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Constructing Tree Decompositions of Graphs with Bounded Gonality^{*}

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Abstract. In this paper, we give a constructive proof of the fact that the treewidth of a graph is at most its divisorial gonality. The proof gives a polynomial time algorithm to construct a tree decomposition of width at most k , when an effective divisor of degree k that reaches all vertices is given. We also give a similar result for two related notions: stable divisorial gonality and stable gonality.

1 Introduction

In this paper, we investigate the relation between well-studied graph parameters: treewidth and divisorial gonality. In particular, we give a constructive proof that the treewidth of a graph is at most its divisorial gonality.

Treewidth is a graph parameter with a long history. Its first appearance was under the name of *dimension*, in 1972, by Bertele and Briochi [4]. It was rediscovered several times since, under different names (see e.g. [5]). Robertson and Seymour introduced the notions of *treewidth* and *tree decompositions* in their fundamental work on graph minors; these notions became the dominant terminology.

The notion of divisorial gonality finds its origin in algebraic geometry. Baker and Norine [2] developed a divisor theory on graphs in analogy with divisor theory on curves, proving a Riemann–Roch theorem for graphs. The graph analog of gonality for curves was introduced by Baker [1]. To distinguish it from other notions of gonality (which we discuss briefly in Section 5), we denote the version we study by *divisorial gonality*. Divisorial gonality can be described in terms of a chip firing game. A placement of k chips on the vertices of a graph (where

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vertices can have 0 or more chips) is called an *effective divisor* of *degree* k . Under certain rules (see Section 2), sets of vertices can *fire*, causing some of the chips to move to different vertices. The divisorial gonality of a graph is the minimum degree of an effective divisor such that for each vertex v , there is a firing sequence ending with a configuration with at least one chip at v .

The treewidth of a graph is never larger than its divisorial gonality⁴. A non-constructive proof of this fact was given by van Dobben de Bruyn and Gijswijt [10]. Their proof is based on the characterization of treewidth in terms of brambles, due to Seymour and Thomas [13]. In this paper, we give a constructive proof of the same fact. We formulate our proof in terms of a search game characterization of treewidth, but with small modifications, we can also obtain a corresponding tree decomposition. The proof also yields a polynomial time algorithm that, when given an effective divisor of degree k , constructs a search strategy with at most $k + 1$ searchers and a tree decomposition of width at most k of the input graph.

This paper is organized as follows. Some preliminaries are given in Section 2. In Section 3, we prove the main result with help of a characterization of treewidth in terms of a search game and discuss that we also can obtain a tree decomposition of width equal to the degree of a given effective divisor that reaches all vertices. An example is given in Section 4. In Section 5, we give constructive proofs that bound the treewidth of a graph in terms of two related other notions of gonality.

2 Preliminaries

2.1 Graphs

In this paper, all graphs are assumed to be finite. We allow multiple edges, but no loops. Let $G = (V, E)$ be a graph. For disjoint $U, W \subseteq V$ we denote by $E(U, W)$ the set of edges with one end in U and one end in W , and use the shorthand $\delta(U) = E(U, V \setminus U)$. The *degree* of a vertex $v \in V$ is $\deg(v) = |\delta(\{v\})|$, and given $v \in U \subseteq V$ we denote by $\text{outdeg}_U(v) = |E(\{v\}, V \setminus U)|$ the number of edges from v to $V \setminus U$. By $N(U)$ we denote the set of vertices in $V \setminus U$ that have a neighbor in U . The *Laplacian* of G is the matrix $Q(G) \in \mathbb{R}^{V \times V}$ given by

$$Q_{uv} = \begin{cases} \deg(u) & \text{if } u = v, \\ -|E(\{u\}, \{v\})| & \text{otherwise.} \end{cases}$$

2.2 Divisors and gonality

Let $G = (V, E)$ be a connected graph with Laplacian matrix $Q = Q(G)$. A *divisor* on G is an integer vector $D \in \mathbb{Z}^V$. The *degree* of D is $\deg(D) = \sum_{v \in V} D(v)$. We say that a divisor D is *effective* if $D \geq 0$, i.e., $D(v) \geq 0$ for all $v \in V$.

The divisorial gonality can be defined in a number of equivalent ways. Most intuitive is the definition in terms of a chip firing game. An effective divisor D

⁴ Conversely, graphs of treewidth 2 can have arbitrarily high divisorial gonality, which can be seen by considering ‘chains of circuits’. See for instance [7,11].

can be viewed as a chip configuration with $D(v)$ chips on vertex v . If $U \subset V$ is such that $\text{outdeg}_U(v) \leq D(v)$ for every $v \in U$ (i.e., each vertex has at least as many chips as it has edges to vertices outside U), then we say that U can be fired. If this is the case, then *firing* U means that every vertex in U gives chips to each of its neighbors outside U , one chip for every edge connecting to that neighbor. The resulting chip configuration is the divisor $D' = D - Q\mathbf{1}_U$. The assumption $\text{outdeg}_U(v) \leq D(v)$ guarantees that the number of chips on each vertex remains nonnegative, i.e. that D' is effective. Now, the divisorial gonality of a graph is the minimum number k such that there is a starting configuration (divisor) with k chips, such that for each vertex $x \in V$ there is a sequence of sets we can fire such that x receives a chip.

We now give the more formal definition, that is needed in our proofs. Two divisors D and D' are *equivalent* (notation: $D \sim D'$) if $D' = D - Qx$ for some $x \in \mathbb{Z}^V$. Note that equivalent divisors have the same degree since $Q^T \mathbf{1} = 0$. If D and D' are equivalent, then, since the null space of Q consists of all scalar multiples of $\mathbf{1}$, $D' = D - Qx$ has a unique solution $x \in \mathbb{Z}^V$ that is nonnegative and has $x_v = 0$ for at least one vertex v . We denote this x by $\text{script}(D, D')$ and write $\text{dist}(D, D') = \max\{x_v : v \in V\}$. Note that if $t = \text{dist}(D, D')$, then $\text{script}(D', D) = t\mathbf{1} - x$ and thus $\text{dist}(D', D) = \text{dist}(D, D')$. If D, D', D'' are pairwise equivalent, then we have the triangle inequality $\text{dist}(D, D'') \leq \text{dist}(D, D') + \text{dist}(D', D'')$ as $\text{script}(D, D'') = \text{script}(D, D') + \text{script}(D', D'') - c\mathbf{1}$ for some nonnegative integer c .

Let D be a divisor. If D is equivalent to an effective divisor, then we define

$$\text{rank}(D) = \max\{k \in \mathbb{Z}_{\geq 0} : D - E \text{ is equivalent to an effective divisor} \\ \text{for every effective divisor } E \text{ of degree at most } k\}.$$

If D is not equivalent to an effective divisor, we set $\text{rank}(D) = -1$. The *divisorial gonality* of a graph G is defined as

$$\text{dgon}(G) = \min\{\text{deg}(D) : \text{rank}(D) \geq 1\}.$$

In the remainder of the paper, we will only consider effective divisors. If we can go from D to D' by sequentially firing a number of subsets, then clearly $D \sim D'$. The converse is also true (part (i) of the next lemma) as was shown in [10, Lemma 1.3]. (The proof can also be found in [6].)

Lemma 1. *Let D and D' be equivalent effective divisors.*

- (i) *There is a unique increasing chain $\emptyset \subsetneq U_1 \subsetneq U_2 \subsetneq \dots \subsetneq U_t \subsetneq V$ of subsets on which we can fire in sequence to obtain D' from D . That is, setting $D_0 = D$ and $D_i = D_{i-1} - Q\mathbf{1}_{U_i}$ for $i = 1, \dots, t$ we have $D_t = D'$ and D_i is effective for all $i = 0, \dots, t$.*
- (ii) *We have $t = \text{dist}(D, D') \leq \text{deg}(D) \cdot |V|$.*

We see that the two definitions of divisorial gonality are equivalent. Lemma 1 shows that we even can require the sets of vertices that are fired to be increasing.

For a given vertex q , a divisor $D \geq 0$ is called *q-reduced* if there is no nonempty set $U \subseteq V \setminus \{q\}$ such that $D - Q\mathbf{1}_U \geq 0$.

Lemma 2 ([2, Proposition 3.1]). *Let D be an effective divisor and let q be a vertex. There is a unique q -reduced divisor equivalent to D .*

Let D be an effective divisor and let D_q be the q -reduced divisor equivalent to D . Suppose that $D \neq D_q$. By Lemma 1 we obtain D_q from D by firing on a chain of sets $U_1 \subseteq \dots \subseteq U_t$ and, conversely, we obtain D from D_q by firing on the complements of U_t, \dots, U_1 . Since D_q is q -reduced, it follows that q is in the complement of U_t , and hence $q \notin U_1$. It follows that $x = \text{script}(D, D_q)$ satisfies $x_q = 0$ and $D_q(q) \geq D(q)$. In particular, a divisor D has positive rank if and only if for every $q \in V$ the q -reduced divisor equivalent to D has at least one chip on vertex q .

Given an effective divisor D and a vertex q , Dhar's algorithm [9] finds in polynomial time a nonempty subset $U \subseteq V \setminus \{q\}$ on which we can fire, or concludes that D is q -reduced.

Algorithm 1: Dhar's burning algorithm

Input : Divisor $D \geq 0$ on G and vertex q .
Output : Nonempty subset $U \subseteq V(G) \setminus \{q\}$ s.t. $D - Q\mathbf{1}_U \geq 0$ or $U = \emptyset$ if none exists.
 $U \leftarrow V \setminus \{q\};$
while $\text{outdeg}_U(v) > D(v)$ for some $v \in U$ **do**
 $U \leftarrow U \setminus \{v\}$
end
return U

Lemma 3. *Dhar's algorithm is correct, and the output is the unique inclusion-wise maximal subset $U \subseteq V \setminus \{q\}$ that can be fired.*

Proof. The set returned by Algorithm 1 can be fired, as it satisfies the requirement $\text{outdeg}_U(v) \leq D(v)$ for every $v \in U$. To complete the proof it therefore suffices to show that U contains every subset $W \subseteq V \setminus \{q\}$ that can be fired.

Let $W \subseteq V \setminus \{q\}$ be any such subset. At the start of the algorithm $U = V \setminus \{q\}$ contains W . While $U \supseteq W$, we have $\text{outdeg}_U(v) \leq \text{outdeg}_W(v) \leq D(v)$ for any $v \in W$, so the algorithm never removes a vertex $v \in W$ from U . \square

Note: in particular, Lemma 3 shows that the output of Algorithm 1 does not depend on the order in which vertices are selected for removal.

If throughout the algorithm we keep for every vertex v the number $\text{outdeg}_U(v)$ and a list of vertices for which $\text{outdeg}_U(v) > D(v)$, then we need only $O(|E|)$ updates, and we can implement the algorithm to run in time $O(|E|)$.

Lemma 4. *Let D be an effective divisor on the graph $G = (V, E)$, let $q \in V$, and let D_q be the q -reduced divisor equivalent to D . Let U be the set returned by Dhar's algorithm when applied to D and q , and suppose that $U \neq \emptyset$. Let $D' = D - Q\mathbf{1}_U$. Then $\text{dist}(D', D_q) = \text{dist}(D, D_q) - 1$.*

Proof. Let $x = \text{script}(D, D_q)$. Since D_q is q -reduced, we have $x_q = 0$. On the other hand, since $D \neq D_q$ (as we can fire on U), the number $t = \max\{x_v : v \in V\}$ is positive. Let $W = \{v \in V : x_v = t\}$. By Lemma 1, we can fire on W , so by Lemma 3 we have $W \subseteq U$.

Let $x' = \text{script}(D', D_q)$ and let $t' = \max\{x'_v : v \in V\}$. As D_q is q -reduced, we have $x'_q = 0$. Since there is a unique nonnegative $y \in \mathbb{Z}^V$ with $y_q = 0$ and $D_q = D - Qy$, and we have $D - Qx = D_q = (D - Q\mathbf{1}_U) - Qx'$, it follows that $x = x' + \mathbf{1}_U$. Since $U \supseteq W$, it follows that $x - \mathbf{1}_W \geq x'$, and hence $t - 1 \geq t'$. We find that $\text{dist}(D', D_q) \leq \text{dist}(D, D_q) - 1$. Since $\text{dist}(D, D') = 1$, equality follows by the triangle inequality. \square

Since $\text{dist}(D, D_q) \leq \deg(D) \cdot |V(G)|$, we can find a q -reduced divisor equivalent to D using no more than $\deg(D) \cdot |V|$ applications of Dhar's algorithm.

2.3 Treewidth and tree decompositions

The notions of treewidth and tree decomposition were introduced by Robertson and Seymour [12] in their fundamental work on graph minors.

Let $G = (V, E)$ be a graph, let $T = (I, F)$ be a tree, and let $X_i \subseteq V$ be a set of vertices (called *bags*) associated to i for every node $i \in I$. The pair $(T, (X_i)_{i \in I})$ is a *tree decomposition* of G if it satisfies the following conditions:

1. $\bigcup_{i \in I} X_i = V$;
2. for all $e = vw \in E$, there is an $i \in I$ with $v, w \in X_i$;
3. for all $v \in V$, the set of nodes $I_v = \{i \in I \mid v \in X_i\}$ is connected (it induces a subtree of T).

The *width* of the tree decomposition is $\max_{i \in I} |X_i| - 1$. The *treewidth* of a G is the minimum width of a tree decomposition of G . Note that the treewidth of a multigraph is equal to the treewidth of the underlying simple graph.

There are several notions that are equivalent to treewidth. We will use a notion that is based on a Cops and Robbers game, introduced by Seymour and Thomas [13]. Here, a number of searchers need to catch a fugitive. Searchers can move from a vertex in the graph to a 'helicopter', or from a helicopter to any vertex in the graph. Between moves of searchers, the fugitive can move with infinite speed in the graph, but may not move over or to vertices with a searcher. The fugitive is captured when a searcher moves to the vertex with the fugitive, and there is no other vertex without a searcher that the fugitive can move to. The location of the fugitive is known to the searchers at all times. We say that k searchers can capture a fugitive in a graph G , if there is a strategy for k searchers on G that guarantees that the fugitive is captured. In the initial configuration, the fugitive can choose a vertex, and all searchers are in a helicopter. A search strategy is *monotone* if it is never possible for the fugitive to move to a vertex that had been unreachable before. In particular, in a monotone search strategy, there is never a path without searchers from the location of the fugitive to a vertex previously occupied by a searcher.

Theorem 1 (Seymour and Thomas [13]). *Let G be a graph and k a positive integer. The following statements are equivalent.*

1. *The treewidth of G is at most k .*
2. *$k + 1$ searchers can capture a fugitive in G .*
3. *$k + 1$ searchers can capture a fugitive in G with a monotone search strategy.*

3 Construction of a search strategy

In this section, we present a polynomial time algorithm that, given an effective divisor D of degree k as input, constructs a monotone search strategy with $k + 1$ searchers to capture the fugitive.

We start by providing a way to encode monotone search strategies. Let G be a graph. For $X \subseteq V(G)$, the vertex set of a component of $G - X$ is called an X -flap. A *position* is a pair (X, R) , where $X \subseteq V(G)$ and R is a union⁵ of X -flaps (we allow $R = \emptyset$). The set X represents the vertices occupied by searchers, and the fugitive can move freely within some X -flap contained in R (if $R = \emptyset$, then the fugitive has been captured). In a monotone search strategy, the fugitive will remain confined to R , so placing searchers on vertices other than R is of no use. Therefore, it suffices to consider three types of moves for the searchers: (a) remove searchers that are not necessary to confine the fugitive to R ; (b) add searchers to R ; (c) if R consists of more than one X -flap, restrict attention to the X -flap $R_i \subset R$ containing the fugitive. This leads us to the following definition.

Definition 1. *Let G be a graph and let k be a positive integer. A monotone search strategy (MSS) with k searchers for G is a directed tree $T = (\mathcal{P}, F)$ where \mathcal{P} is a set of positions with $|X| \leq k$ for every $(X, R) \in \mathcal{P}$, and the following hold:*

- (i) *The root of T is (\emptyset, V) .*
- (ii) *If (X, R) is a leaf of T , then $R = \emptyset$.*
- (iii) *Let (X, R) be a non-leaf of T . Then $R \neq \emptyset$ and there is a set $X' \subseteq X \cup R$ such that exactly one of the following applies:*
 - (a) *$X' \subset X$ and position (X', R) is the unique out-neighbor of (X, R) .*
 - (b) *$X' \supset X$ and position (X', R') is the unique out-neighbor of (X, R) , where $R' = R \setminus X'$.*
 - (c) *$X' = X$ and the out-neighbors of (X, R) are the positions $(X, R_1), \dots, (X, R_t)$ where $t \geq 2$ and R_1, \dots, R_t are the X -flaps contained in R .*

If condition (ii) does not necessarily hold, we say that T is a partial MSS. Note that we do not consider the root node to be a leaf even if it has degree 1.

It is clear that if T is an MSS for k searchers, then, as the name suggests, k searchers can capture the fugitive, the fugitive can never reach a vertex that it could not reach before, and a searcher is never placed on a vertex from which a searcher was previously removed.

⁵ Here we deviate from the definition of position as stated in [13] in that we allow R to consist of zero X -flaps or more than one X -flap.

Lemma 5. *Let G be a graph on n vertices and let T be a (partial) MSS with k searchers for G . Then T has no more than $n^2 + 1$ nodes.*

Proof. For any position (X, R) , define $f(X, R) = |R|(|X| + |R|)$. For any leaf node (X, R) we have $f(X, R) \geq 0$. For any non-leaf node (X, R) , the value $f(X, R)$ is at least the sum of the values of its children plus the number of children. Indeed, in case (a) and (b) we have $f(X, R) \geq f(X', R') + 1$, and in case (c) we have $f(X, R) \geq f(X, R_1) + \dots + f(X, R_k) + k$ as can be easily verified. It follows that $f(X, R)$ is an upper bound on the number of descendants of (X, R) in T . Since every non-root node is a descendant of the root, it follows that the total number of nodes is at most $1 + f(\emptyset, V) = 1 + n^2$. \square

In the construction of an MSS we will use the following lemma.

Lemma 6. *Let R be an X -flap. Let D be a positive rank effective divisor such that $X \subseteq \text{supp}(D)$ and $R \cap \text{supp}(D) = \emptyset$. Then we can find in polynomial time an effective divisor $D' \sim D$ such that $X \subseteq \text{supp}(D')$, $R \cap \text{supp}(D') = \emptyset$, and such that from D' we can fire a subset U with $U \cap R = \emptyset$ and $U \cap X \neq \emptyset$.*

Proof. Let $q \in R$. Let U be the set found by Dhar's algorithm. Since R is connected and U does not contain R , it follows that $U \cap R = \emptyset$ (otherwise $\text{outdeg}_U(r) \geq 1 > D(r)$ for some $r \in U \cap R$). If $U \cap X$ is nonempty, we set $D' = D$ and we are done. Otherwise, we set $D \leftarrow D - \mathbf{1}_U$. Then $X \subseteq \text{supp}(D)$, $R \cap \text{supp}(D) = \emptyset$ and we iterate. We must finish in no more than $\text{deg}(D) \cdot |V|$ iterations by Lemma 1 and Lemma 4. Hence, we can find the required D' and U in time $|E(G)| \cdot |V(G)| \text{deg}(D)$. \square

Construction of a monotone search strategy. Let G be a connected graph and let D be an effective divisor on G of positive rank. Let $k = \text{deg}(D)$. We will construct an MSS for $k + 1$ searchers on G . We do this by keeping a partial MSS, starting with only the root node (\emptyset, V) and an edge to the node $(X, V \setminus X)$, where $X = \text{supp}(D)$. Then, we iteratively grow T at the leaves (X, R) with $R \neq \emptyset$ until T is an MSS. At each step, we also keep, for every leaf (X, R) of T , an effective divisor $D' \sim D$ such that $X \subseteq \text{supp}(D')$ and $R \cap \text{supp}(D') = \emptyset$. We now describe the iterative procedure.

While T has a leaf (X, R) with $R \neq \emptyset$, let D' be the divisor associated to (X, R) and perform one of the following steps.

- I. If R consists of multiple X -flaps R_1, \dots, R_t , then we add nodes $(X, R_1), \dots, (X, R_t)$ as children of (X, R) and associate D' to each. Iterate.
- II. If $X' = N(R)$ is a strict subset of X , then add the node (X', R) as a child of (X, R) , associate D' to this node and iterate.
- III. The remaining case is that $N(R) = X$ and R is a single X -flap. By Lemma 6 we can find an effective divisor $D'' \sim D'$ such that $X \subseteq \text{supp}(D'')$, $R \cap \text{supp}(D'') = \emptyset$ and from D'' we can fire on a set U such that $U \cap R = \emptyset$ and $U \cap X \neq \emptyset$. We set $U \cap X = \{s_1, s_2, \dots, s_t\}$. That we can fire on U implies that

$$D''(s_i) \geq |N(s_i) \cap R| \quad \text{for } i = 1, \dots, t. \quad (1)$$

For $i = 1, \dots, t$ we define positions (X_i, R_i) and (X'_i, R_i) as follows:

$$X_i = X'_{i-1} \cup (N(s_i) \cap R), \quad R_i = R \setminus X_i, \quad \text{and} \quad X'_i = X_i \setminus \{s_i\},$$

where we set $X'_0 = X$. Using (1) and the fact that $X'_0 \subseteq \text{supp}(D'')$, it is easy to check that $|X'_i| \leq k$ and $|X_i| \leq k + 1$ for every i . Since every edge in $\delta(R)$ has at least one endpoint in every X'_i , it follows that indeed R_i is a union of X'_i -flaps (and of X_i -flaps). We add the path $(X, R) \rightarrow (X_1, R_1) \rightarrow (X'_1, R_1) \rightarrow \dots \rightarrow (X'_t, R_t)$ to T (it may happen that $(X_i, R_i) = (X'_{i-1}, R_{i-1})$ in which case we leave out one of the two). We associate $D'' - Q\mathbf{1}_U$ to the leaf (X'_t, R_t) .

By Lemma 5, we are done in at most $|V(G)|^2$ steps. This completes the construction. By combining the construction described above with that of the lemma below, we obtain Theorem 2. Note that so far only a non-constructive proof that the divisorial gonality of a graph is an upper bound for the treewidth was known [10]. See [6] for the proof of the next lemma.

Lemma 7. *Let $T' = (\mathcal{P}, F)$ be a monotone search strategy for k searchers in the connected graph G and let T be the undirected tree obtained by ignoring the orientation of edges in T' . Then $(T, \{X\}_{(X,R) \in \mathcal{P}})$ is a tree decomposition of G of width at most $k - 1$.*

Theorem 2. *There is a polynomial time algorithm that, when given a graph G and an effective divisor of degree k , finds a tree decomposition of G of width at most k .*

4 An example

We apply the constructions of the previous section to a relatively small example. Let G be the graph as in Figure 1. Let D be the divisor on G that has value 3 on vertex a and value 0 elsewhere. If we follow the construction of Section 3, we will end up with the monotone search strategy found in Figure 2. We start with the root node (X, R) with $X = \emptyset$ and $R = V$ and connect it to the node $(\text{supp}(D), V \setminus \text{supp}(D))$. The three ways of growing the tree (steps I, II, III) are indicated in the picture. The four occurrences of step III are explained below.

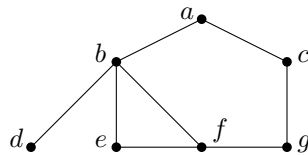


Fig. 1. An example graph G . It has divisorial gonality equal to 3.

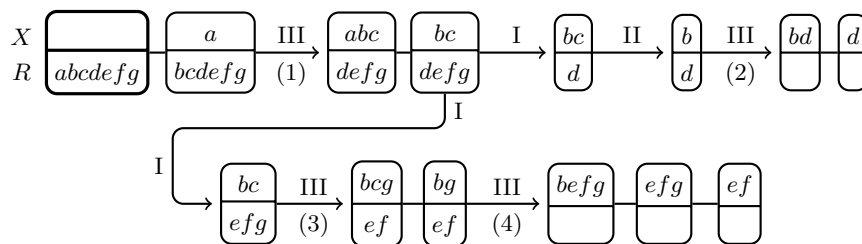


Fig. 2. The monotone search strategy obtained from G with divisor $D = 3a$. Each node shows the corresponding pair (X, R) with the root being $(\emptyset, \{a, b, c, d, e, f, g\})$. The labels I–III refer to the steps in the construction.

For compactness of notation, we write the divisors as a formal sum. For instance, if D' has 2 chips on b and 1 chip on g , we write $D' = 2b + g$.

- (1) Divisor D' is equal to $3a$. We fire the set $\{a\}$ and obtain the new divisor $a + b + c$.
- (2) Divisor D' is equal to $a + b + c$. We fire the set $\{a, b, c, e, f, g\}$ and obtain the new divisor $a + c + d$.
- (3) Divisor D' is equal to $a + b + c$. We fire the set $\{a, c\}$ and obtain the new divisor $2b + g$.
- (4) Divisor D' is equal to $2b + g$. We fire the set $\{a, b, c, d, g\}$ and obtain the new divisor $e + 2f$.

5 Other notions of gonality

5.1 Stable divisorial gonality

The *stable divisorial gonality* of a graph G is the minimum of $\text{dgon}(H)$ over all subdivisions H of G (i.e., graphs H that can be obtained by subdividing zero or more edges of G). The bound for divisorial gonality can easily be transferred to one for stable divisorial gonality. If G is simple, then the treewidth of G equals the treewidth of any of its subdivisions. (This is well known.) If G is not simple, then either the treewidth of G equals the treewidth of all its subdivisions, or G is obtained by adding parallel edges to a forest (i.e., the treewidth of G equals 1), and we subdivide at least one of these parallel edges (thus creating a graph with a cycle; the treewidth will be equal to 2 in this case.) In the latter case, the (stable) divisorial gonality will be at least two. Thus, we have the following easy corollary.

Corollary 1. *The treewidth of a graph G is at most the stable divisorial gonality of G .*

Standard treewidth techniques allow us to transform a tree decomposition of a subdivision of G into a tree decomposition of G of the same width. (For each subdivided edge $\{v, w\}$ replace each occurrence of a vertex representing a subdivision of this edge by v in each bag.)

5.2 Stable gonality

Related to (stable) divisorial gonality is the notion of *stable gonality*; see [8]. This notion is defined using finite harmonic morphisms to trees.

Let G and H be undirected nonempty graphs. We allow G and H to have parallel edges but not loops. A graph homomorphism from G to H is a map $f : V(G) \cup E(G) \rightarrow V(H) \cup E(H)$ that maps vertices to vertices, edges to edges, and preserves incidences of vertices and edges:

- $f(V(G)) \subseteq V(H)$,
- if e is an edge between vertices u and v , then $f(e)$ is an edge between $f(u)$ and $f(v)$.

A *finite morphism* from G to H (notation: $f : G \rightarrow H$) is graph homomorphism f from G to H together with an *index function* $r_f : E(G) \rightarrow \mathbb{Z}_{>0}$.

A finite morphism $f : G \rightarrow H$ with index function r_f is *harmonic* if for every vertex $v \in V(G)$, there is a constant $m_f(v)$ such that for each edge $e \in E(H)$ incident to $f(v)$, we have

$$\sum_{e' \text{ incident to } v; f(e')=e} r_f(e') = m_f(v)$$

If H is connected and $|E(G)| \geq 1$, then there is a positive integer $\deg(f)$, the *degree* of f , such that for all vertices $w \in V(H)$ and edges $e \in E(H)$, we have

$$\deg(f) = \sum_{v \in V(G); f(v)=w} m_f(v) = \sum_{e' \in E(G); f(e')=e} r_f(e');$$

see [14, Lemma 2.12] and [3, Lemma 2.3]. In particular, f is surjective in this case.

A *refinement* of a graph G is a graph G' that can be obtained from G by zero or more of the following two operations: subdivide an edge; add a leaf (i.e., add one new vertex and an edge from that vertex to an existing vertex).

The *stable gonality* of a connected non-empty graph G is the minimum degree of a finite harmonic morphism of a refinement of G to a tree.

Lemma 8. *Let G be an undirected connected graph without loops and at least one edge. Given a tree T and a finite harmonic morphism $f : G \rightarrow T$ of degree k , a tree decomposition of G of width at most k can be constructed in $O(k^2|V(G)|)$ time.*

Before proving the lemma, we make some simple observations. Recall that indices $r_f(e)$ are positive integers. We thus have for each edge $e \in E(T)$:

$$|\{e' \in E(G) \mid f(e') = e\}| \leq \sum_{e' \in E(G); f(e')=e} r_f(e') = \deg(f).$$

Since G is connected and has at least one edge, it follows that $m_f(v) \geq 1$ for every $v \in V(G)$. Hence, for each vertex $i \in V(T)$:

$$|\{v \in V(G) \mid f(v) = i\}| \leq \sum_{v \in V(G); f(v)=i} m_f(v) = \deg(f).$$

Proof (of Lemma 8). We build a tree decomposition of G in the following way. For each edge $e \in E(T)$, we have that $|\{e' \in E(G) \mid f(e') = e\}| \leq k$. Call this number $\ell(e)$. We subdivide e precisely $\ell(e)$ times; that is, we add $\ell(e)$ new vertices on this edge. Let T' be the tree that is obtained in this way.

To the nodes i of T' , we associate sets X_i in the following way. If i is a node of T (i.e., not a node resulting from the subdivisions), then $X_i = f^{-1}(i)$, i.e., all vertices mapped by the morphism to i . By the observation above, we have that $|X_i| \leq \deg(f) = k$.

Consider an edge $\{i, j\}$ in T . Write $k' = \ell(\{i, j\})$. Recall that there are $k' \leq k$ edges of G that are mapped to $\{i, j\}$. Suppose these are $e_1 = \{v_1, w_1\}, \dots, e_{k'} = \{v_{k'}, w_{k'}\}$ with $f(v_1) = f(v_2) = \dots = f(v_{k'}) = i$ and $f(w_1) = f(w_2) = \dots = f(w_{k'}) = j$. Let $i_1, i_2, \dots, i_{k'}$ be the subdivision nodes of the edge $\{i, j\}$, with i_1 incident to i and $i_{k'}$ incident to j . Set $X_{i_r} = \{v_s \mid r \leq s \leq k'\} \cup \{w_t \mid 1 \leq t \leq r\}$ for $r \in \{1, \dots, k'\}$. The construction is illustrated in Figure 3. We claim that this yields a tree decomposition of G of width at most k .

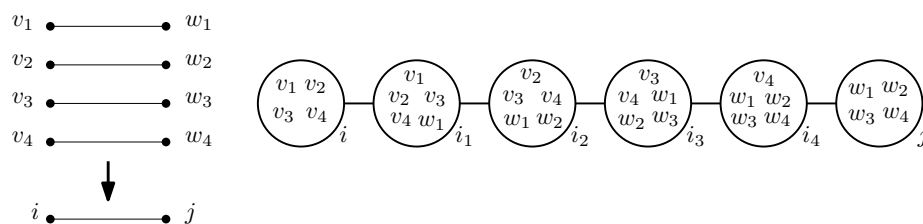


Fig. 3. Example of a step in the proof of Lemma 8. Here $k' = 4$. Left: four edges are mapped to the edge $\{i, j\}$ by the finite harmonic morphism. Right: the corresponding bags in the tree decomposition.

For all edges $\{v, w\} \in E(G)$, we have $\{f(v), f(w)\} \in E(T)$. Suppose without loss of generality that $f(v)$ has the role of i , $f(w)$ the role of j , $v = v_r$ and $w = w_r$ in the construction above. Then $v, w \in X_{i_r}$.

Finally, for all $v \in V$, the sets X_i to which v belongs are the following: v is in $X_{f(v)}$, and for each edge incident to $f(v) \in T$, v is in zero or more successive bags of subdivision nodes of this edge, with the first one (if existing), incident to $f(v)$. Thus, the bags to which v belongs form a connected subtree.

The first condition of tree decompositions follows from the second and the fact that G is connected. Hence T' , with bags as defined above, yields a tree decomposition of G .

Finally, note that each set X_i is of size at most $k + 1$: vertices in T have a bag of size k and subdivision vertices have a bag of size $k' + 1 \leq k + 1$. So, we have a tree decomposition of G of width at most k .

It is straightforward to see that the construction in the proof can be carried out in $O(k^2|V(G)|)$ time. (Use that $|V(T)| \leq |V(G)|$, since f is surjective.) \square

Theorem 3. *Let G be an undirected connected graph without loops. Suppose that G has stable gonality k . Then G has treewidth at most k . Given a refinement G' of G and a finite harmonic morphism $f : G' \rightarrow T$ of degree k , a tree decomposition of G of width at most k can be constructed in $O(k^2|V(G')|)$ time.*

Proof. The degenerate case that G has no edges must be handled separately; here we have that the treewidth of G is 0, which is equal to its stable gonality.

Suppose G has at least one edge. By Lemma 8, we obtain a tree-decomposition of G' of width k in $O(k^2|V(G')|)$ time. Standard treewidth techniques allow us to transform a tree decomposition of a refinement of G into a tree decomposition of G of the same or smaller width. Added leaves can just be removed from all bags where they occur. For each subdivided edge $\{v, w\}$, replace each occurrence of a vertex representing a subdivision of this edge by v in each bag. \square

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References

1. M. Baker. Specialization of linear systems from curves to graphs. *Algebra & Number Theory*, 2(6):613–653, 2008.
2. M. Baker and S. Norine. Riemann–Roch and Abel–Jacobi theory on a finite graph. *Advances in Mathematics*, 215(2):766–788, 2007.
3. M. Baker and S. Norine. Harmonic morphisms and hyperelliptic graphs. *International Mathematics Research Notices*, 2009(15):2914–2955, 2009.
4. U. Bertele and F. Brioschi. *Nonserial Dynamic Programming*. Academic Press, New York, 1972.
5. H. L. Bodlaender. A partial k -arboretum of graphs with bounded treewidth. *Theoretical Computer Science*, 209:1–45, 1998.
6. H. L. Bodlaender, J. van Dobben de Bruyn, D. Gijswijt, and H. Smit. Constructing tree decompositions of graphs with bounded gonality, 2020. ArXiv e-prints, 2005.05569.
7. F. Cools, J. Draisma, S. Payne, and E. Robeva. A tropical proof of the Brill–Noether theorem. *Advances in Mathematics*, 230(2):759–776, 2012.
8. G. Cornelissen, F. Kato, and J. Kool. A combinatorial Li–Yau inequality and rational points on curves. *Mathematische Annalen*, 361(1-2):211–258, 2015.
9. D. Dhar. Self-organized critical state of sandpile automaton models. *Physical Review Letters*, 64(14):1613, 1990.
10. J. van Dobben de Bruyn and D. Gijswijt. Treewidth is a lower bound on graph gonality, 2014. ArXiv e-prints, 1407.7055v2.
11. K. Hendrey. Sparse graphs of high gonality. *SIAM Journal on Discrete Mathematics*, 32(2):1400–1407, 2018.
12. N. Robertson and P. D. Seymour. Graph minors. II. Algorithmic aspects of treewidth. *Journal of Algorithms*, 7:309–322, 1986.
13. P. D. Seymour and R. Thomas. Graph searching and a minimax theorem for tree-width. *Journal of Combinatorial Theory, Series B*, 58:239–257, 1993.
14. H. Urakawa. A discrete analogue of the harmonic morphism and Green kernel comparison theorems. *Glasgow Mathematical Journal*, 42(3):319–334, 2000.