as if there were only one label, although this requires some cognitive effort. Because human beings are unable to process spaces of much higher dimensionality than our fictitious example [78], we expect $D + 1$ to be a relatively low number. It follows that, in markets that are sufficiently diverse for products to span a large number of type-based categories, some products may span enough for this cognitive mechanism to produce its effect. Therefore, we should be able to witness a relative increase in the average evaluation as the number of type-based labels approaches $D + 1$. Notice that this does not mean that

the evaluations received by very atypicals product will be ever as high as those received by very typical ones, because on the one hand it remains the case that category spanning decreases a product's fit with consumers' tastes [17], and on the other hand, it is likely that consumers will devalue atypical products for the cognitive effort they require. Still, evaluators should incur lower uncertainty when they are able to derive the products' features with a greater level of accuracy, and this should be sufficient to cause an inflection in the relationship between the number of type-based categories spanned by a product and its average evaluation.

We thus predict that the relationship between a product's number of type-based category labels and its average evaluation by consumers is U-shaped. Thus, we expect products with a single type-based label to be evaluated best, but products with a sufficiently large number of labels should be evaluated better than those with an intermediate number. One can think of this relationship as the result of the sum of two cost functions [cf. 79]. First, evaluations reflect the "actual" [13] cost of having a lower fit with consumers' preferences: this function is monotonic because the more a product deviates from the prototypes, the less its average fit [17]. Second, evaluations reflect the perceptual cost of ambiguity: this function is non-monotonic because ambiguity is minimal either when the number of categories is one or when it approaches $D+1$. The sum of the two functions results in a U-shaped effect on consumers' evaluations. Therefore:

Hypothesis 3.1. *The relationship between a product's number of type-based category labels and its average evaluation by consumers is U-shaped.*

3.2.3. Suitability for Multiple Goals
We now turn our attention to the consequences of spanning categories that revolve around goals as opposed to prototypes. As mentioned before, the crucial distinction between type- and goal-based categories is that the goodness of membership in a goal-based category does not increase with centrality or resemblance to a prototype [67]. Given a set of objects that belong to the same goal-based category, computing an average of the objects' features will not necessarily result in an object that is consistent with membership in the goal-based category, let alone one that may be considered its best representative. This is because the suitability of an object for a particular purpose depends on its features but not on its distance from a specific reference point [cf. 14]: in Barsalou's classic example [80], the category of THINGS TO TAKE FROM ONE'S HOME DURING A FIRE can include as diverse members as dogs, children, and blankets. Likewise, the category of FOODS TO EAT WHILE DRIVING [33] can include relatively dissimilar

objects like apples and granola bars. It does not include oranges, though these are closer to apples in feature space than apples are to granola bars.

Even if they are unrelated to prototypes, the members of a particular goal-based category are likely to have some features in common: the things one may salvage from home during a fire, for instance, tend to be either moving or easily movable, and the things one finds convenient to eat at the wheel do not usually require peeling. Such regularities exist not because the category members resemble some central instantiation of a concept, but because particular features or combinations thereof are instrumental to achieving certain goals. For this reason, two nearly identical objects will most likely belong to the same goal-based category, but objects in the same goal-based category are not necessarily similar to one another. This asymmetry has an important implication for categorization in markets: relatively dissimilar products can belong to the same goal-based category as long as they help consumers address the same need [65, 68]. They would remain members of this particular category even if they differed in every other respect. Arguably, this perspective explains demand elasticities that would hardly make sense from a prototype-centered view [81].

In our geometric model, this key distinction between type- and goal-based categories implies that goal-based categories do not necessarily map to convex regions of the feature space. To illustrate this argument, consider again the four-dimensional market presented in Figure 3.1. Suppose that a certain goal were relevant for consumers in this context that required products to fulfill a specific condition, namely that the two left-hand quadrants on its surface have exactly the same color ($f_2 = f_3$). Because it is a goal rather than a prototype serving as the engine of categorization, we refer to the category in notation as GOAL($f_2 = f_3$).[1] Eight feature profiles are consistent with membership in this category, including $\vec{f}_1$, $\vec{f}_{4-6}$, $\vec{f}_{11-13}$, and $\vec{f}_{16}$. However, these are not necessarily separated by profiles that are themselves included in the category, as would be required by convexity. The absence of convexity affects the information consumers can derive from products' category labels: if all they know about a product is that it is a member of GOAL($f_2 = f_3$), then they are unable to pinpoint its location in the feature space as accurately as if they had a prototype. They can infer that the two left-hand quadrants have the same color, but they do not know which color it is—the actual coordinates thus remain uncertain.

Unlike the case of type-based categories, this ambiguity is not necessarily reduced if the product's number of goal-based category labels is

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[1]Although this notation is slightly cumbersome, we find it appropriate because it reflects the notion that goal-based categories depend on features but not on a feature profile.

sufficiently high. In fact, no minimum number of labels in a goal-based system guarantees that the product's position is geometrically derivable due to the absence of reference points. For example, suppose that a second goal-based category GOAL($f_1 = f_4$) were relevant for consumers, which required products to have the same color on the two right-hand quadrants. If a product were labeled both GOAL($f_2 = f_3$) and GOAL($f_1 = f_4$), its feature profile could be any of the following: $\vec{f_1}$, $\vec{f_6}$, $\vec{f}_{11}$, or $\vec{f}_{16}$. In fact, these are the only profiles that satisfy both goal-driven conditions. Once again, these positions are not adjacent in the feature space; in other words, they do constitute a convex set. Now suppose that a third goal-based category existed in this market, namely GOAL($\sum_{i=1}^{4} f_i = 2$), which required products to have the same color on no more than two quadrants (whichever they are). A product labeled GOAL($f_2 = f_3$), GOAL($f_1 = f_4$), and GOAL($\sum_{i=1}^{4} f_i = 2$) could be traced to either $\vec{f_6}$ or $\vec{f}_{11}$: again, these positions are not adjacent to each other, and while it is true that having additional labels allows consumers to rule out some options, the number of labels required to obtain a unique solution is not strictly constrained by the space's dimensionality. Ultimately, this is because goal-based categories do not convey information about distances: therefore, the trilateration-like mechanism described in the previous section is not available to evaluators. With the exception of some limit cases,[2] having a greater number of labels in a goal-based system does not decrease consumers' uncertainty.

This argument does not imply that no relationship exists between a product's number of goal-based category labels and its average evaluations by consumers. As in the case of type-based categories, having multiple category memberships in a goal-based system results in a lower fit with individual needs because "producers face technological barriers to serve multiple consumer goals optimally" [68, p. 242]. This is the reason why, for example, high-technology start-ups that pursue fundamental science are deemed unlikely to produce commercial applications [35], and foods considered appropriate for breakfast tend to be inopportune at a dinner party [34]. Consistently with the ecological principle of allocation, multiple goal-based category labels imply that the product is less effective on average at each of the needs it purports to fulfill. Because a greater number of labels does not simultaneously contribute to evaluations by way

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[2]Two cases exist whereby consumers can locate a product thanks to a greater number of goal-based labels: (1) if one of the goals is so difficult to fulfill that only one position in the feature space is suitable, and (2) if the goals are so difficult to fulfill concurrently that only one position is even partly suitable. These are extreme cases and they are unlikely to occur in any complex real-world domain.

of uncertainty reduction, we expect a linear negative effect on consumers' evaluations. This leads to our second hypothesis:

Hypothesis 3.2. *The relationship between a product's number of goal-based category labels and its average evaluation by consumers is linearly negative.*

3.2.4. THE EFFECT OF SPANNING IN DIFFERENT SYSTEMS

In the previous section, we explained why having multiple goal-based category labels does not normally contribute to uncertainty reduction. We now turn to explaining why spanning goal-based categories can still have positive consequences for product evaluations. So far, we implicitly assumed evaluators to consider products' category memberships according to either a type- or a goal-based system, but the cornerstone of this chapter is precisely that both classification systems jointly contribute to the valuation of products. This is because consumers can derive information from both prototypes and goals in order to pinpoint a product in the feature space. The way in which the information available from these two systems is combined by evaluators deserves additional consideration: the net effect of spanning in both systems is not merely the sum of the two effects.

Consider again the four-dimensional example in Figure 3.1: suppose that a product were labeled TYPE($\vec{f_1}$), TYPE($\vec{f}_{15}$), GOAL($f_2 = f_3$), and GOAL($f_1 = f_4$). What consumers can infer from these category labels is that, on the one hand, the product must be located somewhere between $\vec{f_1}$ and $\vec{f}_{15}$, and on the other hand that the product's features must fulfill the conditions $f_2 = f_3$ and $f_1 = f_4$. Merging this information allows a single feature profile to emerge as plausible, namely $\vec{f}_{11}$, because this is the only option located between the two prototypes that is simultaneously consistent with the goal-driven requirements. The level of accuracy in prediction that consumers could achieve with three prototypes is now achievable with as little as two. The presence of multiple goal-based labels thus makes additional memberships in a type-based system no longer necessary to extrapolate the product's coordinates, and the trilateration-like mechanism described in Section 3.2.2 allows evaluators to find a solution with fewer type-based labels than would be required otherwise. Generalizing to spaces of $D > 2$ dimensions, the number of goal-based labels required for this mechanism to yield a unique solution is not necessarily as low; nevertheless, agents will be allowed to rule out some options with each additional label.

To make a more realistic example, consider the case of a music product spanning the genres JAZZ and ELECTRONIC. Suppose that the feature space of music were simple enough that only two features could not be predicted on the basis of these two type-based labels, namely the presence of vocals

and the music's tempo. Even in this simple case, there are several ways in which a product could span these two categories and still be consistent with the category labels, some of which include vocals whereas others do not, and some of which involve a fast tempo whereas others do not. Further assume for the sake of simplicity that these features are binary. In this case, there are exactly four options: (1) fast with vocals, (2) fast without vocals, (3) slow with vocals, and (4) slow without vocals. Having a third type-based category label, such as AMBIENT or RAP, would be sufficient to dispel uncertainty one way or another, because the prototype of RAP is only consistent with (1), and the prototype of AMBIENT is only consistent with (4).[3] However, consumers do not have any additional type-based label: instead, they know that the product is suitable for a particular goal, i.e., STUDYING. What they may infer from this goal-based label is that the product is unlikely to have features that interfere with concentration and other high-order cognitive skills. Vocals tend to be distracting [82], and hence options (1) and (3) appear less probable. The relation between concentration and tempo, however, is far less predictable, so that (2) and (4) are still equally plausible. Suppose that consumers also knew that the product is suitable for EXERCISING. What they can infer from this additional goal-based label is that the product is unlikely to be slow-paced because such music is hardly appropriate to enhance physical exertion [83]. Given this extra piece of information, (2) appears decidedly more likely than any other option.

Through this conceptual integration, having more category labels in a goal-based classification system can moderate the negative consequences of spanning type-based categories. Of the two cost functions underlying the U-shaped relationship predicted in Hypothesis 3.1, spanning goal-based categories should not affect the "actual" cost that is due to the allocation principle—this ensues regardless of the products' number of goal-based category memberships—however, it should mitigate the perceptual cost of ambiguity. By facilitating the identification of a product's feature profile among a set of alternatives that would be equally plausible in the presence of type-based category labels alone, having more goal-based labels can help consumers predict the features of atypical objects. This can induce a subtle but significant change in the U-shaped relationship: the curvature should remain the same, meaning that the relative difference between products with the same number of goal-based category labels but a different number of type-based ones is not expected to change; the slope of the

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[3] From the liner notes of Brian Eno's *Ambient 1/Music for Airports* 1978 US release: "Ambient music must be able to accomodate many levels of listening attention without enforcing one in particular; it must be as ignorable as it is interesting."

curve, however, should turn from negative to positive at a comparatively low number of type-based category labels. In other words, the turning point of the U-shape is expected to shift toward the left [cf. 79]. This leads to our third and final hypothesis:

Hypothesis 3.3. *Having more goal-based category labels causes the turning point of the U-shaped relationship between a product's number of type-based category labels and its average evaluation by consumers to shift toward the left.*

3.3. METHODOLOGY

3.3.1. EMPIRICAL SETTING

We test our three hypotheses on data collected from *AllMusic.com*, a popular online platform that provides editorial information, user ratings, and category memberships for thousands of music products. The market for recorded music, and especially popular music [84], makes a suitable setting for our empirical analysis because the endemic conditions of oversupply and uncertainty make category labels crucial to consumers' decision-making. Moreover, this context is germane because different classification systems are simultaneously relevant to audiences. The most widely known system is the genre taxonomy [85]: as explained in Chapter 2, these categories encode resemblance to specific combinations of features [37]. Genre categories play a key role in organizing the music landscape both from a supply- [86] and a demand-side perspective [87]. Because music is consumed for the purpose of affirming one's social identity [88], audiences care about products' conformity to certain aesthetic canons [89]. Consistently with this rule, prototypical music is generally found more appealing among non-expert audiences: for example, Smith and Melara [90] found that among novices, undergraduate students, and graduate students of music, all three groups were highly sensitive to atypicality but only graduate students preferred atypical compositions.

Genres are not the only relevant classification system in this context, however. Beyond the categories and subcategories that comprise the genre taxonomy, music is widely cross-classified according to the emotions or moods it evokes [39–42]. "That music is an especially powerful stimulus for affecting moods is no revelation; it is attested to throughout history by poets, playwrights, composers, and, in the last two centuries, researchers" [38, p. 94]. The longstanding evidence of this connection that offered in cognitive psychology [91] is corroborated by many observational studies in marketing and consumer research [92–94]. Music moods can be considered

goal-based categories because, like art in general, music is a hedonic good and its consumption is driven by emotional arousal [72]. A product's capacity to evoke particular moods clearly depends on its features [95]; however, it does not normally depend on conformity to a particular genre. Mood categories like MELANCHOLY, ENERGETIC, and PLAYFUL can encompass products from as diverse genre categories as ROCK, JAZZ, and LATIN.

AllMusic is a well-established source of category information within the music business. It was originally launched in 1991 as a physical encyclopedia, but its content soon acquired proportions unfit for printing and the database became freely accessible online as early as 1995. At the end of 2007, both the website and the database were acquired by Rovi Corporation. In 2013, ownership of the website was sold to a spin-off company, All Media Network, which is currently licensed by Rovi to use the data. The same license is granted to other companies, including Amazon, Microsoft, and Apple. The website currently receives more than eight million visits per month on average, making it the eighth music website worldwide by traffic. Its editorial content is relayed by "virtually all digital music services," [96] including iTunes, Napster, Pandora, Shazam, Slacker, and Spotify. At present, the website is maintained by some 900 critics, who assign category labels to products as part of their editorial routine. The reviewing process also involves rating the product on a scale from one-half to five stars. As of 2013, registered users can also rate products on the website.

Using this data for our empirical analysis entails some unavoidable limitations. One of these is that we analyze products released since 1995, the year the database was put online, but the genre taxonomy can change over time, as does the meaning of individual categories [97, 98]. These changes may have induced AllMusic editors to revisit the labels previously assigned to some products. To assess whether this is the case, we systematically cross-referenced our data with the 2001 paperback edition of *The All Music Guide*. For the purpose of this check, we randomly selected 200 records in our sample that were released before the guidebook's publication: these include relatively popular products, like Eminem's *The Slim Shady LP* (633 ratings), as well as products that were never rated by AllMusic users at all. In 93.4-percent of cases, the categories attributed to products on the website were found to be consistent with the print edition. Some categories were merged or split over time: for example, PROG-ROCK/ART ROCK is a single category in the guidebook, but PROG-ROCK and ART ROCK are separate categories on the website.[4]

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[4] We account for this empirically by controlling for the categories' similarity.

Another possible limitation is that AllMusic lists category labels only at the level of full-length albums, or shorter products like singles and EPs, but not individual tracks. This concerns our analysis because an album can span categories either by having a tracklist of atypical songs or by having a tracklist of highly typical songs that belong to different categories. While it is impossible to account for this heterogeneity due to unavailability of data, we can at least control for the number of tracks included in each album and use an empirical strategy that accounts for artists' idiosyncrasies.

Finally, our analysis can be affected by the fact that user ratings were only added to the website in 2013, one year before data collection. This is an advantage inasmuch as all the ratings we observe were cast by users who had access to the same information; however, it can be a drawback because users may have held prior beliefs about the value of products. We are unable to rule out this possibility by restricting our sample only to the most recent products because this would limit sample size and make it impossible to properly capture artist-level effects.

Notwithstanding these limitations, AllMusic satisfies some important desiderata. First, it explicitly lists category labels in the form of tags, which is important because, as noted in Section 3.2.1, consumers may only engage in multiple-category induction if they are explicitly "reminded" of the different category labels [74]. A review describing the music in a narrative fashion may not serve this purpose equally well. Second, the category labels are listen in alphabetical order: therefore, the first label is not necessarily the one where the product is more typical. This alleviates the concern that in the presence of a long list of labels consumers may default to the first. Third, the AllMusic data is reliable in that the website employs human experts to assign mood category labels as opposed to automated methods [99]. For this reason, AllMusic moods are frequently used as a ground truth corpus in music information retrieval to train and benchmark various classifier algorithms [e.g., 42]. Researchers in this nascent field argue that category labels assigned to products by experts are more reliable and consistent than those assigned by consumers themselves [41].

3.3.2. SAMPLE AND VARIABLES

We focus our analysis on products released in the US during the years 1995–2014. Although the AllMusic database includes hundreds of thousands of records, we restrict our preliminary sample to 135,536 unique releases that could be cross-referenced with other large databases, including Discogs and MusicBrainz (cf. Chapter 2). This allows us to filter out inaccurate or duplicate entries as well as reissues, remastered versions, and limited, international,

or deluxe editions. After dropping these observations, we are left with 80,917 original products. We further exclude 11,105 that are either anonymous or credited to more than one artist (or band), leaving us with 69,812 records. Most of these records were never rated by AllMusic users: only 4,708 or 6.7-percent have at least one rating. Because consumers' evaluations are unavailable for non-rated products, we drop these observations from our sample. It is legitimate to wonder whether this leads to selection bias: we perform t-tests to verify that the mean number of category labels in our sample does not significantly change after non-rated products are dropped. The results of these tests show that the null hypothesis of equal variance cannot be rejected ($p = 0.104$ in the case of type-based category labels; $p = 0.541$ in the case of goal-based ones). We thus conclude that selection bias is not of substantive concern. As a last step, we drop 1,320 observations for which the category labels are missing.

Our final sample thus includes 3,388 products. These are authored by 1,665 distinct artists (or bands). Consistently with the industry's reputation as a highly concentrated business [100], about half of the products in our sample were released by one of the major record companies, including Universal, Sony, Warner, PolyGram (until 1999), Bertelsmann (until 2004), and EMI (until 2012), or by any of their subsidiaries and imprints. All other products were released by independent firms or self-released by the artists.

Dependent variable.   Our outcome of interest is the average evaluation of products by AllMusic users (MeanRating). Registration on the website is free and the ratings are expressed on a 10-point scale, with the average being automatically approximated to the nearest integer. The products in our sample have 577,922 ratings in total, with 83.5-percent having at least 10 ratings, 53.3-percent having at least 50, 36-percent having at least 100, and 9.3-percent having 500 or more. With a mean of 8.14, the distribution of ratings shows a tendency towards higher scores, which seems to be common in online evaluations of cultural products [cf. 101]; however, 88.9-percent of observations are within one std. dev. of the mean and no more than 5.6-percent have extreme values. Figure 3.2 plots the distribution of mean user rating against the natural logarithm of the number of ratings, with the local regression (LOESS) curve and its 95-percent confidence interval. The scale of MeanRating is strictly discrete: the jitter along the horizontal axis in this figure is only added to enhance visualization.

Independent variables.   Every product in the AllMusic database is associated with a variable number of genre, style, and mood category labels.
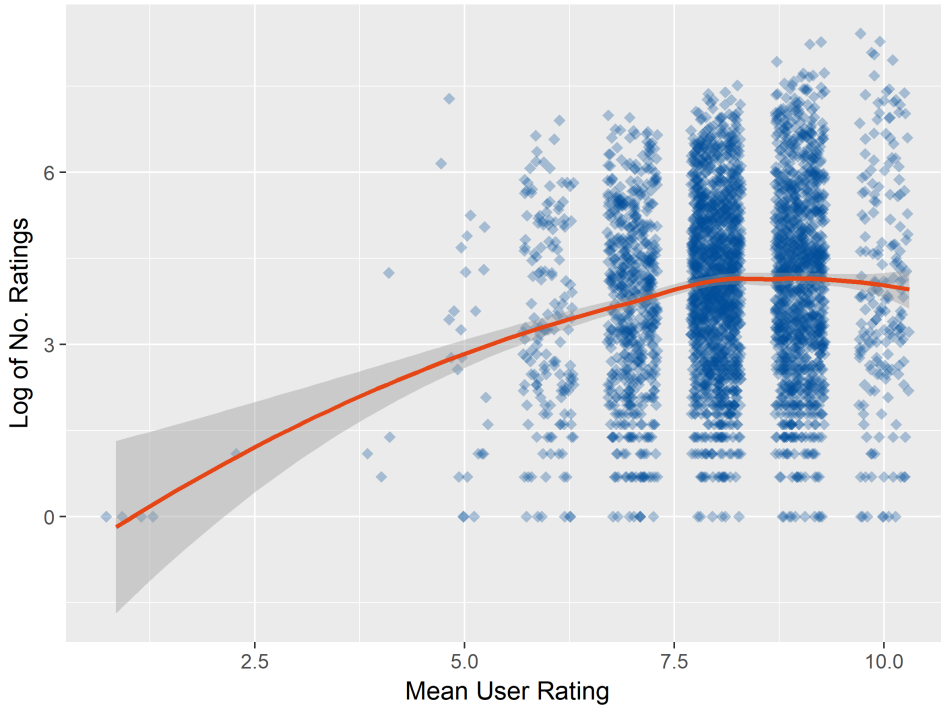
Figure 3.2: Distribution of the mean rating by number of ratings

Genres and styles represent two levels of a nested classification system (cf. Chapter 2): in this study, we use styles rather than genres as the type-based classification system because these capture family resemblance at a much finer level of detail [37]; however, we control for systematic differences in the evaluation of products that belong to different genres through binary variables. The count of products in each genre is visualized in Figure 3.3. A total of 21 genres, 509 styles, and 278 moods are represented in our sample. As a word of caution, we do not assume AllMusic users (or editors) to know all of these categories: our arguments only require each user to be familiar with some of those associated with the products they rate. Moreover, we do not assume the users to agree with the categorization chosen by the editors; only that they are able to interpret some of these category judgments. Both assumptions seem reasonable in our setting because the users ought to register on the website to rate the products, and this basic act of engagement suggests a minimal level of familiarity with the way AllMusic works. Table 3.1 reports the top 20 style and mood labels by frequency in our sample.

3
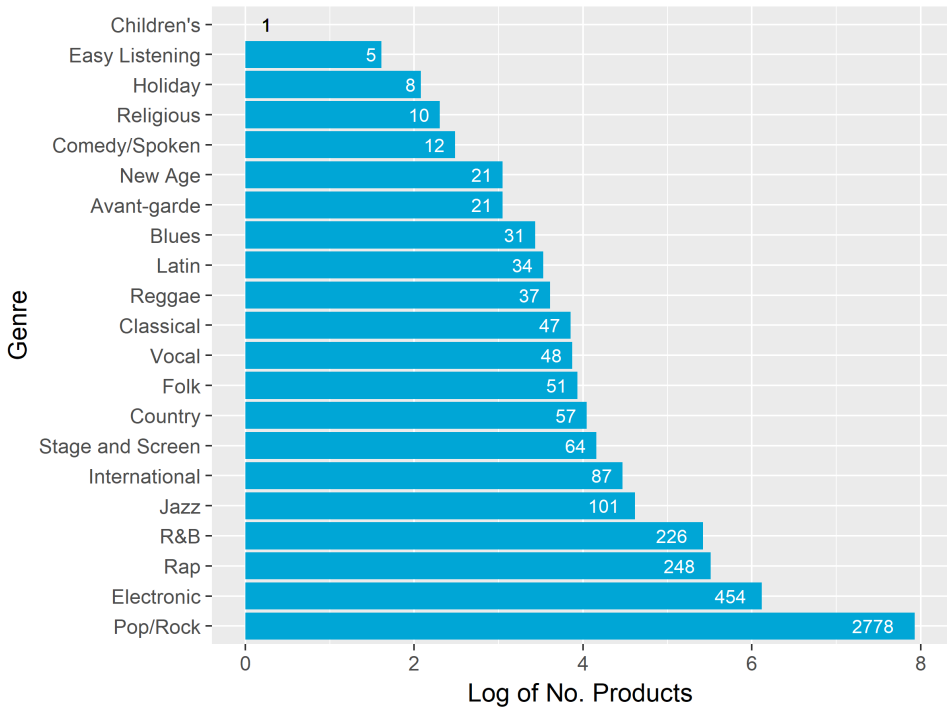
Figure 3.3: Frequencies of products in each genre

Some pairs between these labels, like the styles ALTERNATIVE/INDIE ROCK and INDIE ROCK, or the moods ROUSING and ENERGETIC, are likely to have similar meanings. It is necessary to account for this overlap because a product with multiple labels is not much of a spanner if the categories map to the same positions in the feature space [19]. Consistently with previous research [e.g., 9, 29], we control for category similarity by calculating the pairwise Jaccard coefficients between category labels. Table 3.2 reports some of the category pairs in each system, along with their similarity scores. In the case of styles, the coefficients range between zero and one, whereas in the case of moods they only range between zero and 0.4, which implies lesser overlap on average. This makes sense given that there are nearly twice as many styles than moods in our sample. We use the coefficients to compute two control variables (StyleSimilarity and MoodSimilarity). If a product spans exactly two categories, these variables are equal to the coefficient of the two labels; if it spans more than two categories, they are equal to the mean coefficient of all label pairs; if it does not span categories, the variables are set to one.

Table 3.1: Top style and mood category labels by frequency

Styles		Moods	
Category label	Freq.	Category label	Freq.
Alternative/Indie Rock	1939	Rousing	1176
Alternative Pop/Rock	925	Energetic	1135
Indie Rock	579	Reflective	1101
Heavy Metal	422	Stylish	941
Adult Alternative Pop/Rock	303	Earnest	915
Contemporary Pop/Rock	297	Playful	899
Club/Dance	279	Confident	893
Punk Revival	274	Aggressive	889
Alternative Metal	218	Passionate	842
Hard Rock	217	Dramatic	835
Punk-pop	177	Intense	813
Post-grunge	170	Brash	756
Dance-pop	169	Bittersweet	716
Indie Pop	160	Theatrical	709
Punk/New Wave	144	Freewheeling	682
Album Rock	139	Cathartic	675
Electronica	139	Intimate	666
Pop	133	Brooding	653
Indie Electronic	124	Literate	652
Contemporary R&B	118	Exuberant	649

In addition to similarity, it is necessary to account for the fact that some categories, like POP or INTENSE, have relatively fuzzy meanings, whereas others, like THIRD WAVE SKA REVIVAL or CELEBRATORY, are relatively specific. Categories that have more more clearly defined meanings stand out more sharply from other categories in the same classification system—in ecological terms, they have greater contrast [11]. As explained above, contrast can affect the information consumers derive from products' category memberships because higher-contrast categories encompass fewer positions in the feature space. We compute two separate variables (StyleContrast and MoodContrast) to control for the average contrast of the categories spanned by the products in each system. Each variable is calculated on the basis of products' grades of membership, as is common in the ecological literature [28, 77]: to this purpose, we first compute the mean grade of membership

Table 3.2: Style and mood pairs by Jaccard similarity

Style A	Style B	Coef.
Early American Blues	Work Songs	1.00
Indian Classical	Raga	0.92
Bulgarian	Bulgarian Folk	0.80
Country Blues	Acoustic Blues	0.71
Southern Gospel	Traditional Gospel	0.60
Sound Art	Sound Sculpture	0.50
Contemporary Reggae	Reggae-pop	0.40
Uptown Soul	Chicago Soul	0.30
East Coast Blues	Vaudeville Blues	0.20
Mexican Traditions	Madagascan	0.10

Mood A	Mood B	Coef.
Demonic	Macabre	0.40
Reflective	Intimate	0.36
Confrontational	Aggressive	0.32
Laid Back/Mellow	Soothing	0.28
Celebratory	Exuberant	0.24
Brooding	Bittersweet	0.20
Whimsical	Playful	0.16
Harsh	Paranoid	0.12
Theatrical	Somber	0.08
Druggy	Irreverent	0.04

of each product as the reciprocal of the number of categories to which the product belongs. Then, we calculate the contrast of each category by taking the mean grade of membership of category members (cf. Chapter 3). Finally, for each product, we calculate the mean contrast of the categories spanned in each classification system. The resulting measure increases if the categories to which the product belongs tend to be more exclusive.

We use the category labels assigned to each product to create two main predictors (NoStyles and NoMoods). These are count variables representing the total number of categories spanned in each classification system. Though their distribution is skewed, we avoid transforming these variables (e.g., by computing the inverse, the square root, or the natural logarithm) because our hypotheses directly concern the number of category labels and keeping this value untransformed allows for a more adequate test.
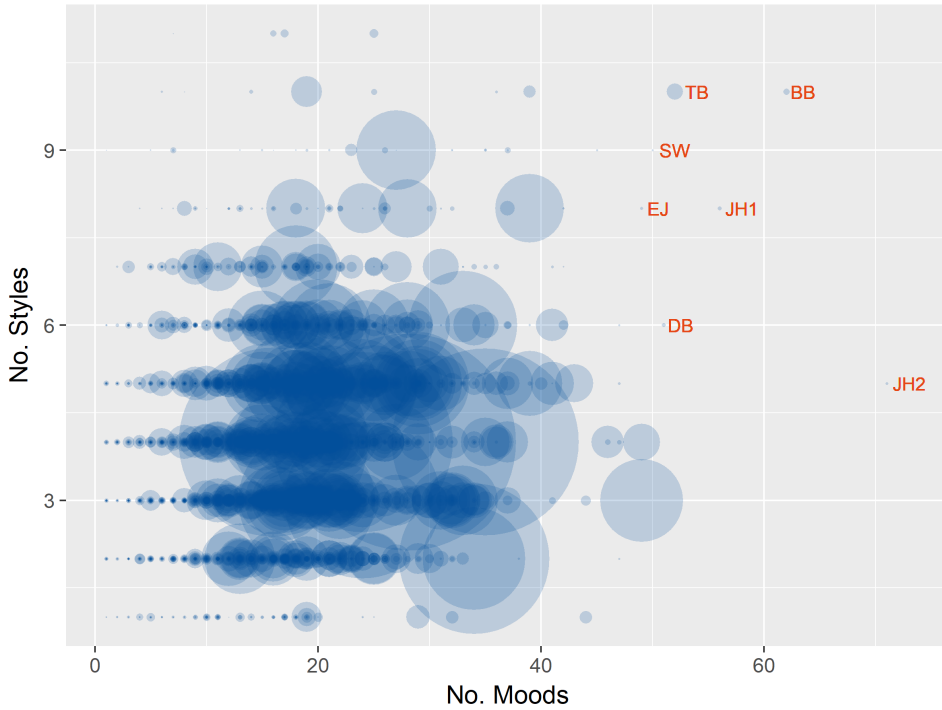
Figure 3.4: Distribution of the number of moods by number of styles

Figure 3.4 presents the two variables' distribution. In this figure, the size of each circle increases with the number of users who rated the products with a particular combination of values, and a darker hue indicates a greater concentration of products at particular values.

Seven observations, marked by initials in the figure, stand out for their very high number of moods. Closer inspection of these data points reveals the reason for their anomaly: these are long, celebratory compilations by very well-established and often very eclectic artists. Namely, EJ stands for Elton John's *The Greatest Hits, 1970–2002* (34 tracks, 157 minutes); SW stands for Stevie Wonder's *At the Close of a Century* (70 tracks, 312 minutes); DB represents David Bowie's *Bowie at the Beeb: The Best of the BBC Radio Sessions, 1968–1972* (53 tracks, 218 minutes); TB represents *One* by The Beatles (27 tracks, 79 minutes); BB stands for *Sounds of Summer: The Very Best of the Beach Boys* (30 tracks, 76 minutes); JH1 represents The Jimi Hendrix Experience's self-titled compilation (56 tracks, 259 minutes); and JH2 represents their *Winterland* box set (36 tracks, 278 minutes). Although these observations are far from the median of NoMoods, they do not have a high

leverage on the regression lines and their exclusion does not substantively affect our estimates.

We compute a binary variable (Compilation) to control for whether the focal product is a box set, an anthology, a collection of greatest hits. Moreover, we control for differences in the number of tracks by regressing on the natural logarithm of the size of the tracklist (NoTracks). We also compute a binary control variable to capture possible identity or authenticity effects in the evaluation of products released by independent vs. major record companies (IndieRelease). We control for the products' release year through a set of 19 dummy variables.[5] Consistently with previous research [e.g., 17], we also control for differences in the size of the audience by regressing on the natural logarithm of the number of ratings (NoRatings). Finally, we control for differences in the products' editorial evaluation (EditorRating): on the one hand, this is necessary because consumers' evaluations can be influenced by the editors' judgment; on the other, it is useful to capture quality-related heterogeneity among products and decouple the perceptual from the "actual" effects of category spanning [13].

3.3.3. Estimation Procedure

It is relevant to note that a considerable share of the variance we wish to explain in MeanRating can be due to unobserved artist-level characteristics, such as status [26] or tenure in the industry [20]. We must account for this heterogeneity because higher-status or longer-established artists may afford to span categories more than the average, and conversely, lower-status or younger artists may be subject to greater constraints. Hierarchical or multilevel models [102] are explicitly designed to handle this source of bias: using a restricted maximum likelihood (REML) estimator, these allow the isolation of variance components at different levels of analysis.

In our study, we use a two-level, mixed model specification with fixed effects at the level of products and random effects at the level of artists. This model accounts for the correlations among regression residuals for products "nested" within the same artist, thereby ruling out the effects of unmeasured artist-level determinants. To capture the hypothesized curvilinear effect of NoStyles on MeanRating, we use a quadratic polynomial for the independent variable [103]. To capture the interaction between NoStyles and NoMoods, both terms of the polynomial are multiplied with the moderating variable [79, 104]. The full model, which includes the quadratic and interaction terms, takes the following form:

---

[5]Including ReleaseYear as a continuous variable does not change our model results.

$$\begin{aligned}
\text{MeanRating} = {} & \alpha + \beta_1\text{ReleaseYear}_2 \cdots \beta_{19}\text{ReleaseYear}_{20} + \beta_{20}\text{Genre}_1 \cdots \beta_{40}\text{Genre}_{21} \\
& + \beta_{41}\text{Compilation} + \beta_{42}\text{NoRatings} + \beta_{43}\text{NoTracks} + \beta_{44}\text{EditorRating} \\
& + \beta_{45}\text{IndieRelease} + \beta_{46}\text{StyleSimilarity} + \beta_{47}\text{StyleContrast} \\
& + \beta_{48}\text{MoodSimilarity} + \beta_{49}\text{MoodContrast} + \beta_{50}\text{NoStyles} \\
& + \beta_{51}\text{NoStyles}^2 + \beta_{52}\text{NoMoods} + \beta_{53}\text{NoStyles} \times \text{NoMoods} \\
& + \beta_{54}\text{NoStyles}^2 \times \text{NoMoods} + v + \varepsilon
\end{aligned}$$

$$(3.2)$$

where v is an artist-level error drawn from a distribution $N(0, \sigma_v)$ and ε is a product-level error drawn from a distribution $N(0, \sigma_\varepsilon)$. The intra-class correlation (ICC) coefficient suggests that this model is appropriate: about 32.5-percent of the variance in MeanRating is due to artist-level differences and is thus captured by v. Consistently with this, likelihood-ratio tests show that our mixed models fit the data better than generalized linear models without the artist-level component ($p < 0.001$).

3.4. RESULTS

Table 3.3 reports the descriptive statistics for the main variables involved in our analysis. For brevity, we omit the 21 binary variables that capture genre memberships. Although our dataset includes a much greater number of styles than moods, the mean number of category labels for products in our sample is much larger in the goal-based system ($p < 0.001$). The contrast and the similarity of styles spanned by each products also tends to be higher than that of spanned moods ($p < 0.001$).

Table 3.4 reports the pairwise correlations between these variables. Our two predictors of theoretical interest are positively but weakly correlated with one another ($r = 0.15$, $p < 0.001$), which suggests that, although atypical products tend to be suitable for a greater number of moods, the observations in our sample can span styles and moods independently. To assess the risk of multicollinearity, we perform conditioning diagnostics on the model matrix. The condition number of the matrix is 67.03, which is well above the recommended threshold of 30 [105], but standardizing the variables helps addressing this problem and reduces the number to a much more acceptable 3.45. Thus, we are reassured that the interdependencies between our regressors do not affect our estimates. In the tables below, we report the estimated coefficients for standardized variables: the intrepetation of these parameters is relatively straightforward because the dependent variable has a std. dev. of approximately one; therefore, any

Table 3.3: Descriptive statistics

	Mean	Std. dev.	Min.	Max.
MeanRating	8.14	1.01	1	10
ReleaseYear	2004.24	5.43	1995	2014
Compilation	0.10	0.30	0	1
NoRatings	4.00	1.61	0	8.42
NoTracks	2.60	0.38	0	5.59
EditorRating	6.32	1.36	2	9
IndieRelease	0.47	0.50	0	1
StyleSimilarity	0.15	0.16	0	1
StyleContrast	0.28	0.04	0.18	1
MoodSimilarity	0.12	0.09	0	1
MoodContrast	0.08	0.03	0.06	0.35
NoStyles	3.85	1.56	1	11
NoMoods	15.14	7.45	1	71

coefficient can be interpreted as the effect of a one-std. dev. increase in the predictor on the original value of the dependent.

The results our mixed models are reported in Tables 3.5 and 3.6. We begin our analysis in Model 1 by regressing MeanRating on all the non-category-related variables as well as the 19 dummies for ReleaseYear. The model results suggest that compilations tend to be evaluated better by users than ordinary albums, as the estimated rating is approximately 0.6-point higher ($p < 0.001$). This effect is unsurprising if we consider that compilations usually include an artist's most successful tracks. Products rated by a greater number of users also tend to be rated better on average ($\hat{\beta} = 0.148$, $p < 0.001$), as do products released by independent record companies ($\hat{\beta} = 0.076$, $p = 0.020$). A higher evaluation by AllMusic editors is also associated with a higher mean rating ($\hat{\beta} = 0.418$, $p < 0.001$).

In subsequent models, we add the category-related control variables to our list of regressors. This includes the 21 binary variables for genre categories (Model 2) as well as the four contrast and similarity variables (Model 3). The results of systematic likelihood-ratio tests suggest that the contrast and similarity variables do not significantly affect MeanRating (Model 3 vs. Model 2: $p = 0.507$); the genre controls, however, decisively contribute to model fit (Model 2 vs. Model 1: $p < 0.001$). All the estimates reported above remain consistent in size and direction throughout the fol-

Table 3.4: Pairwise correlations matrix

		1	2	3	4
1	MeanRating				
2	ReleaseYear	−0.15•••			
3	Compilation	0.23•••	−0.10•••		
4	NoRatings	0.14•••	0.08•••	−0.23•••	
5	NoTracks	0.13•••	−0.05••	0.41•••	−0.08•••
6	EditorRating	0.51•••	−0.04••	0.16•••	0.20•••
7	IndieRelease	0.03	0.16•••	−0.11•••	−0.19•••
8	StyleSimilarity	−0.01	−0.05••	−0.05••	−0.11•••
9	StyleContrast	−0.06••	0.04•	−0.17•••	−0.08•••
10	MoodSimilarity	0.00	−0.15•••	−0.03	−0.10•••
11	MoodContrast	0.00	−0.15•••	−0.05••	−0.16•••
12	NoStyles	0.07•••	−0.22•••	0.22•••	0.06•••
13	NoMoods	0.01	0.22•••	0.21•••	0.36•••

		5	6	7	8
6	EditorRating	0.12•••			
7	IndieRelease	−0.12•••	0.06•••		
8	StyleSimilarity	0.04•	−0.04•	−0.05••	
9	StyleContrast	0.03	−0.01	−0.14•••	0.27•••
10	MoodSimilarity	−0.01	−0.04•	−0.03	0.04•
11	MoodContrast	−0.02	−0.05••	−0.05••	0.07•••
12	NoStyles	0.11•••	0.09•••	−0.02	−0.40•••
13	NoMoods	0.13•••	0.15•••	−0.04•	−0.09•••

		9	10	11	12
10	MoodSimilarity	0.01			
11	MoodContrast	0.05••	0.33•••		
12	NoStyles	−0.42•••	−0.02	−0.04•	
13	NoMoods	−0.12•••	−0.28•••	−0.28•••	0.15•••

Note: • $p < 0.05$; •• $p < 0.01$; ••• $p < 0.001$

lowing models, and the effects of NoRatings and Compilation further increase in their level of statistical significance.

In Model 4, we add our predictors of theoretical interest, NoStyles and NoMoods. At this stage, we include only the first-order variable of NoStyles. The goodness of fit of our model significantly increases (Model 4 vs. Model 3:

Table 3.5: Mixed model results: Controls

	Model 1		Model 2		Model 3	
	Dependent variable:					
			MeanRating			
Intercept	0.17 •	(0.07)	0.31 •••	(0.08)	0.31 •••	(0.08)
Compilation	0.59 •••	(0.06)	0.62 •••	(0.06)	0.63 •••	(0.06)
NoRatings	0.15 •••	(0.02)	0.18 •••	(0.02)	0.18 •••	(0.02)
NoTracks	0.02	(0.02)	0.02	(0.02)	0.02	(0.02)
EditorRating	0.42 •••	(0.01)	0.41 •••	(0.01)	0.41 •••	(0.01)
IndieRelease	0.08 •	(0.03)	0.09 ••	(0.03)	0.09 ••	(0.03)
StyleSimilarity					−0.01	(0.01)
StyleContrast					−0.00	(0.02)
MoodSimilarity					−0.00	(0.01)
MoodContrast					0.03	(0.01)
Intercept σ_v	0.51		0.48		0.48	
Residual σ_v	0.67		0.68		0.68	
Genre variables	Not included		Included		Included	
Release year dummies	Included		Included		Included	
No. products	3388		3388		3388	
No. artists	1665		1665		1665	
Log likelihood	−4096.39		−4075.73		−4086.82	

Note: • $p < 0.05$; •• $p < 0.01$; ••• $p < 0.001$; std. errors in parentheses

$p < 0.001$), and the estimates suggest a negative and highly significant effect for NoMoods ($\hat{\beta} = -0.116$, $p < 0.001$) as well as a small but highly significant negative effect for NoStyles ($\hat{\beta} = -0.064$, $p < 0.001$). We suspect that the small size of this effect is because a straight regression line does not adequately capture the relationship between a product's number of styles and its average evaluation by AllMusic users.

We continue our analysis in Model 5 by adding the quadratic transformation of NoStyles to our list of regressors. The estimates from this model specification confirm the suspicion above, as the inclusion of NoStyles^2 results in highly significant coefficients for both terms of the polynomial ($\hat{\beta} = -0.111$, $p < 0.001$, and $\hat{\beta} = 0.040$, $p < 0.001$, respectively) and a better overall fit for our model (Model 5 vs. Model 4: $p < 0.001$). We formally test the significance of the U-shaped relationship between NoStyles and

Table 3.6: Mixed model results: Main effects and interaction

	Dependent variable:					
	MeanRating					
	Model 4		Model 5		Model 6	
Intercept	0.23 ••	(0.08)	0.20 •	(0.08)	0.21 •	(0.08)
Compilation	0.75 •••	(0.06)	0.75 •••	(0.06)	0.74 •••	(0.06)
NoRatings	0.23 •••	(0.02)	0.24 •••	(0.02)	0.24 •••	(0.02)
NoTracks	0.02	(0.02)	0.02	(0.02)	0.02	(0.02)
EditorRating	0.41 •••	(0.01)	0.41 •••	(0.01)	0.41 •••	(0.01)
IndieRelease	0.10 •••	(0.03)	0.10 •••	(0.03)	0.10 •••	(0.03)
StyleSimilarity	0.02	(0.02)	−0.05 ••	(0.02)	−0.05 •••	(0.02)
StyleContrast	−0.03	(0.02)	−0.04	(0.02)	−0.05 ••	(0.02)
MoodSimilarity	−0.02	(0.01)	−0.02	(0.01)	−0.02 ••	(0.01)
MoodContrast	0.01	(0.01)	0.01	(0.01)	0.01	(0.01)
NoStyles	−0.06 •••	(0.02)	−0.11 •••	(0.02)	−0.11 •••	(0.22)
NoStyles^2			0.04 •••	(0.01)	0.03 ••	(0.01)
NoMoods	−0.12 •••	(0.02)	−0.12 •••	(0.02)	−0.14 •••	(0.02)
NoStyles $\times$ NoMoods					0.06 •••	(0.02)
NoStyles^2 $\times$ NoMoods					−0.01	(0.01)
Intercept σ_v	0.46		0.46		0.46	
Residual σ_v	0.68		0.68		0.68	
Genre variables	Included		Included		Included	
Release year dummies	Included		Included		Included	
No. products	3388		3388		3388	
No. artists	1665		1665		1665	
Log likelihood	−4062.90		−4050.44		−4039.83	

Note: • $p < 0.05$; •• $p < 0.01$; ••• $p < 0.001$; std. errors in parentheses

MeanRating using the three-step procedure recommended by Lind and Mehlum [103]. According to this method, three conditions must hold for the curvilinear effect to be safely interpretable: first, the coefficients of NoStyles and NoStyles^2 must be significant and in the expected direction; second, the slope of the curve must be significantly different from zero at both ends of the observed range of NoStyles; third, and related to the above,

the turning point of the curve must be located within the observed range of NoStyles. The second and third conditions are important because, if they do not hold, there is not enough evidence to accept the hypothesis of a U-shaped relationship. The true relationship may actually be only half of a U-shape, which could be more parsimoniously fitted through a logarithmic function of the independent variable.

In our case, the relationship fulfills the three aforementioned criteria. The coefficients are in the expected direction: negative for the first-order term of the polynomial and positive for the second-order term. Two t-tests performed according to Lind and Mehlum's procedure [103] suggest that the slope of the curve is significantly smaller than zero when the variable is at its lowest ($\widehat{m}_L = -0.257$, $p < 0.001$) but significantly greater than zero when the variable is at its highest ($\widehat{m}_H = 0.255$, $p = 0.001$). The point where the slope switches from negative to positive, i.e., the turning point of the U-shape, is located at 1.38 std. dev. above the mean of NoStyles, which corresponds to approximately six categories. Hence, the effect reversal occurs well within the variable's range. This implies that MeanRating does not monotonically decrease when the value of NoStyles increases: it decreases at first, but after a large enough number of styles the average evaluation begins to increase again. To verify that this non-monotonic effect is unique to type-based categories, as implied by our theory, we estimate additional models where a similar quadratic relationship is specified for NoMoods. In this case, we find no evidence of a U-shape ($p = 0.159$). We report on these models in greater detail at the end of this section.

We continue our analysis in Model 6 by estimating the interaction effect. Consistently with Aiken and West [104], we account for this effect by including the multiplicative terms NoStyles $\times$ NoMoods and NoStyles$^2 \times$ NoMoods to the list of regressors. The model fits the data significantly better than the one without interactions (Model 5 vs. Model 4: $p < 0.001$). The results of Lind and Mehlum's test of the U-shaped relationship [103] continue to hold ($p = 0.023$). Model estimates indicate a highly significant effect for the interaction with the first-order variable ($\hat{\beta} = 0.062$, $p < 0.001$), but not for the one with the second-order variable ($\hat{\beta} = -0.007$, $p = 0.286$), which suggests that an increase in the value of NoMoods does not significantly affect the curvature of the U-shaped relationship between NoStyles and MeanRating. In a linear model, in fact, the curvature of such a relationship is only affected by the moderating variable if the coefficient of its interaction with the quadratic term is significantly different from zero [79]. Yet it is not necessary for the curvature to change in order for a significant moderation to occur: even if the slope stays the same, a shift in the curve's turning
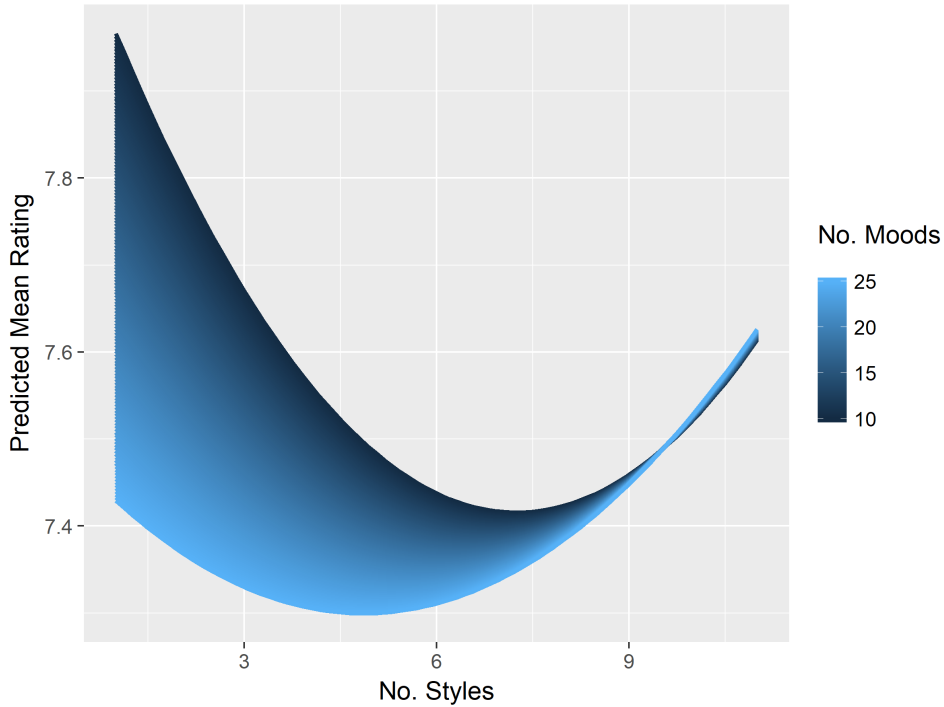
Figure 3.5: Effect of spanning in both classification systems

point remains possible. We formally test this by estimating the derivative of the U-shape's turning point with respect to the moderating variable. This partial derivative is computed as follows [79, p. 1187]:

$$\frac{\delta X^*}{\delta Z} = \frac{\beta_{50}\beta_{53} - \beta_{51}\beta_{52}}{2\left(\beta_{51} + \beta_{53}Z\right)^2},\tag{3.3}$$

where X^* represents the value of NoStyles where the U-shape turns upward, β_{50-53} are the coefficients of NoStyles, NoMoods, NoStyles $\times$ NoMoods, and NoStyles^2 $\times$ NoMoods, respectively (cf. Equation 3.2), and Z is an arbitrary but meaningful value within the observed range of NoMoods.

We perform this test repeatedly by setting Z at different values located within one std. dev. of the mean of NoMoods. Throughout our tests, we obtain negative and significant estimates ($p < 0.05$), which constitutes formal evidence that the turning point shifts to the left when the value of NoMoods increases. The interaction between these two variables is graphically presented in Figure 3.5. In this plot, the gradient represents the changing effect of NoStyles as the value of NoMoods increases from 10 to 25,

which covers more than 70-percent of observations in our sample. Notice that the change in the curvature is not statistically significant: however, the leftward shift of the turning point is.

Robustness tests.   Post-estimation diagnostics (available upon request) suggest that our estimates are safely interpretable. The exclusion of outliers does not substantively affect the size or the significance of our results.

As mentioned before, we perform additional tests to rule out the possibility that a U-shaped relationship exists between NoMoods and MeanRating, which would be incompatible with our theoretical argument that type- and goal-based categories have different internal structures. For the purpose of this validation, we estimate two additional models: one analogous to Model 4, where we also include NoMoods^2, and one analogous to Model 5, where we also include NoMoods^2 and its interactions with NoStyles and NoStyles^2. In the first model, i.e., the one without interactions, the U-shaped effect of NoStyles is significant ($\widehat{m}_L = -0.243$, $p < 0.001$, and $\widehat{m}_H = 0.226$, $p = 0.004$) and the putative U-shaped effect of NoMoods seems to be significant as well ($\widehat{m}_L = -0.276$, $p < 0.001$, and $\widehat{m}_H = 0.283$, $p = 0.004$). In the second model, i.e., the one with interactions, the U-shaped effect of NoStyles is still strongly significant ($\widehat{m}_L = -0.257$, $p < 0.001$, and $\widehat{m}_H = 0.215$, $p = 0.009$) but the U-shaped effect of NoMoods is only marginally so ($\widehat{m}_L = -0.257$, $p < 0.001$, and $\widehat{m}_H = 0.189$, $p = 0.069$).

Not only is this effect marginally significant, but it appears to be entirely driven by outliers: re-estimating the model after dropping as little as three observations whose value of NoMoods is more than five std. dev. away from the variable's mean (denoted JH1, JH2, and BB in Figure 3.4) causes the curvilinear effect of NoMoods to be no longer significant at all ($\widehat{m}_L = -0.242$, $p < 0.001$, and $\widehat{m}_H = 0.147$, $p = 0.159$). We thus conclude that there is no evidence of a U-shaped relationship between NoMoods and MeanRating. This is fully consistent with our expectations.

3.5. DISCUSSION

In this chapter, we proposed a conceptual model whereby consumers rely on different classification systems to localize products in a cognitive space. We explained the distinction between type- and goal-based categories, and illustrated how the fundamental difference in their internal structure affects the information consumers are allowed to derive from products' category labels. Building on these considerations, we predicted that the effect of spanning type-based categories on product evaluations follows a U-shape

(Hypothesis 3.1), whereas the effect of spanning goal-based categories is linearly negative (Hypothesis 3.2). Further, we predicted that the U-shaped effect of spanning type-based categories is moderated by the effect of spanning goal-based categories in such a way that the turning point of the U-shape moves leftward if a product spans in both systems (Hypothesis 3.3). Our empirical analysis of product ratings on an online music platform yielded sufficient evidence to accept all hypotheses.

With regard to Hypothesis 3.1, we found that having more type-based category labels has a negative effect on consumers' evaluations, but consistently with our prediction, this effect reverses if the product's number of type-based labels is sufficiently large. We estimated this reversal to occur at approximately six labels on average. If we assumed that (a) AllMusic users were familiar with every prototype associated with the products they rate, and (b) that they did not devalue products that require cognitive effort, this particular estimate would suggest that consumers are unable to distinguish more than 12 features or dimensions in the feature space of popular music. This is because the average evaluation of a product with 13 ($D + 1$) type-based category labels is predicted to be equal to that of a product with only one type-based label, all else being equal. These assumptions are hardly realistic, however: the users may not know all the prototypes associated with a product according to AllMusic editors, and even if they did, they would still penalize records whose labels are difficult to understand [cf. 106]. In light of this, 12 is a rather generous estimate of the space's dimensionality: the actual number of features that consumers can distinguish is probably lower. This is consistent with psychological research, according to which people tend to reduce even relatively complex domains to spaces of less than a dozen dimensions [78].

It is worth noting that the number of dimensions of the feature space has no direct relationship with the number of category labels the audience is familiar with. A space of as little as one dimension can be partitioned into an arbitrarily large number of categories if audience members are willing to make very fine-grained distinctions—in other words, the dimensionality of the feature space and that of category labels is not directly related [57]. The fact that AllMusic includes more than 500 styles does not necessarily mean that the feature space of popular music is particularly complex [cf. 107, 108], nor that consumers are able to make very fine-grained distinctions.

Our findings are compatible with previous research on the consequences of category spanning inasmuch as products partly conforming to different prototypes tend to receive lower evaluations by consumers, at least up to a threshold. This agrees with the categorical-imperative perspective in

organization theory [50]. Our findings partly diverge from this perspective, however, as we find evidence that very atypical products receive better evaluations than moderately atypical ones. The explanation we proposed for this phenomenon hinges on the idea that type-based categories map to convex regions of the feature space [76]: because people are already familiar with these categories' geometric centers (the prototypes), they can make relatively accurate inferences about products either when these are highly typical or when they are so atypical that combining the concepts [51–53] yields a unique solution. As a result of this geometric process, consumers' evaluations of products can be relatively high at both ends of the typicality distribution. Conforming to a specific prototype may still be the optimal strategy, however, because consumers are averse to mental effort and they are likely to penalize products if a great deal of cognitive effort is needed to pinpoint their location in the feature space.

With regard to Hypothesis 3.2, we found that the effect of spanning goal-based categories is linearly negative. This implies that products are better off focusing on fewer goals, all else being equal. Our tests returned no evidence of a U-shaped relationship in this case, confirming that the information consumers can derive from type- and goal-based categories is radically different. This is consistent with Barsalou's [66] research in cognitive psychology, according to which goal-based categories do not coalesce around some central instantiation of a concept [cf. 67] or reference point [14]. As a result, the region of feature space to which goal-based categories map is not necessarily convex. Because of non-convexity, audience members find it harder to predict which feature profiles are compatible with certain labels and cannot normally determine the coordinates of products that span goal-based categories by way of distance-based reasoning [cf. 29]. As there is ultimately no effect on uncertainty, the consequences of spanning categories in a goal-based system entirely depend on producer-side constraints and the relationship with consumers' evaluation is monotonically negative. This agrees both with the organizational literature, which suggests that goal-consistency is desirable for evaluators [24, 31], and with marketing research, according to which products perceived to be suitable for multiple goals are perceived to be less suitable for each [68].

Finally, with regard to Hypothesis 3.3, we found that the effect of spanning type-based categories shifts from negative to positive after fewer category memberships if the product also spans categories in a goal-based system. This supports our conjecture that evaluators tend to merge information from different classification systems in order to make sense of products. It is easy to imagine what kind of considerations or thought

processes underpin this interaction effect: for example, consumers may think that addressing several needs justifies deviance from prototypes. If a phone has a display wider than six inches, for instance, thereby deviating from the prototype of its category, then it better be useful for more than just calling and sending text messages. Likewise, if consumers know that a phone is good for more than calling and sending text messages, e.g., because it is advertised as suitable for watching videos or playing games, then they are likely to infer that its screen is wider than six inches and they will be less inclined to devalue the product.

The arguments presented in this chapter complement the various explanations for the positive consequences of atypicality already offered by organization scholars, yet they allow for a more parsimonious explanation because they do not rely on higher-level theoretical constructs like producer attributes [26, 27] or category properties [28]. Moreover, they do not require audience members to have particular goals in mind when evaluating products [23, 24]: in this sense, our arguments are generalizable to an audience with heterogeneous needs and desires. Nevertheless, evaluators are required to possess some minimal level of experience with the domain at hand (i.e., they should know the positions of at least some prototypes) for our geometric arguments to hold, hence our findings may not extend to settings where consumers are insufficiently familiar with the objects on offer. In addition, the question remains of whether our arguments apply to the valuation of organizations as well as their products. While the consumers face a similar uncertainty problem, these two cases are different because products do not move around in the feature space: firms do, albeit with some difficulties [109–111], and this may make locating their (current) position in the feature space a rather different exercise.

The assumptions we made about consumer behavior limit the generalizability of our results. Though there is no shortage of contexts where consumers tend to anchor their decision on third-party information, in many of these contexts the category labels may be presented in a way that makes multiple-category induction less likely. The objective of our research, however, was not to argue that the dominant wisdom about category spanning in organization theory is faulty, but rather to offer proof of the different information that type- and goal-based category labels convey to the audience. An implication of this structural difference is that spanning type-based categories has a U-shaped effect on evaluations under the conditions we assumed (cf. Section 3.2.1); another is that this effect is moderated by the number of goal-based categories. These conditions do not necessarily hold in other contexts, so that a linear relationship between atypicality and eval-

uations can still be perfectly adequate. Nonetheless, goal-based categories may still be relevant in these contexts and a prototype-centered perspective on categorization is still likely to engender an incomplete account of the role of categories in the ordering of markets.

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II

LOGICAL FORMALIZATIONS

4

Classification Systems as Concept Lattices

4.1. MOTIVATION

Though the use of logical methods is perceived to be somewhat exotic in the social sciences [1], their application to organizational research is neither recent [e.g., 2] nor confined to the periphery of the discipline [3]. Many studies deployed logical formalization to access, repair, and improve the content of organizational theories: examples include Kamps and Pólos' [4] reconstruction of Thompson's classic propositions from *Organizations in Action* [5], and works by Hannan [6, 7], Péli [8, 9], and colleagues [10–12] in the context of organizational ecology. With respect to the research on categories, modal logic has been especially appreciated for its power to express the agents' beliefs as well as their factual knowledge [13]. From a purely mathematical perspective, much of the interdisciplinary success of modal logic is due to to the well-known and highly celebrated theory of correspondence [14, 15]. Indeed, the foundational results of this theory underpin the diffusion of modalities in fields as diverse as computer science [16], artificial intelligence [17], game theory [18, 19], and—most important to the aims of this chapter—economic sociology [20–22].

Correspondence theory originates from the observation that relational structures known as *Kripke frames*, which consist of tuples of sets and relations, serve as models for both first-order sentences and modal formulas. A modal and a first-order formula are said to *correspond* if they are valid in exactly the same class of Kripke frames. The correspondence theory developed by Sahlqvist [14] offers a method for computing the first-order correspondent of certain special formulas, i.e., Sahlqvist formulas, and makes it possible to understand the "meaning" of a modal axiom in terms of the condition expressed by its first-order correspondent. This is precisely the key that made modal logic an exceptionally intuitive tool: for instance, by allowing $\Box A \to A$ to be understood as the *reflexivity axiom*, and $\Box A \to \Box\Box A$ as the *transitivity axiom*.

In recent years, an encompassing perspective has emerged that, building on duality-theoretic insights [23], made it possible to export the state-of-the-art in Sahlqvist theory from the original context of modal logic to a wide spectrum of logics associated with algebras known as normal (distributive) lattice expansions.[1] In addition to intuitionistic and distributive lattice-based (normal modal) logics [24], this also extends to non-normal

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[1]A *normal (distributive) lattice expansion* is a bounded (distributive) lattice endowed with operations of finite arity, where each coordinate is either positive, i.e., order-preserving, or negative, i.e., order-reversing. These operations are either finitely join-preserving (resp. meet-reversing) in their positive (resp. negative) coordinates, or finitely meet-preserving (resp. join-reversing) in their positive (resp. negative) coordinates.

(regular) modal logics [25], substructural logics [26], hybrid logics [27], and modal mu-calculi [28, 29]. Many applications were stimulated by this comprehensive research program, some of which relate to the basic concerns of Sahlqvist theory, like the understanding of relationships between different methods for obtaining canonicity results [30–35]. Other applications concern the theory of finite lattices in universal algebra [36] and the theory of analytic calculi in structural proof theory [37, 38]. These results gave rise to a coherent framework termed *unified correspondence* [39].

The cornerstone of unified correspondence is the realization that the mechanisms underlying Sahlqvist's correspondence [14] are algebraic, order-theoretic, and duality-theoretic. Just like the Sahlqvist theory for modal logic builds on the duality between Kripke frames and their associated algebras, correspondence theory for lattice-based modal logic builds on the duality between perfect lattices and RS-polarities, first suggested by Birkhoff [40] and later discussed by Gehrke [41]. Recent research [26] has shown that this duality can be expanded so as to add normal modal operators on the side of the algebras and relations on the side of the polarities, thereby obtaining what we refer to as *RS-frames*. While this theory works excellently from a mathematical perpective, the resulting correspondences have proven difficult to understand intuitively. In this chapter, we propose a novel and intuitive interpretation of these mathematical results using the notion of categories or concepts as they are studied in psychology [42] and organization theory [13]. Our interpretation pivots on the dual view of categories as sets of objects and sets of features proposed by Ganter and Wille [43, 44] within the framework of Formal Concept Analysis (FCA). From this original perspective, the normal modal operators acquire a natural epistemic interpretation that captures important properties like the factivity and the positive introspection of knowledge.

The link between RS-frames and FCA is given by the fact that RS-frames arise from polarities [41], i.e., tuples (A, X, I) such that A and X are sets, and $I \subseteq A \times X$, and that polarities can be interpreted as *formal contexts* [44], which consist of objects $a \in A$, features $x \in X$, and a relation I connecting every object with the features it possesses. As noted by Birkhoff [40], any polarity induces a Galois connection between the powersets of A and X, the stable sets of which form a complete lattice. Indeed, by Birkhoff's representation theorem, any complete lattice is isomorphic to one arising from some polarity. This representation theory for general lattices provides the polarity-to-lattice direction of the duality discussed in this chapter, and it is also at the heart of FCA, because the Galois-stable sets arising from formal contexts can be interpreted as *formal concepts* or categories. One

of the most felicitous insights of FCA is that formal concepts are endowed by construction with a double interpretation: an *extensional* one, specified by the objects that are instances of the concept, and an *intensional* one, specified by the features shared by these objects.

While the logical contribution of our proposal is to attain an intuitive grasp of a class of mysterious models with excellect mathematical properties, the conceptual connection we establish can be used to better understand certain aspects of categorization in markets. In economic sociology, categories constitute collective identities for groups of individuals or organizations, such as COMEDY ACTORS [45] or NOUVELLE CUISINE RESTAURANTS [46], as well as sets of products with some distinguishing characteristics, like BIODYNAMIC WINES [47] and LIGHT CIGARETTES [48]. The labels associated with these categories [49] are essential to economic decision-making because they induce expectations and default beliefs about the features of objects [13], on which boundedly rational agents like investors, consumers, and firm managers base their choices. This view is consistent with cognitive-psychological research on categorization (cf. Chapter 3), according to which categories are cognitive sieves that preserve relevant distinctions and suppress whatever information is deemed inconsequential or redundant [50]. Taken together, categories form classification systems, which consist of multiple levels of abstraction [51]. This agrees with the FCA treatment, according to which concepts arise embedded in their concept lattice. The extensional and intensional interpretations of concepts in FCA find a very suitable ground for application because categories can be equivalently defined as groups of objects, which represent the category members, or as lists of features, which represent the membership requirements.

As the organizational research on categories matured and progressed, organization scholars became increasingly interested in the more dynamic aspects of categorization [52]. Considerable attention has been devoted to the processes whereby new market categories are born, either *ex nihilo* or through the recombination of existing features [53], and to the role played by agents' communication in these dynamic processes [54]. This is important because, even though market categories arise from factual information about products or organizations, a critical component of their nature cannot be reduced to sheer truth. Like all categories, market categories are ultimately social constructs [55–57], and reasoning about them requires a peculiar combination of factual truth, subjective perception, and social interaction. This tripartition is the centerpiece of our formal proposal, which contributes to the field of organization theory precisely by formalizing the objective, subjective, and social aspects of categorization.

In this chapter, we develop a formal theory where the agents are allowed to entertain idiosyncratic beliefs about the objects' category memberships [cf. 58, 59]. We accomplish this by associating each agent with a binary relation $R \subseteq A \times X$ on the polarity (A, X, I). Intuitively, the incidence relation I of the polarity is taken to represent factual relationships between objects and features, whereas the agent-specific relation R represents the subjective perception of this information, which can be partial or even grossly mistaken. For every object $a \in A$ and every feature $x \in X$, we read aRx as "object a has feature x according to the agent." By general order-theoretic facts, these relations induce normal modal operators that have a natural epistemic interpretation: namely, $\Box\varphi$ denotes the category φ as this is understood or perceived by the agent. In this new logical language, it is easy to distinguish between factual information, encoded by the formulas of the modal-free fragment of the logic, and its subjective interpretation, encoded by the formulas where the modal operators occur.

This language is expressive enough to capture agents' beliefs, but also their beliefs about the beliefs of other agents, and so forth. We define *fixed points* of these iterations in a way similar to how common knowledge is defined in classical epistemic logic [60]. In our theory, these points represent convergence in a process of social interaction: for example, the consensus reached by a group of agents vis-à-vis the objects that have certain features and thus belong to a particular category. The addition of new objects or new features to the market can destabilize this consensus, triggering a new round of interaction that ultimately begets a new equilibrium. Hence, we trace the origin of classification systems both to factual information about the objects on the market and the features they possess, which can be updated via the addition of new elements to A and X, and to the agents' idiosyncratic apprehension of this information, which can change even if the elements of A and X remain the same.

Our exposition of this formal theory is structured as follows: In Section 4.2, we recall some preliminaries about perfect lattices, RS-polarities, generalized Kripke frames, and Formal Concept Analysis. In addition, we provide a brief technical explanation of the dual interpretation of RS-semantics. We assume familiarity with the basics of lattice theory [61]. In Section 4.3, we discuss how our algebraic semantic structure can be understood in terms of categories and classification systems. Further, we show that the normal modal operators on lattices can support an epistemic interpretation. We build on this interpretation to introduce a common knowledge-type construction that accounts for the emergence of categories by way of social interaction. In Section 4.4, we discuss the strengths of our logical framework,

propose its application to other fields of research, and describe possible extensions to be pursued in future study.

4.2. PRELIMINARIES

4.2.1. PERFECT LATTICES AND BIRKHOFF'S THEOREM

A bounded lattice $\mathbb{L} = (L, \wedge, \vee, 0, 1)$ is *complete* if all subsets $S \subseteq L$ have both a supremum $\bigvee S$ and an infimum $\bigwedge S$. An element a in $\mathbb{L}$ is *completely join-irreducible* if, for any $S \subseteq \mathbb{L}$, $a = \bigvee S$ implies $a \in S$. *Complete meet-irreducibility* is defined order-dually. The sets of completely join- and meet-irreducible elements of $\mathbb{L}$ are denoted by $J^\infty(\mathbb{L})$ and $M^\infty(\mathbb{L})$, respectively.

A complete lattice $\mathbb{L}$ is *perfect* if it is join-generated by its completely join-irreducibles, and meet-generated by its completely meet-irreducibles. That is, $\mathbb{L}$ is perfect if for any $u \in \mathbb{L}$ we have:

$$\bigvee \{j \in J^\infty(\mathbb{L}) \mid j \leqslant u\} = u = \bigwedge \{m \in M^\infty(\mathbb{L}) \mid u \leqslant m\}. \qquad (4.1)$$

Definition 4.2.1. *A polarity is a triple $\mathbb{P} = (A, X, I)$ where A and X are sets, and $I \subseteq A \times X$ is a relation. For every polarity $\mathbb{P}$, we define the functions $(\cdot)^\uparrow$ (upper) and $(\cdot)^\downarrow$ (lower)[2] between the posets $(\mathcal{P}(A), \subseteq)$ and $(\mathcal{P}(X), \subseteq)$ as:*

$$for \quad U \in \mathcal{P}(A) \quad let \quad U^\uparrow := \{x \in X \mid \forall a \, (a \in U \rightarrow a I x)\}, \qquad (4.2)$$

$$for \quad V \in \mathcal{P}(X) \quad let \quad V^\downarrow := \{a \in A \mid \forall x \, (x \in V \rightarrow a I x)\}. \qquad (4.3)$$

The two maps $(\cdot)^\uparrow$ and $(\cdot)^\downarrow$ form a Galois connection between $(\mathcal{P}(A), \subseteq)$ and $(\mathcal{P}(X), \subseteq)$, i.e., $V \subseteq U^\uparrow$ iff $U \subseteq V^\downarrow$ for all $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(X)$. This connection has important and well-known consequences, including:

(a) The composition maps $(\cdot)^{\uparrow\downarrow} := (\cdot)^\downarrow \circ (\cdot)^\uparrow$ and $(\cdot)^{\downarrow\uparrow} := (\cdot)^\uparrow \circ (\cdot)^\downarrow$ are closure operators on $(\mathcal{P}(A), \subseteq)$ and $(\mathcal{P}(X), \subseteq)$, respectively.[3]

(b) The set of all Galois-stable subsets of A, i.e., those $U \in \mathcal{P}(A)$ such that $U^{\uparrow\downarrow} = U$, forms a complete sub-semilattice of $(\mathcal{P}(A), \bigcap)$. Likewise, the set of all Galois-stable subsets of X, i.e., those $V \in \mathcal{P}(X)$ such that $V^{\downarrow\uparrow} = V$, forms a complete sub-semilattice of $(\mathcal{P}(X), \bigcap)$. We denote this semilattice by $\mathbb{P}^+$.

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[2]In what follows, we simplify notation wherever possible and write $a^\uparrow$ for $\{a\}^\uparrow$ and $x^\downarrow$ for $\{x\}^\downarrow$ for every $a \in A$ and $x \in X$.

[3]Recall that a closure operator on a poset $(S, \leqslant)$ is a map $f : S \rightarrow S$, which is extensive ($\forall a \in S[a \leqslant f(a)]$), monotone ($\forall a, b \in S[a \leqslant b \Rightarrow f(a) \leqslant f(b)]$), and idempotent ($\forall a \in S[f(a) = f(f(a))]$).

(c) Because it is complete, the semilattice $\mathbb{P}^+$ is in fact a lattice, where meet is the set-theoretic intersection and join is the closure of the set-theoretic union.

Birkhoff [40] showed that every complete lattice is isomorphic to $\mathbb{P}^+$ for some polarity $\mathbb{P}$. In the terminology of FCA, this complete lattice represents the concept lattice arising from $\mathbb{P}$, i.e., all the tuples (C, D) such that $C \subseteq A$, $D \subseteq X$, $D^{\downarrow} = C$, and $C^{\uparrow} = D$. The concepts, i.e., the Galois-stable subsets of X and A, can be characterized as (members of) tuples $(U^{\uparrow\downarrow}, U^{\uparrow})$ and $(V^{\downarrow}, V^{\downarrow\uparrow})$ for any $U \subseteq A$ and $V \subseteq X$. The sets C and D are respectively referred to as the "extension" and the "intension" of a concept.

A polarity (A, X, I) induces specialization pre-orders on A and X defined as follows: $x \leqslant y$ iff $\forall a(aIx \rightarrow aIy)$ for all $x, y \in X$, and $a \leqslant b$ iff $\forall x(bIx \rightarrow aIx)$ for all $a, b \in A$. Clearly, $\leqslant \circ I \circ \leqslant \subseteq I$. For every $b \in A$ and $z \in X$, let $z{\uparrow} := \{x \mid z \leqslant x\}$, and $b{\downarrow} := \{a \mid a \leqslant b\}$.

Lemma 4.2.2. $z{\uparrow}$ and $b{\downarrow}$ are Galois-stable for all $b \in A$ and $z \in X$.

Proof. As the two parts of the proof are symmetric, we only prove the part concerning z. Let $x \in z{\uparrow}^{\downarrow\uparrow}$ and let us show that $z \leqslant x$. That is, let us fix a such that aIz and show that aIx. Because $I \circ \leqslant \subseteq I$, from aIz it follows that $\forall y(z \leqslant y \rightarrow aIy)$, which means that $a \in z{\uparrow}^{\downarrow}$. Because $x \in z{\uparrow}^{\downarrow\uparrow}$ by assumption, this implies aIx, as required.   □

Corollary 4.2.3. $z^{\downarrow\uparrow} = z{\uparrow}$ and $b^{\uparrow\downarrow} = b{\downarrow}$ for all $b \in A$ and $z \in X$.

Proof. Because $z{\uparrow}$ is Galois-stable and contains z, and $z^{\downarrow\uparrow}$ is the smallest such set by definition, it follows that $z^{\downarrow\uparrow} \subseteq z{\uparrow}$. For the converse inclusion, let $z \leqslant y$ and aIz. Because $I \circ \leqslant \subseteq I$, this implies that aIy and thus that $y \in z^{\downarrow\uparrow}$, as required.   □

In summary, the formal concepts generated by each $a \in A$ and $x \in X$ are defined as $(a{\downarrow}, a^{\uparrow})$ and $(x^{\downarrow}, x{\uparrow})$, respectively.

4.2.2. DUALITY WITH RS-POLARITIES

As mentioned above, every complete lattice is isomorphic to $\mathbb{P}^+$ for some polarity $\mathbb{P}$. When specializing to distributive lattices and Boolean algebras, well-known dualities exist between set-theoretic structures and *perfect algebras*. In particular, perfect distributive lattices are dual to posets, and perfect (i.e., complete and atomic) Boolean algebras are dual to sets. The question arises of which polarities are dual to perfect lattices. Gehrke [41] offered an answer in the form of reduced and separated polarities, or

RS-polarities, by rephrasing the duality for perfect lattices [62] in a model-theoretic fashion. In this section, we recall what it means for a polarity to be reduced and separated, and briefly explain how these two properties guarantee the perfection of the dual lattice.

The route from perfect lattices to polarities is given by the following:

Definition 4.2.4. *For every perfect lattice $\mathbb{L}$, the polarity associated with $\mathbb{L}$ is the triple $\mathbb{L}_+ := (J^\infty(\mathbb{L}), M^\infty(\mathbb{L}), I_+)$ where I_+ is the lattice order $\leqslant_\mathbb{L}$ restricted to $J^\infty(\mathbb{L}) \times M^\infty(\mathbb{L})$.*

Definition 4.2.5. *[cf. 41, Definitions 2.3 and 2.12] A polarity $\mathbb{P} = (A, X, I)$ is*

1. *separated, if the following conditions are satisfied:*

   (S1) for all $a, b \in A$, if $a \neq b$ then $a^\uparrow \neq b^\uparrow$,
   (S2) for all $x, y \in Y$, if $x \neq y$ then $x^\downarrow \neq y^\downarrow$;

2. *reduced, if the following conditions are satisfied:*

   (R1) for every $a \in A$, some $x \in X$ exists such that a is $\leqslant$-minimal in $\{b \in A \mid (b, x) \notin I\}$,
   (R2) for every $x \in X$, some $a \in A$ exists such that x is $\leqslant$-maximal in $\{y \in X \mid (a, x) \notin I\}$;

3. *an RS-polarity, if it is reduced and separated.[4]*

Let us denote $S := \{b \mid b \in A \text{ and } b < a\} = a{\downarrow} \setminus \{a\}$ for each $a \in A$. If $\mathbb{P}$ is separated, then $a{\downarrow}$ is completely join-irreducible in $\mathbb{P}^+$ iff $\bigvee_{b \in S} b{\downarrow} \subsetneq a{\downarrow}$ iff $a^\uparrow \subsetneq \bigcap_{b \in S} b^\uparrow$, that is, iff some $x \in X$ exists such that bIx for all $b \in S$ and $(a, x) \notin I$. This corresponds to the first reducing condition (R1). Similarly, the second reducing condition (R2) dually characterizes the notion that, for every $x \in X$, the subset $x{\uparrow}$ is completely meet-irreducible in $\mathbb{P}^+$, represented as a sub meet-semilattice of $\mathcal{P}(X)$.

Proposition 4.2.6. *[cf. 62, Proposition 4.7 and Corollary 4.9] For every perfect lattice $\mathbb{L}$ and RS-polarity $\mathbb{P}$:*

1. *$\mathbb{L}_+$ is an RS-polarity and $(\mathbb{L}_+)^+ \cong \mathbb{L}$,*

2. *$\mathbb{P}^+$ is a perfect lattice and $(\mathbb{P}^+)_+ \cong \mathbb{P}$.*

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[4]Gehrke [41] refers to RS-polarities as RS-frames. In this chapter, we distinguish between these two terms and reserve RS-frames for RS-polarities endowed with additional relations used to interpret operations on the lattice expansion.

This duality serves to generalize the Kripke semantics of modal logic to logics with possibly non-distributive propositional base. As in the dual correspondence between Kripke frames and complete atomic Boolean algebras with operators (BAOs), one would want a dual correspondence between perfect normal lattice expansions and RS-polarities endowed with additional relations. In previous work, Conradie and Palmigiano [26, Section 2] introduced a method for computing the definition of relations dually corresponding to normal modal operators for a certain modal signature consisting of unary and binary modal operators. We apply this method here to obtain an expansion $\mathcal{L}$ of the basic lattice language with a unary normal box-type connective $\Box$, which is canonically interpreted on lattices endowed with a completely meet-preserving operation.

Taking the connection between the satisfaction relation $\Vdash$ in Kripke frames and the interpretation of modal formulas in BAOs as our guideline, let $\mathbb{F} = (W, R)$ be a Kripke frame. From the satisfaction relation $\Vdash \subseteq W \times \mathcal{L}$ between states of $\mathbb{F}$ and formulas, we define an interpretation $\overline{v} : \mathcal{L} \to \mathbb{F}^+$ into the complex algebra of $\mathbb{F}$. This is an $\mathcal{L}$-homomorphism, and it is obtained as the unique homomorphic extension of the equivalent functional representation of the relation $\Vdash$ as a map $v : \mathrm{Prop} \to \mathbb{F}^+$, defined as $v(p) = \Vdash^{-1}[p]$.[5] This makes it possible to derive interpretations from satisfaction relations, so that for any $a \in J^{\infty}(\mathbb{F}^+)$ and any formula φ,

$$a \Vdash \varphi \quad \text{iff} \quad a \leqslant \overline{v}(\varphi), \tag{4.4}$$

where, on the left-hand side of the condition, $a \in J^{\infty}(\mathbb{F}^+)$ is identified with a state of $\mathbb{F}$ via the isomorphism $\mathbb{F} \cong (\mathbb{F}^+)_+$.

Conversely, consider a perfect lattice with completely meet-preserving operation $\mathbb{C} = (\mathbb{L}, \Box)$ and a homomorphic assignment $\overline{v} : \mathcal{L} \to \mathbb{C}$. Recall that the complete lattice $\mathbb{L}$ can be identified with the lattice $\mathbb{P}^+$ arising from some RS-polarity $\mathbb{P} = (A, X, I)$. We want to define a suitable relation $R = R_{\Box}$ and satisfaction relation $\Vdash_{\overline{v}}$ that fulfills Condition 4.4. Our method hinges on the dual characterization of $\overline{v}$ as a pair of relations $(\Vdash_{\overline{v}}, \succ_{\overline{v}})$ such that $\Vdash_{\overline{v}} \subseteq J^{\infty}(\mathbb{L}) \times \mathcal{L} \cong A \times \mathcal{L}$ and $\succ_{\overline{v}} \subseteq M^{\infty}(\mathbb{L}) \times \mathcal{L} \cong X \times \mathcal{L}$. This dual characterization is established by induction on formulas. The base of the

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[5]Notice that, in order for this equivalent functional representation to be well defined, we are required to assume that the relation $\Vdash$ is $\mathbb{F}^+$-*compatible*, i.e., that $\Vdash^{-1}[p] \in \mathbb{F}^+$ for every $p \in \mathrm{Prop}$. In the Boolean case, every relation from W to LML is clearly $\mathbb{F}^+$-compatible, but this is not necessarily so in the distributive case because $\Vdash^{-1}[p]$ needs to be an upward- or downward-closed subset of $\mathbb{F}$. This gives rise to the *persistency* condition, e.g., in the relational semantics of intuitionistic logic.

induction is clear: for every $a \in J^{\infty}(\mathbb{P}^+)$ and every $p \in \mathrm{Prop} \cup \{0, 1\}$, we have

$$a \Vdash_{\overline{v}} p \quad \text{iff} \quad a \leqslant \overline{v}(p). \tag{4.5}$$

Let us now turn to the inductive step for the box. Because $\overline{v} : \mathcal{L} \to \mathbb{P}^+$ is a homomorphism, $\overline{v}(\Box\varphi) = \Box^{\mathbb{P}^+}\overline{v}(\varphi)$. Suppose that Condition 4.4 holds for φ. Because $\mathbb{P}^+$ is perfect, $\overline{v}(\varphi) = \bigwedge\{x \in M^{\infty}(\mathbb{L}) \mid \overline{v}(\varphi) \leq x\}$. Therefore,

$$
\begin{align}
a \leqslant \overline{v}(\Box\varphi) \quad &\text{iff} \quad a \leqslant \Box^{\mathbb{P}^+}\overline{v}(\varphi) \tag{4.6}\\
&\text{iff} \quad a \leqslant \Box^{\mathbb{P}^+}\bigwedge\{x \in M^{\infty}(\mathbb{P}^+) \mid \overline{v}(\varphi) \leq x\} \tag{4.7}\\
&\text{iff} \quad a \leqslant \bigwedge\{\Box^{\mathbb{P}^+}x \mid x \in M^{\infty}(\mathbb{P}^+) \text{ and } \overline{v}(\varphi) \leqslant x\} \tag{4.8}\\
&\text{iff} \quad \forall x\left[(x \in M^{\infty}(\mathbb{L}) \text{ and } \overline{v}(\varphi) \leqslant x) \to a \leqslant \Box^{\mathbb{P}^+}x\right]. \tag{4.9}
\end{align}
$$

At the end of this chain, we have equivalently reduced the whole information on $\Box$ to the information of whether $a \leqslant \Box^{\mathbb{P}^+}x$ for each a and x. Consequently, this can be taken as the definition of the relation $R \subseteq A \times X$: we let aRx iff $a \leqslant \Box^{\mathbb{P}^+}x$.

To turn the last clause above into a satisfaction clause for $\Box$, we first replace $M^{\infty}(\mathbb{L})$ with X, which we identify via the isomorphism $\mathbb{P} \cong (\mathbb{P}^+)_+$. Then, we recall the second relation $\succ_{\overline{v}}$ between elements of X and formulas obeying the following condition, which is to be defined by induction on the structure of the formulas, analogously to Condition 4.4:

$$x \succ \varphi \quad \text{iff} \quad \overline{v}(\varphi) \leqslant x. \tag{4.10}$$

These considerations produce the following satisfaction clause for $\Box$:

$$
\begin{align}
a \Vdash_{\overline{v}} \Box\varphi \quad &\text{iff} \quad a \leqslant \overline{v}(\Box\varphi) \tag{4.11}\\
&\text{iff} \quad \forall x[(x \in X \text{ and } x \succ \varphi) \to aR_{\Box}x]. \tag{4.12}
\end{align}
$$

The co-satisfaction relation $\succ$ deserves some additional comments. In the Boolean and distributive settings, $\succ$ is completely determined by $\Vdash$ and hence it is not mentioned explicitly. In the non-distributive setting, however, the relation needs to be defined along with $\Vdash$. Condition 4.10 determines the base case:

$$y \succ \overline{v}(p) \quad \text{iff} \quad \overline{v}(p) \leqslant y. \tag{4.13}$$

If we specialize the clause above to powerset algebras $\mathcal{P}(W)$, then we have $y \succ_{V} p$ iff $V(p) \leqslant y$ iff $V(p) \subseteq W/\{x\}$ for some $x \in W$ iff $\{x\} \nsubseteq V(p)$ iff

$x \notin V(p)$ iff $x \nVdash p$. This goes to show that the relation $\succ$ can be regarded as an upside-down description of the satisfaction relation $\Vdash$, which we refer to as a *co-satisfaction* or *refutation*. The inductive step for the derivation of the co-satisfaction clause for $\square$ is:

$$\overline{v}(\square\varphi) \leqslant x \qquad \text{iff} \qquad \bigvee \{a \in J^{\infty}(\mathbb{L}) \mid a \leqslant \overline{v}(\square\varphi)\} \leqslant x \qquad (4.14)$$

$$\text{iff} \qquad \forall a\,[(a \in J^{\infty}(\mathbb{L}) \text{ and } a \leqslant \overline{v}(\square\varphi)) \rightarrow a \leqslant x] \qquad (4.15)$$

$$\text{iff} \qquad \forall a\,[(a \in A \text{ and } a \Vdash \square\varphi) \rightarrow aIx]. \qquad (4.16)$$

The last line follows from Condition 4.4 for $\square\varphi$, from the identification of $J^{\infty}(\mathbb{L})$ with A via the isomorphism $\mathbb{P} \cong (\mathbb{P}^{+})_{+}$, and from the identification of the lattice order $\leqslant$ restricted to $J^{\infty}(\mathbb{L}) \times M^{\infty}(\mathbb{L})$ with the relation I.

4.2.3. RS-FRAMES AND MODELS
In addition to the semantics for $\square$ discussed above, we define relational semantics for a further expansion of $\mathcal{L}$ with a unary normal diamond-type connective $\blacklozenge$ and with two special sorts of variables $\mathbf{i}, \mathbf{j}$, termed *nominals*, and $\mathbf{m}, \mathbf{n}$, termed *co-nominals*. These semantics are the outcome of a dual characterization similar to the one used for $\square$.

Definition 4.2.7. *An RS-frame for an expansion $\mathcal{L}$ of the basic lattice language is a structure $\mathbb{F} = (\mathbb{P}, R)$, where $\mathbb{P} = (A, X, I)$ is an RS-polarity and $R \subseteq A \times X$ so that the (pre-)images of singletons under R are Galois-closed, i.e., for every $x \in X$, $R^{-1}[x]^{\uparrow\downarrow} \subseteq R^{-1}[x]$, where $R^{-1}[x] := \{a \mid aRx\}$, and for every $a \in A$, $R[a]^{\downarrow\uparrow} \subseteq R[a]$, where $R[a] := \{x \mid aRx\}$. The relations R satisfying this condition are termed RS-compatible.*

The additional conditions on R are compatibility conditions, which guarantee that the following assignments respectively define the operations $\square$ and $\blacklozenge$ associated with R on the lattice $\mathbb{P}^{+}$. Thus, for every $U \in \mathbb{P}^{+}$, we have:

$$\square U := \bigcap \{R^{-1}[x] \mid U \subseteq x^{\downarrow}\}, \qquad (4.17)$$

$$\blacklozenge U := \bigvee \{R[a] \mid a^{\uparrow\downarrow} \subseteq U\}. \qquad (4.18)$$

Definition 4.2.8. *For every RS-frame $\mathbb{F} = (\mathbb{P}, R)$, its complex algebra is the lattice expansion $\mathbb{F}^{+} := (\mathbb{P}^{+}, \square)$ where $\square$ is defined as above.*

Lemma 4.2.9. $\leqslant \circ R \circ \leqslant\, \subseteq R$ *for every RS-frame $\mathbb{F} = (\mathbb{P}, R)$.*

Proof. Assume that aRz and $z \leqslant y$. To show that $y \in R[a]$, by the second compatibility condition it is enough to show that $y \in R[a]^{\downarrow\uparrow}$. That is, let

Table 4.1: Satisfaction and co-satisfaction relations on $\mathbb{M}$

$\mathbb{M}, a \Vdash 0$	never
$\mathbb{M}, x > 0$	always
$\mathbb{M}, a \Vdash 1$	always
$\mathbb{M}, x > 1$	never
$\mathbb{M}, a \Vdash p$	iff $a \in V_1(p)$
$\mathbb{M}, x > p$	iff $x \in V_2(p)$
$\mathbb{M}, a \Vdash \mathbf{i}$	iff $a \in V_1(\mathbf{i})$
$\mathbb{M}, x > \mathbf{i}$	iff $x \in V_2(\mathbf{i})$
$\mathbb{M}, a \Vdash \mathbf{m}$	iff $a \in V_1(\mathbf{m})$
$\mathbb{M}, x > \mathbf{m}$	iff $x \in V_2(\mathbf{m})$
$\mathbb{M}, a \Vdash \varphi \wedge \psi$	iff $\mathbb{M}, a \Vdash \varphi$ and $\mathbb{M}, a \Vdash \psi$
$\mathbb{M}, x > \varphi \wedge \psi$	iff for all $a \in A$, if $\mathbb{M}, a \Vdash \varphi \wedge \psi$, then aIx
$\mathbb{M}, a \Vdash \varphi \vee \psi$	iff for all $x \in X$, if $\mathbb{M}, x > \varphi \vee \psi$, then aIx
$\mathbb{M}, x > \varphi \vee \psi$	iff $\mathbb{M}, x > \varphi$ and $\mathbb{M}, x > \psi$
$\mathbb{M}, a \Vdash \Box\varphi$	iff for all $x \in X$, if $\mathbb{M}, x > \varphi$, then aRx
$\mathbb{M}, x > \Box\varphi$	iff for all $a \in A$, if $\mathbb{M}, a \Vdash \Box\varphi$, then aIx
$\mathbb{M}, a \Vdash \blacklozenge\varphi$	iff for all $x \in X$, if $\mathbb{M}, x > \blacklozenge\varphi$, then aIx
$\mathbb{M}, x > \blacklozenge\varphi$	iff for all $a \in A$, if $\mathbb{M}, a \Vdash \varphi$, then aRx

us fix $b \in R[a]^{\downarrow}$ and show that bIy. From $b \in R[a]^{\downarrow}$ and aRz it follows that bIz. Given that $I \circ \leqslant \; \subseteq I$, bIz and $z \leqslant y$ imply that bIy. The rest is proven in a similar fashion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

An RS-model for $\mathcal{L}$ on $\mathbb{F}$ is a structure $\mathbb{M} = (\mathbb{F}, v)$ such that $\mathbb{F}$ is an RS-frame for $\mathcal{L}$ and v is a variable assignment mapping each $p \in \mathsf{Prop}$ to a pair $(V_1(p), V_2(p))$ of Galois-stable sets in $\mathcal{P}(A)$ and $\mathcal{P}(X)$, respectively. Given a model for the expanded language with $\blacklozenge$, nominals, and co-nominals, the variable assignments also map nominals $\mathbf{j}$ to $(j^{\uparrow\downarrow}, j^{\uparrow})$ for some j in A, and co-nominals $\mathbf{m}$ to $(m^{\downarrow}, m^{\downarrow\uparrow})$ for some m in X. Table 4.1 presents the recursive definitions of the satisfaction and co-satisfaction relations on $\mathbb{M}$. The following lemma is easily proven by simultaneous induction on φ and ψ using these definitions. The base cases for 0 and 1 use conditions (R1) and (R2), whereas those for proposition letters, nominals, and co-nominals follow from the way valuations are defined.

Lemma 4.2.10. *For all formulas φ and ψ, it holds that:*

1. $\mathbb{M}, a \Vdash \varphi$ iff for all $x \in X$, if $\mathbb{M}, x \succ \varphi$ then aIx,

2. $\mathbb{M}, x \succ \psi$ iff for all $a \in A$, if $\mathbb{M}, a \Vdash \psi$ then aIx.

An inequality $\varphi \leqslant \psi$ is *true* in $\mathbb{M}$, denoted by $\mathbb{M} \Vdash \varphi \leqslant \psi$, iff, for all $a \in A$ and all $x \in X$, if $\mathbb{M}, a \Vdash \varphi$ and $\mathbb{M}, x \succ \psi$ then aIx.

Remark 4.2.11. *It follows from Lemma 4.2.10 that $\mathbb{M} \Vdash \varphi \leqslant \psi$ iff, for all $a \in A$, if $\mathbb{M}, a \Vdash \varphi$ then $\mathbb{M}, a \Vdash \psi$. It also follows that $\mathbb{M} \Vdash \varphi \leqslant \psi$ iff, for all $x \in X$, if $\mathbb{M}, x \succ \psi$ then $\mathbb{M}, x \succ \varphi$. We will find these equivalent characterizations of truth in $\mathbb{M}$ useful when discussing examples.*

4.2.4. STANDARD TRANSLATION

As in the Boolean case, each RS-model $\mathbb{M}$ for $\mathcal{L}$ can be viewed as a first-order structure, albeit two-sorted. Accordingly, we define correspondence languages. Let L_1 be the two-sorted first-order language with equality built over the denumerable and disjoint sets of individual variables A and X, with the binary relation symbol I, R, and two unary predicate symbols P_1, P_2 for each $p \in \mathsf{Prop}$. The intended interpretation links P_1 and P_2 in the way suggested by the definition of $\mathcal{L}$-valuations. Every $p \in \mathsf{Prop}$ maps to a pair $(V_1(p), V_2(p))$ of Galois-stable sets, as explained in Section 4.2.3. The interpretation of pairs (P_1, P_2) of predicate symbols is thus restricted to pairs of Galois-stable sets, and the interpretation of universal second-order quantification is also restricted to range over such sets.

We assume that L_1 contains denumerably many individual variables $i, j, \ldots$, which correspond to the nominals $\mathbf{i}, \mathbf{j}, \ldots \in \mathsf{Nom}$, and denumerably many individual variables $n, m, \ldots$, which correspond to the co-nominals $\mathbf{n}, \mathbf{m}, \ldots \in \mathsf{CoNom}$. Let L_0 be the sub-language that does not contain the unary predicate symbols corresponding to the propositional variables. Table 4.2 presents the recursive definitions of the *standard translation* of $\mathcal{L}^+$ into L_1. In reading the table, recall that $a \leqslant j$ abbreviates $\forall x (jIx \rightarrow aIx)$ and $m \leqslant x$ abbreviates $\forall a (aIm \rightarrow aIx)$.

Lemma 4.2.12. *[cf. 26, Lemma 2.5] For any $\mathcal{L}$-model $\mathbb{M}$ and any $\mathcal{L}^+$-inequality $\varphi \leqslant \psi$, it holds that*

$$\mathbb{M} \Vdash \varphi \leqslant \psi \quad \textit{iff} \quad \mathbb{M} \models \forall a \forall x \left[\mathrm{ST}_a(\varphi) \wedge \mathrm{ST}_x(\psi) \rightarrow aIx \right] \tag{4.19}$$
$$\textit{iff} \quad \mathbb{M} \models \forall a \left[\mathrm{ST}_a(\varphi) \rightarrow \mathrm{ST}_a(\psi) \right] \tag{4.20}$$
$$\textit{iff} \quad \mathbb{M} \models \forall x \left[\mathrm{ST}_x(\psi) \rightarrow \mathrm{ST}_x(\varphi) \right]. \tag{4.21}$$

By virtue of this translation, RS-frames provide an algebraically motivated generalization of correspondence theory. As anticipated at the beginning

Table 4.2: Standard translation on RS-frames

$\mathrm{ST}_a(0)$	$:= a \neq a$
$\mathrm{ST}_x(0)$	$:= x = x$
$\mathrm{ST}_a(1)$	$:= a = a$
$\mathrm{ST}_x(1)$	$:= x \neq x$
$\mathrm{ST}_a(p)$	$:= P_1(a)$
$\mathrm{ST}_x(p)$	$:= P_2(x)$
$\mathrm{ST}_a(\mathbf{j})$	$:= a \leqslant j$
$\mathrm{ST}_x(\mathbf{j})$	$:= j I x$
$\mathrm{ST}_a(\mathbf{m})$	$:= a I m$
$\mathrm{ST}_x(\mathbf{m})$	$:= m \leqslant x$
$\mathrm{ST}_a(\varphi \wedge \psi)$	$:= \mathrm{ST}_a(\varphi) \wedge \mathrm{ST}_a(\psi)$
$\mathrm{ST}_x(\varphi \wedge \psi)$	$:= \forall a[\mathrm{ST}_a(\varphi \wedge \psi) \to a I x]$
$\mathrm{ST}_a(\varphi \vee \psi)$	$:= \forall x[\mathrm{ST}_x(\varphi \vee \psi) \to a I x]$
$\mathrm{ST}_x(\varphi \vee \psi)$	$:= \mathrm{ST}_x(\varphi) \wedge \mathrm{ST}_x(\psi)$
$\mathrm{ST}_a(\Box\varphi)$	$:= \forall x[\mathrm{ST}_x(\varphi) \to a R x]$
$\mathrm{ST}_x(\Box\varphi)$	$:= \forall a[\mathrm{ST}_a(\Box\varphi) \to a I x]$
$\mathrm{ST}_a(\blacklozenge\varphi)$	$:= \forall x[\mathrm{ST}_x(\blacklozenge\varphi) \to a I x]$
$\mathrm{ST}_x(\blacklozenge\varphi)$	$:= \forall a[\mathrm{ST}_a(\varphi) \to a R x]$

of this chapter, one of the objectives of our study is to understand whether such generalized environment retains some of the intuition that made Kripke semantics and modal logic so appealing and well-suited for a number of interdisciplinary applications, including the study of categories in organization theory [13].

Let us start with the inequality $\Box 0 \leqslant 0$, which corresponds on Kripke frames to the condition that every state has a successor.

$$\Box 0 \leqslant 0 \quad \text{iff} \quad \forall a \left[\mathrm{ST}_a(\Box 0) \to \forall x (\mathrm{ST}_x(0) \to a I x)\right] \tag{4.22}$$

$$\text{iff} \quad \forall a \left[\forall y (y = y \to a R y) \to \forall x (x = x \to a I x)\right] \tag{4.23}$$

$$\text{iff} \quad \forall a \left[\forall y (a R y) \to \forall x (a I x)\right] \tag{4.24}$$

$$\text{iff} \quad \forall a \exists y (\neg (a R y)). \tag{4.25}$$

To justify the last equivalence recall that, by definition, no object a verifies $\forall x(a I x)$ in an RS-polarity (cf. Definition 4.2.5). Therefore, the condition in the penultimate line is true precisely when the premise of the implication, namely $\forall y (a R y)$, is false. This means that every state is not R-related to

some co-state. The condition on Kripke frames is recognizable given the suitable insertion of negations.

Next, let us consider the inequality $\Box p \leqslant p$, which corresponds on Kripke frames to the condition that R is reflexive.

$$\forall p\,(\Box p \leqslant p) \quad \text{iff} \quad \forall \mathbf{m}\,(\Box \mathbf{m} \leqslant \mathbf{m}) \tag{4.26}$$
$$\text{iff} \quad \forall a \forall \mathbf{m}\,[\mathrm{ST}_a\,(\Box \mathbf{m}) \to \mathrm{ST}_a\,(\mathbf{m})] \tag{4.27}$$
$$\text{iff} \quad \forall a \forall m\,(aRm \to aIm). \tag{4.28}$$

By definition, $\mathrm{ST}_a(\mathbf{m}) = aIm$ and $\mathrm{ST}_a(\Box \mathbf{m}) = \forall y(m \leqslant y \to aRy)$ can be rewritten as $m{\uparrow} \subseteq R[a]$, which is equivalent to aRm because $R \circ \leqslant \, \subseteq R$ (cf. Lemma 4.2.9). To recognize the connection with the usual reflexivity condition, observe that $\forall a \forall m(aRm \to aIm)$ is equivalent to $R \subseteq I$, and the reflexivity of a relation $R \subseteq A \times A$ can be written as $\mathsf{Id} := \, \subseteq R$, where for every A, $\mathsf{Id}_A := \{(a,a) \mid a \in A\}$, which is equivalent to $R^c \subseteq \mathsf{Id}^c$.

Clearly, $\Box p \leqslant p$ implies $\Box\Box p \leqslant \Box p$. Let us then consider the converse inequality, which in the classical setting corresponds to transitivity:

$$\forall p\,(\Box p \leqslant \Box\Box p) \quad \text{iff} \quad \forall \mathbf{m}\,(\Box \mathbf{m} \leqslant \Box\Box \mathbf{m}) \tag{4.29}$$
$$\text{iff} \quad \forall a \forall \mathbf{m}\,(\mathrm{ST}_a\,(\Box \mathbf{m}) \to \mathrm{ST}_a\,(\Box\Box \mathbf{m})) \tag{4.30}$$
$$\text{iff} \quad \forall a \forall m\left(aRm \to R^{-1}\,[m]^{\uparrow} \subseteq R\,[a]\right). \tag{4.31}$$

where

$$\mathrm{ST}_a\,(\Box\Box \mathbf{m}) = \forall y\left[\mathrm{ST}_y\,(\Box \mathbf{m}) \to aRy\right] \tag{4.32}$$
$$= \forall y\left[\forall b\,(\mathrm{ST}_b\,(\Box \mathbf{m}) \to bIy) \to aRy\right] \tag{4.33}$$
$$= \forall y\left[\forall b\,(bRm \to bIy) \to aRy\right] \tag{4.34}$$
$$= R^{-1}\,[m]^{\uparrow} \subseteq R\,[a]. \tag{4.35}$$

Although it is possible to retrieve the transitivity condition in this new interpretation, already with a relatively simple inequality like $\Box p \leqslant \Box\Box p$ this is hardly useful to gain a better understanding of this semantics, because the accessibility relation on states is encoded here into a "non-inaccessibility" relation between states and co-states. As a result, the condition quickly becomes awkward and unintuitive. In the next section, we propose a conceptual interpretation grounded in organization theory and show that better results can be achieved by taking this condition as primitive rather than as the generalization of some other semantics.

4.3. Application to Organization Theory

4.3.1. Categorization via RS-Semantics

At first sight, the mathematical framework presented above may seem far too abstract to capture notions of practical relevance to the sociology of markets: however, this framework can be easily understood by organization scholars as the order-theoretic representation of a classification system. The value of this abstraction goes beyond the mere appreciation of formal theory—though this too can sometimes be helpful [1, 3]—because thinking of classification systems as RS-frames allows one to better understand how market categories arise from a mix of factual information, subjective beliefs, and interaction the agents that populate the market.

The cornerstone of this sociological application of RS-frames is the notion, also core to FCA, that polarities (A, X, I) can be taken to represent databases where A is a set of objects, such as products or organizations in a market, X is a set of features that the agents find relevant to categorization, and I encodes whether a particular object possesses a particular feature. The specialization pre-order on objects $a \leqslant b$ can thus be interpreted as "a is at least as specified as b," meaning that a has at least all the feature of b. The pre-order on features $x \leqslant y$, instead, can be interpreted as "y is at least as generic as x," meaning that any object with x is bound to have y as well. With this interpretation in mind, the reducing and separating conditions (cf. Definition 4.2.5) that $\mathbb{P}$ must fulfill in order to dually correspond to a perfect lattice $\mathbb{P}^+$ can be rephrased as follows:

(S1) any two objects can be told apart by some feature;

(S2) for any two features, there is an object having one but not the other;

(R1) for any object a, if there are strictly more specified objects than a (i.e., objects that have all the features of a but also some more), then all of these objects share some feature x which a does not have;

(R2) for any feature x, if there are strictly more generic features than x (i.e., features shared by all the objects with x but also by others), then some object a exists that has all of these features but not x.

Conditions (S1) and (S2) are intuitive and do not require much explanation. They will be satisfied as long as no two objects have exactly the same features and no two features are shared by exactly the same objects. Condition (R1) can be enforced by adding *ad hoc* features to the database. For example, consider a market for songs with the following three features:

$x :=$ "composed in D minor," $y :=$ "sung in mezzo-soprano," and $z :=$ "lyrics with political undertones." Suppose that three songs a, b, c share feature x but b also has y and c also has z. Such a database violates (R1), because b and c are strictly more specified than a but they do not have features in common that a does not also possess. This can be remedied by adding a feature $w :=$ "cover art different from a". Although this feature may be utterly irrelevant to the agents, it does encode a factual truth about the objects and thus it can be safely added to the database. In other words, its relationships with the objects are encoded by I but they need not be encoded by R. Finally, (R2) can be enforced by removing particular features when they are the intersection of two, more generic features: for example, if the database included $x :=$ "composed in D minor," $y :=$ "sung in mezzo-soprano," and $z :=$ "composed in D minor and sung in mezzo-soprano." In this case, (R2) would be violated because no object exists that has both x and y while not having z. This can be easily remedied by excluding z from the database. Removing such features can always be done without loss of descriptive power: in point of fact, we can always enforce the separating and reducing conditions because the finite polarities we consider are a subclass of *doubly founded* polarities, for which this operation is always possible [43]. In Chapter 5, we will remove the RS-conditions so as to accommodate any polarity, but for the moment we restrict our consideration to those where these conditions are enforced. Figure 4.1 provides a visual example of a small database that abides by the RS-conditions, as well as the corresponding RS-polarity and perfect lattice.

We propose to understand the lattice $\mathbb{P}^+$ generated by the RS-polarity $\mathbb{P}$ as the partially ordered collection of all the *candidate categories* in the market. Formally, each element of $\mathbb{P}^+$ is a set of objects that is completely identified by a set of features: any object with these features is a member of the candidate category. We refer to these categories as "candidate" because they are purely implicit in the database and do not necessarily enjoy social recognition. Only a very limited subset of candidate categories will support the interpretation of *real categories*, i.e., those that are actually deemed meaningful by agents.[6] Only these categories, which of course are much fewer than the set of all candidate categories the agents could use to sort objects and features, will be assigned a category label.

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[6]Notice that our definition of "real" categories is not the same as the one given by Hsu and Hannan [55, p. 478] for "sociologically real categories." Sociologically real categories are those where membership has tangible consequences in terms of competitive outcomes [e.g., 47]. The categories we define as "real" are merely those whose intension and extension are agreed upon by the agents. In Chapter 5, we will further refine this definition.
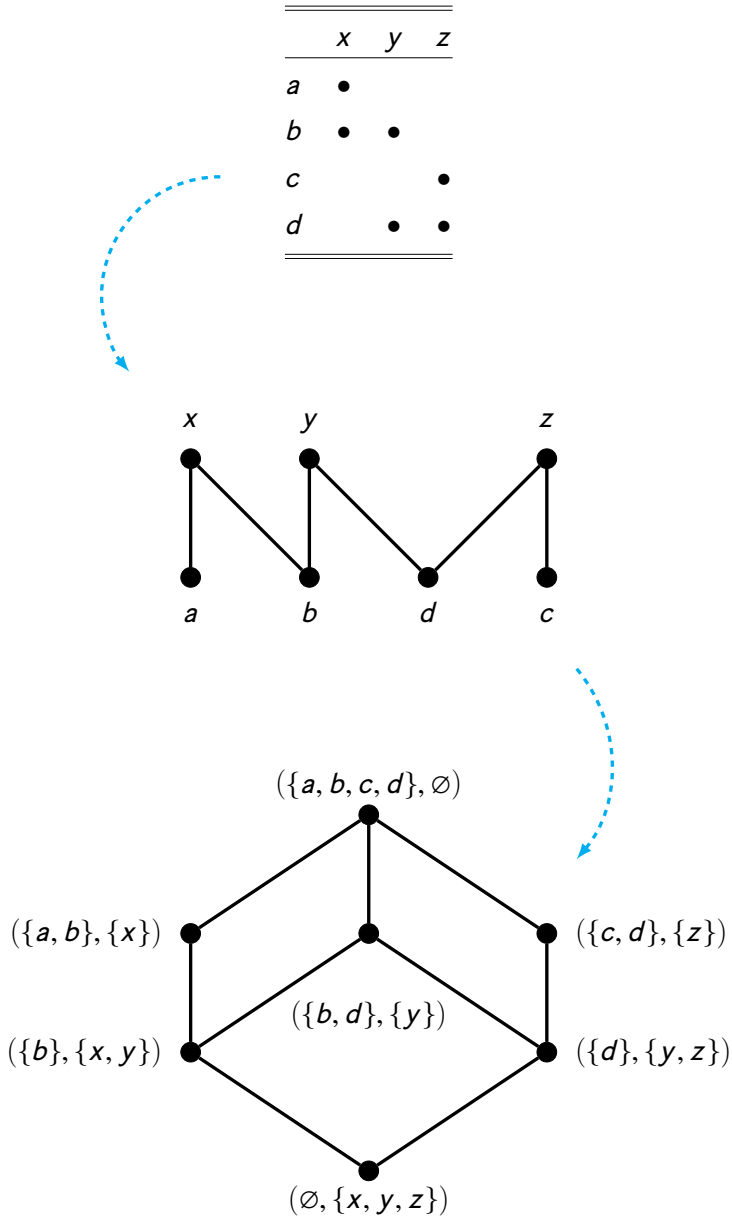
Figure 4.1: Database, RS-polarity, and perfect lattice

The labels of real categories can be attached to candidate categories by means of an assignment v that links each *atomic category label* $p \in$ Prop to a category viewed both extensionally as $V_1(p) \subseteq X$ and intensionally as $V_2(p) \subseteq A$.[7] Notice the perfect match between the encoding of the meaning of atomic propositions on Kripke models and that of atomic category labels on RS-models: on Kripke models, the meaning of atomic proposition p is given as the set of states at which p holds true; on RS-models, the meaning of atomic category label p is given as the set of objects that constitute the extension of p, or equivalently, as the set of features that constitute the intension of p. For convenience, we refer to a category's intension as its *description*, and we say that a feature *describes* a category if it belongs to the category's description. Given this assignment, the database is endowed with the structure of an $\mathcal{L}$-model $\mathbb{M}$ in such a way that, for any formula or category label $\varphi \in \mathcal{L}$, any object $a \in A$, and any feature $x \in X$, the expressions $\mathbb{M}, a \Vdash \varphi$ and $\mathbb{M}, x > \varphi$ respectively read as "object a is a member of category φ" and "feature x describes category φ."

One advantage of this conceptualization is that it provides an intuitive way to understand $>$ from first principles rather than as the negative counterpart of $\Vdash$. Another advantage concerns the understanding of the connectives $\wedge$ and $\vee$ in the general lattice environment. These operators are useful to identify categories that result from conceptual combinations [63–66]: there is a problem, however, in that their standard interpretation as conjunction and disjunction is unintuitive because distributivity seems to be hardwired in the way we understand "and" and "or" in natural language. In our framework, the satisfaction and co-satisfaction clauses for $\wedge$ and $\vee$ formulas are as follows (cf. Table 4.1):

$$\mathbb{M}, a \Vdash \varphi \wedge \psi \quad \text{iff} \quad \mathbb{M}, a \Vdash \varphi \text{ and } \mathbb{M}, a \Vdash \psi \tag{4.36}$$
$$\mathbb{M}, x > \varphi \wedge \psi \quad \text{iff} \quad \text{for all } a \in A, \text{ if } \mathbb{M}, a \Vdash \varphi \wedge \psi \text{ then } aIx \tag{4.37}$$
$$\mathbb{M}, a \Vdash \varphi \vee \psi \quad \text{iff} \quad \text{for all } x \in X, \text{ if } \mathbb{M}, x > \varphi \vee \psi \text{ then } aIx \tag{4.38}$$
$$\mathbb{M}, x > \varphi \vee \psi \quad \text{iff} \quad \mathbb{M}, x > \varphi \text{ and } \mathbb{M}, x > \psi. \tag{4.39}$$

These definitions imply that the category $\varphi \wedge \psi$ is the one whose extension corresponds to the intersection of the extensions of φ and of ψ. In other words, the members of $\varphi \wedge \psi$ are the objects that belong both to φ and to ψ. These satisfy at least the descriptions of φ and of ψ, and therefore the description of $\varphi \wedge \psi$ contains at least the union of the two descriptions, but it will usually contain additional features. For example, the category BLACK CARS $\wedge$ LUXURY CARS includes all the cars that qualify as black and

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[7]Recall that, for such an assignment, $V_1(p) = V_2(p)^{\downarrow}$ and $V_2(p) = V_1(p)^{\uparrow}$.

luxury. However, suppose that every member of BLACK CARS ∧ LUXURY CARS has leather seats, which is neither the case for every black car nor for every luxury car: in this case, having leather seats will be part of the description of BLACK CARS ∧ LUXURY CARS and hence this description is strictly larger than the union of the descriptions of BLACK and of LUXURY.

Conversely, the category $\varphi \vee \psi$ is the one whose description is equal to the intersection of the descriptions of φ and of ψ. Because in this case the objects are only required to possess the features common to φ and to ψ, the extension of $\varphi \vee \psi$ will include at least the union of the two extensions, but it will usually contain additional objects. For example, the category BLACK CARS ∨ LUXURY CARS includes cars of other colors in addition to black, because color is not a defining feature of LUXURY CARS, and of other vehicle classes in addition to luxury (e.g., sedan, economy, off-road), because pertaining to a particular vehicle class is not a defining feature of BLACK CARS. That is, this category includes virtually every car.

This interpretation of the connectives ∧ and ∨ makes it extremely easy to understand why distributivity fails. Suppose that cars were divided into three colors: black, red, and white. A member of the composite category BLACK CARS ∨ (WHITE CARS ∧ RED CARS) must have all the features in the description of BLACK CARS, because WHITE CARS ∧ RED CARS is so general that it does not add anything to this list. However, this is not the same as having membership in (BLACK CARS ∨ WHITE CARS) ∧ (BLACK CARS ∨ RED CARS), because both BLACK CARS ∨ WHITE CARS and BLACK CARS ∨ WHITE CARS include every car and thus their composition via ∧ does not restrict the extension of the resulting category to cars that are black.

With this working understanding of ⊩ and ≻, we can recognize the normal box-type operator on $\mathbb{P}^+$ as capturing the beliefs of individual agents vis-à-vis the objects or the features that belong to a given category. Consistently with this, we read $\mathbb{M}, a \Vdash \Box\varphi$ as "a belongs to φ according to the agent," and $\mathbb{M}, x \succ \Box\varphi$ as "x describes φ according to the agent." The normality conditions $\Box\top = \top$ and $\Box(\varphi \wedge \psi) = \Box\varphi \wedge \Box\psi$ can be understood as rationality requirements, i.e., the agent correctly recognizes the (uninformative) category $\top$ as such, and her understanding of the greatest common subcategory of any two categories φ and ψ is the greatest common subcategory of the categories she understands as φ and ψ.

On the database side, the agent's subjective perception of the incidence between objects and features is modeled by a relation $R \subseteq A \times X$, so that aRx intuitively reads "object a has feature x according to the agent." Unsurprisingly, the additional properties of R (cf. Lemma 4.2.9) can also be viewed as rationality requirements: if aRx then aRy for every $y \geqslant x$,

which means that, if the agent attributes feature x to object a then she also attributes to a all the features that follow from x. Likewise, if aRx then bRx for every $b \leqslant a$, which means that if the agent attributes x to a then she also attributes x to all the objects that possess at least the same features as a. Analogously to the classical case, two modal operators $\Box$ and $\blacklozenge$ are associated with the same relation R, but unlike the classical case these operations are not dual to one another in the sense of, e.g., $\blacklozenge := \neg\Box\neg$; instead, they are adjoints, which means that, for all $u, v \in \mathbb{P}^+$, $\blacklozenge u \leqslant v$ iff $u \leqslant \Box v$. Therefore, rather than encoding the dual perspective on the subjective information encoded by $\Box$, $\blacklozenge$ encodes exactly the same perspective as as $\Box$, except that $\blacklozenge$ is geared towards the objects whereas $\Box$ is geared towards the features. If we denote by $\mathbf{j}$ and $\mathbf{m}$ the categories generated by object j and feature m, respectively, for every j, m we have:

$$\blacklozenge\mathbf{j} \leqslant \mathbf{m} \quad \text{iff} \quad jRm \quad \text{iff} \quad \mathbf{j} \leqslant \Box\mathbf{m}. \tag{4.40}$$

As a result, the information jRm, which reads "the agent attributes feature m to object j," is equivalently encoded on the side of categories by saying that m describes the category $\blacklozenge\mathbf{j}$, i.e., the one the agent understands as the category generated by j, and by saying that j is a member of the category $\Box\mathbf{m}$, i.e., the one the agent understands as the category generated by m.

With regard to the clauses of the recursive definition of $\Vdash$ and $\succ$, $\mathbb{M}, a \Vdash \Box\varphi$ is the case iff, for all the features $x \in X$, if $\mathbb{M}, x \succ \varphi$ then aRx. That is, object a is recognized by the agent as a member of φ iff the agent attributes to a all the features that belong to the description of φ. Similarly, $\mathbb{M}, x \succ \Box\varphi$ is the case iff, for all the objects $a \in A$, if $\mathbb{M}, a \Vdash \Box\varphi$ then aIx. That is, feature x pertains to the description of φ according to the agent iff x is shared by every object the agent recognizes as a member of φ.

It is worth considering how two modal axioms that are relatively common in epistemic logic, namely those referred to at the beginning of this chapter as reflexivity ($\Box p \leqslant p$) and transitivity ($\Box p \leqslant \Box\Box p$), can be interpreted in our logical framework. The axiom $\Box p \leqslant p$ is interpreted epistemically as the *factivity* of knowledge, which means that "if the agent knows p then p is true." The first-order correspondent of the factivity axiom on RS-frames is $\forall a \forall x(aRx \rightarrow aIx)$. This expresses a form of factivity because it requires that, whenever the agent attributes feature x to object a, then it is indeed the case that x is a feature of a. The axiom $\Box p \leqslant \Box\Box p$ is interpreted epistemically as the *positive introspection* of knowledge: "if the agent knows p then she knows that she knows p." The first-order correspondent of the positive introspection axiom on RS-frames is $\forall a \forall m(aRm \rightarrow R^{-1}[m]^{\uparrow} \subseteq$

$R[a]$).[8] This means that, if an agent attributes feature m to object a, then she also attributes to a all the features shared by the objects to which she attributes m.

To better understand the link between this condition and the positive introspection of knowledge, consider the category $\Box\mathbf{m}$, i.e., the category the agent understands as generated by feature m.[9] This category can be identified with the tuple $(R^{-1}[m], R^{-1}[m]^{\uparrow})$: that is, the members of $\Box\mathbf{m}$ are the objects to which the agent attributes m[10] and the description of $\Box\mathbf{m}$ is the set of the features that the objects in $R^{-1}[m]$ have in common. By definition, bIz for every $b \in R^{-1}[m]$ and $z \in R^{-1}[m]^{\uparrow}$. The first-order correspondent of $\Box p \leqslant \Box\Box p$ requires that bRz for such b and z. Therefore, while factivity corresponds to $R \subseteq I$, i.e., what the agent believes is objectively the case, positive introspection yields the reverse inclusion restricted to objects and features within "boxed categories," i.e., the agent is aware of the features shared by the objects in the categories she knows.

Whether factivity and positive introspection are incorporated into this framework or not, the result of our characterization is a formal language that can describe categorization in real-world contexts while fully preserving the distinction between categories that are only possible (i.e., candidate) vs. those that are actually meaningful (i.e., real). Further, this system accounts for conceptual combination through different mechanisms, $\wedge$ and $\vee$, which respectively return the greatest common subordinate category and the smallest common superordinate category of any two given categories. All of this is achieved without imputing excessive computational ability to the agents, who can be aware of only a small number of objects and features and even be wrong in attributing certain features to objects. Although different agents are not required to agree as to which objects or features constitute a category, it is reasonable to presume that they derive utility from coordination [cf. 54, 67] and thus seek to reach some sort of consensus with regard to their category definitions. To this end, they can interact with one another and learn about each other's beliefs. We now turn to characterizing this interaction, which culminates in the identification of a subset of categories whose intension and extension are agreed upon by the agents. These categories are assigned a label [cf. 49] and become the building blocks of discourse. Thus emerges a classification system from three fundamental determinants: factual information, subjective perception, and social interaction.

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[8]Recall that $R^{-1}[x] \coloneqq \{a \mid aRx\}$ (cf. Definition 4.2.7).
[9]The same argument would hold more generally for any category $\Box\varphi$.
[10]Recall that $R^{-1}[m]$ is a Galois-stable set (cf. Definition 4.2.7).

4.3.2. CATEGORY EMERGENCE

In the organizational literature, *category emergence* refers to the sociocognitive mechanism whereby groups of objects (or lists of features) turn from being merely possible to being actually meaningful to agents in a market [22]. Researchers' interest in the processes through which new market categories are born [e.g., 68] has grown substantially in recent years, as it has become increasingly important to explain how classification systems change over time. As a first step toward this purpose, we offer an account of category emergence as the outcome of social interaction [54, 69]. For simplicity we consider a setting with only two agents, but the theory can be extended to include any finite number.

In this minimal setting, we introduce a bimodal logic $\mathcal{L}$ that extends the basic normal lattice expansion logic with two unary normal box-type modal operators 1 and 2 as well as the axioms $ip \leqslant p$ and $ip \leqslant iip$ for $1 \leqslant i \leqslant 2$. Models for this logic are structures $(\mathbb{P}, R_1, R_2, v)$ such that

(a) $\mathbb{P} = (A, X, I)$ is an RS-polarity;

(b) $R_i \subseteq A \times X$ for $1 \leqslant i \leqslant 2$ such that the following conditions hold:

    1. $\forall x (R_i^{-1}[x]^{\uparrow\downarrow} \subseteq R_i^{-1}[x])$,

    2. $\forall a (R_i[a]^{\downarrow\uparrow} \subseteq R_i[a])$,

    3. $R_i \subseteq I$,

    4. $\forall a \forall x (aR_i x \rightarrow R_i^{-1}[x]^{\uparrow} \subseteq R_i[a])$;

(c) and v is an assignment that associates each $p \in \text{Prop}$ to an element of $\mathbb{P}^+$, viewed both extensionally as $V_1(p) \subseteq A$ and intensionally as $V_2(p) \subseteq X$, in such a way that $V_1(p) = V_2(p)^{\downarrow}$ and $V_2(p) = V_1(p)^{\uparrow}$.

We proceed by implementing a construction reminiscent of what classical epistemic logic refers to as "common knowledge" [60]. When applied to our framework, this construction gives an expansion $\mathcal{L}_C$ of the bimodal lattice expansion logic above with a normal box-type operator C. The interpretation of this modal operator on $\mathbb{P}^+$, given the additional axioms, is as follows: for any $u \in \mathbb{P}^+$,

$$C(u) := \bigwedge_{s \in S} su, \tag{4.41}$$

where S is the set of all compound modalities of the forms $(ij)^n$ and $i(ji)^n$, for $1 \leqslant i \neq j \leqslant 2$ and for some $n \in \mathbb{N}$.

Lemma 4.3.1. $C(u) \leqslant u$ and $C(u) \leqslant C(C(u))$ for any $u \in \mathbb{P}^+$.

Proof. Clearly, $C(u) \leqslant 1u \leqslant u$, which proves the first inequality.

$$C(C(u)) = \bigwedge_{s \in S} sC(u) = \bigwedge_{s \in S} s\left(\bigwedge_{t \in S} tu\right) = \bigwedge_{s \in S}\bigwedge_{t \in S} stu \geqslant \bigwedge_{s' \in S} s'u = C(u) \quad (4.42)$$

$\square$

Let $R_C, R_s \subseteq A \times X$ for any $s \in S$ be defined as follows: $aR_s x$ iff $a \leqslant sx$ and $aR_C x$ iff $a \leqslant C(x)$. Clearly, $R_C = \bigcap_{s \in S} R_s$. In the standard setting of epistemic logic, the accessibility relations associated with the agents do not encode their knowledge but rather their uncertainty: therefore, the relation associated with the common knowledge operator is defined as the reflexive transitive closure of the union of the relations associated with individual agents. Normally, this is bigger than any relation associated with an individual agent. In our setting, however, the relations associated with each agent encode what the agents positively know, rather than what they are uncertain about. For this reason, the common knowledge relation R_C is the intersection of the relations R_s encoding the finite iterations, which is normally smaller than any individual agent's relation.

Because C and every $s \in S$ are compositions of normal box-operators, they are themselves normal box-operators. Consequently, the relations R_C and R_s to which they give rise are RS-compatible (cf. Definition 4.2.7). The correspondence reductions discussed in Section 4.2.4 can thus be applied to C and R_C, yielding the following:

Lemma 4.3.2. *The relation R_C defined above verifies the following conditions: $R_C \subseteq I$, and $\forall a \forall x (aR_C x \rightarrow R_C^{-1}[x]^{\uparrow} \subseteq R_C[a])$.*

For any category label φ, the category $C(\varphi) = \bigwedge\{C(\mathbf{m}) \mid \varphi^{\mathbb{P}^+} \leqslant \mathbf{m}\}$. In what follows, we restrict our attention to categories $C(\mathbf{m})$ for some feature $m \in X$. The extension of $C(\mathbf{m})$ comprises the objects belonging to $R_C^{-1}[m] = (\bigcap_{s \in S} R_s)^{-1}[m]$, whereas the description of $C(\mathbf{m})$ comprises the features belonging to $R_C^{-1}[m]^{\uparrow} = ((\bigcap_{s \in S} R_s)^{-1}[m])^{\uparrow}$. The category $C(\mathbf{m})$ can be understood as a socially constructed category, for which the category members (i.e., the extension) and the membership requirements (i.e., the intension or description) are agreed upon by the agents. Clearly, there are fewer such categories than there are candidate categories in the database. This is consistent with the intuition that not all the possible categories whereby the objects or the features could be sorted are actually meaningful to decision-makers in a market [e.g., 70].

We conclude this section with an important consideration about our interpretation of C. Although we describe category emergence as a sociocognitive process [cf. 69], our formalization does not (yet) fully capture epistemic dynamicity. In fact, while we characterize agents as inclined to social interaction, we do not (yet) account for the fact that interaction can amount to more than a simple exchange of beliefs. Through communication, the agents may also end up influencing each other's perspectives: for example, by convincing their peers that certain beliefs are more accurate or simply more useful [cf. 54]. Hence, their epistemic iterations may not only lead to the identification of categories whose meaning is shared but also to updates in beliefs. In our discussion, we will explain how these dynamics can be incorporated in our theory.

4.4. DISCUSSION

In this chapter, we presented a formal theory of classification systems that extends the basic framework of Formal Concept Analysis [44] building on the idea that a database with information about objects and features can be "coerced" into an RS-polarity and hence an RS-frame by enforcing the RS-conditions [cf. 41], which allows for an algebraic representation of classification systems as perfect lattices [40]. As in FCA, this lattice represents the hierarchy of formal concepts generated by objects and features in the database; unlike FCA, however, our theory accounts for the possibility that the agents may have incomplete perceptions of objects, features, and their incidence relation. As a result, they may have idiosyncratic views of the perfect lattice. Moreover, we fashioned an epistemic framework in which some of the concepts or categories are recognized by every agent and through social interaction they are found to be consensual, but some are recognized only by a few agents and some are not recognized by at all, and thus they remain fully implicit in the system. By allowing for such different levels of acknowledgment, our theory captures the distinction between categories that are meaningful and those that are merely possible.

The contribution of this chapter is twofold: To logicians, we offer an original interpretation of RS-semantics in terms of agents' reasoning about objects, about features, and about categories induced by their accompanying relation. We showed that RS-semantics lend themselves well to such an epistemic interpretation, and that this interpretation, in turn, enables a much more intuitive understanding of RS-semantics. To the organization theorists, instead, we offer a logical framework capable of capturing key aspects of categorization in markets, such as the subjective nature of

category representations [59], the role played by audience interaction [54], and the sociocognitive mechanism underlying category emergence [69]. As this framework allows one to compute the consequences of adding or removing objects and features, whether these are new or already associated with some other category, it allows for a very fine-grained analysis of the changes induced by different kinds of innovation [cf. 71, 72].

Although our proposal has a distinctly epistemic character, it differs from standard epistemic logic in at least two respects. First, the relations used to interpret the epistemic operators are intended to capture positive knowledge rather than uncertainty. Second, these relations link objects to features and vice versa rather than possible worlds to one another. We considered two classical principles of epistemic logic, namely factivity and positive introspection of knowledge, and using correspondence theory [26], we computed the relational properties that correspond to these principles. These are necessary and sufficient conditions on the agents' subjective perception of the incidence relation between objects and features that ensure the agents' understanding of categories verifies these common epistemic principles. These conditions may or may not be added to the system, depending on the computational ability and degree of access to reality one wishes to impute to the agents. Many research directions connected to epistemic logic are available for further study: in addition to most standard logical questions concerning axiomatization, proof system, decidability, and complexity, one can ask what is the meaning of other classical epistemic principles, like negative introspection, in our original setting. It is also reasonable to ask whether additional principles exist that should be included in a minimal logic of categorization.

Because this chapter is a foray into the use of RS-frames to model agents' reasoning about categories in real-world contexts, it remains quite general in its assumptions. To be of greater practical relevance, our formalism should be specialized to individual fields of research where categorization is important. These are not necessarily related to the study of organizations or economic activity: for example, RS-frames can also be useful in the study of natural language semantics. This is because the assignments of RS-models support a notion of meaning different from the one normally used in classical modal logic, but this is arguably closer to what categories represent in natural language, that is, semantic groups that can be specified both intensionally and extensionally. In natural language semantics, linguistic utterances are assigned a meaning in the same spirit, generalizing the truth-based semantics of sentences. Categorization is key to the construction of meaning because each word is associated with a category: for this reason,

exploring systematic connections between categories and natural language can be a very fruitful endeavor.

Categories are also central to the study of knowledge representation. Description logics [73] are the dominant paradigm for logical reasoning in this context, but our formalism can provide a complementary perspective on the formal ontologies, classification systems, and taxonomies studied in this field. In particular, the non-distributive nature of category combination thorugh $\wedge$ and $\vee$ and the double interpretation of categories as sets of objects and set of features are foreign to description logics, but they can enhance our understanding of formal ontologies. The question remains of whether some of the expressive features of description logic, e.g., uniqueness quantification and qualified cardinality restrictions, can be accommodated in our framework.

In conclusion, we suggest two possible extensions that concern the application of our framework to the study of categories in organization theory. First, it is necessary to account for the fact that the membership of products and organizations in particular categories is not necessarily crisp: instead, it is often a matter of degree [74]. This calls for quantitative, possibly many-valued generalizations of our semantics. Second, agents' beliefs are not necessarily static: they continuously change as new objects, new features, and new combinations of existing features appear on the market. While our theory accounts for category emergence given a set of agents with certain beliefs, it does not (yet) account for belief updates. Dynamic versions of our formalism ought to be developed in order to properly deal with such inherently changeable systems. In recent work [75, 76], logicians proposed a methodology for developing dynamic versions of nonclassical epistemic logics and successfully applied this to settings in which the agents' beliefs are probabilistic [77]. We plan to apply this methodology to more fully account for category dynamics and track changes in the way the agents perceive them.

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5

TOWARD AN EPISTEMIC LOGIC OF CATEGORIES

5.1. Motivation

In the previous chapter, we introduced a formal theory of classification systems that builds on the fundamental mathematical notion of order [1]. As discussed before, this order-theoretic representation can be of great value to organizational research, where the need for an "ontological turn" [2] in the study of categories has been pointed out with increasing frequency. Yet this framework can also be valuable to other domains of scholarship as the literature on categories is expanding rapidly in many disciplines, motivated by (and in connection with) theories and methodologies that span the social and the exact sciences. In linguistics, for example, categories are traditionally examined for their relationship with grammar [3]; in psychology, they have been analyzed extensively for their role in learning and induction [4–6]; in artificial intelligence, categorization is recognized as key to pattern recognition [7], text mining [8], and knowledge representation in databases [9]; in management science, categories are known to serve as cognitive infrastructures for consumers [10] and managers alike [11, 12], and thus exert an enormous influence on competition [13]

This range of interdisciplinary applications calls for logical formalisms that achieve generality without losing any of their intuitiveness. The framework presented in Chapter 4 may not entirely fulfill these desiderata because it requires technical restrictions on Kripke-style models, i.e., the RS-conditions, which limit its scope to those settings where such conditions can be reasonably enforced. This chapter lays the groundwork for broader applications of our formal theory by considering a simpler and more general class of models for the logic defined in Chapter 4. Because it is free from the RS-conditions, this class of models more naturally allows for the representation of objects, features, and concepts as these occur in the disciplines above. The outcome of our study is a new and improved logical framework that can synthesize various perspectives on categorization and facilitate the transfer of results across settings. We demonstrate the utility of this refined machinery by formalizing certain theoretical constructs from the organizational literature on categories—typicality [14], similarity [15, 16], contrast [17–19], and leniency [20, 21]—that are considered important both in sociology [22] and in management science [23].

The structure of this chapter is as follows: In Section 5.2, we review the foundational insights about categories in cognitive psychology and examine their links with the formal approaches currently available to researchers. In Section 5.3 we define our epistemic logic of categories, for which we introduce a refined Kripke-style semantics and an axiomatization as well as two language enrichments, including a common knowledge-type construc-

tion (cf. Chapter 4), and hybrid-style nominal and co-nominal variables. In Section 5.4 we prove the soundness and completeness of this logic. In Section 5.5, we propose formalizations for the aforementioned theoretical constructs. Finally, in Section 5.6, we discuss additional applications of our formalism and identify directions for further research.

5.2. Theoretical Foundations

5.2.1. Cognitive Perspectives on Categorization

The literature on categories in cognitive psychology provides a multitude of definitions, theories, and models, each of which appears best suited to capture a facet of the complex epistemic mechanisms underlying the formation of categories. The oldest perspective, known as the *classical view* [24], traces its roots back to Aristotle. Although this is arguably the more restrictive approach, it has been very influential in computer science [25]. This view hinges on the idea that all the members of a category have some features in common: hence, categorization is viewed as a deductive process geared toward the satisfaction of necessary and sufficient conditions. This engenders categories with crisp boundaries, where no partial memberships are possible and where all the members are considered to be equally representative. Clearly, this view fails to account for the fact that people can assign objects to particular categories even if they do not consider them to be highly representative category members. For example, most people agree that robins and penguins are members of the category BIRDS, but robins are more typical BIRDS than penguins and thus they are considered "better" category members [4]. Instead of being crisp, category memberships tend to be a matter of degree [cf. 14].

Not only is this view oblivious to the graded structure of categories [26], but it is also restrictive because it requires the agents to know exhaustive lists of features in order to determine category memberships. In most real-world settings, this assumption is unwarranted: human beings must cope with considerable cognitive limitations and they are unable to recall long lists of feature requirements. Such considerations motivated the development of *prototype theory*, which is especially represented by the work of Rosch [27]. According to this perspective, categorization is an inductive process that requires finding the closest match between an object and a cognitive reference point stored in the agent's memory [28, 29]. Many limitations of the classical view are addressed by relaxing the requirement that category memberships be decided via the satisfaction of necessary and sufficient conditions: as a result, prototype theory allows for partial

and ambiguous cases. Nevertheless, this theory has been found wanting because it requires prototypes to be defined *ex ante* and does not fully explain how these reference points are acquired by the agents. To explain how categorization occurs in the absence of prototypes, cognitive psychologists proposed the *exemplar view* [30]. In this view, category memberships are determined by comparing newly encountered objects with instances of concepts that accumulate in the agent's memory by way of experience. Prototypes for concepts may emerge over time as the agent's understanding of the membership requirements consolidates [31].

Although the exemplar view does not necessitate *ex ante* reference points to explain how people sort objects, it is still grounded on the notion of similarity. As noted by Douglas Medin [32], similarity-based accounts of categorization have the power to explain how categories are internally structured but do not fully explain why "we have the categories we have," or why some categories seem to be more cogent and coherent than others. Moreover, both prototype and exemplar theory assume similarity to be a determinant of conceptual coherence rather than its consequence: in some cases, however, the similarity between objects may be imposed rather than discovered [e.g., 33]. For this reason, it cannot be regarded as the sole criterion for determining category memberships [34].

Pivoting on the notion of imposed coherence, the *theory-based view* [35] maintains that categories arise in connection with theories (broadly understood so as to include informal explanations). The coherence of a category in the eyes of an agent follows from the coherence of the theory whereon the category is constructed. This allows the agent to sort together objects that would be considered too distant from a similarity-based perspective: for example, an agent may lump together objects like a gold watch, a family portrait, and a deed to a piece of land into the category of THINGS SHE WISHES HER CHILD TO INHERIT. Goal-based categories (cf. Chapter 3) can be considered a specific kind of categories that arise from theories, namely the agents' theories of what objects are suited to particular goals [36, 37]. Although this view allows for considerable freedom in determining category memberships, it can lead to a circularity problem because the categories themselves can serve as building blocks in theory formation. The theory of what constitutes a HEIRLOOM, for example, may itself depend on categories like FAMILY and PRIVATE PROPERTY.

In summary, the extant theoretical insights on categorization in cognitive psychology focus on different aspects of this complex phenomenon and they are difficult to reconcile into a satisfactory, overarching perspective. This chapter represents an early step within a broader research program

aimed at clarifying the notions related to categorization and developing a unified theory. An important step toward this goal is defining an adequate mathematical representation of categories or concepts, upon which an adequate semantic framework can be built that is capable of describing categories regardless of how they are formed. In the next section, we discuss two of the most promising mathematical approaches.

5.2.2. EXTANT FORMAL APPROACHES

The formal method for category representation that is perhaps most widely adopted in sociology [16, 22, 38–41] is the one developed by Peter Gärdenfors [42, 43] and based on the notion of conceptual spaces. These are multidimensional metric spaces where each axis or dimension represents a feature along which the objects in a particular domain, like items of clothing [44], software companies [38], or pieces of music [45], can meaningfully differ according to the agents. Concepts, or formal categories, are modeled according to the rules of similarity: the smaller the distance between any two objects, the greater the likelihood that the objects are considered instances of the same concept. From this perspective, categories can be viewed as convex subsets or regions of the conceptual space [46].[1] The geometric center of any such region corresponds to the category prototype, and if an object is closer to this geometric average, it tends to be perceived as a more typical (representative) member of the category.

Another approach, still relatively foreign to sociologists, is the one pioneered by Bernhard Ganter and Rudolf Wille [47] and termed Formal Concept Analysis (FCA). This builds directly on Birkhoff's theory of complete lattices [48]: in FCA, databases are viewed as *formal contexts*, i.e., structures (A, X, I) such that A and X are sets and $I \subseteq A \times X$. Intuitively, A can be interpreted as a set of objects, X can be interpreted as a set of features, and for any object $a \in A$ and feature $x \in X$, the tuple $(a, x) \in I$ exactly when object a has feature x. Every formal context is associated with a collection of formal concepts (or categories): as explained in Chapter 4, these are tuples (B, Y) such that $B \subseteq A$, $Y \subseteq X$, and $B \times Y$ is a maximal rectangle included in I.[2] The set B is commonly referred to as the extension of the formal concept, whereas Y is referred to as its intension or description.[3] Because of maximality, the extension of a formal concept uniquely identifies

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[1]Recall that a region of space is convex if it includes the segments between any two of its points (cf. Chapter 3). In a Euclidean plane, squares are convex but stars are not.

[2]That is, one cannot enlarge B to B' or Y to Y' in such a way that $B' \times Y \in I$ or $B \times Y' \in I$.

[3]This is an informal way to account for the genesis of Galois-stable sets (cf. Chapter 4, Definition 4.2.1).

and is identified by its description. We say that an object is a member of a category if it belongs to the category's extension and that a feature describes a category if it belongs to the category's description.

Compared to conceptual spaces, the FCA framework is more general in nature and more strongly oriented towards qualitative comparisons between objects or between categories. The reason why FCA generalizes conceptual spaces can be explained in precise mathematical terms. Formal concepts in conceptual spaces are convex closures of regions: therefore, they represent the fixed points of a special closure operator (the convex closure) that arises from the metric of the space. As discussed in Chapter 4, the formation of concepts in FCA is also modeled via a closure operator (the Galois closure); however, this is more general and it is defined in purely order-theoretic terms. As to why FCA provides a more natural support for qualitative comparisons, notice that formal concepts are by nature partially ordered: namely, (B, Y) is a *subconcept* of (C, Z) exactly when $B \subseteq C$, or equivalently, when $Z \subseteq Y$.[4] Moreover, objects can be ordered in terms of the features they have, whereas features can be ordered in terms of the objects that share them. An object is regarded as more specific than another if it possesses all of its features, as well as some others, while a feature is regarded as more generic than another if it is shared by all of the objects that share the other, as well as some more. This (partial) ordering makes FCA better equipped than conceptual spaces to examine categories as elements of a multilevel classification system [cf. 49].

In Chapter 4, we developed a formal connection between FCA and modal logic based on the idea that formal contexts enriched with additional relations can be taken as models of an epistemic modal logic of categories. The formulas of this logic are constructed out of a set of atomic variables using the standard positive propositional connectives $\wedge, \vee, \top, \bot$ and modal operators $\square_i$ associated with each agent $i \in$ Ag. The formulas thusly generated do not denote states of affairs to which a truth value can be assigned, but rather categories or concepts. In this modal language, it is easy to distinguish between the objective or factual information agents have access to, which is encoded by the formulas of the modal-free fragment of the language, and their subjective interpretation of this information, which is encoded by the formulas with modal operators. One can easily describe an agent's beliefs about the category that another agent believes to be the category that yet another agent believes to be the category of, e.g., CLASSICAL

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[4]In this sense, the concepts of a formal context represent a complete lattice, meaning that any collection of formal concepts has a least upper bound and a greatest lower bound. By Birkhoff's theorem [48], every complete lattice is isomorphic to some concept lattice.

MUSIC. We defined fixed points of these iterations in similar way as common knowledge is defined in classical epistemic logic [50], and proposed an interpretation of this common knowledge-type construction as the end point in a process of social interaction; for example, the consensus reached by a group of agents about the intension and extension of CLASSICAL MUSIC. In this chapter, we will further elaborate on this interpretation and connect it to the notion of typicality.

5.3. BUILDING AN EPISTEMIC-LOGICAL LANGUAGE

5.3.1. BASIC LOGIC AND INTENDED MEANING

Let Prop be a finite set of atomic propositions and Ag be a finite set of agents. The basic language Ł of our epistemic logic of categories is:

$$\varphi := \bot \mid \top \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box_i \varphi, \tag{5.1}$$

where $p \in$ Prop. As mentioned above, the formulas in this language denote categories or concepts. While the atomic propositions provide a vocabulary of category labels, such as CLASSICAL MUSIC, compound formulas $\varphi \wedge \varphi$ and $\varphi \vee \psi$ respectively denote the greatest common subordinate category and the smallest common superordinate category of φ and ψ. For any agent $i \in$ Ag, the formula $\Box_i \varphi$ denotes what is category φ according to i. At this stage, we are deliberately vague as to the precise meaning of "according to." Depending on the properties of $\Box_i$, the formula $\Box_i \varphi$ can denote the category i believes or perceives to be φ. The selection of a particular meaning can be left up to the specifics of the context to which our logic is deployed: in sociological applications, agents' beliefs are especially important [cf. 51], but with regard to psychology and cognition, the notion of perception may be decidedly more relevant [cf. 52].

The *basic* or *minimal normal $\mathcal{L}$-logic* is a set **L** of sequents $\varphi \vdash \psi$ (which reads "φ is a subordinate category of ψ") with $\varphi, \psi \in \mathcal{L}$, containing

(a) the following sequents for propositional connectives:

$$p \vdash p, \tag{5.2}$$
$$\bot \vdash p, \tag{5.3}$$
$$p \vdash \top, \tag{5.4}$$
$$p \vdash p \vee q, \tag{5.5}$$
$$q \vdash p \vee q, \tag{5.6}$$
$$p \wedge q \vdash p, \tag{5.7}$$
$$p \wedge q \vdash q; \tag{5.8}$$

(b) the following sequents for modal operators:

$$\top \vdash \Box_i \top, \tag{5.9}$$

$$\Box_i p \wedge \Box_i q \vdash \Box_i (p \wedge q); \tag{5.10}$$

and closed under the following inference rules:

$$\frac{\varphi \vdash \chi \quad \chi \vdash \psi}{\varphi \vdash \psi}, \tag{5.11}$$

$$\frac{\varphi \vdash \psi}{\varphi (\chi/p) \vdash \psi (\chi/p)}, \tag{5.12}$$

$$\frac{\chi \vdash \varphi \quad \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi}, \tag{5.13}$$

$$\frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}, \tag{5.14}$$

$$\frac{\varphi \vdash \psi}{\Box_i \varphi \vdash \Box_i \psi}. \tag{5.15}$$

The modal fragment of **L** incorporates agents' beliefs (or perceptions) into the syllogistic reasoning supported by the propositional fragment of **L**. By an *$\mathcal{L}$-logic*, we refer to any extension of **L** with $\mathcal{L}$-axioms $\varphi \vdash \psi$.

5.3.2. Interpretation in Enriched Formal Contexts

We now turn to discussing the semantic structures that, in our formal theory, play the role of Kripke frames. An *enriched formal context* is defined as a tuple $\mathbb{F} = (\mathbb{P}, \{R_i \mid i \in \mathrm{Ag}\})$ such that $\mathbb{P} = (A, X, I)$ is a formal context and $R_i \subseteq A \times X$ for every $i \in \mathrm{Ag}$ satisfying certain additional properties, which guarantee their associated modal operators are well defined (cf. Definition 5.4.6). As mentioned above, formal contexts represent databases that contain information about objects $a \in A$ and features $x \in X$, as well as an incidence relation $I \subseteq A \times X$. Intuitively, aIx reads "object a has feature x." In addition to this factual relation, enriched formal contexts contain information about the epistemic attitudes of individual agents. Thus, $aR_i x$ reads "object a has feature x according to agent i."

We define *valuation* on $\mathbb{F}$ as a map $V : \mathrm{Prop} \to \mathcal{P}(A) \times \mathcal{P}(B)$, with the restriction that $V(p)$ is a formal concept of $\mathbb{P} = (A, X, I)$. This means that every $p \in \mathrm{Prop}$ maps to $V(p) = (B, Y)$ in such a way that $B \subseteq A$, $Y \subseteq X$, and $B \times Y$ is a maximal rectangle contained in I. For example, if p is a category label denoting CLASSICAL MUSIC and $\mathbb{P}$ is a database of musical tracks (stored in A) and musical features (stored in X), then V interprets the category

label p in the model $\mathbb{M} = (\mathbb{F}, V)$ as the formal concept $V(p) = (B, Y)$ that is equivalently specified by the set of tracks B, i.e., all the CLASSICAL tracks in the database, and by the set of features Y, i.e., all the features in the database shared by CLASSICAL tracks. The elements of B are the members of p in $\mathbb{M}$, whereas the elements of Y describe p in $\mathbb{M}$. The set B (resp. Y) is the extension (resp. intension or description) of p in $\mathbb{M}$, and we denote it $[\![p]\!]_{\mathbb{M}}$ (resp. $\{\![p]\!\}_{\mathbb{M}}$) or simply $[\![p]\!]$ (resp. $\{\![p]\!\}$). Alternatively, we write:

$$\mathbb{M}, a \Vdash p \quad \text{iff} \quad a \in [\![p]\!]_{\mathbb{M}}, \tag{5.16}$$

$$\mathbb{M}, x > p \quad \text{iff} \quad x \in \{\![p]\!\}_{\mathbb{M}}, \tag{5.17}$$

and read $\mathbb{M}, a \Vdash p$ as "object a is a member of category p," and $\mathbb{M}, x > p$ as "feature x describes category p."

The interpretation of atomic propositions can be extended to propositional $\mathcal{L}$-formulas as follows (cf. Table 4.1):

$$\mathbb{M}, a \Vdash \top \qquad \text{always}, \tag{5.18}$$

$$\mathbb{M}, x > \top \qquad \text{iff} \quad a I x \text{ for all } a \in A, \tag{5.19}$$

$$\mathbb{M}, x > \bot \qquad \text{always}, \tag{5.20}$$

$$\mathbb{M}, a \Vdash \bot \qquad \text{iff} \quad a I x \text{ for all } x \in X, \tag{5.21}$$

$$\mathbb{M}, a \Vdash \varphi \wedge \psi \qquad \text{iff} \quad \mathbb{M}, a \Vdash \varphi \text{ and } \mathbb{M}, a \Vdash \psi, \tag{5.22}$$

$$\mathbb{M}, x > \varphi \wedge \psi \qquad \text{iff} \quad \text{for all } a \in A, \text{ if } \mathbb{M}, a \Vdash \varphi \wedge \psi \text{ then } a I x, \tag{5.23}$$

$$\mathbb{M}, x > \varphi \vee \psi \qquad \text{iff} \quad \mathbb{M}, x > \varphi \text{ and } \mathbb{M}, x > \psi, \tag{5.24}$$

$$\mathbb{M}, a \Vdash \varphi \vee \psi \qquad \text{iff} \quad \text{for all } x \in X, \text{ if } \mathbb{M}, x > \varphi \vee \psi \text{ then } a I x. \tag{5.25}$$

Therefore, in each model $\mathbb{M}$, $\top$ is interpreted as the category generated by the set of all the objects, i.e., the broadest possible category or the one with the laxest (possibly empty) description, whereas $\bot$ is interpreted as the category generated by the set of all the features, i.e., the most specific (possibly empty) category or the one with the most restrictive description. Further, $\varphi \wedge \psi$ is interpreted as the category generated by the intersection of the extensions of φ and ψ; hence, the description of $\varphi \wedge \psi$ certainly includes $\{\![\varphi]\!\} \cup \{\![\psi]\!\}$, but it can also be larger. Conversely, $\varphi \vee \psi$ is interpreted as the category generated by the intersection of the intensions of φ and ψ; hence, the objects in $[\![\varphi]\!] \cup [\![\psi]\!]$ are certainly members of $\varphi \vee \psi$, but there can also be others. As to the interpretation of modal formulas:

$$\mathbb{M}, a \Vdash \Box_i \varphi \quad \text{iff} \quad \text{for all } x \in X, \text{ if } \mathbb{M}, x > \varphi \text{ then } a R_i x, \tag{5.26}$$

$$\mathbb{M}, x > \Box_i \varphi \quad \text{iff} \quad \text{for all } a \in A, \text{ if } \mathbb{M}, a \Vdash \Box \varphi \text{ then } a I x. \tag{5.27}$$

Therefore, in each model $\mathbb{M}$, $\square_i \varphi$ is interpreted as the category whose members are the objects to which agent i attributes every feature in the description of φ. Finally, as to the interpretation of sequents:

$$\mathbb{M} \models \varphi \vdash \psi \quad \text{iff} \quad \text{for all } a \in A, \text{ if } \mathbb{M}, a \Vdash \varphi \text{ then } \mathbb{M}, a \Vdash \psi. \tag{5.28}$$

5.3.3. INTRODUCING COMMON KNOWLEDGE

In Chapter 4, we observed that the logical framework we presented is well-equipped to capture not only the factual information and the epistemic attitudes of individual agents, but also the outcome of social interaction. To this end, we introduced an expansion $\mathcal{L}_C$ of $\mathcal{L}$ with a common knowledge-type operator C. Given Prop and Ag as above, the language $\mathcal{L}_C$ of the epistemic logic of categories with "common knowledge" is:

$$\varphi := \bot \mid \top \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \square_i \varphi \mid C(\varphi). \tag{5.29}$$

and C-formulas are interpreted in each model $\mathbb{M}$ as follows:

$$\mathbb{M}, a \Vdash C(\varphi) \quad \text{iff} \quad \text{for all } x \in X, \text{ if } \mathbb{M}, x \succ \varphi \text{ then } a R_C x, \tag{5.30}$$

$$\mathbb{M}, x \succ C(\varphi) \quad \text{iff} \quad \text{for all } a \in A, \text{ if } \mathbb{M}, a \Vdash C(\varphi) \text{ then } a I x, \tag{5.31}$$

where $R_C \subseteq A \times X$ is defined as $R_C = \bigcap_{s \in S} R_s$, and $R_s \subseteq A \times X$ is the relation associated with the modal operator $\square_s := \square_{i_1} \ldots \square_{i_n}$ for any element $s = i_1 \ldots i_n$ in the set S of finite sequences of elements of Ag.

The basic logic of categories with "common knowledge" is a set $\mathbf{L}_C$ of sequents $\varphi \vdash \psi$, with $\varphi, \psi \in \mathcal{L}_C$, which contains the axioms and is closed under the rules of $\mathbf{L}$, and further contains the following axioms:

$$\top \vdash C(\top), \tag{5.32}$$

$$C(p) \wedge C(q) \vdash C(p \wedge q), \tag{5.33}$$

$$C(p) \vdash \bigwedge \{\square_i p \wedge \square_i C(p) \mid i \in \text{Ag}\}; \tag{5.34}$$

and is closed under the following rules:

$$\frac{\varphi \vdash \psi}{C(\varphi) \vdash C(\psi)}, \tag{5.35}$$

$$\frac{\chi \vdash \bigwedge_{i \in \text{Ag}} \square_i \varphi \quad \{\chi \vdash \square_i \chi \mid i \in \text{Ag}\}}{\chi \vdash C(\varphi)}. \tag{5.36}$$

5.3.4. Hybrid Expansions of the Basic Language

In many settings, it is convenient to be able to reason about sets of objects or sets of features even if these do not constitute real categories in the sense described in Chapter 4 and are not assigned any particular label. For this reason, the languages $\mathcal{L}$ and $\mathcal{L}_C$ can be further enriched with dedicated sets of variables in the style of hybrid logic. As before, let Prop be a finite set of atomic propositions and Ag be a finite set of agents. Given Prop, Ag, and the finite sets Nom and CoNom of nominals and co-nominals, respectively, the language $\mathcal{L}_H$ of the hybrid logic of categories is:

$$\varphi := \bot \mid \top \mid p \mid \mathbf{a} \mid \mathbf{x} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box_i \varphi, \tag{5.37}$$

where $i \in$ Ag, $p \in$ Prop, $\mathbf{a} \in$ Nom, and $\mathbf{x} \in$ CoNom.

A *hybrid valuation* on an enriched formal concept $\mathbb{F}$ maps atomic propositions to formal concepts, nominal variables to the formal concepts generated by individual elements of the object domain A, and co-nominal variables to the formal concepts generated by individual elements of the feature domain X. Denoting $V(\mathbf{a})$ as the category generated by $a \in A$, and $V(\mathbf{x})$ as the category generated by $x \in X$, nominals and co-nominals are interpreted as follows:

$$\mathbb{M}, y \succ \mathbf{a} \quad \text{iff} \quad a I y, \tag{5.38}$$
$$\mathbb{M}, b \Vdash \mathbf{a} \quad \text{iff} \quad \text{for all } y \in X, \text{ if } a I y \text{ then } b I y, \tag{5.39}$$
$$\mathbb{M}, y \succ \mathbf{x} \quad \text{iff} \quad \text{for all } b \in A, \text{ if } b I x \text{ then } b I y, \tag{5.40}$$
$$\mathbb{M}, b \Vdash \mathbf{x} \quad \text{iff} \quad b I x. \tag{5.41}$$

5.4. Soundness and Completeness

5.4.1. Definition of I-Compatible Relations

Throughout this section, we fix two sets A and X, and use a, b (resp. x, y) for elements of A (resp. X), and B, C, A_j (resp. Y, W, X_j) for subsets of A (resp. X). For any relation $S \subseteq A \times X$, let

$$S^{\uparrow}[B] := \{x \mid \forall a \, (a \in B \Rightarrow a S x)\}, \tag{5.42}$$
$$S^{\downarrow}[Y] := \{a \mid \forall x \, (x \in Y \Rightarrow a S x)\}. \tag{5.43}$$

The following lemma recaps some well-known properties of this construction [cf. 1, Sections 7.22–7.29]:

Lemma 5.4.1.

1. *$B \subseteq C$ implies $S^{\uparrow}[C] \subseteq S^{\uparrow}[B]$, and $Y \subseteq W$ implies $S^{\downarrow}[W] \subseteq S^{\downarrow}[Y]$.*

2. $B \subseteq S^{\downarrow}\left[S^{\uparrow}[B]\right]$ and $Y \subseteq S^{\uparrow}\left[S^{\downarrow}[Y]\right]$.

3. $S^{\uparrow}[B] = S^{\uparrow}\left[S^{\downarrow}\left[S^{\uparrow}[B]\right]\right]$ and $S^{\downarrow}[Y] = S^{\downarrow}\left[S^{\uparrow}\left[S^{\downarrow}[Y]\right]\right]$.

4. $S^{\downarrow}[\bigcup \mathcal{Y}] = \bigcap_{Y \in \mathcal{Y}} S^{\downarrow}[Y]$ and $S^{\uparrow}[\bigcup \mathcal{B}] = \bigcap_{B \in \mathcal{B}} S^{\uparrow}[B]$.

For any formal context $\mathbb{P} = (A, X, I)$, we sometimes use $B^{\uparrow}$ for $I^{\uparrow}[B]$ and $Y^{\downarrow}$ for $I^{\downarrow}[Y]$, and say that B (resp. Y) is *Galois-stable* if $B = B^{\uparrow\downarrow}$ (resp. $Y = Y^{\downarrow\uparrow}$). When $B = \{a\}$ (resp. $Y = \{x\}$), we write $a^{\uparrow\downarrow}$ for $\{a\}^{\uparrow\downarrow}$ (resp. $x^{\downarrow\uparrow}$ for $\{x\}^{\downarrow\uparrow}$). The Galois-stable sets represent projections of some maximal rectangle (i.e., formal concept) of $\mathbb{P}$. The lemma below reports additional facts:

Lemma 5.4.2.

1. $B^{\uparrow}$ and $Y^{\downarrow}$ are Galois-stable.

2. $B = \bigcup_{a \in B} a^{\uparrow\downarrow}$ and $Y = \bigcup_{y \in Y} y^{\downarrow\uparrow}$ for any Galois-stable B and Y.

3. Galois-stable sets are closed under arbitrary intersections.

Proof. With regard to the second item, because $a^{\uparrow\downarrow} \supseteq \{a\}$, we have that $B \subseteq \bigcup_{a \in B} a^{\uparrow\downarrow}$. For the other direction, if $\{a\} \subseteq B$ then $a^{\uparrow\downarrow} \subseteq B^{\uparrow\downarrow}$. Because B is Galois-stable, we have that $B = B^{\uparrow\downarrow}$. Hence, $a^{\uparrow\downarrow} \subseteq B$ for any $a \in B$, which implies that $\bigcup_{a \in B} a^{\uparrow\downarrow} \subseteq B$. The proof for Y is analogous. ☐

Definition 5.4.3. *For any $\mathbb{P} = (A, X, I)$, any $R \subseteq A \times X$ is I-compatible if $R^{\downarrow}[x]$ and $R^{\uparrow}[a]$ are Galois-stable for all x and all a.*

By Lemma 5.4.1 (3), I is an I-compatible relation.

Lemma 5.4.4. *If $R \subseteq A \times X$ is I-compatible, then $R^{\downarrow}[Y] = R^{\downarrow}[Y^{\downarrow\uparrow}]$ and $R^{\uparrow}[B] = R^{\uparrow}[B^{\uparrow\downarrow}]$.*

Proof. By Lemma 5.4.1 (2) we have $Y \subseteq Y^{\downarrow\uparrow}$, which implies by Lemma 5.4.1 (1) that $R^{\downarrow}[Y^{\downarrow\uparrow}] \subseteq R^{\downarrow}[Y]$. Conversely, if $a \in R^{\downarrow}[Y]$, i.e., $Y \subseteq R^{\uparrow}[a]$, then $Y^{\downarrow\uparrow} \subseteq (R^{\uparrow}[a])^{\downarrow\uparrow} = R^{\uparrow}[a]$, with the last identity holding because R is I-compatible. Thus, $a \in R^{\downarrow}[Y^{\downarrow\uparrow}]$. The proof of the second identity is analogous. ☐

Lemma 5.4.5. *If R is I-compatible and Y is Galois-stable, then $R^{\downarrow}[Y]$ is Galois-stable.*

Proof. Because $Y = \bigcup_{y \in Y}\{y\}$, by Lemma 5.4.1 (4),

$$R^{\downarrow}[Y] = R^{\downarrow}\left[\bigcup_{y \in Y} \{y\}\right] = \bigcap_{y \in Y} R^{\downarrow}[\{y\}] = \bigcap_{y \in Y} R^{\downarrow}[y]. \qquad (5.44)$$

By the I-compatibility of R, the last term is an intersection of Galois-stable sets, which is itself Galois-stable (cf. Lemma 5.4.2, 3). □

The lemma above ensures that the interpretation of $\mathcal{L}$-formulas on enriched formal contexts defines a compositional semantics on formal concepts if the relations R_i are I-compatible. Indeed, for every enriched formal context $\mathbb{F} = (\mathbb{P}, \{R_i \mid i \in \text{Ag}\})$, every valuation V on $\mathbb{F}$ extends to an interpretation map of $\mathcal{L}$-formulas defined as follows:

$$V(p) = (\llbracket p \rrbracket, \{\!\{p\}\!\}), \tag{5.45}$$

$$V(\top) = \left(A, A^{\uparrow}\right), \tag{5.46}$$

$$V(\bot) = \left(X^{\downarrow}, X\right), \tag{5.47}$$

$$V(\varphi \wedge \psi) = \left(\llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket, (\llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket)^{\uparrow}\right), \tag{5.48}$$

$$V(\varphi \vee \psi) = \left((\{\!\{\varphi\}\!\} \cap \{\!\{\psi\}\!\})^{\downarrow}, \{\!\{\varphi\}\!\} \cap \{\!\{\psi\}\!\}\right), \tag{5.49}$$

$$V(\Box_i \varphi) = \left(R_i^{\downarrow}[\{\!\{\varphi\}\!\}], \left(R_i^{\downarrow}[\{\!\{\varphi\}\!\}]\right)^{\uparrow}\right). \tag{5.50}$$

By Lemma 5.4.5, if $V(\varphi)$ is a formal concept, then so is $V(\Box_i \varphi)$.

Definition 5.4.6. *An enriched formal context* $\mathbb{F} = (\mathbb{P}, \{R_i \mid i \in \text{Ag}\})$ *is compositional if* R_i *is* I-*compatible (cf. Definition 5.4.3) for every* $i \in$ Ag. *A model* $\mathbb{M} = (\mathbb{F}, V)$ *is compositional if so is* $\mathbb{F}$.

5.4.2. INTERPRETATION OF C

For any formal context $\mathbb{P} = (A, X, I)$ the I-*product* of the relations $R_s, R_t \subseteq A \times X$ is the relation $R_{st} \subseteq A \times X$ defined as follows:

$$a \in R_{st}^{\downarrow}[x] \text{ iff } a \in R_s^{\downarrow}\left[I^{\uparrow}\left[R_t^{\downarrow}\left[x^{\downarrow\uparrow}\right]\right]\right]. \tag{5.51}$$

Lemma 5.4.7. *If* R_s *and* R_t *are* I-*compatible, then* R_{st} *is* I-*compatible.*

Proof. That $R_{st}^{\downarrow}[x]$ is Galois-stable follows from the definition of R_{st}, from Lemma 5.4.5, and from the I-compatibility of R_s and R_t. To show that $R_{st}^{\uparrow}[a]$ is Galois-stable, i.e., that $(R_{st}^{\uparrow}[a])^{\downarrow\uparrow} \subseteq R_{st}^{\uparrow}[a]$, by Lemma 5.4.2 (2) it suffices to show that if $y \in R_{st}^{\uparrow}[a]$ then $y^{\downarrow\uparrow} \subseteq R_{st}^{\uparrow}[a]$. Let $y \in R_{st}^{\uparrow}[a]$, i.e., $a \in R_{st}^{\downarrow}[y] = R_s^{\downarrow}[I^{\uparrow}[R_t^{\downarrow}[y^{\downarrow\uparrow}]]]$. If $x \in y^{\downarrow\uparrow}$, then $x^{\downarrow\uparrow} \subseteq y^{\downarrow\uparrow}$, which by the antitonicity of $R_s^{\downarrow}$, $I^{\uparrow}$, and $R_t^{\downarrow}$ (cf. Lemma 5.4.1, 1) implies that $R_s^{\downarrow}[I^{\uparrow}[R_t^{\downarrow}[y^{\downarrow\uparrow}]]] \subseteq R_s^{\downarrow}[I^{\uparrow}[R_t^{\downarrow}[x^{\downarrow\uparrow}]]]$. Therefore, $a \in R_{st}^{\downarrow}[x]$, i.e., $x \in R_{st}^{\uparrow}[a]$, as required. □

The definition of I-product serves to semantically characterize the relation associated with the modal operators $\Box_s := \Box_{i_1} \ldots \Box_{i_n}$ for any finite non-empty sequence $s := i_1 \ldots i_n \in S$ of elements of Ag in terms of the relations associated with each primitive modal operator. For any such s, let R_s be defined recursively as follows: if $s = i$, then $R_s = R_i$, and if $s = it$, then $R_s^{\downarrow}[x] = R_i^{\downarrow}[I^{\uparrow}[R_t^{\downarrow}[x^{\downarrow\uparrow}]]]$. Lemma 5.4.7 immediately implies:

Corollary 5.4.8. *For every $s \in S$, the relation R_s is I-compatible.*

Lemma 5.4.9. *If Y is Galois-stable and R_s, R_t are I-compatible, then $R_{st}^{\downarrow}[Y] = R_s^{\downarrow}[I^{\uparrow}[R_t^{\downarrow}[Y]]]$.*

Proof.

$$R_s^{\downarrow}\left[I^{\uparrow}\left[R_t^{\downarrow}[Y]\right]\right] = R_s^{\downarrow}\left[I^{\uparrow}\left[R_t^{\downarrow}\left[\bigcup_{x \in Y} x^{\downarrow\uparrow}\right]\right]\right] \qquad \text{Lemma 5.4.2 (2)} \quad (5.52)$$

$$= R_s^{\downarrow}\left[I^{\uparrow}\left[\bigcap_{x \in Y} R_t^{\downarrow}\left[x^{\downarrow\uparrow}\right]\right]\right] \qquad \text{Lemma 5.4.1 (4)} \quad (5.53)$$

$$= R_s^{\downarrow}\left[I^{\uparrow}\left[\bigcap_{x \in Y} I^{\downarrow}\left[I^{\uparrow}\left[R_t^{\downarrow}\left[x^{\downarrow\uparrow}\right]\right]\right]\right]\right] \qquad\qquad (5.54)$$

$$= R_s^{\downarrow}\left[I^{\uparrow}\left[I^{\downarrow}\left[\bigcup_{x \in Y} I^{\uparrow}\left[R_t^{\downarrow}\left[x^{\downarrow\uparrow}\right]\right]\right]\right]\right] \qquad \text{Lemma 5.4.1 (4)} \quad (5.55)$$

$$= R_s^{\downarrow}\left[I^{\uparrow}\left[I^{\downarrow}\left[\bigcup_{x \in Y} I^{\uparrow}\left[R_t^{\downarrow}\left[x^{\downarrow\uparrow}\right]\right]\right]\right]\right] \qquad \text{Lemma 5.4.1 (4)} \quad (5.56)$$

$$= R_s^{\downarrow}\left[\bigcup_{x \in Y} I^{\uparrow}\left[R_t^{\downarrow}\left[x^{\downarrow\uparrow}\right]\right]\right] \qquad \text{Lemma 5.4.4} \quad (5.57)$$

$$= \bigcap_{x \in Y} R_s^{\downarrow}\left[I^{\uparrow}\left[R_t^{\downarrow}\left[x^{\downarrow\uparrow}\right]\right]\right] \qquad \text{Lemma 5.4.1 (4)} \quad (5.58)$$

$$= \bigcap_{x \in Y} R_{st}^{\downarrow}[x] \qquad \text{Definition of } R_{st} \quad (5.59)$$

$$= R_{st}^{\downarrow}\left[\bigcup_{x \in Y} x\right] \qquad \text{Lemma 5.4.1 (4)} \quad (5.60)$$

$$= R_{st}^{\downarrow}[Y]. \qquad Y = \bigcup_{x \in Y} x \quad (5.61)$$

Identity 5.54 follows from the fact that $R_t^{\downarrow}[x^{\downarrow\uparrow}]$ is Galois-stable. $\qquad\square$

Lemma 5.4.10. *If R_s, R_t, R_w are I-compatible, then $R_{s(tw)} = R_{(st)w}$.*

Proof. For any x,

$$R^{\downarrow}_{s(tw)}[x] = R^{\downarrow}_s\left[I^{\uparrow}\left[R^{\downarrow}_{tw}\left[x^{\downarrow\uparrow}\right]\right]\right] \qquad\qquad \text{Equation 5.51} \qquad (5.62)$$

$$= R^{\downarrow}_s\left[I^{\uparrow}\left[R^{\downarrow}_t\left[I^{\uparrow}\left[R^{\downarrow}_w\left[x^{\downarrow\uparrow}\right]\right]\right]\right]\right] \qquad \text{Lemma 5.4.9} \qquad (5.63)$$

$$= R^{\downarrow}_{st}\left[I^{\uparrow}\left[R^{\downarrow}_w\left[x^{\downarrow\uparrow}\right]\right]\right] \qquad\qquad \text{Lemma 5.4.9} \qquad (5.64)$$

$$= R^{\downarrow}_{(st)w}[x]. \qquad\qquad\qquad\qquad \text{Equation 5.51} \qquad (5.65)$$

□

Let $s = i_1 \ldots i_n \in S$ and $\square_s := \square_{i_1} \ldots \square_{i_n}$.

Lemma 5.4.11. *For any model $\mathbb{M} = (\mathbb{F}, V)$,*

$$\mathbb{M}, a \Vdash \square_s\varphi \quad \textit{iff} \quad \textit{for all } x \in X, \textit{ if } \mathbb{M}, x > \varphi, \textit{ then } aR_s x, \qquad (5.66)$$
$$\mathbb{M}, x > \square_s\varphi \quad \textit{iff} \quad \textit{for all } a \in A, \textit{ if } \mathbb{M}, a \Vdash \square_s\varphi, \textit{ then } aIx. \qquad (5.67)$$

Proof. By induction on the length of s. The base case of the induction is immediate: let $s = it$; it follows that $[\![\square_i\square_t\varphi]\!] = R^{\downarrow}_i[\{\![\square_t\varphi]\!\}] = R^{\downarrow}_i[I^{\uparrow}[[\![\square_t\varphi]\!]]] = R^{\downarrow}_i[I^{\uparrow}[R^{\downarrow}_t[\{\![\varphi]\!\}]]] = R^{\downarrow}_s[\{\![\varphi]\!\}]$. The last equality holds by Lemma 5.4.9. The second item of the lemma is trivially true. □

Lemma 5.4.12. *For any family $\mathcal{R}$ of I-compatible relations,*

1. *$\bigcap\mathcal{R}$ is an I-compatible relation.*

2. *$(\bigcap\mathcal{R})^{\downarrow}[Y] = \bigcap_{T\in\mathcal{R}} T^{\downarrow}[Y]$ for any $Y \subseteq X$.*

Proof. As to the first item, let $R = \bigcap\mathcal{R}$. It follows that $R^{\downarrow}[x] = \bigcap_{T\in\mathcal{R}} T^{\downarrow}[x]$ and $R^{\uparrow}[a] = \bigcap_{T\in\mathcal{R}} T^{\uparrow}[a]$. Thus, the statement follows from Lemma 5.4.2 (3). As to the second item,

$$\bigcap_{T\in\mathcal{R}} T^{\downarrow}[Y] = \bigcap_{T\in\mathcal{R}} T^{\downarrow}\left[\bigcup_{y\in Y} y\right] \qquad Y = \bigcup_{y\in Y} y \qquad (5.68)$$

$$= \bigcap_{T\in\mathcal{R}}\bigcap_{y\in Y} T^{\downarrow}[y] \qquad \text{Lemma 5.4.1 (4)} \qquad (5.69)$$

$$= \bigcap_{y \in Y} \bigcap_{T \in \mathcal{R}} T^{\downarrow}[y] \tag{5.70}$$

$$= \bigcap_{y \in Y} \left(\bigcap \mathcal{R} \right)^{\downarrow}[y] \qquad \text{Definition of } (\cdot)^{\downarrow} \tag{5.71}$$

$$= \left(\bigcap \mathcal{R} \right)^{\downarrow} \left[\bigcup_{y \in Y} y \right] \qquad \text{Lemma 5.4.1 (4)} \tag{5.72}$$

$$= \left(\bigcap \mathcal{R} \right)^{\downarrow}[Y]. \qquad Y = \bigcup_{y \in Y} y \tag{5.73}$$

Identity 5.71 follows from the associativity and commutativity of $\bigcap$.   □

The lemmas above ensure that, in enriched formal contexts where the relations R_i are I-compatible, the relation $R_C := \bigcap_{s \in S} R_s$ is likewise I-compatible. Therefore, the interpretation of $\mathcal{L}_C$-formulas on the model based on these enriched formal contexts defines a compositional semantics on formal concepts. Indeed, for every such enriched formal context $\mathbb{F} = (\mathbb{P}, \{R_i \mid i \in \text{Ag}\})$, every valuation V on $\mathbb{F}$ extends to the following interpretation map of C-formulas:

$$V(C(\varphi)) = \left(R_C^{\downarrow}[\{\![\varphi]\!\}], \left(R_C^{\downarrow}[\{\![\varphi]\!\}] \right)^{\uparrow} \right), \tag{5.74}$$

so that, if $V(\varphi)$ is a formal concept, then so is $V(\square_i\varphi)$. In addition, the following identity is semantically supported:

$$C(\varphi) = \bigwedge_{s \in S} \square_s\varphi, \tag{5.75}$$

where $s := i_1 \dots i_n$ is any finite non-empty string of elements of Ag, and $\square_s := \square_{i_1} \dots \square_{i_n}$.

5.4.3. Soundness
Proposition 5.4.13. *For any compositional model $\mathbb{M}$ and any $i \in$ Ag,*

1. *if $\mathbb{M} \models \varphi \vdash \psi$, then $\mathbb{M} \models \square_i\varphi \vdash \square_i\psi$;*

2. *$\mathbb{M} \models \top \vdash \square_i\top$;*

3. *$\mathbb{M} \models \square_i\varphi \wedge \square_i\psi \vdash \square_i(\varphi \wedge \psi)$.*

Proof. As to the first item, by Lemma 5.4.1 (1), if $[\![\varphi]\!] \subseteq [\![\psi]\!]$ then

$$[\![\Box_i\varphi]\!] = R_i^\downarrow\left[I^\uparrow[\![\varphi]\!]\right] \subseteq R_i^\downarrow\left[I^\uparrow[\![\psi]\!]\right] = [\![\Box_i\psi]\!], \qquad (5.76)$$

As to the second item, it suffices to show that $[\![\Box_i\top]\!] = A$. By definition,

$$[\![\Box_i\top]\!] = R_i^\downarrow[\{\top\}] = R_i^\downarrow\left[A^\uparrow\right]. \qquad (5.77)$$

For this reason, it is enough to show that $R_i^\downarrow[A^\uparrow] = A$. The assumption of I-compatibility implies that $R_i^\uparrow[a]$ is Galois-stable for every $a \in A$. Therefore, $A^\uparrow \subseteq R_i^\uparrow[a]$. Thus, by adjunction, $a \in R_i^\downarrow[A^\uparrow]$ for every $a \in A$, which implies that $R_i^\downarrow[A^\uparrow] = A$, as required. Finally, with regard to the third item,

$$
\begin{align}
[\![\Box(\varphi) \wedge \Box(\psi)]\!] &= R^\downarrow[\{\varphi\}] \cap R^\downarrow[\{\psi\}] &&\text{Definition of } [\![\cdot]\!] &&(5.78)\\
&= R^\downarrow[\{\varphi\} \cup \{\psi\}] &&\text{Lemma 5.4.1 (4)} &&(5.79)
\end{align}
$$

$$
\begin{align}
&= R^\downarrow\left[I^\uparrow\left[I^\downarrow[\{\varphi\} \cup \{\psi\}]\right]\right] &&\text{Lemma 5.4.4} &&(5.80)\\
&= R^\downarrow\left[I^\uparrow\left[I^\downarrow[\{\varphi\}] \cap I^\downarrow[\{\psi\}]\right]\right] &&\text{Lemma 5.4.1 (4)} &&(5.81)\\
&= R^\downarrow\left[I^\uparrow[\![\varphi]\!] \cap [\![\psi]\!]\right] &&&&(5.82)\\
&= R^\downarrow\left[I^\uparrow[\![\varphi \wedge \psi]\!]\right] &&\text{Definition of } [\![\cdot]\!] &&(5.83)\\
&= [\![\Box(\varphi \wedge \psi)]\!]. &&\text{Definition of } [\![\cdot]\!] &&(5.84)
\end{align}
$$

Identity 5.82 follows from the fact that $V(\varphi), V(\psi)$ are formal concepts. □

Proposition 5.4.14 (Soundness)**.** *For any compositional model* $\mathbb{M}$,

1. $\mathbb{M} \models C(\varphi) \vdash \bigwedge\{\Box_i\varphi \wedge \Box_i C(\varphi) \mid i \in \mathrm{Ag}\}$;

2. *if* $\mathbb{M} \models \chi \vdash \bigwedge_{i \in \mathrm{Ag}} \Box_i\varphi$ *and* $\mathbb{M} \models \chi \vdash \bigwedge_{i \in \mathrm{Ag}} \Box_i\chi$, *then* $\mathbb{M} \models \chi \vdash C(\varphi)$.

Proof. With regard to the first item, by definition and by Lemma 5.4.12 (2) it holds that $[\![C(\varphi)]\!] = R_C^\downarrow[\{\varphi\}] = \bigcap_{s \in S} R_s^\downarrow[\{\varphi\}] \subseteq \bigcap_{i \in \mathrm{Ag}} R_i^\downarrow[\{\varphi\}]$, which proves $\mathbb{M} \models C(\varphi) \vdash \bigwedge\{\Box_i\varphi \mid i \in \mathrm{Ag}\}$. Let $i \in \mathrm{Ag}$. The following chain of

(in)equalities completes the proof:

$$\llbracket \Box_i C(\varphi) \rrbracket = R_i^{\downarrow}\left[I^{\uparrow}\left[R_C^{\downarrow}\left[\{\!\{\varphi\}\!\}\right]\right]\right] \qquad\qquad \text{Definition of } \llbracket \cdot \rrbracket \quad (5.85)$$

$$= R_i^{\downarrow}\left[I^{\uparrow}\left[\bigcap_{s\in S} R_s^{\downarrow}\left[\{\!\{\varphi\}\!\}\right]\right]\right] \qquad\qquad \text{Lemma 5.4.12 (2)} \quad (5.86)$$

$$= R_i^{\downarrow}\left[I^{\uparrow}\left[\bigcap_{s\in S} I^{\downarrow}\left[I^{\uparrow}\left[R_s^{\downarrow}\left[\{\!\{\varphi\}\!\}\right]\right]\right]\right]\right] \qquad\qquad (5.87)$$

$$= R_i^{\downarrow}\left[I^{\uparrow}\left[I^{\downarrow}\left[\bigcup_{s\in S} I^{\uparrow}\left[R_s^{\downarrow}\left[\{\!\{\varphi\}\!\}\right]\right]\right]\right]\right] \qquad \text{Lemma 5.4.1 (4)} \quad (5.88)$$

$$= R_i^{\downarrow}\left[\bigcup_{s\in S} I^{\uparrow}\left[R_s^{\downarrow}\left[\{\!\{\varphi\}\!\}\right]\right]\right] \qquad\qquad \text{Lemma 5.4.4} \quad (5.89)$$

$$= \bigcap_{s\in S} R_i^{\downarrow}\left[I^{\uparrow}\left[R_s^{\downarrow}\left[\{\!\{\varphi\}\!\}\right]\right]\right] \qquad\qquad \text{Lemma 5.4.1 (4)} \quad (5.90)$$

$$= \bigcap_{s\in S} R_{is}^{\downarrow}\left[\{\!\{\varphi\}\!\}\right] \qquad\qquad \text{Lemma 5.4.9} \quad (5.91)$$

$$\supseteq \bigcap_{s\in S} R_s^{\downarrow}\left[\{\!\{\varphi\}\!\}\right] \qquad\qquad \{is \mid s\in S\}\subseteq S \quad (5.92)$$

$$= \llbracket C(\varphi) \rrbracket. \qquad\qquad \text{Lemma 5.4.12 (2)} \quad (5.93)$$

Identity 5.87 follows from the fact that $R_s^{\downarrow}[\{\!\{\varphi\}\!\}]$ is Galois-stable.

With regard to the second item, using Proposition 5.4.13 (1) and the assumptions, one can show that $\mathbb{M} \models \chi \vdash \Box_s\varphi$ for every $s \in S$. Therefore, $\llbracket\chi\rrbracket \subseteq \bigcap_{s\in S} R_s^{\downarrow}[\{\!\{\varphi\}\!\}] = R_C^{\downarrow}[\{\!\{\varphi\}\!\}] = \llbracket C(\varphi)\rrbracket$, as required.   □

5.4.4. COMPLETENESS
The completeness of $\mathbf{L}$ is proven via a standard canonical model construction. For any lattice $\mathbb{L}$ with normal operators $\Box_i$, let $\mathbb{F}_{\mathbb{L}} = (\mathbb{P}_{\mathbb{L}}, \{R_i \mid i \in \text{Ag}\})$ be defined as follows: $\mathbb{P}_{\mathbb{L}} = (A, X, I)$ where A (resp. X) is the set of lattice filters (resp. ideals) of $\mathbb{L}$, and aIx iff $a \cap x \neq \varnothing$. For every $i \in \text{Ag}$, let $R_i \subseteq A \times X$ be defined by aR_ix iff $\Box_iu \in a$ for some $u \in \mathbb{L}$ such that $u \in x$. In what follows, for any $a \in A$ and any $x \in X$, we let $\Box_ix := \{\Box_iu \in \mathbb{L} \mid u \in x\}$ and $\Box_i^{-1}a := \{u \in \mathbb{L} \mid \Box_iu \in a\}$. Hence, by definition, $R_i^{\downarrow}[x] = \{a \mid a \cap \Box_ix \neq \varnothing\}$ for any $x \in X$ and $R_i^{\uparrow}[a] = \{x \mid x \cap \Box_i^{-1}a \neq \varnothing\}$ for any $a \in A$. Notice also that $\Box_i\top = \top$ implies that $\Box_i^{-1}a \neq \varnothing$ for every $a \in A$.

Lemma 5.4.15. *For $\mathbb{F}_{\mathbb{L}}$ defined as above and any $a \in A$, $x \in X$ and $i \in \text{Ag}$,*

1. $I^\uparrow\left[R_i^\downarrow[x]\right] = \{y \in X \mid \Box_i x \subseteq y\}$;

2. $I^\downarrow[R_i[a]] = \{b \in A \mid \Box_i^{-1} a \subseteq b\}$;

3. $I^\downarrow\left[I^\uparrow\left[R_i^\downarrow[x]\right]\right] = \{b \in A \mid \Box_i x \cap b \neq \varnothing\} = R_i^\downarrow[x]$;

4. $I^\uparrow\left[I^\downarrow\left[R_i^\uparrow[a]\right]\right] = \{y \in X \mid \Box_i^{-1} a \cap y \neq \varnothing\} = R_i^\uparrow[a]$.

Proof. The first and second items immediately follow from the definitions of $\Box_i x$ and $\Box_i^{-1} a$. As to the third and fourth items, from the previous two items it follows that $I^\downarrow[I^\uparrow[R_i^\downarrow[x]]] = \{b \in A \mid \lceil \Box_i x \rceil \cap b \neq \varnothing\}$ and $I^\uparrow[I^\downarrow[R_i^\uparrow[a]]] = \{y \in X \mid \lfloor \Box_i^{-1} a \rfloor \cap y \neq \varnothing\}$, where $\lceil \Box_i x \rceil$ and $\lfloor \Box_i^{-1} a \rfloor$ respectively denote the ideal generated by $\Box_i x$ and the filter generated by $\Box_i^{-1} a$. Via the monotonicity of $\Box_i$, one can show that $\{b \in A \mid \lceil \Box_i x \rceil \cap b \neq \varnothing\} = \{b \in A \mid \Box_i x \cap b \neq \varnothing\} = R_i^\downarrow[x]$, and using the meet preservation of $\Box_i$, one can further show that $\{y \in X \mid \lfloor \Box_i^{-1} a \rfloor \cap y \neq \varnothing\} = \{y \in X \mid \Box_i^{-1} a \cap y \neq \varnothing\} = R_i^\uparrow[a]$, as required. Notice that the last equality holds for every $a \in A$ under the assumption that $\Box_i^{-1} a \neq \varnothing$. As remarked above, this is guaranteed by $\Box_i$ being normal. □

The third and fourth items of the lemma above immediately imply that:

Lemma 5.4.16. $\mathbb{F}_\mathbb{L}$ *is a compositional enriched formal context (cf. Definition 5.4.6).*

Recall that S is the set of non-empty finite sequences of elements of Ag.

Lemma 5.4.17. *If x is the ideal generated by some $u \in \mathbb{L}$, then, for every $s \in S$, $R_s^\downarrow[x] = \{a \mid \Box_s u \in a\}$.*

Proof. By induction on the length of s. If $s = i$ then $aR_i x$ iff $a \in R_i^\downarrow[x]$ iff $a \cap \Box_i x \neq \varnothing$. Because x is the ideal generated by u, we have that u is the greatest element of x. Hence, the monotonicity of $\Box_i$ implies that $\Box_i u$ is the greatest element of $\Box_i x$. Because a is a filter, and thus it is upward-closed, $a \cap \Box_i x \neq \varnothing$ is equivalent to $\Box_i u \in a$, which yields proof of the base case of the induction. Let us now assume that $R_s^\downarrow[x] = \{b \in A \mid \Box_s u \in b\}$ and show that $R_{is}^\downarrow[x] = \{b \in A \mid \Box_{is} u \in b\}$. By Lemma 5.4.15 (3, 4) and Lemma 5.4.7, R_s

is I-compatible for every $s \in S$. Now let z be the ideal generated by $\square_s u$.

$$R^{\downarrow}_{is}[x] = R^{\downarrow}_i \left[I^{\uparrow} \left[R^{\downarrow}_s[x] \right] \right] \qquad \text{Lemmas 5.4.4 and 5.4.9} \qquad (5.94)$$

$$= R^{\downarrow}_i \left[(\{b \in A \mid \square_s u \in b\})^{\uparrow} \right] \qquad \text{Induction hypothesis} \qquad (5.95)$$

$$= R^{\downarrow}_i [\{y \in X \mid \square_s u \in y\}] \qquad\qquad\qquad (5.96)$$

$$= R^{\downarrow}_i [z] \qquad\qquad \text{Definition of } z \qquad (5.97)$$

$$= \{a \mid \square_i \square_s u \in a\} \qquad\qquad \text{Base case} \qquad (5.98)$$

$$= \{a \mid \square_{is} u \in a\} . \qquad\qquad \text{Definition of } \square_{is} \qquad (5.99)$$

Identity 5.96 follows from the fact that the filter generated by $\square_s u$ is the smallest element of $R^{\downarrow}_s[x]$. □

The *canonical enriched formal context* is defined by instantiating the construction above to the Lindembaum-Tarski algebra of **L**. In this case, let V be the valuation such that $[\![p]\!]$ (resp. $\{\![p]\!\}$) is the set of the filters (resp. ideals) to which p belongs, and let $\mathbb{M} = (\mathbb{F}_{\mathbf{L}}, V)$ be the canonical model.

Lemma 5.4.18 (Truth Lemma). *For every $\varphi \in \mathcal{L}$,*

1. *$\mathbb{M}, a \Vdash \varphi$ iff $\varphi \in a$;*

2. *$\mathbb{M}, x > \varphi$ iff $\varphi \in x$.*

Proof. By induction on φ. We only show the inductive step for $\varphi := \square_i \sigma$.

$$\mathbb{M}, a \Vdash \square_i \sigma \quad \text{iff} \quad a \in R^{\downarrow}_i [\{\![\sigma]\!\}] \qquad\qquad\qquad (5.100)$$

$$\text{iff} \quad a \in R^{\downarrow}_i [\{x \mid \sigma \in x\}] \qquad \text{Induction hypothesis} \qquad (5.101)$$

$$\text{iff} \quad a \in \{b \in A \mid \square_i \sigma \in b\} \qquad \text{Definition of } R_i \qquad (5.102)$$

$$\text{iff} \quad \square_i \sigma \in a. \qquad\qquad\qquad (5.103)$$

$$\mathbb{M}, x > \square_i \sigma \quad \text{iff} \quad x \in \{\![\square_i \sigma]\!\} \qquad\qquad\qquad (5.104)$$

$$\text{iff} \quad x \in [\![\square_i \sigma]\!]^{\uparrow} \qquad\qquad\qquad (5.105)$$

$$\text{iff} \quad x \in \{a \in A \mid \square_i \sigma \in a\}^{\uparrow} \qquad \text{Proof above} \qquad (5.106)$$

$$\text{iff} \quad \square_i \sigma \in x. \qquad\qquad\qquad (5.107)$$

□

The weak completeness of **L** follows from the lemma above with the usual argument.

Proposition 5.4.19 (Completeness). *If $\varphi \vdash \psi$ is an $\mathcal{L}$-sequent which is not derivable in* **L**, *then* $\mathbb{M} \not\models \varphi \vdash \psi$.

The weak completeness for $\mathbf{L}_C$ is proven along the lines of Fagin, Halpern, Moses, and Vardi [50, Theorem 3.3.1]. Namely, for any $\mathcal{L}_C$-sequent $\varphi \vdash \psi$ that is not derivable in $\mathbf{L}_C$, we construct a finite model $\mathbb{M}_{\varphi,\psi}$ such that $\mathbb{M}_{\varphi,\psi} \not\models \varphi \vdash \psi$. Let Φ_0 be the set whose elements are $\top$, $\bot$, and all the subformulas of φ and of ψ. Let

$$\Phi_1 := \Phi_0 \cup \bigcup_{i \in \mathrm{Ag}} \{\Box_i \sigma \mid \sigma \in \Phi_0\}, \tag{5.108}$$

$$\Phi := \left\{ \bigwedge \Psi \mid \Psi \subseteq \Phi_1 \right\}. \tag{5.109}$$

By construction, Φ is finite. Consider the canonical model $\mathbb{M}$ defined above, and the following equivalence relations on A and X:

$$a \equiv_\Phi b \quad \text{iff} \quad a \cap \Phi = b \cap \Phi, \tag{5.110}$$

$$x \equiv_\Phi y \quad \text{iff} \quad x \cap \Phi = y \cap \Phi. \tag{5.111}$$

Because Φ is finite, these equivalence relations induce finitely many equivalence classes on A and X. In particular, considering $\vdash$ as a pre-order on Φ, each element $\overline{a}$ of $A/\!\equiv_\Phi$ is uniquely identified by some Φ-*filter*, i.e., a $\vdash$-upward closed subset of Φ that is also closed under existing conjunctions. Analogously, each $\overline{x} \in X/\!\equiv_\Phi$ is uniquely identified by some Φ-*ideal*, i.e., a $\vdash$-downward closed subset of Φ that is also closed under existing disjunctions. In addition, because Φ is closed under conjunctions, the Φ-filter corresponding to each $\overline{a}$ is *principal*, i.e. for each $\overline{a} \in A/\!\equiv_\Phi$, some $\tau_{\overline{a}} \in \Phi$ exists such that $\overline{a}$ can be identified with the set of the formulas $\sigma \in \Phi$ such that $\tau_{\overline{a}} \vdash \sigma$ is an $\mathbf{L}_C$-derivable sequent.

In what follows, we abuse notation and let $\overline{a}$ and $\overline{x}$ respectively denote the principal Φ-filter and the Φ-ideal with which $\overline{a}$ and $\overline{x}$ can be identified, as discussed above. With this convention we can write $\Box_i^* \overline{x} := \{\Box_i \sigma \mid \sigma \in \overline{x}\} \cap \Phi$ and $(\Box_i^{-1})^* \overline{a} := \{\tau \in \Phi \mid \Box_i \tau \in \overline{a}\}$. As a consequence of $\bot, \top \in \Phi_0$, and $\Box_i \top = \top$, we have that $\Box_i^* \overline{x}$ and $(\Box_i^{-1})^* \overline{a}$ are always non-empty. Let us define:

$$\mathbb{M}_{\varphi,\psi} = \left(A/\!\equiv_\Phi, X/\!\equiv_\Phi, I_{\varphi,\psi}, R_i^{\varphi,\psi}, V_{\varphi,\psi} \right), \tag{5.112}$$

where

$$\overline{a} I_{\varphi,\psi} \overline{x} \quad \text{iff} \quad \overline{a} \cap \overline{x} \neq \varnothing \tag{5.113}$$

$$\text{iff} \quad \tau_{\overline{a}} \in \overline{x}, \tag{5.114}$$

$$\overline{a} R_i^{\varphi,\psi} \overline{x} \quad \text{iff} \quad \Box_i^* \overline{x} \cap \overline{a} \neq \varnothing \tag{5.115}$$

$$\text{iff} \quad \tau_{\overline{a}} \vdash \Box_i \tau \text{ is } \mathbf{L}_C\text{-derivable for some } \tau \in \overline{x}, \tag{5.116}$$

and $V_{\varphi,\psi}$ is any valuation such that $[\![p]\!] = \{\overline{a} \mid p \in \overline{a}\}$ and $\{\!\{p\}\!\} = \{\overline{x} \mid p \in \overline{x}\}$ for all $p \in \mathrm{Prop} \cap \Phi$. Henceforth, we abbreviate $I_{\varphi,\psi}$ as I wherever possible. It readily follows from this definition that $[\![p]\!]^{\uparrow\downarrow} = [\![p]\!]$ and $\{\!\{p\}\!\}^{\downarrow\uparrow} = \{\!\{p\}\!\}$ for any $p \in \mathrm{Prop} \cap \Phi$. Moreover, $(R_i^{\varphi,\psi})^{\downarrow}[\overline{x}] = \{\overline{a} \mid \overline{a} \cap \square_i^* \overline{x} \neq \varnothing\}$ and $(R_i^{\varphi,\psi})^{\uparrow}[\overline{a}] = \{\overline{x} \mid \overline{x} \cap (\square_i^{-1})^* \overline{a} \neq \varnothing\}$. From this, analogously to Lemma 5.4.15, it follows that:

Lemma 5.4.20. *For any $\overline{a}, \overline{x}$ and $i \in \mathrm{Ag}$,*

1. $I_{\varphi,\psi}^{\uparrow}\left[\left(R_i^{\varphi,\psi}\right)^{\downarrow}[\overline{x}]\right] = \{\overline{y} \mid \square_i^* \overline{x} \subseteq \overline{y}\}$;

2. $I_{\varphi,\psi}^{\downarrow}[(R_i^{\varphi,\psi})^{\uparrow}[\overline{a}]] = \{\overline{b} \in A \mid (\square_i^{-1})^* \overline{a} \subseteq \overline{b}\}$;

3. $I_{\varphi,\psi}^{\downarrow}\left[I_{\varphi,\psi}^{\uparrow}\left[\left(R_i^{\varphi,\psi}\right)^{\downarrow}[\overline{x}]\right]\right] = \left\{\overline{b} \mid \square_i^* \overline{x} \cap \overline{b} \neq \varnothing\right\} = \left(R_i^{\varphi,\psi}\right)^{\downarrow}[\overline{x}]$;

4. $I_{\varphi,\psi}^{\uparrow}\left[I_{\varphi,\psi}^{\downarrow}\left[\left(R_i^{\varphi,\psi}\right)^{\uparrow}[\overline{a}]\right]\right] = \{\overline{y} \mid (\square_i^{-1})^* \overline{a} \cap \overline{y} \neq \varnothing\} = R_i^{\uparrow}[\overline{a}]$.

The third and fourth items of the lemma immediately imply that:

Lemma 5.4.21. $R_i^{\varphi,\psi}$ *is $I_{\varphi,\psi}$-compatible for any $i \in \mathrm{Ag}$.*

The following is key to the proof of the Truth Lemma (5.4.18).

Lemma 5.4.22. *If $C(\sigma) \in \Phi$, then the following is an $\mathbf{L}_C$-derivable sequent for any $i \in \mathrm{Ag}$:*

$$\bigvee_{\overline{a} \in [\![C(\sigma)]\!]} \tau_{\overline{a}} \vdash \square_i \left(\bigvee_{\overline{a} \in [\![C(\sigma)]\!]} \tau_{\overline{a}} \right). \tag{5.117}$$

Proof. Fix $i \in \mathrm{Ag}$ and $\overline{a} \in [\![C(\sigma)]\!]$. Because $\square_i$ is monotone, it is enough to show that some $\tau \in \Phi$ exists such that

$$\tau_{\overline{a}} \vdash \square_i \tau \quad \text{and} \quad \tau \vdash \bigvee_{\overline{a} \in [\![C(\sigma)]\!]} \tau_{\overline{a}}. \tag{5.118}$$

By the definition of $R_i^{\varphi,\psi}$, this is equivalent to showing that $\overline{a} R_i^{\varphi,\psi} \overline{y}$, where $\overline{y}$ is the Φ-ideal generated by $\bigvee_{\overline{a} \in [\![C(\sigma)]\!]} \tau_{\overline{a}}$. Note that $\{\!\{C(\sigma)\}\!\} = [\![C(\sigma)]\!]^{\uparrow}$ is the collection of all the Φ-ideals $\overline{x}$ such that $\tau_{\overline{b}} \in \overline{x}$ for every $\overline{b} \in [\![C(\sigma)]\!]$. Hence, $\overline{y} \in \{\!\{C(\sigma)\}\!\}$ (and it is in fact the smallest element in $\{\!\{C(\sigma)\}\!\}$). Thus, to prove that $\overline{a} R_i^{\varphi,\psi} \overline{y}$, it is enough to show that $[\![C(\sigma)]\!] \subseteq (R_s^{\varphi,\psi})^{\downarrow}[\{\!\{C(\sigma)\}\!\}]$. This immediately follows from the fact that $(R_s^{\varphi,\psi})^{\downarrow}[\{\!\{C(\sigma)\}\!\}] = [\![\square_i C(\sigma)]\!]$, that $C(\sigma) \vdash \square_i C(\sigma)$ is an $\mathbf{L}_C$-derivable sequent, that $\mathbf{L}_C$ is sound with respect to compositional models (cf. Proposition 5.4.14), and that $\mathbb{M}_{\varphi,\psi}$ is a compositional model (cf. Lemma 5.4.21). $\square$

Lemma 5.4.23 (Truth Lemma). *For every $\tau \in \Phi_0$,*

1. *$M_{\varphi,\psi}, \bar{a} \Vdash \tau$ iff $\tau \in \bar{a}$;*

2. *$M_{\varphi,\psi}, \bar{x} > \tau$ iff $\tau \in \bar{x}$.*

Proof. As to the first item of the lemma, we only show the inductive step for $\tau := C(\sigma)$ for some $\sigma \in \Phi_0$. If $M_{\varphi,\psi}, \bar{a} \Vdash C(\sigma)$, i.e., $\bar{a} \in [\![C(\sigma)]\!] = \bigcap_{s \in S}(R_s^{\varphi,\psi})^{\downarrow}[\{\!\{\sigma\}\!\}]$, then $\bar{a} \in (R_i^{\varphi,\psi})^{\downarrow}[\{\!\{\sigma\}\!\}] = [\![\Box_i\sigma]\!]$ for any $i \in \text{Ag}$. By definition, $\sigma \in \Phi_0$ implies that $\Box_i\sigma \in \Phi$. Moreover,

$$\bar{a} \in [\![\Box_i\sigma]\!] \quad \text{iff} \quad \bar{a} \in \left(R_i^{\varphi,\psi}\right)^{\downarrow}[\{\!\{\sigma\}\!\}] \tag{5.119}$$

$$\text{iff} \quad \bar{a} \in \left(R_i^{\varphi,\psi}\right)^{\downarrow}[\{\bar{x} \mid \sigma \in \bar{x}\}] \quad \text{Induction hypothesis} \tag{5.120}$$

$$\text{iff} \quad \bar{a} \in \left\{\bar{b} \mid \Box_i\sigma \in \bar{b}\right\} \quad\quad\quad \text{Definition of } R_i^{\varphi,\psi} \tag{5.121}$$

$$\text{iff} \quad \Box_i\sigma \in \bar{a}, \tag{5.122}$$

which implies that $\tau_{\bar{a}} \vdash \bigwedge_{i \in \text{Ag}} \Box_i\sigma$. By Lemma 5.4.22 and the fact that $\mathbf{L}_C$ is closed under the following rule:

$$\frac{\chi \vdash \bigwedge_{i \in \text{Ag}} \Box_i\varphi \quad \{\chi \vdash \Box_i\chi \mid i \in \text{Ag}\}}{\chi \vdash C(\varphi)}, \tag{5.123}$$

we conclude that $\tau_{\bar{a}} \vdash C(\sigma)$, i.e., $C(\sigma) \in \bar{a}$.

For the reverse direction, let $\bar{b}$ be the principal Φ-filter generated by $C(\sigma)$. Let us show, by induction on the length of s, that $\bar{b} \in (R_s^{\varphi,\psi})^{\downarrow}[\{\!\{\sigma\}\!\}]$ for all $s \in S$. Indeed, for the base case, $\Box_i\sigma \in \Phi$ and $C(\sigma) \vdash \Box_i\sigma$ being an $\mathbf{L}_C$-derivable sequent imply that $\Box_i\sigma \in \bar{b}$, and thus $\bar{b} \in (R_i^{\varphi,\psi})^{\downarrow}[\{\!\{\sigma\}\!\}]$. For the inductive step, assume that $\bar{b} \in (R_s^{\varphi,\psi})^{\downarrow}[\{\!\{\sigma\}\!\}]$. Then, every element of $I^{\uparrow}[(R_s^{\varphi,\psi})^{\downarrow}[\{\!\{\sigma\}\!\}]]$ contains $C(\sigma)$. Moreover, we have that $\Box_i C(\sigma) \in \bar{b}$ because $\Box_i C(\sigma) \in \Phi$ and $C(\sigma) \vdash \Box_i C(\sigma)$ is an $\mathbf{L}_C$-derivable sequent. Therefore, by Lemma 5.4.9, we have the following:

$$\bar{b} \in \left(R_i^{\varphi,\psi}\right)^{\downarrow}\left[I^{\uparrow}\left[(R_s^{\varphi,\psi})^{\downarrow}[\{\!\{\sigma\}\!\}]\right]\right] = \left(R_{is}^{\varphi,\psi}\right)^{\downarrow}[\{\!\{\sigma\}\!\}], \tag{5.124}$$

which concludes the proof that $\bar{b} \in (R_s^{\varphi,\psi})^{\downarrow}[\{\!\{\sigma\}\!\}]$ for all $s \in S$. To finish the proof, for any $\bar{a}$, if $C(\sigma) \in \bar{a}$, then $\bar{b} \subseteq \bar{a}$. Because $(R_s^{\varphi,\psi})^{\downarrow}[\{\!\{\sigma\}\!\}]$ is Galois-stable for any $s \in S$, this implies that $\bar{a} \in (R_s^{\varphi,\psi})^{\downarrow}[\{\!\{\sigma\}\!\}]$ for every $s \in S$. This goes to show that $M_{\varphi,\psi}, \bar{a} \Vdash C(\sigma)$.

As to the second item of the lemma,

$$
\begin{aligned}
M_{\varphi,\psi}, \overline{x} > C(\sigma) \quad &\text{iff} \quad \overline{x} \in [\![C(\sigma)]\!]^{\uparrow} && (5.125) \\
&\text{iff} \quad \overline{x} \cap \overline{a} \neq \varnothing \text{ for all } \overline{a} \in [\![C(\sigma)]\!] && (5.126) \\
&\text{iff} \quad C(\sigma) \in \overline{x}. && (5.127)
\end{aligned}
$$

□

The weak completeness of $\mathbf{L}_C$ follows from the lemma above with the usual argument.

Proposition 5.4.24 (Completeness)**.** *If $\varphi \vdash \psi$ is an $\mathcal{L}_C$-sequent that is not derivable in $\mathbf{L}_C$, then $M_{\varphi,\psi} \not\models \varphi \vdash \psi$.*

5.5. Proposed Formalizations

Having defined our epistemic-logical language and offered proofs of its soundness and completeness, we now turn to demonstrating its usefulness for the interdisciplinary research on categories by proposing formalizations of some of the most important notions about categorization in the social sciences. Restricting our focus to the extant literature in sociology [22] and management science [23], our choice falls on the notions of typicality [e.g., 14], contrast [e.g., 18], leniency [e.g., 21], and similarity [e.g., 16]. These represent important theoretical constructs that exert demonstrable influence on economic and strategic decision-making (cf. Chapters 2 and 3). By now, they have entered the common lexicon of organization theory. In what follows, we propose formalizations for these constructs using the languages $\mathcal{L}$, $\mathcal{L}_C$, and $\mathcal{L}_H$ previously defined in this chapter. These formalizations should not be interpreted as guidelines on how to measure these constructs empirically, as they rely on information concerning what agents believe about (or how they perceive) objects and categories that may be difficult to acquire outside experimental settings. Their aim is rather to capture the qualitative content of these constructs: in this sense, they are instruments of theory construction, not empirical inquiry.

Typicality. The issue of whether an object *a* is a representative member of a category *φ*, that is, the extent to which *a is typical of φ*, is key to the similarity-based perspectives on categorization [27, 32]. By extension, it is also central to the literature on conceptual spaces [42, 43, 53], where prototypes play a critical role in determining category memberships [46]. In this literature, the prototype of a category (or concept) is defined as the

geometric center of that category: the closer an object is to this point, the greater its typicality as a member of the category (or an instance of the concept). Although FCA is not as naturally suited as conceptual spaces to reason about metric distance,[5] the notion of typicality can be captured through the common knowledge-type construction introduced above.

The interpretation of C-formulas on models is such that, for every category φ, the members of $C(\varphi)$ are those that belong to φ according to every agent; moreover, every agent believes that these objects belong to φ according to every other agent. This motivates our proposal to regard the members of $C(\varphi)$ as the prototypical members of φ. In geometric terms, $C(\varphi)$ is the *center* of φ [cf. 22, p. 42]. The main advantage of this approach is that it directly links typicality to the agents' perception or beliefs, so that consistently with probabilistic perspectives [e.g., 56], the prototypical objects are those for which membership in the category is never in question. Compare this interpretation of C to the one we proposed in Chapter 4: in that case, the operator was used to characterize the outcome of social interaction, whereby the extension and intension of a particular category are agreed upon by the agents. Our current proposal enhances this epistemic interpretation by tying it directly to the psychological [57] and sociological [14] notion of typicality. This is consistent with a probabilistic approach [e.g., 22, p. 40–42] inasmuch as the prototypes of a category (or concept) are those for which the likelihood of being considered a member of the category (or an instance of the concept) is one.

There is a number of reasons why an object may fail to be considered a prototypical member of φ, the most severe being that some agents do not recognize its membership in φ. Alternatively, it may occur that some agents do not believe that every other agent recognizes the object as a member of φ. This provides a purely qualitative route to encoding the gradedness of category memberships. That is, the typicality of two objects a and b, represented in the language $\mathcal{L}_H$ as nominal variables, can be compared in terms of the minimum number of epistemic iterations between the agents that are needed for their typicality test to fail, so that b can be regarded as more typical than a if more iterations are needed for b than for a. This definition can also be used to compare objects that belong to different categories, so as to say that b is more typical of ψ than a is of φ.

Similarity.   The extent to which two categories are similar to one another can be defined in multiple ways. One approach, very naturally applicable to

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[5]It is still suitable to reason about distance [e.g., 54, 55], but not directly from a geometric standpoint [cf. 16].

conceptual spaces [e.g., 38], is to calculate the Hausdorff distance between the subsets of space that correspond to the two categories, which is equal to the maximum of the two minimal point-to-set distances. Another approach, distinctly set-theoretic in nature, is to use a Jaccard coefficient [e.g., 16, 41, 58], which is a ratio of the number of objects in the categories' intersection over the number of objects in their union. In our algebraic framework, the first method is not easy to implement but, compatibly with the second, the fact that two categories can have a greater vs. a lower number of members (or features) in common allows us to define similarity based on the overlap between the categories' extensions (or descriptions) [cf. 54, 55].

For four categories φ, ψ, χ, ξ, we say that *φ is more similar to ψ than χ is to ξ* by means of the sequent $\varphi \vee \psi \vdash \xi \vee \chi$, the sequent $\xi \wedge \chi \vdash \varphi \wedge \psi$, or by requiring the two sequents to hold simultaneously. The first sequent intuitively means that φ and ψ have more features in common than ξ and χ have. The second means that φ and ψ have more members in common than ξ and χ have. Because neither sequent implies nor is implied by the other, it can be useful to consider the information encoded in both, and it may be an interesting question to address in empirical research whether the implications of similarity are identical in these two cases. When instantiated to $\varphi = \xi$, these conditions can be used to express that φ is more similar to ψ than it is to χ.

Contrast.   The contrast of a category indicates the extent to which the category stands out from other categories in the same domain [59]. Intuitively, it quantifies the sharpness of category boundaries and it is empirically defined as a function of the typicality of category members [e.g., 19]. In a high-contrast category, objects are considered either very typical members or not members at all, wheraes in a low-contrast one small grades of memberships are frequent [14]. Contrast is an important property of categories to account for in sociological and management research because the sharpness of category boundaries affects the appeal of product and organizations for their target audiences. In fact, objects that neatly fit into sharply bounded categories tend to be more visible and more positively valued [18], whereas those that straddle such boundaries (cf. Chapter 3) tend to incur greater penalties [17].

We propose a definition of contrast that relies on the same common knowledge-type construction underpinning the definition of typicality. If $\varphi \vdash C(\varphi)$ holds for a category φ then every member of φ is a prototypical member of φ in the sense discussed above, and therefore φ has maximal contrast. The contrast of φ gradually decreases as the category includes

more and more members that do not belong to $C(\varphi)$. Using the formal definitions of typicality and similarity, for any two categories φ and ψ we say that φ *has higher contrast than* ψ if φ is more similar to $C(\varphi)$ than ψ is to $C(\psi)$. That is, we require that $\varphi \vee C(\varphi) \vdash \psi \vee C(\psi)$, that $\psi \vee C(\psi) \vdash \varphi \vee C(\varphi)$, or that both sequents hold simultaneously.

Leniency. By the common sociological definition of contrast [14], the objects in a high-contrast category are highly typical members of the category on average; conversely, the objects in a low-contrast category tend to be more atypical. This can occur either if there are many other categories where the objects also have some positive grade of membership, or if there are few such categories (possibly none). It is important to distinguish between these two cases, as this is indicative of the category's capacity to accommodate deviations. The notion of leniency serves exactly this purpose [23]: given a category, its leniency is defined as the extent to which the category members tend to belong to additional categories in the same domain. Like contrast, this is an important property to consider in organizational research because a category's leniency can have an effect on economic [20] and strategic decisions [21].

We say that a category φ *is non-lenient* if its members do not simultaneously belong to any other category in the formal context. This property is captured by the following condition: for any ψ and χ, if $\psi \vdash \varphi$ and $\psi \vdash \chi$, then either $\varphi \vdash \chi$ or $\chi \vdash \varphi$. To better understand this condition, let us instantiate ψ as the nominal category a, i.e., the category generated by the object a. The sequent $\mathsf{a} \vdash \varphi$ means that a is a member of φ, and the non-leniency of φ requires that a does not belong to any other category. However, the order-theoretic nature of our logical framework requires a to be a member of every category χ such that $\varphi \vdash \chi$. As a result, a must at least belong to these logically unavoidable categories. All the categories χ such that $\mathsf{a} \vdash \chi \vdash \varphi$ cannot be excluded either, because the possibility that intermediate categories exist does not solely depend on a and φ but also on other objects and features in the formal context. Non-leniency implies that no other categories have a as a member than those within this minimal set of categories that cannot be excluded. For example, a member of the category CHAMBER MUSIC is necessarily a member of the superordinate category of CLASSICAL MUSIC, but this does mean that CHAMBER MUSIC is lenient. Similarly, the fact that some members of CLASSICAL MUSIC also belong to CHAMBER MUSIC does not affect the leniency of CLASSICAL.

This definition can also be used for the purpose of comparison. For any two categories φ and ψ, we say that φ is more lenient than ψ if, for every

nominal a, if $a \vdash \psi$ and $a \vdash \chi$ for some χ such that $\chi \nvdash \psi$ and $\psi \nvdash \chi$, then $a \vdash \varphi$; moreover, $a \vdash \xi$ for some category ξ such that $\xi \nvdash \varphi$ and $\varphi \nvdash \xi$. These conditions can be alternatively given in terms of features (using co-nominal variables) and in terms of modal operators.

5.6. Discussion

This chapter presented a basic epistemic logic of categories, expanded it with a common knowledge-type construction and hybrid-style variables, proved its soundness and completeness, and deployed the resulting formal machinery to capture some of the most fundamental notions about categories in sociology and management science. The characteristics of our logical formalism are well-suited to reason about the hierarchical nature of classification systems, and in addition to factual information about objects and features, it can represent the subjective perspectives of individual agents as well as social interaction. The propositional base of our logic is the positive (i.e., negation-free and implication-free) fragment of classical propositional logic, which is devoid of distributivity laws. The Kripke-style semantics are given by structures known as formal contexts in FCA [47], which we enriched with binary relations to account for the epistemic interpretation of the modal operators. An important difference between this semantics and the usual Kripke semantics for epistemic logics is that the relations directly encode the viewpoint of the individual agents.

The starting point of this chapter was the observation that logic can decisively contribute to the growing research on categories in the social sciences, especially with regard to the analysis of various types of social interaction (e.g., epistemic, dynamic, strategic). The prospective contributions are both technical and conceptual in nature. From a technical perspective, this chapter refined the foundational work presented in Chapter 4 linking epistemic logic and FCA. This preliminary work offered an intuitive explanation of the definition of the interpretation clauses of $\mathcal{L}$-formulas on certain enriched formal contexts, whereas the treatment we presented here adapts these clauses to the more general and natural class of arbitrary (enriched) formal contexts. One of the novel aspects of this proposal is that the typicality of objects is captured via a common knowledge-type operator that is semantically equivalent to the usual greatest fixed point construction. This paves the way to the use of logical languages expanded with fixed point operators to model the more subtle aspects of agents' knowledge, such as introspection. In addition, our framework makes it possible to blend syllogistic and epistemic reasoning. Specific proof calculi are needed to

further analyze the aspects connected to reasoning and deduction in $\mathbf{L}$ and $\mathbf{L}_C$, and to explore the computational properties of these logics. As these calculi can be used to draw conclusions from formal inferences, they can also be helpful to empirical research on categories by allowing the derivation of testable hypotheses.

From a more conceptual standpoint, the formalism introduced in Chapter 4 and further refined in this chapter can be instrumental to the growing research on categories in sociology and management science. An adequate account of the dynamic nature of categories is one of the major challenges faced by researchers in these fields [13, 23]: by enabling classification systems to be represented as partial orders where changes due to the addition of new products or features can be formally computed, our framework can illuminate the mechanisms whereby new market categories emerge [60–62] or disappear [2, 63]. Another challenge is that categories both shape and are shaped by social interaction [64]. While this bidirectional causality is empirically problematic to account for, it is theoretically interesting and arguably essential to understand just how categories contribute to the functioning of markets. A compelling direction for further study concerns precisely the way categories affect agents' patterns of interaction [cf. 65], and how these patterns, in turn, modify the agents' beliefs. An important step in this direction would be to expand our framework with dynamic modalities and to extend the construction of dynamic updates [66, 67] to models based on enriched formal contexts. Recent work [68] has shown that these dynamic formalisms can also be incorporated in settings where the agents' beliefs are probabilistic [e.g., 22].

Another avenue for further study relates to the observation that categorizing agents, such as investors, analysts, or consumers in a market, tend to be motivated by different goals [69]. The needs and desires agents have in mind when sorting products or organizations can influence the features they find worthy to consider [70, 71]. Moreover, goals can play a crucial role in category emergence by shaping consensus about which objects deserve membership in a new category [72]. As explained in Chapter 3, it can be difficult to properly account for goal-based categories given their lack of a prototypical structure [26]: extant formal approaches that build on similarity-based views of categorization, like conceptual spaces, are not necessarily the best option. The framework we proposed can be much more suitable because, while it takes prototypes into account by allowing formal concepts to be generated by specific objects, it does not assume prototypes to serve as the sole or even the most important drivers of category generation [cf. 46]. Indeed, formal concepts can also coalesce

around specific features or arbitrary combinations thereof: for this reason, our logic holds promise to reconcile different theories of categorization.

5.7. References

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CONCLUSIONS

6.1. Summary of Findings

This dissertation sought to contribute to the growing research on categories in organization theory by examining how the information encoded by category labels is decoded by agents for the purpose of decision-making. In Chapters 2 and 3, we engaged with this question empirically by analyzing how the categories' internal structures, i.e., the rules underlying objects' category memberships [1], and their external structures, i.e., their hierarchical ordering into a classification system [2], affect the strategic behavior of organizations and the evaluation of products by customers. These studies equipped us a more thorough understanding of how agents with limited cognitive resources use category labels to make better decisions. In Chapters 4 and 5, we built on these insights to develop a formal theory of categorization inspired by Formal Concept Analysis (FCA) [3, 4] and similarly based on the notions of lattices and order [5]. After enriching FCA's basic framework with additional constructions that allow an epistemic interpretation, we introduced a sound and complete logical language capable of describing categories in terms of their hierarchical linkage and informational content. In the next few paragraphs, we recap the main findings from these chapters, discuss their implications for the themes outlined in Chapter 1, and identify directions for further research.

Chapter 2 dealt with the consequences of product proliferation. This strategy has been analyzed extensively in industrial economics for its capacity to deter competition in (sub)markets [6]; however, the recurrent suggestion from game-theoretic models that product proliferation is effective at deterring rival product introductions [7–12] was not supported by previous empirical studies [13]. Suspecting this could be due to structural variation across submarkets or categories, we proposed that the complexity of the region of feature space to which a category maps, i.e., the level of heterogeneity in the features of category members [cf. 14], weakens the deterrent power of proliferation strategies and can ultimately cancel their effect on competition. To measure category complexity, we observed the hierarchical relations within the industry's classification system [cf. 15]: from this perspective, a more complex category is one where the category members are scattered across a more diverse set of subcategories.

Consistently with our predictions, quasi-Poisson estimates suggested that the more a category is complex, the less a firm's strategic behavior affects the positioning choices of rivals. Additional tests revealed that this effect mainly concerns organizations who can safely shift their attention to other regions of the feature space, whereas those that are constrained to particular regions (e.g., by size or specialization) remain undeterred.

There is thus a second reason why product proliferation can sometimes appear to be ineffective at keeping rivals at bay, which relates to the rivals' capacity to handle environmental change [16]. Beyond suggesting theoretically-grounded explanations for previous studies' failure to support the "deterrence hypothesis" [13, p. 149], our study demonstrates that the information encoded into category labels—in this case, the variance of features among category members—affects competitive interactions.

In Chapter 3, we turned our attention to the effects of category spanning. Organizational research on this topic has been feverish ever since Zuckerman's compelling study [17]. Although many analyzed the consequences of category spanning before, particularly within the domain of organizational ecology [18], most scholars assumed categories to possess a specific internal structure, namely a prototypical one [19], and did not consider the possibility that categorization in markets may also depend on other criteria [20–23]. Building on the psychological literatures on goal-based categories [1] and cross-classification [24], we dissected the process whereby consumers combine information from multiple (and internally different) category labels to derive expectations about products. In particular, we considered the implications of multiple memberships in a system where categories are based on prototypes and one where they are based on consumers' goals. Tracing the negative effect of uncertainty on the perceived value of products [cf. 25, 26], we identified conditions in which multiple category labels enhance or impair the consumers' ability to make accurate inferences about the products' location in a feature space.

Our mixed-effects regression models returned evidence that having a single category label in each classification system minimizes consumers' uncertainty about the product's location and thus maximizes their average evaluation, which is consistent with extant theory [27]. However, we also found evidence that, because of the geometric convexity of type-based categories, it is possible to pinpoint a product's location if there is a sufficiently large number of type-based category labels. Further, we found that the actual number of type-based labels required for this geometric derivation decreases with the number of goal-based categories to which the product is assigned. These results point to the possibility of reconciling divergent findings about the effects of category spanning [e.g., 28, 29] under a relatively parsimonious set of theoretical assumptions.

The principal implication of these two empirical chapters is that organization theorists cannot afford to ignore the influence of categories' hierarchical arrangement into a classification system, nor the different cognitive mechanisms through which category labels encode factual in-

formation. Accordingly, in the following chapters we set out to develop a formal theory of categorization that properly captures these key aspects. With regard to formal models, a trend is currently building in sociological research to rely on the notion of conceptual spaces [30, 31] and thus embrace a geometric approach to category representation in social domains like markets [32–35]. There is great merit to this framework as it allows for an intuitive depiction of categories as multidimensional concepts; however, it is heavily geared toward prototype theory [36] and consequently it is ill-equipped to accommodate other rationales for categorization, such as the goal-driven [37] or the theory-based [38], which can be just as crucial as prototypes to explain agents' reasoning [39]. Perhaps unbeknown to sociologists, an alternative framework exists that is better suited to account for different mechanisms for categorization, i.e., Formal Concept Analysis [4]. This approach is the keystone of our logical formalism.

Chapter 4 was entirely devoted to connecting the mathematical machinery underlying FCA to the theoretical insights on categories in organizational contexts. As discussed in previous research [40], the basic FCA framework already allows for the characterization of categories as elements of a formal context, or concept lattice, by virtue of Birkhoff's representation theorem [41]. Our adaptation to organization theory involved the definition of relational semantics that enrich the FCA framework with relations used to interpret modal operators on normal lattice expansions. These modalities capture the categories' informational content by encoding the incomplete, idiosyncratic, and possibly mistaken beliefs of agents vis-à-vis the objects that belong to a given category or, equivalently, the features that the category members share. We further reasoned that social interaction allows agents in a market to learn about the beliefs of other agents, eventually leading to the identification of a small set of categories whose meaning is consensual. We formalized the emergence of real categories from the much broader set of candidate (i.e., merely possible) categories by means of a fixed-point construction that is reminiscent of the definition of common knowledge in classical epistemic logic [42].

Owing to these language enrichments, our formal theory represents market categories as naturally embedded in a hierarchical system, as per the basic FCA framework, and it enables one to describe their meaning in a way that is independent from (but compatible with) the notion of prototypes. In addition, and consistently with extant research [43], our theory describes category emergence as simultaneously contingent on factual information, subjective beliefs or perceptions, and social interaction. In future research, this formalism will be extended so as to allow for epistemic updates [44–46],

which enables us to capture category emergence in a truly dynamic fashion, compatibly with how this process is described by researchers in sociology [e.g., 47] and management [e.g., 48].

Chapter 5 rounded off the formal part of this dissertation by generalizing our theory and developing the epistemic logic associated with our lattice-based framework. After clarifying the semantics, axiomatization, and language enrichments of this logic, we offered proof of its soundness and completeness.[1] Our epistemic logic can be fruitfully applied to real-world contexts: to demonstrate the value of this application, we proposed formalizations for four theoretical constructs relevant to the research on categories in sociology [cf. 35] and management science [cf. 49]: typicality, contrast, leniency, and similarity. Given their order-theoretic flavor, we believe these formalizations can be useful to generate advanced theory about the role of these constructs in economic and strategic decision-making.

6.2. IMPLICATIONS FOR THE THEMES

Some general conclusions can be drawn with regard to the four themes presented in Chapter 1. The first theme relates to the principle of cognitive economy, which states that the primary objective of any categorization is to reduce the potentially overwhelming influx of stimuli to behaviorally and cognitively manageable proportions [50]. Consistently with this rule, our empirical results corroborate the argument that firm managers use categories to organize their view of the competitive landscape. To this end, they sort consumers into segments, or niches [51], and products into categories, so as to obtain a cognitive map that facilitates their strategic decisions. At the same time, consumers use category labels to predict the features of new products, hence lowering the cost of the information they require to make optimal purchasing decisions [52]. Even when considering products with multiple labels, their inferences about features obey the principle of cognitive economy. In this sense, our findings are consistent with a rational model of categorization [53, 54]. We also found support for the notion that, when confronted with multiple labels, consumers reason about the necessity and the impossibility of certain feature combinations [55], looking for an "economical" explanation of their category labels.

With respect to our second theme, which concerns the structure of classification systems, our research suggests that agents in a market can exploit their knowledge of the categories' external [2] and internal struc-

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[1]These two properties of the logic respectively imply that all which can be logically derived within our formal system is true, and that all which is true can be logically derived.

tures [1] to make better decisions. In particular, firm managers can leverage the categories' hierarchical relations to predict whether their competitive actions will exert the effect they intend. Consumers, instead, can consider the rules that govern category memberships to more accurately infer the characteristics of products. When the categories are governed by family resemblance, reasoning about distance and its implications [cf. 33] can be especially useful to derive information about a product's features. Nevertheless, categories that do not possess a prototypical structure can still play a role because they are often deployed in conjunction with those based on prototypes [cf. 24] so as to complement their information. It is important for organization theorists to take these different classification systems into consideration. Admittedly, goal-based categories can be difficult to observe empirically as they are seldom institutionalized and thus, for the most part, they remain implicit; however, social scientists' familiarization with machine learning [56] will undoubtedly open up unprecedented opportunities to extract these categories from discourse. On a similar note, goal-based categories can be immensely relevant to innovation research as they have been suggested to drive the consolidation of new markets [22]. Further, they are relevant to the field of strategic management because their non-prototypical structure can explain why firms that offer relatively different products still tend to perceive one another as competitors [57]. Unfortunately, while the limitations of a prototype-centered approach have long been acknowledged in cognitive psychology [38, 39] and consumer research [58], the alternatives proposed over the years remain heavily under-represented in management-related disciplines [20].

The third theme addressed in this dissertation relates to category dynamics. Sociologists concur that the meaning of market categories can change over time to keep up with shifts in the domain the category labels are intended to map [32, 47, 59–61]. This has a twofold implication for organizational research: First, considering the informational content of classification systems as time-variant is necessary to correctly estimate the effects of categorization on competitive and strategic outcomes. In accord with this necessity, the empirical part of our research leaned toward a dynamic treatment of category properties, but we also decoupled properties that are subject to periodic change, such as complexity, from those that are relatively enduring, like the presence or the absence of a geometric center. Second, changes in classification systems represent sociologically relevant events in and of themselves, and thus they are worthy of researchers' attention. The formal part of this dissertation aimed to advance our understanding of category emergence precisely by formalizing

its sociocognitive determinants [43, 62]. While our formalism does not (yet) account for dynamic updates, it already highlights that new categories may arise from changes in agents' beliefs about the features shared by the objects in the domain—regardless of whether these beliefs are correct—and by their patterns of interaction. Firm managers are well advised to monitor these aspects in the markets or submarkets where they operate, as changes in categories can generate variance to the effects of their strategies and in the perceived value of their offerings.

The fourth and final theme concerns the utility of logical methods in organization theory. Although logical formalizations are not new to the literature on categories [63], they are hardly considered essential to the purpose of theory construction. Contrariwise, we argued that these methods are especially important at this stage of theory development, as the field is slowly but surely consolidating into a self-standing research domain [49]. Precisely at this moment, the task of formalizing divergent theories of categorization, assessing their compatibility, and evaluating their potential for future unification appears urgent. In this thesis, we laid the foundations of an epistemic-logical theory of categorization that builds on formal methods commonly used in information science [64–66]. The approach we proposed is advantageous in that it naturally accounts for categories that do not possess a prototypical structure, provides a flexible way to represent conceptual combinations [cf. 67, 68], and favors an ontological perspective on classification systems [3]. For obvious reasons, we believe this proposal addresses organization theorists' call for an "ontological turn in categories research" [69]. By providing an order-theoretic view that elegantly captures hierarchical relations, our formalism also complements the increasingly popular approach based on conceptual spaces, which is rather more geared toward prototypes and distance.

It is worth remarking once again that the cross-fertilization between logic and organization theory is not only generative for organization scholars but also for logicians, because the difficulties inherent to the formalization of social phenomena [70–72] can lead to nontrivial mathematical results. This dissertation attests to such mutual benefits: in fact, our application of Formal Concept Analysis to the study of categories in markets and its subsequent enrichment with additional relations to interpret modal operators contributed to research in logic by yielding a much more intuitive characterization of the RS-semantics of lattice-based modal logic [40]. Moreover, our effort to free this lattice-based framework from the restrictions imposed by the RS-conditions helped clarifying the link between RS-frames and more general Kripke-style structures.

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6.3. Limitations and Further Research

Some empirical and theoretical limitations arose in the course of our research that could not be resolved within the scope of this dissertation but nonetheless require further attention. To begin with, category emergence was considered in our logical formalizations but it was not studied empirically. To some extent, this is because the appearance of new categories tends to be a sparse and subtle phenomenon—indeed, classification systems would not be very useful to decision-making if they were subject to frequent or sudden revision. Still, new categories regularly emerge and consolidate over time, however slowly, and this can complicate their empirical treatment in studies using archival data. Controlling for category properties in a time-variant fashion, as we did in our empirical chapters, partly addresses this problem by allowing researchers to account for subtle changes in category meanings, but it may not be enough when studying markets where the classification system itself can be emergent or "in flux" [73]. In these contexts, extracting categories from agents' discourse [e.g., 74] can be more effective than relying on archival sources.

A second empirical limitation is that some of the arguments presented in the first half of this thesis, such as the geometric derivation of products' location in a feature space based on third-party categorization, concern cognitive mechanisms that are difficult to test directly with our data. In point of fact, auxiliary assumptions were required in Chapter 3 to derive testable hypotheses. This is not necessarily a problem as long as the assumptions are reasonable in the chosen empirical setting; however, stronger evidence could be garnered for our arguments via experimental research designs. Testing cognate hypotheses in controlled experiments thus represents a promising next step.

On the formal side, there are two main shortcoming to be addressed in future research: First, as mentioned before, our proposal to model the emergence of categories through social interaction necessitates the addition of dynamic updates [44, 46]. In this regard, further extensions of our formalism are already underway. Second, we did not yet establish a systematic connection between the order-theoretic representation of categories in FCA, which builds on the theory of lattices, and their geometric representation in conceptual spaces, which pivots on the notion of distance. We believe these two approaches are complementary insofar as they emphasize different but non-mutually exclusive aspects of categorization. More than this, we suspect they can be unified into a single, comprehensive framework [cf. 75] using techniques from correspondence theory [76]. In fact, previous studies were successful in developing distance measures

for concepts in FCA [64, 66]: the establishment of a connection, therefore, appears both possible and mathematically interesting.

Another purely theoretical direction for further research would be to examine the implications of our formal theory with the aid of computational models. For example, agent-based simulations [77] could be used to study how the attributes of classification systems, such as the number of categories they include or the number of hierarchical levels they comprise, depend on the number of objects, features, and agents, but also on the exactness of the agents' perceptions, their degree of access to reality, and their level of consensus. For practical purposes, such simulations would allow us to analyze the effects on a classification system's stability of, *inter alia*, various kinds of innovation, represented by the addition of new objects, new features, or new combinations of features, and market growth, which can be represented via the addition of new agents. Finally, it could be helpful to explore the effects wrought by increases or decreases in the relative importance of particular groups of agents, who can be taken to represent influencers or selectors [78].

Given our express interest in formal theory, and especially logic, a final reflection is in order with regard to the use of logical methods in organizational research. As noted by many previous studies in sociology [e.g., 72, 79] and management [e.g., 80], the role of pure mathematics in this field can be invaluable. However, the application and appreciation of logical tools requires training that is not normally available to social scientists. On the contrary, students of these disciplines are encouraged early on in their education to dismiss technical knowledge [81]: this is regrettable, not only because the abstraction enabled by logic contributes to the discipline's scientific rigor [82], but also because depriving social science of the vital iteration between (formal) theory and empirical inquiry can cause researchers to miss important opportunities for improvement. The contribution of this feedback loop to the generation of scientific knowledge was perhaps most effectively described by statistician George Box [83, p. 792]:

> For the theory-practice iteration to work, the scientist must be, as it were, mentally ambidextrous; fascinated equally on the one hand by possible meanings, theories, and tentative models to be induced from data and the practical reality of the real world, and on the other with the factual implications deducible from tentative theories, models, and hypotheses. [. . .]

> Mathematics artfully employed can then enable him to derive the logical consequences of his tentative hypotheses and his

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strategically selected environment will allow him to compare
these consequences with practical reality.

Such continuous iteration is crucial to the production of theory that is
"conceptually interesting, empirically generative, or practically successful"
[82, p. 118]. In fact, it is precisely the discrepancy between what a theory
predicts to be true and what empirical inquiry shows to be the case that
fuels the development of better theory and more accurate tests. Moreover,
having formalized theoretical arguments guide the design of empirical analyses helps ensuring that the evidence uncovered is relevant to the theory
in question. In this dissertation, we offered a demonstration of this dual
approach by suggesting a novel, unconventional application for advanced
mathematical formalisms to the organizational research on categories. We
hope that the "marriage" hereby proposed between logic and organization
theory piques the curiosity of logicians and organization scholars alike.

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ACKNOWLEDGEMENTS

Finishing a Ph.D. can be a lonely and grueling process, but I was lucky to embark on this journey under the wing of two supportive advisors. Over the past four years, Alessandra and Nachoem have been friends to me as well as mentors, and they have complemented one another to perfection in helping me become an independent researcher. I am grateful to Alessandra for being an endless source of inspiration to me, not only because she is supremely competent, but also because she is implacable in the face of adversity. Prone as I am to having lapses of faith, I may never have completed my dissertation without her sterling example.

I am grateful to Nachoem, instead, for being a tirelessly patient teacher. He accorded me a lot of freedom in the pursuit of my research interests, sometimes against his own judgment, yet he always remained available to help me deal with setbacks. I am thankful for his pragmatism, intelligence, and dark sense of humor; most of all, however, I am indebted to Nachoem for his relentless criticism toward my writing and lack of tolerance for poor argumentation. Entering his office to talk about my research often felt like taking a stroll into a minefield, but each time I walked away a slightly better thinker. I know clarity is not an easy quality to pass on, and I am extremely proud to have been Nachoem's apprentice.

I extend my deepest gratitude to all the members of the Applied Logic Group at the Delft University of Technology. Apostolos, Fan, Fei, Giuseppe, Sabine, Umberto, and Zhiguang: thank you for introducing me to the beauty of logic. Your patience paid off—before joining you I was barely aware of what formal logic looked like and now I find it puzzling that the whole realm of social science does not revolve around it. Thank you also for giving me the chance to present some of our work in Guangzhou. I was terrified, but I appreciated the challenge. On this note, I would like to thank Minghui Ma for accepting to be on my committee and for being an impeccable host at the Institute of Logic and Cognition of Sun Yat-sen University.

Speaking of exotic places, a sizable chunk of this dissertation was written on the terrace of a beach house in KwaZulu-Natal, South Africa, a venue we have grown accustomed to calling the *Leisure Bay Institute for Advanced Study*. I was fortunate to have this opportunity: not all Ph.D. students get to write up their research while listening to the Indian Ocean and watching

monkeys scurry around the lawn. I am forever indebted to Dion and Willem for making this possible, and for offering me shelter in Johannesburg during the dreadful Dutch winters. They are among the kindest, most welcoming people that I have ever met, and I am sincerely honored by their friendship. I am grateful that Willem could make the trip in the reverse direction in order to join my doctoral committee. I would also like to express my gratitude to Andrew Craig, Drew Moshier, Peter Jipsen, and all the other colleagues who partook in our South African adventures, sharing knowledge, jokes, recipes, and trips to national parks. So long, and thanks for all the *braai*.

During my studies, I have had the privilege of meeting many scholars in my field whom I admire, and whose work I find outstanding. Most of them were kind enough to help me with their comments, but some went out of their way to be encouraging and point out areas of improvement. In this regard, I am especially grateful to Giacomo Negro, whose research on categories shaped my way of reasoning about this topic and inspired many of the arguments presented in this dissertation. I would like him to know that, in addition to his papers, I admire his degree of self-control, of which he offered demonstration when I met him for the first time and immediately proceeded to interrogate him. He traveled all the way to Amsterdam in the middle of his holidays only to be met with an endless barrage of questions, but he withstood this ordeal with remarkable poise.

For his monumental contribution to the field and approachable nature, I am very grateful to Mike Hannan, who allowed me to visit him at Stanford and agreed to serve on my committee. Having him on board was perhaps the most rewarding moment of my Ph.D., which, in all fairness, did not really take off until I started reading Mike's publications. The past four years have taught me that, in academia, good researchers write good papers and tend to be acknowledged for it. The most inventive craft pioneering research programs and, over time, they build a following. Only the best and brightest, however, ever manage to create a school of thought, and this is usually the achievement of their lifetime. Mike is the peculiar kind of scholar who creates a school of thought, outgrows it, and moves on to create another. I suspect that few researchers of this caliber are alive at any given time and I am humbled to have my thesis reviewed by one of them.

Many others deserve credit for being kind to me and offering insightful feedback at different stages of my research, including Alex van Venrooij, Andrej Srakar, Balázs Kovács, Cameron Verhaal, Christophe van den Bulte, Filippo Wezel, Gaël Le Mens, Greta Hsu, Jerker Denrell, László Pólos, Marco Giarratana, Marilena Vecco, Min Liu, Ming Leung, Martina Montauti, Olav Sorenson, Özgecan Koçak, Stoyan Sgourev, and a number of anonymous

reviewers and unknown conference attendants. I am thankful for the opportunity I had to learn from such an impressive bunch of people and I sincerely hope to have done justice to their efforts.

While working at TU Delft, I had the pleasure of sharing my workplace with delightful colleagues at the Faculty of Technology, Policy, and Management. Behnam, Diana, Filippo, Georgy, Gert-Jan, Helle, Jelle, Jonas, Maarten, Mark, Nathalie, Peter, Taylor, and Udo: thank you for being such amiable coworkers. Special credit is due to Jan, Philip, Shannon, and Zoë for making me feel welcome since day one. Moreover, I would like to express my utmost gratitude to Jeroen van den Hoven and Neelke Doorn for accepting to be part of my committee, and to Ibo van de Poel for acting as promotor. I hope to have provided for an interesting if unusual read.

Having spent a considerable amount of time informally at the University of Amsterdam, I am also grateful to many current or former members of the Entrepreneurship and Innovation section of the Amsterdam Business School. Alex, Balazs, Bram, Frederik, Ieva, Jan, Joris, Monika, Rens, and Yuval: thank you for all the hard questions, the engaging discussions, and the drinks in between. I look forward to working with you in the years to come. Very special thanks are due to Angelo, my "partner in crime" (as he was called once by an Assistant Professor of Accounting), with whom I shared many a moment of uncertainty about our future in academia.

On a final and more personal note, I would like to thank Marcello and Valerio, my lifelong friends and fellow academics, for being a most trusted inner circle. I could not ask for better paranymphs on the day of my defense. I am also deeply thankful for Chiara, who graces me with her support despite the innumerable flaws of my character. She is a truly miraculous bundle of brilliance and charm. Further, I am grateful for the support of my parents Anna and Luciano, my brother Saverio, and all the members of my extended family. Finally, I would like to acknowledge the subtle but decisive contribution to this dissertation of my brainstorming companion Monte Carlo, a distinctively affectionate specimen of *Pantherophis emoryi* who is now terrorizing mice in heaven. His stochastic swirls served as a reminder of all the things in life and research I find hard to predict.

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